Step 4: Assume $K = 4, \cdots$

$$\vdots$$

Continue until K is large enough!

Remark:

- RELAX is found to perform <u>better than existing</u> <u>high-resolution algorithms</u>, especially in obtaining better $\hat{\beta}_k$, $k = 1, \cdots, K$

- RELAX is more robust to the choice of K and the data model errors.

The **RELAX** Algorithm

Step 1: Assume $K = 1$. Obtain $\hat{\omega}_1$ and $\hat{\beta}_1$ from $y(n)$.

Step 2: Obtain $y_2(n)$ by assuming $K=2$ and using $\hat{\omega}_1$ and $\hat{\beta}_1$ obtained from Step 1.

Iterate until converg. $\begin{cases} \text{Obtain } \hat{\omega}_2 \text{ and } \hat{\beta}_2 \text{ from } y_2(n) \\ \text{Obtain } y_1(n) \text{ by using } \hat{\omega}_2 \text{ and } \hat{\beta}_2 \\ \text{and reestimate } \hat{\omega}_1 \text{ and } \hat{\beta}_1 \text{ from } y_1(n) \end{cases}$

Step 3: Assume $K = 3$.

Obtain $y_3(n)$ from $\hat{\omega}_1, \hat{\beta}_1, \hat{\omega}_2, \hat{\beta}_2$. Obtain $\hat{\omega}_3$ and $\hat{\beta}_3$ from $y_3(n)$.

Obtain $y_1(n)$ from $\hat{\omega}_2, \hat{\beta}_2, \hat{\omega}_3, \hat{\beta}_3$. Reestimate $\hat{\omega}_1$ and $\hat{\beta}_1$ from $y_1(n)$.

Obtain $y_2(n)$ from $\hat{\omega}_1, \hat{\beta}_1, \hat{\omega}_3, \hat{\beta}_3$. Reestimate $\hat{\omega}_2$ and $\hat{\beta}_2$ from $y_2(n)$.

Iterate until g does not decrease "significantly" anymore !

Remarks:

$$\sum_{n=0}^{N-1} y_k(n)e^{-j\omega_k n} \quad \underline{\text{is the DTFT of} \quad y_k(n)}!$$

(can be computed via FFT and zero-padding.)

- $\underline{\hat{\omega}_k \text{ corresponds to the peak of the Periodogram}}!$

- $\hat{\beta}_k$ is the $\underline{\text{peak height}}$ (complex number!) of $\underline{\text{the DTFT of } y_k(n)}$ (at $\hat{\omega}_k$) divided by N.

Preparation:

$$\text{Let} \quad y_k(n) = y(n) - \sum_{i=1, i \neq k}^{K} \hat{\beta}_i e^{j\hat{\omega}_i n}$$

* $\hat{\beta}_i$ and $\hat{\omega}_i$, $i \neq k$, are <u>assumed</u> given, <u>known</u>, or <u>estimated</u>.

$$\text{Let} \quad g_k = \sum_{n=0}^{N-1} \left| y_k(n) - \beta_k e^{j\omega_k n} \right|^2.$$

* Minimizing g_k gives:

$$\hat{\omega}_k = \quad \text{argmax}_{\omega_k} \left| \sum_{n=0}^{N-1} y_k(n) e^{-j\omega_k n} \right|^2.$$

$$\hat{\beta}_k = \frac{1}{N} \sum_{n=0}^{N-1} y_k(n) e^{-j\omega_k n} \Bigg|_{\omega_k = \hat{\omega}_k}.$$

Remarks:

- Concentrating out $\{\beta_k\}$ gives

$$\hat{\boldsymbol{\omega}} = \quad \text{argmax}_{\omega} \left[\mathbf{y}^H \mathbf{B} \left(\mathbf{B}^H \mathbf{B} \right)^{-1} \mathbf{B}^H \mathbf{y} \right]$$

$$\hat{\boldsymbol{\beta}} = \left. \left(\mathbf{B}^H \mathbf{B} \right)^{-1} \mathbf{B}^H \mathbf{y} \right|_{\omega = \hat{\omega}}.$$

- Concentrating out $\{\beta_k\}$, instead of simplifying the problem, actually complicates the problem.

- The RELAX algorithm is a <u>relaxation - based</u> optimization approach.

- RELAX is both <u>computationally and conceptually simple</u>.

Remarks:

- $\hat{\beta}_k$ and $\hat{\omega}_k,\quad k = 1, \cdots, K$ are found by minimizing g .

- When $e(n)$ is zero mean Gaussian white noise, this NLS method is the ML method.

- When $e(n)$ is non-white noise, NLS method gives asymptotically $(N \to \infty)$ statistically efficient estimates of $\hat{\omega}_k$ and $\hat{\beta}_k$ despite the fact that NLS is not an ML method for this case.

- The non-linear minimization is a difficult problem.

Sinusoidal Parameter Estimation in the Presence
of Colored Noise via RELAX

$$y(n) = \sum_{k=1}^{K} \beta_k e^{j\omega_k n} + e(n)$$

- β_k = Complex amplitudes, unknown.
- ω_k = Unknown frequencies.
- $e(n)$ = Unknown AR or ARMA noise.

Consider the Non-linear least-squares (NLS) method.

$$g = \sum_{n=0}^{N-1} \left| y(n) - \sum_{k=1}^{K} \beta_k e^{j\omega_k n} \right|^2$$

Remarks:

- $\hat{\mathbf{S}}_2 \mathbf{\Psi} \approx \hat{\mathbf{S}}_1$

can also be solved with <u>Total Least Squares Method</u>

- Since $\mathbf{\Psi}$ is $K \times K$ matrix, we <u>do not</u> need to pick K roots nearest the unit circle, which could be wrong roots.

- ESPRIT does not require the search over parameter space, as required by Spectral MUSIC.

All of these remarks make ESPRIT a recommended method !

Let $\mathbf{S}_1$ and $\mathbf{S}_2$ be formed from $\mathbf{S}$ the same way as $\mathbf{B}_1$ and $\mathbf{B}_2$ from $\mathbf{A}_M$

Recall: $\mathbf{S} = \mathbf{A}_M \mathbf{C}$

$$\Rightarrow \begin{cases} \mathbf{S}_1 = \mathbf{B}_1\mathbf{C} = \mathbf{B}_2\mathbf{D}\mathbf{C}. \\ \mathbf{S}_2 = \mathbf{B}_2\mathbf{C} \end{cases}$$

$$\mathbf{S}_2\mathbf{C}^{-1} = \mathbf{B}_2$$

$$\Rightarrow \quad \mathbf{S}_1 = \mathbf{S}_2\mathbf{C}^{-1}\mathbf{D}\mathbf{C} \triangleq \mathbf{S}_2\boldsymbol{\Psi}.$$

$$\Rightarrow \qquad \boxed{\boldsymbol{\Psi} = \left(\mathbf{S}_2^H\mathbf{S}_2\right)^{-1}\mathbf{S}_2^H\mathbf{S}_1.}$$

The diagonal elements of $\mathbf{D}$ are the eigenvalues of $\boldsymbol{\Psi}$.

Steps of ESPRIT: Step 1: $\hat{\boldsymbol{\Psi}} = \left(\hat{\mathbf{S}}_2^H\hat{\mathbf{S}}_2\right)^{-1}\hat{\mathbf{S}}_2^H\hat{\mathbf{S}}_1$

Step 2: Frequency estimates are angular positions of the eigenvalues of $\hat{\boldsymbol{\Psi}}$.

ESPRIT Method

(Estimation of Signal Parameters by Rotational Invariance Techniques)

$$\mathbf{A}_M = \begin{bmatrix} 1 & \cdots & 1 \\ e^{-j\omega_1} & \cdots & e^{-j\omega_K} \\ \vdots & & \\ e^{-j(M-1)\omega_1} & \cdots & e^{-j(M-1)\omega_K} \end{bmatrix}$$

Let $\mathbf{B}_1$ = first $M-1$ rows of $\mathbf{A}_M$, $\quad \mathbf{B}_2$ = last $M-1$ rows of $\mathbf{A}_M$.

$$\mathbf{B}_2 \mathbf{D} = \mathbf{B}_1,$$

$$\mathbf{D} = \begin{bmatrix} e^{j\omega_1} & & 0 \\ & \ddots & \\ 0 & & e^{j\omega_K} \end{bmatrix}$$

Pisarenko Method = (MUSIC with $M = K + 1$)

Remarks:

- Pisarenko method is not as good as MUSIC.

- M in MUSIC should not be too large due to poor accuracy of $\hat{r}(k)$ for large k.

Step 2a (Spectral MUSIC): Determine the frequency estimates as the locations of the K highest peaks of the MUSIC spectrum

$$\frac{1}{\mathbf{a}_M^H(\omega)\hat{\mathbf{G}}\hat{\mathbf{G}}^H\mathbf{a}_M(\omega)}, \quad \omega \in [-\pi, \pi]$$

Step 2b (Root MUSIC): Determine the frequency estimates as angular positions (phases) of K (pairs of reciprocal) roots of equation

$$\mathbf{a}_M^H\left(z^{-1}\right)\hat{\mathbf{G}}\hat{\mathbf{G}}^H\mathbf{a}_M(z) = 0$$

that are closest to the unit circle

$$\mathbf{a}_M(z) = \begin{bmatrix} 1 & z^{-1} & \cdots & z^{-M+1} \end{bmatrix}^T, i.e., \quad \mathbf{a}_M(z)|_{z=e^{j\omega}} = \mathbf{a}_M(\omega)$$

MUSIC:

The true frequency values $\{\omega_k\}_{k=1}^{K}$ are the only solutions of

$$\mathbf{a}_M^H(\omega)\mathbf{G}\mathbf{G}^H\mathbf{a}_M(\omega) = 0.$$

$$\mathbf{a}_M(\omega) = \begin{bmatrix} 1 \\ e^{-j\omega} \\ \vdots \\ e^{-j\omega(M-1)} \end{bmatrix}.$$

Steps in MUSIC:

Step 1: Compute $\hat{\mathbf{R}} = \frac{1}{N}\sum_{n=M}^{N} \tilde{\mathbf{y}}_M(n)\tilde{\mathbf{y}}_M^H(n)$, and its eigendecomposition.

Form $\hat{\mathbf{G}}$ whose columns are the eigenvectors of $\hat{\mathbf{R}}$ that correspond to the $M - K$ smallest eigenvalues of $\hat{\mathbf{R}}$.

Remark:

Let the linearly independent K columns of $\mathbf{A}_M$ define $\underline{K\text{-dimensional signal subspace}}$

* Then the eigenvectors of $\mathbf{R}$ that correspond to the $M - K$ $\underline{\text{smallest}}$ eigenvalues are $\underline{\text{orthogonal}}$ to the signal subspace.

* The eigenvectors of $\mathbf{R}$ that correspond to the K largest eigenvalues of $\mathbf{R}$ span the same signal subspace as $\mathbf{A}_M$.

$\Rightarrow \mathbf{A}_M = \mathbf{SC}$ for a $K \times K$ non-singular $\mathbf{C}$.

Let $\mathbf{s}_1, \cdots, \mathbf{s}_K$ be the eigenvectors of $\mathbf{R}$ that correspond to $\lambda_1, \cdots, \lambda_K$.

Let $\mathbf{S} = [\mathbf{s}_1, \cdots, \mathbf{s}_K]$

Let $\mathbf{s}_{K+1}, \cdots, \mathbf{s}_M$ be the eigenvectors of $\mathbf{R}$ that correspond to $\lambda_{K+1}, \cdots, \lambda_M$.

