Connections between the Total Least Squares and the correction of an infeasible system of Linear Inequalities

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Abstract

Given an infeasible system of linear inequalities, $Ax \leq b$, we address the problem of correcting both the matrix of coefficients A by $A + H$ and vector b by $b + p$ to minimize the Frobenius norm of $[H, p]$. For a system of linear equations this problem can be solved by an algebraic and well-studied method known as the Total Least Squares. For inequalities, Vatolin [?] was the first to approach this problem, presenting a result with necessary and sufficient conditions for local minimizers. Unfortunately the direct application of these results is impracticable for large problems. Since the sufficient conditions are not necessary, in case of their failure one is unable to draw conclusions on a search path for a local minimizer. We have analyzed the problem using the KKT conditions and derived necessary and sufficient conditions which enabled us to unequivocally characterize local optima in terms of the solution of the Total Least Squares and the set of active constraints. Establishing the common features between these two problems is not only important from a theoretical point of view, but it opens the possibility of using theoretical developments related with the Total Least Squares to solve the problem with inequalities.

1 Introduction

The motivation for developing a study for linear inequalities relies on the many practical situations that can be modeled by a linear system involving inequalities. Among others, linear programming is one field where this kind of models are so common. Inconsistency in linear models can be explained by many different situations, from which we can mention a few: complexity of the problem, conflicting goals in different groups of decision-makers, lack of communication between different groups that define the constraints, different views over the problem, partial information, wrong or inaccurate estimates, over-optimistic purposes, errors in data, integration of different formulations and the update of old models. In general, inconsistency of a model can be overcome by changes in the coefficients of the constraints. Those are the situations we are interested in, and for which the study in this paper applies. We analyze the problem of finding the minimal correction, in the Frobenius norm, for an inconsistent system of inequalities. For this problem Vatolin [**?**] introduced a result, no proof presented, of necessary and also sufficient conditions. How to make use of it is unclear, since it does not leave an open door on the way to meet the requirements of the result. His work implies the existence of a subset of rows and columns and upon their choice there are some conditions that should be met. We show that the sufficient conditions are not necessary and under failure of the conditions imposed by the result of Vatolin, for a "guessed" subset of rows and columns, we are left without any clue on how to proceed, since no information can be driven from that.

If instead of inequalities we were dealing with equalities, the problem is well-studied and known as the "Total Least Squares" (TLS). In a previous work [**?**] we have used the TLS as a framework in a heuristic approach. In [**?**] the authors combined a gradient projection method and the TLS, but could not provide any guarantees on the quality of the solution. This paper presents results that consubstantiate this previous work. Specifically, this paper shows how the TLS relates with the problem of correction of inequalities. Through the KKT conditions we derive necessary and sufficient conditions for local optima and we prove that if the problem has a minimum, then a correction corresponds to a local optima iff the solution of the corrected system is the TLS solution for the set of active constraints and is feasible for the non active constraints. In practice, this relation enables to check whether the set of constraints we select are in fact the active set of constraints; otherwise we are able to propose a new set. Such relation also enables to rely on efficient methods developed for the TLS that can be incorporated in a procedure to find local optima and it gives support to the heuristic procedures that we have previously presented. Relating the solution of these two problems is quite interesting from a theoretical viewpoint and opens new ways for the search of global optima. We also analyze, in a comprehensive way, the case where there is no minimum and the infimum is zero.

This paper is organized as follows. In the next section we make an overview on the work related with the problem of inconsistency in models and present some views on how it can be tackled. In section 3 we formulate the problem and in section 4 we present the Total Least Squares problem. The theoretical developments regarding the correction of the system of inequalities minimizing the Frobenius norm, including the derivation of necessary and sufficient conditions for local minimizers, are presented in section 5. Section 6 is dedicated to conclusions and further work.

2 Work on inconsistent systems

In a 1979 paper [?] Roodman stressed how little had been done on "pos-infeasibility analysis" and proposed the first known approaches regarding the correction of models, which accounted only for changes in the right-hand side of constraints. A method based on the analysis of the final Phase I solution in the Simplex method allowed to estimate lower bounds on the amount of change in the right hand side of each constraint to attain feasibility, assuming that the model could be made feasible by changing one constraint alone. Some additional results based on the analysis of the final solution in the Dual Simplex were also presented. Insights on how to work with multiple changes in constraints were given in a sort of parametric approach although guided by one parameter alone. To our knowledge, the first to propose a study for the correction of both the coefficient matrix and the right hand side was Vatolin. In [?] we can find the English version of the original paper in Russian, which deals with a special type of objective function as the correction criteria, which did not include the Frobenius norm of the correction matrix. He showed that for a system of equalities and inequalities, with nonnegativity constraints for variables and for a certain class of objective functions, the optimal correction of the model could be solved by a finite number of linear programming problems. Erimin and Vatolin [?] developed a broad theory on the study of inconsistent mathematical programming problems, presenting duality concepts for corrected systems.

Considering the minimization of the Frobenius norm of the correction matrix of the constraints, we can find in [?] necessary and sufficient conditions for a local minimizer. These conditions depend on a convenient choice of a submatrix, but no indication is given on how to find such submatrix. A complete search would require the study of a large number of possible combinations of rows and columns which is impracticable for large problems. Related with the analysis of inconsistent systems in general we would like to point out the work of Van Loon [?] and Wang and Huang [?], on results that permit the identification of Irreducible Inconsistent Systems (IIS) in a quasi-Simplex tableau. A system of linear relations is an IIS if it is non solvable, but each of its proper subsystems is solvable. Identifying ISS is an important tool for analyzing inconsistencies, since they allow us to focus on small sets of constraints to understand the causes of the inconsistency. Also related to ISS, Chinneck and Dravnieks [?] presented the "deletion filtering" method, a simple algorithm that guarantees that an IIS is always found. This method consists of creating a set S, initiated as the set of all constraints, from which constraints are deleted if their removal does not make the set S feasible. In the area of Constraint Programming the problem of finding Irreducible Inconsistent Systems was addressed in the context of an incremental addition of constraints, typical in (Hierarchical) Constraint Logic Programming [?]. The authors showed how to find minimal conflict sets (equivalent to IISs) upon the addition to a set of feasible constraints of a further constraint that makes the new set infeasible. A hierarchy manager of the system would then remove one of the constraints of the set, but the method does not guarantee minimality of the set of removed constraints.

