Problem 1: (12pts)

Part 1: Determine if the following statements are TRUE or FALSE (2 pts each)

a) [FALSE] The convergence of the LMS is adversely effected by the noise level on the desired signal.

b) [FALSE] LMS converges faster when the condition number of the input auto-correlation matrix increases.

c) [TRUE] Excess error of the LMS filters is proportional to the power of input signal.

Part 2: Answer the following questions. (3pts each)

e) Explain how to implement a channel equalizer with LMS adaptive filter.

Training sequence is transmitted; the received signal is filtered through LMS filter to minimize the MSE error between received signal and transmitted sequence. After the adaptation, Decision-Feedback-Equalization can be applied to track channel variations.

f) Explain the relation between the shape of the equi-error level curves of LMS cost function (ellipsis, circular etc.) and the convergence of the algorithm.

Circular shaped level curves imply that the convergence will be uniform for the entries of coefficient vector.

Ellipsis indicate that the condition number of $\underline{R}_x$ is large since the step size (μ) has to be selected to guarantee the convergence of all modes; μ will be selected to guarantee the convergence of slowest mode, therefore overall convergence will be slow.

Problem 2: (20pts)

A parameter to be estimated is modeled as Gaussian distributed with mean 1 and variance $\sigma_x^2 = 2$.

a) Determine the minimum mean square error estimate of the parameter from the statistical model.

b) A noisy measurement on the estimation paramater is given as $y = 2x + v$, where v is Gaussian with the mean 2 and variance σ_v^2.

Find the minimum MSE estimate of x in the form $\hat{x} = a + by$.

What is the value of $\hat{x}$ when the measurement noise has negligible small power ? What is the value of $\hat{x}$ when measurement noise is large ? Compare your results with part a.

c) How would you modify the parameters of estimator in part b if noise source was uniform ?

a) $\hat{x} = c$, $\frac{d}{dc} E\{(x-\hat{x})^2\} = 0 \rightarrow E\{\hat{x}\} = E\{x\} \rightarrow c = \bar{x} = 1.$

b) $\hat{x} = a + by$, $\frac{d}{da} E\{(x-\hat{x})^2\} = 0 \rightarrow E\{(x-\hat{x}).1\} = 0 \rightarrow$

$\frac{d}{db} E\{(x-\hat{x})^2\} = 0 \rightarrow E\{(x-\hat{x})y\} = 0 \rightarrow$

$\rightarrow E\{\hat{x}\} = 1 \rightarrow a + bE\{y\} = 1$

$\rightarrow E\{\hat{x}y\} = E\{xy\} \rightarrow aE\{y\} + bE\{y^2\} = E\{xy\}$

$E\{y\} = 2\underbrace{E\{x\}}_{1} + \underbrace{E\{v\}}_{2} = 4$

$E\{y^2\} = E\{4x^2 + 4xv + v^2\} = 4\underbrace{E\{x^2\}}_{(2+1^2)} + 4\underbrace{E\{x\}E\{v\}}_{2} + \underbrace{E\{v^2\}}_{\sigma_v^2+4} = 24 + \sigma_v^2$

$E\{xy\} = E\{2x^2 + xv\} = 2(2+1^2) + 2 = 8$

$$\begin{bmatrix} 1 & 4 \\ 4 & 24+\sigma_v^2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \end{bmatrix} \rightarrow \begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{8+\sigma_v^2} \begin{bmatrix} 24+\sigma_v^2 & -4 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{8+\sigma_v^2} \begin{bmatrix} \sigma_v^2 - 8 \\ 4 \end{bmatrix}$$

$$\hat{x} = \frac{\sigma_v^2 - 8}{8+\sigma_v^2} + \frac{4}{8+\sigma_v^2} \cdot y$$

$$\hat{x} = 1 + \frac{4}{8+\sigma_v^2}(y-4) \longrightarrow \boxed{\hat{x} = \bar{x} + \frac{4}{8+\sigma_v^2}(y-\bar{y})}$$

If $\sigma_v^2 \to 0 \longrightarrow \underset{\text{constant bias}}{\hat{\theta}} = \bar{\theta} = 2 \qquad y = 2x + 2 \rightarrow x = \frac{y-2}{2}$

(no noise)

$$\hat{x} = 1 + \frac{4}{8}(y-4) = \frac{y-2}{2} \rightarrow \hat{x} = x$$

$\sigma_v^2 \to \infty \longrightarrow \hat{x} = \bar{x}$

(unreliable "y")

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c) No modification. Part b) depends on first and second order statistics, not on p.d.f.'s.