Let $\mathbf{G} = [\mathbf{s}_{K+1}, \cdots, \mathbf{s}_M]$

$$\mathbf{RG} = \mathbf{G} \begin{bmatrix} \sigma^2 & & 0 \\ & \ddots & \\ 0 & \cdots & \sigma^2 \end{bmatrix} = \sigma^2 \mathbf{G}$$

$$
\begin{aligned}
\mathbf{RG} &= \left(\mathbf{A}_M \mathbf{P} \mathbf{A}_M^H + \sigma^2 \mathbf{I} \right) \mathbf{G} \\
&= \mathbf{A}_M \mathbf{P} \mathbf{A}_M^H \mathbf{G} + \sigma^2 \mathbf{G} \\
&\Rightarrow \mathbf{A}_M \mathbf{P} \mathbf{A}_M^H \mathbf{G} = 0 \quad \Rightarrow \mathbf{A}_M^H \mathbf{G} = 0
\end{aligned}
$$

$$\mathbf{R} = \mathbf{A}_M \mathbf{P} \mathbf{A}_M^H + \sigma^2 \mathbf{I},$$

$\Rightarrow$

$$\mathbf{P} = \begin{bmatrix} \alpha_1^2 & & 0 \\ & \ddots & \\ 0 & & \alpha_K^2 \end{bmatrix}.$$

Remarks: • rank $(\mathbf{A}_M \mathbf{P} \mathbf{A}_M^H) = K$ if $M \geq K$.

• If $M \geq K, \mathbf{A}_M \mathbf{P} \mathbf{A}_M^H$ has K positive eigenvalues and $M - K$ zero eigenvalues. We shall consider $M \geq K$ below.

• Let the positive eigenvalues of $\mathbf{A}_M \mathbf{P} \mathbf{A}_M^H$ be denoted

$$\tilde{\lambda}_1 \geq \tilde{\lambda}_2 \geq \cdots \geq \tilde{\lambda}_K$$

The eigenvalues of $\mathbf{R}$ are:

$$\text{Two groups} \begin{cases} \lambda_k = \tilde{\lambda}_k + \sigma^2, & k = 1, \cdots, K. \\ \lambda_k = \sigma^2, & k = K+1, \cdots, M \end{cases}$$

$$\tilde{\mathbf{x}}(n) = \begin{bmatrix} x_1(n) \\ \vdots \\ x_K(n) \end{bmatrix},$$

$$\tilde{\mathbf{w}}_M(n) = \begin{bmatrix} w(n) \\ \vdots \\ w(n-M+1) \end{bmatrix}$$

$$\boxed{\tilde{\mathbf{y}}_M(n) = \mathbf{A}_M \tilde{\mathbf{x}}(n) + \tilde{\mathbf{w}}_M(n)}$$

$$
\begin{aligned}
\text{Let} \quad \mathbf{R} &= E\left\{ \tilde{\mathbf{y}}_M(n) \tilde{\mathbf{y}}_M^H(n) \right\} \\
&= E\left\{ \mathbf{A}_M \tilde{\mathbf{x}}(n) \tilde{\mathbf{x}}^H(n) \mathbf{A}_M^H \right\} \\
&\quad + E\left\{ \tilde{\mathbf{w}}_M(n) \tilde{\mathbf{w}}_M^H(n) \right\}
\end{aligned}
$$

Pisarenko and MUSIC Methods

Remark: Pisarenko method is a special case of MUSIC (<u>Mu</u>ltiple <u>S</u>ignal <u>C</u>lassification) method.

Recall:

$$
\tilde{\mathbf{y}}_M(n) = \begin{bmatrix} y(n) \\ y(n-1) \\ \vdots \\ y(n-M+1) \end{bmatrix}
$$

$$
\mathbf{A}_M = \begin{bmatrix} 1 & \cdots & 1 \\ e^{-j\omega_1} & \cdots & e^{-j\omega_K} \\ \vdots & \cdots & \vdots \\ e^{-j(M-1)\omega_1} & \cdots & e^{-j(M-1)\omega_K} \end{bmatrix},
$$

$$
\begin{bmatrix} \hat{\tilde{a}}_1 \\ \vdots \\ \hat{\tilde{a}}_L \end{bmatrix} = -\mathbf{V}_1\boldsymbol{\Sigma}_1^{-1}\mathbf{U}_1^H \begin{bmatrix} y(L+1) \\ \vdots \\ y(N-1) \end{bmatrix} . \qquad (\dagger)
$$

Summary of SVD Prony Estimator.

Step 1. Form $\mathbf{Y}$ and compute SVD of $\mathbf{Y}$

Step 2. Determine $\hat{\tilde{\mathbf{a}}}$ with $(\dagger)$

Step 3. Compute the roots from $\hat{\tilde{\mathbf{a}}}$. Pick K roots that are nearest the unit circle. Obtain frequency estimates as phases of the roots.

Remark: • Although one cannot guarantee that the K roots nearest the unit circle give the best frequency estimates, empirical results show that this is true most often.

• A more accurate method is obtained by "cleaning" (i.e., rank K approximation of) the matrix $[\mathbf{Y} \;\vdots\; \mathbf{y}]$.

Remark: If $N - L - 1 \geq K$ and $L \geq K$, $\mathbf{X}$ is of rank K.

From (*)

$$
\underbrace{\begin{bmatrix} y(L) & \cdots & y(1) \\ \vdots & & \vdots \\ y(N-2) & \cdots & y(N-L-1) \end{bmatrix}}_{\mathbf{Y}} \begin{bmatrix} \tilde{a}_1 \\ \vdots \\ \tilde{a}_L \end{bmatrix} \approx - \underbrace{\begin{bmatrix} y(L+1) \\ \vdots \\ y(N-1) \end{bmatrix}}_{\mathbf{y}}
$$

Remark: A rank K approximation of $\mathbf{Y}$ has " Noise Cleaning " effect.

$$
\text{Let} \quad \mathbf{Y} = \begin{bmatrix} \mathbf{U}_1 & \mathbf{U}_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{\Sigma}_1 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Sigma}_2 \end{bmatrix} \begin{bmatrix} \mathbf{V}_1^H \\ \mathbf{V}_2^H \end{bmatrix} \begin{matrix} K \\ L-K \end{matrix}
$$

$$\begin{bmatrix} x(1) \\ \vdots \\ x(N-L-1) \end{bmatrix} = \begin{bmatrix} 1 & .. & 1 \\ e^{j\omega_1} & .. & e^{j\omega_K} \\ \vdots & \vdots & \\ e^{j(N-L-2)\omega_1} & .. & e^{j(N-L-2)\omega_K} \end{bmatrix} \begin{bmatrix} \beta_1 e^{j\omega_1} \\ \vdots \\ \beta_K e^{j\omega_K} \end{bmatrix}$$

$$\Rightarrow \mathbf{X} = \begin{bmatrix} 1 & \cdots & 1 \\ e^{j\omega_1} & \cdots & e^{j\omega_K} \\ \vdots & \vdots & \\ e^{j(N-L-2)\omega_1} & \cdots & e^{j(N-L-2)\omega_K} \end{bmatrix} \begin{bmatrix} \beta_1 & & 0 \\ & \ddots & \\ 0 & & \beta_K \end{bmatrix}$$

$$\begin{bmatrix} e^{jL\omega_1} & \cdots & e^{j\omega_1} \\ e^{jL\omega_2} & \cdots & e^{j\omega_2} \\ \vdots & \vdots & \vdots \\ e^{jL\omega_K} & \cdots & e^{j\omega_K} \end{bmatrix}$$

Remark: • If $w(n) = 0$, Eq $(*)$ holds exactly.

• If $w(n) = 0$, Eq $(*)$ gives <u>EXACT</u> frequency estimates.

<u>Consider next the rank of</u>

$$\mathbf{X} = \begin{bmatrix} x(L) & \cdots & x(1) \\ \vdots & & \\ x(N-2) & \cdots & x(N-L-1) \end{bmatrix}$$

Note

$$\begin{bmatrix} x(0) \\ \vdots \\ x(N-L-2) \end{bmatrix} = \begin{bmatrix} 1 & \cdots & 1 \\ e^{j\omega_1} & \cdots & e^{j\omega_K} \\ \vdots & & \vdots \\ e^{j(N-L-2)\omega_1} & \cdots & e^{j(N-L-2)\omega_K} \end{bmatrix} \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_K \end{bmatrix}$$

SVD Prony Method

Recall: $\left(1 + \tilde{a}_1 z^{-1} + \cdots + \tilde{a}_L z^{-L}\right) y(n)$

$$= (1 + \tilde{a}_1 z^{-1} + \cdots + \tilde{a}_L z^{-L}) w(n). \quad (L \geq K)$$

At not too low SNR,

$$\begin{bmatrix} y(n) & y(n-1) & \cdots & y(n-L) \end{bmatrix} \begin{bmatrix} 1 \\ \tilde{a}_1 \\ \vdots \\ \tilde{a}_L \end{bmatrix} \approx 0$$

$$\begin{bmatrix} y(L) & y(L-1) & \cdots & y(0) \\ y(L+1) & y(L) & \cdots & y(1) \\ \vdots & & & \\ y(N-1) & y(N-2) & \cdots & y(N-L-1) \end{bmatrix} \begin{bmatrix} 1 \\ \tilde{a}_1 \\ \vdots \\ \tilde{a}_L \end{bmatrix} \approx 0 \qquad (*)$$

To minimize $||\mathbf{Ax} - \mathbf{b}||_2^2$, we must have,

$$\mathbf{\Sigma}_1 \mathbf{y}_1 = \mathbf{U}_1^H \mathbf{b}$$

$\Rightarrow$
$$\boxed{\mathbf{y}_1 = \mathbf{\Sigma}_1^{-1} \mathbf{U}_1^H \mathbf{b}\ .}$$

Note that $\mathbf{y}_2$ can be anything and $||\mathbf{Ax} - \mathbf{b}||_2^2$ is not affected.

Let $\mathbf{y}_2 = 0$ so that $||\mathbf{y}||_2^2 = ||\mathbf{x}||_2^2 = $ minimum.

$$\Rightarrow \mathbf{V}^H \mathbf{x} = \mathbf{y} = \begin{bmatrix} \mathbf{y}_1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \mathbf{x} = \mathbf{Vy} = \begin{bmatrix} \mathbf{V}_1 & \mathbf{V}_2 \end{bmatrix} \begin{bmatrix} \mathbf{y}_1 \\ 0 \end{bmatrix} = \mathbf{V}_1 \mathbf{y}_1$$

$\Rightarrow$
$$\boxed{\mathbf{x} = \mathbf{V}_1 \mathbf{\Sigma}_1^{-1} \mathbf{U}_1^H \mathbf{b}.}$$

$$||\mathbf{x}||_2^2 = ||\mathbf{y}||_2^2 = \quad \text{minimum}$$

SVD of A:

$$\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^H = \begin{bmatrix} \mathbf{U}_1 & \mathbf{U}_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{\Sigma}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{V}_1^H \\ \mathbf{V}_2^H \end{bmatrix}$$

Goal: Find the minimum-norm $\mathbf{x}$ so that $||\mathbf{Ax} - \mathbf{b}||_2^2 = $ minimum.

$$
\begin{aligned}
||\mathbf{Ax} - \mathbf{b}||_2^2 &= ||\mathbf{U}^H\mathbf{Ax} - \mathbf{U}^H\mathbf{b}||_2^2 \\
&= ||\mathbf{U}^H\mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^H\mathbf{x} - \mathbf{U}^H\mathbf{b}||^2 \\
&= ||\boldsymbol{\Sigma}\underbrace{\mathbf{V}^H\mathbf{x}}_{\mathbf{y}} - \mathbf{U}^H\mathbf{b}||_2^2 \\
&= ||\boldsymbol{\Sigma}\mathbf{y} - \mathbf{U}^H\mathbf{b}||_2^2 \\
&= \left|\left| \begin{bmatrix} \boldsymbol{\Sigma}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix} - \begin{bmatrix} \mathbf{U}_1^H\mathbf{b} \\ \mathbf{U}_2^H\mathbf{b} \end{bmatrix} \right|\right|_2^2 \\
&= ||\boldsymbol{\Sigma}_1\mathbf{y}_1 - \mathbf{U}_1^H\mathbf{b}||_2^2 + ||\mathbf{U}_2^H\mathbf{b}||_2^2
\end{aligned}
$$

Some Math Background

Lemma: Let $\mathbf{U}$ be a unitary matrix; i.e., $\mathbf{U}^H \mathbf{U} = \mathbf{I}$.

$$\text{Then} \quad ||\mathbf{U}\mathbf{b}||_2^2 = ||\mathbf{b}||_2^2,$$

$$\text{where} \quad ||\mathbf{x}||_2^2 = \mathbf{x}^H \mathbf{x}.$$

Proof:

$$||\mathbf{U}\mathbf{b}||_2^2 = \mathbf{b}^H \mathbf{U}^H \mathbf{U} \mathbf{b} = \mathbf{b}^H \mathbf{b} = ||\mathbf{b}||^2.$$

$$\text{Consider} \quad \mathbf{A}\mathbf{x} \approx \mathbf{b},$$

$$\text{where} \qquad \mathbf{A} \quad \text{is} \quad M \times L,$$

$$\mathbf{x} \quad \text{is} \quad L \times 1,$$

$$\mathbf{b} \quad \text{is} \quad M \times 1,$$

$$\mathbf{A} \quad \text{is of rank} \quad K$$

Summary of HOYW Frequency Estimator

Step 1: Compute $\hat{r}(k), k = 1, 2, \cdots, L + M$.