Another important method to analyze an inconsistent model is the identification of Minimum Cardinality Infeasibility Sets (MCIS). Given a set of relations (C), a Minimum Cardinality Infeasibility Set (M) is a subset of C such that $C \setminus M$ is feasible and among sets that verify this condition, M has smallest cardinality. Unfortunately the MCIS, that can be formulated as a mixed integer problem, is NP-Hard as proved by Chakravatty [**?**].

In [**?**] Harvey Greenberg described the method of "isolation" to detect the sources of infeasibility and to conduct a diagnosis of the model, which he described in conjunction with a stage designated as "explanation". In addition, "Phase I price aggregation" and "Successive bound reduction" were presented as instances of the isolation method.

To conclude we may mention the work of Renegar [**?**], [**?**], Pena [**?**] and Vera [**?**], on the distance of a problem to the set of ill-posed problems. They considered as ill-posed problems those problems for which arbitrarily small changes in the data can alter the classification of the problem form consistent to inconsistent and vice versa. The actual matrix of corrections is not computed.

3 Problem formulation

Consider a linear system that can involve both equalities and inequalities

$$Ax \bowtie b,$$

where $A \in \Re^{m \times n}$ $(m > n)$ is the coefficient matrix for variables, $b \in \Re^m$ is the right-hand side and $\bowtie$ is an ordered set of relations of type $\leq$ or $=$. If $Ax^* \bowtie b$, then x^* is a solution to this system. We say that the system is inconsistent, or infeasible, if there are no such solutions. For an inconsistent system we assume that we can perform small changes in the coefficients $[A, b]$ and we wish to find a set of new coefficients $[\Psi(A), \psi(b)]$, such that the new system is feasible. In order to assure that the modified model is plausible, the new matrix of coefficients must be close to the original one. This measure of closeness can be defined accordingly to a function on $[A, b] - [\Psi(A), \psi(b)]$. This difference is regarded as the correction matrix. A general formulation for the correction problem is

$$\begin{aligned} \hat{f} = \quad & \inf \quad \|[E \otimes H, e \otimes p]\|_k^2 \\ & \text{s.t.} \quad (A + E \otimes H)x \bowtie b + e \otimes p \\ & \qquad H \in \Re^{m \times n}, \quad p \in \Re^m, \quad x \in \Re^n, \end{aligned} \tag{1}$$

where $\otimes$ stands for the Hadamard product of matrices, $\|\cdot\|_k$ is some matrix norm, and $\bowtie$ is a sequence of relations of the type $\leq$ or $=$. The notation $\equiv$ will be used to identify a particular type of relation (for instance, $\bowtie\equiv=$ refers to a system of equalities). The role of E relates to particular structures in A that are allowed to change (the same for e, in regard to b). Different assignments to $(E, e, \bowtie, k)$ result in different correction models. Denoting by $\mathbf{0}_{m\times n}$ and $\mathbf{1}_{m\times n}$ the m by n matrix of zeros and ones respectively, for:

a) $E = \mathbf{0}_{m\times n}$,$e = \mathbf{1}_m$, $\bowtie\equiv=$ and $k = 2$, we have the classic Least Squares Problem;

b) $E = \mathbf{1}_{m\times n}$,$e = \mathbf{1}_m$, $\bowtie\equiv=$ and $k = F$ we have the Total Least Squares Problem;

c) $E = \mathbf{1}_{m\times n}$,$e = -\mathbf{1}_m$, $\bowtie\equiv\leq$ and $k = F$, we have the problem that will be addressed in this paper;

d) $E = A$,$e = b$, $\bowtie\equiv\leq$ and $k = F$, with the substitution $H' = A \otimes H$ and $p' = b \otimes p$, we have a problem similar to (c), but only nonzero coefficients are allowed to change.

The problem described in (c) can be formulated as

$$\begin{array}{rll} \hat{f} = & \inf & \|[H, p]\|_F^2 \\ & \text{s.t.} & (A + H)x \leq b - p \\ & & H \in \Re^{m\times n}, \quad p \in \Re^m, \quad x \in \Re^n. \end{array} \tag{2}$$

For $k = \infty$ (and also for the generalizations of the vector norms l_1 and l_∞ to matrices), taking $[E, e] = [\mathbf{1}_{m\times n}, \mathbf{1}_m]$ and for any sequence $\bowtie$, Vatolin has proved that problem (**??**) can be solved through the solution of a finite number of linear programming problems. For different matrices $[E, e]$ this is still an open problem. Another open problem consists of the TLS when we fix some special structure in $[A, b]$, apart from sets of rows and (or) columns. The correction of a substructure of rows and/or columns in an inconsistent systems of equations, with $m \geq n+1$, can be done using a theoretical result from Demmel [**?**]. In the next section we make a brief introduction to the TLS. A comprehensive theoretical development can be found in [**?**] and in [**?**] we can find some methods for large scale TLS.

