Analysis and Design of Control Laws for Advanced Driver-Assistance Systems Theory and Applications

N. Mimmo

Department of Electrical, Electronic and Information Engineering "G. Marconi"
University of Bologna
Viale del Risorgimento, 2, 40136 Bologna - ITALY

email: nicola.mimmo2@unibo.it

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Notation

This book denotes with $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ the sets of natural, real and complex numbers. The elements of $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ are represented with lowercase letters, *e.g.* $x \in \mathbb{R}$. Sets are denoted by calligraphic letters, *e.g.* $\mathcal{X} \subseteq \mathbb{R}^n$ with $n \in \mathbb{N}$. To denote vector spaces, this book adopt bold capital letters like $\mathbb{V}(\mathbb{C})$. Vectors and matrices are denoted by bold lowercase and uppercase letters, *e.g.* let n, $m \in \mathbb{N}$, then $\mathbf{x} \in \mathbb{R}^n$ denotes a vector whereas $\mathbf{X} \in \mathbb{R}^{n\times m}$ represents a matrix.

Let n matrices $\mathbf{X}_i \in \mathbb{R}^{n_i \times m}$ with n, n_i, $m \in \mathbb{N}$, the column operator $\texttt{col}(\cdot) : \mathbb{R}^{n_1\times m} \times \cdots \times \mathbb{R}^{n_n \times m} \to \mathbb{R}^{\left(\sum_{i=1}^n n_i\right)\times m}$ is defined as

$$\texttt{col}(\mathbf{X}_1, \ldots, \mathbf{X}_n) = \begin{bmatrix} \mathbf{X}_1 \\ \vdots \\ \mathbf{X}_n \end{bmatrix}.$$

Let $\mathbf{x} \in \mathbb{R}^n$ with $\mathbf{x} = \texttt{col}(x_1, \ldots, x_n)$ and $\mathbf{f} : \mathbb{R}^n \mapsto \mathbb{R}^m$, the gradient of $\mathbf{f}$ is denoted as $\nabla\mathbf{f}_{\mathbf{x}} \in \mathbb{R}^{m\times n}$ with $\nabla\mathbf{f}_{\mathbf{x}} := \begin{bmatrix} \dfrac{\partial\mathbf{f}}{\partial x_1} & \cdots & \dfrac{\partial\mathbf{f}}{\partial x_n} \end{bmatrix}$.

Let $\mathbf{x}$, $\mathbf{y}$, $\mathbf{A}$, and $\mathbf{B}$ be two vectors and two matrices of proper dimensions. Then, to reduce the notation complexity, this book denotes with $\mathbf{x}^\top\mathbf{y}$ the inner product $\langle \mathbf{x}, \mathbf{y} \rangle$, with $\mathbf{A}\mathbf{x}$ and $\mathbf{A}\mathbf{B}$ the dot products $\mathbf{A}\cdot\mathbf{x}$ and $\mathbf{A}\cdot\mathbf{B}$.

The distance of a vector $\mathbf{x} \in \mathbb{R}^n$ to a set $\mathcal{X} \subset \mathbb{R}^n$ is denoted as $\|\mathbf{x}\|_{\mathcal{X}}$ with $\|\mathbf{x}\|_{\mathcal{X}} := \inf_{\mathbf{s}\in\mathcal{X}} \|\mathbf{x} - \mathbf{s}\|$.

The lower and upper bounds of matrices are define through $\underline{\sigma}$, $\overline{\sigma} : \mathbb{R}^{n\times m} \to \mathbb{R}^+$ such as for any $\mathbf{X} \in \mathbb{R}^{n\times m}$

$$\begin{aligned} \underline{\sigma}(\mathbf{X}) &= \inf_{\mathbf{x}\in\mathbb{R}^m \,:\, \|\mathbf{x}\|=1} \frac{\|\mathbf{X}\,\mathbf{x}\|}{\|\mathbf{x}\|} \\ \overline{\sigma}(\mathbf{X}) &= \sup_{\mathbf{x}\in\mathbb{R}^m \,:\, \|\mathbf{x}\|=1} \frac{\|\mathbf{X}\,\mathbf{x}\|}{\|\mathbf{x}\|}. \end{aligned}$$

Acronyms

ABS Anti-lock Braking System. 6, 9

ACC Adaptive Cruise Control. 6, 9

ADAS Advanced Driver-Assistance Systems. 6, 9, 10, 27

AEB Automatic Emergency Braking. 9

AS Active Suspensions. 6

BIBS Bounded-Input Bounded-State. 64, 104

BSD Blind Spot Detection. 9

CC Cruise Control. 9

EPS Electric Power Steering. 6, 10

ESP Electonic Stability Program. 6, 9

FWD Forward Collision Warning. 9

GNSS Global Navigation Satellite Systems. 9, 10, 105

IMU Inertial Measurement Unit. 9, 10, 105

LC Lane Changing. 10

LDW Lane Departure Warning. 9

LK Lane Keeping. 10

LTI Linear Time Invariant. 21, 46, 48, 52, 53, 55, 56, 63, 64

MIMO Multi-Input Multi-Output. 12

MISO Multi-Input Single-Output. 12

NHTSA National Highway Traffic Safety Administration. 6–8

PID Proportional-Integral-Derivative. 18, 24–26, 28

SAE Society of Automotive Engineers. 7

SIMO Single-Input Multi-Output. 12

SISO Single-Input Single-Output. 12

SPA Self-Park Assist. 10

TC Traction Control. 9

Chapter 1

Introduction

After the description of the motivations that justify the presence of control systems onboard of modern vehicles (Section 1.1), the nomenclatures and the models required to formalise any control problem are introduced in Section 1.2. The control technique described in this textbook is based on the knowledge of a linear system generated from the nonlinear model of the plant. The technique to generate this linear system is called *linearisation* and it is described in Section 1.3.1. The outcome of the linearisation are exploited for the design of the elementary block constituting the control system architecture described in Section 1.3.2. Moreover, this control architecture is compared with the classic PIDs in Section 1.3.3 while the limitations of the application of controls via linearisation to nonlinear plants is briefly discussed in Section 1.3.4

Contents

1.1 Why ADAS?

Since their affirmation at the end of the XIX century, automobiles and motorcycles have become some of the most used means of transport. Along their glorious history, they have been improved to reach the highest levels of safety and performance.

Early vehicles were predominantly characterised by passive mechanical and hydraulic components designed to accomplish pre-defined tasks, and whose behaviour was inherently defined by their physical structure. As an example, a given vehicle equipped with purely passive mechanical/hydraulic systems, such as the suspensions, the steering, or the braking systems, are designed to perform the best in some nominal conditions. When these vehicles work off their nominal conditions the performances are reduced with, in some cases, a dangerous impact on stability and safety. To solve or at least mitigate these problems, the vehicles have been equipped with more sophisticated systems that actively react to the changed environmental conditions to keep the performance levels uniform, with remarkable benefits for stability and safety. These active systems are composed of three main elements: a sensor suite to be aware of the surrounding environment, a computational unit in which the data of the sensors are processed to elaborate corrective actions based on some pre-defined criteria, and an actuation suite necessary to implement the output of the computational unit. The set of criteria implemented into the computational unit, which represents the *logic* that manages the behaviour of the vehicle, constitutes the focus of this book and it is formally identified as the set of *control laws*.

As far as new sensors and actuators are designed, and the computational power is increased, new control algorithms can be implemented to accomplish more and more complex tasks, ranging from the stability augmentation to fully autonomous functions in which the pilot is no longer involved. This book focuses on the so called Advanced Driver-Assistance Systems (ADAS) whose main task is to aid the drivers in the execution of basic tasks, such as steering, acceleration, and braking, with the objective of keeping the performances to the their nominal values while guaranteeing the stability of the system. Examples of these kind of systems are the Anti-lock Braking System (ABS), the Adaptive Cruise Control (ACC), the Electric Power Steering (EPS), the Electonic Stability Program (ESP), the Active Suspensions (AS), and many others.

Accordingly to the National Highway Traffic Safety Administration (NHTSA), the introduction of automatic control systems could improve the safety, save lives and reduce injuries through the reduction of human errors. Moreover the automation may lead to economic and societal benefits because the reduction of accidents directly implies a reduction of costs like assurance, health-care, maintenance, etc. On the other hand, collaborative vehicles could improve the life quality by reducing the time spent in traffic congestions. Last bu not least, automated vehicles could improve the mobility of aged people or those affected by disabilities.

1.1.1 Levels of Automation & ADAS

Mainly driven by safety needs, an increasing number of automatic control systems have been developed and adopted in modern vehicles. The level of automation can range from a pure warning system that aims to provide safety information to the drives to a complete autonomous vehicle in which the driver represents just a payload. Accordingly to the NHTSA and to the Society of Automotive Engineers (SAE) vehicles can be classified within different levels of automation, see Fig. 1.1. In particular, the following clusters are defined

- **Level 0 - No Automation**: The vehicle is directly driven by the human driver. As an example, the acceleration pedal is mechanically connected to the throttle valve through a steel wire, the brake pedal is directly connected to a hydraulic cylinder that distributes the hydraulic pressure among the brake calipers, the power windows are mechanic, etc. On the other hand, all the warning and advising systems such as the lane departure warning, the parking proximity sensors, the blind spot information system, the forward-collision warning, the traffic signs recognition and many others are automatic systems embedded in vehicles belonging to this automation level.

- **Level 1 - Driver Assistance**: The vehicle is controlled by the human driver but his actions are supported by automatic devices operating only in specific conditions. Examples of control systems equipping vehicles within this automation level are the emergency braking assist, the lane keeping, the adaptive cruise control, etc. It is worth to note at this automation level the above mentioned systems do not operate concurrently.

- **Level 2 - Partial Automation**: The main difference between Level 2 and Level 1 consists in the concurrent operability of the automatic control systems. For some specific driving conditions the automatic systems can take the full control of the vehicle and the driver is tasked to monitor the environment and override/stop the automatic systems if

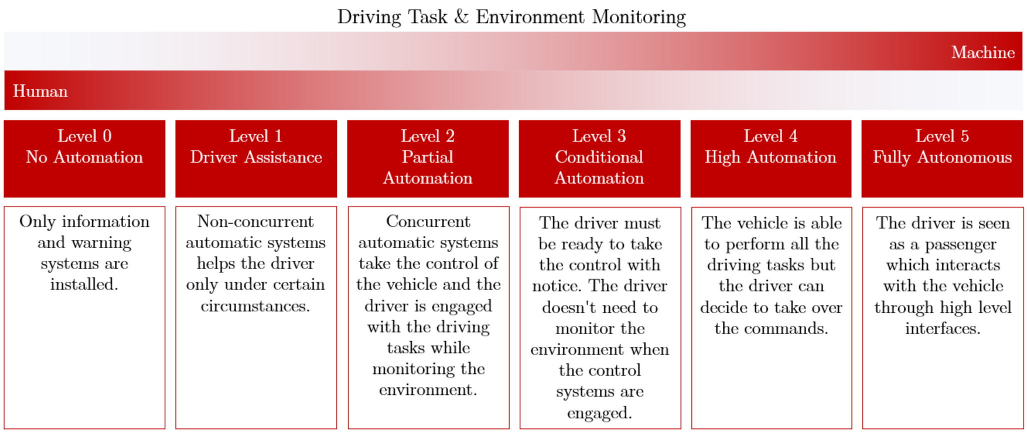


Figure 1.1: The automation levels as conceived by NHTSA.

needed. Examples of systems installed on vehicles belonging to this category are the autonomous obstacle avoidance, the autonomous parking, etc.

- **Level 3 - Conditional Automation**: Under certain circumstances the driver left to the automatic systems the complete control of the vehicle. The driver is called to be ready to take the control with notice by the automatic system. Moreover, the driver does not need to monitor the environment when the automatic systems are working. As an example, the highway chauffeur is an automatic systems for a vehicle of automation level 3.

- **Level 4 - High Automation**: In this class of vehicles the driver is mainly asked to be ready to take back the control and to drive the vehicle in all the conditions in which the automatic systems are not designed to operate. When the automatic control systems are operating the driver does not need to monitor the systems and the environment. The automatic valet parking represents an example of automatic systems featuring vehicle belonging to this category.

- **Level 5 - Full Autonomous**: At this automation level the driver represents a passenger. High level interfaces are provided to the driver which can decide, as an example, the destination and the stop-overs. It is worth to note that at this level of automation the vehicles are able to perform all the driving tasks in any environmental condition (night, fog, crowded city centres, etc.).

In the context of the automation levels described in this section, the

ADAS are systems equipping vehicles belonging vehicles from Level 0 to Level 2. More in details, this book focuses on the study and the design of the control laws of ADAS of Level 1 and Level 2. Indeed, the monitoring systems listed in Level 0 are usually designed through analytical tools (*e.g.* computer vision algorithms, if/then algorithms, etc.) different from those presented in this book.

1.1.2 Brief Historical Notes

The road for automation, started in early 1950, is progressing hand in hand with the evolution of sensors and embedded computation capabilities, see Fig. 1.2. Roughly, the first historical period from 1950 to late 1990 saw the development of systems for the stabilisation of the vehicle dynamics based on proprioceptive sensors as tone-wheels and Inertial Measurement Unit (IMU).

At the opposite from early 2000 to nowadays the focus has been placed on information and warning systems which rely on exteroceptive sensors as Global Navigation Satellite Systems (GNSS), radar, cameras, etc.

The current trends seems to suggest that, in future, vehicles will be more and more automated and cooperative and that this goal is achievable though vehicle-to-vehicle connections and infrastructures like traffic sensor networks.

In more details, it is not surprising that the first applications of automatic controls in automotive were the ABS and the TC. Indeed, these systems rely only on a couple of tone-wheels installed on the front and rear wheels respectively. On the base of the ABS and the TC, the CC came as natural consequence. These control systems represented the main driver assistance technologies from 1950 to 1990.

At early 1990, the integration of the ABS/TC control policies (specialised to modify the vehicle yaw torque) with gyroscopes lead to the development of the ESP.

The years from 2000 to 2010 saw the affirmation of the Forward Collision Warning (FWD), the Blind Spot Detection (BSD) and the Lane Departure Warning (LDW) which rely on sensors as ultrasonic and infrared sensors, radar, lidar, and vision sensors, exploited to get aware of the surrounding environment.

The development of obstacle detection systems jointly with the functionalities of the FWD and ABS were merged to create the Automatic Emergency Braking (AEB), which represented the main innvoative technology for the years 2010-2016.

From 2016 on, new technologies were introduced. Among them the ACC came as natural extension of the CC, based on an more complex use of inter-vehicle distance sensors. On the other hand, the introduction of torque-

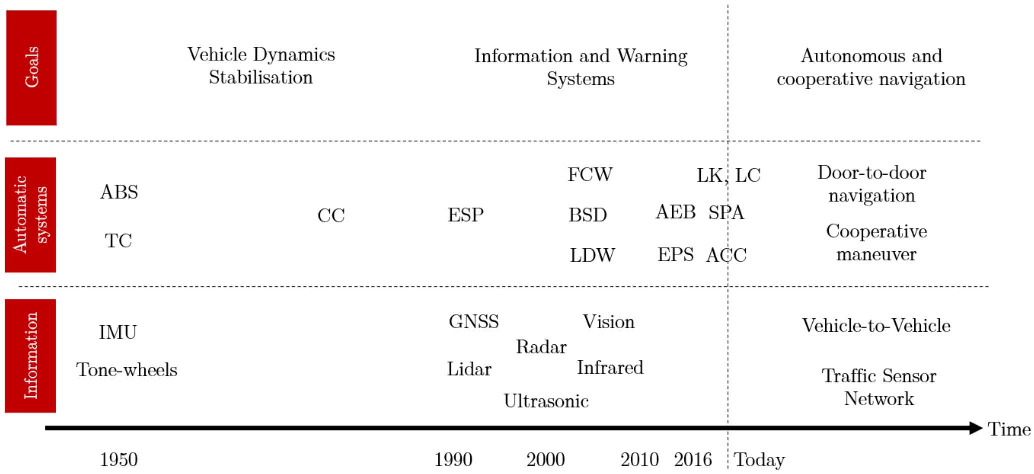


Figure 1.2: Historical evolution of automatic control system in automotive.

meters installed on the steering column supported the evolution of the EPS and so the ability to change the vehicle direction. Thus, recent years saw the affirmation of more complex control systems as the Lane Keeping (LK), the Lane Changing (LC) and the Self-Park Assist (SPA).

Future affirmation of semi-automatic/autonomous navigation systems relying on the GNSS and on the vehicle cooperation through vehicle-to-vehicle and ad-hoc network connections is plausible. Examples of future technologies could be the door-to-door automated driving and the cooperative manoeuvring.

In the context of an historical classification of ADAS (Level 1 & 2), this textbook covers both past and current control systems where, it is worth to note that, most of the ADAS from 1950 to nowadays consist in the stabilisation of a dynamic system around a nominal working point.

1.2 Control Problem Formalisation

In the context of this textbook, an automotive system is conceived as a plant composed of vehicle, driver, actuators, and sensors, see Fig. 1.3. The suite of sensors collect informations from both the plant (IMU, GNSS, etc) and the driver (cameras, steering column torque-meters, etc). These data are elaborated to create higher level information displayed to the driver through the Information & Warning Systems (info messages, lamps, steering wheels vibrations, warning sounds, etc). The driver, whose sensors are kinaesthetic perception and senses, can control the vehicle directly and/or through the automatic control systems. On the other hand, these latter elaborate control

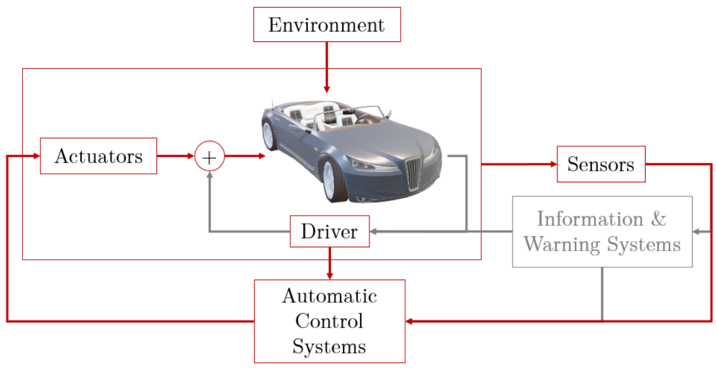


Figure 1.3: Typical control system architecture for automotive applications. The grey parts are out of the scopes of this textbook.

actions on the base of the informations provided by the sensor suite and accordingly to the driver inputs. The automatic control system is completed with dedicated actuators that, applying the control inputs, affect the vehicle dynamics. It is worth to note that for safety reasons the control inputs of the driver and those of the automatic system actuators are parallel. This let the driver to take over the control in case of control system malfunctions.

The goal of any control system is that of *modifying* the natural behaviour of the **plant** under investigation. But, what does **plant** mean in the context of control systems?

We conceive the **plant** as a system, in which we have identified two groups of signals, namely the **inputs** and the **outputs**, as described in Fig. 1.4. The plant is then thought as the link between the inputs (causes) and the outputs (effects). Moreover, the inputs can be further classified as **controls** and **exogenous signals**. The controls are manipulable input signals (that can be exploited to modify the behaviour of the system), whereas the exogenous signals cannot be governed to change the plant behaviour. Like the inputs, also the outputs can be divided in two clusters, namely **measurable outputs** (or simply outputs) and **controlled outputs**, where the first group represents the set of all the data available thanks to the sensors installed on the plant and the second denotes the set of variables we want to modify by means of the control (often the controlled outputs are a subset of the measured output).

It is worth observing that the exogenous signals do not depend on the behaviour of the system. They collect disturbances, namely $\mathbf{d}$, sensor noises, denoted with $\boldsymbol{\nu}$, as well as the reference signal the controlled output should track, identified with the symbol $\mathbf{r}$.

Let $p, r, q, m \in \mathbb{N}$, then we declare with $\mathbf{u} \in \mathbb{R}^p$ the control vector, with $\mathbf{w} \in \mathbb{R}^r$ the exogenous vector, usually defined as $\mathbf{w} = \texttt{col}(\mathbf{d}, \boldsymbol{\nu}, \mathbf{r})$, with $\mathbf{y} \in \mathbb{R}^q$ the output vector of the plant and with $\mathbf{e} \in \mathbb{R}^m$ the controlled outputs. Based on the dimensions of the input and output vectors we classify the plants as Multi-Input Multi-Output (MIMO) if $p, q > 1$ and Single-Input Single-Output (SISO) if $p = q = 1$ with MISO and SIMO representing intermediate configurations.

The plant behaviour is represented by a mathematical model, usually given by a set of ordinary differential nonlinear equations in the so-called Input-State-Output (or State Space) representation. More in detail, given the inputs $\mathbf{u}$, the exogenous $\mathbf{w}$ and the outputs $\mathbf{y}$ and $\mathbf{e}$, the plant dynamics is modelled as

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \mathbf{u}, \mathbf{w}) \qquad \mathbf{x}(t_0) = \mathbf{x}_0 \\ \mathbf{y} &= \mathbf{h}(\mathbf{x}, \mathbf{u}, \mathbf{w}) \\ \mathbf{e} &= \mathbf{h}_e(\mathbf{x}, \mathbf{u}, \mathbf{w}) \end{aligned} \tag{1.1}$$

where $\mathbf{x} \in \mathbb{R}^n$, $n \in \mathbb{N}$, is called **state** vector. As an example, ground vehicles are systems whose inputs are the throttle, the steering angle and the brake pressure and whose exogenous are the wind field, the road inclination, the reference speed, the noises affecting the sensors, etc. On the other hand, the outputs are given by the measurements of the tachometer, GNSS receiver, accelerometers, gyroscopes, magnetometers, potentiometers, engine tone-wheels, engine pressures, airflow sensors, etc. Moreover, the controlled outputs could be the vehicle speed, the turning rate, the wheel speed, the chassis vibrations, etc. Finally, the state could (it indeed depends on the mathematical representation adopted for the description of the dynamics) be represented by the inertial position, the inertial speed, the attitude with respect to the inertial space, the angular speeds of the body, the speed of the wheels, and that of the engine, etc.

Example 1.1 (The cart-pole model).
The aim of this example is to describe the identification process which leads to the definition of the plant model. To achieve this goal we need to identify the state $\mathbf{x}$*, the control* $\mathbf{u}$*, the exogenous signals* $\mathbf{w}$*, the output* $\mathbf{y}$ *as well as the functions* $\mathbf{f}(\mathbf{x}, \mathbf{u}, \mathbf{w})$*,* $\mathbf{h}(\mathbf{x}, \mathbf{u}, \mathbf{w})$*, and* $\mathbf{h}_e(\mathbf{x}, \mathbf{u}, \mathbf{w})$*. This example focuses on the model of a cart-pole which has been obtained through a Lagrangian mechanics approach, see [5]. To describe the dynamics of this plant we rely on two reference frames, the first is conceived as inertial and identified by the axes* $x_I - O_I - z_I$ *and the second is attached to the base of pole and is identified by the axes* $x_B - O_B - z_B$*. The plant represents a cart of mass* $M > 0$ *and a pole of mass* $m > 0$ *and inertia*

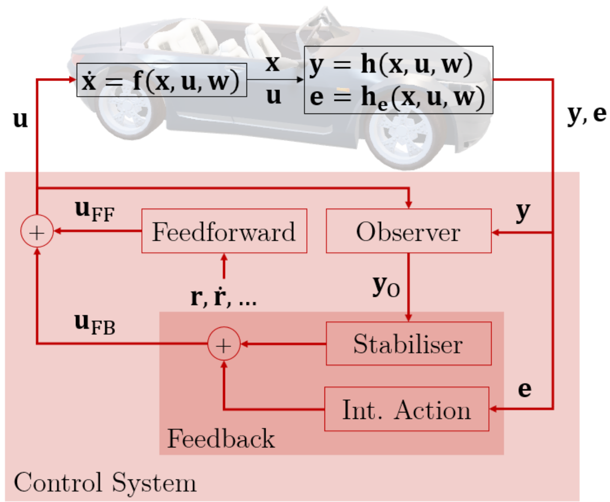
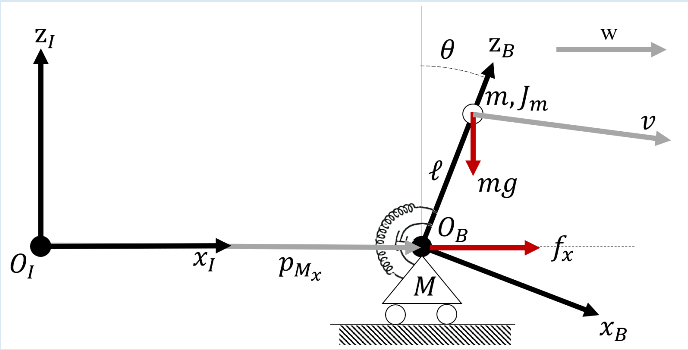


Figure 1.4: Ground vehicles as plant with inputs, disturbances, states and outputs. The control system architecture consists of three blocks. First, the output $\mathbf{y}$ and the input $\mathbf{u}$ are elaborated through an observer that provides $\mathbf{y}_O$ which represents a proxy of $\mathbf{x}$. This information and the couple ($\mathbf{y}$, $\mathbf{e}$) are passed to the feedback control which elaborates the control $\mathbf{u}_{\mathrm{FB}}$. On the other hand, the feedforward control takes as input the signal $\mathbf{y}_O$, the reference $\mathbf{r}$, and eventually its derivatives, and generates the signal $\mathbf{u}_{\mathrm{FF}}$. The two control signal are summed to create $\mathbf{u}$.

$J_m > 0$. The link between these two masses is rigid and its length is denoted by $\ell > 0$. The linear position of the cart is identified by p_{M_x} whereas the attitude of the pole with respect to the vertical axis is θ. The system is subject to the gravity acceleration g and to two external forces, one representing the aerodynamic drag acting on the pole whereas the other, namely f_x, is an horizontal force applied on the cart. The aerodynamic drag is function of the air density $\rho > 0$, the pole cross-section $S > 0$, and the drag coefficient $C_D > 0$. Finally, the horizontal wind speed is denoted by $\mathtt{w}$.



The nonlinear system

$$\begin{bmatrix} m+M & m\ell\cos\theta \\ m\ell\cos\theta & J_m + m\ell^2 \end{bmatrix} \begin{bmatrix} \ddot{p}_{M_x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} m\ell\dot{\theta}^2\sin\theta + f_x - \frac{1}{2}\rho S v C_D(\dot{p}_{M_x} - \mathtt{w} + \ell\cos\theta\dot{\theta}) \\ \ell g m\sin\theta - k\theta - \mu\dot{\theta} - \frac{1}{2}\rho S v C_D \ell(\cos\theta(\dot{p}_{M_x} - \mathtt{w}) + \ell\dot{\theta}) \end{bmatrix} \tag{1.2}$$

represents the dynamics of the cart-pole with

$$v = \sqrt{(\dot{p}_{M_x} + \cos\theta\ell\dot{\theta} - \mathtt{w})^2 + (\sin\theta\ell\dot{\theta})^2}.$$

Given the changes of variable $\dot{p}_{M_x} = v_{M_x}$ and $\dot{\theta} = \omega$, let us define the state $\mathbf{x} \in \mathbb{R}^4$ as the vector $\mathbf{x} := col(p_{M_x}, v_{M_x}, \theta, \omega)$, the control input $u \in \mathbb{R}$ as $u := f_x$. Let us assume that the system is equipped with a sensor suite composed of a exteroceptive sensor providing p_{M_x}, an odometer measuring v_{M_x} and a potentiometer providing θ. Moreover, we assume that the sensors are affected by the noise ν_p, ν_v, $\nu_\theta \in \mathbb{R}$ respectively. Let $\boldsymbol{\nu} := \mathtt{col}(\nu_p, \nu_v, \nu_\theta)$, then the output vector $\mathbf{y} \in \mathbb{R}^3$ is defined as $\mathbf{y} := col(p_{M_x}, v_{M_x}, \theta) + \boldsymbol{\nu}$. Furthermore, the regulated output is defined as $e = (p_{M_x} + \nu_p) + \ell\sin(\theta + \nu_\theta) - p_R$ where $p_{M_x} + \ell\sin\theta$ represents the *horizontal position of the pole and p_R is the reference we want to track. The exogenous $\mathbf{w} \in \mathbb{R}^4$ is defined as $\mathbf{w} := col(\mathtt{w}, \boldsymbol{\nu}, p_R)$ in which $\mathtt{w} \in \mathbb{R}$ represents the horizontal wind. Thanks to the definitions $\dot{p}_{M_x} = v_{M_x}$ and $\dot{\theta} = \omega$, Eq. (1.2) is rewritten as*

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & m+M & 0 & m\ell\cos\theta \\ 0 & 0 & 1 & 0 \\ 0 & m\ell\cos\theta & 0 & J_m + m\ell^2 \end{bmatrix} \begin{bmatrix} \dot{p}_{M_x} \\ \dot{v}_{M_x} \\ \dot{\theta} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} v_{M_x} \\ m\ell\dot{\theta}^2 \sin\theta + f_x - \frac{1}{2}\rho S v C_D(\dot{p}_{M_x} - \mathtt{w} + \ell\cos\theta\dot{\theta}) \\ \omega \\ \ell g m \sin\theta - k\theta - \mu\dot{\theta} - \frac{1}{2}\rho S v C_D \ell(\cos\theta(\dot{p}_{M_x} - \mathtt{w}) + \ell\dot{\theta}) \end{bmatrix}. \tag{1.3}$$

Define

$$\mathbf{F}(\mathbf{x}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & m+M & 0 & m\ell\cos\theta \\ 0 & 0 & 1 & 0 \\ 0 & m\ell\cos\theta & 0 & J_m + m\ell^2 \end{bmatrix}$$

and write (1.3) in the form (1.1) with

$$\mathbf{f}(\mathbf{x}, u, \mathbf{w}) := \mathbf{F}^{-1}(\mathbf{x}) \begin{bmatrix} v_{M_x} \\ m\ell\omega^2 \sin\theta + f_x - \frac{1}{2}\rho S v C_D(v_{M_x} - \mathtt{w} + \ell\cos\theta\omega) \\ \omega \\ \ell g m \sin\theta - k\theta - \mu\omega - \frac{1}{2}\rho S v C_D \ell(\cos\theta(v_{M_x} - \mathtt{w}) + \ell\omega) \end{bmatrix}$$

$$\mathbf{h}(\mathbf{x}, u, \mathbf{w}) := \begin{bmatrix} p_{M_x} + \nu_p \\ v_{M_x} + \nu_v \\ \theta + \nu_\theta \end{bmatrix},$$

$$h_e(\mathbf{x}, u, \mathbf{w}) := (p_{M_x} + \nu_p) + \ell\sin(\theta + \nu_\theta) - p_R,$$

and

$$\mathbf{F}^{-1}(\mathbf{x}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{J_m + m\ell^2}{\Delta} & 0 & -\frac{m\ell\cos\theta}{\Delta} \\ 0 & 0 & 1 & 0 \\ 0 & -\frac{m\ell\cos\theta}{\Delta} & 0 & \frac{m+M}{\Delta} \end{bmatrix}$$

with $\Delta = (J_m + m\ell^2)(m + M) - (m\ell\cos\theta)^2 > 0$.

Keynote

Let the plant be identified, *i.e.* assume that the mathematical model, the control, the exogenous, the state and the outputs are defined and the model (1.1) is available. Then, the following assumptions are made to formalise the control problem.

Assumption 1.1 (Disturbance and Reference)**.** *We assume*

1. *that the disturbance is not observable, i.e. it is not possible to reconstruct* d*. This assumption forces us to create a control system which is robust with respect to plant uncertainties;*

2. *that the disturbance is bounded. This assumption is necessary to guarantee the existence of a bounded control action able to achieve the control goals G1 and G2 described later;*

3. *to know the reference* $\mathbf{r}(t)$ *and its time derivatives up to the order* $r_{\max} > 0$*. We further assume that* $d^i/dt^i\mathbf{r}(t)$ *is continuous and bounded for all* $i = 0, \ldots, r_{\max}$*.*

Assumption 1.2 (Input Redundancy)**.** *The number of inputs is greater or equal to the number of regulated outputs, i.e.* $p \geq m$*. As described in Section 4.6, this assumption is necessary (but not sufficient) to determine a control input able to steer* $\mathbf{e}$ *to zero in presence of a time varying reference* $\mathbf{r}$*.*

Assumption 1.3 (Regulated Output Readibility)**.** *The regulated output* $\mathbf{e}$ *is readible from* $\mathbf{y}$*, i.e. there exists a surjective map* $\mathbf{E} : \mathbb{R}^q \to \mathbb{R}^m$*, with* $q \geq m$*, such that* $\mathbf{e} = \mathbf{E}(\mathbf{y})$*. Moreover, the* q *measurements are linearly independent. This assumption is necessary to achieve the control problem G2) described later.*

Then, the control system is designed to

G1) assure the existence of a non-empty compact set $\mathcal{X}_0 \subset \mathbb{R}$ such that for any $x_0 \in \mathcal{X}_0$ the state $\mathbf{x}(t)$, the regulated output $\mathbf{e}(t)$, and the control $\mathbf{u}(t)$ remains bounded for all times

G2) in case of constant disturbances and absence of measurement noises, assure $\limsup_{t\to\infty} \|\mathbf{e}(t)\| = 0$.

To achieve G1 and G2 we adopt the mixed **closed/open loop** control system architecture highlighted in the light red-filled box in Fig. 1.4. It is composed of a feedback line that provides the outputs $\mathbf{y}$ and $\mathbf{e}$ to the controller. This latter generates a bounded control action $\mathbf{u}$ that both keeps bounded $\mathbf{x}$ and asymptotically steers $\mathbf{e}$ close to zero.

The internal structure of the control system depicted in Fig. 1.4 is complex but the presence of each block, *i.e.* the observer, the feedback, and the feedforward, can be motivated as follows.

The observer represents a tool to gain knowledge, about the plant, which is more than a possible direct inversion of $\mathbf{y} = \mathbf{h}(\mathbf{x}, \mathbf{u}, \mathbf{w})$. Indeed, assuming known the plant model (1.1), the observer exploits the signals $\mathbf{u}$ and $\mathbf{y}$ to generate $\mathbf{y}_O$ which is a supplementary information related to $\mathbf{x}$.

The information extracted from the plant through the observer are fused with the current controlled output $\mathbf{e}$ via the feedback. This block aims to solve the first part of the control problem, *i.e.* that of keeping the state $\mathbf{x}$ and the error $\mathbf{e}$ bounded at any time through the stabiliser. Moreover, the presence of an integral action provides robustness to constant external disturbances.

On the other hand, the feedforward exploits the knowledge of the reference and its time derivative to generate control actions needed to compensate future variations of $\mathbf{e}$. In doing so, the feedforward doesn't rely on either the current values of $\mathbf{y}$ or the supplementary $\mathbf{y}_O$.

Roughly, the feedback and the feedforward are associated to the concepts of stability and robustness (the former) and performance (the latter). In the following, these intuitive connections are given through the description of the control actions we undertake while performing the same turn at higher and higher speeds in presence of a lateral wind. Let us now assume the ideal trajectory is known (by experience).

The correlation between the feedback and the stabilisation and robustness can be explained through the description of the task of performing the turn at low speed. In doing so, we turn the steering wheel, then we feel the acceleration and we observer the position and the direction of our vehicle. The first implicit task is that of keeping the vehicle *under control*, *i.e.* to avoid the vehicle to swerve off the road. While performing this task we compensate extra accelerations, drifts and yaw (measured by our sensory apparatus). Second, we correct wrong vehicle positions and alignments to keep the difference between the actual and the ideal trajectory close to zero. Moreover, the wind induces a side speed which leads to an incremental drift. To compensate this effect we rotate the steering wheel to create a lateral force that cancels the wind effects and eliminates the trajectory tracking error *accumulated* (*i.e.* integrated) up to that time. As result, the ideal trajectory is tracked with bounded errors whose upper values depend on our driving ability. In this experiment we act as a feedback controller which tries to keep both the state and the tracking errors bounded.

Let us now repeat the turn at higher and higher speeds. Intuitively, to keep the trajectory tracking errors confined within the same bounds we should act on the steering wheel more and more aggressively. Moreover, performing this aggressive manoeuvre relying a continuous feedback policy would need our senses and our brain to be more and more responsive. As consequence, the inherent limitations of our sensory apparatus and the finite responsiveness of our brain imply that a pure feedback policy cannot guarantee good performance at high speeds.

On the other hand, let us assume we are able to foresee the evolution of the tracking error (through a model of the phenomenon we built by experience). Then, without waiting for the evidence of the tracking error we anticipate our actions on the steering wheel to keep the tracking error at zero. With this control policy, we do not rely on the continuous check of the tracking error and we do not require the brain to elaborate feedback actions based on this information. Then, in line of principle, this control policy, named feedforward, may lead to superior performances. The drawback of the feedforward is that its reliability is directly associated to our ability to foresee the future which, commonly, is questionable.

To conclude, the presence of the feedback is necessary to assure stability and to face uncertainties (unknown vehicle mass and inertia, ground conditions, wind, etc.) but its trajectory tracking performances are constrained by the bounded ability to elaborate the output and the trajectory tracking error. On the other hand, the feedforward is necessary to guarantee high trajectory tracking performances but its efficacy is constrained by a non perfect knowledge of the process.

The design of observer, feedback, and feedforward is the subject matter of automatic controls. Among all the possible design strategies this textbook focuses on the so called *design via linearisation*. The next section specialises the control architecture of Fig. 1.4 for the design of linear observers and controllers.

1.3 Control via Linearisation

When the vehicle is required to work in a neighbourhood of a stationary operating point, the design of observer, feedback, and feedforward can be successfully achieved via linearisation. This section aims to formalise the linearisation process and to present the control architecture, specialisation of Fig. 1.4, of linear control systems. Then, this control architecture is compared with the PID, one of the most adopted industrial control systems. This section ends with some comments about the limitations of the control systems designed via linearisation.

1.3.1 Linearisation

Let the nonlinear system

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x},\, \mathbf{u},\, \mathbf{w}) \qquad \mathbf{x}(t_0) = \mathbf{x}_0 \\ \mathbf{y} &= \mathbf{h}(\mathbf{x},\, \mathbf{u},\, \mathbf{w}) \\ \mathbf{e} &= \mathbf{h}_e(\mathbf{x},\, \mathbf{u},\, \mathbf{w}) \end{aligned} \tag{1.4}$$

and define $\mathbf{x} \in \mathbb{R}^n$ as the state vector, $\mathbf{y} \in \mathbb{R}^q$ as the output vector, $\mathbf{e} \in \mathbb{R}^m$ as the controlled output vector, $\mathbf{u} \in \mathbb{R}^p$ as the control vector and $\mathbf{w} \in \mathbb{R}^r$ as the exogenous vector. This textbook assumes that $\mathbf{f} : \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^r \to \mathbb{R}^n$ and $\mathbf{h} : \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^r \to \mathbb{R}^q$ are smooth and locally Lipschitz in $\mathbf{x}$ (this latter is needed to guarantee the existence and unicity of the solutions of (1.4)). Moreover, let $\mathbf{u}^\star : \mathcal{T} \subset \mathbb{R} \to \mathbb{R}^p$ and $\mathbf{w}^\star : \mathcal{T} \subset \mathbb{R} \to \mathbb{R}^r$, and assume that there exists a unique integral curve

$$\mathbf{x}^\star : \mathcal{T} \subset \mathbb{R} \to \mathbb{R}^n \tag{1.5}$$

such that

$$\begin{aligned} \dot{\mathbf{x}}^\star &= \mathbf{f}(\mathbf{x}^\star, \mathbf{u}^\star, \mathbf{w}^\star) \qquad \mathbf{x}^\star(t_0) = \mathbf{x}_0^\star \\ \mathbf{y}^\star &= \mathbf{h}(\mathbf{x}^\star, \mathbf{u}^\star, \mathbf{w}^\star) \\ \mathbf{0} &= \mathbf{h}_e(\mathbf{x}^\star, \mathbf{u}^\star, \mathbf{w}^\star). \end{aligned} \tag{1.6}$$

Let the functions $\mathbf{u} : \mathcal{T} \subset \mathbb{R} \to \mathbb{R}^p$, $\mathbf{w} : \mathcal{T} \subset \mathbb{R} \to \mathbb{R}^r$, the state $\mathbf{x}$ and the output $\mathbf{y}$ defined in agreement with (1.4), and define the errors

$$\begin{aligned} \tilde{\mathbf{x}} &= \mathbf{x} - \mathbf{x}^\star, \quad & \tilde{\mathbf{u}} &= \mathbf{u} - \mathbf{u}^\star \\ \tilde{\mathbf{y}} &= \mathbf{y} - \mathbf{y}^\star, \quad & \tilde{\mathbf{w}} &= \mathbf{w} - \mathbf{w}^\star. \end{aligned} \tag{1.7}$$

The dynamics of the error $\tilde{\mathbf{x}}$ is given by

$$\begin{aligned} \dot{\tilde{\mathbf{x}}} &= \dot{\mathbf{x}} - \dot{\mathbf{x}}^\star = \\ &= \mathbf{f}(\mathbf{x}, \mathbf{u}, \mathbf{w}) - \mathbf{f}(\mathbf{x}^\star, \mathbf{u}^\star, \mathbf{w}^\star) = \\ &= \mathbf{f}(\mathbf{x}^\star + \tilde{\mathbf{x}}, \mathbf{u}^\star + \tilde{\mathbf{u}}, \mathbf{w}^\star + \tilde{\mathbf{w}}) - \mathbf{f}(\mathbf{x}^\star, \mathbf{u}^\star, \mathbf{w}^\star). \end{aligned} \tag{1.8}$$

Since $\mathbf{f}$ is differentiable, we can write

$$\begin{aligned}&\mathbf{f}(\mathbf{x}^\star+\tilde{\mathbf{x}},\mathbf{u}^\star+\tilde{\mathbf{u}},\mathbf{w}^\star+\tilde{\mathbf{w}}) = \\ &\quad = \mathbf{f}(\mathbf{x}^\star,\mathbf{u}^\star,\mathbf{w}^\star)+\nabla\mathbf{f}_\mathbf{x}\tilde{\mathbf{x}}+\nabla\mathbf{f}_\mathbf{u}\tilde{\mathbf{u}}+\nabla\mathbf{f}_\mathbf{w}\tilde{\mathbf{w}}+o(\|\tilde{\mathbf{x}}\|^2,\|\tilde{\mathbf{u}}\|^2,\|\tilde{\mathbf{w}}\|^2)\end{aligned} \tag{1.9}$$

The substitution of (1.9) into (1.8) leads to

$$\dot{\tilde{\mathbf{x}}} = \nabla\mathbf{f}_\mathbf{x}\tilde{\mathbf{x}}+\nabla\mathbf{f}_\mathbf{u}\tilde{\mathbf{u}}+\nabla\mathbf{f}_\mathbf{w}\tilde{\mathbf{w}}+o(\|\tilde{\mathbf{x}}\|^2,\|\tilde{\mathbf{u}}\|^2,\|\tilde{\mathbf{w}}\|^2). \tag{1.10}$$

On the other hand, the output error $\tilde{\mathbf{y}}$ is defined as

$$\tilde{\mathbf{y}} = \mathbf{h}(\mathbf{x}^\star+\tilde{\mathbf{x}},\mathbf{u}^\star+\tilde{\mathbf{u}},\mathbf{w}^\star+\tilde{\mathbf{w}})-\mathbf{h}(\mathbf{x}^\star,\mathbf{u}^\star,\mathbf{w}^\star) \tag{1.11}$$

where, as previously done for $\mathbf{f}$, also $\mathbf{h}$ can be written in a Taylor polynomial form as

$$\begin{aligned}&\mathbf{h}(\mathbf{x}^\star+\tilde{\mathbf{x}},\mathbf{u}^\star+\tilde{\mathbf{u}},\mathbf{w}^\star+\tilde{\mathbf{w}}) = \\ &\quad \mathbf{h}(\mathbf{x}^\star,\mathbf{u}^\star,\mathbf{w}^\star)+\nabla\mathbf{h}_\mathbf{x}\tilde{\mathbf{x}}+\nabla\mathbf{h}_\mathbf{u}\tilde{\mathbf{u}}+\nabla\mathbf{h}_\mathbf{w}\tilde{\mathbf{w}}+O(\|\tilde{\mathbf{x}}\|^2,\|\tilde{\mathbf{u}}\|^2,\|\tilde{\mathbf{w}}\|^2).\end{aligned} \tag{1.12}$$

The output error is then rewritten by substituting (1.12) into (1.11)

$$\tilde{\mathbf{y}} = \nabla\mathbf{h}_\mathbf{x}\tilde{\mathbf{x}}+\nabla\mathbf{h}_\mathbf{u}\tilde{\mathbf{u}}+\nabla\mathbf{h}_\mathbf{w}\tilde{\mathbf{w}}+O(\|\tilde{\mathbf{x}}\|^2,\|\tilde{\mathbf{u}}\|^2,\|\tilde{\mathbf{w}}\|^2). \tag{1.13}$$

Moreover, also the regulated output error, *i.e.* $\tilde{\mathbf{e}} = \mathbf{e}-\mathbf{0}$, can be formally expressed as

$$\tilde{\mathbf{e}} = \nabla\mathbf{h}_{e_\mathbf{x}}\tilde{\mathbf{x}}+\nabla\mathbf{h}_{e_\mathbf{u}}\tilde{\mathbf{u}}+\nabla\mathbf{h}_{e_\mathbf{w}}\tilde{\mathbf{w}}+O(\|\tilde{\mathbf{x}}\|^2,\|\tilde{\mathbf{u}}\|^2,\|\tilde{\mathbf{w}}\|^2) \tag{1.14}$$

exploiting the same strategy used to obtain (1.13). For small values of errors (1.7), the linearisation of (1.4) in the neighbourhood of $\mathbf{x}^\star$ solution of (1.6), for $t \in \mathcal{T} \subset \mathbb{R}$, is defined as

Appendix A illustrates how to deal with matrices and vectors.

$$\begin{aligned}\dot{\tilde{\mathbf{x}}} =&\ \mathbf{A}(\mathbf{x}^\star,\mathbf{u}^\star,\mathbf{w}^\star)\tilde{\mathbf{x}}+\mathbf{B}_1(\mathbf{x}^\star,\mathbf{u}^\star,\mathbf{w}^\star)\tilde{\mathbf{u}}+\mathbf{B}_2(\mathbf{x}^\star,\mathbf{u}^\star,\mathbf{w}^\star)\tilde{\mathbf{w}}\\ \tilde{\mathbf{y}} =&\ \mathbf{C}(\mathbf{x}^\star,\mathbf{u}^\star,\mathbf{w}^\star)\tilde{\mathbf{x}}+\mathbf{D}_1(\mathbf{x}^\star,\mathbf{u}^\star,\mathbf{w}^\star)\tilde{\mathbf{u}}+\mathbf{D}_2(\mathbf{x}^\star,\mathbf{u}^\star,\mathbf{w}^\star)\tilde{\mathbf{w}}\\ \mathbf{e} =&\ \mathbf{C}_e(\mathbf{x}^\star,\mathbf{u}^\star,\mathbf{w}^\star)\tilde{\mathbf{x}}+\mathbf{D}_{e_1}(\mathbf{x}^\star,\mathbf{u}^\star,\mathbf{w}^\star)\tilde{\mathbf{u}}+\mathbf{D}_{e_2}(\mathbf{x}^\star,\mathbf{u}^\star,\mathbf{w}^\star)\tilde{\mathbf{w}}\\ \tilde{\mathbf{x}}(t_0) =&\ \tilde{\mathbf{x}}_0\end{aligned} \tag{1.15}$$

where

$$\begin{aligned}\mathbf{A}(\mathbf{x}^\star,\mathbf{u}^\star,\mathbf{w}^\star) &:= \nabla\mathbf{f}_\mathbf{x}|_{\mathbf{x}^\star,\mathbf{u}^\star,\mathbf{w}^\star}\\ \mathbf{B}_1(\mathbf{x}^\star,\mathbf{u}^\star,\mathbf{w}^\star) &:= \nabla\mathbf{f}_\mathbf{u}|_{\mathbf{x}^\star,\mathbf{u}^\star,\mathbf{w}^\star}\\ \mathbf{B}_2(\mathbf{x}^\star,\mathbf{u}^\star,\mathbf{w}^\star) &:= \nabla\mathbf{f}_\mathbf{w}|_{\mathbf{x}^\star,\mathbf{u}^\star,\mathbf{w}^\star}\end{aligned}$$

$$\begin{aligned}\mathbf{C}(\mathbf{x}^\star,\mathbf{u}^\star,\mathbf{w}^\star) &:= \nabla\mathbf{h}_\mathbf{x}|_{\mathbf{x}^\star,\mathbf{u}^\star,\mathbf{w}^\star}\,, & \mathbf{C}_e(\mathbf{x}^\star,\mathbf{u}^\star,\mathbf{w}^\star) &:= \nabla\mathbf{h}_{e_\mathbf{x}}|_{\mathbf{x}^\star,\mathbf{u}^\star,\mathbf{w}^\star}\\ \mathbf{D}_1(\mathbf{x}^\star,\mathbf{u}^\star,\mathbf{w}^\star) &:= \nabla\mathbf{h}_\mathbf{u}|_{\mathbf{x}^\star,\mathbf{u}^\star,\mathbf{w}^\star}\,, & \mathbf{D}_{e_1}(\mathbf{x}^\star,\mathbf{u}^\star,\mathbf{w}^\star) &:= \nabla\mathbf{h}_{e_\mathbf{u}}|_{\mathbf{x}^\star,\mathbf{u}^\star,\mathbf{w}^\star}\\ \mathbf{D}_2(\mathbf{x}^\star,\mathbf{u}^\star,\mathbf{w}^\star) &:= \nabla\mathbf{h}_\mathbf{w}|_{\mathbf{x}^\star,\mathbf{u}^\star,\mathbf{w}^\star}\,, & \mathbf{D}_{e_2}(\mathbf{x}^\star,\mathbf{u}^\star,\mathbf{w}^\star) &:= \nabla\mathbf{h}_{e_\mathbf{w}}|_{\mathbf{x}^\star,\mathbf{u}^\star,\mathbf{w}^\star}\,.\end{aligned}$$

Remark 1.1. *It is worth noting that the matrices* $\mathbf{A}$, $\mathbf{B}_1$, *etc. appearing in* (1.15) *are time varying if at least one element of the triplet* $(\mathbf{x}^\star, \mathbf{u}^\star, \mathbf{w}^\star)$ *is time varying. Thus, the mentioned matrices are constant if and only if all the elements of the triplet* $(\mathbf{x}^\star, \mathbf{u}^\star, \mathbf{w}^\star)$ *are constant. Let us assume that there exists* $\mathbf{u}^\star \equiv \mathbf{u}_0$ *and* $\mathbf{w}^\star \equiv \mathbf{w}_0$ *such that the integral curve* $\mathbf{x}^\star \equiv \mathbf{x}_0$ *for all* $t \in \mathcal{T}$ *then*

$$\begin{aligned} \mathbf{0} &= \mathbf{f}(\mathbf{x}_0, \mathbf{u}_0, \mathbf{w}_0) \quad \mathbf{x}(t_0) = \mathbf{x}_0 \\ \mathbf{y}_0 &= \mathbf{h}(\mathbf{x}_0, \mathbf{u}_0, \mathbf{w}_0) \\ \mathbf{0} &= \mathbf{h}(\mathbf{x}_0, \mathbf{u}_0, \mathbf{w}_0) \end{aligned} \tag{1.16}$$

and the triplet $(\mathbf{x}_0, \mathbf{u}_0, \mathbf{w}_0)$ *is called* ***equilibrium triplet*** *whereas the associated system* (1.15) *is called* **Linear Time Invariant (LTI)**.