Problem 3: (20 pts)

The desired signal $d[n]$ is needed to do the filter coefficients updates. In some applications the exact knowledge of the desired signal may not be available. Blind methods make use of the known $d[n]$ characteristics instead of its exact values to do the coefficient updates. Blinds methods are mostly used in communications applications.

a) Modulation schemes such as BPSK, FSK have constant magnitude signal transmitted signals, $||d[n]||^2 = 1$. For these signals the usual cost function can be modified to

$$J = E\left\{\left((1 - (\mathbf{w}_n{}^T\mathbf{x}_n)^2\right)^2\right\}$$

Derive the corresponding LMS update equations for the cost function.

b) Derive the LMS update equation for the following modification:

$$J = E\left\{(1 - |\mathbf{w}_n{}^T\mathbf{x}_n|)^2\right\}$$

a) $J = E\left\{\big(\overbrace{1 - (\underline{w}_n^T \underline{x}_n)^2}^{e(n)}\big)^2\right\}$

$\nabla_{\underline{w}_n} J = E\left\{ e(n)\left[-2(\underline{w}_n^T \underline{x}_n)\,\underline{x}_n\right]\right\}$

Steepest Descent.

$w_{n+1} = w_n + \mu\left(-\nabla_w J(\underline{w}_n)\right)$

$\underline{w}_{n+1} = \underline{w}_n + 2\mu\, E\left\{ e(n)\, \underline{w}_n^T \underline{x}_n \underline{x}_n\right\}$

LMS

$\underline{w}_{n+1} = \underline{w}_n + \left[2\mu\, e(n)\, \underline{w}_n^T \underline{x}_n\right] \underline{x}_n$

b) $J = E\left\{\big(\overbrace{1 - |\underline{w}_n^T \underline{x}_n|}^{e(n)}\big)^2\right\}$

$\nabla_{\underline{w}_n} J(\underline{w}_n) = E\left\{ e(n)\left[-2\,\underline{x}_n\, \mathrm{sgn}(\underline{w}_n^T \underline{x}_n)\right]\right\}$

Steepest Descent + LMS

$\underline{w}_{n+1} = \underline{w}_n + \left[2\mu\, e(n)\, \mathrm{sgn}(\underline{w}_n^T \underline{x}_n)\right] \underline{x}_n$

Problem 4: (15 pts)

Multiresolution methods produce different resolution representations of an input waveform. The following multiresolution representation technique uses the average of samples in a block as the coarse representation and the difference between the average and the original signal values as the detail representation. The figure given below illustrates the method.

The block length of the method is selected 2. The signals s_1, s_2 denote the original signal values, c denotes the coarse value, d_1, d_2 denote the detail values in a block.

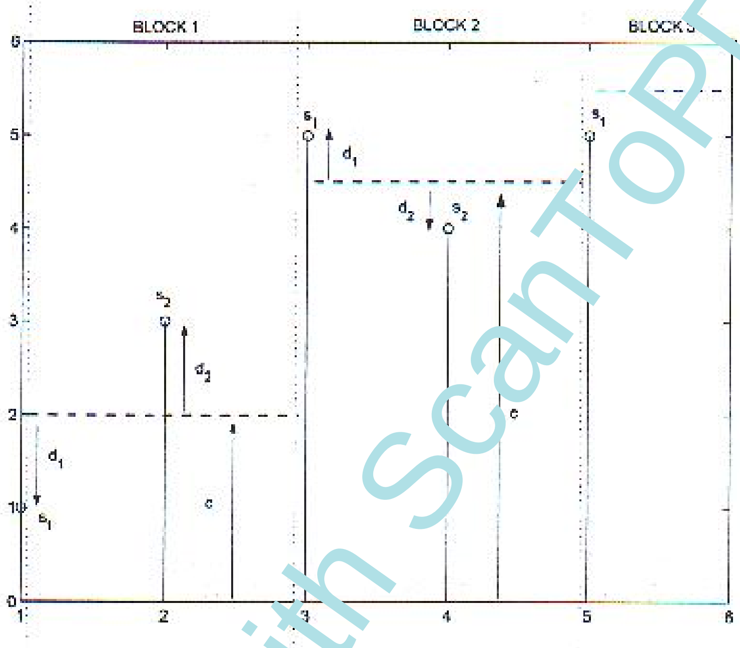


Figure 1: Coarse and Detail Representations

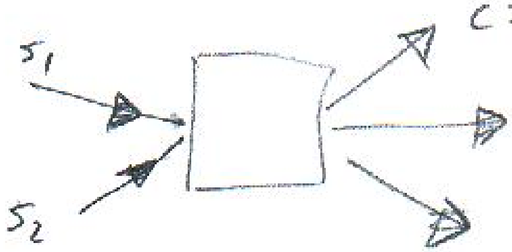
The method produces three output samples for every two samples. Such methods are called over-determined methods. The reconstruction or the synthesis technique of overdetermined methods is not unique.