4 The Total Least Squares

The TLS makes use of the Singular Value Decomposition (SVD). In the following given $C \in \Re^{m\times n}$ $(m \geq n)$, we will write $C = U\Sigma V^T$ as the SVD of C and we will consider

$$\begin{aligned} U &= [u_1, \cdots, u_m] \in \Re^{m\times m}, \\ V &= [v_1, \cdots, v_n] \in \Re^{n\times n}, \\ \Sigma &= diag(\sigma_1, \cdots, \sigma_n) \in \Re^{m\times n}. \end{aligned}$$

If necessary we will write $\sigma_i(C)$ instead of σ_i to specify to which matrix the i-th singular value refers. We know that for an infeasible system $Ax = b$, the optimal solution (H^*, p^*, x_{TLS}) of the problem:

$$\begin{aligned} \min \quad & \|[H, p]\|_F \\ \text{s.t.} \quad & (A + H)x = b - p \end{aligned}$$

is unique if $\sigma_n(A) > \sigma_{n+1}(C)$ $(C = [A, -b])$ and is given by

$$[H^*, p^*] = -\sigma_{n+1} u_{n+1} v_{n+1}^T, \tag{3}$$

$$x_{TLS} = \frac{1}{v_{n+1,n+1}} \begin{bmatrix} v_{1,n+1} \\ \vdots \\ v_{n,n+1} \end{bmatrix}. \tag{4}$$

The condition $\sigma_n(A) > \sigma_{n+1}(C)$ aims at ensuring that $v_{n+1,n+1} \neq 0$, which is not very restrictive since $\sigma_n(A) \geq \sigma_{n+1}(C)$ is already guaranteed by the interlacing property for eigenvalues. In [?] there is a complete work on TLS, and situations where the above requirement is not met are analyzed. The following table summarizes these results.

$\sigma_p > \sigma_{p+1} = \cdots = \sigma_{n+1} = 0$		$\sigma_p > \sigma_{p+1} = \cdots = \sigma_{n+1} > 0$	
$\exists i,\ v_{n+1,i} \neq 0$ $p+1 \leq i \leq n+1$	$\forall i,\ v_{n+1,i} = 0$ $p+1 \leq i \leq n+1$	$\exists i,\ v_{n+1,i} \neq 0$ $p+1 \leq i \leq n+1$	$\forall i,\ v_{n+1,i} = 0$ $p+1 \leq i \leq n+1$
$Ax = b$ is consistent	The problem has no TLS solution $inf = 0$	The problem has a TLS solution, which is unique if only one i can be found in this situation	The problem has no TLS solution $inf = \sigma_{n+1}$

5 An inconsistent problem with inequalities

The solution set of problem (??) is non empty and open. The problem may fail to have a solution altogether. This problem was analyzed by Vatolin [?] who described necessary and sufficient conditions for the existence of local minimizers. He obtained these conditions via a parametric approximation approach but no proofs were presented. There he stated that for a local minimizer (H^*, p^*), there exists a subset of columns of A, say I, and a subset of rows J and based on these sets he defines a matrix D_{IJ}, such that the columns are those in I and the rows in J are replaced by zero. The local optimal correction is given by $D_{IJ} v v^T / \|v\|^2$, where v is a proper vector of

$D_{IJ}^T D_{IJ}$ associated with its smallest eigenvalue. Some additional conditions impose that vector v be feasible for the homogeneous version of the constraints in J and that its last component is not zero. Sufficient conditions depend on the strict verification of the constraints in J. Unfortunately no method is provided, or even hinted, in order to find the subsets I and J. Even if such sets were found, no more details are given about v and in case of multiplicity of the corresponding eigenvalue, or when its value is zero, this question becomes more pertinent. The sufficient conditions are not necessary and the failure of the conditions of the theorem leaves no indication on how to perform an efficient search for a local minimizer. Moreover, it was not clear how nonnegativity constraints for the variables interfered with this scheme. So we present, together with the respective proofs, the theoretical results we obtained when analyzing the problem in the context of a nonlinear programming problem.

5.1 Characterization of local minimizers

Since the problem is not convex, we may derive necessary and sufficient conditions for the existence of local minimizer. Let the problem be written in the form

$$\begin{cases} \min \frac{1}{2} f(h,p,x) \\ \text{s.t. } g_i(h,p,x) \leq 0 \text{ for } i = 1,\ldots,m. \end{cases} \tag{5}$$

We will consider $\frac{1}{2}f(h,p,x)$ instead of $f(h,p,x)$, which is equivalent in terms of optimal solution, in order to obtain simpler expressions when writing down the Karush-Kuhn-Tucker (KKT) conditions. In the following we will write $\mathbf{0}_k$ for a vector of zeros of size k. If in the context it is not relevant the indication of k, for sake of simplicity we will omit it.

The Lagrangian function $L(h,p,x,\lambda) = \frac{1}{2}f(h,p,x) + \lambda^T g(x)$ of problem (**??**), considering $f(h,p,x)$ and $g(x)$ to be respectively the objective function and the constraints of problem (**??**), is given by

$$L(h,p,x,\lambda) = \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n} h_{ij}^2 + \frac{1}{2}\sum_{i=1}^{m} p_i^2 + \sum_{i=1}^{m}\lambda_i \sum_{j=1}^{n}\left(a_{ij}+h_{ij}\right)x_j + p_i - b_i \tag{6}$$

From the KKT conditions we know we must have

$$\frac{\delta L(h,p,x,\lambda)}{h_{ij}} = 0 \quad \Leftrightarrow \quad h_{ij} + \lambda_i x_j = 0, \tag{7}$$

$$\frac{\delta L(h,p,x,\lambda)}{p_i} = 0 \quad \Leftrightarrow \quad p_i + \lambda_i = 0, \tag{8}$$

$$\frac{\delta L(h,p,x,\lambda)}{x_j} = 0 \quad \Leftrightarrow \quad \Sigma_{i=1}^{m}\lambda_i(a_{ij}+h_{ij}) = 0, \tag{9}$$

together with

$$\lambda^T((A+H)x - b + p) = 0 \text{ (complementarity slackness conditions)}, \quad (10)$$
$$\lambda \geq \mathbf{0}. \quad (11)$$

We can express conditions (**??**) to (**??**) by the following relations between variables

$$\begin{aligned} H &= -\lambda x^T, && (12)\\ p &= -\lambda, && (13)\\ \lambda^T(A+H) &= \mathbf{0}. && (14) \end{aligned}$$