For the remaining of this book, the dependency on the triplet $(\mathbf{x}^\star, \mathbf{u}^\star, \mathbf{w}^\star)$ will be omitted and the notation of (1.15) will be shortened as follows

$$\begin{aligned} \dot{\tilde{\mathbf{x}}} &= \mathbf{A}\tilde{\mathbf{x}} + \mathbf{B}_1\tilde{\mathbf{u}} + \mathbf{B}_2\tilde{\mathbf{w}} \qquad \tilde{\mathbf{x}}(t_0) = \tilde{\mathbf{x}}_0 \\ \tilde{\mathbf{y}} &= \mathbf{C}\tilde{\mathbf{x}} + \mathbf{D}_1\tilde{\mathbf{u}} + \mathbf{D}_2\tilde{\mathbf{w}} \\ \tilde{\mathbf{e}} &= \mathbf{C}_e\tilde{\mathbf{x}} + \mathbf{D}_{e_1}\tilde{\mathbf{u}} + \mathbf{D}_{e_2}\tilde{\mathbf{w}}. \end{aligned} \tag{1.17}$$

Example 1.2 (Linearisation of the cart-pole model)**.**
Given the nonlinear system of Example 1.1, let us assume that there exist a reference initial condition $\mathbf{x}_0^\star = \texttt{col}(p_0^\star,\, v_0^\star,\, \theta_0^\star,\, 0)$ *and a* $u^\star(t) : \mathbb{R} \to \mathbb{R}$ *such that, for* $\mathbf{w}^\star \equiv \mathbf{0}$, *the following equality holds for all* $t \geq t_0$

$$\begin{bmatrix} v_0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{f}(\mathbf{x}_0^\star, u^\star, \mathbf{0}).$$

Then, there exists a reference trajectory $\mathbf{x}^\star : \mathbb{R} \to \mathbb{R}^4$ *which is defined as solution of*

$$\begin{aligned} \dot{\mathbf{x}}^\star &= \mathbf{f}(\mathbf{x}_0^\star, u^\star, \mathbf{0}) \qquad \mathbf{x}^\star(t_0) = \mathbf{x}_0^\star \\ \mathbf{y}^\star &= \mathbf{h}(\mathbf{x}^\star, u^\star, \mathbf{0}) \\ \mathbf{0} &= \mathbf{h}_e(\mathbf{x}^\star, u^\star, \mathbf{0}). \end{aligned}$$

Remark. *In this simple case we have that* $\dot{\mathbf{x}}^\star = \texttt{col}(v_0, 0, 0, 0)$ *and* $\mathbf{x}^\star(t) = \texttt{col}(p_0 + v_0(t - t_0), v_0, \theta_0, 0)$. *Note that* $\theta_0 \neq 0$ *due to the aerodynamic drag which creates a torque statically balanced by the rotational spring reaction.*

The linearised system, obtained approximating the nonlinear dynamics (1.2) *in a neighbourhood of the reference trajectory* $\mathbf{x}^\star$ *is given by* (1.15) *in which the following matrices appear*

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & A_{22} & A_{23} & A_{24} \\ 0 & 0 & 0 & 1 \\ 0 & A_{42} & A_{43} & A_{44} \end{bmatrix}, \ \mathbf{B}_1 = \mathbf{F}^{-1}(\mathbf{x}^\star) \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{B}_2 = \mathbf{F}^{-1}(\mathbf{x}^\star) \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \rho S C_D v_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \ \mathbf{D}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \ \mathbf{D}_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\mathbf{C}_e = \begin{bmatrix} 1 & 0 & \ell\cos\theta^\star & 0 \end{bmatrix}, \ D_{e1} = 0, \ D_{e2} = \begin{bmatrix} 1 & 0 & \ell\cos\theta^\star & 0 \end{bmatrix}$$

with

$$\begin{bmatrix} 1 \\ A_{22} \\ 0 \\ A_{42} \end{bmatrix} = \mathbf{F}^{-1}(\mathbf{x}^\star) \begin{bmatrix} 1 \\ -\rho S C_D v_0 \\ 0 \\ -\rho S C_D v_0 \ell\cos\theta_0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ A_{23} \\ 0 \\ A_{43} \end{bmatrix} = \mathbf{F}^{-1}(\mathbf{x}^\star) \begin{bmatrix} 0 \\ 0 \\ 0 \\ \ell m g\cos\theta_0 - k + \frac{1}{2}\rho S C_D v_0^2 \sin\theta_0 \ell \end{bmatrix} + \frac{\partial \mathbf{F}^{-1}}{\partial\theta} \begin{bmatrix} v_0 \\ 0 + u^\star - \frac{1}{2}\rho S v_0^2 C_D \\ 0 \\ \ell g m\sin\theta_0 - k\theta_0 - \frac{1}{2}\rho S v_0^2 C_D \ell\cos\theta_0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ A_{24} \\ 1 \\ A_{44} \end{bmatrix} = \mathbf{F}^{-1}(\mathbf{x}^\star) \begin{bmatrix} 0 \\ -\rho S C_D \ell\cos\theta_0 \\ 1 \\ -\mu - \frac{1}{2}\rho S C_D \ell^2 v_0 (1 + \cos^2\theta_0) \end{bmatrix}$$

$$\frac{\partial \mathbf{F}^{-1}}{\partial \theta} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{m\ell \sin\theta}{\Delta_0} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{m\ell \sin\theta}{\Delta_0} & 0 & 0 \end{bmatrix}$$
$$+ 2m\ell \cos\theta_0 \sin\theta_0 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{J_m + m\ell^2}{\Delta_0^2} & 0 & \frac{m\ell \cos\theta}{\Delta_0^2} \\ 0 & 0 & 1 & 0 \\ 0 & \frac{m\ell \cos\theta}{\Delta_0^2} & 0 & -\frac{m+M}{\Delta_0^2} \end{bmatrix}.$$

1.3.2 Control System Architecture

In the context of linear control systems, the controller is not directly designed on the model (1.4) but rather on its linear approximation (1.6)-(1.15). More in detail, the nonlinear control law $\mathbf{u}$ is decomposed as sum of the reference $\mathbf{u}^\star$ plus the linearised control $\tilde{\mathbf{u}}$, with the former designed on (1.6) and the latter designed on (1.15).

Since the controller is designed on the approximation (1.15) and in order to implement it, it is necessary to make available to the controller the same outputs $\tilde{\mathbf{y}}$ and $\tilde{\mathbf{e}}$ of the system (1.15). For this reason, the feedback line is constituted by a subtraction node which, taking the values $\mathbf{y}$, $\mathbf{e}$ and $\mathbf{y}^\star$, generates the signals $\tilde{\mathbf{y}}$ and $\tilde{\mathbf{e}}$. Then, $\tilde{\mathbf{y}}$ and $\tilde{\mathbf{e}}$ are sent to the controller to elaborate $\tilde{\mathbf{u}}$.

Moreover, in the context of linear control systems is natural to conceive the observer, the feedback, and the feedforward algorithms as further linear systems. In particular, let the observer be defined as

$$\begin{aligned} \dot{\mathbf{x}}_\mathrm{O} &= \mathbf{A}_\mathrm{O}\mathbf{x}_\mathrm{O} + \mathbf{B}_\mathrm{O}\tilde{\mathbf{u}} + \mathbf{K}_\mathrm{O}\tilde{\mathbf{y}} \\ \mathbf{y}_\mathrm{O} &= \mathbf{C}_\mathrm{O}\mathbf{x}_\mathrm{O} + \mathbf{D}_\mathrm{O}\tilde{\mathbf{y}}, \end{aligned} \tag{1.18a}$$

and let the feedback control law be

$$\mathbf{u}_\mathrm{FB} := \mathbf{u}_\mathrm{S} + \mathbf{u}_\mathrm{IA} \tag{1.18b}$$

where $\mathbf{u}_\mathrm{S}$ denotes the stabiliser law

$$\mathbf{u}_\mathrm{S} := \mathbf{K}_\mathrm{S}\mathbf{y}_\mathrm{O}. \tag{1.18c}$$

On the other hand, $\mathbf{u}_\mathrm{IA}$ defines the integral action, which represents the output of the following dynamic system

$$\begin{aligned} \dot{\boldsymbol{\eta}} &= \mathbf{e} \\ \mathbf{u}_\mathrm{IA} &= \mathbf{K}_\mathrm{I}\boldsymbol{\eta}. \end{aligned} \tag{1.18d}$$

Let the feedforward control law be defined as

$$\begin{aligned}\dot{\mathbf{x}}_{\mathrm{FF}} &= \mathbf{A}_{\mathrm{FF}}\mathbf{x}_{\mathrm{FF}} + \sum_{i}^{r_{\max}} \mathbf{B}_{\mathrm{FF}_i}\frac{d^i}{dt^i}\mathbf{r} \\ \mathbf{u}_{\mathrm{FF}} &= \mathbf{C}_{\mathrm{FF}}\mathbf{x}_{\mathrm{FF}} + \sum_{i}^{r_{\max}} \mathbf{D}_{\mathrm{FF}_i}\frac{d^i}{dt^i}\mathbf{r}\end{aligned} \tag{1.18e}$$

for some finite $r_{\max} \in \mathbb{N}$.

The architecture of a control system designed via linearisation is depicted in Fig. 1.5 where the matrices constituting the observer, the feedback, and the feedforward are designed exploiting $\mathbf{A}, \mathbf{B}_1, \ldots$ of (1.15). The rationale behind (1.18) will be clear in next section where a comparison with classic PIDs is provided.

Keynote

A straight implementation of the control law (1.18c) would apparently lead to an algebraic loop. Indeed, $\mathbf{u}_{\mathrm{S}}$ depends on $\tilde{\mathbf{y}}$ which, in turn, depends on $\mathbf{u}_{\mathrm{S}}$. This algebraic loop is broken if either $\mathbf{D}_1 = \mathbf{0}$ or $\mathbf{D}_{\mathrm{O}} = \mathbf{0}$. The matrix $\mathbf{D}_{\mathrm{O}}$ has been introduced to compare (1.18) with the classic PIDs (see Section 1.3.3) but it is not necessary to achieve the control goals G1 and G2. As will be detailed in Section 4.4, the matrix $\mathbf{D}_{\mathrm{O}} \neq \mathbf{0}$ is introduced when $\mathbf{D}_1 = \mathbf{0}$ to reduce the size of $\mathbf{x}_{\mathrm{O}}$.

1.3.3 Comparison with Classic PIDs

The Proportional-Integral-Derivative (PID) controller is widespread in industrial applications. It is composed of a parallel of three control actions, the proportional, the integral and the derivate, all of them elaborating the regulated output, arranged as in Fig. 1.6.

The classic time domain formulation of a PID controller is given by

$$\dot{\mathbf{w}} = \mathbf{A}_{\mathrm{O}}\mathbf{w} + \mathbf{K}_{\mathrm{O}}\dot{\mathbf{e}} \tag{1.19a}$$

$$\mathbf{u}_{\mathrm{PID}} = \mathbf{K}_{\mathrm{P}}\mathbf{e} + \mathbf{K}_{\mathrm{D}}\mathbf{w} + \mathbf{K}_{\mathrm{I}}\int_0^t \mathbf{e}(\tau)d\tau + \mathbf{u}^\star \tag{1.19b}$$

where is it worth to note that $\mathbf{w}$ represents the output of dynamic system necessary to let the control system to be causal. Moreover, $\mathbf{u}^\star$ denotes the initial condition of the integral action. To compare the PID control architecture with that proposed in Fig. 1.5 note that $\mathbf{e} := \mathbf{r} - \mathbf{y}$, assume $\mathbf{r}^\star = \mathbf{y}^\star$, then subtract $\mathbf{r}^\star - \mathbf{y}^\star$ to $\mathbf{e}$ to obtain

$$\mathbf{e} = \mathbf{r} - \mathbf{y} = \tilde{\mathbf{r}} - \tilde{\mathbf{y}}$$

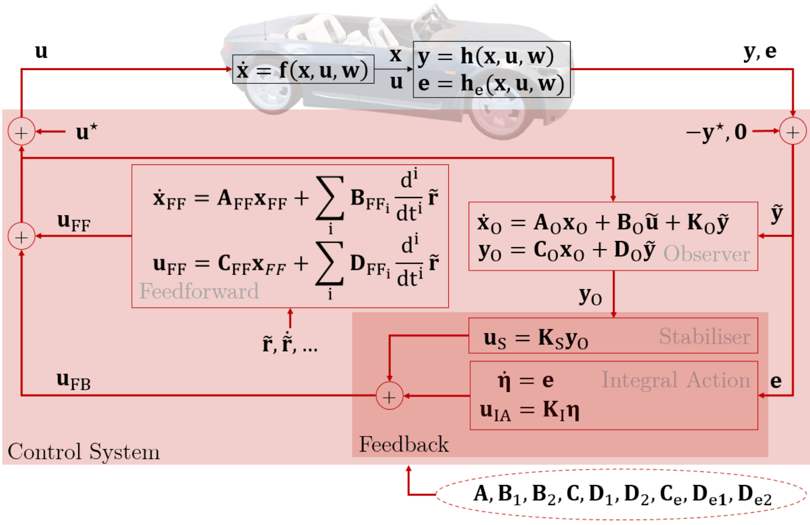


Figure 1.5: This book deals with control systems based on the linearised model of the plant. The control system architecture is composed of sum and subtraction nodes, reference signals and linear dynamic systems.

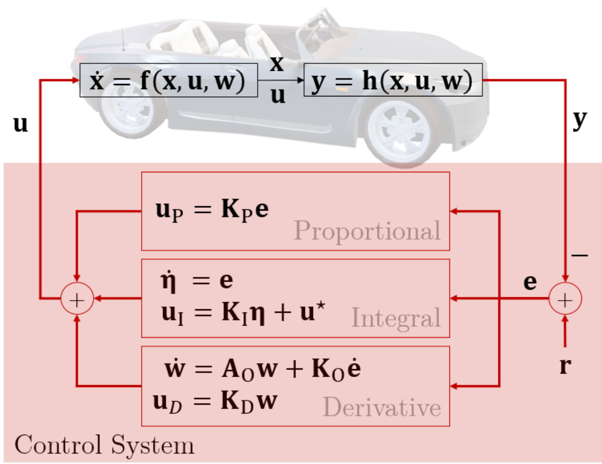


Figure 1.6: Architecture of a basic PID. The control actions is the sum of three parallel lines, the proportional, the derivative and the integral.

where $\tilde{\mathbf{r}} := \mathbf{r} - \mathbf{r}^\star$ and $\tilde{\mathbf{y}} := \mathbf{y} - \mathbf{y}^\star$. Let $\mathbf{w} := \mathbf{x}_{\mathrm{FF}} - \mathbf{x}_{\mathrm{O}}$ and exploit the superposition principle to rewrite the dynamics of $\mathbf{w}$ as

$$\begin{aligned} \dot{\mathbf{x}}_{\mathrm{O}} &= \mathbf{A}_{\mathrm{O}}\mathbf{x}_{\mathrm{O}} + \mathbf{K}_{\mathrm{O}}\dot{\tilde{\mathbf{y}}} \\ \dot{\mathbf{x}}_{\mathrm{FF}} &= \mathbf{A}_{\mathrm{O}}\mathbf{x}_{\mathrm{FF}} + \mathbf{K}_{\mathrm{O}}\dot{\tilde{\mathbf{r}}}. \end{aligned} \tag{1.20a}$$

Define

$$\dot{\boldsymbol{\eta}} = \mathbf{e}, \tag{1.20b}$$

let $\mathbf{K}_{\mathrm{S}} := [-\mathbf{K}_{\mathrm{P}} \;\; -\mathbf{K}_{\mathrm{D}}]$, $\mathbf{y}_O := \mathtt{col}(\tilde{\mathbf{y}}, \mathbf{x}_{\mathrm{O}})$, $\mathbf{C}_O := \mathtt{col}(\mathbf{0}, \mathbf{I})$, and $\mathbf{D}_O := \mathtt{col}(\mathbf{I}, \mathbf{0})$. Substitute all these terms in (1.19) to obtain

$$\mathbf{u}_{\mathrm{PID}} = \mathbf{u}_{\mathrm{FB}} + \mathbf{u}_{\mathrm{FF}} \tag{1.20c}$$

where

$$\begin{aligned} \mathbf{u}_{\mathrm{FF}} &:= \mathbf{K}_{\mathrm{D}}\mathbf{x}_{\mathrm{FF}} + \mathbf{K}_{\mathrm{P}}\tilde{\mathbf{r}} \\ \mathbf{u}_{\mathrm{FB}} &:= \mathbf{K}_{\mathrm{S}}\mathbf{y}_O + \mathbf{K}_{\mathrm{I}}\boldsymbol{\eta} \end{aligned} \tag{1.20d}$$

The architecture of (1.20), depicted in Fig. 1.7, quasi-perfectly matches that proposed in Fig. 1.5 with three remarkable differences

1. if $\mathbf{y}$ depends on $\mathbf{u}$ the classic PID cannot be implemented due to the algebraic loop created by $\mathbf{u}_{\mathrm{P}}$. At the opposite, the observer (1.18a) faces this issue by imposing $\mathbf{D}_{\mathrm{O}} = \mathbf{0}$;

2. to implement (1.20a) the derivative of $\tilde{\mathbf{y}}$ is needed. This is usually achieved via numerical approximations plus noise suppression filters. For this reason, the block above the feedback is not an observer in the sense of (1.18a). At the opposite, (1.18a) only relies on the signal $\tilde{\mathbf{y}}$;

3. the PID adopts only the first time derivative of the reference signal $\mathbf{r}$. As described in Section 4.6, the perfect tracking of a known trajectory may require higher order derivatives of $\mathbf{r}$, as foreseen by the feedforward (1.18e);

4. the dynamics (1.20a), mimicking (1.19a), are driven by the need of making the PID causal. At the opposite, the dynamics of the feedforward (1.18e) and the observer (1.18a) are generic and, as detailed in Section 4, are designed with a different perspective.

In conclusion, we can conceive the controller (1.18) as a generalisation of the classic PID without the issues associated to the derivative of $\mathbf{y}$. Moreover, looking at the PID from the perspective of Fig. 1.7 allows to identify the sub-parts responsible for the stability, those that make the control system robust and those providing performance.

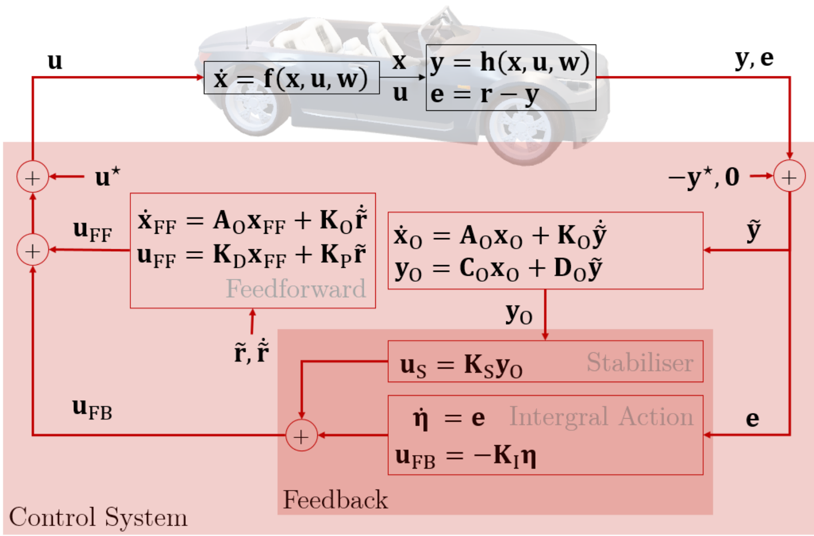


Figure 1.7: Architecture of a PID after some manipulations. This control structure looks like that of Fig. 1.5.

1.3.4 Limitations

Despite their simple structure and an attractive set of design tools, the control via linearisation (PIDs included) are inherently limited by their local nature. More in details, since they are tuned on a nominal design point, their performances may rapidly deteriorate in severe off-design conditions.

As described in Section 4.5.3, when the control via linearisation is applied to nonlinear plants the boundedness of state and control can be guaranteed only if the initial conditions and the exogenous belong to sufficiently small neighbourhoods of the design point. More in details, if the plant is characterised by linear saturated input and linear output, Section 4.5.3 demonstrates that for any linearisation point and any linearised controller, there exists sufficiently small sets of initial conditions and exogenous such that the trajectories of the nonlinear plant are confined in a bounded set. The maximisation of the sets of initial conditions and exogenous that guarantees bounded evolutions of the closed loop nonlinear system is not trivial and it results in a compromise between saturation prevention and the need to cope with plant nonlinearities. Moreover, as depicted in Section 4.5.2, even for linear plants without input saturations the asymptotic bounds on state and regulated output cannot be reduced by increasing the magnitude of the controller gains due to the measurement noise amplification.

In conclusion, the major limitation of linearised controls consists in the impossibility to arbitrary enlarge the domain of initial conditions and exogenous with no compromises between disturbance attenuation, nonlinearities ruling, actuator saturation prevention and noise magnification.

1.4 Summary

The benefits associated with the use of automatic controls in ground vehicles have been briefly recalled in this chapter. Undeniable advantages are the safety improvement, the pollution reduction, the mobility increase and a better inclusion.

Moreover, the control systems representing the target of this book have been identified among sets of automatic algorithms that make a vehicle autonomous. The six levels of automation have been reviewed and the so called ADAS have been identified as some of the control systems of the second and third automation level.

Then, the basic ingredients of any automotive control system architecture have been introduced and the control problem has been defined formally. The control system architecture consists of sensors, computational units, interfaces with the driver and actuators. The plant (actuators+vehicle+sensors) is modelled as a generic nonlinear dynamic system whose state should be guaranteed to be bounded through a bounded control action despite the presence of bounded measurement noises and environmental disturbances. Moreover, the regulated output should be asymptotically vanishing in case of constant disturbances.

A solution to the stated control problem, consisting in a linear dynamic system, has been proposed. The process of obtaining the design plant from the nonlinear system, called linearisation, has been described. The internal architecture of controls via linearisation consists of an observer, a static feedback, and an integral action. Moreover, a feedforward control is added to exploit the knowledge of the reference to be tracked.

The limitations of the proposed solution and its similarities with standard PIDs have been described. The time-domain formulation of PID has been rearranged to identify the same components of a control via linearisation with two remarkable differences: the use of derivative of the output and the absence of higher order derivatives of the reference signals. Then, the limitations of linearised control systems applied to nonlinear plants have been presented.

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Chapter 3

LTI System analysis

This chapter introduces the study of the trajectories of linear systems. In particular, the boundedness of trajectories is linked to the eigenvalues of the matrix **A**. More in detail, Section 3.1 introduces an instrumental mathematical tool called *Jordan canonical form*. This canonical form is then exploited in Section 3.2 to compute the solution of set of ordinary differential linear equations. These solutions are investigated in Section 3.2 where a link with the eigenvalues is formalised. In view of the control goal G1, this link is exploited in Section 3.3 to state and understand the stability criterion which assures the boundedness of trajectories. To conclude, these theoretical tools are exploited to investigate the linearised plants introduced in Chapter 2.

Contents

3.1 Jordan Canonical Form

This section presents the design of a change of coordinate suitable to study the behaviour of the state of LTI systems. The original system is then transformed into the so called *Jordan Canonical Form*. To achieve this result we need two ingredients, *i.e.* a change of coordinates and a design criteria. The design criteria is based on the knowledge of the eigenvectors, whose computation is detailed in Section 3.1.2. The change of coordinates is defined in Section 3.1.1 and in Section 3.1.3 is specialised for the study of the dynamics of LTI systems.

3.1.1 Change of Coordinates

Let $\mathbb{V}(\mathbb{C})$ be an *n-dimensional* vector space with $n \in \mathbb{N}$, see Appendix A. Then, a basis of $\mathbb{V}(\mathbb{C})$ is a finite set of vectors $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, $\mathbf{b}_i \in \mathbb{V}(\mathbb{C})$ for $i = 1, \ldots, n$, such that any vector $\mathbf{v} \in \mathbb{V}(\mathbb{C})$ can be represented as linear combination of $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, *i.e.* there exists n constants $\beta_i \in \mathbb{C}$ such that

$$\mathbf{v} = \begin{bmatrix} \mathbf{b}_1 & \cdots & \mathbf{b}_n \end{bmatrix} \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix}$$

where, β_i represents the i-th component of $\mathbf{v}$ on $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$.

Let us take a second basis $\{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$, such that the vector $\mathbf{v} \in \mathbb{V}(\mathbb{C})$ has components

$$\mathbf{u} = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix} \text{ on } \{\mathbf{b}_1, \ldots, \mathbf{b}_n\} \text{ and } \mathbf{w} = \begin{bmatrix} \gamma_1 \\ \vdots \\ \gamma_n \end{bmatrix} \text{ on } \{\mathbf{c}_1, \ldots, \mathbf{c}_n\}.$$

The vectors $\mathbf{u}$ and $\mathbf{w}$ are related through a linear function $\mathbf{T} : \mathbb{V}(\mathbb{C}) \to \mathbb{V}(\mathbb{R})$ called change of coordinates. It is represented by the matrix $\mathbf{T} \in \mathbb{C}^{n \times n}$ such that

$$\mathbf{u} = \mathbf{T}\,\mathbf{w}$$

where the columns of $\mathbf{T}$ represent the components of vectors $\{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ on the basis $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$. Since the representation of a vector on a given basis is unique, the change of coordinates represents a bijection and so it is invertible and such that

$$\mathbf{w} = \mathbf{T}^{-1}\mathbf{u}.$$

Let $\mathbf{A} : \mathbb{V}(\mathbb{C}) \to \mathbb{V}(\mathbb{C})$ be a linear function represented by $\mathbf{A} \in \mathbb{C}^{n\times n}$. Let us take two vectors $\mathbf{x}$, $\mathbf{y} \in \mathbb{V}(\mathbb{C})$, both defined on the basis $\{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ and such that

$$\mathbf{y} = \mathbf{A}\mathbf{x}. \tag{3.1}$$

Then, let

$$\boldsymbol{\chi} := \mathbf{T}\mathbf{x} \qquad \boldsymbol{\mu} := \mathbf{T}\mathbf{y},$$

pre-multiply both sides of (3.1) by $\mathbf{T}$, and exploit $\mathbf{x} = \mathbf{T}^{-1}\boldsymbol{\chi}$ to obtain

$$\boldsymbol{\mu} = \bar{\mathbf{A}}\boldsymbol{\chi}$$

where $\bar{\mathbf{A}} := \mathbf{T}\mathbf{A}\mathbf{T}^{-1}$ represents the linear function $\mathbf{A}$ described on the basis $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$. This result is directly exploited in Section 3.1.3 to build the Jordan canonical form.

We now introduce instrumental details for the results proposed in Chapter 4.

Let $\mathbb{S}(\mathbb{C}) \subseteq \mathbb{V}(\mathbb{C})$ be a p-dimensional subspace of $\mathbb{V}(\mathbb{C})$ and let $\{\mathbf{s}_1, \ldots, \mathbf{s}_p\}$ be a basis for $\mathbb{S}(\mathbb{C})$. Then, the orthogonal complement of $\mathbb{S}(\mathbb{C})$, denoted with $\mathbb{S}^{\perp}(\mathbb{C})$, is defined as a subspace of $\mathbb{V}(\mathbb{C})$ such that

$$\mathbb{V}(\mathbb{C}) = \mathbb{S}(\mathbb{C}) \oplus \mathbb{S}^{\perp}(\mathbb{C}).$$

Remark 3.1. *The direct sum $\oplus$ means that $\mathbb{S}(\mathbb{C}) \cup \mathbb{S}^{\perp}(\mathbb{C}) = \mathbb{V}(\mathbb{C})$ and $\mathbb{S}(\mathbb{C}) \cap \mathbb{S}^{\perp}(\mathbb{C}) = \{\emptyset\}$, or equivalently that $\mathbb{S}(\mathbb{C})$ and $\mathbb{S}^{\perp}(\mathbb{C})$ do not share common elements but their union corresponds to $\mathbb{V}(\mathbb{C})$.*

Moreover, let $\{\mathbf{s}_1^{\star}, \ldots, \mathbf{s}_q^{\star}\}$, with $q = n-p$, be a basis for $\mathbb{S}^{\perp}(\mathbb{C})$. Then, the composition of the bases associated to $\mathbb{S}(\mathbb{C})$ and $\mathbb{S}^{\perp}(\mathbb{C})$ constitutes a basis for $\mathbb{V}(\mathbb{C})$, *i.e.* the set $\{\mathbf{s}_1, \ldots, \mathbf{s}_p, \mathbf{s}_1^{\star}, \ldots, \mathbf{s}_q^{\star}\}$ represents a basis for $\mathbb{V}(\mathbb{C})$.

Let $\mathbb{X}(\mathbb{C})$ and $\mathbb{Y}(\mathbb{C})$ be two linear vectorial spaces and let $\mathbf{A} : \mathbb{X} \to \mathbb{Y}$ be a linear function. Then, the *range* of $\mathbf{A}$ is the subspace of $\mathcal{Y}$ defined as

$$\mathrm{im}(\mathbf{A}) = \{\mathbf{y} \in \mathbb{Y} \,:\, \mathbf{y} = \mathbf{A}\mathbf{x},\, \mathbf{x} \in \mathbb{X}\}$$

with $\mathrm{im}(\mathbf{A}) \subseteq \mathcal{Y}$. The *kernel* of $\mathbf{A}$ is the subspace of $\mathbb{X}$ defined as

$$\ker(\mathbf{A}) = \{\mathbf{x} \in \mathbb{X} \,:\, \mathbf{0} = \mathbf{A}\mathbf{x}\}$$

with $\ker N(\mathbf{A}) \subseteq \mathbb{X}$.

The kernel and the range are linked through $\ker(\mathbf{A}) = \left(\mathrm{im}(\mathbf{A}^\top)\right)^\perp$, where the symbol $(\cdot)^\perp$ denotes the *orthogonal complement*. Finally, the following equalities hold true

$$\begin{aligned}\mathcal{X} &= \ker(\mathbf{A}) \oplus (\ker(\mathbf{A}))^\perp = \ker(\mathbf{A}) \oplus \mathrm{im}(\mathbf{A}^\top)\\ \mathcal{Y} &= \mathrm{im}(\mathbf{A}) \oplus (\mathrm{im}(\mathbf{A}))^\perp = \mathrm{im}(\mathbf{A}) \oplus \ker(\mathbf{A}^\top).\end{aligned}$$

3.1.2 Eigenvalues and Eigenvectors

Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ be a square matrix and let

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \tag{3.2}$$

be a LTI system. We are interested in finding all the non trivial solutions of (3.2) for which there exists $\lambda \in \mathbb{C}$ such that

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} = \lambda\mathbf{x}, \tag{3.3}$$

i.e. we look for vectors whose direction does not change over time. To solve this problem, we rearrange the most right equality of (3.3) as

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0} \tag{3.4}$$

The definition of the *kernel* is provided in Appendix A.

where the non trivial solutions are those that belong to the kernel of $\mathbf{A} - \lambda\mathbf{I}$. As consequence, we first need to find the values of λ that make $\mathbf{A}-\lambda\mathbf{I}$ singular (*i.e.* those making the kernel of $\mathbf{A}-\lambda\mathbf{I}$ non trivial). These particular λ, called **eigenvalues of A**, are found as solutions of

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

where $\det(\mathbf{A} - \lambda\mathbf{I})$ is a polynomial in λ and it is called **characteristic polynomial**. Let λ_i be the i-th eigenvalue, then the **algebraic multiplicity** is defined as the order of the root λ_i and it is denoted by a_i. Thanks to this definition, and considering p distinct eigenvalues, the characteristic polynomial can be also written as

$$\det(\lambda\mathbf{I} - \mathbf{A}) = \prod_{i=1}^{p} (\lambda - \lambda_i)^{a_i}$$

with $\sum_{i=1}^{p} a_i = n$.

Remark 3.2. *In the special case of block triangular matrices the eigenvalues λ_i correspond to the eigenvalues of the matrices on the main diagonal. This result is used in Chapter 4.*

$$
\det(\mathbf{A}-\lambda\mathbf{I})=\mathbf{0}\begin{cases}\lambda_1\\ \vdots\\ \lambda_i\begin{cases}\mathbf{v}_{i,1;1} & \cdots & \mathbf{v}_{i,1;q_{i,1}}\\ \vdots & & \\ \mathbf{v}_{i,g_i;1} & \cdots & \mathbf{v}_{i,g_i;q_{i,g_i}}\end{cases}\\ \vdots\\ \lambda_p\end{cases}
$$

Figure 3.1: Organisation of eigenvectors of a matrix. First, the p distinct eigenvalues are defined. Second, g_i eigenvectors are associated to each eigenvalue. Third, a chain of eigenvectors of length $q_{i,j}$ is associated to each eigenvector $\mathbf{v}_{i,j,1}$.

The **eigenvectors** associated to each root λ_i are defined as the vectors $\mathbf{v}$ satisfying the following identity

$$(\mathbf{A}-\lambda_i\mathbf{I})^{a_i}\mathbf{v}=\mathbf{0}. \tag{3.5}$$

The dimension of the kernel of $\mathbf{A}-\lambda_i\mathbf{I}$ is defined as **geometric multiplicity** and it is denoted by g_i, with $1 \le g_i \le a_i$. To find the eigenvectors associated to $\mathbf{A}$ we rewrite (3.5) through a recursive definition. More in details, let $\mathbf{v}_{i,j,0} := \mathbf{0}$ and $\mathbf{v}_{i,j,k-1} := (\mathbf{A}-\lambda_i\mathbf{I})\mathbf{v}_{i,j,k}$ for $i = 1, \dots, p$, $j = 1, \dots, g_i$, and $k = 1, \dots, q_{i,j}$ with $1 \le q_{i,j} \le a_i$ such that $\sum_{j=1}^{g_i} q_{i,j} = a_i$. Then (3.5) is equivalent to

$$
\begin{aligned}
(\mathbf{A}-\lambda_i\mathbf{I})\mathbf{v}_{i,j,1} &= \mathbf{0}\\
(\mathbf{A}-\lambda_i\mathbf{I})\mathbf{v}_{i,j,2} &= \mathbf{v}_{i,j,1}\\
&\vdots\\
(\mathbf{A}-\lambda_i\mathbf{I})\mathbf{v}_{i,j,q_{i,j}} &= \mathbf{v}_{i,j,q_{i,j}-1}.
\end{aligned} \tag{3.6}
$$

The term $q_{i,j}$ denotes the **length of the chain of the eigenvectors** associated to $\mathbf{v}_{i,j,1}$ and represents the largest integer such that the vectors $\mathbf{v}_{i,j,q_{i,j}}$ and $\mathbf{v}_{i,j,q_{i,j}-1}$ are linearly independent. The organisation of the eigenvectors of $\mathbf{A}$ is depicted in Fig. 3.1

Remark 3.3. *The length of chains of eigenvectors is equal to 1 when the geometric multiplicity is equal to the algebraic multiplicity. Indeed, assuming* $g_i = a_i$ *and* $q_{i,j} \ge 1$ *we have* $\sum_{j=1}^{a_i} q_{i,j} = a_i$ *if and only if* $q_{i,j} = 1$ *for each* $j = 1, \dots, g_i$.

Example 3.1 (Eigenvalues and Eigenvectors)**.**
Given the 5*-dimensional real matrix*

$$\mathbf{A} = \begin{bmatrix} s_1 & 0 & 0 & 0 & 0 \\ 0 & s_2 & 0 & 0 & 0 \\ 0 & 0 & s_2 & 0 & 0 \\ 0 & 0 & 0 & s_3 & 1 \\ 0 & 0 & 0 & 0 & s_3 \end{bmatrix}$$

the eigenvalues are the roots of $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$*, where*

$$\begin{aligned} \det(\mathbf{A} - \lambda\mathbf{I}) = &\det\left(\begin{bmatrix} s_1-\lambda & 0 & 0 & 0 & 0 \\ 0 & s_2-\lambda & 0 & 0 & 0 \\ 0 & 0 & s_2-\lambda & 0 & 0 \\ 0 & 0 & 0 & s_3-\lambda & 1 \\ 0 & 0 & 0 & 0 & s_3-\lambda \end{bmatrix} \right) \\ &= (s_1-\lambda)(s_2-\lambda)^2(s_3-\lambda)^2. \end{aligned}$$

There are three independent eigenvalues (thus $p = 3$*) which are* $\lambda_1 = s_1$*,* $\lambda_2 = s_2$ *and* $\lambda_3 = s_3$ *with algebraic multiplicity given by* $a_1 = 1$*,* $a_2 = 2$ *and* $a_3 = 2$ *with* $\sum_{i=1}^{3} a_i = 5$*. The eigenvector associated to* λ_1 *is found by solving*

$$\begin{aligned} (\mathbf{A} - \lambda_1\mathbf{I})\mathbf{v} &= \mathbf{0} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & s_2-s_1 & 0 & 0 & 0 \\ 0 & 0 & s_2-s_1 & 0 & 0 \\ 0 & 0 & 0 & s_3-s_1 & 1 \\ 0 & 0 & 0 & 0 & s_3-s_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 \\ (s_2-s_1)x_2 \\ (s_2-s_1)x_3 \\ (s_3-s_1)x_4 + x_5 \\ (s_3-s_1)x_5 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

Assuming $s_1 \neq s_2 \neq s_3 \neq 0$*, the last four equations imply* $x_2 = x_3 =$

$x_4 = x_5 = 0$ then, the eigenvector associated to λ_1 is found as

$$\mathbf{v}_{1,1,1} = \begin{bmatrix} x_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

with $x_1 \neq 0$ free (one possible choice is $x_1 = 1$). The eigenvalues associated to the second eigenvalue, λ_2, are found by solving

$$(\mathbf{A} - \lambda_2 \mathbf{I})\mathbf{v} = \mathbf{0}$$

$$\begin{bmatrix} s_1 - s_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & s_3 - s_2 & 1 \\ 0 & 0 & 0 & 0 & s_3 - s_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} (s_1 - s_2)x_1 \\ 0 \\ 0 \\ (s_3 - s_2)x_4 + x_5 \\ (s_3 - s_2)x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

which leads to $x_1 = x_4 = x_5 = 0$ and $(x_2, x_3) \in \mathbb{R}^2 \setminus \{(0,0)\}$. Then, it is possible to find two eigenvectors (geometric multiplicity $g_2 = 2$), namely $\mathbf{v}_{2,1}$ and $\mathbf{v}_{2,2}$, associated to λ_2 as

$$\mathbf{v}_{2,1,1} = \begin{bmatrix} 0 \\ x_2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_{2,2,1} = \begin{bmatrix} 0 \\ 0 \\ x_3 \\ 0 \\ 0 \end{bmatrix}.$$

The eigenvectors associated to λ_3 are the solution of

$$(\mathbf{A} - \lambda_3\mathbf{I})\mathbf{v} = \mathbf{0}$$

$$\begin{bmatrix} s_1 - s_3 & 0 & 0 & 0 & 0 \\ 0 & s_2 - s_3 & 0 & 0 & 0 \\ 0 & 0 & s_2 - s_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} (s_1 - s_3)x_1 \\ (s_2 - s_3)x_2 \\ (s_2 - s_3)x_3 \\ x_5 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

which leads to $x_1 = x_2 = x_3 = x_5 = 0$ implying the existence of only one eigenvector, namely $\mathbf{v}_{3,1,1}$, with geometric multiplicity $g_3 = 1$, given by

$$\mathbf{v}_{3,1,1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ x_4 \\ 0 \end{bmatrix}.$$

Since $a_3 - g_3 > 0$ there exists a chain of eigenvectors, with length $q_{3,1} = 2$, found as solution of

$$(\mathbf{A} - \lambda_3\mathbf{I})\mathbf{v}_{3,1,2} = \mathbf{v}_{3,1,1}$$

$$\begin{bmatrix} s_1 - s_3 & 0 & 0 & 0 & 0 \\ 0 & s_2 - s_3 & 0 & 0 & 0 \\ 0 & 0 & s_2 - s_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ x_4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} (s_1 - s_3)y_1 \\ (s_2 - s_3)y_2 \\ (s_2 - s_3)y_3 \\ y_5 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ x_4 \\ 0 \end{bmatrix}$$

which implies $y_1 = y_2 = y_3 = 0$ and $y_5 = x_4$ leading to

$$\mathbf{v}_{3,1,2} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ y_4 \\ x_4 \end{bmatrix}.$$

3.1.3 Jordan Transformation

Let $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ be a LTI system with $\mathbf{A} \in \mathbb{R}^{n\times n}$. To investigate the dynamics of this system we rely on a change of coordinates based on the eigenvectors.

Define the set of distinct eigenvalues of $\mathbf{A}$ as

$$\{\lambda_1, \ldots, \lambda_p\} \qquad p \leq n.$$

Let a_i and g_i be the algebraic and the geometric multiplicity associated to λ_i. Moreover, let $q_{i,j}$ be the length of the chain of eigenvectors associated to the eigenvector $\mathbf{v}_{i,j,1}$. Let

$$\mathbf{V}_{i,j} := \begin{bmatrix} \mathbf{v}_{i,j,1} & \mathbf{v}_{i,j,2} & \cdots & \mathbf{v}_{i,j,q_{i,j}} \end{bmatrix} \tag{3.7}$$

be the matrix obtained as composition of the $q_{i,j}$ eigenvectors associated to $\mathbf{v}_{i,j,1}$. On the other hand, for each $i = 1, \ldots, p$ the following relations are true (see (3.6))

$$\mathbf{A}\mathbf{v}_{i,j,k} = \lambda_i \mathbf{v}_{i,j,k} + \mathbf{v}_{i,j,k-1} \qquad j = 1, \ldots, g_i, \quad k = 1, \ldots, q_{i,j}$$

which can be written in compact form as

$$\mathbf{A}\mathbf{V}_{i,j} = \mathbf{V}_{i,j}\mathbf{J}_{i,j}$$

where

$$\mathbf{J}_{i,j} = \begin{bmatrix} \lambda_i & 1 & \ldots & 0 & 0 \\ 0 & \lambda_i & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \lambda_i & 1 \\ 0 & 0 & \ldots & 0 & \lambda_i \end{bmatrix} \in \mathbb{C}^{q_{i,j}\times q_{i,j}}.$$

Moreover, let

$$\mathbf{V}_i := \begin{bmatrix} \mathbf{V}_{i,1} & \mathbf{V}_{i,2} & \cdots & \mathbf{V}_{i,g_i} \end{bmatrix}$$

be the compositions of all the chains of eigenvectors associated to the eigenvalue λ_i. Let

$$\mathbf{V} := \begin{bmatrix} \mathbf{V}_1 & \cdots & \mathbf{V}_p \end{bmatrix}$$

be the composition of all the eigenvectors of $\mathbf{A}$, then $\mathbf{V}$ represents a basis for $\mathbb{C}^n$ thanks to the linear independence of $\mathbf{V}_i$, $\mathbf{V}_j$ for each $i, j \in \{1, \ldots, p\}$ with $i \neq j$. Let now

$$\mathbf{J}_i := \mathrm{blckdiag}\left(\mathbf{J}_{i,1}, \ldots, \mathbf{J}_{i,g_i}\right) \qquad \mathbf{J} := \mathrm{blckdiag}\left(\mathbf{J}_1, \ldots, \mathbf{J}_p\right),$$

then

$$\mathbf{AV} = \mathbf{VJ}.$$

Since $\mathbf{V}$ is composed of linearly independent vectors, its inverse is well posed and can be exploited to define the following transformation

$$\mathbf{V}^{-1}\mathbf{AV} = \mathbf{J} \tag{3.8}$$

which is called *Jordan canonical form* and represents the key tool for the investigation of LTI system dynamics.

In the remaining of this section we present a procedure to reduce $\mathbf{V}$ to a pure real valued matrix.

Let λ be an eigenvalue of $\mathbf{A}$ and let $\mathbf{v}$ an eigenvector associated to λ. Let us assume $\lambda \in \mathbb{C}$ and $\mathbf{v} \in \mathbb{C}^n$. Then there exists $\alpha, \beta \in \mathbb{R}$ and $\mathbf{a}$, $\mathbf{b} \in \mathbb{R}^n$ such that

$$\begin{aligned} \lambda &= \alpha + \mathrm{i}\beta \quad \lambda^\star = \alpha - \mathrm{i}\beta \\ \mathbf{v} &= \mathbf{a} + \mathrm{i}\mathbf{b} \quad \mathbf{v}^\star = \mathbf{a} - \mathrm{i}\mathbf{b}. \end{aligned}$$

where the starred quantities denote the complex conjugate. Then, the following relations hold

$$\begin{aligned} \left[\mathbf{A} - \lambda\mathbf{I}\right]\mathbf{v} &= \left[\mathbf{A} - (\alpha + \mathrm{i}\beta)\,\mathbf{I}\right]\left[\mathbf{a} + \mathrm{i}\mathbf{b}\right] \\ &= \left[\mathbf{A} - \alpha\mathbf{I} - \mathrm{i}\beta\mathbf{I}\right]\mathbf{a} + \left[\mathbf{A} - \alpha\mathbf{I} - \mathrm{i}\beta\mathbf{I}\right]\mathrm{i}\mathbf{b} \\ &= \left[\mathbf{A} - \alpha\mathbf{I}\right]\mathbf{a} - \mathrm{i}\beta_i\mathbf{I}\mathbf{a} + \left[\mathbf{A} - \alpha\mathbf{I}\right]\mathrm{i}\mathbf{b} - \mathrm{i}^2\beta_i\mathbf{I}\mathbf{b} \\ &= \left[\mathbf{A} - \alpha\mathbf{I}\right]\mathbf{a} + \beta_i\mathbf{I}\mathbf{b} + \mathrm{i}\left\{\left[\mathbf{A} - \alpha\mathbf{I}\right]\mathbf{b} - \beta\mathbf{I}\mathbf{a}\right\} = 0. \end{aligned}$$

The latter equality can be split in two components, the first for the real terms and the second for the complex terms as

$$\begin{cases} \left[\mathbf{A} - \alpha\mathbf{I}\right]\mathbf{a} + \beta\mathbf{I}\mathbf{b} = 0 \\ \left[\mathbf{A} - \alpha\mathbf{I}\right]\mathbf{b} - \beta\mathbf{I}\mathbf{a} = 0 \end{cases} \Longleftrightarrow \begin{cases} \mathbf{A}\mathbf{a} = \alpha\mathbf{a} - \beta\mathbf{b} \\ \mathbf{A}\mathbf{b} = \alpha\mathbf{b} + \beta\mathbf{a} \end{cases}$$

which can be compacted in the following matrix notation

$$\mathbf{A}\begin{bmatrix} \mathbf{a} & \mathbf{b} \end{bmatrix} = \begin{bmatrix} \mathbf{a} & \mathbf{b} \end{bmatrix}\begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}.$$

Now, exploit this result to determine a real form for the associated Jordan block as follows. Let $\lambda_i \in \mathbb{C}$ be an eigenvalue of $\mathbf{A}$ and let $\alpha_i, \beta_i \in \mathbb{R}$ such that $\lambda_i = \alpha_i + \mathrm{i}\beta_i$. Let $\mathbf{V}_{i,j}$ be the chain of eigenvectors defined in (3.7) and let $\mathbf{a}_{i,j,k}$, $\mathbf{b}_{i,j,k} \in \mathbb{R}^n$ such that for $j = 1, \ldots, g_i$ and $k = 1, \ldots, q_{i,j}$ we have $\mathbf{v}_{i,j,k} = \mathbf{a}_{i,j,k} + \mathrm{i}\mathbf{b}_{i,j,k}$. Let

$$\bar{\mathbf{V}}_{i,j} := \begin{bmatrix} \mathbf{a}_{i,j,1} & \mathbf{b}_{i,j,1} & \mathbf{a}_{i,j,2} & \mathbf{b}_{i,j,2} & \ldots & \mathbf{a}_{i,j,q_{i,j}} & \mathbf{b}_{i,j,q_{i,j}} \end{bmatrix}$$

$$\bar{\mathbf{J}}_{i,j} := \begin{bmatrix} \begin{matrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{matrix} & \mathbf{I} & \ldots & \mathbf{0} \\ \mathbf{0} & \begin{matrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{matrix} & \ddots & \mathbf{0} \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \ldots & \begin{matrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{matrix} \end{bmatrix},$$

then, $\mathbf{A}\bar{\mathbf{V}}_{i,j} = \bar{\mathbf{V}}_{i,j}\bar{\mathbf{J}}_{i,j}$ holds true.

Example 3.2 (Study of the cart-pole)**.** *We want to study the matrix* $\mathbf{A}$ *defined in the Example 1.2 assuming as linearisation triplet* $(\mathbf{x}^\star, u^\star, d^\star) = (\mathbf{0}, 0, 0)$*. Then, with this assumption the matrix becomes*

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & A_{23} & A_{24} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & A_{43} & A_{44} \end{bmatrix}$$

with

$$A_{23} = \frac{\ell m(k - \ell m g)}{Mm\ell^2 + J_m(M+m)} \qquad A_{24} = \frac{\ell k m}{Mm\ell^2 + J_m(M+m)}$$
$$A_{43} = -\frac{(M+m)(k - \ell m g)}{Mm\ell^2 + J_m(M+m)} \qquad A_{44} = -\frac{\mu(M+m)}{Mm\ell^2 + J_m(M+m)}.$$

The eigenvalues of this matrix and their algebraic multiplicity, obtained by solving the problem det$(\mathbf{A} - \lambda\mathbf{I}) = \mathbf{0}$*, are*

$$\lambda_1 = 0 \qquad\qquad a_1 = 2$$
$$\lambda_{2,3} = \frac{A_{44} \pm \sqrt{A_{44}^2 + 4A_{43}}}{2} \qquad a_{2,3} = 1.$$

Assuming the numerical values for the tuple $(m,\ M,\ \ell,\ g,\ J_m,\ k,\ \mu) = (1000,\ 100,\ 1,\ 9.81,\ 100,\ 16000,\ 1000)$ *(these numerical values model the* *longitudinal dynamics of a car) we obtain that the $\lambda_{2,3}$ are complex conjugate, i.e. $\lambda_2 = \lambda_3^\star$. Moreover, it is easy to check that the geometric multiplicity of λ_1 is $g_1 = 1$, so there exist a chain a generalised eigenvectors (associated to λ_1) of length $a_1 - g_1 = 1$. Let $\alpha_2, \beta_2 \in \mathbb{R}$ such that $\lambda_2 = \alpha_2 + \mathrm{i}\beta_2$, then thanks to the knowledge about the kind of eigenvalues and the chains of generalised eigenvectors we can define the Jordan matrix as*

$$\mathbf{J} = \begin{bmatrix} \mathbf{J}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_2 \end{bmatrix}, \mathbf{J}_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \bar{\mathbf{J}}_2 = \begin{bmatrix} \alpha_2 & \beta_2 \\ -\beta_2 & \alpha_2 \end{bmatrix}.$$

3.2 Dynamics of LTI systems

In this section we exploit the change of coordinates defined on the eigenvectors of $\mathbf{A}$ to study the dynamics of an LTI system. The goal is to transform the system from the original coordinates to some special coordinates in which the dynamics of the whole system can be seen as compositions of independent sub-dynamics, for which an explicit description is available.