Assume that coarse and detail signals are quantized and the quantized versions are used at reconstruction. Assume $\{c_1, d_1, d_2\}$ are quantized with the identical quantizers introducing zero mean and unit variance white noise.

a) Find the MSE error for s_1 and s_2 for the reconstruction method:

$$s_1 = c + d_1$$
$$s_2 = c + d_2$$

b) Find the min MSE reconstructor. Calculate its MSE.

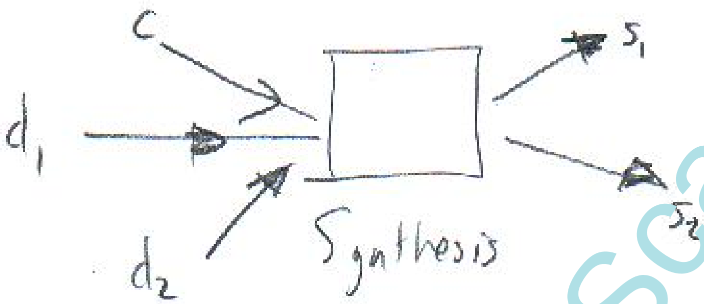


$$c = \frac{s_1 + s_2}{2}$$

$$d_1 = \frac{s_1 - s_2}{2}$$

$$d_2 = \frac{s_2 - s_1}{2}$$

Analysis



Synthesis

$$\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} c \\ d_1 \\ d_2 \end{bmatrix} \quad ; \quad \begin{aligned} s_1 &= c + \alpha d_1 + (\alpha - 1) d_2 \\ s_2 &= c + (\beta - 1) d_1 + \beta d_2 \end{aligned}$$

(α, β any real number)

Analysis

Synthesis (not unique)

Quantization noise

a) $s_1^r = Q[c] + Q[d_1] = (c + v_1) + (d_1 + v_2) \rightarrow E\{(s_1 - s_1^r)^2\} = E\{v_1^2\} + E\{v_2^2\}$

$s_2^r = Q[c] + Q[d_2] = (c + v_1) + (d_1 + v_3)$

$= 2$

b) Method 1: Uses synthesis equations and take derivative wrt α (β for s_2) and select α to minimize the variance.

Method 2:

zero mean, white

$$\underline{A}\begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} c \\ d_1 \\ d_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \rightarrow$$

min MSE constructor is

$$\begin{bmatrix} s_1^r \\ s_2^r \end{bmatrix} = (A^T A)^{-1} A^T \begin{bmatrix} c \\ d_1 \\ d_2 \end{bmatrix}$$

$$\begin{bmatrix} s_1^r \\ s_2^r \end{bmatrix} = \left(\frac{1}{4} \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 1 \end{bmatrix} \right)^{-1} \frac{1}{2} \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} \varphi[c] \\ \varphi[d_1] \\ \varphi[d_2] \end{bmatrix}$$

$$= \left(\frac{1}{4} \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \right)^{-1} \frac{1}{2} \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} \varphi[c] \\ \varphi[d_1] \\ \varphi[d_2] \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} \varphi[c] \\ \varphi[d_1] \\ \varphi[d_2] \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 4 & 2 & -2 \\ 4 & -2 & 2 \end{bmatrix} \begin{bmatrix} \varphi[c] \\ \varphi[d_1] \\ \varphi[d_2] \end{bmatrix}$$

$$\left.\begin{aligned} s_1^r &= c^q + \frac{1}{2} d_1^q - \frac{1}{2} d_2^q \\ s_2^r &= c^q - \frac{1}{2} d_1^q + \frac{1}{2} d_2^q \end{aligned}\right\}$$

hence $\alpha = 1/2$ for $\beta = 1/2$ min MSE reconstruction

$$E\{(s_1^r - s_1)^2\} = E\left\{\left(v_1 + \frac{v_2}{2} - \frac{v_3}{2}\right)^2\right\} = 1 + \frac{1}{2} = \frac{3}{2}$$

$$E\{(s_2^r - s_2)^2\} = 3/2$$

Note that, you can find the error covariance matrix for LS estimator under noisy condition in your notes.

$$(\underline{A}^T \underline{A})^{-1} \underbrace{\sigma_v^2}_{1} = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}$$

variance of estimation error

Problem 5: (20 pts)

The signal $d[n]$ is corrupted by the additive noise source $v[n]$, $x[n] = d[n] + v[n]$. The power spectrum density of the desired signal and noise are given below. Assume that $v[n]$ is zero mean and uncorrelated with $d[n]$.