Some facts can be outlined: the correction matrix is of rank one and the correction in each row is proportional to $[x^T, 1]$. If we replace H and p respectively by (**??**) and (**??**) in (**??**), we can reduce substantially the number of variables in the problem, obtaining the equivalent problem

$$\begin{aligned} \min \quad & \lambda^T\lambda(x^Tx+1) \\ \text{s.t.} \quad & Ax - \lambda x^T x \leq b + \lambda \\ & \lambda \geq \mathbf{0}. \end{aligned} \quad (15)$$

Through the complementarity conditions (**??**) we can further reduce the number of variables, since

$$\lambda_i \left(\sum_{j=1}^{n} a_{ij}x_j - \lambda_i \sum_{j=1}^{n} x_j^2 - b_i - \lambda_i \right) = 0 \Leftrightarrow$$

$$\lambda_i = 0 \text{ or } \lambda_i = \frac{\sum_{j=1}^{n} a_{ij}x_j - b_i}{\sum_{j=1}^{n} x_j^2 + 1} \quad \forall i = 1, \ldots, m.$$

Let $(.)^+$ be the component wise operator on vectors in $\Re^n$ that transforms each component into $\max(0, .)$. The set of Lagrange multipliers corresponding to a local minimizer of (**??**), λ^*, depends on x alone

$$\lambda^* = \frac{(Ax^* - b)^+}{x^{*T}x^* + 1}. \quad (16)$$

Replacing λ in formulation (**??**) by (**??**) we obtain

$$\begin{aligned} \min \quad & \frac{(Ax-b)^{+T}(Ax-b)^+}{(x^Tx+1)^2}(x^Tx+1) \\ \text{s.t.} \quad & \frac{Ax-b}{x^Tx+1} \leq \frac{(Ax-b)^+}{x^Tx+1} \\ & \frac{(Ax-b)^+}{x^Tx+1} \geq \mathbf{0}. \end{aligned}$$

Thus, given that the constraints are trivially satisfied, this result proves that (**??**) can be formulated as the unconstrained problem

$$\min \frac{(Ax-b)^{+T}(Ax-b)^{+}}{x^T x + 1}.$$

We can thus use methods for unconstrained minimization to solve the problem. In [**?**] the authors used a combination of a gradient search to identify the set of active constraints, followed by an algebraic step using the TLS on the active set to find the solution. That work had two important insufficiencies. Firstly, if the mentioned step was successful (in the sense that the TLS solution was feasible for the non active constraints) it was not proved that the solution was in fact a local minimizer. Secondly, all local minimizers could be found by an appropriate choice of the active constraints. This is the subject we will now address.

5.2 Relations with the TLS

Throughout this section, and whenever necessary, we will assume, with no loss of generality, that the active constraints are the first $m_a \leq m$ constraints and that $[A_a, b_a]$ and $[A_r, b_r]$ are, respectively, the first m_a and the remaining rows of $[A, b]$. Let P_a be the problem

$$\begin{array}{ll} \min & \|[H_a, p_a]\|_F \\ \text{s.t.} & (A_a + H_a)x = (b_a - p_a) \\ & A_r x \leq b_r. \end{array} \tag{17}$$

If the TLS solution of

$$\begin{array}{ll} \min & \|[H_a, p_a]\|_F \\ \text{s.t.} & (A_a + H_a)x = (b_a - p_a) \end{array} \tag{18}$$

verifies the additional constraints $A_r x \leq b_r$, there is no doubt that it is also the optimal solution of (**??**) and we will prove that it is also a local minimizer for (**??**). Otherwise what can be inferred? We start to answer this question by characterizing the local minimizers.

Theorem 5.1. *If (H^*, p^*, x^*) is a local minimizer of (**??**) and λ^* the corresponding Lagrange multiplier, then*

$$\left[\begin{array}{c} x^* \\ 1 \end{array}\right] = \left[\begin{array}{c} \frac{A^T\lambda^*}{\lambda^{*T}\lambda^*} \\ -\frac{\lambda^{*T}b}{\lambda^{*T}\lambda^*} \end{array}\right]. \tag{19}$$

Proof - If in (**??**) we replace H for (**??**) then we get

$$\lambda^{*T}(A - \lambda^* x^{*T}) = \mathbf{0} \Leftrightarrow \lambda^{*T}A - \lambda^{*T}\lambda^* x^{*T} = \mathbf{0} \Leftrightarrow x^{*T} = \frac{\lambda^{*T}A}{\lambda^{*T}\lambda^*} \Leftrightarrow x^* = \frac{A^T\lambda^*}{\lambda^{*T}\lambda^*}.$$

The inconsistency of the original system guarantees that $\lambda^T\lambda \neq 0$. To complete the proof, from (**??**) to (**??**) and the complementarity relations (**??**) we obtain the second relation.$\diamondsuit$

Theorem 5.2. *Let (H^*, p^*, x^*) be a local minimizer of (**??**) and λ^* the corresponding Lagrange multiplier. Let $C_a = [A_a, b_a]$ be the coefficient matrix of the active constraints and λ_a the corresponding multipliers . Then $y^{*T} = [x^{*T}, 1]$ and λ_a are the proper vectors respectively of $C_a^T C_a$ and $C_a C_a^T$, associated with the eigenvalue $(\lambda^{*T}\lambda^* y^{*T} y^*)$.*

Proof - If we let $y^T = [x^T, 1]$ and $C = [A, -b]$ from (**??**) and (**??**) we obtain

$$y^* = \frac{C^T\lambda^*}{\lambda^{*T}\lambda^*}, \tag{20}$$

$$\lambda^* = \frac{(Cy^*)^+}{y^{*T}y^*}. \tag{21}$$

The result follows directly by replacing respectively λ^* in the numerator of (**??**) by (**??**) and y^* in the numerator of (**??**) by (**??**). $\diamondsuit$

The last theorem guarantees that if the singular value $\sqrt{(y^{*T}y^*\lambda_a^T\lambda_a)}$ has multiplicity one, then $\lambda^*/\|\lambda^*\|$ and $y^*/\|y^*\|$ are its corresponding left and right singular vectors and so there exists a t such that

$$t \le n+1, \ \ f^* = \sigma_t^2, \ \ v_{n+1,t} \neq 0 \text{ and } (H_a^*, p_a^*) = -\sigma_t u_t v_t^T. \tag{22}$$

The following theorem shows under which conditions $t = n+1$, and so the TLS correction and solution, for the TLS problem defined by the active constraints, is a local minimizer of (**??**).