Let

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \quad \mathbf{x}(t_0) = \mathbf{x}_0 \tag{3.9}$$

be an LTI system, let $\mathbf{V}$ be the set of the eigenvectors of $\mathbf{A}$, and assume $\mathbf{V}$ be in the pure real form. Let $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^n$ be a change of coordinates and assume $\mathbf{T} = \mathbf{V}^{-1}$. Define $\mathbf{z} = \mathbf{T}\mathbf{x}$. Then, the dynamics of $\mathbf{z}$ is obtained by pre-multiplying both members of (3.9) by $\mathbf{T}$ and by exploiting $\mathbf{x} = \mathbf{T}^{-1}\mathbf{z}$

$$\dot{\mathbf{z}} = \bar{\mathbf{A}}\mathbf{z} + \bar{\mathbf{B}}\mathbf{u} \quad \mathbf{z}(t_0) = \mathbf{T}\mathbf{x}_0. \tag{3.10}$$

where $\bar{\mathbf{A}} := \mathbf{T}\mathbf{A}\mathbf{T}^{-1}$ and $\bar{\mathbf{B}} = \mathbf{T}\mathbf{B}$.

Since the new coordinates are chosen accordingly to the Jordan canonical form, the matrix $\bar{\mathbf{A}}$ assumes a block diagonal form

$$\bar{\mathbf{A}} = \begin{bmatrix} \mathbf{J}_1 & 0 & \dots & 0 \\ 0 & \mathbf{J}_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \mathbf{J}_p \end{bmatrix}, \mathbf{J}_i = \begin{bmatrix} \mathbf{J}_{i1} & 0 & \dots & 0 \\ 0 & \mathbf{J}_{i2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \mathbf{J}_{ig_i} \end{bmatrix}. \tag{3.11}$$

Redefine

$$\mathbf{z} := \mathrm{col}(\mathbf{z}_{1,1}, \dots, \mathbf{z}_{1,g_1}, \mathbf{z}_{2,1}, \dots, \mathbf{z}_{2,g_2}, \dots, \mathbf{z}_{p,1}, \dots, \mathbf{z}_{p,g_p}),$$

and exploit (3.11) to rewrite (3.10) as

$$\begin{aligned} \dot{\mathbf{z}}_{i,j} &= \mathbf{J}_{i,j}\mathbf{z}_{i,j} + \bar{\mathbf{B}}_{i,j}\mathbf{u} \quad \mathbf{z}_{i,j}(t_0) = \mathbf{V}_{i,j}\mathbf{x}_0 \qquad i = 1, \dots, p \\ \mathbf{y} &= \bar{\mathbf{C}}_{i,j}\mathbf{z}_{i,j} + \mathbf{D}\mathbf{u} \qquad\qquad\qquad\qquad\quad j = 1, \dots, g_i \end{aligned} \tag{3.12}$$

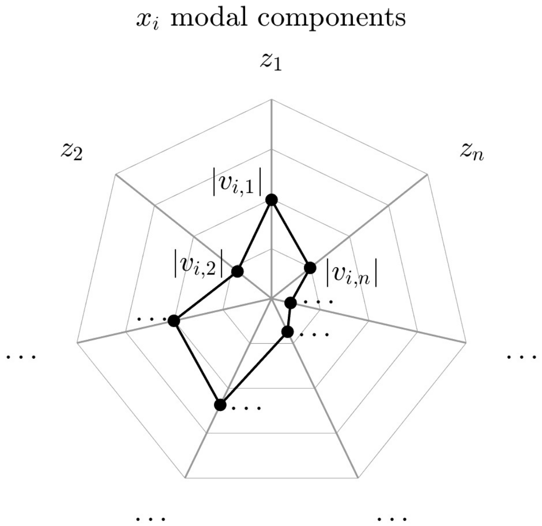

Figure 3.2: The i-th state component is constituted by a linear combination of the modes $\mathbf{z}$. This diagram provides a graphical support to understand which are, if any, the components of $\mathbf{z}$ that principally affects the dynamics x_i.

where $\bar{\mathbf{B}}_{i,j}$ and $\bar{\mathbf{C}}_{i,j}$ are proper sub-parts of $\bar{\mathbf{B}}$ and $\bar{\mathbf{C}}$.

The solution of (3.12) is

$$\mathbf{z}_{i,j}(t) = \exp(\mathbf{J}_{i,j}(t-t_0))\mathbf{z}_{i,j}(t_0) + \int_{t_0}^{t} \exp(\mathbf{J}_{i,j}(t-\tau))\bar{\mathbf{B}}_{i,j}\mathbf{u}(\tau)d\tau \tag{3.13}$$

where $\exp(\mathbf{J}_{i,j}t) := \sum_{k=0}^{\infty} \frac{1}{k!}\mathbf{J}_{i,j}^k t^k$ is called *transition matrix.*

Remark 3.4. *Once determined* $\mathbf{z}(t)$ *the solution of* (3.9) *is obtained as* $\mathbf{x}(t) = \mathbf{V}\,\mathbf{z}(t)$ *for all* $t \geq t_0$*. Let* $\mathbf{z} := \mathtt{col}(z_1, \ldots, z_n)$ *and assume* $\mathbf{V}$ *be written in the pure real form. Denote with* v_{ij} *the* (i,j)*-th element of* $\mathbf{V}$*. Then, the time behaviour of the i-th state component is obtained as linear combination of the modes*

$$x_i(t) = \sum_{j=1}^{n} v_{i,j} z_j(t) \tag{3.14}$$

where $x_i(t)$ *is thought as a weighted mean of* $\mathbf{z}$ *with weights* $v_{i,j}$*. An interesting graphical representation of* (3.14) *is provided in Fig. 3.2.*

It is interesting to note that the evolution of $\mathbf{z}_{i,j}(t)$ can be seen as superposition of two parts

- the *free evolution*, only dependent on the initial conditions,

$$\mathbf{z}_{i,j}^{\text{free}}(t) = \exp(\mathbf{J}_{i,j}(t-t_0))\mathbf{z}_{i,j}(t_0)$$

- the *forced evolution*, only dependent of the input history,

$$\mathbf{z}_{i,j}^{\text{forced}}(t) = \int_{t_0}^{t} \exp(\mathbf{J}_{i,j}(t-\tau))\bar{\mathbf{B}}_{i,j}\mathbf{u}(\tau)d\tau$$

where the term $\mathbf{z}_{i,j}^{\text{free}}(t)$ is null if $\mathbf{z}_{i,j}(t_0) = \mathbf{0}$ whereas the term $\mathbf{z}_{i,j}^{\text{forced}}(t) = \mathbf{0}$ if $\mathbf{u}(\tau) \equiv \mathbf{0}\ \forall \tau \in [t_0, t]$. To compute $\exp(\mathbf{J}_{i,j}t)$ we note that

$$\mathbf{J}_{i,j} = \begin{bmatrix} \lambda_i & 1 & \dots & 0 & 0 \\ 0 & \lambda_i & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \lambda_i & 1 \\ 0 & 0 & \dots & 0 & \lambda_i \end{bmatrix}$$

can be seen as $\mathbf{J}_{i,j} = \lambda_i\mathbf{I} + \mathbf{N}_{i,j}$ with

$$\mathbf{N}_{i,j} = \begin{bmatrix} 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix}.$$

Then, the transition matrix is

$$\exp(\mathbf{J}_{i,j}t) = \exp((\lambda_i\mathbf{I} + \mathbf{N}_{i,j})t) = \exp(\lambda_i t)\exp(\mathbf{N}_{i,j}t)$$

and

$$\exp(\mathbf{N}_{i,j}t) = \sum_{k=0}^{\infty} \frac{1}{k!}\bar{\mathbf{N}}_{i,j}^{k}t^{k} = \mathbf{I} + \mathbf{N}_{i,j}t + \frac{1}{2}\mathbf{N}_{i,j}^{2}t^{2} + \dots$$

$$= \begin{bmatrix} 1 & t & \frac{t^2}{2} & \dots & \frac{t^{q_{i,j}-2}}{(q_{i,j}-2)!} & \frac{t^{q_{i,j}-1}}{(q_{i,j}-1)!} \\ 0 & 1 & t & \ddots & \frac{t^{q_{i,j}-3}}{(q_{i,j}-3)!} & \frac{t^{q_{i,j}-2}}{(q_{i,j}-2)!} \\ 0 & 0 & 1 & \ddots & \frac{t^{q_{i,j}-4}}{(q_{i,j}-4)!} & \frac{t^{q_{i,j}-3}}{(q_{i,j}-3)!} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 & t \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}.$$

Infobox 3.3 (Powers of $\mathbf{N}_{i,j}$). *The k-th power of $\mathbf{N}_{i,j}$ is a null matrix with the k-th upper diagonal composed of "1" elements. As an example, for $k = 2, 3$ we have*

$$\mathbf{N}_{i,j}^2 = \begin{bmatrix} 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}, \mathbf{N}_{i,j}^3 = \begin{bmatrix} 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}.$$

*Let $q_{i,j}$ be the dimension of $\mathbf{N}_{i,j}$, then it can be verified that $\mathbf{N}_{i,j}^k = \mathbf{0}$ for all $k \geq q_{i,j}$. To conclude, $\mathbf{N}_{i,j}$ is said to be **nilpotent** of order $q_{i,j}$.*

In case of complex conjugate eigenvalues, the Jordan block $\mathbf{J}_{i,j}$ can be reformulated as

$$\mathbf{J}_{i,j} = \begin{bmatrix} \mathbf{M} & \mathbf{I} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} & \ddots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{M} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{M} \end{bmatrix}$$

where $\mathbf{M} = \alpha_i \mathbf{I} + \beta_i \mathbf{S}$ with $\mathbf{S} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. So the matrix $\mathbf{J}_{i,j}$ assumes the form $\mathbf{J}_{i,j} = \mathbf{D}_{i,j} + \mathbf{N}_{i,j}$ where

$$\mathbf{D}_{i,j} = \begin{bmatrix} \alpha_i \mathbf{I} + \beta_i \mathbf{S} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \alpha_i \mathbf{I} + \beta_i \mathbf{S} & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \alpha_i \mathbf{I} + \beta_i \mathbf{S} \end{bmatrix}$$

$$\mathbf{N}_{i,j} = \begin{bmatrix} \mathbf{0} & \mathbf{I} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \ddots & \mathbf{0} & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \ddots & \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

The exponential of $\mathbf{D}_{i,j}$ is composed of the exponentials of $\alpha_i\mathbf{I} + \beta_i\mathbf{S}$, where

$$\begin{aligned}\exp\left(\left(\alpha_i\mathbf{I} + \beta_i\mathbf{S}\right)t\right) &= \exp\left(\alpha_i\mathbf{I}t\right)\exp\left(\beta_i\mathbf{S}t\right)\\ &= \exp\left(\alpha_i t\right)\exp\left(\beta_i\mathbf{S}t\right).\end{aligned}$$

It can be demonstrated that the exponential of $\beta_i\mathbf{S}t$ is given by

$$\exp\left(\beta_i\mathbf{S}t\right) = \begin{bmatrix} \cos(\beta_i t) & \sin(\beta_i t) \\ -\sin(\beta_i t) & \cos(\beta_i t) \end{bmatrix}$$

and, on the other hand, the exponential of $\mathbf{N}_{i,j}$ is given by

$$\exp(\mathbf{N}_{i,j}t) = \begin{bmatrix} \mathbf{I} & t\mathbf{I} & \frac{t^2}{2}\mathbf{I} & \dots & \frac{t^{q_{i,j}-2}}{(q_{i,j}-2)!}\mathbf{I} & \frac{t^{q_{i,j}-1}}{(q_{i,j}-1)!}\mathbf{I} \\ \mathbf{0} & \mathbf{I} & t\mathbf{I} & \ddots & \frac{t^{q_{i,j}-3}}{(q_{i,j}-3)!}\mathbf{I} & \frac{t^{q_{i,j}-2}}{(q_{i,j}-2)!}\mathbf{I} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \ddots & \frac{t^{q_{i,j}-4}}{(q_{i,j}-4)!}\mathbf{I} & \frac{t^{q_{i,j}-3}}{(q_{i,j}-3)!}\mathbf{I} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \ddots & \mathbf{I} & t\mathbf{I} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{I} \end{bmatrix}.$$

In conclusion, the exponential of $\mathbf{J}_{i,j}$ in case of complex conjugate eigenvalues is

$$\begin{aligned}\exp(\mathbf{J}_{i,j}t) &= \exp\left(\mathbf{D}_{i,j}t + \mathbf{N}_{i,j}t\right) = \exp\left(\mathbf{D}_{i,j}t\right)\exp\left(\mathbf{N}_{i,j}t\right)\\ &= \left[\exp\left(\alpha_i t\right)\mathbf{I} \otimes \exp\left(\beta_i\mathbf{S}t\right)\right]\,\exp\left(\mathbf{N}_{i,j}t\right).\end{aligned}$$

The symbol $\otimes$ represents the so called *Kroneker* products, see Appendix A for details.

The following subsections investigate the time behaviour associated to the transition matrix of a single block $\mathbf{J}_{i,j}$ for different eigenvalues.

Real eigenvalues

In case of real eigenvalue λ_i, the transition matrix associated to $\mathbf{J}_{i,j}$ is given by

$$\begin{aligned}\exp(\mathbf{J}_{i,j}t) &= \exp(\lambda_i t)\exp(\mathbf{N}_{i,j}t)\\ &= \begin{bmatrix} \exp(\lambda_i t) & t\exp(\lambda_i t) & \dots & \frac{t^{q_{i,j}-1}}{(q_{i,j}-1)!}\exp(\lambda_i t) \\ 0 & \exp(\lambda_i t) & \dots & \frac{t^{q_{i,j}-2}}{(q_{i,j}-2)!}\exp(\lambda_i t) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \exp(\lambda_i t) \end{bmatrix}.\end{aligned}$$

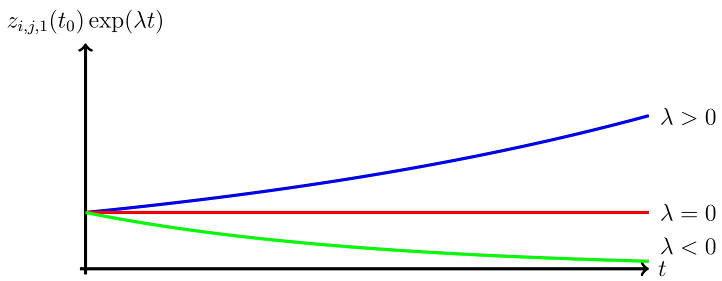

Figure 3.3: Time behaviour of $z_{i,j,1}(t_0)\exp(\lambda t)$

If $\mathbf{u} = \mathbf{0}$, the dynamics of $\mathbf{z}_{i,j}$ is given by

$$\mathbf{z}_{i,j}(t) = \exp(\mathbf{J}_{i,j}t)\mathbf{z}_{i,j}(t_0)$$

and, in particular, the dynamics of the first element of $\mathbf{z}_{i,j}$, namely $z_{i,j,1}$, is

$$z_{i,j,1}(t) = \exp(\lambda_i t)\sum_{k=0}^{q_{i,j}-1}\frac{t^k}{k!}z_{i,j,(k+1)}(t_0)$$

which is bounded if, for each k there exists a finite real M_k such that

$$\lim_{t\to\infty}\exp(\lambda_i t)t^k < M_k \in \mathbb{R},\ \forall k = 0,\dots,q_{i,j}-1$$

thus leading to the following conditions:

$$\mathbf{z}_{i,j}(t)\begin{cases}\text{exponentially convergent to zero} & \lambda_i < 0,\ q_{i,j}\geq 1\\ \text{constant} & \lambda_i = 0,\ q_{i,j} = 1\\ \text{polinomially divergent from } \mathbf{z}_{i,j}(t_0) & \lambda_i = 0,\ q_{i,j} > 1\\ \text{exponentially divergent from } \mathbf{z}_{i,j}(t_0) & \lambda_i > 0,\ q_{i,j}\geq 1\end{cases}.$$

Figures 3.3-3.4 depict the time behaviour of the term $\exp(\lambda_i t)\dfrac{t^k}{k!}z_{i,j,(k+1)}(t_0)$ in case of real λ_i for $k = 0$ and $k > 0$ respectively.

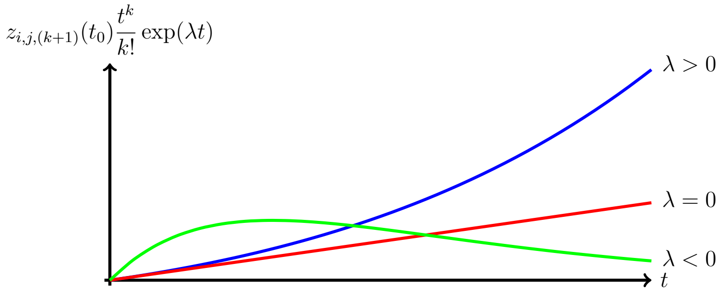

Figure 3.4: Time behaviour of $z_{i,j,(k+1)}(t_0)\dfrac{t^k}{k!}\exp(\lambda t)$ for $k > 0$.

Complex conjugate eigenvalues

In case of complex conjugate eigenvalues, the exponential of $\mathbf{J}_{i,j}t$ becomes

$$\begin{aligned}\exp(\mathbf{J}_{i,j}t) &= [\exp(\alpha_i t)\,\mathbf{I}\otimes\exp(\beta_i\mathbf{S}t)]\,\exp(\mathbf{N}_{i,j}t) = \\ &= \exp(\alpha_i t)\,\mathbf{I}\otimes\begin{bmatrix}\cos(\beta_i t) & \sin(\beta_i t)\\ -\sin(\beta_i t) & \cos(\beta_i t)\end{bmatrix}\cdot \\ &\begin{bmatrix}\mathbf{I} & t\mathbf{I} & \frac{t^2}{2}\mathbf{I} & \frac{t^3}{3!}\mathbf{I} & \dots & \frac{t^{q_{i,j}-2}}{(q_{i,j}-2)!}\mathbf{I} & \frac{t^{q_{i,j}-1}}{(q_{i,j}-1)!}\mathbf{I}\\ \mathbf{0} & \mathbf{I} & t\mathbf{I} & \frac{t^2}{2}\mathbf{I} & \dots & \frac{t^{q_{i,j}-3}}{(q_{i,j}-3)!}\mathbf{I} & \frac{t^{q_{i,j}-2}}{(q_{i,j}-2)!}\mathbf{I}\\ \mathbf{0} & \mathbf{0} & \mathbf{I} & t\mathbf{I} & \dots & \frac{t^{q_{i,j}-4}}{(q_{i,j}-4)!}\mathbf{I} & \frac{t^{q_{i,j}-3}}{(q_{i,j}-3)!}\mathbf{I}\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{I} & t\mathbf{I}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{I}\end{bmatrix}.\end{aligned}$$

Assuming $\mathbf{u} = \mathbf{0}$, the dynamics of $\mathbf{z}_{i,j}$ is given by

$$\mathbf{z}_{i,j}(t) = \exp(\mathbf{J}_{i,j}t)\mathbf{z}_{i,j}(t_0)$$

where, in particular, the dynamics of the first two elements of $\mathbf{z}_{i,j}$, *i.e.* $\mathrm{col}\,(z_{i,j,1}(t), z_{i,j,2}(t))$, is

$$\begin{aligned}\begin{bmatrix} z_{i,j,1}(t)\\ z_{i,j,2}(t)\end{bmatrix} &= \exp(\alpha_i t)\begin{bmatrix}\cos(\beta_i t) & \sin(\beta_i t)\\ -\sin(\beta_i t) & \cos(\beta_i t)\end{bmatrix}\cdot \\ &\sum_{k=0}^{q_{i,j}-1}\frac{t^k}{k!}\begin{bmatrix} z_{i,j,2k+1}(t_0)\\ z_{i,j,2k+2}(t_0)\end{bmatrix}\end{aligned} \tag{3.15}$$

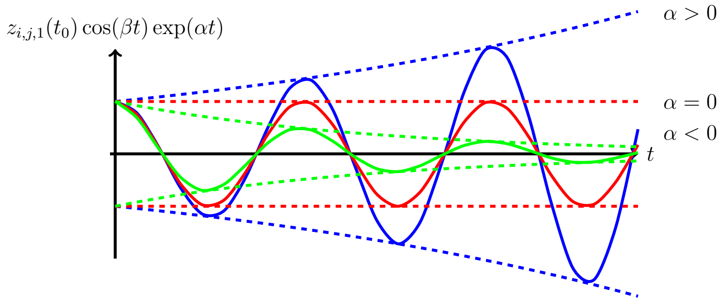

Figure 3.5: Time behaviour of $z_{i,j,1}(t_0)\cos(\beta t)\exp(\alpha t)$

The time behaviour $\mathrm{col}\,(z_{i,j,1}(t), z_{i,j,2}(t))$ is bounded if for each k there exist a finite real M such that

$$\lim_{t\to\infty} \exp\left(\alpha_i t\right) t^k < M_k \in \mathbb{R},\ \forall k = 0, \ldots, q_{i,j} - 1$$

thus leading to the following conditions:

$$\mathbf{z}_{i,j}\begin{cases} \text{exponentially convergent to zero} & \alpha_i < 0,\ q_{i,j} \geq 1 \\ \text{constant} & \alpha_i = 0,\ q_{i,j} = 1 \\ \text{polynomially divergent from } \mathbf{z}_{i,j}(t_0) & \alpha_i = 0,\ q_{i,j} > 1 \\ \text{exponentially divergent from } \mathbf{z}_{i,j}(t_0) & \alpha_i > 0,\ q_{i,j} \geq 1. \end{cases}$$

Figures 3.6-3.8 depict the time behaviour of the terms appearing in (3.15) for both $k = 0$ and $k > 0$.

Infobox 3.4. *With reference to the frequency analysis, the values α_i and β_i, representing the real and the imaginary part of a complex eigenvalue, are usually exploited to define the pulsation ω_i and the damping ratio δ_i as*

$$\begin{aligned} \omega_i &= \sqrt{\alpha_i^2 + \beta_i^2} \\ \delta_i &= \tan^{-1}\left(\alpha_i/\beta_i\right). \end{aligned}$$

Example 3.5 (Cart-pole modes)**.** *Given the Jordan matrix associated to the matrix $\mathbf{A}$ of the Exercise 3.2, the trajectories are defined by*

$$\exp(\mathbf{J}t) = \begin{bmatrix} \exp(\mathbf{J}_1 t) & \mathbf{0} \\ \mathbf{0} & \exp(\mathbf{J}_1 t) \end{bmatrix}, \exp(\mathbf{J}_1 t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$
$$\exp(\mathbf{J}_2 t) = \exp(\alpha_2 t) \begin{bmatrix} \cos(\beta_2 t) & \sin(\beta_2 t) \\ -\sin(\beta_2 t) & \cos(\beta_2 t) \end{bmatrix}.$$

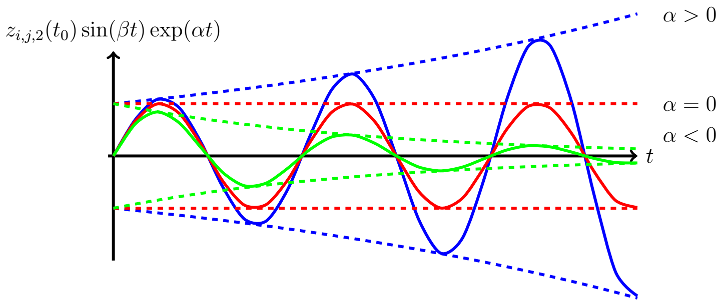

Figure 3.6: Time behaviour of $z_{i,j,2}(t_0)\sin(\beta t)\exp(\alpha t)$

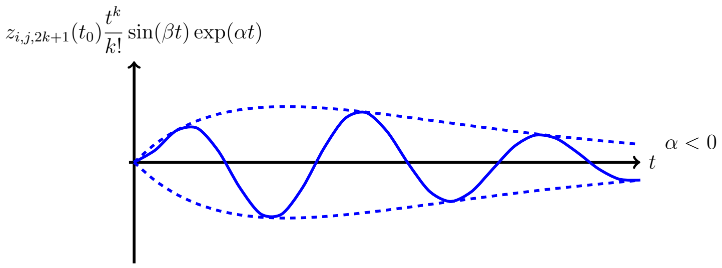

Figure 3.7: Time behaviour of $z_{i,j,2k+1}(t_0)\dfrac{t^k}{k!}\sin(\beta)t)\exp(\alpha t)$ for $k > 0$

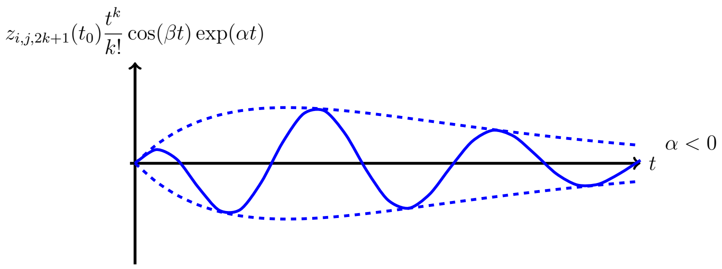

Figure 3.8: Time behaviour of $z_{i,j,2k+1}(t_0)\dfrac{t^k}{k!}\cos(\beta t)\exp(\alpha t)$ for $k > 0$

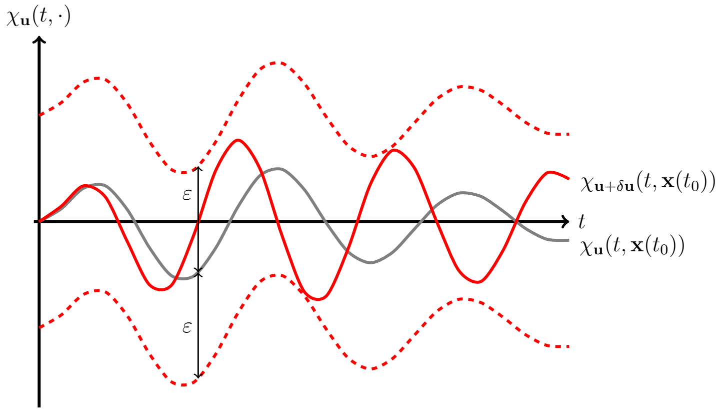

Figure 3.9: Case of BIBS stable system

Since the numerical values specified in Exercise 3.2 lead to $\alpha_2 < 0$, the system (i.e. the linearisation of the cart-pole around the static equilibrium) has one divergent mode (associated to the first line of $\mathbf{J}_1$), one constant mode (associated to the second line of $\mathbf{J}_1$) and two dumped oscillating modes (associated to $\mathbf{J}_2$).

3.3 BIBS Stability

As we have seen in the previous sections, the dynamics of LTI systems is completely described by the eigenvalues and by the lengths of the chains of eigenvectors associated to the matrix $\mathbf{A}$. In particular, the trajectories of the state $\mathbf{x}(t)$ are convergent to the origin of the state space $\mathbb{R}^n$ *if and only if* the real parts of the eigenvalues associated to $\mathbf{A}$ are all strictly negative. On the other hand, null or positive real parts of the eigenvalues may lead to constant or divergent behaviours depending on the length of the chain of eigenvectors. These results are hereafter exploited to state a stability criteria which assures the boundedness of trajectories.

Defintion 3.1. Bounded Input Bounded State (BIBS) Stability. *The LTI system* (3.9) *is BIBS stable if*

$$\forall \varepsilon > 0 \, \exists \, \delta_\varepsilon \, : \, \forall \delta\mathbf{u}(\cdot) \, : \, \|\delta\mathbf{u}(t)\| \leq \delta_\varepsilon \implies \|\chi_{\mathbf{u}+\delta\mathbf{u}}(t,\mathbf{x}(t_0)) - \chi_{\mathbf{u}}(t,\mathbf{x}(t_0))\| \leq \varepsilon$$

for all $t \geq t_0$.

The definition of BIBS stability links the perturbation of the input $\mathbf{u}$, *i.e.* $\delta\mathbf{u}$, with the evolution of the state $\mathbf{x}$, see Fig. 3.9.

We want to demonstrate that the BIBS stability is implied by having negative the real part of all eigenvalues of $\mathbf{A}$. To this end, and without loss of generality, let us assume $\mathbf{x}(t_0) = \mathbf{0}$ and calculate the perturbed and unperturbed trajectories

$$\begin{aligned}\chi_{\mathbf{u}+\delta\mathbf{u}}(t, \mathbf{x}(t_0)) &= \int_{t_0}^{t} \exp(\mathbf{A}(t-\tau))\mathbf{B}\left(\mathbf{u}(\tau) + \delta\mathbf{u}(\tau)\right) d\tau \\ \chi_{\mathbf{u}}(t, \mathbf{x}(t_0)) &= \int_{t_0}^{t} \exp(\mathbf{A}(t-\tau))\mathbf{B}\mathbf{u}(\tau) d\tau\end{aligned}$$

whose difference is

$$\chi_{\mathbf{u}+\delta\mathbf{u}}(t, \mathbf{x}(t_0)) - \chi_{\mathbf{u}}(t, \mathbf{x}(t_0)) = \int_{t_0}^{t} \exp(\mathbf{A}(t-\tau))\mathbf{B}\delta\mathbf{u}(\tau) d\tau$$

with norm given by

$$\begin{gathered}\|\chi_{\mathbf{u}+\delta\mathbf{u}}(t, \mathbf{x}(t_0)) - \chi_{\mathbf{u}}(t, \mathbf{x}(t_0))\| = \left\| \int_{t_0}^{t} \exp(\mathbf{A}(t-\tau))\mathbf{B}\delta\mathbf{u}(\tau) d\tau \right\| \leq \\ \int_{t_0}^{t} \|\exp(\mathbf{A}(t-\tau))\mathbf{B}\delta\mathbf{u}(\tau)\|\, d\tau \leq \int_{t_0}^{t} \|\exp(\mathbf{A}(t-\tau))\| d\tau\, \|\mathbf{B}\| \delta_\varepsilon \leq \\ \int_{t_0}^{t} \max_{\substack{i=1,\ldots,p \\ j=1,\ldots,g_i}} \sum_{k=0}^{q_{i,j}} \left| \frac{t^k}{k!} \exp(\alpha_i t) \right| d\tau \|\mathbf{B}\| \delta_\varepsilon\end{gathered}$$

So, in conclusion, the LTI system (3.9) is BIBS if, for all $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that

$$\delta_\varepsilon \leq \frac{\varepsilon}{\|\mathbf{B}\| \int_{t_0}^{t} \max_{\substack{i=1,\ldots,p \\ j=1,\ldots,g_i}} \sum_{k=0}^{q_{i,j}} \left| \frac{t^k}{k!} \exp(\alpha_i t) \right| d\tau}.$$

The analysis of the denominator of the second member tells us that the integral is finite if

$$\max_{\substack{i=1,\ldots,p \\ j=1,\ldots,a_i}} \sum_{k=0}^{q_{i,j}} \left| \frac{t^k}{k!} \exp(\alpha_i t) \right|$$

is absolutely integrable which, in turn, is true if the real part of all the eigenvalues of $\mathbf{A}$ is strictly negative, *i.e.* if $\mathbf{A}$ is Hurwitz.

Example 3.6 (Cart-pole Stability). *Given the results of the Exercise 3.5 we can assess that the system is not BIBS stable.*

3.6 Exercises

Exercise 3.1. *For $a = 1, 2$, find the eigenvalues and the eigenvectors of the following matrix*

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}$$

Exercise 3.2. *For $a = 1, 2$, find the eigenvalues and the eigenvectors of the following matrix*

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & a \end{bmatrix}.$$

Exercise 3.3. *For $a = 1, 2$, find the eigenvalues and the eigenvectors of the following matrix*

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 0 & a \end{bmatrix}.$$

Exercise 3.4. *For $a = 0, 1$, find the eigenvalues and the eigenvectors of the following matrix*

$$\mathbf{A} = \begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix}.$$

Exercise 3.5. *For $a = 0, 1$, find the eigenvalues and the eigenvectors of the following matrix*

$$\mathbf{A} = \begin{bmatrix} 1 & a/2 & -a/2 \\ -a & 1 & 0 \\ a & 0 & 1 \end{bmatrix}.$$

Exercise 3.6. *For $a = 0, 1$, find the eigenvalues and the eigenvectors of* *the following matrix*

$$\mathbf{A} = \begin{bmatrix} 1 & a/2 & -a/2 \\ 0 & 1+a/2 & a/2 \\ 0 & -a/2 & 1-a/2 \end{bmatrix}.$$

Exercise 3.7. *For $a = 0,\, 1$, find the eigenvalues and the eigenvectors of the following matrix*

$$\mathbf{A} = \begin{bmatrix} 1 & a/2+1/2 & a/2-1/2 & 0 \\ -1 & 1 & 0 & -a-1 \\ 1 & 0 & 1 & a-1 \\ 0 & 1/2 & 1/2 & 1 \end{bmatrix}.$$

Exercise 3.8. *Draw the time evolution and the trajectories of the LTI system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, $\mathbf{x} \in \mathbb{R}^2$, characterised by the following matrix*

$$\mathbf{A} = \begin{bmatrix} a & 1 \\ 0 & 0 \end{bmatrix}$$

for $\mathbf{x}(0) = \texttt{col}(0,\, 1)$ and for $a = -1,\, 0,\, 1$.

Exercise 3.9. *Draw the time evolution and the trajectories of the LTI system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, $\mathbf{x} \in \mathbb{R}^2$, characterised by the following matrix*

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & a \end{bmatrix}$$

for $\mathbf{x}(0) = \texttt{col}(0,\, 1)$, and for $a = \pm 1$.

Exercise 3.10. *Draw the time evolution and the trajectories of the LTI system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, $\mathbf{x} \in \mathbb{R}^2$, characterised by the following matrix*

$$\mathbf{A} = \begin{bmatrix} a & 1 \\ -1 & a \end{bmatrix}$$

for $\mathbf{x}(0) = \texttt{col}(1, 0)$ *and for* $a = -1, 0, 1$.

Exercise 3.11. *Draw the time evolution of the LTI system* $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, $\mathbf{x} \in \mathbb{R}^3$, *characterised by the following matrix*

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & a & 1 \\ 0 & -1 & a \end{bmatrix}$$

for $\mathbf{x}(0) = \texttt{col}(0, 1, 0)$ *and for* $a = -1, 0, 1$.

Exercise 3.12. *Draw the time evolution of the LTI system* $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, $\mathbf{x} \in \mathbb{R}^4$, *characterised by the following matrix*

$$\mathbf{A} = \begin{bmatrix} a & 1 & 1 & 0 \\ -1 & a & 0 & 1 \\ 0 & 0 & a & 1 \\ 0 & 0 & -1 & a \end{bmatrix}$$

for $\mathbf{x}(0) = \texttt{col}(0, 0, 1, 0)$ *and for* $a = -1, 0$.

Exercise 3.13. *Given the LTI system* $\dot{x} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ *with*

$$\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

determine if it is BIBS stable.

Exercise 3.14. *Given the LTI system* $\dot{x} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ *with*

$$\mathbf{A} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$$

determine if it is BIBS stable.

Exercise 3.15. *Given the LTI system* $\dot{x} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ *with*

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$$

determine if it is BIBS stable.

Exercise 3.16. *Given the LTI system* $\dot{x} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ *with*

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

determine if it is BIBS stable.

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Chapter 4

Control System Architecture

The control goal G1 requires the control to keep the state $\mathbf{x}$, the control $\mathbf{u}$ and the regulated output $\mathbf{e}$ bounded for all times under the assumption of bounded exogenous (*i.e.* bounded disturbances, noises and references). Chapter 3 links the boundedness of these signals to the BIBS stability through the eigenvalues of the linearised system. In Chapter 4 the controller architecture is incrementally obtained starting from the simplest control system, *i.e.* the *stabiliser* based on a *state feedback*, see Section 4.2. The robust output regulation of set points in case of unknown but constant disturbances is presented in Section 4.3. The main limitation of the stabiliser stays in the need of the whole state. To overcome this limitation, Section 4.5 describes how to stabilise the plant by means of a feasible *output feedback*. To achieve this result, the concept of the *state observer* described in Section 4.4 plays a fundamental role. Once the output feedback stabiliser is presented, the reader is guided toward the solution of the tracking of time varying references by means of the feed-forward solution proposed in Section 4.6. TBC

Contents

4.1 Closed Loop System

In this section we compose (1.17) and (1.18) to identify an overall linear system whose properties are directly associated to the achievement of the control goal G1 and G2.

By now, to simplify the presentation of topics, we impose $\mathbf{D}_\mathrm{O} = \mathbf{0}$ and $\mathbf{C}_\mathrm{O} = \mathbf{I}$ and in Section 4.4 (Infobox 4.22) we show how to remove this assumption.

Let $\mathbf{e}_x := \mathbf{x}_\mathrm{O} - \tilde{\mathbf{x}}$ and $\boldsymbol{\chi} := \mathtt{col}(\tilde{\mathbf{x}}, \boldsymbol{\eta}, \mathbf{e}_x, \mathbf{x}_\mathrm{FF})$. Then exploit $\mathbf{x}_\mathrm{O} = \mathbf{e}_x + \tilde{\mathbf{x}}$, (1.17) and (1.18) to compute the closed loop dynamics

$$\begin{aligned} \dot{\boldsymbol{\chi}} &= \mathbf{A}_\chi \boldsymbol{\chi} + \mathbf{B}_{\chi_w} \tilde{\mathbf{w}} + \sum_{i=0}^{r_{\max}} \mathbf{B}_{\chi_i} \frac{d^i}{dt^i}\mathbf{r} \\ \tilde{\mathbf{u}} &= \mathbf{C}_\chi \boldsymbol{\chi} + \sum_{i=0}^{r_{\max}} \mathbf{D}_{\mathrm{FF}_i} \frac{d^i}{dt^i}\mathbf{r} \end{aligned} \tag{4.1a}$$

where

$$\mathbf{A}_\chi = \begin{bmatrix} \mathbf{A} + \mathbf{B}_1\mathbf{K}_\mathrm{S} & \mathbf{B}_1\mathbf{K}_\mathrm{I} & \mathbf{B}_1\mathbf{K}_\mathrm{S} & \mathbf{B}_1\mathbf{C}_\mathrm{FF} \\ \mathbf{C}_e + \mathbf{D}_{e_1}\mathbf{K}_\mathrm{S} & \mathbf{D}_{e_1}\mathbf{K}_\mathrm{I} & \mathbf{D}_{e_1}\mathbf{K}_\mathrm{S} & \mathbf{D}_{e_1}\mathbf{C}_\mathrm{FF} \\ \mathbf{A}_\mathrm{O} + \mathbf{K}_\mathrm{O}\mathbf{C} - \mathbf{A} + \mathbf{M}\mathbf{K}_\mathrm{S} & \mathbf{M}\mathbf{K}_\mathrm{I} & \mathbf{A}_\mathrm{O} + \mathbf{M}\mathbf{K}_\mathrm{S} & \mathbf{M}\mathbf{C}_\mathrm{FF} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_\mathrm{FF} \end{bmatrix}, \tag{4.1b}$$

$$\mathbf{B}_{\chi_w} = \begin{bmatrix} \mathbf{B}_2 \\ \mathbf{D}_{e_2} \\ \mathbf{K}_\mathrm{O}\mathbf{D}_2 - \mathbf{B}_2 \\ \mathbf{0} \end{bmatrix}, \qquad \mathbf{B}_{\chi_i} = \begin{bmatrix} \mathbf{B}_1\mathbf{D}_{\mathrm{FF}_i} \\ \mathbf{D}_{e_1}\mathbf{D}_{\mathrm{FF}_i} \\ \mathbf{M}\mathbf{D}_{\mathrm{FF}_i} \\ \mathbf{B}_{\mathrm{FF}_i} \end{bmatrix}, \tag{4.1c}$$

and

$$\mathbf{C}_\chi = \left[\begin{array}{cccc} \mathbf{K}_\mathrm{S} & \mathbf{K}_\mathrm{I} & \mathbf{K}_\mathrm{S} & \mathbf{C}_\mathrm{FF} \end{array} \right] \tag{4.1d}$$

with $\mathbf{M} = \mathbf{B}_\mathrm{O} + \mathbf{K}_\mathrm{O}\mathbf{D}_1 - \mathbf{B}_1$.

First, we observe that the boundedness of $\tilde{\mathbf{u}}$ is implied by the boundedness of $\boldsymbol{\chi}$ and $\mathbf{r}$ and all its time derivatives up to order $r_{\max}$. On one hand, the boundedness of $\mathbf{r}$ is guaranteed by assumption but, on the other hand, exploiting the results of Section 3.3 the boundedness of $\boldsymbol{\chi}$ is linked to the BIBS stability of (4.1a) and thus to the eigenvalues of $\mathbf{A}_\chi$ (which must be Hurwitz). Moreover, the BIBS stability of (4.1a) is twofold because $\mathbf{e}$ is part of $\boldsymbol{\chi}$ and thus the boundedness of the latter implies the boundedness of the former. As consequence, the achievement of the control goals G1 and G2 is related to the ability to make $\mathbf{A}_\chi$ is Hurwitz through the design of the control system.

The remaining sections describe the design of the controller matrices with an incremental approach. More in details, Section 4.2 states the conditions the plant must satisfy to guarantee the existence of $\mathbf{K}_\mathrm{S}$ such that $\mathbf{A} + \mathbf{B}_1\mathbf{K}_\mathrm{S}$ is Hurwitz. Then, Section 4.3 introduces the integral action and designs $\mathbf{K}_\mathrm{I}$ that makes Hurwitz the sub-matrix

$$\left[\begin{array}{cc} \mathbf{A} + \mathbf{B}_1\mathbf{K}_\mathrm{S} & \mathbf{B}_1\mathbf{K}_\mathrm{I} \\ \mathbf{C}_e + \mathbf{D}_{e_1}\mathbf{K}_\mathrm{S} & \mathbf{D}_{e_1}\mathbf{K}_\mathrm{I} \end{array} \right].$$

Section 4.4 describes the conditions the plant must satisfy to let the existence of $\mathbf{K}_\mathrm{O}$ such that $\mathbf{A} - \mathbf{K}_\mathrm{O}\mathbf{C}$ is Hurwitz. Moreover, Section 4.5 provides the design of $\mathbf{A}_\mathrm{O}$ and $\mathbf{B}_\mathrm{O}$, which make Hurwitz the sub-matrix

$$\left[\begin{array}{ccc} \mathbf{A} + \mathbf{B}_1\mathbf{K}_\mathrm{S} & \mathbf{B}_1\mathbf{K}_\mathrm{I} & \mathbf{B}_1\mathbf{K}_\mathrm{S} \\ \mathbf{C}_e + \mathbf{D}_{e_1}\mathbf{K}_\mathrm{S} & \mathbf{D}_{e_1}\mathbf{K}_\mathrm{I} & \mathbf{D}_{e_1}\mathbf{K}_\mathrm{S} \\ \mathbf{A}_\mathrm{O} + \mathbf{K}_\mathrm{O}\mathbf{C} - \mathbf{A} + \mathbf{M}\mathbf{K}_\mathrm{S} & \mathbf{M}\mathbf{K}_\mathrm{I} & \mathbf{A}_\mathrm{O} + \mathbf{M}\mathbf{K}_\mathrm{S} \end{array} \right].$$

Finally, Section 4.6 designs the feedforward control (with $\mathbf{A}_\mathrm{FF}$ Hurwitz) and allows the tracking of time-varying references.

4.2 State Feedback Stabiliser

As introduced in Section 4.1 the achievement of control goals G1 and G2 is obtained through a design strategy whose first step consists in making Hurwitz the matrix $\mathbf{A} + \mathbf{B}_1\mathbf{K}_\mathrm{S}$. First, we need to understand which parts of the plant "can be modified through the control input $\tilde{\mathbf{u}}$"and which are assumptions the plant must verify to let the existence of such $\mathbf{K}_\mathrm{S}$. These results are obtained through the so called reachability Kalman Decomposition. From now on, to keep the notation lighter we refer to the linearised vectors $\tilde{\mathbf{x}}$, $\tilde{\mathbf{y}}$, $\tilde{\mathbf{u}}$, and $\tilde{\mathbf{w}}$ hiding the accent $\tilde{}$.

Let

$$\begin{aligned} \dot{\mathbf{x}} =&\ \mathbf{A}\mathbf{x} + \mathbf{B}_1\mathbf{u} + \mathbf{B}_2\mathbf{w} \quad \mathbf{x}(t_0) = \mathbf{x}_0 \\ \mathbf{y} =&\ \mathbf{C}\mathbf{x} + \mathbf{D}_1\mathbf{u} + \mathbf{D}_2\mathbf{w} \end{aligned} \tag{4.2}$$

be the LTI system obtained by picking the first two equations of (1.17). Assuming $\mathbf{w} \equiv \mathbf{0}$, the integral curve of (4.2) is given by

$$\chi_u(t, \mathbf{x}(t_0)) = \exp(\mathbf{A}(t - t_0))\mathbf{x}(t_0) + \int_{t_0}^{t} \exp(\mathbf{A}(t - \tau))\mathbf{B}_1\mathbf{u}(\tau)d\tau. \tag{4.3}$$

Let us assume that the initial condition is identically null, *i.e.* $\mathbf{x}(t_0) = \mathbf{0}$ and denote with $\mathcal{R}(t)$ the set of the *reachable states from the origin* thanks to any control $\mathbf{u} : [t_0,\ t] \to \mathbb{R}^p$:

$$\mathcal{R}(t) := \{\mathbf{x} \in \mathbb{R}^n\ :\ \mathbf{x} = \chi_u(t, \mathbf{0})\}\,.$$

In practice, the set $\mathcal{R}(t)$ represents the set of states reached from the origin thanks to the whole set of time varying control signals $t \mapsto \mathbf{u}(t)$ (not just one particular control law!).

Remark 4.1. *It can be demonstrated that the reachability set $\mathcal{R}$ is a subspace of $\mathbb{R}^n$, i.e. is a set closed with respect to the multiplication by scalar and with respect to the addition. Therefore, there exists a basis $\{\mathbf{b}_1, \ldots, \mathbf{b}_{\ell_R}\}$ which spans $\mathcal{R}$ where ℓ_R represents the dimension of $\mathcal{R}$. On the other hand, if $\mathcal{R}$ is a subspace of $\mathbb{R}^n$, i.e. if $\mathcal{R} \subset \mathbb{R}^n$, it means that starting from the origin it is impossible to reach the states which do not belong to $\mathcal{R}$. Equivalently, the control $\mathbf{u}(\cdot)$ can not influence the dynamics of the states out of $\mathcal{R}$.*

Theorem 4.1 (Reachability)**.** *Let* (4.2)*, then the matrix*

$$\mathbf{R} := \left[\ \mathbf{B}_1 \quad \mathbf{A}\mathbf{B}_1 \quad \mathbf{A}^2\mathbf{B}_1 \quad \cdots \quad \mathbf{A}^{n-1}\mathbf{B}_1\ \right] \in \mathbb{R}^{n \times np}$$

is such that $im(\mathbf{R}) = \mathcal{R}$. The matrix $\mathbf{R}$ is called **reachability matrix***.*

Infobox 4.2 (Proof of Theorem 4.1)**.**
The proof of the Theorem 4.1 is founded on the Cayley-Hamilton theorem which allows to write

$$\exp(\mathbf{A}t) = \sum_{i=0}^{n-1} \Phi_i(t)\mathbf{A}^i$$

where $\Phi_i : \mathbb{R} \to \mathbb{R}$ *for any* $i = 1, \ldots, n-1$. *Then, write*

$$\begin{aligned}
\mathbf{x}(t) &= \int_0^t \exp\left(\mathbf{A}(t-\tau)\right)\mathbf{B}_1\mathbf{u}(\tau)\,d\tau \\
&= \int_0^t \sum_{i=0}^{n-1} \Phi_i(t-\tau)\mathbf{A}^i\mathbf{B}_1\mathbf{u}(\tau)\,d\tau \\
&= \sum_{i=0}^{n-1} \mathbf{A}^i\mathbf{B}_1 \int_0^t \Phi_i(t-\tau)\mathbf{u}(\tau)\,d\tau \\
&= \mathbf{R} \begin{bmatrix} \int_0^t \Phi_0(t-\tau)\mathbf{u}(\tau)\,d\tau \\ \vdots \\ \int_0^t \Phi_{n-1}(t-\tau)\mathbf{u}(\tau)\,d\tau \end{bmatrix}.
\end{aligned}$$

Conceive the vector of the convolution integrals as a degree of freedom to obtain $\mathbf{x}(t)$ *as linear combination of the elements of* $\mathbf{R}$. *So,* $\mathbf{x}(t)$ *must belong to the image of* $\mathbf{R}$ *to let the equality to be verified, i.e. to allow for the existence of a family of control laws* $\mathbf{u}(t)$ *that verify the equality.*

Infobox 4.3 (Reachability and Controllability).
Beside the ***reachability set****, the control system theory developed the concept of* ***controllability set*** *that, for LTI systems, can be defined as follows. Let* (4.2)*, then* $\bar{\mathbf{x}} \in \mathbb{R}^n$ *belongs to the controllability set, namely* $\mathcal{C}(t)$*, if there exist a control law* $\tau \in [t_0,\, t] \mapsto \mathbf{u}(\tau)$ *such that*

$$\mathbf{0} = \exp(\mathbf{A}(t-t_0))\bar{\mathbf{x}} + \int_{t_0}^t \exp(\mathbf{A}(t-\tau))\mathbf{B}_1\mathbf{u}(\tau)d\tau.$$

Since $\exp(\mathbf{A}(t-t_0))$ *is invertible, the latter equality can be re-arranged as*

$$\bar{\mathbf{x}} = -\exp(-\mathbf{A}(t-t_0))\int_{t_0}^t \exp(\mathbf{A}(t-\tau))\mathbf{B}_1\mathbf{u}(\tau)d\tau.$$

Now, note that $\exp(-\mathbf{A}(t-t_0))$ *is full rank and that*

$$\int_{t_0}^t \exp(\mathbf{A}(t-\tau))\mathbf{B}_1\mathbf{u}(\tau)d\tau \in \mathcal{R}.$$

As consequence, one can conclude that if $\bar{\mathbf{x}} \in \mathcal{C}$ *then* $\bar{\mathbf{x}} \in \mathcal{R}$*, and vice-versa, which demonstrates that the reachability subspace is equivalent to* *the controllability subspace (only for LTI systems!). For this reason, the terms* ***reachability*** *and* ***controllability*** *are interchangeable (for instance this book adopts the former while MATLAB uses the latter, e.g. in the command* `ctrb`*).*

Keynote

An LTI system is said to be **completely reachable** if $\mathrm{rank}(\mathbf{R}) = n$, *i.e.* if $\mathcal{R} \equiv \mathbb{R}^n$.