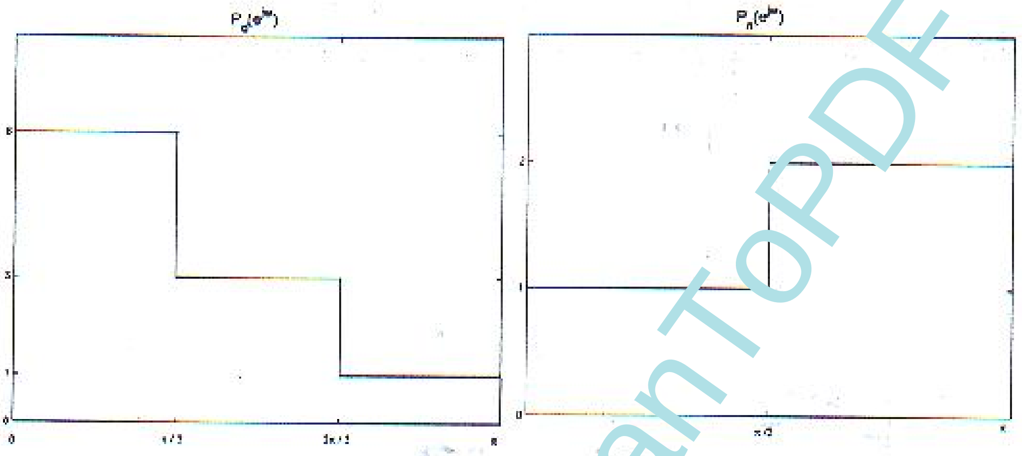


Figure 2: Power Spectrum Density of Desired and Noise Signal

a) Calculate the MSE error and SNR of the signal $x[n]$ before filtering.

b) Find the optimum non-causal Wiener filter to estimate $d[n]$. Calculate MSE error and SNR at the output.

c) Find the optimum scaling parameter for the estimator $\hat{d}[n] = w_0 x[n]$ minimizing the MSE. Calculate the MSE error and SNR.

d) Find the optimum two tap filter to estimate $d[n]$, $\hat{d}[n] = w_0 x[n] + w_1 x[n-1]$. Find the MSE and SNR.

a) $x(n) = d(n) + v(n) \rightarrow$ MSE $\rightarrow E\{[x(n) - d(n)]^2\} = E\{v^2(n)\} = r_v(0)$

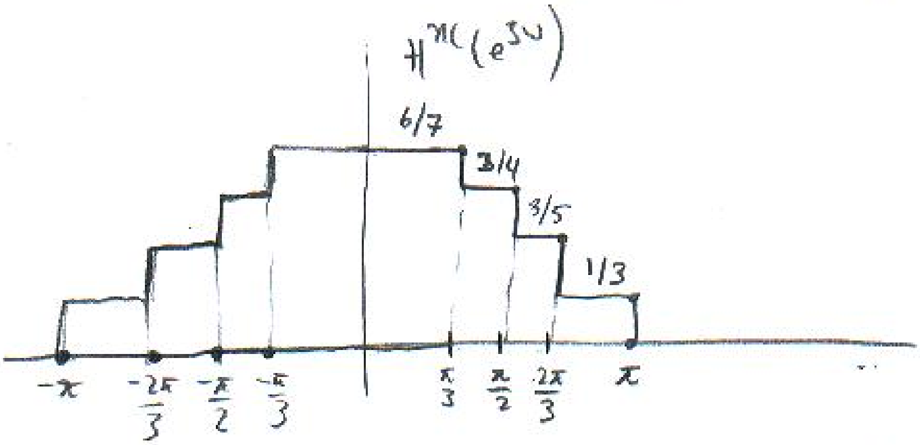
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} P_v(e^{jw})\,dw$$

$$= \frac{1}{2\pi}\left[2\left(\frac{\pi}{2} + \pi\right)\right]$$

$$= 3/2 = 1.5$$

SNR $\rightarrow$ $$\frac{E\{d(n)^2\}}{E\{v^2(n)\}} = \frac{2\left(\frac{1}{2\pi}\left(2\pi + \pi + \frac{\pi}{3}\right)\right)}{3/2} = \frac{20}{9} = 2.\bar{2}$$

b) $H^{nc}(e^{j\omega}) = \dfrac{P_{dy}(e^{j\omega})}{P_x(e^{j\omega})} = \dfrac{P_d(e^{j\omega})}{P_d(e^{j\omega}) + P_v(e^{j\omega})}$