Theorem 5.3. *Let (H^*, p^*, x^*) be a local minimizer of (**??**). Then a TLS solution $x^* = x_{TLS}(C_a)$ for C_a exists. If this solution is unique then*

$$v_{n+1,n+1} \neq 0, \tag{23}$$

$$f^* = \lambda^{*T}\lambda^* y^T y = \sigma_{n+1}^2, \tag{24}$$

$$(H^*, p^*) = \begin{bmatrix} H_a^*, p_a^*, \\ \mathbf{0} \end{bmatrix}, (H_a^*, p_a^*) = -\sigma_{n+1}u_{n+1}v_{n+1}^T, \tag{25}$$

$$\begin{bmatrix} x^* \\ 1 \end{bmatrix} = v_{n+1}/v_{n+1,n+1} \ , \begin{bmatrix} \lambda_1^* \\ \vdots \\ \lambda_a^* \end{bmatrix} = \sigma_{n+1}u_{n+1}v_{n+1,n+1}, \tag{26}$$

where σ_{n+1} is the smallest singular value of C_a, v_{n+1} and u_{n+1} are the corresponding right and left singular vectors.

Proof- Again we assume that the first m_a constraints are active. This means that for $\{i : i = 1, \ldots, m_a\}$, $A_{i,.} + H_{i,.}x^* = b_i - p_i$ and only two possible situations may occur

$$
\begin{aligned}
A_{i,.}x^* &> b_i \Rightarrow [H_{i,.}, p_i] \neq \mathbf{0}, \\
A_{i,.}x^* &= b_i \Rightarrow [H_{i,.}, p_i] = \mathbf{0}.
\end{aligned}
$$

So for $i \in \{k : k = m_a + 1, \ldots, m\}$, $A_{i,.}x^* < b_i$. Let us suppose that in (**??**) $t < n + 1$. If we consider

$$
\beta = \sqrt{1 - \epsilon^2} v_t + \epsilon v_{n+1} \quad , 0 \leq \epsilon \leq 1,
$$

then we have

$$
\|\beta\|^2 = (1 - \epsilon^2)\|v_t\|^2 + 2\epsilon\sqrt{1 - \epsilon^2} v_t^T v_{n+1} + \epsilon^2 \|v_{n+1}\|^2 = 1,
$$

since $\|v_t\| = \|v_{n+1}\| = 1$ and $v_t^T v_{n+1} = 0$, which follows from the fact that they are both singular vectors. For $x(\epsilon) = [\beta_1, \ldots, \beta_n]^T / \beta_{n+1}$, and ϵ small enough $A_{i,.}x(\epsilon) < b_i$, so $-C\beta\beta^T$ ($C = [A, -b]$) is a feasible correction for the active set of constraints with cost $(1 - \epsilon^2)\sigma_t^2 + \epsilon^2 \sigma_{n+1}^2$. For $\epsilon > 0$, if $\sigma_{n+1}^2 < \sigma_t^2$, then

$$
\|C\beta\beta^T\|^2 - \sigma_t^2 = \epsilon^2(\sigma_{n+1}^2 - \sigma_t^2) < 0,
$$

which means that (H^*, p^*, x^*) cannot be a local minimizer. So $t = n + 1$ or $\sigma_t = \sigma_{n+1}$, which means that $x_{TLS}(C_a)$ exists. This solution is unique if $\sigma_n(C_a) > \sigma_{n+1}(C_a)$ and in this case conditions (**??**) to (**??**) come as a direct consequence.$\diamondsuit$

Given this theorem a local minimizer can be obtained algebraically, once the active constraints are known. In practice we do not have this information. In fact we would like for a certain set of equations, to verify whether they define a local minimizer. The following subsection deals with this situation.

5.3 Sufficient conditions based on the TLS

In Theorem **??** we present sufficient conditions for a local minimizer. For sake of organization and readability we present two preliminary results in the next theorems.

Theorem 5.4. *Let J_a be the Jacobian of the non degenerate active constraints of problem (**??**) (we may assume without loss of generality that these are the first m_a constraints) and let $y^T = [x^T, 1]$. Then J_a is given by*

$$
J_a = \left[\begin{array}{cccccc}
\begin{matrix} y^T & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & y^T & \ldots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \ldots & y^T \end{matrix} & \mathbf{0}_{m_a \times (m - m_a)(n+1)} & A_a + H_a
\end{array} \right]. \qquad (27)
$$

Proof - Let the constraints of (**??**) be written as

$$g_i(h_{1.}, p_1, h_{2.}, p_2, \cdots, h_{m.}, p_m, x^T) \le 0 \text{ for } i = 1, \ldots, m.$$

Then

$$g_i(h_{11}, \cdots, h_{1n}, p_1, \cdots, h_{m1}, \cdots, h_{mn}, p_m, x^T) = \sum_{j=1}^{n} (a_{ij} + h_{ij})\, x_j - b_i + p_i$$

and so

$$\frac{\partial g_i}{\partial h_{kj}} = \begin{cases} x_j & \text{if } i = k \\ 0 & \text{otherwise} \end{cases}, \frac{\partial g_i}{\partial p_k} = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{otherwise} \end{cases}, \frac{\partial g_i}{\partial x_j} = (a_{ij} + h_{ij}),$$

which proves (**??**).$\diamondsuit$

Theorem 5.5. *Let J_a be as defined in* **??**. *Let $s_i \in \Re^{n+1}$ $(i = 1, \ldots, n)$ be a set of n independent vectors orthogonal to y, w_j $(j = 1, \ldots, (m-m_a)(n+1))$ a set of independent vectors in $\Re^{(m-m_a)(n+1)}$ and B a base of $\Re^n$. Then for*