Step 2: Compute the SVD of $\hat{\boldsymbol{\Gamma}}$ and determine $\hat{\tilde{\mathbf{a}}}$ with (**)

Step 3: Compute the roots of

$$1 + \hat{\tilde{a}}_1 z^{-1} + \cdots + \hat{\tilde{a}}_L z^{-L} = 0$$

Pick the K roots that are nearest the unit circle and obtain the frequency estimates as the angular positions (phases) of these roots.

Remarks: • Rule of Thumb for selecting L and M:

$$L \approx M$$

$$L + M \approx \frac{N}{3}$$

• Although one cannot guarantee that the K roots nearest the unit circle give the best frequency estimates, empirical evidence shows that this is true most often .

Remark:

- Using $\hat{\boldsymbol{\Gamma}}_K$ to replace $\boldsymbol{\Gamma}$ gives <u>better</u> frequency estimation.

- This result may be explained by the fact that $\hat{\boldsymbol{\Gamma}}_K$ is <u>closer</u> to $\boldsymbol{\Gamma}$ than $\hat{\boldsymbol{\Gamma}}$.

- The rank approximation step is referred as " noise cleaning ".

Let us use this rank information as follows: Let

$$\hat{\mathbf{\Gamma}} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^H$$

$$= [\mathbf{U}_1 \quad \mathbf{U}_2] \begin{bmatrix} \mathbf{\Sigma}_1 & 0 \\ 0 & \mathbf{\Sigma}_2 \end{bmatrix} \begin{bmatrix} \mathbf{V}_1^H \\ \mathbf{V}_2^H \end{bmatrix} \begin{matrix} K \\ L-K \end{matrix}$$

denote the singular value decomposition (SVD) of $\hat{\mathbf{\Gamma}}$. (Diagonal

elements in $\begin{bmatrix} \mathbf{\Sigma}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{\Sigma}_2 \end{bmatrix}$ arranged from large to small).

Since $\hat{\mathbf{\Gamma}}$ is close to rank K, and $\mathbf{\Gamma}$ has rank K,

$$\hat{\mathbf{\Gamma}}_K = \mathbf{U}_1 \mathbf{\Sigma}_1 \mathbf{V}_1^H$$

(The best <u>Rank - K Approximation</u> of $\hat{\mathbf{\Gamma}}$ in the Frobenius Norm
sense) is generally a better estimate of $\mathbf{\Gamma}$ than $\hat{\mathbf{\Gamma}}$.

$$\hat{\mathbf{\Gamma}}_K \tilde{\mathbf{a}} \approx -\hat{\boldsymbol{\gamma}}, \qquad \hat{\tilde{\mathbf{a}}} = -\mathbf{V}_1 \mathbf{\Sigma}_1^{-1} \mathbf{U}_1^H \hat{\boldsymbol{\gamma}} \qquad (**)$$

Consider

$$
\begin{bmatrix}
\hat{r}_{yy}(L) & \cdots & \hat{r}_{yy}(1) \\
\vdots & & \\
\hat{r}_{yy}(L+M-1) & \cdots & \hat{r}_{yy}(M)
\end{bmatrix}
\begin{bmatrix}
\tilde{a}_1 \\
\vdots \\
\tilde{a}_L
\end{bmatrix}
\approx -
\begin{bmatrix}
\hat{r}_{yy}(L+1) \\
\vdots \\
\hat{r}_{yy}(L+M)
\end{bmatrix}
$$

$$\Rightarrow \quad \hat{\boldsymbol{\Gamma}}\tilde{\mathbf{a}} \approx -\hat{\boldsymbol{\gamma}}.$$

Remarks: rank $(\hat{\boldsymbol{\Gamma}}) = \min(M, L)$

almost surely, due to errors in $\hat{r}_{yy}(i)$

• For large N, $\hat{r}_{yy}(i) \approx r_{yy}(i)$ makes $\hat{\boldsymbol{\Gamma}}$ <u>ill conditioned</u>.

• For large N, LS estimates of $\tilde{a}_1, \cdots, \tilde{a}_L$ give poor estimates of $\omega_1, \cdots, \omega_K$.

$$
\begin{aligned}
\mathbf{P}_{L+1} &= E\left\{ \begin{bmatrix} x_1(n-L-1) \\ x_2(n-L-1) \\ \vdots \\ x_K(n-L-1) \end{bmatrix} \begin{bmatrix} x_1^*(n-1) & \cdots & x_K^*(n-1) \end{bmatrix} \right\} \\
&= \begin{bmatrix} \alpha_1^2 e^{-j\omega_1 L} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \alpha_K^2 e^{-j\omega_K L} \end{bmatrix}
\end{aligned}
$$

Remark: For $M \geq K$ and $L \geq K$, $\mathbf{\Gamma}^*$ is of rank K, <u>so is $\mathbf{\Gamma}$</u>.

- $$E\left\{x_i(n)\right\} = E\left\{\alpha_i e^{j(\omega_i n + \phi_i)}\right\}$$
  $$= \int_{-\pi}^{\pi} \alpha_i e^{j\omega_i n} e^{j\phi_i} \frac{1}{2\pi} d\phi_i = 0$$

- $E\left\{x_i(n-k)x_i^*(n)\right\}$
  $$= E\left\{\alpha_i e^{j[\omega_i(n-k)+\phi_i]}\alpha_i e^{-j(\omega_i n + \phi_i)}\right\}$$
  $$= \alpha_i^2 e^{-j\omega_i k}$$

- Since $\phi_i's$ are independent of each other,
  $$E\left\{x_i(n-k)x_j^*(n)\right\} = 0, \quad i \neq j$$

$$\text{Thus} \quad \mathbf{\Gamma}^* = E\left\{ \begin{bmatrix} y(n-L-1) \\ \vdots \\ y(n-L-M) \end{bmatrix} \begin{bmatrix} y^*(n-1) & \cdots & y^*(n-L) \end{bmatrix} \right\}$$

$$= E\left\{ \mathbf{A}_M \tilde{\mathbf{x}}(n-L-1)\tilde{\mathbf{x}}^H(n-1)\mathbf{A}_L^H \right\}$$

$$\triangleq \mathbf{A}_M \mathbf{P}_{L+1} \mathbf{A}_L^H,$$

$$\text{where} \quad \mathbf{P}_{L+1} = E\left\{ \tilde{\mathbf{x}}(n-L)\tilde{\mathbf{x}}^H(n) \right\}$$

$$\tilde{\mathbf{y}}_i(n) = \underbrace{\begin{bmatrix} 1 & 1 & \cdots & 1 \\ e^{-j\omega_1} & e^{-j\omega_2} & \cdots & e^{-j\omega_K} \\ \vdots & & & \\ e^{-j(i-1)\omega_1} & \cdots & & e^{-j(i-1)\omega_K} \end{bmatrix}}_{\mathbf{A}_i} \tilde{\mathbf{x}}(n) + \tilde{\mathbf{w}}_i(n)$$

$\mathbf{A}_i = i \times K$ Vandermonde matrix.

$\text{rank}(\mathbf{A}_i) = K$ if $i \geq K$ and $\omega_k \neq \omega_l$ for $k \neq l$.

$\Rightarrow \tilde{\mathbf{y}}_i(n) = \mathbf{A}_i \tilde{\mathbf{x}}(n) + \tilde{\mathbf{w}}_i(n)$

Γ has rank K, if $M \geq K$ and $L \geq K$

Proof: Let $\quad \tilde{\mathbf{y}}_i(n) = \begin{bmatrix} y(n) \\ y(n-1) \\ \vdots \\ y(n-i+1) \end{bmatrix}, \quad \tilde{\mathbf{w}}_i(n) = \begin{bmatrix} w(n) \\ w(n-1) \\ \vdots \\ w(n-i+1) \end{bmatrix}$

$$\tilde{\mathbf{x}}(n) = \begin{bmatrix} x_1(n) \\ \vdots \\ x_K(n) \end{bmatrix}, \quad \mathbf{x}_k(n) = \alpha_k e^{j(\omega_k n + \phi_k)}$$

Remarks:

- When $y(0), \cdots, y(N-1)$ are the only data available, we first estimate $r_{yy}(i)$ and replace $r_{yy}(i)$ in above equation with estimate $\hat{r}_{yy}(i)$

- $\{\hat{\omega}_K\}$ are the angular positions of the K roots nearest the unit circle

- Increasing L and M will

    * give <u>better</u> performance due to using the information in higher lags of $\hat{r}(i)$

- Increasing L and M 'too much' will

    * give <u>worse</u> performance due to increased variance in $\hat{r}(i)$ for large i

Multiplying both sides by $\begin{bmatrix} y^*(n-L-1) \\ \vdots \\ y^*(n-L-M) \end{bmatrix}$,

we get $\begin{bmatrix} r_{yy}(L+1) & \cdots & r_{yy}(1) \\ & \vdots & \\ r_{yy}(L+M) & \cdots & r_{yy}(M) \end{bmatrix} \begin{bmatrix} 1 \\ \tilde{a}_1 \\ \vdots \\ \tilde{a}_L \end{bmatrix} = 0.$

$$\Rightarrow \begin{bmatrix} r_{yy}(L) & \cdots & r_{yy}(1) \\ & \vdots & \\ r_{yy}(L+M-1) & \cdots & r_{yy}(M) \end{bmatrix} \begin{bmatrix} \tilde{a}_1 \\ \vdots \\ \tilde{a}_L \end{bmatrix} = - \begin{bmatrix} r_{yy}(L+1) \\ \vdots \\ r_{yy}(L+M) \end{bmatrix}$$

$$\Rightarrow \boldsymbol{\Gamma}\tilde{\mathbf{a}} = -\boldsymbol{\gamma}$$

Remark:

- It is tempting to cancel $A(z)$ from both sides above, but this is wrong since $y(n) \neq w(n)$!

Multiplying both sides of $(*)$ by a polynomial $\bar{A}(z)$
of order $L - K$ gives

$$\left(1 + \tilde{a}_1 z^{-1} + \cdots + \tilde{a}_L z^{-L}\right) y(n) = \left(1 + \tilde{a}_1 z^{-1} + \cdots + \tilde{a}_L z^{-L}\right) w(n)$$

$$\text{where} \quad 1 + \tilde{a}_1 z^{-1} + \cdots + \tilde{a}_L z^{-L} = A(z) \bar{A}(z)$$

$$\Rightarrow [y(n) \quad y(n-1) \cdots \quad y(n-L)] \begin{bmatrix} 1 \\ \tilde{a}_1 \\ \vdots \\ \tilde{a}_L \end{bmatrix} = w(n) + \cdots + a_L w(n-L)$$

Higher - Order Yule- Walker (HOYW) Method:

$$\text{Let} \quad x_k(n) = \alpha_k e^{j(\omega_k n + \phi_k)}$$

$$
\begin{aligned}
\left[1 - e^{j\omega_k} z^{-1}\right] x_k(n) &= x_k(n) - e^{j\omega_k} x_k(n-1) \\
&= \alpha_k e^{j(\omega_k n + \phi_k)} - e^{j\omega_k} \alpha_k e^{j[\omega_k(n-1) + \phi_k]} \\
&= 0
\end{aligned}
$$

$\Rightarrow \left[1 - e^{j\omega_k} z^{-1}\right]$ is an Annihilating filter for $x_k(n)$.

$$\text{Let} \quad A(z) = \Pi_{k=1}^{K} \left(1 - e^{j\omega_k} z^{-1}\right)$$

$\Rightarrow$

$$\boxed{A(z)x(n) = 0}$$

$$y(n) = x(n) + w(n)$$

$$\Rightarrow \qquad A(z)y(n) = A(z)w(n) \qquad\qquad (*)$$

High - Resolution Methods

- Statistical Performance <u>Close</u> to <u>ML</u> estimator <u>(or CRB)</u>.