$$\hat{x}(n) = h^{nc}(n) * d(n) + h^{nc}(n) * v(n)$$

$$x(n) \longrightarrow \boxed{H^{nc}(z)} \longrightarrow \hat{x}(n)$$

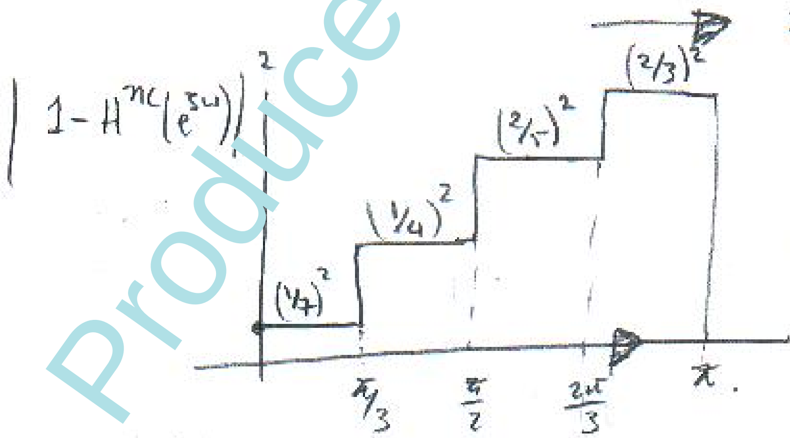
$$d(n) - \hat{x}(n) = \underbrace{\left[\delta(n) - h^{nc}(n)\right] * d(n)}_{A} - \underbrace{h^{nc}(n) * v(n)}_{B}$$

$$E\left\{\left[d(n) - \hat{x}(n)\right]^2\right\} = E\left\{A^2 - 2AB + B^2\right\}$$

$$= E\{A^2\} + E\{B^2\} \quad ; \quad \left(\begin{matrix} v(n) \\ d(n) \\ \text{uncor.} \end{matrix}\right)$$

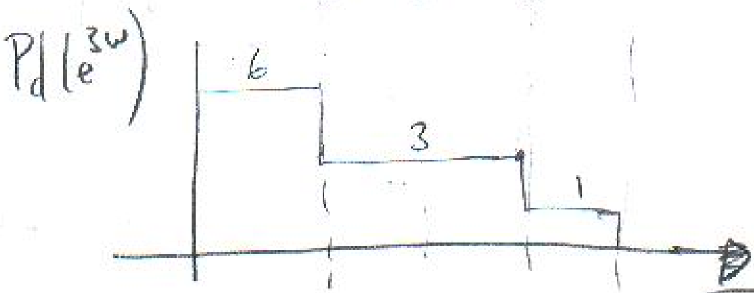
$E\{A^2\} \Rightarrow$ $\quad d(n) \longrightarrow \boxed{\delta(n) - h^{nc}(n)} \longrightarrow e_1(n)$ $\quad E\{A^2\} = E\{e_1^2(n)\} \longrightarrow$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} P_d(e^{j\omega}) \left|1 - H^{nc}(e^{j\omega})\right|^2 d\omega$$



$$= \frac{1}{\cancel{2\pi}} \cdot \cancel{2} \left[\frac{6}{49} \frac{\cancel{\pi}}{3} + \frac{3}{16} \frac{\cancel{\pi}}{6} + \frac{12}{25} \frac{\cancel{\pi}}{6} + \dots \right.$$

$$\left. \dots + \frac{4}{9} \frac{\cancel{\pi}}{3} \right]$$



$$= \frac{2}{49} + \frac{1}{32} + \frac{2}{25} + \frac{4}{27}$$

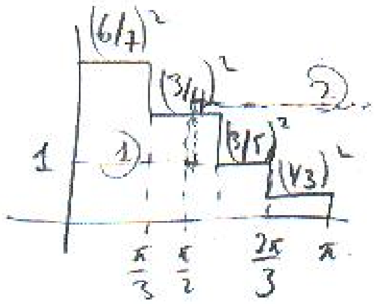
$$= 0{,}3$$

$$\boxed{E\{A^2\} = 0{,}3}$$

$E\{B^2\} \Rightarrow \quad \vartheta(n) \to \boxed{h^{nc}(n)} \to e_2(n)$

$$E\{B^2\} = E\{e_2^2(n)\}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} P_v(e^{j\omega})\,|H^{nc}(e^{j\omega})|^2\, d\omega$$