$$B_1 = \mathit{blockdiag}\left((s_1, \ldots, s_n), \ldots, (s_1, \ldots, s_n)\right),$$

$$B_2 = \left(w_1, \ldots, w_{(m-m_a)(n+1)}\right),$$

$$B_3 = -\frac{1}{\|y\|^2} \left(\mathit{blockdiag}\,(y, \ldots, y)\right) (A_a + H_a) B.$$

the matrix

$$Z = \begin{bmatrix} B_1 & \mathbf{0}_{d_1 \times d_2} & B_3 \\ \mathbf{0}_{d_2 \times d_1} & B_2 & \mathbf{0}_{d_2 \times n} \\ \mathbf{0}_{n \times d_1} & \mathbf{0}_{n \times d_2} & B \end{bmatrix}, \quad \begin{pmatrix} d_1 = m_a(n+1) \\ d_2 = (m - m_a)(n+1) \end{pmatrix}$$

is a base for the null space of J_a.

Proof - The dimension of the null space of J_a is $m(n+1) + n - m_a$. Since

$$J_a \begin{bmatrix} 0 \\ \vdots \\ s_i \\ \vdots \\ 0 \\ \mathbf{0}_{d_2} \\ \mathbf{0}_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ y^T s_i \\ \vdots \\ 0 \end{bmatrix} = \mathbf{0}_{m_a}, \quad \forall i = 1, \ldots, n, \qquad (d_2 = (m - m_a)(n+1)),$$

where s_i may lie between component $k(n+1)+1$ and $(k+1)(n+1)$, for $k = 0, \ldots, m_a - 1$ (remaining components being zero), we construct a total of $m_a n$ independent vectors in $\Re^{m(n+1)+n}$ for Z. We can add to this base $(m-m_a)(n+1)$ independent vectors of $\Re^{m(n+1)+n}$, since

$$J_a \begin{bmatrix} \mathbf{0}_{d_1} \\ w_i \\ \mathbf{0}_n \end{bmatrix} = \mathbf{0}_{m_a}, \quad \forall i = 1, \ldots, (m-m_a)(n+1), \qquad (d_1 = m_a(n+1)).$$

The remaining n vectors are the columns of $[B_3^T, \mathbf{0}_{d_2}, B]^T$, since

$$J_a \begin{bmatrix} B_3 \\ \mathbf{0}_{d_2} \\ B \end{bmatrix} = -I(A_a+H_a)B-(A_a+H_a)B = \mathbf{0}_{m_a}, \qquad (d_2 = (m-m_a)(n+1)).$$

Taking B as a base of $\Re^n$ we ensure the independency of these last n vectors. The independency of the all set of columns of Z can be trivially verified. $\diamondsuit$

Before presenting the next theorem we would like to remark that the second order sufficient conditions for local minimizers impose that the product $Z^T \nabla^2 L Z$ be positive definite on the null space of J_a, where $\nabla^2 L$ is the Hessian matrix [**?**].

Theorem 5.6. *Consider* $[A, -b] = \begin{bmatrix} C_a \\ C_r \end{bmatrix}$, *where* $C_a = [A_a, -b_a]$ *and* $C_r = [A_r, -b_r]$. *Let* u_{n+1} *and* v_{n+1} *be the left and right singular vector of* C_a *corresponding to the minimal singular value* σ_{n+1}, *and* $v_{n+1,n+1} \neq 0$. *For*

$$\begin{bmatrix} x^* \\ 1 \end{bmatrix} = \frac{1}{v_{n+1,n+1}} v_{n+1},$$

if $\lambda_a = \sigma_{n+1} v_{n+1,n+1} u_{n+1} \geq \mathbf{0}$, $A_r x^* \leq b_r$ *and* $\sigma_n > \sigma_{n+1}$, *then* (H^*, p^*, x^*) *is a local minimizer of (**??**), with*

$$(H^*, p^*) = \begin{bmatrix} H_a^*, p_a^* \\ \mathbf{0} \end{bmatrix},$$

$$(H_a^*, p_a^*) = -\sigma_{n+1} u_{n+1} v_{n+1}^T.$$

Proof - The proof that x^* and λ^* verify the necessary conditions is straightforward. Let the singular value decomposition of C_a be given by $U \Sigma V^T$, where $V = [v_1, \ldots, v_n, v_{n+1}]$. We will consider that $V(n)$ consists of the first n rows and n columns of V. From theorem **??** letting

$$y = \frac{1}{v_{n+1,n+1}} v_{n+1} \quad \text{and} \quad s_i = v_i \ (i = 1, \ldots, n),$$

then

$$Z^T \nabla^2 L Z = \begin{bmatrix} I_{d_3} & \mathbf{0}_{d_3 \times d_2} & \Phi B \\ \mathbf{0}_{d_2 \times d_3} & I_{d_2} & \mathbf{0}_{d_2 \times n} \\ B^T \Phi^T & \mathbf{0}_{n \times d_2} & \Upsilon \end{bmatrix}, \begin{pmatrix} d_2 = (m-m_a)(n+1) \\ d_3 = m_a n \end{pmatrix}$$

where

$$\Phi = \begin{bmatrix} v_1^T & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ v_n^T & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & v_1^T & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ \mathbf{0} & v_n^T & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} & v_1^T \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} & v_n^T \end{bmatrix} \begin{bmatrix} \lambda_1 I_n \\ \mathbf{0} \\ \vdots \\ \lambda_a I_n \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \lambda_1 V(n) \\ \vdots \\ \lambda_a V(n) \end{bmatrix}$$

and

$$\Upsilon = \frac{1}{\|y\|^2} B^T (A_a + H_a)^T (A_a + H_a) B + B^T \psi^T B + B^T \psi B,$$

where

$$\begin{aligned} \psi &= \frac{1}{\|y\|^2} \left[\ \lambda_1 I_n, \mathbf{0}, \lambda_2 I_n, \mathbf{0}, \ldots, \lambda_a I_n, \mathbf{0}\ \right] blockdiag(y^T, \ldots, y^T)(A_a + H_a) \\ &= \frac{1}{\|y\|^2} \left[\ \lambda_1 x, \lambda_2 x, \ldots, \lambda_a x\ \right] (A_a + H_a) \\ &= \frac{1}{\|y\|^2} x[\lambda_1, \lambda_2, \ldots, \lambda_a](A_a + H_a). \end{aligned}$$