Intuitively, the reachability matrix form is derived as follows. Assume that $\mathbf{x}(t_0) = \mathbf{0}$ and take a constant control action, $\mathbf{u}$. Select p particular control inputs as

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \ldots, \mathbf{u}_p = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

and evaluate the first time derivatives $\dot{\mathbf{x}}_i(t_0) = \mathbf{B}_1\mathbf{u}_i$, for each of these control inputs, collecting them in the matrix

$$\mathbf{X}^{(1)} := \begin{bmatrix} \frac{d\mathbf{x}_1}{dt} & \frac{d\mathbf{x}_2}{dt} & \cdots & \frac{d\mathbf{x}_p}{dt} \end{bmatrix} = \mathbf{B}_1 \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_p \end{bmatrix}$$

where it is important to remark that $\begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_p \end{bmatrix} = \mathbf{I}$ (hereafter it will be hidden). On the other hand, the higher order time derivatives of $\mathbf{x}$ at time t_0 are obtained as $\frac{d^k\mathbf{x}_i}{dt^k}(t_0) = \mathbf{A}^{k-1}\mathbf{B}_1\mathbf{u}_i$ for $k = 1, \ldots, n$ and $i = 1, \ldots, p$. For each $k = 1, \ldots, n$, build the matrices $\mathbf{X}^{(i)}$ and organize them in a matrix as

$$\begin{bmatrix} \mathbf{X}^{(1)} & \mathbf{X}^{(2)} & \cdots & \mathbf{X}^{(n)} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_1 & \mathbf{A}\mathbf{B}_1 & \cdots & \mathbf{A}^{n-1}\mathbf{B}_1 \end{bmatrix}.$$

In conclusion, the reachability matrix represents the basis for the description of the state time derivatives up to the order n.

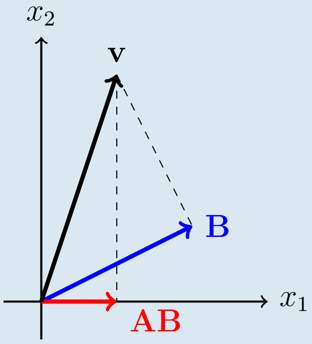
Example 4.4 (Reachability study). *Let the system* $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$ *where* $\mathbf{x} \in \mathbb{R}^2$, $u \in \mathbb{R}$ *and*

$$\mathbf{A} = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

with a, b_1, $b_2 > 0$. *The reachability matrix is then given by*

$$\mathbf{R} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} & \begin{pmatrix} ab_2 \\ 0 \end{pmatrix} \end{bmatrix}.$$

Draw the vectors $\mathbf{B}$ *and* $\mathbf{AB}$ *on the plane* x_1, x_2 *as*



The vectors $\mathbf{B}$ *and* $\mathbf{AB}$ *are linearly independent because they have different directions and thus they represent a valid base for the representation of any vector* $\mathbf{v}$ *in the space* $x_1 - x_2$. *More formally, the vectors* $\mathbf{B}$ *and* $\mathbf{AB}$ *span the whole state space* $\mathbb{R}^2$ *implying a fully reachable system.*

Example 4.5 (Reachability study). *Let the system defined in Example 4.4, and assume* $b_2 = 0$. *The reachability matrix becomes*

$$\mathbf{R} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} b_1 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{bmatrix}$$

which implies the vectors $\mathbf{B}$ *and* $\mathbf{AB}$ *do not span the whole state space and, equivalently, they cannot represent any vector* $\mathbf{v} \in \mathbb{R}^2$ *but only those on the* x_1 *direction. As consequence, the system is not fully reachable and moreover the reachable subspace is spanned by* $\mathbf{B}$ *(which represent the* x_1 *direction). The non-reachable subspace is identified by the direction orthogonal to* $\mathbf{B}$, *i.e. by* $\mathbf{w} = col(0, 1)$.

Example 4.6 (Cart-pole reachability study)**.** *Let*

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & A_{23} & A_{42} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & A_{43} & A_{44} \end{bmatrix}, \; \mathbf{B}_1 = \begin{bmatrix} 0 \\ B_{12} \\ 0 \\ B_{14} \end{bmatrix}$$

defined in Example 1.2, with the assumptions made in Example 3.2. The reachability matrix is obtained as

$$\mathbf{R} = \left[\begin{pmatrix} 0 \\ B_{12} \\ 0 \\ B_{14} \end{pmatrix} \begin{pmatrix} B_{12} \\ A_{24}B_{14} \\ B_{14} \\ A_{44}B_{14} \end{pmatrix} \begin{pmatrix} B_{12} \\ A_{23}B_{14} + A_{24}A_{44}B_{14} \\ A_{44}B_{14} \\ A_{43}B_{14} + A_{44}^2 B_{14} \end{pmatrix} \right.$$
$$\left. \begin{pmatrix} B_{12} \\ A_{23}A_{44}B_{14} + A_{24}(A_{43}B_{14} + A_{44}^2 B_{14}) \\ A_{43}B_{14} + A_{44}^2 B_{14} \\ A_{43}A_{44}B_{14} + A_{44}(A_{43}B_{14} + A_{44}^2 B_{14}) \end{pmatrix} \right].$$

The substitution of the numerical values listed in Example 3.2 leads to a full rank matrix $\mathbf{R}$ *that implies a fully reachable system.*

For the definition of the orthogonal complement of a subspace, $(\cdot)^\perp$, the image $\mathrm{im}(\cdot)$ and the kernel $\ker(\cdot)$, see Appendix A.

Since the reachability subspace $\mathcal{R}$ is a subspace of $\mathbb{R}^n$, there exists also its orthogonal complement, namely $\mathcal{R}^\perp$, such that $\mathcal{R} \oplus \mathcal{R}^\perp = \mathbb{R}^n$. Since the image of $\mathbf{R}$ corresponds to $\mathcal{R}$, then $\ker\left(\mathbf{R}^\top\right) = \mathcal{R}^\perp$. As consequence, the matrices $\mathrm{im}(\mathbf{R})$ and $\ker\left(\mathbf{R}^\top\right)$ can be exploited to identify a change of coordinate, namely $\mathbf{z} = \mathbf{T}_\mathrm{R}\mathbf{x}$, which highlights the reachable part of (4.2). In particular, the transformation matrix $\mathbf{T}_\mathrm{R}$ is defined as

$$\mathbf{T}_\mathrm{R}^{-1} = \left[\; \mathrm{im}\,(\mathbf{R}) \quad \ker\left(\mathbf{R}^\top\right) \; \right]$$

which, applied to (4.2), leads to

$$\begin{aligned} \dot{\mathbf{z}} &= \mathbf{T}_\mathrm{R}\mathbf{A}\mathbf{T}_\mathrm{R}^{-1}\mathbf{z} + \mathbf{T}_\mathrm{R}\mathbf{B}_1\mathbf{u} + \mathbf{T}_\mathrm{R}\mathbf{B}_2\mathbf{w} \quad \mathbf{z}(t_0) = \mathbf{T}_\mathrm{R}\mathbf{x}_0 \\ \mathbf{y} &= \mathbf{C}\mathbf{T}_\mathrm{R}^{-1}\mathbf{z} + \mathbf{D}_1\mathbf{u} + \mathbf{D}_2\mathbf{w}. \end{aligned} \tag{4.4}$$

The study of $\bar{\mathbf{A}} := \mathbf{T}_\mathrm{R}\mathbf{A}\mathbf{T}_\mathrm{R}^{-1}$, $\bar{\mathbf{B}}_1 := \mathbf{T}_\mathrm{R}\mathbf{B}_1$, $\bar{\mathbf{B}}_2 := \mathbf{T}_\mathrm{R}\mathbf{B}_2$, and $\bar{\mathbf{C}} := \mathbf{C}\mathbf{T}_\mathrm{R}^{-1}$ reveals that

$$\bar{\mathbf{A}} = \begin{bmatrix} \bar{\mathbf{A}}_{11} & \bar{\mathbf{A}}_{12} \\ \mathbf{0} & \bar{\mathbf{A}}_{22} \end{bmatrix} \quad \bar{\mathbf{B}}_1 = \begin{bmatrix} \bar{\mathbf{B}}_{11} \\ \mathbf{0} \end{bmatrix} \quad \bar{\mathbf{B}}_2 = \begin{bmatrix} \bar{\mathbf{B}}_{21} \\ \bar{\mathbf{B}}_{22} \end{bmatrix} \quad \bar{\mathbf{C}} = \left[\; \bar{\mathbf{C}}_1 \quad \bar{\mathbf{C}}_2 \; \right]$$

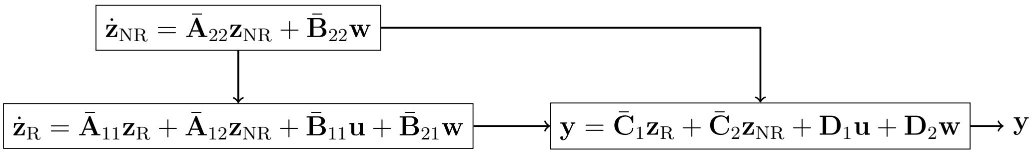


Figure 4.1: Reachability decomposition

so that, if the state is defined as $\mathbf{z} = \mathrm{col}(\mathbf{z}_{\mathrm{R}}, \mathbf{z}_{\mathrm{NR}})$, the dynamics of (4.4) becomes (see Fig. 4.1):

$$
\begin{array}{ccccccccc}
\dot{\mathbf{z}}_{\mathrm{R}} & = & \bar{\mathbf{A}}_{11}\,\mathbf{z}_{\mathrm{R}} & + & \bar{\mathbf{A}}_{12}\,\mathbf{z}_{\mathrm{NR}} & + & \bar{\mathbf{B}}_{11}\mathbf{u} & + & \bar{\mathbf{B}}_{21}\mathbf{w} \\
\dot{\mathbf{z}}_{\mathrm{NR}} & = & & & \bar{\mathbf{A}}_{22}\,\mathbf{z}_{\mathrm{NR}} & & & + & \bar{\mathbf{B}}_{22}\mathbf{w} \\
\mathbf{y} & = & \bar{\mathbf{C}}_{1}\,\mathbf{z}_{\mathrm{R}} & + & \bar{\mathbf{C}}_{2}\,\mathbf{z}_{\mathrm{NR}} & + & \mathbf{D}_{1}\mathbf{u} & + & \mathbf{D}_{2}\mathbf{w} \\
\left[\begin{array}{c} \mathbf{z}_{\mathrm{R}}(t_0) \\ \mathbf{z}_{\mathrm{NR}}(t_0) \end{array}\right] & = & \mathbf{T}_{\mathrm{R}}\,\mathbf{x}_0 & & & & & &
\end{array}
\tag{4.5}
$$

which clearly shows that the dynamics of $\mathbf{z}_{\mathrm{NR}}$ is not (and cannot be) influenced by the control $\mathbf{u}(\cdot)$. It is important to note that the dynamics of

$$\dot{\mathbf{z}}_{\mathrm{NR}} = \bar{\mathbf{A}}_{22}\mathbf{z}_{\mathrm{NR}} + \bar{\mathbf{B}}_{22}\mathbf{w} \quad \mathbf{z}_{\mathrm{NR}}(t_0) = \mathbf{z}_{NR_0}$$

is totally determined by the properties of $\bar{\mathbf{A}}_{22}$.

Keynote

Accordingly to the nature of the eigenvalues and eigenvectors of $\bar{\mathbf{A}}_{22}$, the dynamics of the non reachable sub–part of the state, $\mathbf{z}_{\mathrm{NR}}$, can be either BIBS stable or not and we can do nothing to modify it!

Example 4.7 (Reachability Kalman decomposition)**.** *The reachability matrix associated to the couple*

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

is

$$\mathbf{R} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Since rank$(\mathbf{R}) = 2$*, the reachable subspace has dimensions 2 whereas the non reachable one is one-dimensional. Thus, one possible basis for the* *representation of the reachable subspace is*

$$\{\mathbf{b}_1,\, \mathbf{b}_2\} = \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

while the basis for the representation of the non reachable subspace is obtained as

$$\mathbf{b}_3 := ker\left(\begin{bmatrix} \mathbf{b}_1^\top \\ \mathbf{b}_2^\top \end{bmatrix} \right) = ker\left(\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \right) = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

So, the reachability Kalman decomposition relies on the transformation matrix

$$\mathbf{T}_R^{-1} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{bmatrix} = \left[\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right].$$

Finally, the matrices $\bar{\mathbf{A}} = \mathbf{T}_R \mathbf{A} \mathbf{T}_R^{-1}$ *and* $\bar{\mathbf{B}} = \mathbf{T}_R \mathbf{B}$ *are*

$$\bar{\mathbf{A}} = \begin{bmatrix} 0 & 0 & | & -2 \\ 1 & 1 & | & 0 \\ -- & -- & -- & -- \\ 0 & 0 & | & -1 \end{bmatrix} \qquad \bar{\mathbf{B}} = \begin{bmatrix} 1 \\ 0 \\ -- \\ 0 \end{bmatrix}.$$

Example 4.8 (Cart-pole reach. Kalman decomposition)**.** *Example 4.6 shows that the cart-pole plant of Example 1.2 is fully reachable. As consequence, the non reachable subspace is zero-dimensional and is trivially represented by the origin* $\mathbf{0}$*. Thus, the transformation* $\mathbf{T}_R^{-1} = \mathbf{I}$ *meaning that the linear system of Example 4.6 is already in the reachability Kalman form.*

As for the reachable subsystem, assume $\mathbf{z}_{\mathrm{NR}} = \mathbf{0}$

$$\dot{\mathbf{z}}_{\mathrm{R}} = \bar{\mathbf{A}}_{11}\mathbf{z}_{\mathrm{R}} + \bar{\mathbf{B}}_1\mathbf{u} \quad \mathbf{z}_{\mathrm{R}}(t_0) = \mathbf{z}_{R_0} \tag{4.6}$$

and note that (4.6) is completely reachable by construction.

Theorem 4.9 (Existence of a stabilising state feedback)**.**
Let $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ *be a fully reachable LTI system. Then, there exists a matrix* $\mathbf{K}_R$ *such that* $\mathbf{A} + \mathbf{B}\mathbf{K}_R$ *is Hurwitz.*

Infobox 4.10 (Sketch of the Proof of Theorem 4.9).
To reduce the complexity of this proof, assume that the input is scalar, i.e. $u \in \mathbb{R}$. *Introduce the following system*

$$\dot{\mathbf{z}} = \bar{\mathbf{A}}\mathbf{z} + \bar{\mathbf{B}}u$$

with

$$\bar{\mathbf{A}} := \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \cdots & \ddots & 0 \\ 0 & 0 & \cdots & \cdots & 1 \\ -\alpha_0 & -\alpha_1 & \dots & -\alpha_{n-2} & -\alpha_{n-1} \end{bmatrix}, \qquad \bar{\mathbf{B}} := \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

and where $\alpha_0, \dots, \alpha_{n-1}$ *are the coefficients of the polynomial*

$$\det\left(\bar{\mathbf{A}} - \lambda\mathbf{I}\right) := \alpha_0 + \alpha_1\lambda + \cdots + \alpha_{n-1}\lambda^{n-1} + \lambda^n.$$

It is worth to note that these coefficients are directly linked to the eigenvalues of $\bar{\mathbf{A}}$ *which are the roots of* $\det\left(\bar{\mathbf{A}} - \lambda\mathbf{I}\right) = 0$. *So, any change in* $\alpha_0, \dots, \alpha_{n-1}$ *results in a change of the eigenvalues of the plant. On the other hand, the couple* $(\bar{\mathbf{A}}, \bar{\mathbf{B}})$ *is fully reachable and therefore the matrix*

$$\mathbf{R}_z := \begin{bmatrix} \bar{\mathbf{B}} & \cdots & \bar{\mathbf{A}}^{n-1}\bar{\mathbf{B}} \end{bmatrix}$$

is full rank. Now, any feedback $u := \mathbf{K}_R\mathbf{z}$ *with the matrix*

$$\mathbf{K}_S := \begin{bmatrix} k_0 & \cdots & k_{n-1} \end{bmatrix}$$

leads to a closed loop system $\dot{\mathbf{z}} = \left(\bar{\mathbf{A}} + \bar{\mathbf{B}}\mathbf{K}_R\right)\mathbf{z}$ *with*

$$\bar{\mathbf{A}} + \bar{\mathbf{B}}\mathbf{K}_R := \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \cdots & \ddots & 0 \\ 0 & 0 & \cdots & \cdots & 1 \\ k_0 - \alpha_0 & k_1 - \alpha_1 & \dots & k_{n-2} - \alpha_{n-2} & k_{n-1} - \alpha_{n-1} \end{bmatrix}.$$

As consequence, all the roots of $\det\left(\bar{\mathbf{A}} + \bar{\mathbf{B}}\mathbf{K}_R - \lambda\mathbf{I}\right) = 0$ *can be modified and be made negative real part. The last step in this proof consists in demonstrating that, for any scalar input system* $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$, *fully* *reachable, there exists a change of coordinates* $\mathbf{T}$ *such that* $\mathbf{z} = \mathbf{T}\mathbf{x}$, $\bar{\mathbf{A}} := \mathbf{T}\mathbf{A}\mathbf{T}^{-1}$, *and* $\bar{\mathbf{B}} := \mathbf{T}\mathbf{B}$*. To show the existence of* $\mathbf{T}$ *we note that if* $(\mathbf{A}, \mathbf{B})$ *is fully reachable, the matrix*

$$\mathbf{R}_x := \left[\begin{array}{ccc} \mathbf{B} & \cdots & \mathbf{A}^{n-1}\mathbf{B} \end{array}\right]$$

is full rank. On the other hand, $\mathbf{R}_z$ *is rewritten as*

$$\mathbf{R}_z := \mathbf{T}\left[\begin{array}{ccc} \mathbf{B} & \cdots & \mathbf{A}^{n-1}\mathbf{B} \end{array}\right]$$

which, compared with $\mathbf{R}_x$*, leads to* $\mathbf{R}_z = \mathbf{T}\mathbf{R}_x$*. Then, define* $\mathbf{T} = \mathbf{R}_z\mathbf{R}_x^{-1}$*. This proof strategy can be extended to plants with multidimensional inputs by means of a suitable choices of the reachability subspaces associated to each control input entry with the goal of obtaining a system, in the* $\mathbf{z}$ *coordinates, whose* $\bar{\mathbf{A}} + \bar{\mathbf{B}}\mathbf{K}_R$ *matrix is a block diagonal matrix whose blocks have the structure of the plant exploited in this proof.*

Keynote

The direct consequence of the Theorem 4.9 is that there exists a **state feedback control law** $\mathbf{u} = \mathbf{K}_{\mathrm{R}}\mathbf{z}_{\mathrm{R}}$ which makes the reachable system (4.6) BIBS stable in closed loop.

The substitution of $\mathbf{u} = \mathbf{K}_{\mathrm{R}}\mathbf{z}_{\mathrm{R}}$ into (4.6) leads to

$$\dot{\mathbf{z}}_{\mathrm{R}} = \left(\bar{\mathbf{A}}_{11} + \bar{\mathbf{B}}_{11}\mathbf{K}_{\mathrm{R}}\right)\mathbf{z}_{\mathrm{R}} + \bar{\mathbf{B}}_{21}\mathbf{w} \quad \mathbf{z}_{\mathrm{R}}(t_0) = \mathbf{z}_{R_0} \tag{4.7}$$

whose block scheme is depicted in Fig. 4.2. Moreover, the dynamics of (4.5)

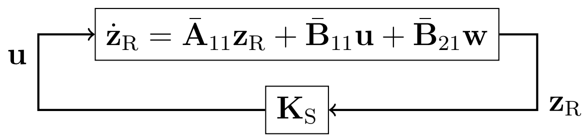


Figure 4.2: The stabiliser based on a state feedback architecture.

subject to $\mathbf{u} = \mathbf{K}_{\mathrm{R}}\mathbf{z}_{\mathrm{R}}$ is given by

$$\begin{array}{ccccccc}
\dot{\mathbf{z}}_{\mathrm{R}} & = & \left(\bar{\mathbf{A}}_{11} + \mathbf{B}_1\mathbf{K}_{\mathrm{R}}\right)\mathbf{z}_{\mathrm{R}} & + & \bar{\mathbf{A}}_{12}\mathbf{z}_{\mathrm{NR}} & + & \bar{\mathbf{B}}_{21}\mathbf{w} \\
\dot{\mathbf{z}}_{\mathrm{NR}} & = & & & \bar{\mathbf{A}}_{22}\mathbf{z}_{\mathrm{NR}} & + & \bar{\mathbf{B}}_{22}\mathbf{w} \\
\mathbf{y} & = & (\bar{\mathbf{C}}_1 + \mathbf{D}_1\mathbf{K}_{\mathrm{R}})\mathbf{z}_{\mathrm{R}} & + & \bar{\mathbf{C}}_2\mathbf{z}_{\mathrm{NR}} & + & \bar{\mathbf{D}}_2\mathbf{w} \\
\left[\begin{array}{c} \mathbf{z}_{\mathrm{R}}(t_0) \\ \mathbf{z}_{\mathrm{NR}}(t_0) \end{array}\right] & = & \mathbf{T}\mathbf{x}_0 & & & &
\end{array} \tag{4.8}$$

in which the dynamics of the reachable part is influenced by the exogenous signal $\mathbf{z}_{\mathrm{NR}}$. Since the reachable subsystem is BIBS thanks to the stability properties of $\bar{\mathbf{A}}_{11} + \mathbf{B}_1\mathbf{K}_{\mathrm{R}}$, the state $\mathbf{z}_{\mathrm{R}}(t)$ is bounded if $\mathbf{z}_{\mathrm{NR}}$ is bounded. On the other hand, the state $\mathbf{z}_{\mathrm{NR}}$ is bounded if $\bar{\mathbf{A}}_{22}$ is Hurwitz.

Keynote

A LTI system is said to be stabilisable if the non–reachable state is BIBS stable, *i.e.* if $\bar{\mathbf{A}}_{22}$ is Hurwitz.

From now on, this book adopts the following assumption.

Assumption 4.1. *The plant* (1.17) *is stabilisable.*

To conclude, with reference to (4.1a), we use Assumptions 4.1, Theorem 4.9, and

$$\mathbf{u} = \begin{bmatrix} \mathbf{K}_{\mathrm{R}} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{z}_{\mathrm{R}} \\ \mathbf{z}_{\mathrm{NR}} \end{bmatrix} = \begin{bmatrix} \mathbf{K}_{\mathrm{R}} & \mathbf{0} \end{bmatrix} \mathbf{T}_{\mathrm{R}}\mathbf{x}$$

to guarantee that $\mathbf{K}_{\mathrm{S}} := \begin{bmatrix} \mathbf{K}_{\mathrm{R}} & \mathbf{0} \end{bmatrix} \mathbf{T}_{\mathrm{R}}$ makes $\mathbf{A} + \mathbf{B}_1\mathbf{K}_{\mathrm{S}}$ Hurwitz.

Before moving to the next session, it is worth to note that the control law $\mathbf{u} = \mathbf{K}_{\mathrm{S}}\mathbf{x}$ applied to (4.2) makes BIBS the closed loop

$$\dot{\mathbf{x}} = (\mathbf{A} + \mathbf{B}_1\mathbf{K}_{\mathrm{S}})\mathbf{x} + \mathbf{B}_2\mathbf{w}$$

which, under Assumption 1.1, achieves the goal G1. Ideally, this law solves all those problems in which the main control goal is represented by the stabilisation of the linearisation point (*e.g.* vibration suppression via active suspensions, yaw damper, lane keeping, etc.). But

- its performance could be improved to achieve an asymptotic tracking, *i.e.* to have $\lim_{t\to\infty} \mathbf{e}(t) = \mathbf{0}$, at least for constant $\mathbf{w}$. Indeed, this scenario (called *set-point regulation*) represents one of the most common operations in automotive applications (*e.g.* speed control, vehicle height regulation, roll control, lane changing, etc.). Section 4.3 extends the control architecture to deal with this requirement

- it is not implementable because it requires a perfect knowledge of $\mathbf{x}$. This problem is overcame in Section 4.5 through the adoption of the observer described in Section 4.4

- the control law can be extended to feedback also the non reachable part, namely as $\mathbf{u} = [\mathbf{K}_{\mathrm{R}}\ \mathbf{K}_{\mathrm{NR}}]\mathbf{z}$. Indeed, even if $\mathbf{K}_{\mathrm{NR}}$ can't contribute to the BIBS stability it can be used to improve the performance of the closed-loop plant. Section 5.1 provides an optimal design criterion for $\mathbf{K}_{\mathrm{NR}}$.

4.3 Integral Action

As anticipated in Section 4.1 and reiterated at the end of Section 4.2, this section increments the controller design complexity by introducing the integral control action. First, this section aims to design $\mathbf{K}_\mathrm{S}$ and $\mathbf{K}_\mathrm{I}$ exploiting the results of Section 4.2. Second, the benefits associated to the introduction to the control action are highlighted. Indeed, with reference to (4.1a) we show that assuming $\mathbf{e}_x \equiv \mathbf{0}$ the integral action asymptotically steers to zero the regulated output for any constant reference $\mathbf{r}$ and despite the presence of unknown but constant disturbances $\mathbf{d}$.

Let

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\boldsymbol{\eta}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{C}_e & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \boldsymbol{\eta} \end{bmatrix} + \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{D}_{e_1} \end{bmatrix} \mathbf{u} + \begin{bmatrix} \mathbf{B}_2 \\ \mathbf{D}_{e_2} \end{bmatrix} \mathbf{w} \tag{4.9}$$

be the LTI system obtained by picking the first and the last equations of (1.17) with $\dot{\boldsymbol{\eta}} := \mathbf{e}$. Then, as done in Section 4.2 we first study the reachability of (4.9) through the matrix

$$\mathbf{R}_e := \begin{bmatrix} \mathbf{B}_1 & \mathbf{A}\mathbf{B}_1 & \cdots & \mathbf{A}^{n-1}\mathbf{B}_1 & \mathbf{A}^{n}\mathbf{B}_1 & \cdots & \mathbf{A}^{n+m-1}\mathbf{B}_1 \\ \mathbf{D}_{e_2} & \mathbf{C}_e\mathbf{B}_1 & \cdots & \mathbf{C}_e\mathbf{A}^{n-2}\mathbf{B}_1 & \mathbf{C}_e\mathbf{A}^{n-1}\mathbf{B}_1 & \cdots & \mathbf{C}_e\mathbf{A}^{n+m-2}\mathbf{B}_1 \end{bmatrix}$$

where the first row represents the reachability matrix of the couple $(\mathbf{A}, \mathbf{B}_1)$, namely $\mathbf{R}$. On the other hand, we note that the second row can be written as $[\mathbf{D}_{e_2}\ \mathbf{C}_e\mathbf{R}]$ and that $\ker([\mathbf{D}_{e_2}\ \mathbf{C}_e\mathbf{R}]^\top) \subseteq \ker((\mathbf{C}_e\mathbf{R})^\top)$. This implies that the non-reachable subspace associated to the second row of $\mathbf{R}_e$ is at most a projection, through $\mathbf{C}_e$, of the non-reachable subspace of the first row of $\mathbf{R}_e$. Then, system (4.9) is stabilisable under Assumption 4.1 and there exists a couple $(\mathbf{K}_\mathrm{S}, \mathbf{K}_\mathrm{I})$ that makes Hurwitz

$$\begin{bmatrix} \mathbf{A} + \mathbf{B}_1\mathbf{K}_\mathrm{S} & \mathbf{B}_1\mathbf{K}_\mathrm{I} \\ \mathbf{C}_e + \mathbf{D}_{e_1}\mathbf{K}_\mathrm{S} & \mathbf{D}_{e_1}\mathbf{K}_\mathrm{I} \end{bmatrix}. \tag{4.10}$$

Thanks to this result, we define

$$\mathbf{u} = \mathbf{K}_\mathrm{S}\mathbf{x} + \mathbf{K}_\mathrm{I}\boldsymbol{\eta} \tag{4.11}$$

and we demonstrate that it asymptotically steers $\mathbf{e}$ to zero if $\mathbf{w}$ is constant. Substitute (4.11) into (4.9)

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\boldsymbol{\eta}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} + \mathbf{B}_1\mathbf{K}_\mathrm{S} & \mathbf{B}_1\mathbf{K}_\mathrm{I} \\ \mathbf{C}_e + \mathbf{D}_{e_1}\mathbf{K}_\mathrm{S} & \mathbf{D}_{e_1}\mathbf{K}_\mathrm{I} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \boldsymbol{\eta} \end{bmatrix} + \begin{bmatrix} \mathbf{B}_2 \\ \mathbf{D}_{e_2} \end{bmatrix} \mathbf{w} \tag{4.12}$$

and assume this system is BIBS. Then, any bounded exogenous $t \mapsto \mathbf{w}(t)$ leads to a bounded trajectory $t \mapsto (\mathbf{x}(t), \boldsymbol{\eta}(t))$. In particular, the integral curve of (4.12) evaluated at $\mathbf{w}$ constant is such that

$$\lim_{t\to\infty}\begin{bmatrix}\mathbf{x}(t)\\ \boldsymbol{\eta}(t)\end{bmatrix}=\begin{bmatrix}\mathbf{A}+\mathbf{B}_1\mathbf{K}_\mathrm{S} & \mathbf{B}_1\mathbf{K}_\mathrm{I}\\ \mathbf{C}_e+\mathbf{D}_{e_1}\mathbf{K}_\mathrm{S} & \mathbf{D}_{e_1}\mathbf{K}_\mathrm{I}\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{B}_2\\ \mathbf{D}_{e_2}\end{bmatrix}\mathbf{w} \tag{4.13}$$

which is well defined and bounded because (4.10) has been assumed Hurwitz (and thus invertible). As consequence $\lim_{t\to\infty}\dot{\boldsymbol{\eta}}(t)=\lim_{t\to\infty}\mathbf{e}(t)=\mathbf{0}$.

Example 4.11 (PI control)**.**
Let

$$\begin{aligned}\dot{x}&=ax+b_1u+b_2d\\ y&=cx\end{aligned}$$

be an LTI system with x, y, u, d, a, b_1, b_2, $c\in\mathbb{R}$, b_1, b_2, $c\neq 0$ *and* $\dot{d}=0$. *We want to asymptotically steer* $x\to 0$ *despite the presence of* d. *We start solving this problem by noting that the couple* (a,b_1) *is fully reachable. So, we design* $u=k_Sx+k_I\eta$ *with* $\dot{\eta}=y$. *Then, the controller takes the form*

$$\begin{aligned}\dot{\eta}&=y\\ u&=k_Sx+k_I\eta\end{aligned}$$

which represents a standard Proportional-Integral controller.

Example 4.12 (Cart-pole set point regulation)**.**
The cart-pole model defined in Example 1.2 is described by

$$\dot{\mathbf{x}}=\mathbf{A}\mathbf{x}+\mathbf{B}_1u+\mathbf{B}_2d$$

in which $\mathbf{x}:=\mathtt{col}(p-p_0-v_0(t-t_0),v-v_0,\theta-\theta_0,0)$, $u=f_x-f_{x0}$, $d=\mathtt{w}$, *and*

$$\mathbf{A}:=\begin{bmatrix}0&1&0&0\\0&A_{22}&A_{23}&A_{24}\\0&0&0&1\\0&A_{42}&A_{43}&A_{44}\end{bmatrix},\ \mathbf{B}_1:=\begin{bmatrix}0\\B_{12}\\0\\B_{14}\end{bmatrix},\ \mathbf{B}_2:=\begin{bmatrix}0\\B_{22}\\0\\B_{24}\end{bmatrix}.$$

Let $p_R(t)$ *be the reference position and* $e:=p-p_R(t)$ *be the tracking error, define* $\dot{\eta}=e$, $\mathbf{C}_e=\begin{bmatrix}1&0&0&0\end{bmatrix}$ *and* $\mathbf{w}:=col(\mathtt{w},p_R(t))$, *and* *extend the plant as*

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{C}_e & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} \mathbf{B}_1 \\ 0 \end{bmatrix} u + \begin{bmatrix} \mathbf{B}_2 & \mathbf{0} \\ 0 & -1 \end{bmatrix} \mathbf{w}.$$

Then, since this system is fully reachable there exists a control law $u = [\mathbf{K}_S \; k_I]$ *that makes Hurwitz*

$$\begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{C}_e & 0 \end{bmatrix} + \begin{bmatrix} \mathbf{B}_1 \\ 0 \end{bmatrix} \begin{bmatrix} \mathbf{K}_S & k_I \end{bmatrix}.$$

4.4 State Observer

Section 4.2 showed that the control law $\mathbf{u} = \mathbf{K}_\mathrm{S}\mathbf{x}$ guarantees the BIBS stability of the closed-loop plant. The same section pointed out that this control law cannot be implemented because $\mathbf{x}$ is unknown. Then, how can we "estimate" $\mathbf{x}$ starting from the available informations $\mathbf{y}$ and $\mathbf{u}$? This section introduces a further dynamic system, called observer, which answers this question. The design of this such a system is achieved through a change of coordinates (dual to the reachability one) which reveals which parts of $\mathbf{x}$ can be "estimated" and which cannot.

Let (4.2) be the plant and assume $\mathbf{u}(\tau), \mathbf{w}(\tau) \equiv \mathbf{0}$ for all $\tau \in [t_0, t]$. As consequence, the integral curve $\boldsymbol{\chi}_\mathbf{0}(t, \mathbf{x}(t_0))$ is only given by the free evolution

$$\boldsymbol{\chi}_\mathbf{0}(t, \mathbf{x}(t_0)) = \exp(\mathbf{A}(t - t_0))\mathbf{x}(t_0) \tag{4.14}$$

which leads to

$$\mathbf{y}(t) = \mathbf{C}\exp(\mathbf{A}(t - t_0))\mathbf{x}(t_0). \tag{4.15}$$

We want to understand if, and eventually which part of, $\mathbf{x}(t_0)$ can be reconstructed from the output $\mathbf{y}(\tau)$ with $\tau \in [t_0, t]$. With this aim we define the *unobservability subspace at time* $t_1 > t_0$, namely $\mathcal{E}(t_1)$, as

$$\mathcal{E}(t_1) = \{\mathbf{x} \in \mathbb{R}^n \; : \; \mathbf{C}\exp(\mathbf{A}(t - t_0))\mathbf{x} \equiv \mathbf{0}, \, \forall t \in [t_0, t_1]\}.$$

In practice, $\mathcal{E}(t_1)$ represents the set of initial conditions which lead to a null output for any time belonging to the interval $[t_0, t_1]$. If $\mathbf{x} \in \mathcal{E}(t_1)$, then $\mathbf{C}\exp(\mathbf{A}(t - t_0))\mathbf{x} = \mathbf{C}\exp(\mathbf{A}(t - t_0))\mathbf{0} = \mathbf{0}$ and so $\mathbf{x}$ is not "distinguishable" from the origin because they produce the same (null) output.

Theorem 4.13 (Unobservability). *Let* (4.2) *be the plant, then its unobservability subspace corresponds to* $\ker(\mathbf{O})$ *where*

$$\mathbf{O} := \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix}$$

is called **observability matrix***.*

Infobox 4.14 (Proof of Theorem 4.13)**.**
The Theorem 4.13 is proved with a joint use of the arguments of the proof of Theorem 4.1, detailed in the Infobox 4.2, and of the concept of duality, presented in Section 5.2.

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A LTI system is said to be **completely observable** if $\ker(\mathbf{O}) \equiv \{\mathbf{0}\}$

Intuitively, the observability matrix is derived through the following procedure. Given the output $\mathbf{y} = \mathbf{Cx}$ and assuming a trivial control input $\mathbf{u} \equiv \mathbf{0}$, let us consider n particular vectors, namely $\mathbf{x}_i$ with $i = 1, \ldots, n$, as

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \ldots, \mathbf{x}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

and collect the relative outputs $\mathbf{y}_i = \mathbf{Cx}_i$ in a matrix as

$$\mathbf{Y}^{(0)} := \begin{bmatrix} \mathbf{y}_1 & \mathbf{y}_2 & \cdots & \mathbf{y}_n \end{bmatrix} = \mathbf{C} \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \end{bmatrix},$$

where $\begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \end{bmatrix} = \mathbf{I}$ will be hidden in what follows. Therefore, let us assume that we can measure (ideally) also the higher order time derivative of $\mathbf{y}$, *i.e.* we can access to $\dfrac{d^k\mathbf{y}_i}{dt^k} = \mathbf{CA}^k\mathbf{x}_i$ for $i = 1, \ldots, n$ and $k = 0, \ldots, n-1$. Let us now collect the terms $\dfrac{d^k\mathbf{y}_i}{dt^k}$ in the matrices $\mathbf{Y}^{(k)}$

and successively collect them in a bigger matrix as

$$\begin{bmatrix} \mathbf{Y}^{(0)} \\ \mathbf{Y}^{(1)} \\ \vdots \\ \mathbf{Y}^{(n-1)} \end{bmatrix} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix}.$$

Thus, intuitively speaking, the kernel of the observability matrix represents the basis for the description of the states that make null the first $n-1$ time derivatives of $\mathbf{y}$.

By looking at this rough explanation and assuming $(\mathbf{A}, \mathbf{C})$ fully observable, one can be tempted to use the vector $\texttt{col}(\mathbf{y}, d\mathbf{y}/dt, \ldots, d^{n-1}\mathbf{y}/\,dt^{n-1})$ to estimate the state $\mathbf{x}$ by means of the pseudo-inverse of $\mathbf{O}$ as

$$\hat{\mathbf{x}} = (\mathbf{O}^\top \mathbf{O})^{-1} \mathbf{O}^\top \begin{bmatrix} \mathbf{y} \\ d\mathbf{y}/dt \\ \vdots \\ d^{n-1}\mathbf{y}/dt^{n-1} \end{bmatrix}.$$

This procedure is feasible only if the signal $\mathbf{y}$ is noise-free. Indeed, let us assume a scalar output $y = \mathbf{Cx} + \nu(t)$ affected by the noise $\nu(t) := \sum_{i=1}^{\infty} \bar{\nu}_i \sin(i\,\omega t)$ with $\omega > 0$ and $\bar{\nu}_i \geq 0$ such that $\sum_i^{\infty} \bar{\nu}_i$ is finite. So, for any $k \geq 1$, the k-th time derivative of y is

$$\frac{d^k y}{dt^k} = \mathbf{CA}^k \mathbf{x} + \frac{d^k \nu}{dt^k}.$$

The application of the pseudo-inverse of $\mathbf{O}$ to $\texttt{col}(y, dy/dt, \ldots, d^{n-1}y/\,dt^{n-1})$ leads to

$$\begin{aligned} \hat{\mathbf{x}} &= (\mathbf{O}^\top \mathbf{O})^{-1} \mathbf{O}^\top \left(\mathbf{Ox} + \begin{bmatrix} \nu \\ d\nu/dt \\ \vdots \\ d^{n-1}\nu/dt^{n-1} \end{bmatrix} \right) \\ &= \mathbf{x} + (\mathbf{O}^\top \mathbf{O})^{-1} \mathbf{O}^\top \begin{bmatrix} \nu \\ d\nu/dt \\ \vdots \\ d^{n-1}\nu/dt^{n-1} \end{bmatrix}. \end{aligned}$$

This expression demonstrates that the estimation $\hat{\mathbf{x}}$ is proportionally affected by the magnitude of the time derivatives of ν. For any $k \geq 1$

these magnitudes are upper bounded by

$$\left\| \frac{d^k \nu}{dt^k} \right\|_\infty = \sum_{i=1}^{\infty} \bar{\nu}_i (i\,\omega)^k$$

which is an increasing function of i and k (for all $i \in \mathbb{N} : i\omega > 1$).

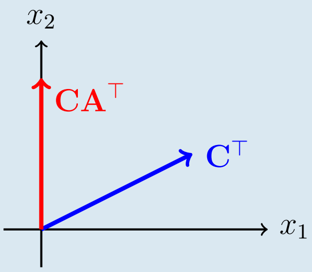
Example 4.15 (Observability study)**.** *Let* $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$*,* $y = \mathbf{C}\mathbf{x}$ *with* $\mathbf{x} \in \mathbb{R}^2$ *and* $y \in \mathbb{R}$ *be the model of the plant. Define* $\mathbf{A}$ *and* $\mathbf{C}$ *as*

$$\mathbf{A} = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} c_1 & c_2 \end{bmatrix}$$

with a*,* c_1*,* $c_2 > 0$*. The observability matrix is build as*

$$\mathbf{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix} = \begin{bmatrix} c_1 & c_2 \\ 0 & c_1 a \end{bmatrix}.$$

Let us now draw the rows of $\mathbf{O}$ *in the plane* $x_1 - x_2$ *as*

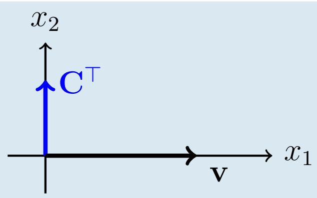


Since the vectors $\mathbf{C}^\top$ *and* $\mathbf{CA}^\top$ *span the whole state space* $\mathbb{R}^2$*, the unique vector which has null projections on* $\mathbf{C}^\top$ *and* $\mathbf{CA}^\top$ *is the trivial vector* $\mathbf{v} = \mathbf{0}$*. Then, since the set of the states which create a null output is only constituted by the origin* $\mathbf{0}$*, the system is fully observable.*

Example 4.16 (Observability Study)**.** *Let the plant be defined in the example 4.15 and consider* $c_1 = 0$*. The observability matrix becomes*

$$\mathbf{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix} = \begin{bmatrix} 0 & c_2 \\ 0 & 0 \end{bmatrix}.$$

whose row vectors are drawn as



Then, since the vector $\mathbf{C}^\top$ *does not span the whole state space* $\mathbb{R}^2$*, the system in not fully observable. Indeed, the unobservable set is composed of vectors* $\mathbf{v} = col(s, 0)$*, with* $s \in \mathbb{R}$*, which have null projection on the direction* $\mathbf{C}^\top$*.*

Example 4.17 (Cart-pole observability)**.** *The couple* $(\mathbf{A}, \mathbf{C})$ *of the linearised cart-pole model of Example 1.2 are reported hereafter*

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & A_{22} & A_{23} & A_{24} \\ 0 & 0 & 0 & 1 \\ 0 & A_{42} & A_{43} & A_{44} \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

The observability matrix associated to this couple is then obtained as

$$\mathbf{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \\ \mathbf{CA}^3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & A_{22} & A_{23} & A_{24} \\ 0 & 0 & 0 & 1 \\ 0 & o_{52} & o_{53} & o_{54} \\ 0 & A_{42} & A_{43} & A_{44} \\ 0 & o_{72} & o_{73} & o_{74} \\ 0 & o_{82} & o_{83} & o_{84} \end{bmatrix}.$$

where

$$\begin{aligned}
o_{52} &= A_{22}^2 + A_{24}A_{42}\\
o_{53} &= A_{22}A_{23} + A_{24}A_{43}\\
o_{54} &= A_{22}A_{24} + A_{23} + A_{24}A_{44}\\
o_{72} &= o_{52}A_{22} + o_{54}A_{42}\\
o_{73} &= o_{52}A_{23} + o_{54}A_{43}\\
o_{74} &= o_{52}A_{24} + o_{53} + o_{54}A_{44}\\
o_{82} &= A_{42}A_{22} + A_{44}A_{42}\\
o_{83} &= A_{42}A_{23} + A_{44}A_{43}\\
o_{84} &= A_{42}A_{24} + A_{43} + A_{44}^2
\end{aligned}$$

Since the first column of $\mathbf{O}$ *is null whereas the transpose of the first, the second and the fourth row span* $\mathbb{R}^3$*, the kernel of* $\mathbf{O}^\top$ *is 1-dimensional and given by*

$$\ker(\mathbf{O}) = \texttt{col}(1,\, 0,\, 0,\, 0)$$

. As consequence, the system of Example 1.2 is not fully observable.

Let $\mathcal{E}$ be an unobservability subspace, then its orthogonal complement $\mathcal{E}^\perp$ is such that $\mathcal{E} \oplus \mathcal{E}^\perp = \mathbb{R}^n$. In terms of basis, if $\ker(\mathbf{O})$ represents a basis for $\mathcal{E}$, then $(\ker(\mathbf{O}))^\perp = \mathrm{im}\left(\mathbf{O}^\top\right)$ defines a basis for $\mathcal{E}^\perp$. In the following, we rely on these bases to define a change of coordinates that decomposes the state in unobservable and observable subparts. The so called "Kalman decomposition of observability" is defined by introducing the transformation $\mathbf{z} = \mathbf{T}_\mathrm{O}\mathbf{x}$ where

$$\mathbf{T}_\mathrm{O}^{-1} = \left[\ \ker(\mathbf{O}) \quad \mathrm{im}\left(\mathbf{O}^\top\right)\ \right] \tag{4.16}$$

that applied to (4.2) leads to

$$\begin{aligned}
\dot{\mathbf{z}} =&\ \mathbf{T}_\mathrm{O}\mathbf{A}\mathbf{T}_\mathrm{O}^{-1}\mathbf{z} + \mathbf{T}_\mathrm{O}\mathbf{B}_1\mathbf{u} + \mathbf{T}_\mathrm{O}\mathbf{B}_2\mathbf{w} \quad \mathbf{z}(t_0) = \mathbf{T}_\mathrm{O}\mathbf{x}_0\\
\mathbf{y} =&\ \mathbf{C}\mathbf{T}_\mathrm{O}^{-1}\mathbf{z} + \mathbf{D}_1\mathbf{u} + \mathbf{D}_2\mathbf{w}
\end{aligned} \tag{4.17}$$

The study of $\bar{\mathbf{A}} := \mathbf{T}_\mathrm{O}\mathbf{A}\mathbf{T}_\mathrm{O}^{-1}$, $\bar{\mathbf{B}}_1 := \mathbf{T}_\mathrm{O}\mathbf{B}_1$, $\bar{\mathbf{B}}_2 := \mathbf{T}_\mathrm{O}\mathbf{B}_2$, and $\bar{\mathbf{C}} := \mathbf{C}\mathbf{T}_\mathrm{O}^{-1}$ reveals that

$$\bar{\mathbf{A}} = \begin{bmatrix} \bar{\mathbf{A}}_{11} & \bar{\mathbf{A}}_{12}\\ \mathbf{0} & \bar{\mathbf{A}}_{22} \end{bmatrix} \quad \bar{\mathbf{B}}_1 = \begin{bmatrix} \bar{\mathbf{B}}_{11}\\ \bar{\mathbf{B}}_{12} \end{bmatrix} \quad \bar{\mathbf{B}}_2 = \begin{bmatrix} \bar{\mathbf{B}}_{21}\\ \bar{\mathbf{B}}_{22} \end{bmatrix} \quad \bar{\mathbf{C}} = \left[\ \bar{\mathbf{0}} \quad \bar{\mathbf{C}}_2\ \right].$$

So, if we define $\mathbf{z} = \mathrm{col}(\mathbf{z}_\mathrm{NO}, \mathbf{z}_\mathrm{O})$ the dynamics of (4.17) becomes

$$\begin{array}{ccccccccc}
\dot{\mathbf{z}}_\mathrm{NO} & = & \bar{\mathbf{A}}_{11}\mathbf{z}_\mathrm{NO} & + & \bar{\mathbf{A}}_{12}\mathbf{z}_\mathrm{O} & + & \bar{\mathbf{B}}_{11}\mathbf{u} & + & \bar{\mathbf{B}}_{21}\mathbf{w}\\
\dot{\mathbf{z}}_\mathrm{O} & = & & & \bar{\mathbf{A}}_{22}\mathbf{z}_\mathrm{O} & + & \bar{\mathbf{B}}_{21}\mathbf{u} & + & \bar{\mathbf{B}}_{22}\mathbf{w}\\
\mathbf{y} & = & & & \bar{\mathbf{C}}_2\mathbf{z}_\mathrm{O} & + & \mathbf{D}_1\mathbf{u} & + & \mathbf{D}_2\mathbf{w}\\
\begin{bmatrix} \mathbf{z}_\mathrm{NO}(t_0)\\ \mathbf{z}_\mathrm{O}(t_0) \end{bmatrix} & = & \mathbf{T}_\mathrm{O}\mathbf{x}_0. & & & & & &
\end{array} \tag{4.18}$$

$$\begin{tikzcd}
\boxed{\dot{\mathbf{z}}_{\mathrm{NO}} = \bar{\mathbf{A}}_{11}\mathbf{z}_{\mathrm{NO}} + \bar{\mathbf{A}}_{12}\mathbf{z}_{\mathrm{O}} + \bar{\mathbf{B}}_{11}\mathbf{u} + \bar{\mathbf{B}}_{21}\mathbf{w}} & & \\
\boxed{\dot{\mathbf{z}}_{\mathrm{O}} = \bar{\mathbf{A}}_{22}\mathbf{z}_{\mathrm{O}} + \bar{\mathbf{B}}_{21}\mathbf{u} + \bar{\mathbf{B}}_{22}\mathbf{w}} \arrow[u] \arrow[r] & \boxed{\mathbf{y} = \bar{\mathbf{C}}_{2}\mathbf{z}_{\mathrm{O}} + \mathbf{D}_1\mathbf{u} + \mathbf{D}_2\mathbf{u}} \arrow[r] & \mathbf{y}
\end{tikzcd}$$

Figure 4.3: Observability decomposition

This shows that $\mathbf{z}_{\mathrm{O}}$ represents the subpart of the system available at output and that its dynamics is not influenced by the unobservable states $\mathbf{z}_{\mathrm{NO}}$. In practice, the dynamics of $\mathbf{z}_{\mathrm{NO}}$ cannot influence the output $\mathbf{y}$ and, assuming $\mathbf{u}, \mathbf{w} \equiv 0$, the output $\mathbf{y}$ is null for any time if and only if $\mathbf{z}_{\mathrm{O}} = \mathbf{0}$ at any time. The system (4.18) is depicted in Fig. 4.3.