$$= \frac{1}{\cancel{2\pi}}\cdot\cancel{2}\left[\frac{36}{49}\frac{\pi}{3} + \frac{9}{16}\frac{\pi}{6} + \frac{18}{25}\frac{\pi}{6} + \frac{2}{9}\frac{\pi}{3}\right]$$

$$= \frac{12}{49} + \frac{3}{32} + \frac{3}{25} + \frac{2}{27}$$

$$E\{B^2\} = 0.53$$

MSE → for $H^{nc}(e^{j\omega})$ filtering → $\boxed{E\{A^2\} + E\{B^2\} = 0.83}$

SNR → $d(n) \to \boxed{H^{nc}} \to \hat{s}(n)$

$\vartheta(n) \to \boxed{H^{nc}} \to \hat{\vartheta}(n)$

$$\frac{E\{[\hat{s}(n)]^2\}}{E\{\hat{\vartheta}(n)^2\}} = \frac{E\{(\hat{s}(n))^2\}}{E\{B^2\}}$$

$$E\{\hat{s}(n)^2\} = \frac{1}{\cancel{2\pi}}\cdot\cancel{2}\left[\frac{36}{49}\cdot 6\frac{\pi}{3} + \frac{9}{16}\cdot 3\cdot\frac{\pi}{6} + \frac{9}{25}\cdot 3\frac{\pi}{6} + \frac{1}{9}\cdot\frac{\pi}{3}\right]$$

$$= \frac{72}{49} + \frac{9}{32} + \frac{9}{50} + \frac{1}{27} = 1.96$$

SNR after $H^{nc}(e^{j\omega})$ filtering → $\boxed{\frac{1.96}{0.53} = 3.70}$

c) $\hat{d}(n) = w_0 x(n) \rightarrow \underline{\underline{R}}_x \cdot \underline{w} = \underline{r}_{dx} \rightarrow r_x(0) \cdot w_0 = r_{dx}(0)$ $\underline{5.4}$

$$\left[r_d(0) + r_v(0) \right] \cdot w_0 = r_d(0)$$

$$r_d(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_d(e^{j\omega}) = 10/3 \quad \text{(found in part a)}$$

$$r_v(0) = \frac{3}{2} \quad \text{(found in part a)}$$

$$W_0 = \frac{10/3}{10/3 + \frac{3}{2}} = \frac{20}{29} \quad \text{(optimal coef)}.$$

min MSE after w_0 scaling: $E\left\{ \left[\hat{d}(n) - d(n) \right]^2 \right\} = E\left\{ \left(\frac{20}{29}(d(n) + v(n)) - d(n) \right)^2 \right\}$

$$= E\left\{ \left[\frac{20}{29} v(n) - \frac{9}{29} d(n) \right]^2 \right\}$$

$$= \left(\frac{20}{29} \right)^2 \cdot E\{v_n^2\} + \left(\frac{9}{29} \right)^2 E\{d_n^2\}$$

$$= \left(\frac{20}{29} \right)^2 \cdot \frac{3}{2} + \left(\frac{9}{29} \right)^2 \cdot \frac{10}{3}$$

$$\boxed{= 1.03}$$

SNR after w_0 multiplication $\rightarrow \boxed{SNR = 20/9} = \dfrac{E\{w_0^2\, d(n)^2\}}{E\{w_0^2\, v(n)^2\}}$

5.5

d) $r_d(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_d(e^{jw})e^{jwk}\,dw$

$r_d(0) = 10/3$

$r_d(1) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_d(e^{jw})\cos(w)\,dw$

$= \frac{1}{\pi}\int_{0}^{\pi} P_d(e^{jw})\cos(w)\,dw$

$= \frac{1}{\pi}\left[\int_0^{\pi/3} 6\cos w\,dw + \int_{\pi/3}^{2\pi/3} 3\cos w\,dw + \int_{2\pi/3}^{\pi} 1\cos w\,dw\right]$

$= \frac{1}{\pi}\left[6\sin\left(\frac{\pi}{3}\right) + 3\sin\left(\frac{2\pi}{3}\right) - 3\sin\frac{\pi}{3} + \cancel{\sin(\pi)}^{0} - \sin\left(\frac{2\pi}{3}\right)\right]$

$= \frac{1}{\pi}\left[3\sin(\pi/3) + 2\sin(2\pi/3)\right] = \frac{5}{\pi}\cdot\frac{\sqrt{3}}{2}$

Similarly

$r_v(0) = 3/2$

$r_v(1) = \frac{1}{\pi}\left[1\sin(\pi/2) + \cancel{2\sin(\pi)}^{0} - 2\sin(\pi/2)\right]$