Since from (**??**) $\lambda^T(A+H) = \mathbf{0}$, then $\psi = \mathbf{0}_{n\times n}$ and

$$Z^T \nabla^2 L Z = \begin{bmatrix} I_{d_3} & \mathbf{0}_{d_3\times d_2} & \Phi B \\ \mathbf{0}_{d_2\times d_3} & I_{d_2} & \mathbf{0}_{d_2\times n} \\ B^T\Phi^T & \mathbf{0}_{n\times d_2} & \frac{1}{\|y\|^2} B^T (A_a + H_a)^T (A_a + H_a) B \end{bmatrix}.$$

Let θ be an eigenvalue of this matrix and $[\alpha_1^T, \alpha_2^T, \ldots, \alpha_a^T, \beta_1^T, \ldots, \beta_{(m-m_a)}^T, \gamma^T]^T$, $\alpha_i \in \Re^n$, $\beta_i \in \Re^{n+1}$, and $\gamma \in \Re^n$ a correspondent proper vector. Then the following equations must hold

$$\begin{aligned} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_a \end{bmatrix} + \Phi B \gamma &= \theta \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_a \end{bmatrix} \\ \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_{m-m_a} \end{bmatrix} &= \theta \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_{m-m_a} \end{bmatrix} \qquad (28) \\ B^T \Phi^T \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_a \end{bmatrix} + \tfrac{1}{\|y\|^2} B^T (A_a + H_a)^T (A_a + H_a) B \gamma &= \theta \gamma. \end{aligned}$$

We can verify that $\theta = 1$ is an eigenvalue with multiplicity

$$n(m_a - 1) + (m - m_a)(n + 1) = m(n + 1) - n - m_a.$$

Now in order to establish that $Z^T \nabla^2 L Z$ is positive definite we will analyze the sign of the remaining $2n$ eigenvalues. For $\theta \neq 1$ the first m_a equations of (**??**) become

$$\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_a \end{bmatrix} = \frac{-\Phi B \gamma}{1 - \theta}.$$

Using this substitution in (**??**) we obtain the following equation

$$-B^T \frac{\Phi^T \Phi}{1 - \theta} B\gamma + \frac{B^T (A_a + H_a)^T (A_a + H_a) B}{\|y\|^2} \gamma = \theta\gamma \Leftrightarrow$$

$$B^T \left(\frac{(A_a + H_a)^T (A_a + H_a)}{\|y\|^2} - \frac{\Phi^T \Phi}{1 - \theta} \right) B\gamma = \theta\gamma.$$

We can conclude that θ must also be an eigenvalue of

$$B^T \left(\frac{(A_a + H_a)^T (A_a + H_a)}{\|y\|^2} - \frac{\Phi^T \Phi}{1 - \theta} \right) B.$$

Since we are only interested in establishing the sign of the remaining eigenvalues, using Sylvester's Law of Inertia and taking into consideration that B is nonsingular, this question tantamounts to study the inertia of

$$\frac{(A_a + H_a)^T (A_a + H_a)}{\|y\|^2} - \frac{\Phi^T \Phi}{1 - \theta}. \tag{29}$$

Since

$$(A_a + H_a)^T (A_a + H_a) = V(n) diag(\sigma_1^2, \ldots, \sigma_n^2) V(n)^T$$

and

$$\Phi^T \Phi = \sum_{i=1}^{q} \lambda_i^2 V(n) V(n)^T,$$

then (**??**) is equivalent to:

$$V(n) \left(\frac{diag(\sigma_1^2, \ldots, \sigma_n^2)}{\|y\|^2} - \frac{\sum_{i=1}^{q} \lambda_i^2 I_n}{1 - \theta} \right) V(n)^T. \tag{30}$$

Since

$$\sum \lambda_i^2 = \sigma_{n+1}^2 v_{n+1,n+1}^2$$

and

$$\|y\|^2 = \frac{1}{v_{n+1,n+1}^2} v_{n+1}^T v_{n+1} = \frac{1}{v_{n+1,n+1}^2},$$

it follows that (**??**) reduces to

$$v_{n+1,n+1}^2 V(n) \left(diag(\sigma_1^2 - \frac{\sigma_{n+1}^2}{1-\theta}, \ldots, \sigma_n^2 - \frac{\sigma_{n+1}^2}{1-\theta}) \right) V(n)^T. \tag{31}$$

In order to use, again, the Sylvester Law of Inertia it is necessary to prove that $V(n)$ is nonsingular, which is not straightforward. This result depends, among others, on the fact that we are assuming that $\sigma_n > \sigma_{n+1}$. Finally we only have to study the sign of the eigenvalues of the diagonal matrix

$$\begin{bmatrix} \sigma_1^2 - \frac{\sigma_{n+1}^2}{1-\theta} & 0 & \ldots & 0 \\ 0 & \sigma_2^2 - \frac{\sigma_{n+1}^2}{1-\theta} & \ldots & 0 \\ 0 & 0 & \ldots & \sigma_n^2 - \frac{\sigma_{n+1}^2}{1-\theta} \end{bmatrix}. \tag{32}$$

We know that θ is real because of symmetry properties. Supposing that $\theta \leq 0$, then (**??**) should have a negative eigenvalue value as well, i.e.

$$\sigma_i^2 - \frac{\sigma_{n+1}^2}{1-\theta} < 0 \Leftrightarrow \theta > 1 - \frac{\sigma_{n+1}^2}{\sigma_i^2},$$

for some $i \in \{1, \ldots, n\}$. On the other hand, we know that

$$\sigma_i^2 \geq \sigma_{n+1}^2 > 0 \Leftrightarrow 1 > 1 - \frac{\sigma_{n+1}^2}{\sigma_i^2} \geq 0.$$