Example 4.18 (Observability Kalman decomposition)**.** *The observability matrix associated to the couple*

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix},$$

is

$$\mathbf{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

The kernel of $\mathbf{O}$ *provides the basis for the unobservable subspace*

$$\mathbf{b}_1 := ker(\mathbf{O}) = ker\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}\right) = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

Besides, the image of $\mathbf{O}$ *leads to the basis of the observable subspace*

$$\{\mathbf{b}_2, \mathbf{b}_3\} := \left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\}.$$

Finally, the observability Kalman decomposition is obtained by means of *the transformation matrix* $\mathbf{T}_O^{-1} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{bmatrix}$ *that leads to*

$$\bar{\mathbf{A}} = \mathbf{T}_O \mathbf{A} \mathbf{T}_O^{-1} = \begin{bmatrix} -1 & -1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{C} = \mathbf{C}\mathbf{T}_O^{-1} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}.$$

Example 4.19 (Cart-pole obsv. Kalman decomposition)**.**
As described in Example 4.17, the system of Example 1.2 is not fully observable. Then, the kernel of $\mathbf{O}$ *provides the basis of the unobservable subspace* $\mathbf{b}_1 := ker(\mathbf{O}) = \texttt{col}(1, 0, 0, 0)$ *whereas the basis of the observable subspace is given by*

$$\{\mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4\} := \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Finally, the observability transformation is $\mathbf{T}_O^{-1} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & \mathbf{b}_4 \end{bmatrix} = \mathbf{I}$ *meaning that the system of Example 1.2 is already in the Kalman form of observability.*

Let us now focus on the observable subsystem of (4.18)

$$\begin{aligned} \dot{\mathbf{z}}_{\mathrm{O}} &= \bar{\mathbf{A}}_{22}\mathbf{z}_{\mathrm{O}} + \bar{\mathbf{B}}_{12}\mathbf{u} + \bar{\mathbf{B}}_{22}\mathbf{w} \quad \mathbf{z}_{\mathrm{O}}(t_0) = \mathbf{z}_{O_0} \\ \mathbf{y} &= \bar{\mathbf{C}}_2\mathbf{z}_{\mathrm{O}} + \mathbf{D}_2\mathbf{w} \end{aligned} \tag{4.19}$$

which is completely observable by definition.

Theorem 4.20 (Existence of a stabilising output feedback)**.**
Let

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} \end{aligned}$$

be a completely observable LTI system. Then, there exists a matrix $\mathbf{K}_O$ *such that* $\mathbf{A} - \mathbf{K}_O\mathbf{C}$ *is Hurwitz.*

Infobox 4.21 (Sketch of the proof of Theorem 4.20)**.**
The Theorem 4.20 is proved through the strategy adopted to prove Theorem 4.9, see Infobox 4.10, with the support of the concept of duality presented in Section 5.2.

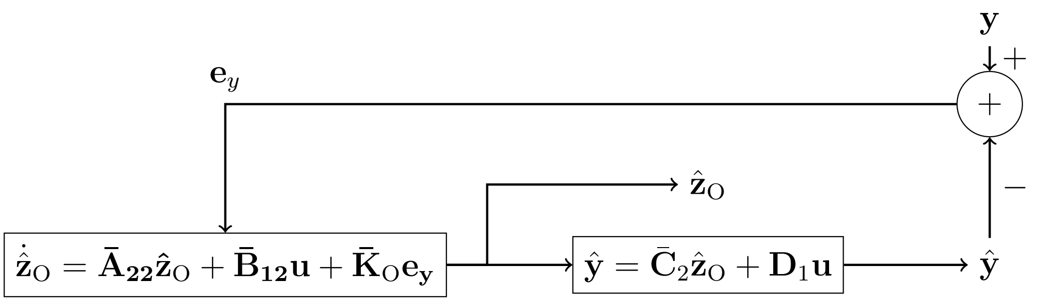


Figure 4.4: State Observer

Keynote

The direct consequence of Theorem 4.20 is that it is possible to design a BIBS stable dynamic system which provides an estimation of $\mathbf{z}_{\mathrm{O}}$ in the following form

$$\begin{aligned} \dot{\hat{\mathbf{z}}}_{\mathrm{O}} =&\ \bar{\mathbf{A}}_{22}\hat{\mathbf{z}}_{\mathrm{O}} + \bar{\mathbf{B}}_{12}\mathbf{u} + \bar{\mathbf{K}}_{\mathrm{O}}\left(\mathbf{y} - \hat{\mathbf{y}}\right) \quad \hat{\mathbf{z}}_{\mathrm{O}}(t_0) = \mathbf{0} \\ \hat{\mathbf{y}} =&\ \bar{\mathbf{C}}_2\hat{\mathbf{z}}_{\mathrm{O}} + \mathbf{D}_1\mathbf{u}. \end{aligned} \tag{4.20}$$

Let $\mathbf{e}_{\mathrm{O}} := \mathbf{z}_{\mathrm{O}} - \hat{\mathbf{z}}_{\mathrm{O}}$ be the estimation error, then the estimator (4.20), whose scheme is illustrated in Fig. 4.4, makes $\mathbf{e}_{\mathrm{O}}$ bounded. Indeed

$$\dot{\mathbf{e}}_{\mathrm{O}} = \left(\bar{\mathbf{A}}_{22} - \bar{\mathbf{K}}_{\mathrm{O}}\bar{\mathbf{C}}_2\right)\mathbf{e}_{\mathrm{O}} + (\bar{\mathbf{B}}_{22} + \bar{\mathbf{K}}_{\mathrm{O}}\mathbf{D}_2)\mathbf{w} \quad \mathbf{e}_z(t_0) = \mathbf{z}_{\mathrm{O}}(t_0) - \hat{\mathbf{z}}_{\mathrm{O}}(t_0) \tag{4.21}$$

whose state asymptotically converges to a neighbourhood of the origin if the matrix $\bar{\mathbf{A}}_{22} - \bar{\mathbf{K}}_{\mathrm{O}}\bar{\mathbf{C}}_2$ is Hurwitz.

Infobox 4.22 (Reduced Order Observer)**.**
The implementation of the observer (4.20) *requires the computation of a potentially large dynamic system whose dimension can be reduced as follows. Let*

$$\begin{aligned} \dot{\mathbf{z}}_O &= \bar{\mathbf{A}}_{22}\mathbf{z}_O + \bar{\mathbf{B}}_{12}\mathbf{u} + \bar{\mathbf{B}}_{22}\mathbf{w} \\ \mathbf{y} &= \bar{\mathbf{C}}_2\mathbf{z}_O + \mathbf{D}_1\mathbf{u} + \mathbf{D}_2\mathbf{w} \end{aligned}$$

be the observable part of (4.18)*. Define* $\mathbf{N}$ *such that*

$$\mathbf{T} := \begin{bmatrix} \bar{\mathbf{C}}_2 \\ \mathbf{N} \end{bmatrix}$$

constitutes a change of coordinates. Then, define $\boldsymbol{\zeta} = \mathbf{T}\mathbf{z}_O$ *with* $\boldsymbol{\zeta} :=$ $\texttt{col}(\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2)$ *and* $\boldsymbol{\zeta}_1 \in \mathbb{R}^q$*. Then, the dynamics of* $\boldsymbol{\zeta}$ *is*

$$\begin{aligned}\dot{\boldsymbol{\zeta}}_1 &= \underline{\mathbf{A}}_{11}\boldsymbol{\zeta}_1 + \underline{\mathbf{A}}_{12}\boldsymbol{\zeta}_2 + \underline{\mathbf{B}}_{11}\mathbf{u} + \underline{\mathbf{B}}_{21}\mathbf{w}\\ \dot{\boldsymbol{\zeta}}_2 &= \underline{\mathbf{A}}_{21}\boldsymbol{\zeta}_1 + \underline{\mathbf{A}}_{22}\boldsymbol{\zeta}_2 + \underline{\mathbf{B}}_{12}\mathbf{u} + \underline{\mathbf{B}}_{22}\mathbf{w}\\ \mathbf{y} &= \boldsymbol{\zeta}_1 + \mathbf{D}_1\mathbf{u} + \mathbf{D}_2\mathbf{w}\end{aligned}$$

where $\underline{\mathbf{A}}_{ij}$ *and* $\underline{\mathbf{B}}_{ij}$*, with* $i, j \in \{1, 2\}$*, denote sub-parts of* $\underline{\mathbf{A}} := \mathbf{T}\bar{\mathbf{A}}_{22}\mathbf{T}^{-1}$, $\underline{\mathbf{B}}_1 := \mathbf{T}\bar{\mathbf{B}}_1$*, and* $\underline{\mathbf{B}}_2 := \mathbf{T}\bar{\mathbf{B}}_2$ *of proper dimensions. Then, since* $\hat{\boldsymbol{\zeta}}_1 :=$ $\mathbf{y} - \mathbf{D}_1\mathbf{u} = \boldsymbol{\zeta}_1 + \mathbf{D}_2\mathbf{w}$ *provides a direct (noisy) estimation of* $\boldsymbol{\zeta}_1$ *we need to design an observer for* $\boldsymbol{\zeta}_2$ *only. To this end we conceive*

$$\boldsymbol{\mu} := \dot{\hat{\boldsymbol{\zeta}}}_1 - \underline{\mathbf{A}}_{11}\hat{\boldsymbol{\zeta}}_1 - \underline{\mathbf{B}}_{11}\mathbf{u} = \underline{\mathbf{A}}_{12}\boldsymbol{\zeta}_2 + (\underline{\mathbf{B}}_{21} - \underline{\mathbf{A}}_{11}\mathbf{D}_2)\mathbf{w} + \mathbf{D}_2\dot{\mathbf{w}}$$

as a potential output and rewrite the plant as

$$\begin{aligned}\dot{\boldsymbol{\zeta}}_2 &= \underline{\mathbf{A}}_{22}\boldsymbol{\zeta}_2 + \underline{\mathbf{A}}_{21}\boldsymbol{\zeta}_1 + \underline{\mathbf{B}}_{12}\mathbf{u} + \underline{\mathbf{B}}_{22}\mathbf{w}\\ \boldsymbol{\mu} &= \underline{\mathbf{A}}_{12}\boldsymbol{\zeta}_2 + (\underline{\mathbf{B}}_{21} - \underline{\mathbf{A}}_{11}\mathbf{D}_2)\mathbf{w} + \mathbf{D}_2\dot{\mathbf{w}}.\end{aligned}$$

Since the couple $(\underline{\mathbf{A}}_{22}, \underline{\mathbf{A}}_{12})$ *is fully observable there exists* $\underline{\mathbf{K}}_O$ *such that* $\underline{\mathbf{A}}_{22} - \underline{\mathbf{K}}_O\underline{\mathbf{A}}_{12}$ *is Hurwitz and*

$$\dot{\hat{\boldsymbol{\zeta}}}_2 = (\underline{\mathbf{A}}_{22} - \underline{\mathbf{K}}_O\underline{\mathbf{A}}_{12})\hat{\boldsymbol{\zeta}}_2 + \underline{\mathbf{A}}_{21}\hat{\boldsymbol{\zeta}}_1 + \underline{\mathbf{B}}_{12}\mathbf{u} + \underline{\mathbf{K}}_O\boldsymbol{\mu}$$

converges to a neighbourhood of $\boldsymbol{\zeta}_2$*. To make this observer implementable we now introduce a change of variables that remove the derivative of* $\hat{\boldsymbol{\zeta}}_1$ *(appearing in* $\boldsymbol{\mu}$*). Let* $\hat{\mathbf{v}} := \hat{\boldsymbol{\zeta}}_2 - \underline{\mathbf{K}}_O\hat{\boldsymbol{\zeta}}_1$*, exploit* $\hat{\boldsymbol{\zeta}}_2 = \hat{\mathbf{v}} + \underline{\mathbf{K}}_O\hat{\boldsymbol{\zeta}}_1$*, and rewrite the observer as*

$$\begin{aligned}\dot{\hat{\mathbf{v}}} = {}&(\underline{\mathbf{A}}_{22} - \underline{\mathbf{K}}_O\underline{\mathbf{A}}_{12})\hat{\mathbf{v}} + ((\underline{\mathbf{A}}_{22} - \underline{\mathbf{K}}_O\underline{\mathbf{A}}_{12})\underline{\mathbf{K}}_O + \underline{\mathbf{A}}_{21} - \underline{\mathbf{K}}_O\underline{\mathbf{A}}_{11})\hat{\boldsymbol{\zeta}}_1\\ &+ (\underline{\mathbf{B}}_{12} - \underline{\mathbf{K}}_O\underline{\mathbf{B}}_{11})\mathbf{u}.\end{aligned}$$

Then the estimation of $\mathbf{z}_O$ *is obtained as*

$$\hat{\mathbf{z}}_O = \mathbf{T}^{-1}\begin{bmatrix}\hat{\boldsymbol{\zeta}}_1\\ \hat{\boldsymbol{\zeta}}_2\end{bmatrix} = \mathbf{T}^{-1}\begin{bmatrix}\mathbf{0} & \mathbf{I}\\ \mathbf{I} & \underline{\mathbf{K}}_O\end{bmatrix}\begin{bmatrix}\hat{\mathbf{v}}\\ \hat{\boldsymbol{\zeta}}_1\end{bmatrix}.$$

To conclude this section we focus on the estimation of the non-observable states.

Keynote

The LTI system of equation (4.18) is said to be **detectable** if the $\bar{\mathbf{A}}_{11}$ is Hurwitz.

Assumption 4.2. *The system* (4.2) *is detectable.*

Let Assumption 4.2 be verified. Then, (4.18) is detectable and the estimation of the non–observable states $\mathbf{z}_{\mathrm{NO}}$ is achieved by the so called *identity observer* simply consisting of a copy of the plant

$$\dot{\hat{\mathbf{z}}}_{\mathrm{NO}} = \bar{\mathbf{A}}_{11}\hat{\mathbf{z}}_{\mathrm{NO}} + \bar{\mathbf{A}}_{12}\hat{\mathbf{z}}_{\mathrm{O}} + \bar{\mathbf{B}}_{12}\mathbf{u} \quad \hat{\mathbf{z}}_{\mathrm{NO}}(t_0) = \hat{\mathbf{z}}_{NO_0}. \tag{4.22}$$

Let $\mathbf{e}_{\mathrm{NO}} := \hat{\mathbf{z}}_{\mathrm{NO}} - \mathbf{z}_{\mathrm{NO}}$ be the estimation error and compute its dynamics exploiting (4.18), and (4.22) as

$$\dot{\mathbf{e}}_{\mathrm{NO}} = \bar{\mathbf{A}}_{11}\mathbf{e}_{\mathrm{NO}} - \bar{\mathbf{A}}_{12}\mathbf{e}_{\mathrm{O}} - \bar{\mathbf{B}}_{22}\mathbf{w} \qquad \mathbf{e}_{\mathrm{NO}}(t_0) = \hat{\mathbf{z}}_{\mathrm{NO}}(t_0) - \mathbf{z}_{\mathrm{NO}}(t_0). \tag{4.23}$$

Then, even if $\mathbf{e}_{\mathrm{NO}}$ is bounded because $\bar{\mathbf{A}}_{22}$ has been assumed Hurwitz, its dynamics cannot be tuned. To conclude the estimation of the state $\mathbf{x}$ is obtained as

$$\hat{\mathbf{x}} = \mathbf{T}_{\mathrm{O}}^{-1}\begin{bmatrix} \hat{\mathbf{z}}_{\mathrm{NO}} \\ \hat{\mathbf{z}}_{\mathrm{O}} \end{bmatrix}. \tag{4.24}$$

4.5 Output Feedback Stabiliser

As described at the end of Section 4.2, the state feedback $\mathbf{u} = \mathbf{K}_{\mathrm{S}}\mathbf{x}$ cannot be implemented because $\mathbf{x}$ is not available. On the other hand, Section 4.4 provides a strategy to estimate $\mathbf{x}$. As consequence, an implementable control law is

$$\mathbf{u}_{\mathrm{S}} = \mathbf{K}_{\mathrm{S}}\hat{\mathbf{x}} \tag{4.25}$$

which matches the architecture (1.18). With reference to (4.1a), this section demonstrates that the use of this latter control law, beside the integral action defined in Section 4.3, makes Hurwitz the matrix

$$\begin{bmatrix} \mathbf{A} + \mathbf{B}_1\mathbf{K}_{\mathrm{S}} & \mathbf{B}_1\mathbf{K}_{\mathrm{I}} & \mathbf{B}_1\mathbf{K}_{\mathrm{S}} \\ \mathbf{C}_e + \mathbf{D}_{e_1}\mathbf{K}_{\mathrm{S}} & \mathbf{D}_{e_1}\mathbf{K}_{\mathrm{I}} & \mathbf{D}_{e_1}\mathbf{K}_{\mathrm{S}} \\ \mathbf{A}_{\mathrm{O}} + \underline{\mathbf{K}}_{\mathrm{O}}\mathbf{C} - \mathbf{A} + \mathbf{M}\mathbf{K}_{\mathrm{S}} & \mathbf{M}\mathbf{K}_{\mathrm{I}} & \mathbf{A}_{\mathrm{O}} + \mathbf{M}\mathbf{K}_{\mathrm{S}} \end{bmatrix} \tag{4.26}$$

where $\mathbf{M} = \mathbf{B}_{\mathrm{O}} + \underline{\mathbf{K}}_{\mathrm{O}}\mathbf{D}_1 - \mathbf{B}_1$.

First, introduce a change of coordinates to exploit the observability Kalman decomposition. Let $\mathbf{T}_{\mathrm{O}}$ as in (4.16) and define $\mathbf{T} = \texttt{blckdiag}(\mathbf{I}, \mathbf{I}, \mathbf{T}_{\mathrm{O}})$. Premultiply and postmultiply (4.26) by $\mathbf{T}$ and $\mathbf{T}^{-1}$ respectively

$$\begin{bmatrix} \mathbf{A}+\mathbf{B}_1\mathbf{K}_{\mathrm{S}} & \mathbf{B}_1\mathbf{K}_{\mathrm{I}} & \mathbf{B}_1\mathbf{K}_{\mathrm{S}}\mathbf{T}_{\mathrm{O}}^{-1} \\ \mathbf{C}_e+\mathbf{D}_{e_1}\mathbf{K}_{\mathrm{S}} & \mathbf{D}_{e_1}\mathbf{K}_{\mathrm{I}} & \mathbf{D}_{e_1}\mathbf{K}_{\mathrm{S}}\mathbf{T}_{\mathrm{O}}^{-1} \\ \mathbf{T}_{\mathrm{O}}(\mathbf{A}_{\mathrm{O}}+\mathbf{K}_{\mathrm{O}}\mathbf{C}-\mathbf{A}+\mathbf{M}\mathbf{K}_{\mathrm{S}}) & \mathbf{T}_{\mathrm{O}}\mathbf{M}\mathbf{K}_{\mathrm{I}} & \mathbf{T}_{\mathrm{O}}(\mathbf{A}_{\mathrm{O}}+\mathbf{M}\mathbf{K}_{\mathrm{S}})\mathbf{T}_{\mathrm{O}}^{-1} \end{bmatrix}. \tag{4.27}$$

Second, use (4.20) and (4.22) to define

$$\dot{\hat{\mathbf{z}}} = \mathbf{A}_{\mathbf{z}}\hat{\mathbf{z}} + \mathbf{B}_{\mathbf{z}}\mathbf{u} + \mathbf{K}_{\mathbf{z}}\mathbf{y} \tag{4.28}$$

where $\hat{\mathbf{z}} = \texttt{col}(\hat{\mathbf{z}}_{\mathrm{NO}}, \hat{\mathbf{z}}_{\mathrm{O}})$ and

$$\mathbf{A}_z = \begin{bmatrix} \bar{\mathbf{A}}_{11} & \bar{\mathbf{A}}_{12} \\ \mathbf{0} & \bar{\mathbf{A}}_{22}-\bar{\mathbf{K}}_{\mathrm{O}}\bar{\mathbf{C}}_2 \end{bmatrix}, \quad \mathbf{B}_z = \mathbf{T}_{\mathrm{O}}\mathbf{B}_1 - \bar{\mathbf{K}}_{\mathrm{O}}\mathbf{D}_1, \quad \mathbf{K}_z = \begin{bmatrix} \mathbf{0} \\ \bar{\mathbf{K}}_{\mathrm{O}} \end{bmatrix}.$$

Then, define $\mathbf{x}_{\mathrm{O}} := \hat{\mathbf{x}}$ and compare (1.18a) with (4.28) to identify $\mathbf{A}_{\mathrm{O}} = \mathbf{T}_{\mathrm{O}}^{-1}\mathbf{A}_z\mathbf{T}_{\mathrm{O}}$, $\mathbf{B}_{\mathrm{O}} = \mathbf{T}_{\mathrm{O}}^{-1}\mathbf{B}_z$, and $\mathbf{K}_{\mathrm{O}} = \mathbf{T}_{\mathrm{O}}^{-1}\mathbf{K}_z$. As consequence $\mathbf{M} = \mathbf{0}$ and $\mathbf{A}_{\mathrm{O}} = \mathbf{A} - \mathbf{K}_{\mathrm{O}}\mathbf{C}$ which reduce (4.27) to

$$\begin{bmatrix} \mathbf{A}+\mathbf{B}_1\mathbf{K}_{\mathrm{S}} & \mathbf{B}_1\mathbf{K}_{\mathrm{I}} & \mathbf{B}_1\mathbf{K}_{\mathrm{S}}\mathbf{T}_{\mathrm{O}}^{-1} \\ \mathbf{C}_e+\mathbf{D}_{e_1}\mathbf{K}_{\mathrm{S}} & \mathbf{D}_{e_1}\mathbf{K}_{\mathrm{I}} & \mathbf{D}_{e_1}\mathbf{K}_{\mathrm{S}}\mathbf{T}_{\mathrm{O}}^{-1} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_z \end{bmatrix}. \tag{4.29}$$

This matrix is Hurwitz because its eigenvalues are given as union of the eigenvalues of the blocks on the main diagonal. Indeed, the matrix $\mathbf{A}_z$ is Hurwitz thanks to Assumption 4.2 and thanks to the design of the observer (4.20), and the matrix

$$\begin{bmatrix} \mathbf{A}+\mathbf{B}_1\mathbf{K}_{\mathrm{S}} & \mathbf{B}_1\mathbf{K}_{\mathrm{I}} \\ \mathbf{C}_e+\mathbf{D}_{e_1}\mathbf{K}_{\mathrm{S}} & \mathbf{D}_{e_1}\mathbf{K}_{\mathrm{I}} \end{bmatrix}$$

is Hurwitz thanks to Assumption 4.1 and thanks to the design of the stabiliser (4.11).

Keynote

The triangular structure of (4.29) let us to design the matrices $\mathbf{K}_{\mathrm{S}}$, $\mathbf{K}_{\mathrm{I}}$ independently of $\bar{\mathbf{K}}_{\mathrm{O}}$. This design feature is called *Separation Principle.*

4.5.1 Minimal Stabiliser

The output feedback stabiliser (4.24), (4.25) and (4.28) requires the implementation of a dynamic system of the same order of the plant. Moreover,

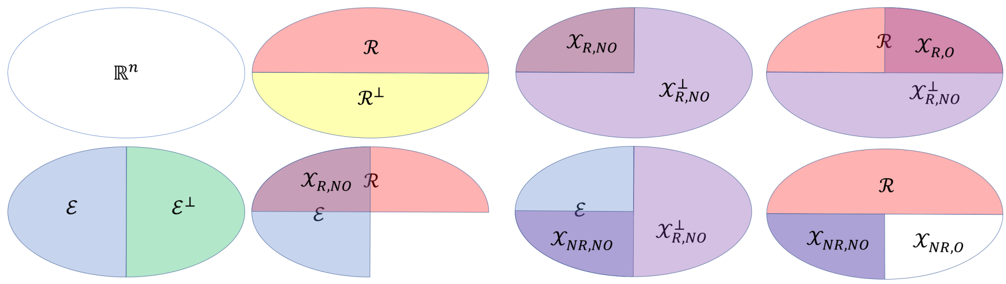


Figure 4.5: Graphical representation of the sequence of intersections exploited to find the ultimate Kalman decomposition of LTI systems.

under Assumptions 4.1 and 4.2 the state feedback can be simplified to assure the stability only of the reachable and observable subpart of $\mathbf{x}$. Thanks to the introduction of a change of coordinates called *Ultimate Kalman Decomposition*, this section highlights that goal G1 is achieved with the observer (4.20) (eventually reduced as in the Infobox 4.22) and by a suitable subpart of $\mathbf{K}_{\mathrm{R}}$.

The concepts of reachability and unobservability and their relative Kalman decompositions can be jointly exploited to find a transformation $\mathbf{z} = \mathbf{T}_{\mathrm{K}}\mathbf{x}$ which highlights the reachable, unreachable, unobservable and observable subsystems of (4.2). Let us identify the following subspaces

- The reachable and unobservable subspace: $\mathcal{X}_{\mathrm{R,NO}} := \mathcal{R} \cap \mathcal{E}$ with basis $X_{\mathrm{R,NO}}$;
- The reachable and observable subspace: $\mathcal{X}_{\mathrm{R,O}} := \mathcal{X}_{\mathrm{R,NO}}^{\perp} \cap \mathcal{R}$ with basis $X_{\mathrm{R,O}}$;
- The unreachable and unobservable subspace: $\mathcal{X}_{\mathrm{NR,NO}} := \mathcal{X}_{\mathrm{R,NO}}^{\perp} \cap \mathcal{E}$ with basis $X_{\mathrm{NR,NO}}$;
- The unreachable and observable subspace: $\mathcal{X}_{\mathrm{NR,O}} := (\mathcal{R} \cup \mathcal{X}_{\mathrm{NR,NO}})^{\perp}$ with basis $X_{\mathrm{NR,O}}$

whose derivation is supported by the Venn diagram of Fig. 4.5.

The transformation $\mathbf{T}_{\mathrm{K}}$ is then defined as:

$$\mathbf{T}_{\mathrm{K}}^{-1} = \left[\begin{array}{cccc} X_{\mathrm{R,NO}} & X_{\mathrm{R,O}} & X_{\mathrm{NR,NO}} & X_{\mathrm{NR,O}} \end{array}\right]$$

which, applied to (4.2), leads to

$$\begin{aligned} \dot{\mathbf{z}} &= \mathbf{T}_{\mathrm{K}}\mathbf{A}\mathbf{T}_{\mathrm{K}}^{-1}\mathbf{z} + \mathbf{T}_{\mathrm{K}}\mathbf{B}_1\mathbf{u} + \mathbf{T}_{\mathrm{K}}\mathbf{B}_2\mathbf{w} \quad \mathbf{z}(t_0) = \mathbf{T}_{\mathrm{K}}\mathbf{x}_0 \\ \mathbf{y} &= \mathbf{C}\mathbf{T}_{\mathrm{K}}^{-1}\mathbf{z} + \mathbf{D}_1\mathbf{u} + \mathbf{D}_2\mathbf{w} \end{aligned} \tag{4.30}$$

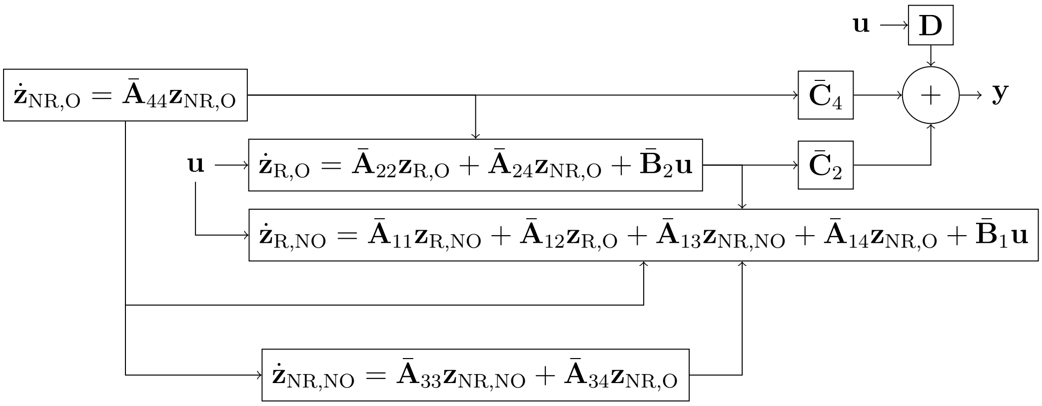


Figure 4.6: The ultimate Kalman decomposition highlights the reachable and the observable parts of the system. The input affects only the reachable parts whereas the observable ones are the only which contribute to the output.

The study of $\bar{\mathbf{A}} := \mathbf{T}_{\mathrm{K}}\mathbf{A}\mathbf{T}_{\mathrm{K}}^{-1}$, $\bar{\mathbf{B}}_1 := \mathbf{T}_{\mathrm{K}}\mathbf{B}$ and $\bar{\mathbf{C}} := \mathbf{C}\mathbf{T}_{\mathrm{K}}^{-1}$ reveals that

$$\bar{\mathbf{A}} = \begin{bmatrix} \bar{\mathbf{A}}_{11} & \bar{\mathbf{A}}_{12} & \bar{\mathbf{A}}_{13} & \bar{\mathbf{A}}_{14} \\ \mathbf{0} & \bar{\mathbf{A}}_{22} & \mathbf{0} & \bar{\mathbf{A}}_{24} \\ \mathbf{0} & \mathbf{0} & \bar{\mathbf{A}}_{33} & \bar{\mathbf{A}}_{34} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \bar{\mathbf{A}}_{44} \end{bmatrix}, \bar{\mathbf{B}}_1 = \begin{bmatrix} \bar{\mathbf{B}}_{11} \\ \bar{\mathbf{B}}_{12} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \bar{\mathbf{B}}_2 = \begin{bmatrix} \bar{\mathbf{B}}_{21} \\ \bar{\mathbf{B}}_{22} \\ \bar{\mathbf{B}}_{23} \\ \bar{\mathbf{B}}_{24} \end{bmatrix}$$
$$\bar{\mathbf{C}} = \begin{bmatrix} \mathbf{0} & \bar{\mathbf{C}}_2 & \mathbf{0} & \bar{\mathbf{C}}_4 \end{bmatrix}.$$

Let the state $\mathbf{z}$ be divided as

$$\mathbf{z} = \mathrm{col}\left(\mathbf{z}_{\mathrm{R,NO}},\, \mathbf{z}_{\mathrm{R,O}},\, \mathbf{z}_{\mathrm{NR,NO}},\, \mathbf{z}_{\mathrm{NR,O}}\right),$$

then the dynamics of $\mathbf{z}$ is given by

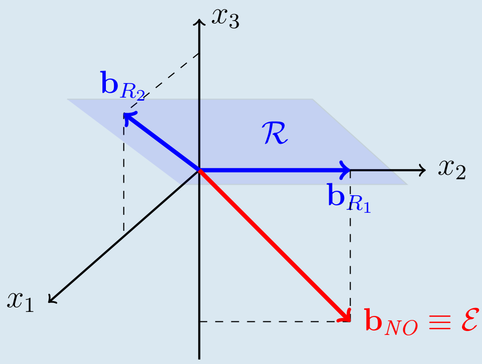
$$\begin{aligned} \dot{\mathbf{z}}_{\mathrm{R,NO}} &= \bar{\mathbf{A}}_{11}\mathbf{z}_{\mathrm{R,NO}} + \bar{\mathbf{A}}_{12}\mathbf{z}_{\mathrm{R,O}} + \bar{\mathbf{A}}_{13}\mathbf{z}_{\mathrm{NR,NO}} + \bar{\mathbf{A}}_{14}\mathbf{z}_{\mathrm{NR,O}} + \bar{\mathbf{B}}_{11}\mathbf{u} + \bar{\mathbf{B}}_{21}\mathbf{w} \\ \dot{\mathbf{z}}_{\mathrm{R,O}} &= \bar{\mathbf{A}}_{22}\mathbf{z}_{\mathrm{R,O}} + \bar{\mathbf{A}}_{24}\mathbf{z}_{\mathrm{NR,O}} + \bar{\mathbf{B}}_{12}\mathbf{u} + \bar{\mathbf{B}}_{22}\mathbf{w} \\ \dot{\mathbf{z}}_{\mathrm{NR,NO}} &= \bar{\mathbf{A}}_{33}\mathbf{z}_{\mathrm{NR,NO}} + \bar{\mathbf{A}}_{34}\mathbf{z}_{\mathrm{NR,O}} + \bar{\mathbf{B}}_{23}\mathbf{w} \\ \dot{\mathbf{z}}_{\mathrm{NR,O}} &= \bar{\mathbf{A}}_{44}\mathbf{z}_{\mathrm{NR,O}} + \bar{\mathbf{B}}_{24}\mathbf{w} \\ \mathbf{y} &= \bar{\mathbf{C}}_2\mathbf{z}_{\mathrm{R,O}} + \bar{\mathbf{C}}_4\mathbf{z}_{\mathrm{NR,O}} + \mathbf{D}_1\mathbf{u} + \mathbf{D}_2\mathbf{w} \end{aligned}$$

Example 4.23 (Ultimate Kalman decomposition)**.** *Given the matrices* $\mathbf{A}$*,* $\mathbf{B}$ *and* $\mathbf{C}$ *defined in Examples 4.7 and 4.18 the basis of the reachable* *and of the unobservable subspaces respectively are*

$$\{\mathbf{b}_{R_1}, \mathbf{b}_{R_2}\} = \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}, \qquad \mathbf{b}_{NO} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

As illustrated in the following picture, the intersection of the reachable and non-observable subspaces is trivial and corresponds to the origin, i.e. $\mathcal{X}_{R,NO} = \mathcal{R} \cup \mathcal{E} = \{0\}$*. As consequence, since* $X_{R,NO}$ *is not defined and is not part of* $\mathbf{T}_K$*, the ultimate Kalman decomposition does not highlight any reachable but non observable subsystem. Moreover, since* $\mathcal{X}_{R,NO}^{\perp} = \mathbb{R}^3$ *we obtain* $\mathcal{X}_{R,O} = \mathcal{X}_{R,NO}^{\perp} \cap \mathcal{R} = \mathcal{R}$ *and* $\mathcal{X}_{NR,NO} = \mathcal{X}_{R,NO}^{\perp} \cap \mathcal{E} = \mathcal{E}$*. In conclusion, since* $\mathcal{R} \cup \mathcal{X}_{NR,NO} = \mathbb{R}^3$*, the subspace* $\mathcal{X}_{NR,O} = \{0\}$ *and then*

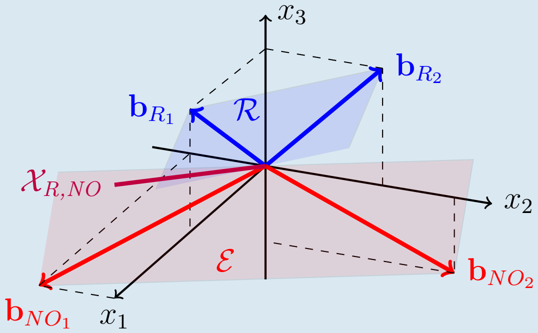
$$\mathbf{T}_K^{-1} = \begin{bmatrix} \mathbf{b}_{R_1} & \mathbf{b}_{R_2} & \mathbf{b}_{NO} \end{bmatrix}.$$



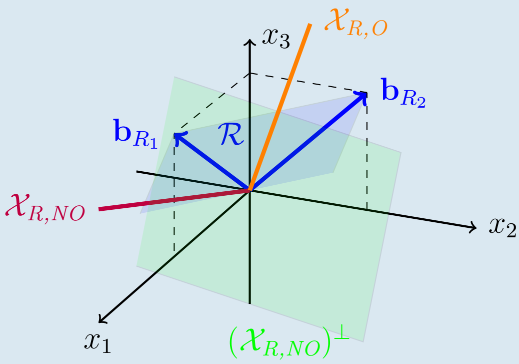
The application of $\mathbf{T}_K$ *to the matrices* $\mathbf{A}$*,* $\mathbf{B}$ *and* $\mathbf{C}$ *defined in the Examples 4.7 and 4.18, leads to*

$$\bar{\mathbf{A}} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \qquad \bar{\mathbf{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}.$$

Example 4.24 (Ultimate Kalman decomposition)**.**
Let us assume that the reachability and unobservability subspaces of an LTI system have as basis $im(\mathbb{R}) = [\mathbf{b}_{R_1}\, \mathbf{b}_{R_1}]$ *and* $ker(\mathbf{O}) = [\mathbf{b}_{NO_1}\, \mathbf{b}_{NO_1}]$ *respectively, with the geometry of the subspaces depicted in figure.*

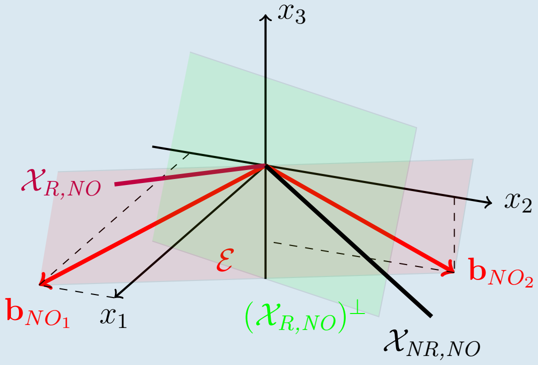


The intersection $\mathcal{X}_{R,NO} = \mathcal{R} \cap \mathcal{E}$ *is represented by unique the 3D-line that belongs to both* $\mathcal{R}$ *and* $\mathcal{E}$*. Moreover, the orthogonal to* $\mathcal{X}_{R,NO}$*, drawn as a green plane in the following figure, intersects the reachable set and provides* $\mathcal{X}_{R,O} = (X_{R,NO})^{\perp} \cap \mathcal{R}^{+}$*, whose basis is* $X_{R,O}$*.*



Finally, the intersection between $(\mathcal{X}_{R,NO})^{\perp}$ *and* $\mathcal{E}$ *generates* $\mathcal{X}_{NR,NO}$ *whose base is* $X_{NR,NO}$*. Then, the transformation matrix is*

$$\mathbf{T}_K^{-1} = \left[\begin{array}{ccc} X_{R,NO} & X_{R,O} & X_{NR,NO} \end{array}\right].$$



Example 4.25 (Ultimate Kalman decomposition)**.** *An LTI system is identified by means of the triplet*

$$\mathbf{A} = \begin{bmatrix} 7 & 5 & -1 \\ -3 & -1 & -1 \\ 2 & 2 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix}.$$

This system is neither fully reachable nor fully observable because the reachability and the observability matrices

$$\mathbf{R} = \begin{bmatrix} 2 & 8 & 24 \\ -1 & -6 & -20 \\ 1 & 2 & 4 \end{bmatrix}, \qquad \mathbf{O} = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 2 & -2 \\ 4 & 4 & -4 \end{bmatrix}$$

have rank $= 2$ *and rank* $= 1$ *respectively. We define im*$(\mathbb{R}) = [\mathbf{b}_{R_1}\, \mathbf{b}_{R_2}]$ *and ker*$(\mathbf{O}) = [\mathbf{b}_{NO_1}\, \mathbf{b}_{NO_2}]$ *with*

$$[\mathbf{b}_{R_1}\, \mathbf{b}_{R_2}] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, \qquad [\mathbf{b}_{NO_1}\, \mathbf{b}_{NO_2}] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

Since these two bases are equal the reachable subspace coincides with the unobservable subspace, i.e. $\mathcal{R} \equiv \mathcal{E}$ *and thus* $\mathcal{X}_{R,NO} = \mathcal{R} \cap \mathcal{E} = \mathcal{R}$. *As consequence* $\mathcal{X}_{R,O} = (\mathcal{X}_{R,NO})^{\perp} \cap \mathcal{R} = \{0\}$ *as well as* $\mathcal{X}_{NR,NO} = (\mathcal{X}_{R,NO})^{\perp} \cap \mathcal{E} = \{0\}$. *The only non trivial subspace further than* $\mathcal{X}_{R,NO}$ *is thus* $\mathcal{X}_{NR,O} = (\mathcal{X}_{R,NO})^{\perp}$. *In this case the basis for* $\mathcal{X}_{NR,O}$ *is found to be* $X_{NR,O} = \texttt{col}(1,\, 1,\, -1)$. *Finally the transformation matrix is*

$$T_K^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

that applied to the matrices $\mathbf{A}$, $\mathbf{B}$ *and* $\mathbf{C}$ *leads to*

$$\bar{\mathbf{A}} = \begin{bmatrix} 6 & 4 & 11 \\ -4 & -2 & -5 \\ 0 & 0 & 2 \end{bmatrix}, \qquad \bar{\mathbf{B}} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} 0 & 0 & 3 \end{bmatrix}.$$

Example 4.26 (Cart-pole ultimate Kalman decomposition)**.**
The bases of the reachability and unobservability subspaces of the cart-

pole modelled in Exercise 1.2 have been studied in Exercise 4.6 and 4.19. More in details, we have

$$\mathtt{Im}(\mathbf{R}) = \mathbf{I} \in \mathbb{R}^{4\times 4}, \quad \mathtt{ker}(\mathbf{O}) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Then the $X_{R,NO}$ is found as the intersection between the 4D line $\mathtt{ker}(\mathbf{O})$ and the whole 4D space. This intersection leads to

$$X_{R,NO} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad X_{R,NO}^{\perp} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The intersection of $\mathcal{X}_{R,NO}^{\perp}$ with $\mathcal{R}$ and $\mathcal{E}$ leads to

$$X_{R,O} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad X_{NR,NO} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Finally, $\mathcal{X}_{NR,O}$ is found as the orthogonal to the union of $\mathcal{R}$ with $\mathcal{X}_{NR,NO}$ whose base is $\mathtt{Im}(\mathcal{R} \cup \mathcal{X}_{NR,NO}) = \mathbf{I} \in \mathbb{R}^{4\times 4}$. Then the bases of its orthogonal is represented by the origin, i.e. $X_{NR,O} = \mathbf{0}$. The ultimate Kalman decomposition is finally obtained as

$$\mathbf{T}^{-1} = \begin{bmatrix} X_{R,NO} & X_{R,O} \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{bmatrix}.$$

The subsystem

$$\begin{aligned} \dot{\mathbf{z}}_{\mathrm{R,O}} &= \bar{\mathbf{A}}_{22}\mathbf{z}_{\mathrm{R,O}} + \bar{\mathbf{A}}_{24}\mathbf{z}_{\mathrm{NR,O}} + \bar{\mathbf{B}}_{12}\mathbf{u} + \bar{\mathbf{B}}_{22}\mathbf{w} \\ \dot{\mathbf{z}}_{\mathrm{NR,O}} &= \bar{\mathbf{A}}_{44}\mathbf{z}_{\mathrm{NR,O}} + \bar{\mathbf{B}}_{24}\mathbf{w} \\ \mathbf{y} &= \bar{\mathbf{C}}_{2}\mathbf{z}_{\mathrm{R,O}} + \bar{\mathbf{C}}_{4}\mathbf{z}_{\mathrm{NR,O}} + \mathbf{D}_{1}\mathbf{u} + \mathbf{D}_{2}\mathbf{w} \end{aligned} \tag{4.31}$$

is of major interest because it is the only part which influences the output and, under Assumption 1.3, the regulated output.

Infobox 4.27 (Transfer Matrix)**.** *The transfer functions representation only takes into account for the reachable and observable part of LTI systems. Indeed, the term* $\mathbf{z}_{NR,O}$ *is considered an external input for the system* (4.31) *which is clearly not involved into the input–output relation between* $\mathbf{u}$ *and* $\mathbf{y}$*. So, the transfer matrix is given by:*

$$\mathbf{G}(s) := \frac{\mathbf{Y}(s)}{\mathbf{U}(s)} = \bar{\mathbf{C}}_2 \left(s\mathbf{I} - \bar{\mathbf{A}}_{22}\right)^{-1} \bar{\mathbf{B}}_{12} + \mathbf{D}_1$$

It is worth noting that the output is influenced by $\mathbf{z}_{\mathrm{R,O}}$ and by $\mathbf{z}_{\mathrm{NR,O}}$ with the latter that can be considered as a known signal (thanks to the existence of the observer) but whose dynamics cannot be modified by the control action $\mathbf{u}$. Since the couple $(\bar{\mathbf{A}}_{22}, \bar{\mathbf{B}}_{12})$ is fully reachable by definition there exists a matrix $\mathbf{K}_{\mathrm{R,O}}$ such that $\bar{\mathbf{A}}_{22}+\bar{\mathbf{B}}_{12}\mathbf{K}_{\mathrm{R,O}}$ is Hurwitz thus implying that the state $\mathbf{z}_{\mathrm{R,O}}$ is bounded if $\mathbf{z}_{\mathrm{NR,O}}$ is bounded which, in turn, is true if Assumption 4.1 is verified. Then, the minimal stabiliser is obtained as

$$\begin{aligned}\begin{bmatrix} \dot{\mathbf{z}}_{\mathrm{R,O}} \\ \dot{\mathbf{z}}_{\mathrm{NR,O}} \end{bmatrix} &= \left(\begin{bmatrix} \bar{\mathbf{A}}_{22} & \bar{\mathbf{A}}_{24} \\ \mathbf{0} & \bar{\mathbf{A}}_{44} \end{bmatrix} - \bar{\mathbf{K}}_{\mathrm{O}} \begin{bmatrix} \bar{\mathbf{C}}_2 & \bar{\mathbf{C}}_4 \end{bmatrix} \right) \begin{bmatrix} \mathbf{z}_{\mathrm{R,O}} \\ \mathbf{z}_{\mathrm{NR,O}} \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{B}}_{12} \\ \mathbf{0} \end{bmatrix} \mathbf{u} + \bar{\mathbf{K}}_{\mathrm{O}}\mathbf{y} \\ \mathbf{u}_{\mathrm{S}} &= \mathbf{K}_{\mathrm{R,O}}\mathbf{z}_{\mathrm{R,O}}. \end{aligned} \tag{4.32}$$

Keynote

The overall closed loop system can be made BIBS stable if the matrices $\bar{\mathbf{A}}_{11}$, $\bar{\mathbf{A}}_{33}$ and $\bar{\mathbf{A}}_{44}$ are Hurwitz, *i.e.* if the plant is detectable and stabilisable.

Example 4.28 (Cart-pole stabiliser)**.**
The ultimate Kalman decomposition of Example 4.26 applied to the plant of Example 1.2 shows that the first state is reachable but not observable whereas the last three states are reachable and observable. Formally, we rewrite the system in the Kalman decomposition as

$$\begin{aligned}\begin{bmatrix} \dot{z}_{R,NO} \\ \dot{\mathbf{z}}_{R,O} \end{bmatrix} &= \begin{bmatrix} \bar{a}_{11} & \bar{\mathbf{A}}_{12} \\ \mathbf{0} & \bar{\mathbf{A}}_{22} \end{bmatrix} \begin{bmatrix} z_{R,NO} \\ \mathbf{z}_{R,O} \end{bmatrix} + \begin{bmatrix} 0 \\ \bar{\mathbf{B}}_2 \end{bmatrix} u \\ \mathbf{y} &= \begin{bmatrix} 0 & \bar{\mathbf{C}}_2 \end{bmatrix} \begin{bmatrix} z_{R,NO} \\ \mathbf{z}_{R,O} \end{bmatrix}\end{aligned}$$

in which the contribution of the disturbances has been neglected. The *output feedback stabiliser is given by*

$$\begin{aligned}\dot{\hat{\mathbf{z}}}_{R,O} &= (\bar{\mathbf{A}}_{22} - \mathbf{K}_O\bar{\mathbf{C}}_2 + \bar{\mathbf{B}}_2\mathbf{K}_S)\hat{\mathbf{z}}_{R,O} + \mathbf{K}_O\mathbf{y}\\ u &= \mathbf{K}_{R,O}\hat{\mathbf{z}}_{R,O}.\end{aligned}$$

4.5.2 Robustness to Disturbances and Noise

Section 4.5 demonstrates that the controller (4.24), (4.25) and (4.28), extended with the integral action (4.11), makes BIBS stable the closed loop plant. Moreover, assuming a fully reachable and observable plant, all the eigenvalues of (4.2) can be assigned through the design of the matrices $\mathbf{K}_\mathrm{S}$, $\mathbf{K}_\mathrm{I}$, and $\mathbf{K}_\mathrm{O}$. Thus, one could be tempted to push on the feedback gains to make the control system as reactive as desired and to reduce the asymptotic bound of the regulated output (see control goal G2). Unfortunately, the presence of measurement noises deeply impact on the design of $\mathbf{K}_\mathrm{S}$, $\mathbf{K}_\mathrm{I}$, and $\mathbf{K}_\mathrm{O}$, as well as on the asymptotic values of the state, and the regulated output. In more details, let

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}_1\mathbf{u} + \mathbf{B}_2\mathbf{d}\\ \dot{\hat{\mathbf{x}}} &= (\mathbf{A} - \mathbf{K}_\mathrm{O}\mathbf{C})\hat{\mathbf{x}} + (\mathbf{B}_1 - \mathbf{K}_\mathrm{O}\mathbf{D}_1)\mathbf{u} + \mathbf{K}_\mathrm{O}\mathbf{y}\\ \dot{\boldsymbol{\eta}} &= \mathbf{C}_e\mathbf{x} + \mathbf{D}_{e1}\mathbf{u} + \mathbf{E}\boldsymbol{\nu}\\ \mathbf{u} &= \mathbf{K}_\mathrm{S}\hat{\mathbf{x}} + \mathbf{K}_\mathrm{I}\boldsymbol{\eta}\\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}_1\mathbf{u} + \boldsymbol{\nu}.\end{aligned} \tag{4.33}$$

be a fully observable and reachable LTI plant, subject to disturbances and measurement noises, controlled by an output feedback stabiliser plus an integral action where $\mathbf{d}$ denotes the exogenous disturbance and $\boldsymbol{\nu}$ is the measurement noise.