- <u>Avoid</u> Multidimensional <u>search</u> over parameter space.

- Do not depend on Resolution condition.

- All provide <u>consistent estimates</u>

- All <u>give similar performance,</u> especially for large N.

- Method of choice is a <u>" Matter - of - Taste "</u>.

$\Rightarrow$
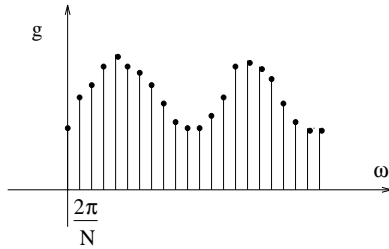
> The K $\tilde{\omega}_i$ that maximizes g_1 correspond to the largest K peaks of the Periodogram.

Remarks: $\bullet$ $\hat{\omega}_k$ estimates obtained by using the K largest peaks of Periodogram have accuracy $\underline{\hat{\omega}_k - \omega_k \propto \frac{2\pi}{N}}$

$\bullet$ The periodogram is a good frequency estimator. (This was introduced by Schuster a century ago !)

2.)

$$\triangle\omega = inf_{i \neq k} |\omega_i - \omega_k| > \frac{2\pi}{N}.$$

$$\text{Since} \quad Var\left(\hat{\omega}_k - \omega_k\right) \propto \frac{1}{N^3}$$

$$\Rightarrow \quad \hat{\omega}_k - \omega_k \propto \frac{1}{N^{\frac{3}{2}}}.$$

$$\Rightarrow \quad inf_{i \neq k} |\hat{\omega}_i - \hat{\omega}_k| > \frac{2\pi}{N}.$$

$\Rightarrow$ We can resolve all K sine waves by evaluating g_1 at FFT points:

$$\tilde{\omega}_i = \frac{2\pi}{N}i, \quad i = 0, \cdots, N - 1$$

Any $\quad K \quad$ of these $\quad \tilde{\omega}_i \quad$ gives $\quad \mathbf{B}^H \mathbf{B} = N\mathbf{I}, \quad \mathbf{I} = \text{Identity matrix.}$

$$\Rightarrow \quad g_1 = \sum_{k=1}^{K} \frac{1}{N} \left| \sum_{n=0}^{N-1} y(n)e^{-j\tilde{\omega}_k n} \right|^2.$$

$$\mathbf{B}^H \mathbf{y} = \begin{bmatrix} 1 & e^{-j\omega} & \cdots & e^{-j(N-1)\omega} \end{bmatrix} \begin{bmatrix} y(0) \\ \vdots \\ y(N-1) \end{bmatrix}$$

$$= \sum_{n-0}^{N-1} y(n) e^{-j\omega n}$$

$$\Rightarrow \quad \hat{\omega} = \operatorname{argmax}_\omega \frac{1}{N} \left| \sum_{n=0}^{N-1} y(n) e^{-j\omega n} \right|^2$$

$\hat{\omega}$ corresponds to the highest peak of the Periodogram !

Special Cases:

1.) $K = 1$

$$\hat{\omega} = \text{argmax}_\omega \underbrace{\left[\mathbf{y}^H \mathbf{B} \left(\mathbf{B}^H \mathbf{B} \right)^{-1} \mathbf{B}^H \mathbf{y} \right]}_{\mathbf{g_1}},$$

$$\mathbf{B} = \begin{bmatrix} 1 \\ e^{j\omega} \\ \vdots \\ e^{j(N-1)\omega} \end{bmatrix}, \mathbf{B}^H \mathbf{B} = N.$$

- For large N,

$$E\left[(\hat{\boldsymbol{\omega}} - \boldsymbol{\omega})(\hat{\boldsymbol{\omega}} - \boldsymbol{\omega})^H\right] = \frac{6\sigma^2}{N^3}\begin{bmatrix} \frac{1}{\alpha_1^2} & & \\ & \ddots & \\ & & \frac{1}{\alpha_K^2} \end{bmatrix}$$

$$= \text{CRB}$$

However,

- The maximization to obtain $\hat{\boldsymbol{\omega}}$ is difficult to implement.

* The search may not find global maximum.

* Computationally expensive.

$$\begin{aligned} g &= (\mathbf{y} - \mathbf{B}\boldsymbol{\beta})^H (\mathbf{y} - \mathbf{B}\boldsymbol{\beta}). \\ &= \left[\boldsymbol{\beta} - \left(\mathbf{B}^H \mathbf{B} \right)^{-1} \mathbf{B}^H \mathbf{y} \right]^H \left(\mathbf{B}^H \mathbf{B} \right) \left[\boldsymbol{\beta} - \left(\mathbf{B}^H \mathbf{B} \right)^{-1} \mathbf{B}^H \mathbf{y} \right] \\ &+ \mathbf{y}^H \mathbf{y} - \mathbf{y}^H \mathbf{B} \left(\mathbf{B}^H \mathbf{B} \right)^{-1} \mathbf{B}^H \mathbf{y}. \end{aligned}$$

$$\Rightarrow \boxed{\begin{aligned} \hat{\boldsymbol{\omega}} &= \operatorname{argmax}_{\boldsymbol{\omega}} \left[\mathbf{y}^H \mathbf{B} \left(\mathbf{B}^H \mathbf{B} \right)^{-1} \mathbf{B}^H \mathbf{y} \right]. \\ \hat{\boldsymbol{\beta}} &= \left(\mathbf{B}^H \mathbf{B} \right)^{-1} \mathbf{B}^H \mathbf{y} \Big|_{\boldsymbol{\omega} = \hat{\boldsymbol{\omega}}}. \end{aligned}}$$

Remarks: $\bullet$ $\hat{\boldsymbol{\omega}}$ is a consistent estimate of $\boldsymbol{\omega}$

Remarks: If $w(n)$ is neither Gaussian nor white, minimizing g is called the non-linear least-squares method, in general.

- Let $\mathbf{y} = \begin{bmatrix} y(0) \\ \vdots \\ y(N-1) \end{bmatrix}, \boldsymbol{\beta} = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_K \end{bmatrix}, \boldsymbol{\omega} = \begin{bmatrix} \omega_1 \\ \vdots \\ \omega_K \end{bmatrix}$

$$\mathbf{B} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ e^{j\omega_1} & e^{j\omega_2} & \cdots & e^{j\omega_K} \\ \vdots & & & \vdots \\ e^{j(N-1)\omega_1} & \cdots & & e^{j(N-1)\omega_K} \end{bmatrix}$$

$$f\left(w(0), \cdots, w(N-1)\right) = \frac{1}{\left(\pi\sigma^2\right)^N} \exp\left\{-\frac{\sum_{n=0}^{N-1}|w(n)|^2}{\sigma^2}\right\}$$

The likelihood function of $y(0), \quad \cdots, \quad y(N-1)$ is

$$f = f\left(y(0), \cdots, y(N-1)\right) = \frac{1}{\left(\pi\sigma^2\right)^N} \exp\left\{-\frac{\sum_{n=0}^{N-1}|y(n)-x(n)|^2}{\sigma^2}\right\}$$

Remark: The ML estimates of

$\omega_1, \cdots, \omega_K, \quad \alpha_1, \cdots, \alpha_K, \quad \phi_1, \cdots, \phi_K$ are found by maximizing f with respect to $\omega_1, \cdots, \omega_K, \alpha_1, \cdots, \alpha_K, \phi_1, \cdots, \phi_K$.

Equivalently, we minimize

$$g = \sum_{n=0}^{N-1}\left|y(n) - \sum_{k=1}^{K}\alpha_k e^{j(\omega_k n + \phi_k)}\right|^2$$

Maximum Likelihood Estimator

$w(n)$ is assumed to be zero-mean circularly symmetric complex Gaussian random variable with variance σ^2.

The pdf of $w(n)$ is $N(0, \sigma^2)$

$$f\left(w(n)\right) = \frac{1}{\pi \sigma^2} \exp\left\{ -\frac{|w(n)|^2}{\sigma^2} \right\}.$$

Remark: • The real and imaginary parts of $w(n)$ are real Gaussian random variables with zero-mean and variance $\frac{\sigma^2}{2}$.

• The two parts are independent of each other.

Remarks:

$$
\begin{aligned}
r_{yy}(k) &= E\left\{y^*(n)y(n+k)\right\} \\
&= \sum_{i=1}^{K} \alpha_i^2 e^{j\omega_i k} + \sigma^2 \delta(k) \\
P_{yy}(\omega) &= 2\pi \sum_{i=1}^{K} \alpha_i^2 \delta(\omega - \omega_i) + \sigma^2.
\end{aligned}
$$
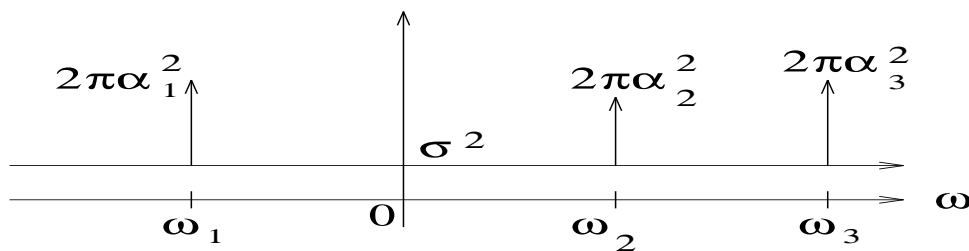
- Recall that the resolution limit of Periodogram is $\frac{1}{N}$

- The Parametric methods below have resolution better than $\frac{1}{N}$. (These methods are the so-called High - Resolution or Super - Resolution methods)

Remarks:

- Applications: Radar, Communications, $\cdots$.

- We are <u>mostly interested</u> in estimating $\omega_1, \cdots, \omega_K$.

- Once $\omega_1, \cdots, \omega_K$ are estimated, $\hat{\alpha}_1, \cdots, \hat{\alpha}_K, \quad \hat{\phi}_1, \cdots, \hat{\phi}_K$ can be found readily from $\hat{\omega}_1, \cdots, \hat{\omega}_K$

Let $\quad \alpha_k e^{j\phi_k} = \beta_k$

$$
\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(N-1) \end{bmatrix} \approx \begin{bmatrix} 1 & 1 & \cdots & 1 \\ e^{j\hat{\omega}_1} & e^{j\hat{\omega}_2} & \cdots & e^{j\hat{\omega}_K} \\ \vdots & & & \vdots \\ e^{j(N-1)\hat{\omega}_1} & \cdots & & e^{j(N-1)\hat{\omega}_K} \end{bmatrix} \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \vdots \\ \hat{\beta}_k \end{bmatrix}
$$

The amplitude of $\hat{\beta}_k$ is α_k. The phase of $\hat{\beta}_k$ is ϕ_k.