$r_v(1) = -1/\pi$

$$\underbrace{\begin{bmatrix} \frac{10}{3}+\frac{3}{2} & \frac{5}{\pi}\frac{\sqrt{3}}{2}-\frac{1}{\pi} \\ \frac{5}{\pi}\frac{\sqrt{3}}{2}-\frac{1}{\pi} & \frac{10}{3}+\frac{3}{2} \end{bmatrix}}_{\underline{R}_x}\underbrace{\begin{bmatrix} w_0 \\ w_1 \end{bmatrix}}_{\underline{w}} = \underbrace{\begin{bmatrix} 10/3 \\ \frac{5}{\pi}\frac{\sqrt{3}}{2} \end{bmatrix}}_{\underline{r}_{dy}} \rightarrow \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} = \begin{bmatrix} 0.65 \\ 0.14 \end{bmatrix}$$

5.6

min MSE after $\{w_0, w_1\}$ filtering $= r_d(0) - \underline{r}_d^T \cdot \underline{w}$

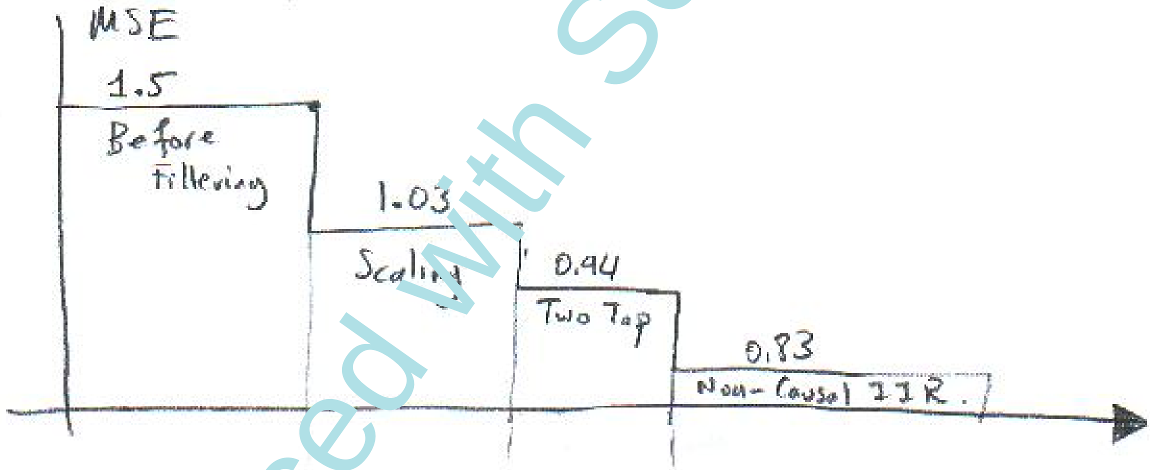
$$\boxed{= 0.94}$$

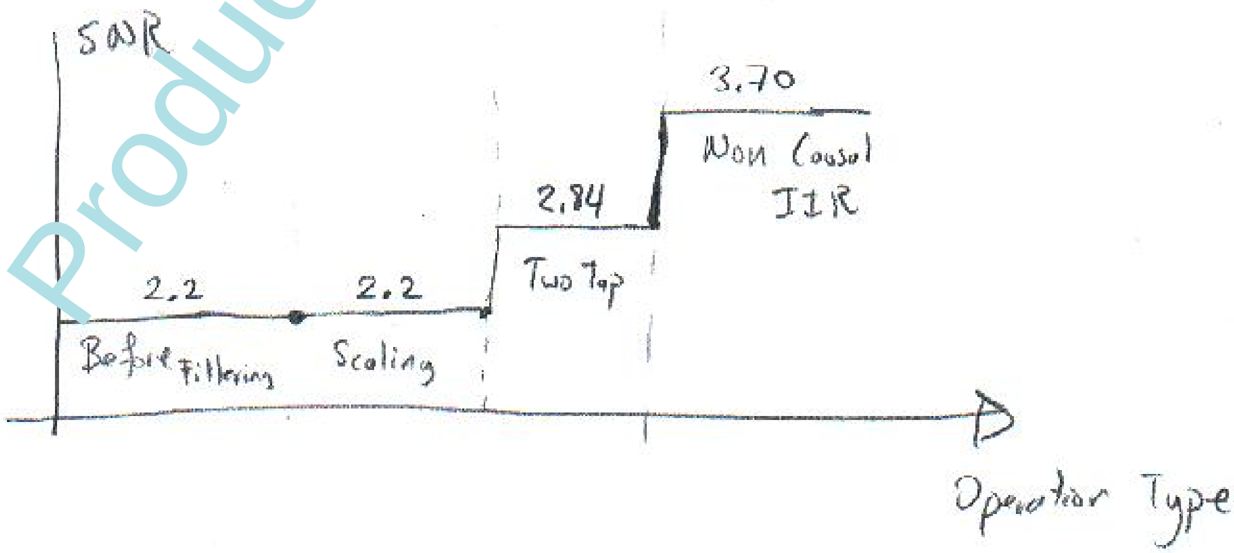
SNR after $\{w_0, w_1\}$ filtering $\Rightarrow$ $SNR = \dfrac{\underline{w}^T \underline{\underline{R}}_d \underline{w}}{\underline{w}^T \underline{\underline{R}}_v \underline{w}}$