Remark 4.2. *With respect to the classical formulation* (4.2)*, the model* (4.33) *specialises the exogenous as disturbance and measurement noise to make evident how they affect the state and the regulated output.*

Define the estimation error as $\mathbf{e}_x = \hat{\mathbf{x}} - \mathbf{x}$, let

$$\mathbf{A}_\mathrm{cl} := \begin{bmatrix} \mathbf{A} + \mathbf{B}_1\mathbf{K}_\mathrm{S} & \mathbf{B}_1\mathbf{K}_\mathrm{I}\\ \mathbf{C}_e + \mathbf{D}_{e1}\mathbf{K}_\mathrm{S} & \mathbf{D}_{e1}\mathbf{K}_\mathrm{I}\end{bmatrix},$$

and use the expressions of $\mathbf{y}$ and $\mathbf{u}$ to rewrite (4.33) as

$$\begin{aligned}\begin{bmatrix}\dot{\mathbf{x}}\\ \dot{\boldsymbol{\eta}}\end{bmatrix} &= \mathbf{A}_\mathrm{cl}\begin{bmatrix}\mathbf{x}\\ \boldsymbol{\eta}\end{bmatrix} + \begin{bmatrix}\mathbf{B}_1\mathbf{K}_\mathrm{S}\\ \mathbf{D}_{e1}\mathbf{K}_\mathrm{S}\end{bmatrix}\mathbf{e}_x + \begin{bmatrix}\mathbf{B}_2 & \mathbf{0}\\ \mathbf{0} & \mathbf{E}\end{bmatrix}\begin{bmatrix}\mathbf{d}\\ \boldsymbol{\nu}\end{bmatrix}\\ \dot{\mathbf{e}}_x &= (\mathbf{A} - \mathbf{K}_\mathrm{O}\mathbf{C})\mathbf{e}_x + \mathbf{K}_\mathrm{O}\boldsymbol{\nu}\end{aligned}$$

whose solution is

$$\begin{aligned}
\begin{bmatrix} \mathbf{x}(t) \\ \boldsymbol{\eta}(t) \end{bmatrix} &= \int_0^t e^{\mathbf{A}_{\rm cl}(t-\tau)} \left(\begin{bmatrix} \mathbf{B}_1\mathbf{K}_{\rm S} \\ \mathbf{D}_{e1}\mathbf{K}_{\rm S} \end{bmatrix} \mathbf{e}_x(\tau) + \begin{bmatrix} \mathbf{B}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{E} \end{bmatrix} \begin{bmatrix} \mathbf{d}(\tau) \\ \boldsymbol{\nu}(\tau) \end{bmatrix} \right) d\tau \\
\mathbf{e}_x(t) &= \int_0^t e^{(\mathbf{A}-\mathbf{K}_{\rm O}\mathbf{C})(t-\tau)} \mathbf{K}_{\rm O}\boldsymbol{\nu}(\tau) d\tau
\end{aligned}$$

where without loss of generality we assumed $(\mathbf{x}(0), \boldsymbol{\eta}(0)) = (\mathbf{0}, \mathbf{0})$ and $\mathbf{e}_x(0) = \mathbf{0}$. Then assume the norms of disturbance and noise uniformly bounded by $\bar{d}$ and $\bar{\nu}$ respectively, *i.e.* $\|\mathbf{d}(t)\| \leq \bar{d}$ and $\|\boldsymbol{\nu}(t)\| \leq \bar{\nu}$ for all $t \geq 0$. Then, we can bound the norm of $\mathbf{x}(t)$, $\boldsymbol{\eta}(t)$ and $\mathbf{e}_x(t)$ as

$$\begin{aligned}
\left\| \begin{bmatrix} \mathbf{x}(t) \\ \boldsymbol{\eta}(t) \end{bmatrix} \right\| &\leq \frac{1}{\underline{\sigma}\left(\mathbf{A}_{\rm cl}\right)} \left(\overline{\sigma}\left(\begin{bmatrix} \mathbf{B}_1\mathbf{K}_{\rm S} \\ \mathbf{D}_{e1}\mathbf{K}_{\rm S} \end{bmatrix} \right) \bar{e} + \overline{\sigma}\left(\mathbf{B}_2\right) \bar{d} + \overline{\sigma}\left(\mathbf{E}\right) \bar{\nu} \right) \\
\|\mathbf{e}_x(t)\| &\leq \bar{e}
\end{aligned} \tag{4.34}$$

with $\bar{e} = [\underline{\sigma}(\mathbf{A} - \mathbf{K}_{\rm O}\mathbf{C})]^{-1}\overline{\sigma}(\mathbf{K}_{\rm O})\bar{\nu}$. It is clear that the attenuation of the effects of $\boldsymbol{\nu}$ on $\mathbf{e}_x$ cannot be achieved through a more aggressive observer feedback $\mathbf{K}_{\rm O}$. Indeed, a larger $\mathbf{K}_{\rm O}$ lead to a larger $\underline{\sigma}(\mathbf{A} - \mathbf{K}_{\rm O}\mathbf{C})$ as well as to a larger $\overline{\sigma}(\mathbf{K}_{\rm O})$. Then, increasing $\mathbf{K}_{\rm O}$ such that $\underline{\sigma}(\mathbf{A} - \mathbf{K}_{\rm O}\mathbf{C}) \approx \underline{\sigma}(\mathbf{K}_{\rm O}\mathbf{C})$ and assuming $\underline{\sigma}(\mathbf{K}_{\rm O}) = \overline{\sigma}(\mathbf{K}_{\rm O})$ leads to $\bar{e} = [\underline{\sigma}(\mathbf{K}_{\rm O}\mathbf{C})]^{-1}\overline{\sigma}(\mathbf{K}_{\rm O})$ which cannot be arbitrary reduced. The same argument applies to demonstrate that increasing $\mathbf{K}_{\rm S}$ does not attenuate the affects of $\boldsymbol{\nu}$ on $\|(\mathbf{x}(t), \boldsymbol{\eta}(t))\|$. At the opposite, more aggressive feedback $\mathbf{K}_{\rm S}$ and $\mathbf{K}_{\rm I}$ attenuate the effects of $\mathbf{d}$ on the norm of the states $\mathbf{x}$ and $\boldsymbol{\eta}$ because, accordingly to the first of (4.34), the disturbance amplification is inversely proportional to $\overline{\sigma}(\mathbf{B}_2)/\underline{\sigma}\left(\mathbf{A}_{\rm cl}\right)$.

In conclusion, the BIBS stability properties guaranteed by the output feedback (4.11), (4.24), (4.25), and (4.28) make both the state and the estimation error bounded in presence of disturbances and noises. Unfortunately, $\|(\mathbf{x}(t), \boldsymbol{\eta}(t))\|$ and $\|\mathbf{e}_x(t)\|$ cannot be arbitrary reduced through an high-gain policy. In more details, the effects of disturbances can be attenuated via a more aggressive $\mathbf{K}_{\rm S}$ and $\mathbf{K}_{\rm I}$ whereas increasing $\mathbf{K}_{\rm O}$ may not lead to any noise attenuation. Chapter 5 presents an optimal criterion for the design of $\mathbf{K}_{\rm S}$, $\mathbf{K}_{\rm I}$, and $\mathbf{K}_{\rm O}$.

4.5.3 Limitations on the Stabilisation of Nonlinear Systems

The output feedback controller (4.11), (4.24), (4.25) and (4.28) and guarantees the origin $\tilde{\mathbf{x}} = \mathbf{0}$ to be a globally exponentially stable equilibrium point for the linearised plant (1.15). On the other hand we know that (1.15) is generated as linearisation of (1.4) in the neighbourhood of $\mathbf{x}^\star$. Then, it seems natural to try to transfer the stability properties of the origin of the linearised system to the equilibrium of the nonlinear one.

Conservatively, if (1.4) is a generic nonlinear system no stability properties can be transferred from the origin of the linearised system to the equilibrium of the nonlinear one.

On the other hand, most of the automotive plants posses two interesting properties that make the control via linearisation attractive: linearity of (saturated) inputs and outputs, and complete observability and reachability. As for the linearity of inputs and outputs, we note that most of the actuators adopted in automotive control systems (see Fig. 1.3) are linearly dependent on the control inputs, *e.g.* electric motors and electromechanical/pneumatic/hydraulic servos are linearly dependent on the input voltage, etc. It is worth to note that usually the admissible inputs are bounded by physical constraints such as maximum currents, pressures, strains, etc. Moreover, typically automotive sensors provide linear combinations of the state or directly parts of the state vector, *e.g.* vehicle inertial speeds provided by the GNSS, wheel speeds measured by tonewheels, potentiometers sensing the length of suspensions, rotational speeds captured through IMUs, etc. Usually, the sensors are selected to have a range of linearity sufficiently larger than the expected magnitude of the measured quantities. Thanks to these kind of actuators and sensors the original nonlinear system (1.1) can be specialised as

$$\begin{aligned} \dot{\mathbf{x}} =&\ \mathbf{f}(\mathbf{x},\, \mathbf{w}) + \mathbf{B}_1\mathtt{sat}(\mathbf{u}) \quad \mathbf{x}(t_0) = \mathbf{x}_0 \\ \mathbf{y} =&\ \mathbf{C}\mathbf{x} + \mathbf{D}_1\mathtt{sat}(\mathbf{u}) + \boldsymbol{\nu} \end{aligned} \tag{4.35}$$

in which $\mathtt{sat}(\cdot) : \mathbb{R}^p \to \mathcal{U} \subset \mathbb{R}^p$ denotes the input saturation. Furthermore as described in this textbook, sensor and actuator suites are usually designed to guarantee complete observability and reachability. Then, for systems (4.35) locally completely reachable and observable we can demonstrate that the stability properties of the origin of the linearised system can be transferred to the equilibrium point of the nonlinear system on domains of initial conditions and exogenous whose size depends on the controller parameters. On the other hand, the controller gains cannot be made too large as described in Section 4.5.2. The result is that is not possible to tune independently the domain of attraction and the maximum bound on $\|\mathbf{x} - \mathbf{x}^\star\|$.

To demonstrate this result we assume to control (4.35) with (4.11), (4.24), (4.25), and (4.28) designed on the equilibrium tuple $(\mathbf{x}^\star, \mathbf{u}^\star, \mathbf{w}^\star, \mathbf{y}^\star)$. First we change the coordinates as

$$\tilde{\mathbf{x}} := \mathbf{x} - \mathbf{x}^\star,\ \tilde{\mathbf{u}} := \mathbf{u} - \mathbf{u}^\star,\ \tilde{\mathbf{w}} := \mathbf{w} - \mathbf{w}^\star,\ \tilde{\mathbf{y}} := \mathbf{y} - \mathbf{y}^\star$$

and we rewrite (4.35) as

$$
\begin{aligned}
\dot{\tilde{\mathbf{x}}} &= \tilde{\mathbf{f}}(\tilde{\mathbf{x}},\, \tilde{\mathbf{w}}) + \mathbf{B}_1 \tilde{\mathtt{sat}}(\tilde{\mathbf{u}}) \quad \tilde{\mathbf{x}}(t_0) = \tilde{\mathbf{x}}_0 \\
\tilde{\mathbf{y}} &= \mathbf{C}\tilde{\mathbf{x}} + \mathbf{D}_1 \tilde{\mathtt{sat}}(\tilde{\mathbf{u}}) + \boldsymbol{\nu}
\end{aligned}
\tag{4.36}
$$

in which $\tilde{\mathbf{f}}(\tilde{\mathbf{x}},\, \tilde{\mathbf{w}}) := \mathbf{f}(\tilde{\mathbf{x}} + \mathbf{x}^\star,\, \tilde{\mathbf{w}} + \mathbf{w}^\star)$ and $\tilde{\mathtt{sat}}(\tilde{\mathbf{u}}) := \mathtt{sat}(\tilde{\mathbf{u}} + \mathbf{u}^\star)$. Second, we write the closed loop as

$$
\begin{aligned}
\dot{\tilde{\mathbf{x}}} &= \tilde{\mathbf{f}}(\tilde{\mathbf{x}},\, \tilde{\mathbf{w}}) + \mathbf{B}_1 \tilde{\mathtt{sat}}(\tilde{\mathbf{u}}) \\
\dot{\boldsymbol{\eta}} &= \mathbf{C}_e \hat{\tilde{\mathbf{x}}} + \mathbf{D}_{\mathbf{e1}} \tilde{\mathtt{sat}}(\tilde{\mathbf{u}}) + \mathbf{E}\boldsymbol{\nu} \\
\dot{\hat{\tilde{\mathbf{x}}}} &= (\mathbf{A} - \mathbf{K}_\mathrm{O}\mathbf{C})\hat{\tilde{\mathbf{x}}} + (\mathbf{B_1} - \mathbf{K}_\mathrm{O}\mathbf{D_1})\tilde{\mathtt{sat}}(\tilde{\mathbf{u}}) + \mathbf{K}_\mathrm{O}\tilde{\mathbf{y}} \\
\tilde{\mathbf{u}} &= \mathbf{K}_\mathrm{S}\hat{\tilde{\mathbf{x}}} + \mathbf{K}_\mathrm{I}\boldsymbol{\eta} \\
\tilde{\mathbf{y}} &= \mathbf{C}\tilde{\mathbf{x}} + \mathbf{D}_1 \tilde{\mathtt{sat}}(\tilde{\mathbf{u}}) + \boldsymbol{\nu}.
\end{aligned}
\tag{4.37}
$$

Then define $\mathbf{e}_x = \hat{\tilde{\mathbf{x}}} - \tilde{\mathbf{x}}$, exploit the formulation of $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{y}}$, and add and subtract $\mathbf{A}\tilde{\mathbf{x}} + \mathbf{B}_1\tilde{\mathbf{u}}$ and $\mathbf{D}_{e1}\tilde{\mathbf{u}}$ to the first and second of (4.37) respectively. Then, let $\bar{\mathbf{x}} := \mathtt{col}(\tilde{\mathbf{x}},\, \boldsymbol{\eta},\, \mathbf{e}_x)$ with dynamics given by

$$
\dot{\bar{\mathbf{x}}} = \bar{\mathbf{A}}\bar{\mathbf{x}} + \mathbf{G}(\bar{\mathbf{x}}, \tilde{\mathbf{w}}) \tag{4.38}
$$

where

$$
\bar{\mathbf{A}} := \begin{bmatrix} \mathbf{A} + \mathbf{B}_1\mathbf{K}_\mathrm{S} & \mathbf{B}_1\mathbf{K}_\mathrm{I} & \mathbf{B}_1\mathbf{K}_\mathrm{S} \\ \mathbf{C}_{e1} + \mathbf{D}_{e1}\mathbf{K}_\mathrm{S} & \mathbf{D}_{e1}\mathbf{K}_\mathrm{I} & \mathbf{D}_{e1}\mathbf{K}_\mathrm{S} \\ \mathbf{0} & \mathbf{0} & \mathbf{A} - \mathbf{K}_\mathrm{O}\mathbf{C} \end{bmatrix},
$$

and

$$
\mathbf{G}(\bar{\mathbf{x}}, \tilde{\mathbf{w}}) := \begin{bmatrix} \tilde{\mathbf{f}}(\tilde{\mathbf{x}},\, \tilde{\mathbf{w}}) - \mathbf{A}\tilde{\mathbf{x}} \\ \mathbf{0} \\ \mathbf{A} - \tilde{\mathbf{f}}(\tilde{\mathbf{x}},\, \tilde{\mathbf{w}})\tilde{\mathbf{x}} \end{bmatrix} + \begin{bmatrix} \mathbf{B}_1 & \mathbf{0} \\ \mathbf{D}_{e1} & \mathbf{E} \\ \mathbf{0} & \mathbf{K}_\mathrm{O} \end{bmatrix} \begin{bmatrix} \tilde{\mathtt{sat}}(\tilde{\mathbf{u}}) - \tilde{\mathbf{u}} \\ \boldsymbol{\nu} \end{bmatrix}.
$$

Define $\mathcal{V}_{\mathbf{K}_\mathrm{FB}} \subset \mathbb{R}^{2n+m}$ as the set in which the input is not saturated, *i.e.* let

$$
\mathcal{V}_{\mathbf{K}_\mathrm{FB}} := \{(\tilde{\mathbf{x}}, \boldsymbol{\eta}, \mathbf{e}_x) \in \mathbb{R}^{2n+m} \,:\, \mathtt{sat}(\tilde{\mathbf{u}}) = \tilde{\mathbf{u}}\}.
$$

To compute the bound of the trajectories of (4.38) we rely on a support function $V(\bar{\mathbf{x}}) := \bar{\mathbf{x}}^\top\bar{\mathbf{x}}/2$. Then we exploit (4.38) to compute the time derivative of V as

$$
\begin{aligned}
\dot{V} &= \frac{\partial V}{\partial \bar{\mathbf{x}}}\dot{\bar{\mathbf{x}}} \\
&= \bar{\mathbf{x}}^\top\dot{\bar{\mathbf{x}}} \\
&= \bar{\mathbf{x}}^\top(\bar{\mathbf{A}}\bar{\mathbf{x}} + \mathbf{G}(\bar{\mathbf{x}}, \tilde{\mathbf{w}}))
\end{aligned}
\tag{4.39}
$$

and we note that, since $\bar{A}$ is Hurwitz by design of $\mathbf{K}_\mathrm{S}$, $\mathbf{K}_\mathrm{I}$ and $\mathbf{K}_\mathrm{O}$, we have $\bar{\mathbf{x}}^\top \bar{\mathbf{A}}\bar{\mathbf{x}} \preceq -\underline{\sigma}(\bar{\mathbf{A}})\|\bar{\mathbf{x}}\|^{\mathbf{2}}\mathbf{I}$. Let

$$\bar{\mathbf{B}} := \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{D}_{e1} \end{bmatrix}, \bar{\mathbf{K}}_\mathrm{FB} := \begin{bmatrix} \mathbf{K}_\mathrm{S} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_\mathrm{I} \end{bmatrix}, \bar{\mathbf{K}}_\mathrm{O} := \begin{bmatrix} \mathbf{E} \\ \mathbf{K}_\mathrm{O} \end{bmatrix},$$

then since $\tilde{\mathbf{f}}(\cdot,\cdot)$ is locally Lipschitz continuous, for any $\epsilon > 0$ there exists $L_\epsilon > 0$ such that for any $\|\mathtt{col}(\tilde{\mathbf{x}},\,\tilde{\mathbf{w}})\| \leq \epsilon$

$$\begin{aligned} \|\mathbf{G}(\bar{\mathbf{x}},\tilde{\mathbf{w}})\| &\leq L_\epsilon \|\mathtt{col}(\tilde{\mathbf{x}},\,\tilde{\mathbf{w}})\| + \overline{\sigma}\left(\bar{\mathbf{K}}_\mathrm{O}\right)\|\boldsymbol{\nu}\| + \overline{\sigma}\left(\bar{\mathbf{B}}\right)\overline{\sigma}\left(\mathbf{K}_\mathrm{FB}\right)\mathtt{dz}(\bar{\mathbf{x}}) \\ &\leq L_\epsilon(\|\mathtt{col}(\tilde{\mathbf{x}},\boldsymbol{\eta})\| + \|\tilde{\mathbf{w}}\|) + \overline{\sigma}(\bar{\mathbf{K}}_\mathrm{O})\|\boldsymbol{\nu}\| + \overline{\sigma}(\mathbf{B})\overline{\sigma}(\bar{\mathbf{K}}_\mathrm{FB})\mathtt{dz}(\bar{\mathbf{x}}) \end{aligned}$$

in which $\mathtt{dz} : \mathbb{R}^{2n} \to \mathbb{R}^+$ represents a dead-zone function with

$$\mathtt{dz}(\bar{\mathbf{x}}) = \begin{cases} 0 & \bar{\mathbf{x}} \in \mathcal{V}_{\mathbf{K}_\mathrm{FB}} \\ \|\mathtt{col}(\tilde{\mathbf{x}},\boldsymbol{\eta})\|_{\mathcal{V}_{\mathbf{K}_\mathrm{FB}}} + \|\mathbf{e}\|_{\mathcal{V}_{\mathbf{K}_\mathrm{FB}}} & \bar{\mathbf{x}} \notin \mathcal{V}_{\mathbf{K}_\mathrm{FB}}. \end{cases}$$

As consequence, exploiting $\|\mathtt{col}(\tilde{\mathbf{x}},\boldsymbol{\eta})\| \leq \|\bar{\mathbf{x}}\|$ and $\|\bar{\mathbf{x}}\|^2 = 2V$, and assuming $\bar{w} > 0$ such that $\|\boldsymbol{\nu}(t)\| \leq \|\tilde{\mathbf{w}}(t)\| \leq \bar{w}$ for all $t \geq 0$, $\dot{V}$ can be bounded from above as

$$\dot{V} \leq 2(L_\epsilon - \underline{\sigma}(\bar{\mathbf{A}}))V + \|\bar{\mathbf{x}}\|(L_\epsilon + \overline{\sigma}(\bar{\mathbf{K}}_\mathrm{O}))\bar{w} + \|\bar{\mathbf{x}}\|\overline{\sigma}(\bar{\mathbf{B}})\overline{\sigma}(\mathbf{K}_\mathrm{FB})\mathtt{dz}(\bar{\mathbf{x}}). \tag{4.40}$$

Let $\mathcal{E} := \{\bar{\mathbf{x}} \in \mathbb{R}^{2n+m} : \|\tilde{\mathbf{x}}\| \leq \epsilon\} \cap \mathcal{V}_{\mathbf{K}_\mathrm{FB}}$ and define $\overline{\rho} = \sup_{\bar{\mathbf{x}}\in\mathcal{E}} \|\bar{\mathbf{x}}\|$. Then, for any $\bar{\mathbf{x}} \in \mathcal{E}$ the saturation is inactive (*i.e.* $\mathtt{dz}(\bar{\mathbf{x}}) = 0$) and equation (4.40) represents a linear system whose solution is

$$V(t) \leq V(\bar{\mathbf{x}}(0))e^{2(L_\epsilon - \underline{\sigma}(\bar{\mathbf{A}}))t} + \overline{\rho}\frac{L_\epsilon + \overline{\sigma}(\bar{\mathbf{K}}_\mathrm{O})}{2\underline{\sigma}(L_\epsilon - \underline{\sigma}(\bar{\mathbf{A}}))}\bar{w}. \tag{4.41}$$

Use $\|\bar{\mathbf{x}}\| = \sqrt{2V}$ and $V(\bar{\mathbf{x}}(0)) = \|\bar{\mathbf{x}}_0\|^2/2$ to bound the trajectories of $\bar{\mathbf{x}}$ as

$$\begin{aligned} \|\bar{\mathbf{x}}(t)\| &\leq \sqrt{2V(\bar{\mathbf{x}}(0))e^{2(L_\epsilon - \underline{\sigma}(\bar{\mathbf{A}}))t} + \overline{\rho}\frac{L_\epsilon + \overline{\sigma}(\bar{\mathbf{K}}_\mathrm{O})}{\underline{\sigma}(L_\epsilon - \underline{\sigma}(\bar{\mathbf{A}}))}\bar{w}} \\ &\leq \|\bar{\mathbf{x}}_0\|e^{(L_\epsilon - \underline{\sigma}(\bar{\mathbf{A}}))t} + \sqrt{\overline{\rho}\frac{L_\epsilon + \overline{\sigma}(\bar{\mathbf{K}}_\mathrm{O})}{\underline{\sigma}(L_\epsilon - \underline{\sigma}(\bar{\mathbf{A}}))}\bar{w}}. \end{aligned} \tag{4.42}$$

To assure $\|\bar{\mathbf{x}}(t)\| \in \mathcal{E}$ for all $t \geq 0$ (and thus to make valid our investigation) we define $\underline{\rho} > 0$ as the largest positive such that $\{\bar{\mathbf{x}} \in \mathbb{R}^{2n} : \|\bar{\mathbf{x}}\| \leq \underline{\rho}\} \subseteq \mathcal{E}$. Then, we impose the following bounds on the initial conditions $\bar{\mathbf{x}}_0$ and on the exogenous $\tilde{\mathbf{w}}$

$$\begin{aligned} \|\bar{\mathbf{x}}_0\| &\leq \frac{\underline{\rho}}{2} \\ \bar{w} &\leq \left\{\frac{\underline{\rho}^2}{4\overline{\rho}}\frac{\underline{\sigma}(L_\epsilon - \underline{\sigma}(\bar{\mathbf{A}}))}{L_\epsilon + \overline{\sigma}(\mathbf{K}_\mathrm{O})}, \epsilon\right\}. \end{aligned}$$

A couple of important remarks can be made on (4.42):

1. for any ϵ, $q > 0$ there always exist $\mathbf{K}_\mathrm{S}$, $\mathbf{K}_\mathrm{I}$ and $\mathbf{K}_\mathrm{O}$ such that $L_\epsilon - \underline{\sigma}(\bar{\mathbf{A}}) = -q$. This result is guaranteed by complete observability and reachability assumptions. As consequence, the equilibrium $\mathbf{x}^\star$ can be made exponentially stable and the size of its domain of attraction is upper bounded by ϵ. In practice, to assure these results we tune the control system with sufficiently large $\mathbf{K}_\mathrm{S}$ and $\mathbf{K}_\mathrm{O}$. This *high-gain* policy comes as natural consequence of the input-output linearity of (4.36)

2. the presence of saturations inherently limit the validity of the linearisation to $\mathcal{E} \subseteq \mathcal{V}$. This latter domain is directly dependent on $\mathbf{K}_\mathrm{S}$ and $\mathbf{K}_\mathrm{I}$ and, in particular, its size decreases if the magnitude of $\mathbf{K}_\mathrm{S}$ and $\mathbf{K}_\mathrm{I}$ increases. The constraints induced by the saturations and the nonlinearities are translated into constraints on the initial conditions and maximum exogenous norms.

3. as showed in Section 4.5.2 the effect of the exogenous disturbances can be reduced by increasing the magnitude of $\mathbf{K}_\mathrm{S}$ and $\mathbf{K}_\mathrm{I}$ whereas, any increase in $\mathbf{K}_\mathrm{O}$ leads to a measurement noise amplification which, in turn, makes larger $\|\bar{\mathbf{x}}\|$

To conclude, we demonstrated that if the nonlinear plant is locally completely observable and reachable and in the form (4.35), then the control via linearisation can make the equilibrium $\mathbf{x}^\star$ exponentially stable with restrictions on the initial conditions and on the exogenous. Then, the trajectories of (4.38) are guaranteed to live into a bounded domain containing $\mathbf{x}^\star$.

4.6 Feedforward Control

As described in Section 1.2, the control system is completed by a feedforward action that, independently on the current state, forces the plant to *anticipate* the variation of the reference. This section presents a design strategy applicable to BIBS stable plants or plants made BIBS stable by a stabiliser. To reduce the topic presentation complexity and without loss of generality, we let the stabiliser be a pure state feedback. Thus, let

$$\mathbf{u} = \mathbf{K}_\mathrm{S}\mathbf{x} + \mathbf{u}_\mathrm{FF} \tag{4.43}$$

be the whole control law whose $\mathbf{K}_\mathrm{S}$ is designed accordingly to Section 4.2. Assuming $\mathbf{d} = \mathbf{0}$, the goals now is to design $\mathbf{u}_\mathrm{FF}$ to assure $\lim_{t\to\infty} \mathbf{e}(t) = \mathbf{0}$. To achieve this result we introduce a further hypothesis, namely Assumption 4.3. If this latter is not verified the set of the trackable references is reduced as detailed in Infobox 4.30.

Let

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}_1\mathbf{u} \qquad\qquad \mathbf{x}(t_0) = \mathbf{x}_0 \\ \mathbf{e} &= \mathbf{C}_e\mathbf{x} + \mathbf{D}_1\mathbf{u} + \mathbf{D}_2\mathbf{r} \end{aligned} \tag{4.44}$$

be a stabilisable LTI system. Let e_i, with $i = 1, \ldots, m$, be the i-th entry of $\mathbf{e}$. Then, for each $i = 1, \ldots, m$ determine $r_{\max_i}$ as the smallest integer such that

$$\frac{d^{r_{\max_i}}}{dt^{r_{\max_i}}} e_i = \mathbf{m}_i\mathbf{u}$$

for some $\mathbf{m}_i \neq \mathbf{0}$.

Assumption 4.3 (Vector Relative Degree)**.** *Let* (4.44) *be the plant, then assume* $\mathbf{D}_1 = \mathbf{0}$ *and* $\sum_{i=1}^{m} r_{\max_i} = n$.

Remark 4.3. *The applications detailed in this book verify Assumption 4.3 when the reference to be tracked is time-varying. On the other hand, for constant references the integral action control (see Section 4.3) guarantees the asymptotic tracking regardless of Assumption 4.3.*

Then substitute (4.43) into (4.44)

$$\begin{aligned} \dot{\mathbf{x}} &= (\mathbf{A} + \mathbf{B}_1\mathbf{K}_{\mathrm{S}})\mathbf{x} + \mathbf{B}_1\mathbf{u}_{\mathrm{FF}} \quad \mathbf{x}(t_0) = \mathbf{x}_0 \\ \mathbf{e} &= \mathbf{C}_e\mathbf{x} + \mathbf{D}_2\mathbf{r}. \end{aligned} \tag{4.45}$$

Let $\mathbf{c}_{e,i}$ be the i-th row of $\mathbf{C}_e$ and define $\boldsymbol{\zeta} = \mathbf{T}_\zeta\mathbf{x}$ with

$$\mathbf{T}_\zeta := \texttt{col}(\mathbf{c}_{e,1}, \cdots, \mathbf{c}_{e,1}\mathbf{A}^{r_{\max_1}-1}, \ldots, \mathbf{c}_{e,m}, \cdots, \mathbf{c}_{e,m}\mathbf{A}^{r_{\max_m}-1}). \tag{4.46}$$

Apply the change of coordinates (4.46) to (4.45) to obtain

$$\begin{aligned} \dot{\boldsymbol{\zeta}} =\ & \mathbf{T}_\zeta(\mathbf{A} + \mathbf{B}_1\mathbf{K}_{\mathrm{S}})\mathbf{T}_\zeta^{-1}\boldsymbol{\zeta} + \mathbf{T}_\zeta\mathbf{B}_1\mathbf{u}_{\mathrm{FF}} \quad \boldsymbol{\zeta}(t_0) = \mathbf{T}_\zeta\mathbf{x}_0 \\ \mathbf{e} =\ & \mathbf{C}_e\mathbf{T}_\zeta^{-1}\boldsymbol{\zeta} + \mathbf{D}_2\mathbf{r}. \end{aligned} \tag{4.47}$$

Let

$$\boldsymbol{\zeta} := \texttt{col}(\zeta_1, \cdots, \zeta_m) \tag{4.48}$$

and for each $i = 1, \ldots, m$ define

$$\boldsymbol{\zeta}_i = \texttt{col}(\zeta_{i,1}, \cdots, \zeta_{i,r_{\max_i}}). \tag{4.49}$$

Then, let $\mathbf{d}_{2,i}$ be the i-th row of $\mathbf{D}_2$ and compute the dynamics of $\boldsymbol{\zeta}_i$ as

$$\begin{aligned} \dot{\zeta}_{i,1} &= \zeta_{i,2} \\ \dot{\zeta}_{i,2} &= \zeta_{i,3} \\ &\ \vdots \\ \dot{\zeta}_{i,r_{\max_i}-1} &= \zeta_{i,r_{\max_i}} \\ \dot{\zeta}_{i,r_{\max_i}} &= \mathbf{c}_{e,i}\mathbf{A}^{r_{\max_i}-1}((\mathbf{A} + \mathbf{B}_1\mathbf{K}_{\mathrm{S}})\mathbf{T}_\zeta^{-1}\boldsymbol{\zeta} + \mathbf{B}_1\mathbf{u}_{\mathrm{FF}}) \\ e_i &= \zeta_{i,1} + \mathbf{d}_{2,i}\mathbf{r}. \end{aligned} \tag{4.50}$$

We now exploit (4.50) to define the reference state $\boldsymbol{\zeta}_i^\star$ such that $\boldsymbol{\zeta}_i = \boldsymbol{\zeta}_i^\star \implies e_i = 0$, *i.e.*

$$\begin{aligned}
\zeta_{i,1}^\star &:= -\mathbf{d}_{2,i}\mathbf{r} \\
\zeta_{i,2}^\star &:= -\mathbf{d}_{2,i}\frac{d}{dt}\mathbf{r} \\
&\vdots \\
\zeta_{i,r_{\max_i}}^\star &:= -\mathbf{d}_{2,i}\frac{d^{r_{\max_i}-1}}{dt^{r_{\max_i}-1}}\mathbf{r}.
\end{aligned} \tag{4.51}$$

To conclude, let $\boldsymbol{\zeta}^\star := \texttt{col}(\boldsymbol{\zeta}_1^\star, \cdots, \boldsymbol{\zeta}_m^\star)$, define $\mathbf{H}$ such that

$$\mathbf{H}\dot{\boldsymbol{\zeta}} = \texttt{col}(\dot{\zeta}_{1,r_{\max_1}}^\star, \ldots, \dot{\zeta}_{m,r_{\max_m}}^\star),$$

take

$$\overline{\mathbf{M}} := \begin{bmatrix} \mathbf{c}_{e,1}\mathbf{A}^{r_{\max_1}-1} \\ \vdots \\ \mathbf{c}_{e,m}\mathbf{A}^{r_{\max_m}-1} \end{bmatrix},$$

and exploit (4.50) for $i = 1, \ldots, m$ to write

$$\mathbf{H}\dot{\boldsymbol{\zeta}}^\star = \overline{\mathbf{M}}((\mathbf{A} + \mathbf{B}_1\mathbf{K}_\mathrm{S})\mathbf{T}_\zeta^{-1}\boldsymbol{\zeta}^\star + \mathbf{B}_1\mathbf{u}_\mathrm{FF}). \tag{4.52}$$

Assumption 4.4. *The matrix* $\mathbf{M} := \overline{\mathbf{M}}\mathbf{B}_1$ *has* m *rows linearly independent. As consequence the Moore-Penrose right pseudo-inverse* $\mathbf{M}^+ := \mathbf{M}^\top(\mathbf{M}\mathbf{M}^\top)^{-1}$ *is well defined.*

If Assumption 4.4 is verified, define $r_{\max} = \max\{r_{\max_1}, \ldots, r_{\max_m}\}$ and compute the feedforward control law as

$$\begin{aligned}
\mathbf{u}_\mathrm{FF} &= \sum_{i=1}^{r_{\max}} \mathbf{D}_{\mathrm{FF}_i}\frac{d^i}{dt^t}\mathbf{r} \\
&:= \mathbf{M}^+\left(\mathbf{H}\dot{\boldsymbol{\zeta}}^\star - \overline{\mathbf{M}}(\mathbf{A} + \mathbf{B}_1\mathbf{K}_\mathrm{S})\mathbf{T}_\zeta^{-1}\boldsymbol{\zeta}^\star\right)
\end{aligned} \tag{4.53}$$

Remark 4.4. *The feedforward control law* (4.53) *represents an algebraic sum of the reference and its time derivatives. This constitutes a subpart of* (1.18e) *whose remaining elements are detailed in Infobox 4.30.*

Example 4.29. *Let*

$$\begin{aligned}
\dot{x} &= ax + bu \\
y &= x \\
e &= cx - dr(t)
\end{aligned}$$

be an LTI system where a, b, c, $d > 0$, x *is the state,* y *the output,* e *the* *controlled output, and $r(t)$ represents the known reference value. Since we want to steer $e \to 0$ we must define $x^\star = d_{e_2}/c_{e_1} r(t)$. We note that $r_{\max} = 1$ and Assumption 4.3 is verified. Thus, we define $u = k_R x + v$ with $k_R : a + b_1 k_R < 0$ and v as*

$$v = [c(a + bk_R)^{-1} b]^{-1} dr(t) + b^{-1} \dot{x}^\star.$$

Infobox 4.30 (Zero Dynamics & Reference Generator).
If Assumption 4.3 is not verified but $\mathbf{D}_1 = \mathbf{0}$ we can still solve the asymptotic tracking problem. As detailed hereafter we rely on a modified change of coordinates. As consequence, the feedforward results to be the output of a dynamic system called Zero Dynamics. Moreover, if some stabilisation conditions are met the reference is designed to make BIBS stable the Zero Dynamics. Let $\mathbf{T}_\zeta$ be defined in (4.46) and define $\mathbf{T}_{\zeta_\perp}$ such that $\mathbf{T} := \texttt{col}(\mathbf{T}_\zeta, \mathbf{T}_{\zeta_\perp})$ is invertible and $\mathbf{T}_{\zeta_\perp}\mathbf{B}_1 = \mathbf{0}$. Then, let $\texttt{col}(\boldsymbol{\zeta}, \boldsymbol{\zeta}_\perp) = \mathbf{T}\mathbf{x}$, define

$$\begin{bmatrix} \bar{\mathbf{A}}_{11} & \bar{\mathbf{A}}_{12} \\ \bar{\mathbf{A}}_{21} & \bar{\mathbf{A}}_{22} \end{bmatrix} = \mathbf{T}(\mathbf{A} + \mathbf{B}_1 \mathbf{K}_S)\mathbf{T}^{-1},$$

and compute the dynamics of $\boldsymbol{\zeta}$, $\boldsymbol{\zeta}_\perp$ as

$$\begin{aligned} \dot{\boldsymbol{\zeta}} &= \bar{\mathbf{A}}_{11}\boldsymbol{\zeta} + \bar{\mathbf{A}}_{12}\boldsymbol{\zeta}_\perp + \mathbf{T}_\zeta \mathbf{B}_1 \mathbf{u}_{FF} \\ \dot{\boldsymbol{\zeta}}_\perp &= \bar{\mathbf{A}}_{21}\boldsymbol{\zeta} + \bar{\mathbf{A}}_{22}\boldsymbol{\zeta}_\perp \\ \mathbf{e} &= \mathbf{C}_e \mathbf{T}_\zeta^{-1}\boldsymbol{\zeta} + \mathbf{D}_2 \mathbf{r}. \end{aligned}$$

We exploit this system to generate the references $\boldsymbol{\zeta}^\star$ and $\boldsymbol{\zeta}_\perp^\star$. Let $\boldsymbol{\zeta}^\star := \texttt{col}(\zeta_1^\star, \cdots, \zeta_m^\star)$ with $\zeta_i^\star$ be defined in (4.51) for $i = 1, \ldots, m$. Then $\boldsymbol{\zeta} = \boldsymbol{\zeta}^\star$ and $\boldsymbol{\zeta}_\perp = \boldsymbol{\zeta}_\perp^\star$ imply

$$\dot{\boldsymbol{\zeta}}_\perp^\star = \bar{\mathbf{A}}_{21}\boldsymbol{\zeta}^\star + \bar{\mathbf{A}}_{22}\boldsymbol{\zeta}_\perp^\star$$

which is called "Zero Dynamics". If the couple $(\bar{\mathbf{A}}_{22}, \bar{\mathbf{A}}_{21})$ is stabilisable there exists $\mathbf{K}$ such that $\bar{\mathbf{A}}_{22} + \bar{\mathbf{A}}_{21}\mathbf{K}$ is Hurwitz. Then, the reference $\mathbf{r}$ generated as $\boldsymbol{\zeta}^\star = \mathbf{K}\boldsymbol{\zeta}_\perp^\star + \mathbf{v}$, where $\mathbf{v}$ represents the actual reference to be tracked, makes BIBS stable the dynamics of $\boldsymbol{\zeta}_\perp^\star$. To conclude, use the *same steps exploited in* (4.52)-(4.53) *to obtain* (1.18e) *with* $\mathbf{x}_{FF} := \boldsymbol{\zeta}_{\perp}^{\star}$, $\mathbf{A}_{FF} := \bar{\mathbf{A}}_{22} + \bar{\mathbf{A}}_{21}\mathbf{K}$, $\sum_{i=1}^{r_{\max}} \mathbf{B}_{FF_i} d^i\mathbf{r}/dt^i := \bar{\mathbf{A}}_{21}\mathbf{v}$,

$$\mathbf{C}_{FF} := \mathbf{M}^{+}\mathbf{H}\left[\mathbf{K}\left(\bar{\mathbf{A}}_{22} + \bar{\mathbf{A}}_{21}\mathbf{K}\right) - \bar{\mathbf{A}}_{12} - \bar{\mathbf{A}}_{11}\mathbf{K}\right],$$

and

$$\sum_{i=1}^{r_{\max}} \mathbf{D}_{FF_i}\frac{d^i}{dt^i}\mathbf{r} := \mathbf{M}^{+}\mathbf{H}\left[\dot{\mathbf{v}} + \left(\mathbf{K}\bar{\mathbf{A}}_{21} - \bar{\mathbf{A}}_{11}\right)\mathbf{v}\right].$$

Example 4.31 (Cart-pole: speed tracking)**.**
As shown in Example 4.28, the reachable and observable subsystem of 1.2 is

$$\dot{\mathbf{z}}_{R,O} = \mathbf{A}\mathbf{z}_{R,O} + \mathbf{B}u$$

in which we set $\mathbf{d} = \mathbf{0}$ *and*

$$\mathbf{A} = \begin{bmatrix} A_{22} & A_{23} & A_{24} \\ 0 & 0 & 1 \\ A_{42} & A_{43} & A_{44} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} b_1 \\ 0 \\ b_3 \end{bmatrix}.$$

We want to design a control law $u = \mathbf{K}_S\mathbf{z}_{R,O} + u_{FF}$ *that makes the cart following a known speed profile, namely* $\dot{p}^{\star}(t)$. *As first step we define the stabilising feedback* $\mathbf{K}_s = \begin{bmatrix} k_{S_1} & k_{S_2} & k_{S_3} \end{bmatrix}$. *Then, let the speed error* $e := \mathbf{C}_e\mathbf{z}_{R,O} - \dot{p}^{\star}(t)$ *where* $\mathbf{C}_e := \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$. *Compute*

$$\dot{e} = \mathbf{C}_e\dot{\mathbf{z}}_{R,O} - \ddot{p}^{\star}(t) = \mathbf{C}_e\mathbf{A}\mathbf{z}_{R,O} + \mathbf{C}_e\mathbf{B}u - \dot{p}^{\star}(t),$$

and observe that $\bar{m} := \mathbf{C}_e\mathbf{B} = b_1 \neq 0$. *Define with* $n_{R,O}$ *the dimension of* $\mathbf{z}_{R,O}$, *then* $r_{\max} = 1 < n_{R,O}$ *implies that Assumption 4.3 is not verified. As consequence, we proceed accordingly to Infobox 4.30 and we identify*

$$\mathbf{T}_{\zeta} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{T}_{\zeta_{\perp}} = \begin{bmatrix} 0 & 1 & 0 \\ -b_3 & 0 & b_1 \end{bmatrix}$$

such that $\mathbf{T} := \texttt{col}(\mathbf{T}_{\zeta}, \mathbf{T}_{\zeta_{\perp}})$ *is full rank and* $\mathbf{T}_{\zeta_{\perp}}\mathbf{B} = \mathbf{0}$. *Compute*

$$\begin{aligned} \begin{bmatrix} \bar{A}_{11} & \bar{\mathbf{A}}_{12} \\ \bar{\mathbf{A}}_{21} & \bar{\mathbf{A}}_{22} \end{bmatrix} :=& \mathbf{T}(\mathbf{A} + \mathbf{B}\mathbf{K}_S)\mathbf{T}^{-1} \\ =& \mathbf{T}\begin{bmatrix} A_{22} + b_1k_{S_1} & A_{23} + b_1k_{S_2} & A_{24} + b_1k_{S_3} \\ 0 & 0 & 1 \\ A_{42} + b_2k_{S_1} & A_{43} + b_2k_{S_2} & A_{44} + b_2k_{S_3} \end{bmatrix}\mathbf{T}^{-1} \end{aligned}$$

with

$$\bar{A}_{11} = A_{22} + b_1 k_{S_1} + \frac{(A_{24} + b_1 k_{S_3})\, b_3}{b_1}$$

$$\bar{\mathbf{A}}_{12} = \begin{bmatrix} A_{23} + b_1 k_{S_2} & \frac{A_{24}+b_1 k_{S_3}}{b_1} \\ 0 & \frac{1}{b_1} \end{bmatrix}$$

$$\bar{\mathbf{A}}_{21} = \begin{bmatrix} \frac{b_3}{b_1} \\ (A_{42} + b_2 k_{S_1})\, b_1 - (A_{22} + b_1 k_{S_1})\, b_3 - \frac{b_3 \left((A_{24}+b_1 k_{S_3})\, b_3 - (A_{44}+b_2 k_{S_3})\, b_1\right)}{b_1} \end{bmatrix}$$

$$\bar{\mathbf{A}}_{22} = \begin{bmatrix} 0 & \frac{1}{b_1} \\ (A_{43} + b_2 k_{S_2})\, b_1 - (A_{23} + b_1 k_{S_2})\, b_3 & -\frac{(A_{24}+b_1 k_{S_3})\, b_3 - (A_{44}+b_2 k_{S_3})\, b_1}{b_1} \end{bmatrix}.$$

Note that the couple $(\bar{\mathbf{A}}_{22}, \bar{\mathbf{A}}_{21})$ *is fully reachable and thus there exists* $\mathbf{K}$ *such that* $\bar{\mathbf{A}}_{22} + \bar{\mathbf{A}}_{21}\mathbf{K}$ *is Hurwitz. Define* $\texttt{col}(\zeta,\, \boldsymbol{\zeta}_\perp) = \mathbf{T}\mathbf{z}_{R,O}$ *whose dynamics is*

$$\begin{aligned} \dot{\zeta} &= \bar{A}_{11}\zeta + \bar{\mathbf{A}}_{12}\boldsymbol{\zeta}_\perp + b_{21} u_{FF} \\ \dot{\boldsymbol{\zeta}}_\perp &= \bar{\mathbf{A}}_{21}\zeta + \bar{\mathbf{A}}_{22}\boldsymbol{\zeta}_\perp \\ \mathbf{e} &= \zeta - \dot{p}^\star(t). \end{aligned}$$

Let $v : \mathbb{R} \to \mathbb{R}$ *be a continuous and bounded function, and define the reference generator as*

$$\begin{aligned} \dot{\boldsymbol{\zeta}}^\star_\perp &= (\bar{\mathbf{A}}_{22} + \bar{\mathbf{A}}_{21}\mathbf{K})\boldsymbol{\zeta}^\star_\perp + \bar{\mathbf{A}}_{21} v(t) \\ \zeta^\star &= \mathbf{K}\boldsymbol{\zeta}^\star_\perp + v(t). \end{aligned}$$

To conclude, the feed-forward control law is

$$\begin{aligned} u_{FF} &= b_1^{-1}\left(\dot{\zeta}^\star - \bar{A}_{11}\zeta^\star - \bar{\mathbf{A}}_{12}\boldsymbol{\zeta}^\star_\perp\right) \\ &= b_1^{-1}\left(\mathbf{K}\left(\bar{\mathbf{A}}_{22} + \bar{\mathbf{A}}_{21}\mathbf{K}\right) - \bar{A}_{11}\mathbf{K} - \bar{\mathbf{A}}_{12}\right)\boldsymbol{\zeta}^\star_\perp \\ &\quad + b_1^{-1}\left(\dot{v}(t) + \left(\mathbf{K}\bar{\mathbf{A}}_{21} - \bar{A}_{11}\right) v(t)\right). \end{aligned}$$

4.7 ADAS Architecture

This section translates into practice the theoretical achievements presented in Sections 4.2-4.6. In particular, thought reachability and observability analysis, this section aims to define the system architecture to achieve the control goals described in Chapter 2.

4.9 Exercises

Exercise 4.1. *Is the couple*

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

fully reachable? Which column of $\mathbf{B}$ *can be eliminated without modifying the original reachability properties?*

Exercise 4.2. *Is the couple*

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

stabilisable? How large is the non-reachable subspace?

Exercise 4.3. *Is the couple*

$$\mathbf{A} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

fully reachable? How large is the unstable part of the non-reachable subspace?

Exercise 4.4. *Is the couple*

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

fully reachable? Which column could be added to $\mathbf{B}$ *to achieve fully reachability? Which element of* $\mathbf{A}$ *could be changed to obtain the original couple* $(\mathbf{A}, \mathbf{B})$ *stabilisable?*

Exercise 4.5. *Is the couple*

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

fully observable? How large is the unobservable subsystem?

Exercise 4.6. *Is the couple*

$$\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

detectable? Which element of $\mathbf{A}$ *could be changed to obtain the detectability?*

Exercise 4.7. *Is the couple*

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$$

detectable? Which row could be added to $\mathbf{C}$ *to make the couple* $(\mathbf{A}, \mathbf{C})$ *fully observable?*

Exercise 4.8. *Is the couple*

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 1 & 1 & 0 & 1 \end{bmatrix}$$

fully observable? Which element of $\mathbf{C}$ *could be changed to lose the fully observability while keeping the detectability?*

Exercise 4.9. *Given the couple*

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

determine the reachability Kalman decomposition.

Exercise 4.10. *Given the couple*

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & -1 & 1 & -1 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$$

determine the observability Kalman decomposition.

Exercise 4.11. *Given the triplet*

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}$$

determine the ultimate Kalman decomposition.

Exercise 4.12. *Given the LTI system*

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} \end{aligned}$$

and assuming the couple $(\mathbf{A}, \mathbf{B})$ *fully reachable and the couple* $(\mathbf{A}, \mathbf{C})$ *fully observable, demonstrate the separation principle.*

Exercise 4.13. *Given the LTI system*

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (u + d)$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

with $\dot{d} = 0$, design a controller able to asymptotically steer $\mathbf{x} = \mathtt{col}(x_1, x_2)$ to the origin of $\mathbb{R}^2$. Taking as reference the state x_1, compare the result with a Proportional-Integral-Derivative standard controller.

Exercise 4.14. *Prove the separation principle for the regulator described in Section 4.3.*

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Chapter 5

Optimal Control and Kalman Filter

Chapter 4 describes the control architecture based on the linearisation of the plant, investigates which part of the plant can be modified and estimated, and provides the existence of the feedback matrices required to implement a dynamic output feedback controller (composed of a chain of an observer and a static controller). This chapter describes one among the possible answers to the question: "How can we design the controller and observer's matrices?". In more details, the state feedback matrices are designed through the *optimal control technique* described in Section 5.1. All the results presented in Section 5.1 are exploited, thanks to the *duality* principle presented in Section 5.2, to design the observer in Section 5.3. TBC

Contents

5.1 Robust Stationary Optimal Control

Sections 4.2 and 4.3 detail the feedback control structure as composition of an ideal state feedback plus an integral action. More in details, let

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}_1\mathbf{u} \qquad \mathbf{x}(t_0) = \mathbf{x}_0 \\ \mathbf{e} &= \mathbf{C}_e\mathbf{x} + \mathbf{D}_{e1}\mathbf{u} \end{aligned} \tag{5.1}$$

be a stabilisable LTI system, then there exist two matrices $\mathbf{K}_\mathrm{S}$ and $\mathbf{K}_\mathrm{I}$ such that the controller

$$\begin{aligned} \dot{\boldsymbol{\eta}} &= \mathbf{e} \\ \mathbf{u} &= \mathbf{K}_\mathrm{S}\mathbf{x} + \mathbf{K}_\mathrm{I}\boldsymbol{\eta} \end{aligned} \tag{5.2}$$

makes (5.1) BIBS stable. This Section provides an optimal criterion for the design of $\mathbf{K}_\mathrm{S}$ and $\mathbf{K}_\mathrm{I}$.

Let $\mathbf{x}_e := \texttt{col}(\mathbf{x}, \boldsymbol{\eta})$,

$$\mathbf{A}_e := \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{C}_e & \mathbf{0} \end{bmatrix}, \quad \mathbf{B}_e := \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{D}_{e1} \end{bmatrix},$$

and substitute (5.2) into (5.1) to obtain

$$\dot{\mathbf{x}}_e = \mathbf{A}_e\mathbf{x}_e + \mathbf{B}_e\mathbf{u} \qquad \mathbf{x}_e(t_0) = \mathbf{x}_{e0} \tag{5.3}$$

To keep the notation clean, in the remaining of this chapter we drop the subscripts from $\mathbf{x}_e$, $\mathbf{A}_e$, and $\mathbf{B}_e$.