Parametric Methods for Line Spectra

$$y(n) = x(n) + w(n)$$

$$x(n) = \sum_{k=1}^{K} \alpha_k e^{j(\omega_k n + \phi_k)}$$

ϕ_k = Initial phases, independent of each other,

uniform distribution on $[-\pi, \pi]$

α_k = amplitudes, constants, > 0

ω_k = angular frequencies

$w(n)$ = zero-mean white Gaussian Noise,

independent of $\phi_1, \cdots, \phi_K$

Case 3: FIR Filter:

$$H(z) = \sum_{k=0}^{p} h_k z^{-k}$$

Again, we can show similarly

$$r_{dy}(i) = \sum_{k=0}^{p} h_k^o r_{yy}(i - k).$$

$$\begin{bmatrix} r_{dy}(0) \\ r_{dy}(1) \\ \vdots \\ r_{dy}(p) \end{bmatrix} = \begin{bmatrix} r_{yy}(0) & r_{yy}(-1) & \cdots & r_{yy}(-p) \\ r_{yy}(1) & r_{yy}(0) & \cdots & \\ \vdots & & \ddots & \\ r_{yy}(p) & r_{yy}(p-1) & \cdots & r_{yy}(0) \end{bmatrix} \begin{bmatrix} h_0^o \\ h_1^o \\ \vdots \\ h_p^o \end{bmatrix}$$

Remark: The Minimum error E is the smallest in case (1) and largest in case (3).

$$P_{dy}(z)B^*\left(\frac{1}{z^*}\right) = \frac{0.36}{\sqrt{1.6}}\left(\frac{\frac{5}{3}}{1-0.8z^{-1}} + \frac{\frac{5}{6}z}{1-0.5z}\right)$$

$$\left[P_{dy}(z)B^*\left(\frac{1}{z^*}\right)\right]_+ = \frac{0.36}{\sqrt{1.6}}\frac{\frac{5}{3}}{1-0.8z^{-1}} = G^o(z)$$

$$H^o(z) = \frac{0.36}{\sqrt{1.6}}\frac{\frac{5}{3}}{1-0.8z^{-1}}\frac{1}{\sqrt{1.6}}\frac{1-0.8z^{-1}}{1-0.5z^{-1}}$$

$$= 0.375\frac{1}{1-0.5z^{-1}}$$

$$\Rightarrow h^o(k) = \frac{3}{8}\left(\frac{1}{2}\right)^k, \quad k=0,1,2,\cdots$$

Ex. (Same as previous one)

$$P_{xx}(z) = \frac{0.36}{(1 - 0.8z^{-1})(1 - 0.8z)},$$

$$P_{ww}(z) = 1. \quad x(n) \quad \text{and} \quad w(n) \quad \text{independent}$$

x(n)

x(n) + w(n)

$\overset{\circ}{H(z)}$

+

−

e(n)
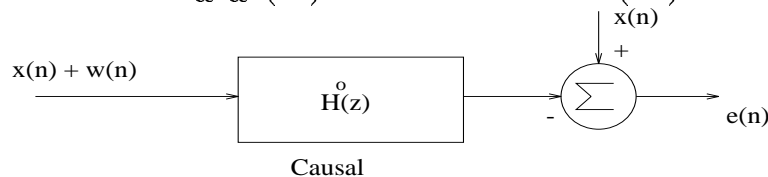
Causal

$$P_{dy}(z) = P_{xy}(z) = P_{xx}(z)$$

$$P_{yy}(z) = \frac{1.6\left(1 - 0.5z^{-1}\right)(1 - 0.5z)}{(1 - 0.8z^{-1})(1 - 0.8z)}.$$

$$B(z) = \frac{1}{\sqrt{1.6}} \frac{1 - 0.8z^{-1}}{1 - 0.5z^{-1}} \quad (\text{ stable and causal })$$

$$P_{dy}(z) B^*\left(\frac{1}{z^*}\right) = \frac{0.36}{(1 - 0.8z^{-1})(1 - 0.8z)} \frac{1}{\sqrt{1.6}} \frac{1 - 0.8z}{1 - 0.5z}$$

$$= \frac{0.36}{\sqrt{1.6}} \frac{1}{(1 - 0.8z^{-1})(1 - 0.5z)}.$$

Then

$$P_{dy}(z) = \underbrace{H^o(z)B^{-1}(z)}_{\text{causal}} B^{-*}\left(\frac{1}{z^*}\right)$$

$$\Rightarrow \quad \frac{H^o(z)}{B(z)} = \left[P_{dy}(z)B^*\left(\frac{1}{z^*}\right)\right]_+ ,$$

where

$$[X(z)]_+ = \left[\sum_{k=-\infty}^{\infty} x_k z^{-k}\right]_+ = \sum_{k=0}^{\infty} x_k z^{-k}.$$

$$\Rightarrow \quad \boxed{H^o(z) = B(z)\left[P_{dy}(z)B^*\left(\frac{1}{z^*}\right)\right]_+}$$

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Case 2: Causal Filter.

$$H(z) = \sum_{k=0}^{\infty} h_k z^{-k}$$

Through similar derivations as for Case 1, we have

$$r_{dy}(i) = \sum_{k=0}^{\infty} h_k^o r_{yy}(i - k),$$

$$h_k^o = ?$$

Let

$$B(z) B^* \left(\frac{1}{z^*} \right) = \frac{1}{P_{yy}(z)}$$

Pick $B(z)$ to be a <u>stable</u>, <u>causal</u>, <u>minimum phase</u> system.

$$
\begin{aligned}
r_{dy}(k) &= E\left[d(n+k)y^*(n)\right] \\
&= E\left\{x(n+k)\left[x^*(n) + w(n)\right]\right\} \\
&= r_{xx}(k). \\
P_{dy}(z) &= P_{xx}(z) \\
H^o(z) &= \frac{P_{yy}(z)}{P_{dy}(z)} \\
&= \frac{0.36}{1.6\left(1 - 0.5z^{-1}\right)\left(1 - 0.5z\right)} \\
h^o(k) &= 0.3\left(\frac{1}{2}\right)^{|k|}
\end{aligned}
$$
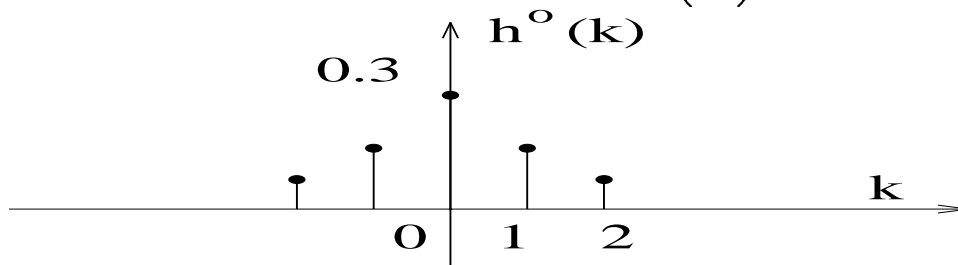
In Z - domain

$$P_{dy}(z) = H^o(z)P_{yy}(z)$$

which is the optimum Non-causal Wiener Filter.

$$Ex: \quad d(n) \quad = \quad x(n), \quad y(n) = x(n) + w(n),$$

$$P_{xx}(z) \quad = \quad \frac{0.36}{\left(1 - 0.8z^{-1}\right)\left(1 - 0.8z\right)}$$

$$P_{ww}(z) \quad = \quad 1.$$

$$x(n) \quad \text{and} \quad w(n) \quad \text{are uncorrelated.}$$

Optimal filter ?

$$P_{yy}(z) \quad = \quad P_{xx}(z) + P_{ww}(z)$$

$$= \quad \frac{0.36}{\left(1 - 0.8z^{-1}\right)\left(1 - 0.8z\right)} + 1$$

$$= \quad 1.6\frac{\left(1 - 0.5z^{-1}\right)\left(1 - 0.5z\right)}{\left(1 - 0.8z^{-1}\right)\left(1 - 0.8z\right)}$$

Case 1: Non-causal Filter.

$$E = E\left\{\left|e(n)\right|^2\right\}$$

$$= E\left\{\left[d(n) - \sum_{k=-\infty}^{\infty} h_k y(n-k)\right]\left[d(n) - \sum_{l=-\infty}^{\infty} h_l y(n-l)\right]^*\right\}$$

$$= r_{dd}(0) - \sum_{l=-\infty}^{\infty} h_l^* r_{dy}(l) - \sum_{k=-\infty}^{\infty} h_k r_{dy}^*(k)$$

$$+ \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} r_{yy}(l-k)h_k h_l^*$$

Remark: For Causal and FIR filters, only limits of sums differ.

$$\text{Let} \quad h_i = \alpha_i + j\beta_i \qquad \frac{\partial E}{\partial \alpha_i} = 0, \quad \frac{\partial E}{\partial \beta_i} = 0.$$

$$\Rightarrow r_{dy}(i) = \sum_{k=-\infty}^{\infty} h_k^o r_{yy}(i-k), \quad \forall i$$

Three common filters:

1.) General Non-causal:

$$H(z) = \sum_{k=-\infty}^{\infty} h_k z^{-k}.$$

2.) General Causal:

$$H(z) = \sum_{k=0}^{\infty} h_k z^{-k}$$

3.) Finite Impulse Response (FIR):

$$H(z) = \sum_{k=0}^{p} h_k z^{-k}$$

General Filtering Problem: (Complex Signals)
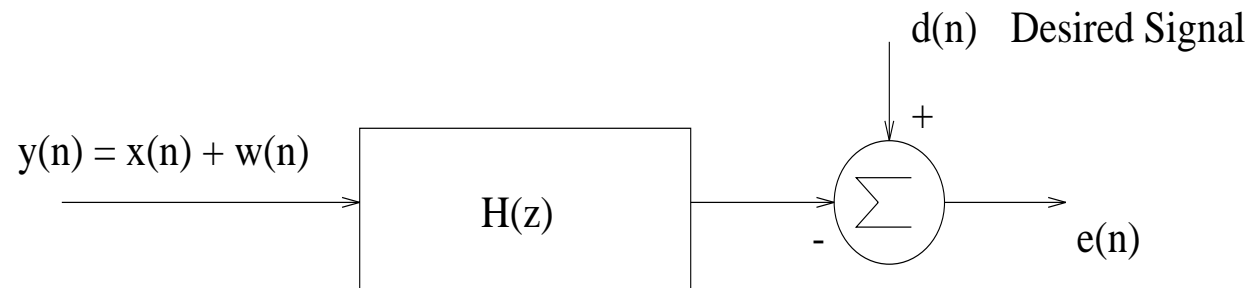
Special case of $d(n)$: $d(n) = x(n + m)$:

1.) $m > 0$, m - step ahead prediction.

2.) $m = 0$,   filtering problem

3.) $m < 0$,   smoothing problem.

Wiener Filter: (Wiener-Hopf Filter)

x(n)   Desired Signal

+

y(n) = x(n) + w(n)

H(z)

$\sum$

-

e(n)

$\hat{x}$(n)
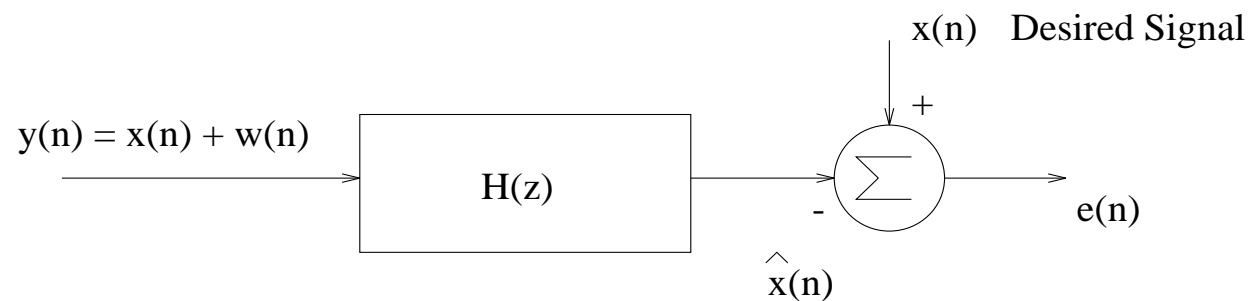
- $H(z)$ is found by minimizing $E\left[|e(n)|^2\right]$.

- $H(z)$ depends on knowing $P_{xy}(\omega)$.

Remarks: • $y(n)$ is an ARMA signal

• $a_1, \cdots, a_p, \quad \sigma^2, \sigma_w^2$ may be estimated by

  * ARMA methods.

  * A large order AR approximation.

  * Compensating the effect of $w(n)$.

  * Bootstrap or adaptive filtering and AR methods.

Noisy AR Processes:

$$y(n) = x(n) + w(n)$$

- $x(n) = \mathrm{AR}(p)$ process.

- $w(n) = $ White Gaussian noise with zero-mean and variance σ_w^2

- $x(n)$ and $w(n)$ are Independent of each other.

$$
\begin{aligned}
P_{yy}(\omega) &= P_{xx}(\omega) + P_{ww}(\omega) \\
&= \frac{\sigma^2}{|A(\omega)|^2} + \sigma_w^2 \\
&= \frac{\sigma^2 + \sigma_w^2 |A(\omega)|^2}{|A(\omega)|^2}.
\end{aligned}
$$

Minimum Description Length (MDL) Criterion

minimizes

$$\mathrm{MDL}(k) = N \ln \hat{\delta}_k + k \ln N.$$

Remark: As $N \to \infty$, MDL's probability of error $\to 0$. (consistent!).