$$\underline{\underline{R}}_d = \begin{bmatrix} r_d(0) & r_d(1) \\ r_d(1) & r_d(0) \end{bmatrix}, \quad \underline{\underline{R}}_v = \begin{bmatrix} r_v(0) & r_v(1) \\ r_v(1) & r_v(0) \end{bmatrix}$$

$$w = [0.65 \quad 0.14]^T$$

$$\boxed{SNR = 2.84}$$





Problem 6: (15 pts)

a) Set-up and run Kalman filtering iterations to find the solution of the equation system:

$$x + y = 4$$
$$x - y = 2$$

use a, b instead of x, y
$a + b = 4$
$a - b = 2$

Take $x(0) = y(0) = 0$ and $\mathbf{P}(0) = P\mathbf{I}$. ($P$ is an arbitrary scalar, $\mathbf{I}$ is the identity matrix)

b) The signal x[n] is periodic with fundamental period of 100 samples. We would like to estimate first 3 harmonics of the signal. Set-up a Kalman filter to estimate the harmonics. (Do not run any iterations)

a) $\underline{x}(n) = \underline{x}(n-1)$ where $\underline{x}(n) = \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{\text{unknowns of the system.}}$

$y(1) \overset{=4}{=} [1 \quad 1]\, x(1)$

$y(2) = [1 \quad -1]\, x(2)$
$\Rightarrow 2$

Init: $\hat{\underline{x}}(0) = \underline{0}$ $\quad P(0) = P\underline{\underline{I}}$

Iterations:

$\underbrace{x(n|n-1)}_{x(n)} = \underbrace{x(n-1|n-1)}_{x(n-1)}$ $\rightarrow$ call $x(n|n-1) = x(n|n-1) = x(n)$

(1) $\underbrace{x(1) = x(0) \qquad P(1|0) = \underbrace{P(0|0)}_{P\underline{\underline{I}}} + 0}_{\text{Time Update.}}$

$K(1) = P\underline{\underline{I}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \left[[1 \ 1] P \underline{\underline{I}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right]^{-1}$

$K(1) = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$

$P(1|1) = \left[I - \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} [1 \ 1] \right] P\underline{\underline{I}}$

$\underbrace{P(1|1) = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & +1/2 \end{bmatrix} P}_{\text{Measurement update.}}$

$$\boxed{\hat{\underline{x}}(1) = \hat{x}(0) + \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} [4 - 0] = \begin{bmatrix} 2 \\ 2 \end{bmatrix}}$$

2) $x(2) = x(1)$

$\underbrace{P(2|1) = P(1|1)}_{\text{Time Update}}$

$$K(2) = \frac{\begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix} P \begin{bmatrix} 1 \\ -1 \end{bmatrix}}{\left([1 \ -1] \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix} P \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right)}$$

$$= \frac{\begin{bmatrix} 1 \\ -1 \end{bmatrix} P}{2P}$$

$$K(2) = \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix}$$

$$P(2|2) = \left[I - \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix} [1 \ -1] \right] \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \frac{P}{2}.$$

$$= \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} P/2$$

$$P(2|2) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \leftarrow$$ (exact solution no error on estimation)

$$\hat{x}(2) = \hat{x}(1) + \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix} \left(2 - [1 \ -1] \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right)$$

$$\hat{x}(2) = \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\boxed{\hat{x}(2) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}}$$

b) $y(n) = c_1 e^{j\omega_0 n} + c_2 e^{j2\omega_0 n} + c_3 e^{j3\omega_0 n}$

$$x(n) = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$\boxed{\begin{aligned} x(n) &= x(n-1) \\ y(n) &= [e^{j\omega_0 n} \ e^{j2\omega_0 n} \ e^{j3\omega_0 n}] x(n) + v(n) \end{aligned}}$$

State Space Model for estimation.