To introduce a robust optimality criterion we alter (5.3) and define a cost function $J > 0$ as follows

$$\dot{\mathbf{x}} = (\mathbf{A} + \alpha\mathbf{I})\mathbf{x} + \mathbf{B}\mathbf{u} \tag{5.4a}$$

$$\boldsymbol{\epsilon} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \tag{5.4b}$$

$$J = \int_{t_0}^{\infty} \boldsymbol{\epsilon}^\top \mathbf{Q}\boldsymbol{\epsilon} + \mathbf{u}^\top\mathbf{R}\mathbf{u}\, dt \tag{5.4c}$$

in which $\alpha \geq 0$, $\boldsymbol{\epsilon}$ denotes any linear combination of states and control we want to penalise, $\mathbf{Q} = \mathbf{Q}^\top \succeq \mathbf{0}$ and $\mathbf{R} = \mathbf{R}^\top \succ \mathbf{0}$ constitute the cost to pay if $\boldsymbol{\epsilon}(t) \neq \mathbf{0}$ and $\mathbf{u}(t) \neq \mathbf{0}$.

The definitions of semi-positive ($\succeq$) and positive ($\succ$) definite matrices are given in Appendix A

Keynote

The error $\boldsymbol{\epsilon}$ is not necessarily a measurable output and therefore does not need to be neither a part of $\mathbf{y}$ nor obtainable from $\mathbf{y}$. The matrices $\mathbf{C}$ and $\mathbf{D}$ are not necessarily those defined in (1.17).

Problem 5.1. *Let* (5.4) *be the optimisation problem, then design a control law* $\mathbf{u}(t)$ *which minimizes the cost* (5.4c) *under the constraint* (5.4a)*.*

Roughly, the tunable parameters appearing in (5.4) can be interpreted as follows. If we do not like to have the error $\boldsymbol{\epsilon}(t)$ far from the origin we increase $\mathbf{Q}$ to associate to the condition $\boldsymbol{\epsilon}(t) \neq \mathbf{0}$ a higher cost. At the opposite, if we do not care about the distance that $\boldsymbol{\epsilon}(t)$ has from the origin we can take $\mathbf{Q} = \mathbf{0}$. Moreover, the cost $\mathbf{Q}$ must be $\succeq \mathbf{0}$ to be well posed: indeed, a negative cost means a "gain" which encourages the error $\boldsymbol{\epsilon}(t)$ to be infinite! As for $\mathbf{R}$, the higher is the magnitude of $\mathbf{R}$ the more $\mathbf{u}$ is considered expensive. So, if we want to limit the control action we can increase the magnitude of $\mathbf{R}$. On the other hand, if the control law $\mathbf{u}$ is "for free"(leading to a null $\mathbf{R}$) it means that we can design a control law $\mathbf{u}$ which can have infinite magnitude. This is clearly unfeasible due to the finite power of actuators and, for this reason, the cost $\mathbf{R}$ must be positive defined, *i.e.* $\mathbf{R} \succ \mathbf{0}$. To conclude α is introduced to design the control system on an " "apparently less" stable plant. This leads to robustness to model uncertainties and to extreme selection of gains $\mathbf{Q}$ and $\mathbf{R}$.

The determination of $\mathbf{u}^\star$ that minimises (5.4c) under the constraints (5.4a) represents, in general, a challenging task. To face with this complex assignment, a good practice consists in the reformulation of this constrained minimisation problem as an unconstrained one through the use of Lagrange multipliers. Basically, we introduce the trivial term $\boldsymbol{\lambda}^\top (\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} - \dot{\mathbf{x}})$ in (5.4c) to obtain

$$J = \int_{t_0}^{\infty} H(\mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}) - \boldsymbol{\lambda}^\top \dot{\mathbf{x}}\, dt \tag{5.5}$$

where

$$H(\mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}) := \boldsymbol{\epsilon}^\top \mathbf{Q}\, \boldsymbol{\epsilon} + \mathbf{u}^\top \mathbf{R}\, \mathbf{u} + \boldsymbol{\lambda}^\top \left((\mathbf{A} + \alpha \mathbf{I})\mathbf{x} + \mathbf{B}\mathbf{u}\right) \tag{5.6}$$

is called Hamiltonian function and where $\boldsymbol{\lambda} : \mathbb{R} \to \mathbb{R}^n$ represents the Lagrange multiplier, also called *co-state*. We integrate (5.5) by parts as

$$J = -\left.\boldsymbol{\lambda}^\top \mathbf{x}\right|_{t_0}^{\infty} + \int_{t_0}^{\infty} H(\mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}) + \dot{\boldsymbol{\lambda}}^\top \mathbf{x}\, dt \tag{5.7}$$

and we introduce the variation of J with respect to $\mathbf{x}$ and $\mathbf{u}$ as

$$\begin{aligned}\delta J = & -\boldsymbol{\lambda}^\top(\infty)\delta\mathbf{x}(\infty) + \boldsymbol{\lambda}^\top(t_0)\delta\mathbf{x}(t_0) + \\ & \int_{t_0}^{\infty} \frac{\partial H(\mathbf{x},\mathbf{u},\boldsymbol{\lambda})}{\partial \mathbf{u}}\delta\mathbf{u} + \left(\frac{\partial H(\mathbf{x},\mathbf{u},\boldsymbol{\lambda})}{\partial \mathbf{x}} + \dot{\boldsymbol{\lambda}}^\top\right)\delta\mathbf{x}\,dt.\end{aligned} \tag{5.8}$$

The variation of J can be made independent on $\delta\mathbf{x}$ during the transient from t_0 to ∞ if we choose

$$\dot{\boldsymbol{\lambda}} = -\left(\frac{\partial H(\mathbf{x},\mathbf{u},\boldsymbol{\lambda})}{\partial \mathbf{x}}\right)^\top, \qquad \boldsymbol{\lambda}(\infty) = \mathbf{0}. \tag{5.9}$$

The condition (5.9) simplifies (5.8) as

$$\delta J = \boldsymbol{\lambda}^\top(t_0)\delta\mathbf{x}(t_0) + \int_{t_0}^{\infty} \frac{\partial H(\mathbf{x},\mathbf{u},\boldsymbol{\lambda})}{\partial \mathbf{u}}\delta\mathbf{u}\,dt. \tag{5.10}$$

Then, if $\mathbf{u}$ evaluated at $\mathbf{u}^\star$ is a minimiser for J, then it must be such that

$$\int_{t_0}^{\infty} \left.\frac{\partial H(\mathbf{x},\mathbf{u},\boldsymbol{\lambda})}{\partial \mathbf{u}}\right|_{\mathbf{u}=\mathbf{u}^\star} \delta\mathbf{u}^\star\,d\tau = \mathbf{0} \tag{5.11}$$

which leads to

$$\left.\frac{\partial H(\mathbf{x},\mathbf{u},\boldsymbol{\lambda})}{\partial \mathbf{u}}\right|_{\mathbf{u}=\mathbf{u}^\star} = 2\mathbf{x}^\top\mathbf{C}^\top\mathbf{Q}\mathbf{D} + \boldsymbol{\lambda}^\top\mathbf{B} + 2[\mathbf{u}^\star]^\top\bar{\mathbf{R}} = \mathbf{0}. \tag{5.12}$$

where $\bar{\mathbf{R}} := \mathbf{D}^\top\mathbf{Q}\mathbf{D} + \mathbf{R}$. From (5.12) we find

$$\mathbf{u}^\star = -\frac{1}{2}\bar{\mathbf{R}}^{-1}\left(2\mathbf{D}^\top\mathbf{Q}\mathbf{C}\mathbf{x} + \mathbf{B}^\top\boldsymbol{\lambda}\right). \tag{5.13}$$

Then, in order to make $\mathbf{u}^\star$ linearly dependent on $\boldsymbol{\lambda}$, we design the Lagrange multiplier as a linear function of $\mathbf{x}$. Let $\boldsymbol{\lambda} := 2\mathbf{S}\mathbf{x}$ with $\mathbf{S} = \mathbf{S}^\top \succ \mathbf{0}$ and write (5.13) as

$$\begin{aligned}\mathbf{u}^\star &= \mathbf{K}\mathbf{x} \\ \mathbf{K} &= -\bar{\mathbf{R}}^{-1}\left(\mathbf{D}^\top\mathbf{Q}\mathbf{C} + \mathbf{B}^\top\mathbf{S}\right).\end{aligned} \tag{5.14}$$

To conclude the design of the optimal control $\mathbf{u}^\star$ we find an expression for $\mathbf{S}$. The solution of the stationary optimal control problem is found by noting that $\dot{\boldsymbol{\lambda}} = 2\mathbf{S}\dot{\mathbf{x}}$ must agree with (5.9) evaluated at $\mathbf{u}^\star$, *i.e.* $\mathbf{S}$ must verify

$$\begin{aligned}&\mathbf{S}\mathbf{B}\bar{\mathbf{R}}^{-1}\mathbf{B}^\top\mathbf{S} - \mathbf{S}(\mathbf{A} + \alpha\mathbf{I} - \mathbf{B}\bar{\mathbf{R}}^{-1}\mathbf{D}^\top\mathbf{Q}\mathbf{C}) \\ &- (\mathbf{A} + \alpha\mathbf{I} - \mathbf{B}\bar{\mathbf{R}}^{-1}\mathbf{D}^\top\mathbf{Q}\mathbf{C})^\top\mathbf{S} - \mathbf{C}^\top\mathbf{Q}[\mathbf{I} - \mathbf{D}\bar{\mathbf{R}}^{-1}\mathbf{D}^\top\mathbf{Q}]\mathbf{C} = \mathbf{0}.\end{aligned} \tag{5.15}$$

Keynote

It is worth noting that the solution of the Problem 5.1 must also stabilise the plant.

Expression (5.15), called Algebraic Riccati Equation (ARE), has one stabilising solution, namely $\mathbf{S}_\infty$, positive semi-definite ($\mathbf{S}_\infty = (\mathbf{S}_\infty)^\top \succeq \mathbf{0}$) if the couple $(\mathbf{A} + \alpha\mathbf{I}, \mathbf{B})$ is stabilisable and the couple $(\mathbf{A} + \alpha\mathbf{I}, \mathbf{C})$ is detectable (these two conditions must be jointly verified!).

Moreover, $(\mathbf{A} + \alpha\mathbf{I}, \mathbf{B})$ stabilisable, $\alpha > 0$, $\mathbf{Q} \succ 0$, and $(\mathbf{A}, \mathbf{C})$ fully observable represent sufficient (but not necessary) conditions to have $\mathbf{S}_\infty \succ \mathbf{0}$.

Let

$$\begin{aligned}\dot{x} &= (a+\alpha)x + bu \\ \varepsilon &= cx \\ 0 &= s^2\frac{b^2}{r} - 2(a+\alpha)s - c^2q\end{aligned}$$

be a scalar plant with the associated ARE. The solutions of the ARE are

$$s_{1,2} = \frac{(a+\alpha)r}{b^2} \pm \sqrt{\frac{(a+\alpha)^2r^2}{b^4} + \frac{c^2qr}{b^2}}$$

with $s_1 \geq 0$ and $s_2 \leq 0$. The optimal control law is $u^\star = ks_1$ with $k := -r^{-1}bs_1$. Then, the solution s_1 approaches zero for $r \to 0$ and the gain k for $r \to 0$ is given by

$$\lim_{r\to 0} k = -\lim_{r\to 0}\frac{bs_1}{r} = -\lim_{r\to 0}\left(\frac{a+\alpha}{b} + b\sqrt{\frac{(a+\alpha)^2}{b^4} + \frac{c^2q}{rb^2}}\right) = -\infty$$

which confirms the intuition that if r is trivial, the control can be infinite. On the other hand, we have that for $r \to \infty$ the solution s_1 goes to infinite but the control gain k results to be finite

$$\begin{aligned}\lim_{r\to\infty} k &= -\lim_{r\to\infty}\frac{bs_1}{r} = -\lim_{r\to\infty}\left(\frac{a+\alpha}{b} + b\sqrt{\frac{(a+\alpha)^2}{b^4} + \frac{c^2q}{rb^2}}\right) \\ &= -\frac{a+\alpha+\sqrt{(a+\alpha)^2}}{b}.\end{aligned}$$

In practice, when $r \to \infty$ the cost associated to the control is so higher than the cost q associated to the error ϵ that the latter can be neglected.

In this configuration, the main objectives are the reduction of the control action and the stability of the closed loop system. As consequence, for any $a \leq -\alpha$ we have $k = 0$ because the system is already BIBS stable with eigenvalue less or equal to $-\alpha$. On the other hand, for any $a > -\alpha$ the feedback gain is $k = -2(a+\alpha)/b$ which assures a closed loop eigenvalue less or equal to $-(a+2\alpha)$.

Infobox 5.1. *The stationary optimal control based on the quadratic cost function represents one of the most attractive control solutions for linear systems. Indeed, as described in Appendix C, this control strategy possesses a surprising robustness to model uncertainties that, in the frequency domain, are translated into a phase margin of at least 60º and a gain margin between* $1/2$ *and* ∞.

To conclude, we want to investigate the solution to Problem 5.1 with respect to the detectability of the couple $(\mathbf{A}+\alpha\mathbf{I}, \mathbf{C})$ and the stabilisability of the couple $(\mathbf{A}+\alpha\mathbf{I}, \mathbf{B})$. Let $\mathbf{T}_{\mathrm{K}}$ be the transformation that changes (5.4a)-(5.4b) into the ultimate Kalman decomposition form. Then, define $\mathbf{z} = \mathbf{T}_{\mathrm{K}}\mathbf{x}$ and determine the dynamics of $\mathbf{z}$ as

$$\begin{aligned} \dot{\mathbf{z}} &= \mathbf{A}^{\star}\mathbf{z} + \mathbf{B}^{\star}\mathbf{u} \quad \mathbf{x}(t_0) = \mathbf{x}_0 \\ \boldsymbol{\epsilon} &= \mathbf{C}^{\star}\mathbf{z} + \mathbf{D}\mathbf{u} \end{aligned}$$

in which $\mathbf{z} := \mathtt{col}(\mathbf{z}_{\mathrm{R,NO}}, \mathbf{z}_{\mathrm{R,O}}, \mathbf{z}_{\mathrm{NR,NO}}, \mathbf{z}_{\mathrm{NR,O}})$, $\mathbf{A}^{\star} := \mathbf{TAT}^{-1}$, $\mathbf{B}^{\star} := \mathbf{TB}$ and $\mathbf{C}^{\star} := \mathbf{CT}^{-1}$. Let $\mathbf{S}^{\star}$ be partitioned in 16 parts as

$$\mathbf{S}^{\star} = \begin{bmatrix} \mathbf{S}^{\star}_{11} & \mathbf{S}^{\star}_{12} & \mathbf{S}^{\star}_{13} & \mathbf{S}^{\star}_{14} \\ (\mathbf{S}^{\star}_{12})^{\top} & \mathbf{S}^{\star}_{22} & \mathbf{S}^{\star}_{23} & \mathbf{S}^{\star}_{24} \\ (\mathbf{S}^{\star}_{13})^{\top} & (\mathbf{S}^{\star}_{23})^{\top} & \mathbf{S}^{\star}_{33} & \mathbf{S}^{\star}_{34} \\ (\mathbf{S}^{\star}_{14})^{\top} & (\mathbf{S}^{\star}_{24})^{\top} & (\mathbf{S}^{\star}_{34})^{\top} & \mathbf{S}^{\star}_{44} \end{bmatrix},$$

then it can be demonstrated that the matrix $\mathbf{S}^{\star}$ reduces to

$$\mathbf{S}^{\star} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}^{\star}_{22} & \mathbf{0} & \mathbf{S}^{\star}_{24} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & (\mathbf{S}^{\star}_{24})^{\top} & \mathbf{0} & \mathbf{S}^{\star}_{44} \end{bmatrix}. \tag{5.16}$$

To prove (5.16) calculate the closed loop $\mathbf{A}^{\star} - \mathbf{B}^{\star}\bar{\mathbf{R}}^{-1}(\mathbf{B}^{\star})^{\top}\mathbf{S}^{\star}$ and note that the ultimate Kamlan decomposition is kept if and only if $\mathbf{S}^{\star}_{11}$, $\mathbf{S}^{\star}_{12}$, $\mathbf{S}^{\star}_{13}$, $\mathbf{S}^{\star}_{23} = 0$. As consequence, the solution of the ARE imposes $\mathbf{S}^{\star}_{14}$, $\mathbf{S}^{\star}_{33}$, $\mathbf{S}^{\star}_{34} = 0$.

The substitution of (5.16) into (5.14), leads to the computation of the optimal control law $\mathbf{u}^\star$ as

$$\mathbf{u}^\star = -\bar{\mathbf{R}}^{-1} \begin{bmatrix} \mathbf{0} & (\mathbf{B}_2^\star)^\top \mathbf{S}_{22}^\star & \mathbf{0} & (\mathbf{B}_2^\star)^\top \mathbf{S}_{24}^\star \end{bmatrix} \begin{bmatrix} \mathbf{z}_{\mathrm{R,NO}} \\ \mathbf{z}_{\mathrm{R,O}} \\ \mathbf{z}_{\mathrm{NR,NO}} \\ \mathbf{z}_{\mathrm{NR,O}} \end{bmatrix}. \tag{5.17}$$

Finally, the closed loop dynamics is given by

$$\begin{bmatrix} \dot{\mathbf{z}}_{\mathrm{R,NO}} \\ \dot{\mathbf{z}}_{\mathrm{R,O}} \\ \dot{\mathbf{z}}_{\mathrm{NR,NO}} \\ \dot{\mathbf{z}}_{\mathrm{NR,O}} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11}^\star + \alpha \mathbf{I} & \mathbf{A}_{12}^\star - \mathbf{B}_1^\star \bar{\mathbf{R}}^{-1} (\mathbf{B}_2^\star)^\top \mathbf{S}_{22}^\star & \mathbf{A}_{13}^\star & \mathbf{A}_{14}^\star - \mathbf{B}_1^\star \bar{\mathbf{R}}^{-1} (\mathbf{B}_2^\star)^\top \mathbf{S}_{24}^\star \\ \mathbf{0} & \mathbf{A}_{22}^\star + \alpha \mathbf{I} - \mathbf{B}_2^\star \bar{\mathbf{R}}^{-1} (\mathbf{B}_2^\star)^\top \mathbf{S}_{22}^\star & \mathbf{0} & \mathbf{A}_{24}^\star - \mathbf{B}_2^\star \bar{\mathbf{R}}^{-1} (\mathbf{B}_2^\star)^\top \mathbf{S}_{24}^\star \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_{33}^\star + \alpha \mathbf{I} & \mathbf{A}_{34}^\star \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{44}^\star + \alpha \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{z}_{\mathrm{R,NO}} \\ \mathbf{z}_{\mathrm{R,O}} \\ \mathbf{z}_{\mathrm{NR,NO}} \\ \mathbf{z}_{\mathrm{NR,O}} \end{bmatrix}$$

which must be stabilisable and detectable to let the ARE having a stabilising solution. Since $\mathbf{A}_{22}^\star + \alpha \mathbf{I} - \mathbf{B}_2^\star \bar{\mathbf{R}}^{-1} (\mathbf{B}_2^\star)^\top \mathbf{S}_{22}^\star$ can be made Hurwitz because $(\mathbf{A}_{22}^\star + \alpha \mathbf{I}, \mathbf{B}_2^\star)$ is fully reachable and observable, we constrain $\alpha > 0$ to be smaller than the smallest absolute value of $\mathbf{A}_{11}^\star$, $\mathbf{A}_{33}^\star$, and $\mathbf{A}_{44}^\star$ eigenvalues' real part.

Example 5.2 (Non detectable model)**.** *Let us consider a point with mass $m > 0$ which moves on a straight path. Define $\mathbf{x} = col(p, v)$ as the state in which $p, v \in \mathbb{R}$ are the position and the speed respectively. Let $u \in \mathbb{R}$ be the force acting on the mass and $y \in \mathbb{R}$ be the speed measurement. Then, the dynamics is modelled through a linear system as*

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}u \quad \mathbf{x}(0) = \mathbf{x}_0 \\ e &= \mathbf{C}\mathbf{x} \end{aligned}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 \\ m^{-1} \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

It is worth noting that the couple $(\mathbf{A}, \mathbf{B})$ is fully reachable whereas the couple $(\mathbf{A}, \mathbf{C})$ is not fully observable. Moreover, the unobservable part of the system is not BIBS stable thus leading to a non detectable system. Let $\alpha = 0$, $r > 0$, $q \geq 0$, and

$$\mathbf{S} := \begin{bmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{bmatrix},$$

then the ARE associated to this problem becomes

$$\frac{1}{r\,m^2}\begin{bmatrix} s_{12}^2 & s_{12}s_{22} \\ s_{12}s_{22} & s_{22}^2 \end{bmatrix} - \begin{bmatrix} 0 & s_{11} \\ s_{11} & 2s_{12}+q \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

which leads to a positive semi-definite $\mathbf{S}$

$$\mathbf{S} = \begin{bmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & m\sqrt{q\,r} \end{bmatrix}.$$

As consequence, the static state feedback control is given by $u = \mathbf{K}\mathbf{x}$ *with*

$$\mathbf{K} = -\frac{1}{r}[0\ m^{-1}]\begin{bmatrix} 0 & 0 \\ 0 & m\sqrt{q\,r} \end{bmatrix} = -\begin{bmatrix} 0 & \sqrt{q/r} \end{bmatrix}.$$

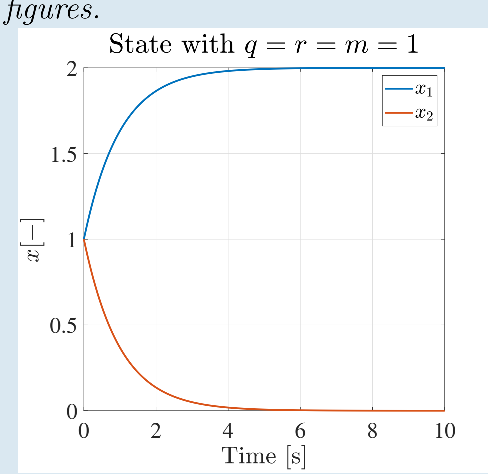
Finally, the closed loop system has a dynamics described by the matrix

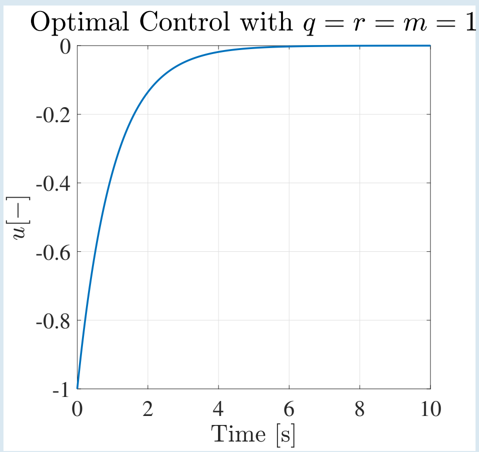
$$\mathbf{A}+\mathbf{B}\mathbf{K} = \begin{bmatrix} 0 & 1 \\ 0 & -\sqrt{q/r}/m \end{bmatrix}$$

which is not BIBS stable. The time evolution of the state is obtained by means of the Lagrange formula (3.13) *that, applied to* $\mathbf{A}+\mathbf{B}\mathbf{K}$, *leads to*

$$\begin{aligned} x_1(t) &= \frac{m}{\sqrt{q/r}}\left(1 - e^{-t\sqrt{q/r}/m}\right)x_2(0) + x_1(0) \\ x_2(t) &= e^{-t\sqrt{q/r}/m}x_2(0) \end{aligned}$$

The time evolution of the state and the control are shown in the following figures.





Example 5.3 (Non stabilisable model). *With reference to the plant of Example 5.2, we now assume that* $\mathbf{C} = [1\ 0]$ *and* $\mathbf{B} = \texttt{col}(1, 0)$. *Due to these variations, the plant becomes fully observable but not fully reachable. Moreover, the non reachable subsystem is characterised by a null eigenvalue and so the system is not stabilisable. Assuming* $\alpha = 0$, *the ARE associated to the optimal control problem becomes*

$$\begin{bmatrix} s_{11}^2/r - q & s_{11}(s_{12}/r - 1) \\ s_{11}(s_{12}/r - 1) & s_{12}(s_{12}/r - 2) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

that has no solutions for $q > 0$. *Therefore, let us assume* $q = 0$ *and calculate the solutions as* $s_{11} = 0$, $s_{22} \in \mathbb{R}$, *and* $s_{12} \in \{0, 2r\}$.

We observe that

$$\texttt{det}\,(\mathbf{S} - \lambda I) = \texttt{det}\left(\begin{bmatrix} -\lambda & s_{12} \\ s_{12} & s_{22} - \lambda \end{bmatrix}\right) = \lambda^2 - \lambda s_{22} - s_{12}^2 = 0 \implies$$
$$\lambda_{1,2} = \frac{s_{22} \pm \sqrt{s_{22}^2 + 4 s_{12}^2}}{2}.$$

The roots $\lambda_{1,2}$ *are opposite in sign and different from zero for any* $s_{22} \in \mathbb{R}$. *Hence, the ARE is solved by an* $\mathbf{S}$ *which is positive definite. The optimal control is obtained as* $u = \mathbf{Kx}$ *with*

$$\mathbf{K} = -\frac{1}{r}\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & s_{12} \\ s_{12} & s_{22} \end{bmatrix} = -\frac{1}{r}\begin{bmatrix} 0 & s_{12} \end{bmatrix}.$$

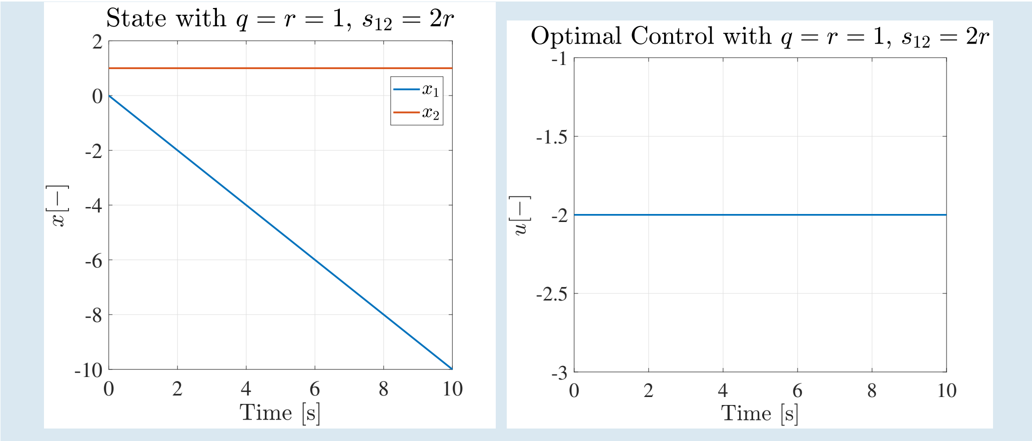
Finally, the closed loop transition matrix becomes

$$\mathbf{A} + \mathbf{BK} = \begin{bmatrix} 0 & 1 - s_{12}/r \\ 0 & 0 \end{bmatrix}$$

which indeed is not BIBS stable. In conclusion, the time evolution of the state is described as

$$\begin{aligned} x_1(t) &= (1 - s_{12}/r) x_2(0)\, t + x_1(0) \\ x_2(t) &= x_2(0) \end{aligned}$$

that leads to the control $u = -x_2(0) s_{12}/r$.



Example 5.4 (Observable and reachable model)**.** *With reference to the plant of Example 5.2 we now assume that* $\mathbf{C} = [1\ 0]$*. Thanks to this new* $\mathbf{C}$ *the couple* $(\mathbf{A}, \mathbf{C})$ *becomes fully observable. Relying on the fully reachability of the couple* $(\mathbf{A}, \mathbf{B})$ *we know that there exists stabilising optimal control* $u = -r^{-1}\mathbf{B}^\top \mathbf{S}\mathbf{x}$ *with* $\mathbf{S}$ *solving the following ARE*

$$\begin{bmatrix} s_{12}^2/(m^2 r) - q & s_{12}s_{22}/(m^2 r) - s_{11} \\ s_{12}s_{22}/(m^2 r) - s_{11} & s_{22}^2/(m^2 r) - 2s_{12} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

which leads to

$$s_{11} = \sqrt{2m}\, q^{3/4}\, r^{1/4}, \quad s_{12} = m\sqrt{rq}, \quad s_{22} = \sqrt{2}\, m^{3/2}\, r^{3/4}\, q^{1/4}.$$

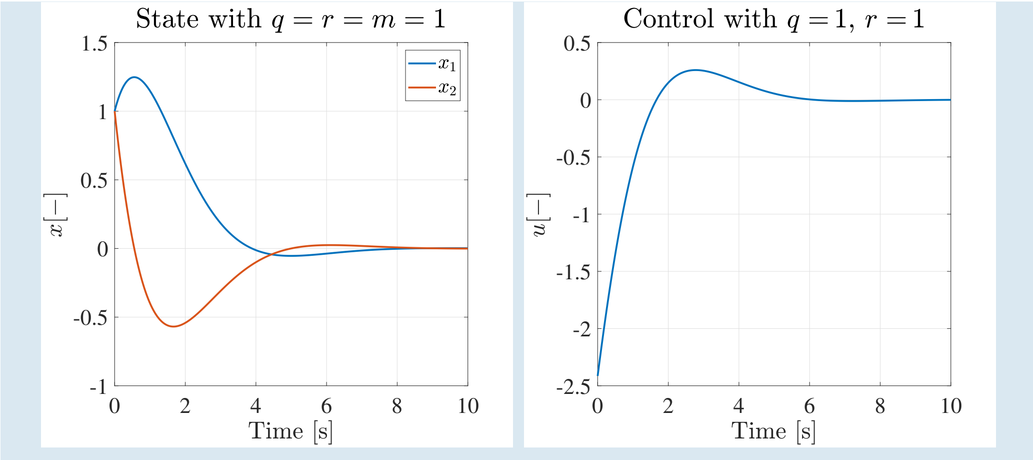
Then, the static state feedback control is given by $u = \mathbf{K}\mathbf{x}$ *with*

$$\mathbf{K} = -\frac{1}{r}[0\ m^{-1}] \begin{bmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{bmatrix} = -\frac{1}{rm} \begin{bmatrix} s_{12} & s_{22} \end{bmatrix}.$$

Finally, the closed loop system has a dynamics described by the matrix

$$\mathbf{A} + \mathbf{B}\mathbf{K} = \begin{bmatrix} 0 & 1 \\ -\dfrac{s_{12}}{rm} & -\dfrac{s_{22}}{rm} \end{bmatrix}$$

which is BIBS stable.



Example 5.5 (Cart-pole set point regulator design).
The constants $\mathbf{K}_S$ *and* k_I*, defined in Example 4.12, are designed by means of the stationary optimal control technique. Let the system be*

$$\begin{bmatrix} \dot{\mathbf{z}}_{R,O} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{A}}_{22} & \mathbf{0} \\ \bar{\mathbf{C}}_e & 0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_{R,O} \\ \eta \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{B}}_2 \\ 0 \end{bmatrix} u$$
$$\mathbf{e}_r = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{z}_{R,O} \\ \eta \end{bmatrix},$$

then define the cost function

$$J = \int_0^\infty \mathbf{e}_r^\top \mathbf{Q} \mathbf{e}_r + u^2 R \, dt$$

in which $\mathbf{Q} = \mathbf{Q}^\top \succ 0$ *and* $R > 0$*. Assume*

$$\mathbf{S} := \begin{bmatrix} \mathbf{S}_{11} & \mathbf{S}_{12} \\ \mathbf{S}_{12}^\top & s_{22} \end{bmatrix},$$

in which $\mathbf{S}_{12} \in \mathbb{R}^{1\times 3}$ *and* $\mathbf{S}_{11} \in \mathbb{R}^{3\times 3}$*, then the stationary optimal control law is obtained as*

$$u^\star = -R^{-1} \begin{bmatrix} \bar{\mathbf{B}}_2^\top & 0 \end{bmatrix} \mathbf{S} \begin{bmatrix} \mathbf{z}_{R,O} \\ \eta \end{bmatrix} = -R^{-1}\bar{\mathbf{B}}_2^\top \mathbf{S}_{11}\mathbf{z}_{R,O} - R^{-1}\bar{\mathbf{B}}_2^\top \mathbf{S}_{12}^\top \eta.$$

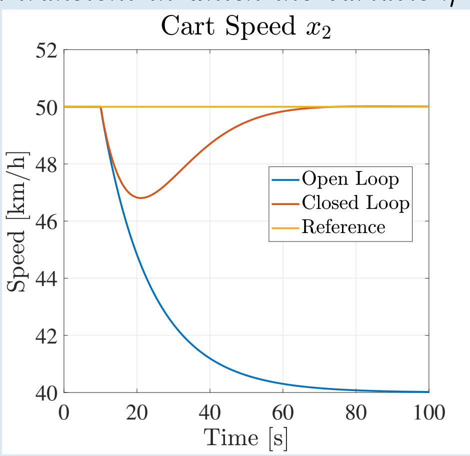
The matrices $\mathbf{S}_{12}$ *and* $\mathbf{S}_{11}$ *can be found through numerical solvers. As an example, we exploit the following Matlab listing.*

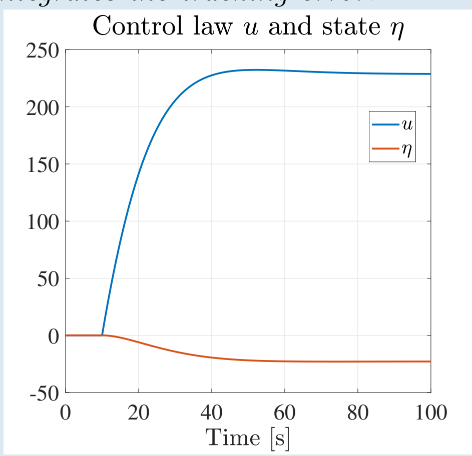
```
% Plant
Ac = [barA22 zeros(3,1); barCe 0];
Bc = [barB2; 0];
Cc = eye(4);

% Costs
Q = % any 4x4 matrix positive definite
R = % any positive scalar

% Solve the ARE
[K,S,e] = lqr(Ac,Bc,Q,R,zeros(4,1)); % WARNING! The control
    law is u = -K*x. S is the positive definite solution of
    the ARE.
```

The following results have been obtained by imposing $\mathbf{Q} = 100\,\mathbf{I}$ *and* $R > 1$. *The cart starts the simulation at the equilibrium speed of* 50 *km/h and at time* $t = 10$s *a constant wind of* -10 *km/h appears and brakes the system. The target speed, thanks to the control action, is recovered after a transient in which the variable* η *integrates the tracking error.*





5.1.1 Gain Selection

Matrices $\mathbf{Q}$ and $\mathbf{R}$ represent tunable gains at disposal of the designer. A rule of thumb for the design of these matrices is driven by the desire of limiting the error $\boldsymbol{\epsilon}(t)$ and the control $\mathbf{u}(t)$. Let

$$\boldsymbol{\epsilon} := \mathtt{col}(\epsilon_1, \ldots, \epsilon_m), \quad \mathbf{u} := \mathtt{col}(u_1, \ldots, u_p)$$

and define $\epsilon_{i_{\max}}, u_{j_{\max}} > 0$ for $i = 1, \ldots, m$ and $j = 1, \ldots, p$. Let

- $|\epsilon_i(t)| \leq \epsilon_{i_{\max}} \in \mathbb{R}_{>0}$ for $i = 1, \ldots, m$;
- $|u_i(t)| \leq u_{i_{\max}} \in \mathbb{R}_{>0}$ for $i = 1, \ldots, p$;

be the constraints the controlled plant should verify. Then, the matrices $\mathbf{Q}$ and $\mathbf{R}$ are defined as

$$\mathbf{Q}^{-1} = q \begin{bmatrix} \epsilon_{1_{\max}}^2 & 0 & \dots & 0 \\ 0 & \epsilon_{2_{\max}}^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \epsilon_{q_{\max}}^2 \end{bmatrix}$$

$$\mathbf{R}^{-1} = p \begin{bmatrix} u_{1_{\max}}^2 & 0 & \dots & 0 \\ 0 & u_{2_{\max}}^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & u_{p_{\max}}^2 \end{bmatrix}.$$

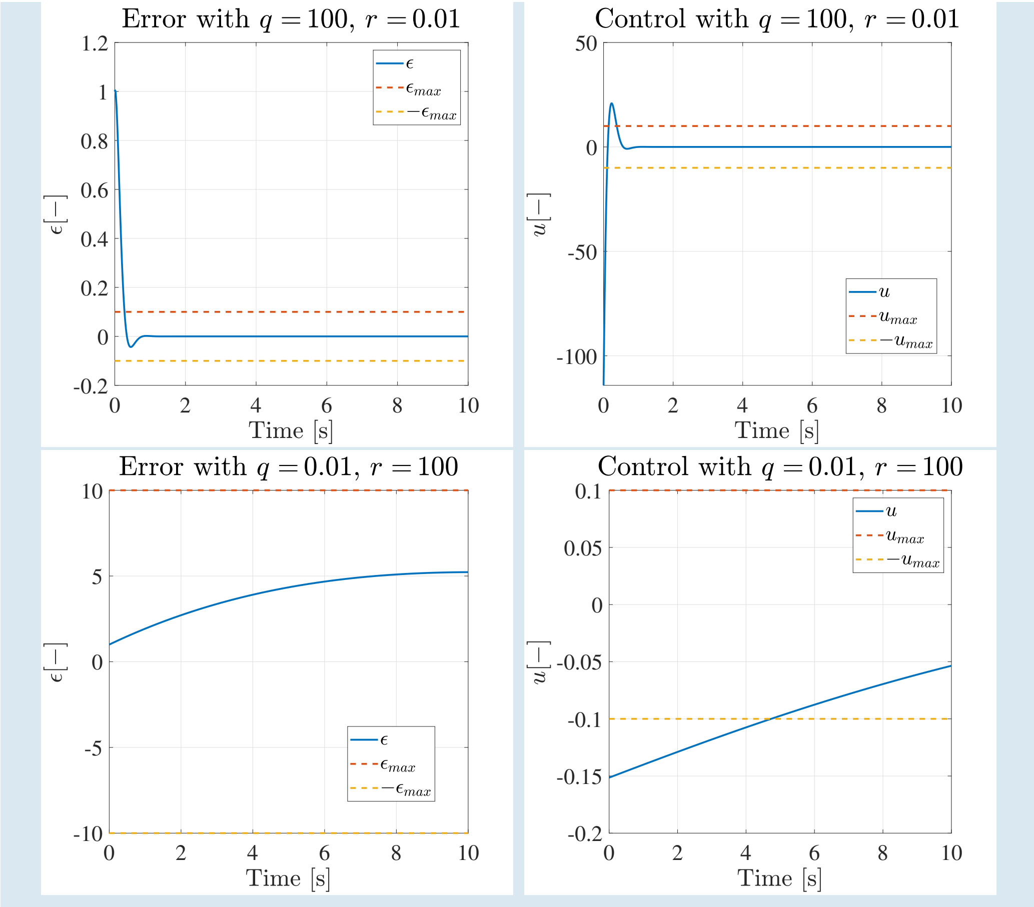
It is worth noting that the above defined $\mathbf{Q}$ and $\mathbf{R}$ do not guarantee that $\boldsymbol{\epsilon}$ and $\mathbf{u}$ remain bounded within the prescribed constraints. Indeed, this design criterion just constitutes a first attempt of design and iterations based on the "trial and error" approach could be necessary. The terms m and p are normalization weights which make the matrices $\mathbf{Q}$ and $\mathbf{R}$ independent on the dimensions of the vectors $\boldsymbol{\epsilon}$ and $\mathbf{u}$.

Example 5.6 (Tuning of $\mathbf{Q}$ and $\mathbf{R}$). *With reference to the model of Example 5.4 we imposed $m = 1$ and tried different settings, one for each combination of ϵ_{max}, $u_{max} \in \{0.1, 10\}$. First of all, we note that since*

$$\frac{s_{12}}{mr} = \sqrt{q/r}, \qquad \frac{s_{22}}{mr} = \sqrt{2m}\,(q/r)^{1/4},$$

the closed loop system behaves the same for any q/r constant. For this reason, the plots for $\epsilon_{max} = u_{max}$ are not shown. It is clear that for these cases $|\epsilon| < \epsilon_{max}$ and $|u| < u_{max}$ only if ϵ_{max} and u_{max} are sufficiently large to bound the time evolutions shown at the end of Example 5.4.

Moreover, with respect to the time evolutions of ϵ and u of Example 5.4, we observe that increasing q and decreasing r leads to a faster response that relies on larger values of the control, which becomes able to rapidly steer the error to zero. At the opposite, decreasing q and increasing r induces the controller to tolerate larger errors with a "lazier" control action.



5.2 Duality

Section 5.1 presents a constructive procedure for the design of the state feedback control gain, namely $\mathbf{K}$. It is possible to apply the same approach for the design of the observer matrix $\mathbf{K}_O$. To understand how, we need to investigate the duality properties of linear systems.

Let

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \quad \mathbf{x}(t_0) = \mathbf{x}_0 \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \end{aligned} \tag{5.18}$$

be an LTI system with $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{u} \in \mathbb{R}^p$ and $\mathbf{y} \in \mathbb{R}^q$, then the *dual system* associated to the *primary system* (5.18) is defined as

$$\begin{aligned} \dot{\boldsymbol{\chi}} &= \mathbf{A}^\top \boldsymbol{\chi} + \mathbf{C}^\top \boldsymbol{\nu} \quad \boldsymbol{\chi}(t_f) = \boldsymbol{\chi}_f \\ \boldsymbol{\mu} &= \mathbf{B}^\top \boldsymbol{\chi} + \mathbf{D}^\top \boldsymbol{\nu} \end{aligned} \tag{5.19}$$

with $\boldsymbol{\chi} \in \mathbb{R}^n$, $\boldsymbol{\nu} \in \mathbb{R}^q$ and $\boldsymbol{\mu} \in \mathbb{R}^p$.

The primary and the dual systems are two equivalent representation of the same mathematical model and, thus, any property possessed by the primary system can be translated in an equivalent property of its dual. Let us start with the properties of reachability and observability.

Lemma 5.1. *Let the primary system* (5.18) *be fully reachable, then the dual system is fully observable and viceversa.*

Proof. The reachability subset of the primary system (5.18) has as basis the image of the reachability matrix $\mathbf{R}_p$

$$\mathbf{R}_p = \begin{bmatrix} \mathbf{B} & \mathbf{A}\mathbf{B} & \mathbf{A}^2\mathbf{B} & \dots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix},$$

whereas the unobservability subspace of the dual system $\mathcal{E}_d$ has as basis the kernel of the observability matrix $\mathbf{O}_d$

$$\mathbf{O}_d = \begin{bmatrix} \mathbf{B}^\top \\ \mathbf{B}^\top\mathbf{A}^\top \\ \mathbf{B}^\top\left(\mathbf{A}^\top\right)^2 \\ \dots \\ \mathbf{B}^\top\left(\mathbf{A}^\top\right)^{n-1} \end{bmatrix}.$$

If the primary system is fully reachable, then $\mathrm{im}(\mathbf{R}_p) = \mathbb{R}^n$, whereas if the dual system is fully observable, then $\ker(\mathbf{O}_d) = \{\mathbf{0}\}$ or equivalently $(\ker(\mathbf{O}_d))^\perp = \mathbb{R}^n$. In particular,

$$(\ker(\mathbf{O}_d))^\perp = \mathrm{im}\left(\mathbf{O}_d^\top\right) = \mathrm{im}\left(\mathbf{R}_p\right)$$

which demonstrates that if the primary system is fully reachable, then the dual system is fully observable and vice-versa. □

Lemma 5.2. *Let the primary system* (5.18) *be fully observable, then the dual system is fully reachable and vice-versa.*

Proof. The reachability subset of the dual system (5.18) has as basis the image of the reachability matrix $\mathbf{R}_d$

$$\mathbf{R}_d = \begin{bmatrix} \mathbf{C}^\top & \mathbf{A}^\top\mathbf{C}^\top & (\mathbf{A}^\top)^2\mathbf{C}^\top & \dots & (\mathbf{A}^\top)^{n-1}\mathbf{C}^\top \end{bmatrix},$$

whereas the unobservability subspace of the primary system $\mathcal{E}_p$ has as basis the kernel of the observability matrix $\mathbf{O}_p$

$$\mathbf{O}_p = \begin{bmatrix} \mathbf{C} \\ \mathbf{C}\mathbf{A} \\ \mathbf{C}\mathbf{A}^2 \\ \dots \\ \mathbf{C}\mathbf{A}^{n-1} \end{bmatrix}.$$

If the dual system is fully reachable, then $\mathrm{im}(\mathbf{R}_d) = \mathbb{R}^n$, whereas if the primary system is fully observable, then $\ker(\mathbf{O}_p) = \{\mathbf{0}\}$ or equivalently $(\ker(\mathbf{O}_p))^\perp = \mathbb{R}^n$. In particular

$$(\ker(\mathbf{O}_p))^\perp = \mathrm{im}\left(\mathbf{O}_p^\top\right) = \mathrm{im}\left(\mathbf{R}_d\right)$$

which demonstrates that if the dual system is fully reachable, then the primary system is fully observable and vice-versa. □

Lemma 5.3. *Let the primary model* (5.18) *be BIBS stable, then also the dual model* (5.19) *is BIBS stable and vice-versa.*

Proof. If the primary model is BIBS stable, the real part of the eigenvalues of the matrix $\mathbf{A}$ is strictly negative. Let $\sigma_{\mathbf{A}}$ be the set of the eigenvalues of $\mathbf{A}$ and let $\mathrm{Re}(\sigma_{\mathbf{A}})$ denote the real part of each eigenvalue of $\mathbf{A}$. Since $\mathbf{A}$ and its transpose have the same eigenvalues, we have

$$\mathrm{Re}(\sigma_{\mathbf{A}}) < 0 \iff \mathrm{Re}(\sigma_{\mathbf{A}^\top}) < 0.$$

□

Let us now try to understand if also the control and the observation problems are dual and then, if the existence of the solution of one problem implies the existence of the solution of its dual.

Let the primary system (5.18) be fully observable, then in agreement with the results of Section 4.4 there exists $\mathbf{K}_{\mathrm{O}_p}$ such that the state estimation error $\mathbf{e}_{\mathbf{x}} := \hat{\mathbf{x}} - \mathbf{x}$ is governed by the dynamics

$$\dot{\mathbf{e}}_{\mathbf{x}} = (\mathbf{A} - \mathbf{K}_{\mathrm{O}_p}\mathbf{C})\mathbf{e}_{\mathbf{x}} \tag{5.20}$$

whose dual is

$$\dot{\boldsymbol{\epsilon}} = (\mathbf{A}^\top - \mathbf{C}^\top\mathbf{K}_{\mathrm{O}_p}^\top)\boldsymbol{\epsilon}. \tag{5.21}$$

Now, since (5.19) is fully reachable, there exists $\mathbf{K}_{\mathrm{S}_d}$ such that $\boldsymbol{\nu} := \mathbf{K}_{\mathrm{S}_d}\boldsymbol{\chi}$ leads to

$$\dot{\boldsymbol{\chi}} = \left(\mathbf{A}^\top + \mathbf{C}^\top\mathbf{K}_{\mathrm{S}_d}\right)\boldsymbol{\chi}. \tag{5.22}$$

Compare Eq.s (5.21) and (5.22) to note that the matrices $\mathbf{A}^\top - \mathbf{C}^\top\mathbf{K}_{\mathrm{O}_p}^\top$ and $\mathbf{A}^\top + \mathbf{C}^\top\mathbf{K}_{\mathrm{S}_d}$ are equal if

$$\mathbf{K}_{\mathrm{O}_p} = -\mathbf{K}_{\mathrm{S}_d}^\top.$$

Then, the solution of the observation problem associated to the primary system is implied by the solution of the control problem associated to the dual system.