Criterion Autoregressive Transfer (CAT) Method

minimizes

$$\mathrm{CAT}(k) = \frac{1}{N} \sum_{i=1}^{k} \frac{1}{\tilde{\delta}_i} - \frac{1}{\tilde{\delta}_k},$$

$$\tilde{\delta}_i = \frac{N}{N-i} \hat{\delta}_i$$

Remarks: • None of the above methods works well for small N

• Use these methods to initially estimate orders. (Practical experience needed).

Final Prediction Error (FPE) Method

minimizes

$$\mathrm{FPE}(k) = \frac{N+k}{N-k} \hat{\delta}_k \ .$$

Akaike Information Criterion (AIC) Method

minimizes

$$\mathrm{AIC}(k) = N \ln \hat{\delta}_k + 2k \ .$$

Remarks:

- As $N \to \infty$, AIC's probability of error in choosing correct order does $\underline{\mathrm{NOT}} \to 0$.

- As $N \uparrow$, AIC tends to overestimate model order.

AR Model Order Selection

Remarks:

- Order too low yields smoothed/biased PSD estimate.

- Order too high yields spurious peaks/large variance in PSD estimate

- Almost all model order estimators are based on the estimate of the power of linear prediction error, denoted $\hat{\delta}_k$, where k is the model order chosen.

Accuracy of AR PSD Estimators

- Accuracy Analysis is <u>difficult</u>.

- Results for large N are available due to Central Limit Theorem.

- For large N, the variances for $\hat{a}_1, \cdots, \hat{a}_p, \quad \hat{k}_1, \cdots, \hat{k}_p, \quad \sigma^2, \hat{P}(\omega)$ are all proportional to $\frac{1}{N}$. Biases $\propto \frac{1}{N}$.

- **An Approximate ML Estimator**

$\hat{a}_1, \cdots, \hat{a}_p, \hat{\sigma}^2$ are found by maximizing f_2.

$\Rightarrow \hat{a}_1, \cdots, \hat{a}_p$ are found by $\underline{\text{minimizing } \mathbf{\bar{a}}^T \mathbf{X}^T \mathbf{X} \mathbf{\bar{a}} = \mathbf{u}^T \mathbf{u}}$

$$
\begin{bmatrix}
x(p) & \cdots & x(0) \\
x(p+1) & \cdots & x(1) \\
\vdots & & \\
x(N-1) & \cdots & x(N-p-1)
\end{bmatrix}
\begin{bmatrix}
1 \\
a_1 \\
\vdots \\
a_p
\end{bmatrix}
=
\begin{bmatrix}
u(p) \\
u(p+1) \\
\vdots \\
u(N-1)
\end{bmatrix}.
$$

$\Rightarrow$ This is exactly $\underline{\text{Prony's Method !}}$

$$
\hat{\sigma}^2 = \frac{1}{N-p} \sum_{n=p}^{N-1} \left[x(n) + \sum_{j=1}^{p} \hat{a}_j x(n-j) \right]^2.
$$

Again, exactly Prony's Method !

$$\overline{\mathbf{a}} = \begin{bmatrix} 1 \\ a_1 \\ \vdots \\ a_p \end{bmatrix}.$$

$$\mathbf{u} = \mathbf{X}\overline{\mathbf{a}}$$

$$f_2 = \frac{1}{(2\pi\sigma^2)^{\frac{N-p}{2}}} \exp\left[-\frac{1}{2\sigma^2}\overline{\mathbf{a}}^T \mathbf{X}^T \mathbf{X}\overline{\mathbf{a}}\right].$$

Remark: Maximizing $f = f_1.f_2$ with respect to $a_1, \cdots, a_p, \sigma^2$ is highly <u>non-linear</u>!

$$
\begin{aligned}
f(u) &= \frac{1}{(2\pi\sigma^2)^{\frac{N-p}{2}}} \exp\left[-\frac{1}{2\sigma^2}\mathbf{u}^T\mathbf{u}\right] \\
f_2 &= f[u(x)]\,|\det(\mathbf{J})| \\
&= f[u(x)].
\end{aligned}
$$

$$
\text{Let} \quad \mathbf{X} = \begin{bmatrix} x(p) & x(p-1) & \cdots & x(0) \\ x(p+1) & x(p) & \cdots & x(1) \\ \vdots & & & \\ x(N-1) & x(N-2) & \cdots & x(N-p-1) \end{bmatrix}
$$

$$\text{Let} \quad \mathbf{u} = \begin{bmatrix} u(p) \\ \vdots \\ u(N-1) \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x(p) \\ \vdots \\ x(N-1) \end{bmatrix}$$

Given $x(0), \cdots, x(p-1), a_1, \cdots, a_p, \sigma^2$, x and u are related by linear transformation.

The Jacobian of the transformation

$$\mathbf{J} = \begin{bmatrix} 1 & 0 & \cdots & & 0 \\ a_1 & 1 & & & \vdots \\ \vdots & & \ddots & & \\ 0 & \cdots & a_p & \cdots & 1 \end{bmatrix}$$

$$\det(\mathbf{J}) = 1$$

$$\begin{bmatrix} u(p) \\ \vdots \\ u(N-1) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ a_1 & 1 & 0 & \cdots & 0 \\ a_2 & a_1 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \\ 0 & \cdots & a_p & \cdots & 1 \end{bmatrix} \begin{bmatrix} x(p) \\ x(p+1) \\ \vdots \\ x(N-1) \end{bmatrix}$$

$$+ \begin{bmatrix} a_1 x(p-1) + \cdots + a_p x(0) \\ a_2 x(p-1) + \cdots + a_p x(1) \\ \vdots \\ a_p x(p-1) \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

* Consider next

$$f_2 = f\left[x(p), \cdots, x(N-1) \mid x(0), \cdots, x(p-1), a_1, \cdots, a_p, \sigma^2\right]$$

$$x(n) + \sum_{k=1}^{p} a_k x(n-k) = u(n)$$

$$\begin{cases} u(p) = x(p) + a_1 x(p-1) + \cdots + a_p x(0). \\ u(p+1) = x(p+1) + a_1 x(p) + \cdots + a_p x(1) \\ \vdots \\ u(N-1) = x(N-1) + a_1 x(N-2) + \cdots + a_p x(N-p-1). \end{cases}$$

$$f = f\left[x(p), \cdots, x(N-1) \mid x(0), \cdots, x(p-1), a_1, \cdots, a_p, \sigma^2\right]$$
$$f\left[x(0), \cdots, x(p-1) \mid a_1, \cdots, a_p, \sigma^2\right]$$

* Consider first $f_1 = f\left[x(0), \cdots, x(p-1) \mid a_1, \cdots, a_p, \sigma^2\right]$

$$f_1 = \frac{1}{(2\pi)^{\frac{p}{2}} \det^{\frac{1}{2}}(\mathbf{R}_p)} \exp\left[-\frac{1}{2}\left(\mathbf{x}_0^T \mathbf{R}_p^{-1} \mathbf{x}_0\right)\right].$$

$$\mathbf{x}_0 = \begin{bmatrix} x(0) \\ \vdots \\ x(p-1) \end{bmatrix}, \quad \mathbf{R}_p = \begin{bmatrix} r(0) & \cdots & r(p-1) \\ \vdots & \ddots & \vdots \\ r(p-1) & \cdots & r(0) \end{bmatrix}.$$

Remark: $r(0), \cdots, r(p-1)$ are functions of $a_1, \cdots, a_p, \sigma^2$. (see, e.g., the YW system of equations)

Maximum Likelihood Estimators:

- ## Exact ML Estimator:
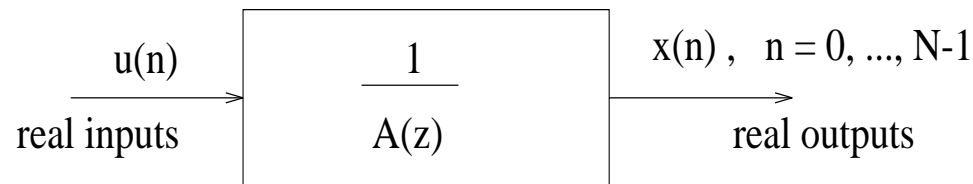
$u(n)$ is Gaussian white noise with zero-mean.

$$\Rightarrow \begin{cases} E[u(n)] = 0, \\ Var[u(n)] = \sigma^2 \\ E[u(i)u(j)] = 0, i \neq j, \end{cases}$$

The likelihood function is

$$f = f\left[x(0), \cdots, x(N-1) | a_1, \cdots, a_p, \sigma^2\right]$$

The ML estimates of $a_1, \cdots, a_p, \sigma^2$ are found by maximizing f.

For Gaussian random variables,

$$\mathbf{x} = \begin{bmatrix} x(0) \\ \vdots \\ x(N-1) \end{bmatrix} \sim \quad N(0, \mathbf{R}_N)$$

$$H_N = \frac{1}{2}\ln(\det\mathbf{R}_N).$$

Since $\quad H_N \to \infty \quad$ as $\quad N \to \infty, \quad$ we consider $\underline{\text{Entropy Rate}}$:

$$h = \lim_{N\to\infty} \frac{H_N}{N+1}$$

h is maximized with respect to $r(p+1), r(p+2), \cdots$.

Remark: For Gaussian case, we obtain Yule-Walker equations .... !

Maximum Entropy Spectral Estimation

Given $r(0), \cdots, r(p)$. The remaining $r(p+1), \cdots$ are extrapolated to <u>maximize</u> entropy.

<u>Entropy</u>: Let Sample space for discrete random variable x be $x_1, \cdots, x_N$. The entropy $H(x)$ is

$$H(x) = -\sum_{i=1}^{N} P(x_i) \ln P(x_i),$$

$$P(x_i) = \text{prob}(x = x_i)$$

For continuous random variable,

$$H(x) = -\int_{-\infty}^{\infty} f(x) \ln f(x) dx.$$

$$f(x) = \text{pdf of} \quad x.$$

Properties of AR(p) Signals:

- Extension of $r(k)$:

* Given $r(0), r(1), \cdots, r(p)$.

* From YW - Equations we can calculate $a_1, a_2, \cdots, a_p, \sigma^2$

* $r(k) = -\sum_{l=1}^{p} a_l r(k-l), \quad k > p$

- Another point of view:

* Given $r(0), \cdots, r(p)$.

* Calculate $a_1, \cdots, a_p, \sigma^2$.

* Obtain $P(\omega)$

* $r(k) \overset{DTFT}{\longleftrightarrow} P(\omega)$.

For $m = 1, 2, \cdots, p$,

- Calculate $\hat{k}_m$ with (*)

- $\hat{\delta}_m = \hat{\delta}_{m-1}(1 - \hat{k}_m^2)$

- $\hat{\boldsymbol{\theta}}_m = \begin{bmatrix} \hat{\boldsymbol{\theta}}_{m-1} \\ 0 \end{bmatrix} + \hat{k}_m \begin{bmatrix} \hat{\tilde{\boldsymbol{\theta}}}_{m-1} \\ 1 \end{bmatrix}, (\quad \hat{\theta}_1 = \hat{k}_1).$

- Update $\hat{e}_m^f(n)$ and $\hat{e}_m^b(n)$ with ($\ddagger$)

Remarks: • $\hat{\delta}_p = \hat{\sigma}^2$.

- Since $a^2 + b^2 \geq 2ab, \quad \left| \hat{k}_m \right| < 1$,

$\Rightarrow$ Burg Method gives poles that are inside unit circle.

- Different ways of calculating $\hat{k}_m$ are available.