If

$$\sigma_n > \sigma_{n+1}$$

then

$$1 - \frac{\sigma_{n+1}^2}{\sigma_i^2} > 0$$

which contradicts $\theta \leq 0$. $\diamondsuit$

One small remark must be made before the next section. We have presented necessary conditions based on the SVD of the matrix of the active constraints, while the sufficient conditions depend on the SVD of the matrix of active non degenerate constraints. Yet this is of no consequence because the singular value and right singular vector do not change if a row orthogonal to this vector is added to the matrix. The left singular vector will be increased by a zero component. This shows that the sufficient conditions presented by Vatolin are not necessary. Another question not answered yet has to do with the failure of some of the hypotheses of the previous theorem, such as the occurrence of $\sigma_{n+1} = 0$ and $v_{n+1,n+1} = 0$. The next theorem analyzes these questions and is an useful tool in the search for local minimizers.

Theorem 5.7. *The number of active constraints in problem (??) is no less than $n+1$.*

Proof- This proof can be restated as $m_a \geq n+1$ in the previous theorem. If $m_a < n+1$ and if r is the rank of C_a, then $r \leq m_a < n+1$, $\sigma_{r+1}, \ldots, \sigma_{n+1} = 0$ and $(v_{r+1}, \ldots, v_{n+1})$ span the null space of C_a. Then $\tilde{z} = \sum_{i=r+1}^{n} \alpha_i v_i$ is a solution of the homogeneous system $C_a z = \mathbf{0}$. If $\tilde{z}_{n+1} \neq 0$, then $[\tilde{z}_1, \ldots, \tilde{z}_n]/\tilde{z}_{n+1}$ is a solution of $A_a x = b_a$ and since it is feasible for the remaining constraints then the original problem is consistent. Else, if $v_{n+1,r+1} = \cdots = v_{n+1,n+1} = 0$ $(v_{n+1,r} \neq 0)$ then, as proved for the TLS, this implies that the problem (??) has no solution (inf $= 0$). Corrections of arbitrarily small cost to $A_a x = b_a$ can render this system feasible. For

$$v = \sum_{j=\mathbf{r}}^{n+1} \epsilon_j v_j,$$

with $\sum_{j=r}^{n+1} \epsilon_j^2 = 1$, $\|v\| = 1$, and because $(C_a - C_a v v^T)v = \mathbf{0}$, $-C_a v v^T$ is a feasible correction for $A_a x = b_a$. Considering that $C_a = U \Sigma V^T$, then the correction, cost and solution can be given respectively by

$$C_a v v^T = \sum_{i=1}^{n+1} \sigma_i u_i v_i^T (\sum_{j=r}^{n+1} \epsilon_j v_j) = \sum_{i=r}^{n+1} \sigma_i u_i \epsilon_i,$$

$$\|C_a v v^T\|^2 = \sum_{i=r}^{n+1} \sigma_i^2 \epsilon_i^2 = \sigma_r^2 \epsilon_r^2,$$

and

$$\begin{aligned} x(\epsilon) &= \frac{\sum_{j=r}^{n+1} \epsilon_j v_j'}{\epsilon_r v_{n+1,r}}, \text{ with } v_i^T = \begin{cases} [v_r'^T, v_{n+1,r}] \text{ for } i = r \\ [v_i'^T, 0] \text{ for } i = r+1, \ldots, n+1 \end{cases} \\ &= \frac{v_r'}{v_{n+1,r}} + \frac{\sum_{j=r+1}^{n+1} \epsilon_j v_j'}{\epsilon_r v_{n+1,r}}. \end{aligned}$$

When $\epsilon_r \to 0$, $x(\epsilon) \to \pm\infty$, and $\|C_a v v^T\|^2 \to 0$. If for the remaining constraints $A_{i,.} x(\epsilon) \leq b_i$, when $\epsilon_r \to 0$, then the problem with inequalities also has no minimum and $inf = 0$. If the problem

$$\begin{array}{rl} \min_{\epsilon_r > 0} & \sigma_r^2 \epsilon_r^2 \\ \text{s.t.} & A_{i.}x(\epsilon) \leq b_i \text{ for } i = q+1, \ldots, m, \\ & \sum_{j=r}^{n+1} \epsilon_j^2 = 1, \end{array}$$

has a solution, then there is $i \in \{q+1, \ldots, m\}$ such that the i-th constraint is blocking the growth of $x(\epsilon)$. This constraint is then active and should be in the set of active constraints, which contradicts the definition of the active set of constraints. So, we conclude that if the system as a local optima then $\sigma_{n+1} > 0$.$\diamondsuit$

6 Conclusion and further work

In this paper we studied the problem of the minimal correction of an infeasible set of linear inequalities accordingly to the Frobenius norm. Although a result on this problem regarding necessary and sufficient conditions is mentioned on a previous work, we were able to complete this work with an approach using the KKT conditions. We presented sufficient conditions different than those presented in [**?**] and established the connections to the similar problem with only equalities constraints, known as the TLS problem. We proved that all local minimizers correspond to a TLS solution applied to some set of active constraints, of size no less then $n+1$, and we studied under which conditions the problem fails to have a minimum. In [**?**] we have presented an heuristic approach to solve the problem, based on the TLS, and in [**?**] an hybrid method combining a gradient search to identify the set of active constraints, and then the application of the TLS to these constraints. With these new results we prove that this methodology can lead us to a local optimal solution, which before we were not able to classify as such. Now we know exactly which conditions should be met in order to classify a set of constraints as the active set for a local minimizer. This constitutes a indispensable tool for a tree search procedure for global optima, which is under development. In general the problem of inconsistency of linear systems opens a broad class of problems worth investigate: inconsistent problems whit some substructure that cannot be changed and problems were we have integrality constraints on the coefficients. We believe this work is an important step to achieve the mentioned goals.

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