5.3 Kalman Filter

Section 5.2 shows that control and observation properties (and problems) associated to linear systems are dual. Therefore, the solution of the observation problem of the primary system is implied by the solution of the control problem of the dual system. Besides, Section 5.1 shows how to design a state feedback gain solving the optimal control problem. This section formulates and solves the optimal observation problem for the primary system through the statement and the solution of the optimal control problem for the dual system. Let

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}_1\mathbf{u} + \mathbf{B}_2\mathbf{w} \quad \mathbf{x}(t_0) = \mathbf{x}_0 \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}_1\mathbf{u} + \mathbf{D}_2\mathbf{w} \end{aligned} \tag{5.23}$$

be a detectable LTI system for which we want to design the observer. In agreement with Section 4.4 we define an identity observer as

$$\begin{aligned} \dot{\hat{\mathbf{x}}} &= \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}_1\mathbf{u} \quad \hat{\mathbf{x}}(t_0) = \hat{\mathbf{x}}_0 \\ \hat{\mathbf{y}} &= \mathbf{C}\hat{\mathbf{x}} + \mathbf{D}_1\mathbf{u}. \end{aligned} \tag{5.24}$$

Let $\mathbf{e}_x := \hat{\mathbf{x}} - \mathbf{x}$ and $\tilde{\mathbf{y}} := \hat{\mathbf{y}} - \mathbf{y}$, then the dynamics of the estimation error is

$$\begin{aligned} \dot{\mathbf{e}}_x &= \mathbf{A}\mathbf{e}_x - \mathbf{B}_2\mathbf{w} \quad \mathbf{e}_x(t_0) = \mathbf{e}_{x0} \\ \tilde{\mathbf{y}} &= \mathbf{C}\mathbf{e}_x - \mathbf{D}_2\mathbf{w}. \end{aligned} \tag{5.25}$$

Let

$$\begin{aligned} \dot{\boldsymbol{\chi}} &= \mathbf{A}^\top\boldsymbol{\chi} + \mathbf{C}^\top\boldsymbol{\nu} \quad \boldsymbol{\chi}(t_f) = \boldsymbol{\chi}_f \\ \boldsymbol{\mu} &= \mathbf{B}_2^\top\boldsymbol{\chi} + \mathbf{D}_2^\top\boldsymbol{\nu} \end{aligned} \tag{5.26}$$

be the dual model associated to (5.25). We now define and solve the robust optimal control problem for (5.26) through the same steps depicted in Section 5.1. Let $\alpha > 0$, alter (5.26) as

$$\begin{aligned} \dot{\boldsymbol{\chi}} &= (\mathbf{A}^\top + \alpha\mathbf{I})\boldsymbol{\chi} + \mathbf{C}^\top\boldsymbol{\nu} \quad \boldsymbol{\chi}(t_f) = \boldsymbol{\chi}_f \\ \boldsymbol{\mu} &= \mathbf{B}_2^\top\boldsymbol{\chi} + \mathbf{D}_2^\top\boldsymbol{\nu} \end{aligned} \tag{5.27a}$$

and define the cost function

$$J_d = \int_{t_0}^{\infty} \boldsymbol{\mu}^\top\mathbf{Q}_d\boldsymbol{\mu} + \boldsymbol{\nu}^\top\mathbf{R}_d\boldsymbol{\nu}\, dt \tag{5.27b}$$

where $\mathbf{Q}_d \succeq 0$ and $\mathbf{R}_d \succ 0$. Apply the steps (5.5)-(5.15) to the constrained optimisation problem (5.27) to obtain $\boldsymbol{\nu}^\star = \mathbf{K}_{\mathrm{S}_d}\boldsymbol{\chi}$ where

$$\begin{aligned} \mathbf{K}_{\mathrm{S}_d} &= -\bar{\mathbf{R}}_d^{-1}\left(\mathbf{D}_2\mathbf{Q}_d\mathbf{B}_2^\top + \mathbf{C}\mathbf{S}\right) \\ \mathbf{0} &= \mathbf{S}\mathbf{C}^\top\bar{\mathbf{R}}_d^{-1}\mathbf{C}\mathbf{S} - \mathbf{S}(\mathbf{A}^\top + \alpha\mathbf{I} - \mathbf{C}^\top\bar{\mathbf{R}}_d^{-1}\mathbf{D}_2\mathbf{Q}_d\mathbf{B}_2^\top) \\ &\quad - (\mathbf{A}^\top + \alpha\mathbf{I} - \mathbf{C}^\top\bar{\mathbf{R}}_d^{-1}\mathbf{D}_2\mathbf{Q}_d\mathbf{B}_2^\top)^\top\mathbf{S} - \mathbf{B}_2\mathbf{Q}_d[\mathbf{I} - \mathbf{D}_2^\top\bar{\mathbf{R}}_d^{-1}\mathbf{D}_2\mathbf{Q}_d]\mathbf{B}_2^\top \end{aligned} \tag{5.28}$$

in which $\bar{\mathbf{R}}_d := \mathbf{D}_2\mathbf{Q}_d\mathbf{D}_2^\top + \mathbf{R}_d$. Then, as described in Section 5.2 the observer for (5.23) is given by

$$\begin{aligned}\dot{\hat{\mathbf{x}}} &= \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}_1\mathbf{u} - \mathbf{K}_{\mathrm{S}_d}^\top(\mathbf{y} - \hat{\mathbf{y}}) \quad \hat{\mathbf{x}}(t_0) = \hat{\mathbf{x}}_0 \\ \hat{\mathbf{y}} &= \mathbf{C}\hat{\mathbf{x}} + \mathbf{D}_1\mathbf{u}.\end{aligned} \tag{5.29}$$

where $\mathbf{K}_{\mathrm{S}_d}$ is defined in (5.28). The solution (5.28) can be specialised by noting that in most of practical cases $\mathbf{B}_2 := \begin{bmatrix} \mathbf{B}_{21} & \mathbf{0} & \mathbf{0} \end{bmatrix}$ and $\mathbf{D}_2 := \begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} \end{bmatrix}$ (remember from Section 1.2 that $\mathbf{w} := \mathtt{col}(\mathbf{d}, \mathbf{v}, \mathbf{r})$). Let $\mathbf{Q}_d$ be divided in nine parts as

$$\mathbf{Q}_d = \begin{bmatrix} \mathbf{Q}_{d_{11}} & \mathbf{Q}_{d_{12}} & \mathbf{Q}_{d_{13}} \\ \mathbf{Q}_{d_{12}}^\top & \mathbf{Q}_{d_{22}} & \mathbf{Q}_{d_{23}} \\ \mathbf{Q}_{d_{13}}^\top & \mathbf{Q}_{d_{23}}^\top & \mathbf{Q}_{d_{33}} \end{bmatrix},$$

then we have

$$\begin{aligned}\bar{\mathbf{R}}_d &= \mathbf{Q}_{d_{22}} + \mathbf{R}_d \\ \mathbf{D}_2\mathbf{Q}_d\mathbf{B}_2^\top &= \mathbf{Q}_{d_{12}}^\top\mathbf{B}_{21}^\top \\ \mathbf{B}_2\mathbf{Q}_d\mathbf{B}_2^\top &= \mathbf{B}_{21}\mathbf{Q}_{d_{11}}\mathbf{B}_{21}^\top.\end{aligned}$$

Infobox 5.7 (Kalman Filter). *It has been demonstrated [?] that the dynamic observer constituted by the composition of the equations* (5.28)–(5.29) *represents an optimal observer in stochastic terms, i.e. an observer which minimizes the mean of the squared estimation error*

$$E[(\mathbf{x}(t) - \hat{\mathbf{x}}(t))^\top\mathbf{M}(\mathbf{x}(t) - \hat{\mathbf{x}}(t))],$$

for some $\mathbf{M} = \mathbf{M}^\top \succ 0$*, if the following conditions are verified:*

- $\mathbf{d}(t)$ *and* $\mathbf{v}(t)$ *are white stochastic processes with covariance given by*
$$E[\mathbf{d}(t)\mathbf{d}^\top(\tau)] = \mathbf{Q}_{d_{11}}\delta(t-\tau) \qquad \forall\, t, \tau \geq t_0$$
$$E[\mathbf{v}(t)\mathbf{v}^\top(\tau)] = (\mathbf{Q}_{d_{22}} + \mathbf{R}_d)\delta(t-\tau) \quad \forall\, t, \tau \geq t_0$$
- *the processes* $\mathbf{d}(t)$ *and* $\mathbf{v}(t)$ *are correlated, i.e.*
$$E[\mathbf{d}(t)\mathbf{v}^\top(\tau)] = \mathbf{Q}_{d_{12}}\delta(t-\tau) \quad \forall\, t, \tau \geq t_0$$

- *the state vector* $\mathbf{x}_0$ *is a random variable with mean and covariance given by*

$$E[\mathbf{x}_0] = \bar{\mathbf{x}}_0 \qquad E[(\mathbf{x}_0 - \bar{\mathbf{x}}_0)(\mathbf{x}_0 - \bar{\mathbf{x}}_0)^\top] = \mathbf{P}_0$$

- *the stochastic processes* $\mathbf{d}(t)$ *and* $\mathbf{v}(t)$ *are not correlated with respect to the random variable* $\mathbf{x}_0$*, i.e.*

$$E[\mathbf{x}_0\mathbf{v}^\top(t)] = \mathbf{0} \quad E[\mathbf{x}_0\mathbf{d}^\top(t)] = \mathbf{0} \quad \forall\, t \geq t_0.$$

Roughly, the gain matrix $\mathbf{K}_{\mathrm{S}_d}$ represents a compromise between two opposite approaches. From one hand the observer exploits the model $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ but from the other hand, the observer corrects the prediction made with $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ through a feedback of the measurement $\mathbf{y}$. These two informations, *i.e.* the model $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ and the measurement $\mathbf{y}$ are corrupted by the unknowns $\mathbf{d}$ and $\mathbf{v}$ respectively. So, the gain $\mathbf{K}_{\mathrm{S}_d}$ represents a compromise between the reliability of the model and the reliability of the measurement. If the model is "perfect" or the measurement is totally unreliable the gain $\mathbf{K}_{\mathrm{S}_d}$ can be "any" in the family of matrices stabilizing the couple $(\mathbf{A}^\top + \alpha\mathbf{I}, \mathbf{C}^\top)$. At the opposite, if the model is totally unreliable or the measurement is perfect, then the gain $\mathbf{K}_{\mathrm{S}_d}$ should go to infinite. To confirm this intuition, it is worth noting that the matrix $\mathbf{K}_{\mathrm{S}_d}$ is directly proportional to $\mathbf{S}$ (which is directly proportional to $\mathbf{Q}_{d_{11}}$) and to $(\mathbf{Q}_{d_{22}} + \mathbf{R}_d)^{-1}\mathbf{C}$. So, high (low) magnitudes of $\mathbf{Q}_{d_{11}}(\mathbf{Q}_{d_{22}} + \mathbf{R}_d)^{-1}\mathbf{C}$ mean that the model is less (more) reliable than the measurement. The designer can choose to tune the magnitudes of $\mathbf{Q}_{d_{11}}$ and $\mathbf{Q}_{d_{22}} + \mathbf{R}_d$ to balance the exploitation of the model and the measurement. To conclude, the term $(\mathbf{Q}_{d_{22}} + \mathbf{R}_d)^{-1}\mathbf{C}$ can be interpreted as a signal-to-noise ratio between the measurement $\mathbf{y}$ and the noise $\mathbf{v}$.

Example 5.8 (Optimal observer)**.** *We want to design an observer which minimises the cost*

$$J = \int_0^\infty \boldsymbol{\mu}^\top \begin{bmatrix} q & 0 \\ 0 & 0 \end{bmatrix} \boldsymbol{\mu} + r\nu^2\, dt$$

under the dynamic constraint constituted by the plant

$$\dot{\boldsymbol{\chi}} = \mathbf{A}^\top \boldsymbol{\chi} + \mathbf{C}^\top \nu$$
$$\boldsymbol{\mu} = \mathbf{B}^\top \boldsymbol{\chi} + \mathbf{D}^\top \nu$$

with

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$
$$\mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

Let $\mathbf{S} := [s_{ij}]$ *for* $i,\ j\ = 1, 2$*, then the ARE in* (5.28) *is specialised as*

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = - \begin{bmatrix} s_{12}^2/r - 2s_{12} - q \\ s_{11} - s_{22} + s_{12}s_{22}/r \\ 2s_{12} + s_{22}^2/r \end{bmatrix}$$

whereas the observer feedback matrix $\mathbf{K}_O = -\mathbf{K}_{S_d}^\top$ *is given by the first of* (5.28) *with*

$$\mathbf{K}_O = -\mathbf{K}_{S_d}^\top = \frac{1}{r} \begin{bmatrix} p_{11} \\ p_{12} \end{bmatrix}.$$

Since the system in the primary space is

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$
$$y = \mathbf{C}\mathbf{x},$$

then the observer takes the form

$$\begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{bmatrix} = \begin{bmatrix} -p_{11}/r & 1 \\ -1 - p_{12}/r & 0 \end{bmatrix} \begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{bmatrix} + \frac{1}{r} \begin{bmatrix} p_{11} \\ p_{12} \end{bmatrix} y.$$

It is worth noting that the eigenvalues of

$$\begin{bmatrix} -p_{11}/r & 1 \\ -1 - p_{12}/r & 0 \end{bmatrix},$$

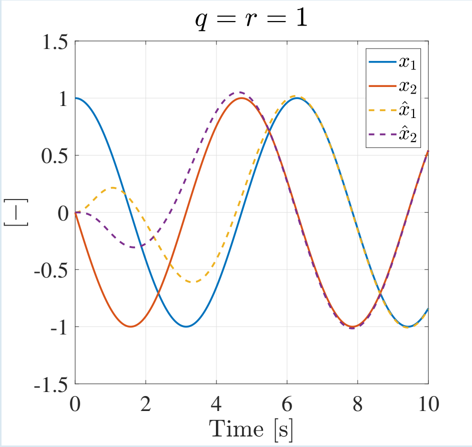
given as solution of

$$\lambda^2 + \lambda p_{11}/r + 1 + p_{12}/r = 0,$$

are

$$\lambda_{1,2} = \frac{-p_{11} \pm \sqrt{p_{11}^2 - 4(r + p_{12})}}{2r}$$

whose real part is negative for $p_{11} > 0$ *and* $p_{12} > -r$. *Finally, the estimation asymptotically tracks the state as depicted in the following figure.*



Example 5.9 (Tuning of $\mathbf{Q}_{d_{11}}$). *This example investigates the tuning of* $\mathbf{Q}_{d_{11}}$ *when this matrix, beside* $\mathbf{R}_d$, *is interpreted in the sense of Kalman as a covariance matrix. To keep clear how* $\mathbf{Q}_{d_{11}}$ *affects the estimation performance, we deal with the scalar system*

$$\begin{aligned}\dot{x} =& ax + bu + w\\ y =& cx + v\end{aligned}$$

to which we associate the covariances

$$\begin{aligned}\mathtt{E}[w(t)w(\tau)] &= q_{d_{11}}\delta(t-\tau)\\ \mathtt{E}[v(t)v(\tau)] &= r_d\delta(t-\tau)\\ \mathtt{E}[w(t)v(\tau)] &= 0.\end{aligned}$$

The observer is given as

$$\begin{aligned}\dot{\hat{x}} &= a\hat{x} + bu + k_O(y - \hat{y})\\ \hat{y} &= c\hat{x}\end{aligned}$$

with the gain

$$\begin{aligned}k_O &= sc/r_d\\ s &= \frac{ar_d}{c^2} + \sqrt{\frac{a^2r_d^2}{c^4} + \frac{r_d q_{d_{11}}}{c^2}}.\end{aligned}$$

obtained thanks to the solution of an optimal observation problem. The feedback gain is directly proportional to $q_{d_{11}}$ *and this is interpreted (in the sense of Kalman) as: "the model* $\dot{x} = ax + bu$ *becomes more and more uncertain as* $q_{d_{11}}$ *increases and thus it is better to rely on the output* y *and so to increase the feedback gain* k_O*."*

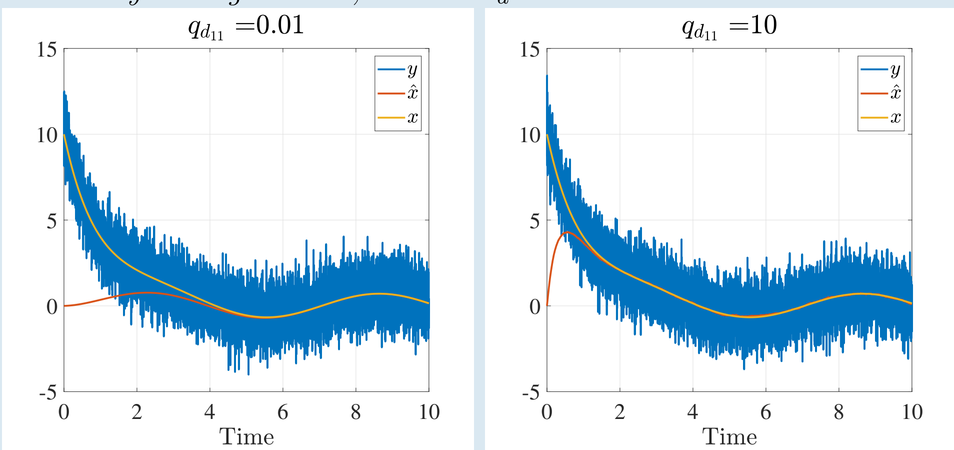
On the other hand, the dynamics of the observation error $e := x - \hat{x}$ *is given by*

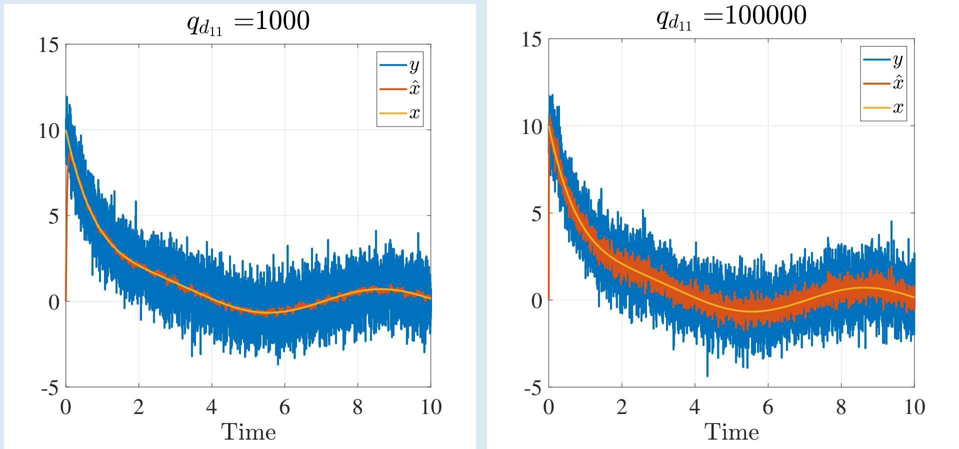
$$\dot{e} = (a - k_O c)e + k_O v$$

whose eigenvalue is

$$\lambda = a - k_O c = -\sqrt{a^2 + \frac{q_{d_{11}}}{r_d} c^2}$$

that is strictly negative and whose norm is directly proportional to $q_{d_{11}}$*. So, a larger* $q_{d_{11}}$ *leads to a lager* k_O *which, from one hand, makes the observer more reactive (thanks to a more negative* λ*) but, on the other hand, magnifies the effects of the measurement noise* v*. For this reason, an intuitive tuning strategy for* $q_{d_{11}}$ *could be guided by the compromise between the observer reactiveness and the noise sensitivity. The observation performance for different tunings is depicted in the following figures, obtained by setting* $a = -1$, $b = c = r_d = 1$ *and* $u = \sin t$*.*





Example 5.10 (Cart-pole observer design).
Let

$$\dot{\mathbf{z}}_{R,O} = \bar{\mathbf{A}}_{22}\mathbf{z}_{R,O} + \bar{\mathbf{B}}_{12}u + \bar{\mathbf{B}}_{22}d$$
$$\mathbf{y} = \bar{\mathbf{C}}_2\mathbf{z}_{R,O} + \mathbf{v}$$

be the dynamics of the reachable and observable part of the model presented in Exercise 1.2. More in detail, in agreement with the results of Exercises 4.17 and 4.19 and thanks to the definition of the plant in Exercise 1.1 we have $\mathbf{z}_{R,O} := col(v_{M_x}, \theta, \omega)$. *Then, we define the dual system*

$$\dot{\boldsymbol{\chi}} = \bar{\mathbf{A}}_{22}\boldsymbol{\chi} + \bar{\mathbf{C}}_2^\top\boldsymbol{\nu}$$
$$\boldsymbol{\mu} = \bar{\mathbf{B}}_{22}^\top\boldsymbol{\chi} + \mathbf{v},$$

and the cost function

$$J_d = \int_0^\infty \boldsymbol{\mu}^\top\mathbf{Q}_d\boldsymbol{\mu} + \boldsymbol{\nu}^\top\mathbf{R}\boldsymbol{\nu}\,dt$$

in which $\mathbf{Q}_d = \mathbf{Q}_d^\top \succ 0$ *and* $\mathbf{R}_d = \mathbf{R}_d^\top \succ 0$. *The optimal control law for the dual is obtained as*

$$\boldsymbol{\nu}^\star = -\mathbf{R}_d^{-1}\bar{\mathbf{C}}_2\mathbf{S}\boldsymbol{\chi}$$

while the observer is

$$\dot{\hat{\mathbf{z}}}_{R,O} = \bar{\mathbf{A}}_{22}\hat{\mathbf{z}}_{R,O} + \bar{\mathbf{B}}_{12}u + \mathbf{S}\bar{\mathbf{C}}_2^\top\mathbf{R}_d^{-1}(\mathbf{y} - \hat{\mathbf{y}})$$
$$\hat{\mathbf{y}} = \bar{\mathbf{C}}_2\hat{\mathbf{z}}_{R,O}.$$

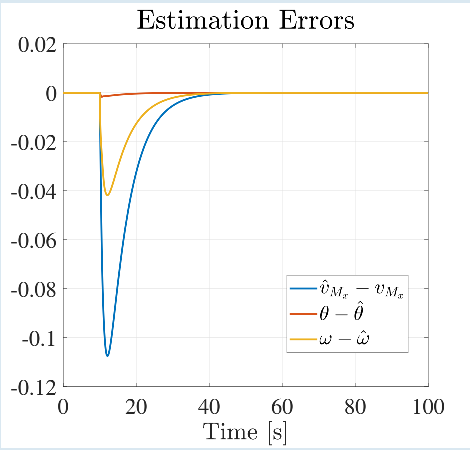
The matrix $\mathbf{S}$ *can be found through numerical solvers. As an example, we exploit the following MATLAB listing.*

```
% Plant

% Costs
Qd11 = % any 3x3 positive definite matrix
Rd = % any 2x2 positive scalar

% Solve the ARE
[K,S,e] = lqr(barA22.',barC2.',Qd11,Rd,zeros(3,2)); % WARNING! The control law is u = -Kc*x. S is the positive definite solution of the ARE.
% Use duality results
Ko = K.';
```

With $\mathbf{Q}_{d11} = \mathbf{I}$ *and* $\mathbf{R}_d = \mathbf{I}$ *the observer demonstrates the performance depicted in the following figure, under the same simulation conditions of Example 5.5.*



It is worth to note that the estimation errors are not vanishing and this is due to the model mismatch. Indeed, since the model is highly non-linear, the linearisation matrices $\bar{\mathbf{A}}_{22}$ *and* $\bar{\mathbf{B}}_2$ *do not accurately approximate the plant.*

5.4 ADAS Design

This section aims to setup the optimal control and observation problems. Then, with respect to the control architectures identified in Section 4.7, the matrices $\mathbf{K}_\mathrm{S}$ and $\mathbf{K}_\mathrm{O}$ will be designed numerically. Lastly, the performance of the control systems are investigated in simulation.

5.5 Exercises

Exercise 5.1. *Let* $\dot{x} = ax + bu$ *be an LTI system, with* x, $u \in \mathbb{R}$, *to which is associated the cost function* $J = \int_0^\infty u^2(t)\, dt$. *Let* $b \neq 0$ *and design the optimal state feedback control* $u = k_S x$.

Exercise 5.2. *Let* $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$ *be an LTI system, with* $\mathbf{x} \in \mathbb{R}^2$ *and* $u \in \mathbb{R}$,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

to which is associated the cost function $J = \int_0^i nftyu^2(t)\, dt$. *Design a state feedback control,* $u = \mathbf{K}_S\mathbf{x}$, *through the optimal control.*

Exercise 5.3. *Let* $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$ *be an LTI system, with* $\mathbf{x} \in \mathbb{R}^2$ *and* $u \in \mathbb{R}$,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

to which is associated the cost function $J = \int_0^\infty u^2(t)\, dt$. *Design an optimal state feedback control* $u = \mathbf{K}_S\mathbf{x}$.

Exercise 5.4. *Let* $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$ *be an LTi system, with* $\mathbf{x} \in \mathbb{R}^2$ *and* $u \in \mathbb{R}$,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

to which is associated the cost function $J = \int_0^\infty \mathbf{x}^\top(t)\mathbf{Q}\mathbf{x}(t) + u^2(t)\, dt$. *Design an optimal state feedback control,* $u = \mathbf{K}_S\mathbf{x}$, *for the cases*

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \mathbf{Q} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Exercise 5.5. *Let*

$$\begin{aligned} \dot{x} &= ax + u \\ e &= x \\ J &= \int_0^\infty qe^2 + u^2\, dt \end{aligned}$$

be a optimisation problem with constraints in which x, u, e, a, $q \in \mathbb{R}$. *Demonstrate that if* $q > 0$ *the optimal control law makes the closed loop BIBS stable for any* $a \in \mathbb{R}$.

Exercise 5.6. *Let* $\dot{x} = ax + b_1 u + b_2 d$ *be an LTI system with* x, u, $d \in \mathbb{R}$. *Design an optimal state feedback control* $u = k_S x$ *assuming as cost function* $J = \int_0^\infty qx^2(t) + u^2(t)\, dt$. *Assume* $a < 0$, *and compare the results of the cases* $q = 0$ *and* $q > 0$. *How does the controlled system react to* d *for* $\lim\limits_{a \to 0^-}$ *?*

Exercise 5.7. *Let* $\dot{x} = ax$ *and* $y = cx$ *to compose an LTI system with* x, $y \in \mathbb{R}$. *Define its dual as* $\dot{\chi} = a\chi + c\nu$. *Design an observer assuming as cost function* $J = \int_0^\infty \chi^2(t) + \nu^2(t)\, dt$.

Exercise 5.8. *Let* $\dot{x} = ax$ *and* $y = cx$ *to represent an LTI system with* x, $y \in \mathbb{R}$. *Define its dual as* $\dot{\chi} = a\chi + c\nu$. *Design an observer assuming as cost function* $J = \int_0^\infty \nu^2(t)\, dt$. *Compare the results for the cases* $a \geq 0$ *and* $a < 0$.

Exercise 5.9. *Let* $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ *and* $\mathbf{y} = \mathbf{C}\mathbf{x}$ *to create an LTI system with* $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^2$ *and*

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Define its dual as $\dot{\boldsymbol{\chi}} = \mathbf{A}^\top \boldsymbol{\chi} + \mathbf{A}^\top \boldsymbol{\nu}$. *Design an observer assuming as* *cost function* $J = \int_0^\infty \boldsymbol{\chi}^\top(t)\mathbf{Q}_d\boldsymbol{\chi}(t) + \boldsymbol{\nu}^\top(t)\mathbf{R}_d\boldsymbol{\nu}(t)\,dt$ *in which*

$$\mathbf{Q}_d = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \mathbf{R}_d = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

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Appendix A

Linear Algebra

This Appendix recalls only the base notions of matrices, vectors, linear spaces and the operations between matrices and vectors exploited in this book. These notions are provided beside MATLAB listings.

Matrices and vectors

Let $a_{ij} \in \mathbb{C}$, with $i = 1, \ldots, n$, $j = 1, \ldots, m$, and $n, m \in \mathbb{N}$, then

$$\mathbf{A} := \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} \tag{A.1}$$

is called *matrix*. A matrix of m rows and n columns, whose entries belong to $\mathbb{C}$, represents an element of the space $\mathbb{C}^{m \times n}$, *i.e.* $\mathbf{A} \in \mathbb{C}^{m \times n}$. If $n = m$ the matrices are said to be *square*.

```
% Declaration and assignment of a m-by-n matrix, A, of random real numbers
m = randi(10); % number of rows as a random integert between 1 and 10
n = randi(10); % number of columns as a random integert between 1 and 10
A = rand(m,n); % Matrix declaration and assignment
```

When $n = 1$, a matrix is said to be a *vector* (or column vector) and it is denotes as $\mathbf{x} \in \mathbb{C}^m$. Let $\mathbf{A}$ be a matrix, then its *transpose*, namely $\mathbf{A}^\top$, is given by

$$\mathbf{A}^\top = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{nm} \end{bmatrix} \tag{A.2}$$

where $\left(\mathbf{A}^\top\right)^\top = \mathbf{A}$.

```
% Transpose of a matrix
B = A.'; % returns the nonconjugate transpose of A, that is, interchanges the row and column index for each element. If A contains complex elements, then A.' does not affect the sign of the imaginary parts.
```

A matrix $\mathbf{A}$ is said to be *skew symmetric* if $\mathbf{A}^\top = -\mathbf{A}$.

Furthermore, a square matrix $\mathbf{A} \in \mathbb{C}^{n\times n}$ is said to be *diagonal* if $a_{ij} = 0$ for any $i, j = 1, \dots, n$, *i.e.* if

$$\mathbf{A} = \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix}. \tag{A.3}$$

```
% Diagonal matrix
v = rand(1,m); % creates a vector of m-columns of real random numbers
D = diag(v);  % returns a square diagonal matrix with the elements of vector v on the main diagonal
```

A diagonal matrix with $a_{ii} = 1$ for all $i = 1, \dots, n$ is called *identity* and is denoted with $\mathbf{I}$. A *null matrix*, denoted with $\mathbf{O}$, is defined as a matrix whose elements are (all) null. The null vectors are denoted with $\mathbf{0}$.

```
% Null matrix
X = zeros(m,n); % returns an n-by-m matrix of zeros
```

A square matrix $\mathbf{A} \in \mathbb{C}^{n\times n}$ is said to be *upper(lower) triangular* if the elements below(above) the principal diagonal are all null, *i.e.*

$$\mathbf{A}_{\text{up}} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{mn} \end{bmatrix} \qquad \mathbf{A}_{\text{low}} = \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}.$$

Matrix sum and product

Let $\mathbf{A}, \mathbf{B} \in \mathbb{C}^{m\times n}$, with $m, n \in \mathbb{N}$, whose elements are a_{ij} and b_{ij} respectively. Then, the sum $\mathbf{A}+\mathbf{B}$ is defined as

$$\mathbf{A}+\mathbf{B} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{bmatrix} =$$

$$= \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \dots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \dots & a_{2n}+b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \dots & a_{mn}+b_{mn} \end{bmatrix}.$$

```
% Sum of matrices
C = A+B; % is the matrix sum of A and B. If A and B are an n-by-m matrices, then C is an n-by-m matrix
```

Let $\mathbf{A} \in \mathbb{C}^{m\times n}$, with $m, n \in \mathbb{N}$, and $\alpha \in \mathbb{C}$, then the product $\alpha\mathbf{A}$ is defined as

$$\alpha\mathbf{A} = \begin{bmatrix} \alpha a_{11} & \alpha a_{12} & \dots & \alpha a_{1n} \\ \alpha a_{21} & \alpha a_{22} & \dots & \alpha a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha a_{m1} & \alpha a_{m2} & \dots & \alpha a_{mn} \end{bmatrix}.$$

```
% Multiplication by a scalar
alpha = rand(1); % generates a random real number
B = alpha*A; % is the matrix product of alpha and A.
```

The operator "$\cdot$" defines the dot matrix product, representative of the famous rule *row-by-column*. In details, let $\mathbf{A} \in \mathbb{C}^{m\times n}$, $\mathbf{B} \in \mathbb{C}^{n\times p}$, and $\mathbf{C} \in \mathbb{C}^{m\times p}$, with $m, n, p \in \mathbb{N}$. Let a_{ik}, b_{kj}, and c_{ij} be the elements of $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$, with $i = 1, \dots, m$, $k = 1, \dots, n$, and $j = 1, \dots, p$. Then, $\mathbf{C} = \mathbf{A}\cdot\mathbf{B}$ if

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}. \tag{A.4}$$

Remark A.1. *It is worth noting that the matrix product is well posed if the number of columns of* $\mathbf{A}$ *is equal to the number of rows of* $\mathbf{B}$.

```
% Matrices multiplications
C = A*B; % is the matrix product of A and B. If A is an m-by-n and B is a n-by-p matrix, then C is an m-by-p matrix
```

Let $\alpha \in \mathbb{C}$ and $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbb{C}^{m \times n}$, then the following equalities are true

$$\begin{gathered}
\mathbf{A} - \mathbf{B} = \mathbf{A} + (-1)\mathbf{B} \\
(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}) \\
\alpha\,(\mathbf{A} \cdot \mathbf{B}) = (\alpha\mathbf{A}) \cdot \mathbf{B} = \mathbf{A} \cdot (\alpha\mathbf{B}) \\
\mathbf{A} \cdot \mathbf{B} \cdot \mathbf{C} = (\mathbf{A} \cdot \mathbf{B}) \cdot \mathbf{C} = \mathbf{A} \cdot (\mathbf{B} \cdot \mathbf{C}) \\
\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C} \\
(\mathbf{B} + \mathbf{C}) \cdot \mathbf{A} = \mathbf{B} \cdot \mathbf{A} + \mathbf{C} \cdot \mathbf{A} \\
(\mathbf{A} \cdot \mathbf{B})^\top = \mathbf{B}^\top \cdot \mathbf{A}^\top \\
(\mathbf{A} + \mathbf{B})^\top = \mathbf{A}^\top + \mathbf{B}^\top.
\end{gathered}$$

Anyway, it is worth noting that in general

- $\mathbf{A} \cdot \mathbf{B} \neq \mathbf{B} \cdot \mathbf{C}$;
- $\mathbf{A} \cdot \mathbf{B} = \mathbf{A} \cdot \mathbf{C} \;\not\Rightarrow\; \mathbf{B} = \mathbf{C}$;
- $\mathbf{A} \cdot \mathbf{B} = \mathbf{O} \;\not\Rightarrow\; \mathbf{B} = \mathbf{O}$ or $\mathbf{A} = \mathbf{O}$.

The identity matrix represents the neutral element for the dot product, *i.e.* it is such that

$$\mathbf{A} \cdot \mathbf{I} = \mathbf{A} \qquad \mathbf{I} \cdot \mathbf{A} = \mathbf{A}.$$

Let $\mathbf{A} \in \mathbb{C}^{n \times n}$, then its k-th power is defined as

$$\mathbf{A}^k = \prod_{i=1}^{k} \mathbf{A} = \mathbf{A} \cdot \mathbf{A} \cdot \ldots \cdot \mathbf{A} \qquad k \text{ times.}$$

Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{B} \in \mathbb{R}^{p \times q}$. Let a_{ij} be the entries of $\mathbf{A}$, with $i = 1, \ldots, m$ and $j = 1, \ldots, n$, then the *Kroneker product* is denoted by the operator $\otimes$ and is such that

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \ldots & a_{1n}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \ldots & a_{2n}\mathbf{B} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}\mathbf{B} & a_{m2}\mathbf{B} & \ldots & a_{mn}\mathbf{B} \end{bmatrix}.$$

```
% Kroneker product
K = kron(A,B) % returns the Kroneker tensor product of
    matrices A and B. If A is an m-by-n matrix and B is
    a p-by-q matrix, then kron(A,B) is an m*p-by-n*q
    matrix formed by taking all possible products
    between the elements of A and the matrix B.
```

Let $\mathbf{A}_i, \mathbf{A} \in \mathbb{C}^{m\times n}$, with $i = 1, \ldots, p$. Then $\mathbf{A}$ is said to be a *linear combination* of $\{\mathbf{A}_1, \ldots, \mathbf{A}_p\}$ if it can be obtained through

$$\mathbf{A} = \alpha_1\mathbf{A}_1 + \cdots + \alpha_p\mathbf{A}_p$$

for some $\alpha_i \in \mathbb{C}$. The matrices in the set $\{\mathbf{A}_1, \ldots, \mathbf{A}_p\}$ are said to be *linearly independent* if each of them can not be written as linear combination of the remaining. Equivalently, $\mathbf{A}_1, \ldots, \mathbf{A}_p$ are linearly independent if and only if

$$\alpha_1\mathbf{A}_1 + \cdots + \alpha_p\mathbf{A}_p = \mathbf{0}$$

implies $\alpha_i = 0$ for any $i = 1, \ldots, p$.

Vector products

Let $\mathbf{u}_1, \mathbf{u}_2 \in \mathbb{C}^n$ be two vectors, then we define the *inner product* $\langle\mathbf{u}_1, \mathbf{u}_2\rangle \in \mathbb{C}$ with

$$\langle\mathbf{u}_1, \mathbf{u}_2\rangle := \mathbf{u}_1^\top \cdot \mathbf{u}_2 = \sum_k u_{1k}u_{2k}.$$

Let $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \in \mathbb{C}^n$ and $\alpha \in \mathbb{C}$, then the inner product between vectors satisfies the following properties

- distribution with respect to the sum: $\langle(\mathbf{u}_1 + \mathbf{u}_2), \mathbf{u}_3\rangle = \langle\mathbf{u}_1, \mathbf{u}_3 + \mathbf{u}_2 \cdot \mathbf{u}_3\rangle$;
- distribution with respect to the scalar product: $\alpha\left(\langle\mathbf{u}_1, \mathbf{u}_2\rangle\right) = \langle(\alpha\mathbf{u}_1), \mathbf{u}_2\rangle = \langle\mathbf{u}_1, (\alpha\mathbf{u}_2)\rangle$;
- commutation: $\langle\mathbf{u}_1, \mathbf{u}_2\rangle = \langle\mathbf{u}_2, \mathbf{u}_1\rangle$;
- positive definition: $\langle\mathbf{u}_1, \mathbf{u}_1\rangle > 0$, $\forall\mathbf{u}_1 \neq \mathbf{0}$.

The (Euclidean) norm of a vector is defined as $\|\mathbf{u}\| = \sqrt{\langle\mathbf{u}, \mathbf{u}\rangle}$. A vector is called *versor* if its norm is equal to 1, *i.e.* if $\|\mathbf{u}\| = 1$.

```
% Inner product and norm of vectors
u3 = dot(u1,u2); %returns the scalar dot
    product of the vectors u1 and u2 which must
    have the same length
```

```
n = norm(u); % returns the Euclidean norm of
    vector u. This norm is also called the 2-
    norm, vector magnitude, or Euclidean length.
```

A square matrix $\mathbf{A} \in \mathbb{C}^{n\times n}$ is said *(semi-)positive definite* if for any vector $\mathbf{x} \in \mathbb{C}^n$, $\mathbf{x} \neq \mathbf{0}$, the scalar $\langle \mathbf{x}, (\mathbf{A}\cdot\mathbf{x})\rangle\, (\geq) > 0$.

Let $\mathbf{u} := \mathtt{col}(u_1, u_2, u_3) \in \mathbb{R}^3$ and define the function $S : \mathbb{R}^3 \to \mathbb{R}^{3\times 3}$ with

$$\mathbf{S}(\mathbf{u}) = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix}.$$

Let $\mathbf{u}_1$, $\mathbf{u}_2 \in \mathbb{R}^3$, then the *outer product* is defined as

$$\mathbf{u}_1 \times \mathbf{u}_2 = \mathbf{S}(\mathbf{u}_1)\cdot \mathbf{u}_2$$

The cross product is

- non–commutative: $\mathbf{u}_1 \times \mathbf{u}_2 = -(\mathbf{u}_2 \times \mathbf{u}_1)$;
- distributive with respect to the sum: $\mathbf{u}_1 \times (\mathbf{u}_2 + \mathbf{u}_3) = \mathbf{u}_1 \times \mathbf{u}_2 + \mathbf{u}_1 \times \mathbf{u}_3$;
- distributive with respect to the product by a scalar: $\alpha(\mathbf{u}_1 \times \mathbf{u}_2) = (\alpha\mathbf{u}_1) \times \mathbf{u}_2 = \mathbf{u}_1 \times (\alpha\mathbf{u}_2)$;
- non–associative: $\mathbf{u}_1 \times (\mathbf{u}_2 \times \mathbf{u}_3) \neq (\mathbf{u}_1 \times \mathbf{u}_2) \times \mathbf{u}_3$.

```
% cross product of vectors
u3 = cross(u1,u2); % returns the cross product
    of the vectors u1 and u2 which must have a
    length of 3.
```

Matrix inverse

Let $\mathbf{A} \in \mathbb{R}^{n\times n}$, then the square matrix obtained from $\mathbf{A}$ by the cancellation of the i-th row and the j-th column is denoted with $\mathbf{A}^{(ij)} \in \mathbb{R}^{n-1\times n-1}$. It is worth observing that the i-th row and the j-th column share one common element, which is a_{ij}. The determinant $\det(\mathbf{A}) : \mathbb{R}^{n\times n} \mapsto \mathbb{R}$ is defined as

$$\det(\mathbf{A}) := \sum_j a_{ij}\, C_{ij}\, (-1)^{i+j}$$

where C_{ij} is the *cofactor* associated to a_{ij}. Iteratively, the cofactors C_{ij} are defined as

$$C_{ij} = \det\left(\mathbf{A}^{(ij)}\right).$$

The following properties hold true

- $\det(\mathbf{A}\cdot\mathbf{B}) = \det(\mathbf{A})\det(\mathbf{B})$
- $\det\left(\mathbf{A}^\top\right) = \det(\mathbf{A})$
- $\det(\mathbf{A}) = 0$ if the matrix $\mathbf{A}$ has a row (column) which is a linear combination of the remaining rows (columns)
- $\det(\mathbf{A}) = \prod_{i=1}^{n} a_{ii}$ if the matrix is either triangular or diagonal.

If $\det(\mathbf{A}) = 0$ the matrix $\mathbf{A}$ is said to be *singular*.

```
% Determinant of a matrix
d = det(A); % returns the determinant of square
    matrix A.
```

The **rank** of a square matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ is defined as a function $\rho : \mathbb{R}^{n\times n} \mapsto \mathbb{N}$ and it corresponds to the maximum integer, namely p, for which there exists at least one sub–matrix of dimensions p (built selecting p rows and p columns of $\mathbf{A}$) whose determinant is not null. The following properties are valid

- $\rho(\mathbf{A}\cdot\mathbf{B}) \le \rho(\mathbf{A})\cdot\rho(\mathbf{B})$
- $\rho\left(\mathbf{A}^\top\right) = \rho(\mathbf{A})$
- $\rho\left(\mathbf{A}^\top\cdot\mathbf{A}\right) = \rho\left(\mathbf{A}\cdot\mathbf{A}^\top\right) = \rho(\mathbf{A})$.

Furthermore, if $\rho(\mathbf{A}) = n$ the matrix $\mathbf{A}$ is said to be *full rank* or, equivalently, *invertible*.

Let $\mathbf{A} \in \mathbb{R}^{n\times n}$, the *adjoint matrix* is defined as

$$\mathrm{Adj}(\mathbf{A}) = \begin{bmatrix} C_{11}(-1)^{(1+1)} & C_{12}(-1)^{(1+2)} & \dots & C_{1n}(-1)^{(1+n)} \\ C_{21}(-1)^{(2+1)} & C_{22}(-1)^{(2+2)} & \dots & C_{2n}(-1)^{(1+n)} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1}(-1)^{(n+1)} & C_{n2}(-1)^{(n+2)} & \dots & C_{nn}(-1)^{(n+n)} \end{bmatrix}^\top .$$

The *matrix inversion* is an operator which applies to square matrices. In particular, let $\mathbf{A} \in \mathcal{R}^{n\times n}$, then if $\mathbf{A}$ is invertible then its inverse is defined as

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\mathrm{Adj}(\mathbf{A})$$

with

$$\mathbf{A}^{-1}\cdot\mathbf{A} = \mathbf{A}\cdot\mathbf{A}^{-1} = \mathbf{I}.$$

Finally, let $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n}$, then $(\mathbf{A}\cdot\mathbf{B})^{-1} = \mathbf{B}^{-1}\cdot\mathbf{A}^{-1}$.

```
% Inverse of a matrix
Y = inv(X); % computes the inverse of square
    matrix X.
```

Matrix pseudo-inverses (Moore-Penrose)

Let $\mathbf{A} \in \mathbb{R}^{m\times n}$ with $\rho(\mathbf{A}) = \min\{m, n\}$, then the *left* and *right pseudo-inverse of* $\mathbf{A}$ are defined as follows

$$\mathbf{A}^+ = \begin{cases} \left(\mathbf{A}^\top \cdot \mathbf{A}\right)^{-1} \cdot \mathbf{A}^\top & \rho(\mathbf{A}) = n \quad \text{left pseudo-inverse} \\ \mathbf{A}^\top \cdot \left(\mathbf{A} \cdot \mathbf{A}^\top\right)^{-1} & \rho(\mathbf{A}) = m \quad \text{right pseudo-inverse} \end{cases}$$

```
% Pseudo-Inverse of a matrix
B = pinv(A); % returns the Moore-Penrose
    Pseudoinverse of matrix A.
```

Vector Space

Let $\mathbb{V}(\mathbb{C}) := \{\mathbf{x} \in \mathbb{C}^n\}$ with $n \in \mathbb{N}$. Then $\mathbb{V}$ is a *vector space* if the following conditions hold true

1. for any $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \in \mathbb{V}$ then
$$(\mathbf{u}_1 + \mathbf{u}_2) + \mathbf{u}_3 = \mathbf{u}_1 + (\mathbf{u}_2 + \mathbf{u}_3)$$
2. for any $\mathbf{u} \in \mathbb{V}$ the null vector $\mathbf{0}$ is such that
$$\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u}$$
3. for any $\mathbf{u}_1 \in \mathbb{V}$ there exists $\mathbf{u_2} \in \mathbb{V}$ such that
$$\mathbf{u}_1 + \mathbf{u}_2 = \mathbf{0} \implies \mathbf{u}_2 = -\mathbf{u}_1$$
4. for any $\mathbf{u}_1, \mathbf{u}_2 \in \mathbb{V}$
$$\mathbf{u}_1 + \mathbf{u}_2 = \mathbf{u}_2 + \mathbf{u}_1$$
5. for any $\mathbf{u} \in \mathbb{V}$ and for any $\alpha, \beta \in \mathbb{C}$
$$\alpha(\beta\mathbf{u}) = (\alpha\beta)\mathbf{u}$$
6. there exists a neutral element, namely 1, such that
$$1\mathbf{u} = \mathbf{u}$$
7. for any $\mathbf{u}_1, \mathbf{u}_2 \in \mathbb{V}$ and for any $\alpha \in \mathbb{C}$
$$\alpha(\mathbf{u}_1 + \mathbf{u}_2) = \alpha\mathbf{u}_1 + \alpha\mathbf{u}_2$$
8. for any $\mathbf{u} \in \mathbb{V}$ and any $\alpha, \beta \in \mathbb{C}$
$$(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}.$$

Linear functions and Matrices

Let $\mathbb{U}(\mathbb{C})$ and $\mathbb{V}(\mathbb{C})$ be to vector spaces, then a function $A : \mathbb{U} \to \mathbb{V}$ is said to be *linear* if for any $\mathbf{u}_1, \mathbf{u}_2 \in \mathbb{U}$ and for any $\alpha \in \mathbb{C}$ the following relations hold

$$\begin{aligned} L(\mathbf{u}_1 + \mathbf{u}_2) &= L(\mathbf{u}_1) + L(\mathbf{u}_2) \\ L(\alpha\mathbf{u}_1) &= \alpha L(\mathbf{u}_1). \end{aligned}$$

Let $\mathbf{x} \in \mathbb{U}$ and $\mathbf{y} \in \mathbb{V}$, then a linear transformation between the two vector spaces can be represented through a matrix $\mathbf{A} \in \mathbb{C}^{m \times n}$, as follows

$$\mathbf{y} = \mathbf{A} \cdot \mathbf{x}.$$

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Appendix C

Optimal Control Robustness

The dynamics of an LTI plant subject to the optimal stationary control based on a quadratic cost is described as

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}u \\ \mathbf{e} &= \mathbf{C}\mathbf{x} \\ u &= -\mathbf{K}_R\mathbf{x} \end{aligned} \tag{C.1a}$$

with the control gain defined as $\mathbf{K}_R = \frac{1}{r}\mathbf{B}^\top\mathbf{S}$ and $\mathbf{S}$ solution of the following ARE

$$\frac{1}{r}\mathbf{S}\mathbf{B}(\mathbf{B})^\top\mathbf{S} - \mathbf{S}\mathbf{A} - (\mathbf{A})^\top\mathbf{S} - \mathbf{C}^\top\bar{\mathbf{Q}}\mathbf{C} = \mathbf{0}. \tag{C.1b}$$

in which, for sake of simplicity and without loss of generality, we assumed $u \in \mathbb{R}$.

The Laplace transform applied to (C.1a) leads to

$$\begin{aligned} \mathbf{X} &= \mathbf{G}(s)U \\ \mathbf{E} &= \mathbf{C}\mathbf{X} \\ U &= -\mathbf{K}_R\mathbf{X} \end{aligned} \tag{C.2}$$

in which the column vector $\mathbf{G}(s) := \mathbf{\Phi}(s)\mathbf{B}$ represents the open loop transfer matrix with $\mathbf{\Phi}(s) := (s\mathbf{I} - \mathbf{A})^{-1}$. To analyse the stability of the closed loop, which is schematically represented in Figure C.1, we define $L(s) = \mathbf{K}_R\mathbf{G}(s)$ and determine the transfer function between the fictitious reference input U_R and the state $\mathbf{X}$

$$\frac{\mathbf{X}}{U_R} = \frac{\mathbf{G}(s)}{1 + L(s)} \tag{C.3}$$

whose stability properties are provided by the so called *return difference* $1 + L(s)$. On the other hand, the ARE (C.1b) can be exploited to provide a

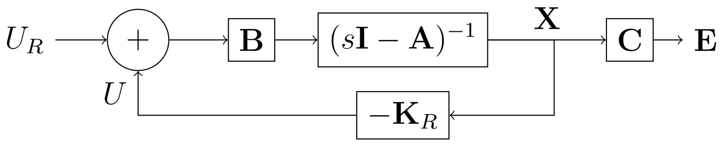

Figure C.1: Control loop architecture

lower bound for the norm of $L(s)$. In particular, after a premultiplication by $\mathbf{G}^\top(-s)$ and a post multiplication by $\mathbf{G}(s)$ and some technicalities we obtain

$$(1+L(-s))\,(1+L(s)) = 1 + \frac{1}{r}\mathbf{G}^\top(-s)\mathbf{C}^\top\bar{\mathbf{Q}}\mathbf{C}\mathbf{G}(s) \tag{C.4}$$

which, evaluated at $s = j\omega$ leads to

$$(1+L(j\omega))^2 = 1 + \frac{1}{r}\mathbf{G}^\top(j\omega)\mathbf{C}^\top\bar{\mathbf{Q}}\mathbf{C}\mathbf{G}(j\omega). \tag{C.5}$$

we now note that $\mathbf{G}^\top(j\omega)\mathbf{C}^\top\bar{\mathbf{Q}}\mathbf{C}\mathbf{G}(j\omega)$ is a positive semi-definite quadratic form thanks to $\mathbf{C}^\top\bar{\mathbf{Q}}\mathbf{C} \geq 0$. As consequence

$$|1+L(j\omega)| = \sqrt{1 + \frac{1}{r}\mathbf{G}^\top(j\omega)\mathbf{C}^\top\bar{\mathbf{Q}}\mathbf{C}\mathbf{G}(j\omega)} \geq 1. \tag{C.6}$$

The (C.6), known as *return difference inequality*, implied that $L(j\omega)$ lives out of the unitary circle centred in the complex plane at $-1+j0$, for any $\omega \in \mathbb{R}$, see Figure C.2. Finally, in the worst case of $|1+L(j\omega)| = 1$ when the phase of $L(j\omega)$ is $-\pi$, the gain margin ranges from $1/2$ to ∞. On the other hand, even assuming $|1+L(j\omega)| = 1$, the intersection with the unitary circle centred at the origin occurs at maximum at phases equal to $\pm 60^o$.

The term $1+L(j\omega)$ can be interpreted as $L(j\omega) - (-1+j0)$, *i.e.* as a change of coordinate that moves the origin of the reference system from $0+j0$ to $-1+j0$.

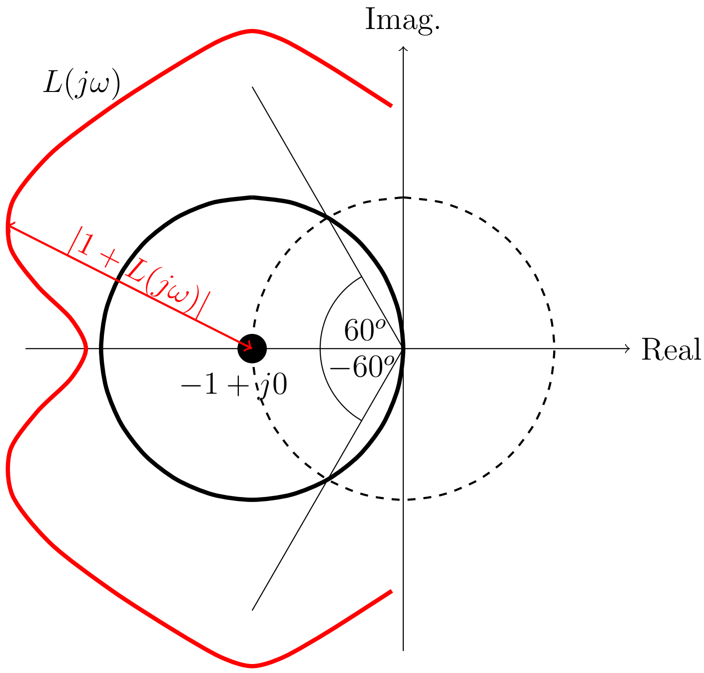

Figure C.2: Graphical representation of the *return difference inequality*

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