$\hat{k}_m$ is found by minimizing (for $\boldsymbol{\theta}_{m-1}$ given)

$$\frac{1}{2} \sum_{n=m}^{N-1} \left\{ \left[\hat{e}_m^f(n) \right]^2 + \left[\hat{e}_m^b(n) \right]^2 \right\}.$$

$$\boxed{\hat{k}_m = \frac{-2 \sum_{n=m}^{N-1} \hat{e}_{m-1}^f(n) \hat{e}_{m-1}^b(n-1)}{\sum_{n=m}^{N-1} \left\{ \left[\hat{e}_{m-1}^f(n) \right]^2 + \left[\hat{e}_{m-1}^b(n-1) \right]^2 \right\}}.} \qquad (*)$$

Steps in Burg method:

Initialization $\begin{cases} \bullet & \hat{r}(0) = \frac{1}{N} \sum_{n=0}^{N-1} x^2(n) \\ \bullet & \hat{\delta}_0 = \hat{r}(0) \\ \bullet & \hat{e}_0^f(n) = x(n), \quad n = 1, 2, \cdots, N-1 \\ \bullet & \hat{e}_0^b(n) = x(n), \quad n = 0, 1, \cdots, N-2. \end{cases}$

Burg Method

Consider real data and real model. Recall LDA:

$$\boldsymbol{\theta}_{n+1} = \begin{bmatrix} \boldsymbol{\theta}_n \\ 0 \end{bmatrix} + k_{n+1} \begin{bmatrix} \tilde{\boldsymbol{\theta}}_n \\ 1 \end{bmatrix}$$

Thus, if we know $\boldsymbol{\theta}_n$ and k_{n+1}, we can find $\boldsymbol{\theta}_{n+1}$.

Recall

$$(\ddagger) \qquad \begin{cases} \hat{e}_m^f(n) = \hat{e}_{m-1}^f(n) + k_m \hat{e}_{m-1}^b(n-1) \\ \hat{e}_m^b(n) = \hat{e}_{m-1}^b(n-1) + k_m \hat{e}_{m-1}^f(n), \end{cases}$$

$$\text{where} \quad \hat{e}_{m-1}^f(n) = x(n) + \sum_{k=1}^{m-1} \hat{a}_{m-1,k} x(n-k).$$

$$\hat{e}_{m-1}^b(n) = x(n-m+1) + \sum_{k=1}^{m-1} \hat{a}_{m-1,k} x(n-m+1+k)$$

- In F/B Method,

$$\text{Let} \quad \mathbf{e}_{LS} = \begin{bmatrix} e^f(p) \\ \vdots \\ e^f(N-1) \\ e^b(0) \\ \vdots \\ e^b(N-p-1) \end{bmatrix}$$

$$\hat{\sigma}^2 = \frac{1}{2(N-p)} \left\{ \sum_{n=p}^{N-1} \left| e^f(n) \right|^2 + \sum_{n=0}^{N-p-1} \left| e^b(n) \right|^2 \right\}$$

Remarks on $\hat{\sigma}^2$:

- In YW method,

$$\hat{\sigma}^2 = \hat{r}(0) + \sum_{k=1}^{p} \hat{a}_k \hat{r}(k).$$

- In Prony Method,

$$\text{Let} \quad \mathbf{e}_{LS} = \begin{bmatrix} e(p) \\ \vdots \\ e(N-1) \end{bmatrix}$$

$$\sigma^2 = \frac{1}{N-p} \sum_{n=p}^{N-1} |e(n)|^2$$

- For complex data and complex model,

$$a_k = a_k^f = \left(a_k^b\right)^*, \quad k = 1, \cdots, p$$

Then F/B solves:

$$-\begin{bmatrix} x(p-1) & \cdots & x(0) \\ \vdots & & \vdots \\ x(N-2) & \cdots & x(N-p-1) \\ x^*(1) & \cdots & x^*(p) \\ \vdots & & \vdots \\ x^*(N-p) & \cdots & x^*(N-1) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{bmatrix} = \begin{bmatrix} x(p) \\ \vdots \\ x(N-1) \\ x^*(0) \\ \vdots \\ x^*(N-p-1) \end{bmatrix}$$

$$
- \begin{bmatrix} x(p-1) & \cdots & x(0) \\ \vdots & & \vdots \\ x(N-2) & \cdots & x(N-p-1) \\ x(1) & \cdots & x(p) \\ \vdots & & \vdots \\ x(N-p) & \cdots & x(N-1) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{bmatrix} = \begin{bmatrix} x(p) \\ \vdots \\ x(N-1) \\ x(0) \\ \vdots \\ x(N-p-1) \end{bmatrix}
$$

Remarks: • The F/B method does not guarantee poles inside the unit circle. In Practice, the poles are <u>usually inside</u> the unit circle.

Modified Covariance or Forward Backward (F/B) Method

Recall Backward Linear Prediction:

$$x(n) = -\sum_{k=1}^{p} a_k^b x(n+k) + e^b(n).$$

For real data and real AR coefficients,

$$a_k^f = a_k^b = a_k, \quad k = 1, \cdots, p$$

$$\begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-p-1) \end{bmatrix} \approx \begin{bmatrix} x(1) & x(2) & \cdots & x(p) \\ x(2) & x(3) & \cdots & x(p+1) \\ \vdots & & & \\ x(N-p) & \cdots & & x(N-1) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{bmatrix}$$

In the F/B method, this backward prediction equation is combined with the forward prediction equation and LS solution is found.

- The Autocorrelation Method or YW-Method minimizes

$$\hat{\sigma}^2 = \frac{1}{N} \sum_{n=-\infty}^{\infty} \left[x(n) + \sum_{k=1}^{p} \hat{a}_k x(n-k) \right]^2$$

where those $x(n)$ that are NOT available are set to <u>zero</u>.

- For large N, the YW and Prony methods yield similar results.

- For small N, YW method gives poor performance. The Prony method can give good estimates $\hat{a}_1, \cdots, \hat{a}_p$ for small N. The Prony method gives <u>exact</u> estimates for $x(n)$ =sum of sinusoids.

- Since biased $\hat{r}(k)$ are used in YW method, the estimated poles are inside unit circle. Prony method does not guarantee stability.

The Prony Method is to find LS solution to the overdetermined equation

$$-\begin{bmatrix} x(p-1) & \cdots & x(0) \\ & \vdots & \\ x(N-2) & \cdots & x(N-p-1) \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_p \end{bmatrix} \approx \begin{bmatrix} x(p) \\ \vdots \\ x(N-1) \end{bmatrix}.$$

Remarks:

- The Covariance or Prony Method minimizes

$$\hat{\sigma}^2 = \frac{1}{N-p} \sum_{n=p}^{N-1} \hat{u}^2(n) = \frac{1}{N-p} \sum_{n=p}^{N-1} \left[x(n) + \sum_{k=1}^{p} \hat{a}_k x(n-k) \right]^2$$

- **Covariance or Prony Method**

Consider the AR(p) signal,

$$x(n) = -\sum_{k=1}^{p} a_k x(n-k) + u(n), \quad n = 0, 1, \cdots, N-1$$

In matrix form, $\begin{bmatrix} x(p) \\ x(p+1) \\ \vdots \\ x(N-1) \end{bmatrix} =$

$$-\begin{bmatrix} x(p-1) & x(p-2) & .. & x(0) \\ x(p) & x(p+1) & .. & x(1) \\ \vdots & & & \\ x(N-2) & & .. & x(N-p-1) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{bmatrix} + \begin{bmatrix} u(p) \\ u(p+1) \\ \vdots \\ u(N-1) \end{bmatrix}$$

AR Spectral Estimation Methods

- **Autocorrelation or Yule-Walker method:** Recall that YW-Equation may be obtained by minimizing

$$E\left[e^2(n)\right] = E\left\{[x(n) - \hat{x}(n)]^2\right\},$$

where

$$\hat{x}(n) = -\sum_{k=1}^{p} a_k x(n-k).$$

The autocorrelation or YW method replaces $r(k)$ in the YW equation with <u>biased</u> $\hat{r}(k)$

$$\begin{bmatrix} \hat{r}(0) & \cdots & \hat{r}(p-1) \\ \vdots & \ddots & \vdots \\ \hat{r}(p-1) & \cdots & \hat{r}(0) \end{bmatrix} \begin{bmatrix} \hat{a}_1 \\ \vdots \\ \hat{a}_p \end{bmatrix} = - \begin{bmatrix} \hat{r}(1) \\ \vdots \\ \hat{r}(p) \end{bmatrix}.$$

Lattice Filter for Linear Prediction Error
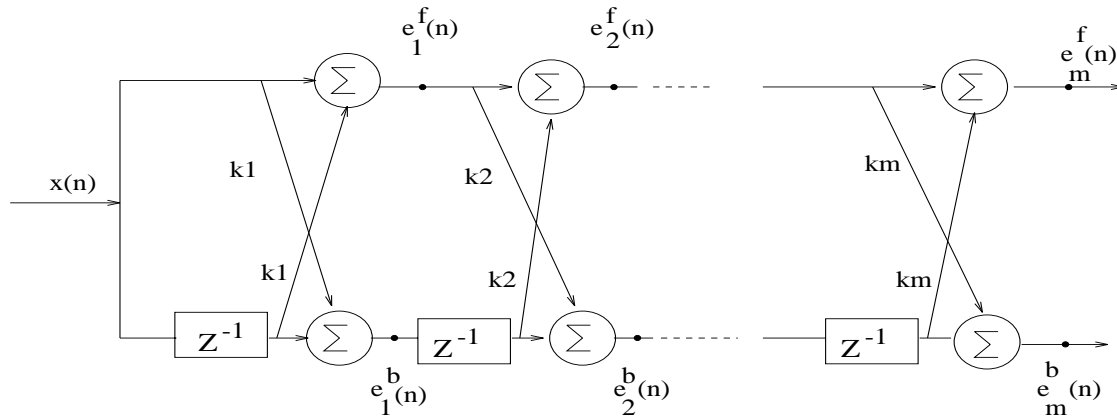
Remarks:• The implementation advantage of lattice filters is that they suffer from less round-off noise and are less sensitive to coefficient errors.

• If $x(n)$ is $AR(p)$ and $m = p$, then
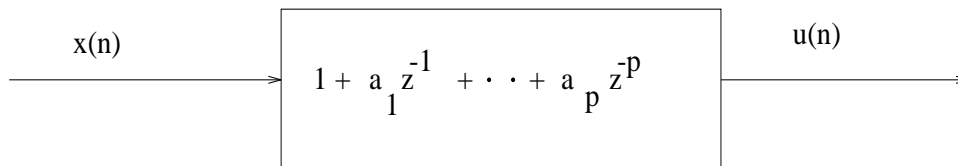
Whitening Filter

Similarly,

$$e_m^f(n) = e_{m-1}^f(n) + k_m e_{m-1}^b(n-1).$$

$$e_m^b(n) = e_{m-1}^b(n-1) + k_m e_{m-1}^f(n).$$

Recall LDA:

$$\boldsymbol{\theta}_m = \begin{bmatrix} \boldsymbol{\theta}_{m-1} \\ 0 \end{bmatrix} + k_m \begin{bmatrix} \tilde{\boldsymbol{\theta}}_{m-1} \\ 1 \end{bmatrix}.$$

$e_m^f(n) =$

$$[x(n) \quad x(n-1) \quad \cdots \quad x(n-m)] \left\{ \begin{bmatrix} 1 \\ \boldsymbol{\theta}_{m-1} \\ 0 \end{bmatrix} + k_m \begin{bmatrix} 0 \\ \tilde{\boldsymbol{\theta}}_{m-1} \\ 1 \end{bmatrix} \right\}$$

$$= [x(n) \quad x(n-1) \quad \cdots \quad x(n-m+1)] \begin{bmatrix} 1 \\ \boldsymbol{\theta}_{m-1} \end{bmatrix}$$

$$+ k_m [x(n-1) \quad x(n-2) \quad \cdots \quad x(n-m)] \begin{bmatrix} \tilde{\boldsymbol{\theta}}_{m-1} \\ 1 \end{bmatrix}$$

Consider an AR(p) model and the notation in LDA:

Let $m = 1, 2, \cdots, p$

$$e_m^f(n) = x(n) + \sum_{i=1}^{m} a_{m,i}^f x(n-i)$$

$$= \begin{bmatrix} x(n) & x(n-1) & \cdots & x(n-m) \end{bmatrix} \begin{bmatrix} 1 \\ \boldsymbol{\theta}_m \end{bmatrix}.$$

$$e_m^b(n) = x(n-m) + \sum_{i=1}^{m} a_{m,i}^b x(n-m+i)$$

$$= \begin{bmatrix} x(n-m) & x(n-m+1) & \cdots & x(n) \end{bmatrix} \begin{bmatrix} 1 \\ \boldsymbol{\theta}_m \end{bmatrix}$$

$$= \begin{bmatrix} x(n) & \cdots & x(n-m+1) & x(n-m) \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{\theta}}_m \\ 1 \end{bmatrix}$$

To minimize δ^b, we obtain

$$\begin{bmatrix} r_{xx}(0) & r_{xx}(1) & \cdots & r_{xx}(m) \\ \vdots & & & \\ r_{xx}(m) & r_{xx}(m-1) & \cdots & r_{xx}(0) \end{bmatrix} \begin{bmatrix} 1 \\ a_1^b \\ \vdots \\ a_m^b \end{bmatrix} = \begin{bmatrix} \delta^b \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

$$\Rightarrow \qquad \boxed{\begin{aligned} a_i^f &= a_i^b, \quad \text{for all} \quad i \\ \delta^f &= \delta^b. \end{aligned}}$$

Backward Linear prediction
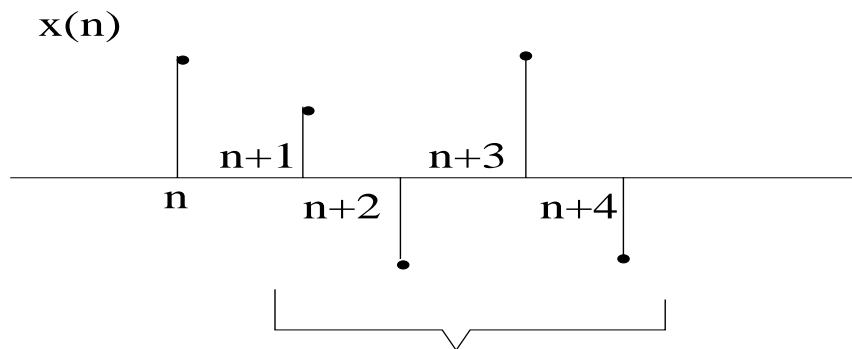
Samples used to predict x(n)

$$\hat{x}^b(n) = -\sum_{i=1}^{m} a_i^b x(n + i).$$

$$e^b(n) = x(n - m) - \hat{x}^b(n - m)$$

$$\delta^b = E\left[\left(e^b(n)\right)^2\right].$$

$\Rightarrow$

$$\begin{bmatrix} r_{xx}(0) & r_{xx}(1) & \cdots & r_{xx}(m) \\ r_{xx}(1) & r_{xx}(0) & \cdots & r_{xx}(m-1) \\ \vdots & & & \\ r_{xx}(m) & r_{xx}(m-1) & \cdots & r_{xx}(0) \end{bmatrix} \begin{bmatrix} 1 \\ a_1^f \\ \vdots \\ a_m^f \end{bmatrix} = \begin{bmatrix} \delta^f \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Remarks: • This is exactly the YW - Equation.

• δ^f decreases as m increases.

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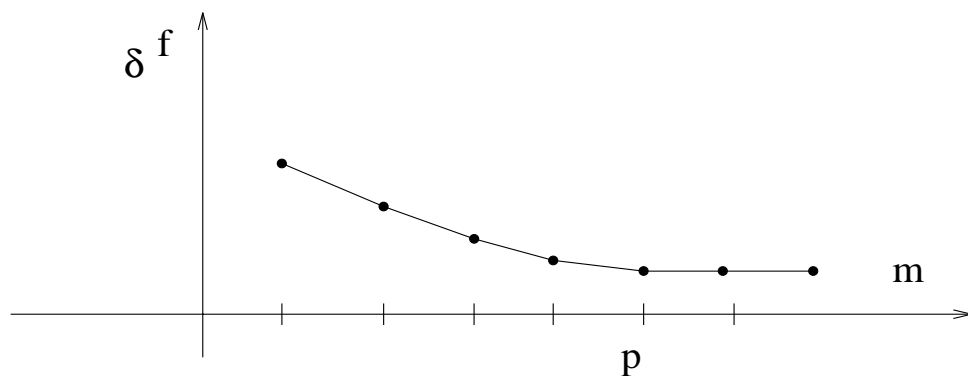
$$e^f(n) = x(n) - \hat{x}^f(n).$$

$$\delta^f = E\left[\left(e^f(n)\right)^2\right]$$

Goal: Minimize δ^f

$$
\begin{aligned}
\delta^f &= E\left[\left(e^f(n)\right)^2\right] \\
&= E\left[\left(x(n) + \sum_{i=1}^{m} a_i^f x(n-i)\right)^2\right] \\
&= r_{xx}(0) + \sum_{i=1}^{m} a_i^f r_{xx}(i) \\
&+ \sum_{j=1}^{m} a_j^f r_{xx}(j) + \sum_{i=1}^{m}\sum_{j=1}^{m} a_i^f a_j^f r_{xx}(j-i)
\end{aligned}
$$

$$\frac{\partial \delta^f}{\partial a_i^f} = 0 \quad \Rightarrow \quad r_{xx}(i) + \sum_{j=1}^{m} a_j^f r_{xx}(j-i) = 0.$$

Further Topics on AR Signals:

Linear prediction of AR Processes

- Forward Linear Prediction

Samples used to predict x(n)
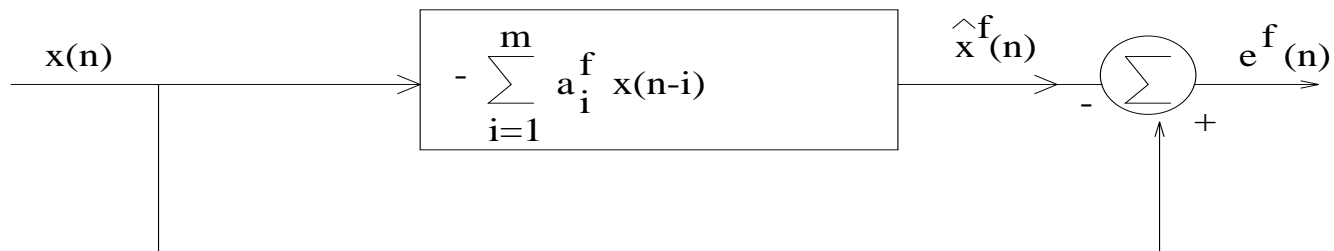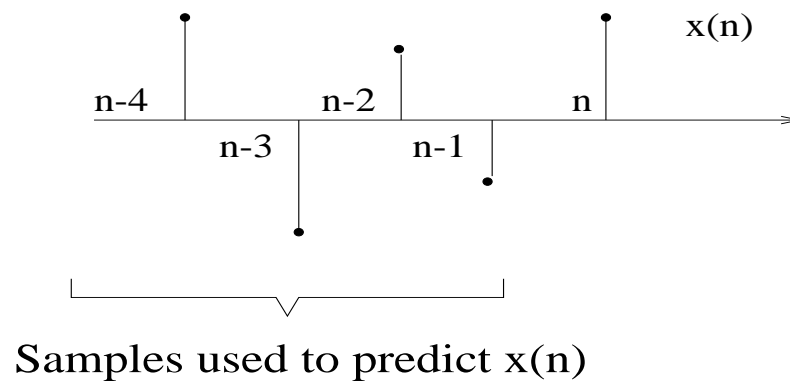
$$\mathbf{x} = \mathbf{H}\boldsymbol{\theta} + \hat{\mathbf{u}} \quad \text{(real signals)} .$$

LS Solution $\Rightarrow$

$$\boxed{\hat{\boldsymbol{\theta}} = \left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{H}^T\left(\mathbf{x} - \hat{\mathbf{u}}\right)}$$

Remarks:

- Any elements in $\mathbf{H}$ that are unknown are set to zero.

- QR Method may be used to solve the LS problem.

Step 3:

$$\hat{P}(\omega) = \hat{\sigma}^2 \left| \frac{1 + \hat{b}_1 e^{-j\omega} + \cdots + \hat{b}_q e^{-j\omega q}}{1 + \hat{a}_1 e^{-j\omega} + \cdots + \hat{a}_p e^{-j\omega p}} \right|^2$$

Remark: The difficult case for this method is when ARMA zeroes are near unit circle.

$$\boldsymbol{\theta} = \begin{bmatrix} -a_1 \\ -a_2 \\ \vdots \\ -a_p \\ b_1 \\ \vdots \\ b_q \end{bmatrix} .$$

$$\mathbf{H} = \begin{bmatrix} x(-1) & \cdots & x(-p) & \hat{u}(-1) & \cdots & \hat{u}(-q) \\ x(0) & \cdots & x(-p+1) & \hat{u}(0) & \cdots & \hat{u}(-q+1) \\ \vdots & & & & & \\ x(N-2) & \cdots & x(N-p-1) & \hat{u}(N-2) & \cdots & \hat{u}(N-q-1) \end{bmatrix}$$

Step 2: System Identification

$$\hat{u}(n) \longrightarrow \boxed{\dfrac{B(z)}{A(z)}} \longrightarrow x(n)$$
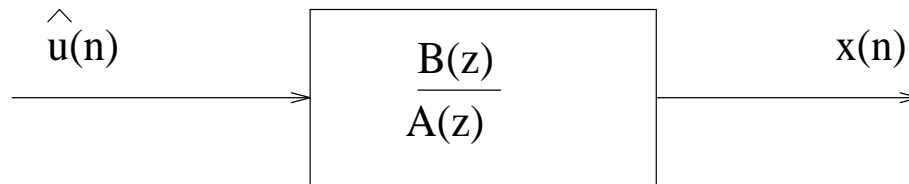
$$\text{Let} \quad \mathbf{x} = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}, \quad \hat{u} = \begin{bmatrix} \hat{u}(0) \\ \hat{u}(1) \\ \vdots \\ \hat{u}(N-1) \end{bmatrix}.$$

ARMA Signals:

Two Stage Least Squares Method

Step 1: Approximate $ARMA(p,q)$ with $AR(L)$ for a large L.

YW Equation may be used to estimate $\hat{a}_1, \hat{a}_2, \cdots, \hat{a}_L$.

$$\hat{u}(n) = x(n) + \hat{a}_1 x(n-1) + \cdots + \hat{a}_L x(n-L).$$

$$\hat{\sigma}^2 = \frac{1}{N-L} \sum_{n=L+1}^{N} \hat{u}^2(n).$$

Let

$$
\mathbf{V} = \begin{array}{c} \\ \end{array}
\begin{bmatrix}
\mathbf{V}_{11} & \vdots & \mathbf{V}_{12} \\
\mathbf{V}_{21} & \vdots & \mathbf{V}_{22}
\end{bmatrix}
\begin{array}{c} n \\ 1 \end{array}
$$

$$
\boxed{\mathbf{x}_{TLS} = -\mathbf{V}_{12}\mathbf{V}_{22}^{-1}}
$$

Remarks:

- At low SNR, TLS may be better than LS.

- At high SNR, TLS and LS yield similar results.

Markov Estimate:

If the <u>statistics</u> of $\mathbf{e} = \mathbf{A}\mathbf{x} - \mathbf{b}$ is known,
<u>the statistics may be used to obtain better solution</u> to $\mathbf{A}\mathbf{x} = \mathbf{b}$.

Remarks: • The columns in $\mathbf{V}$ are the eigenvectors of $\mathbf{C}^H\mathbf{C}$.

• Both $\mathbf{U}$ and $\mathbf{V}$ are unitary matrices, i.e,

$$\mathbf{U}\mathbf{U}^H = \mathbf{U}^H\mathbf{U} = \mathbf{I}, \quad \mathbf{V}\mathbf{V}^H = \mathbf{V}^H\mathbf{V} = \mathbf{I}.$$

• $\boldsymbol{\Sigma}$ is diagonal and the diagonal elements are the $\sqrt{\text{eigenvalues}}$ of $\mathbf{C}^H\mathbf{C}$

$$\begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_{n+1} \\ \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix}.$$

• $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{n+1} \geq 0, \sigma_i$ are real

Illustration of TLS solution
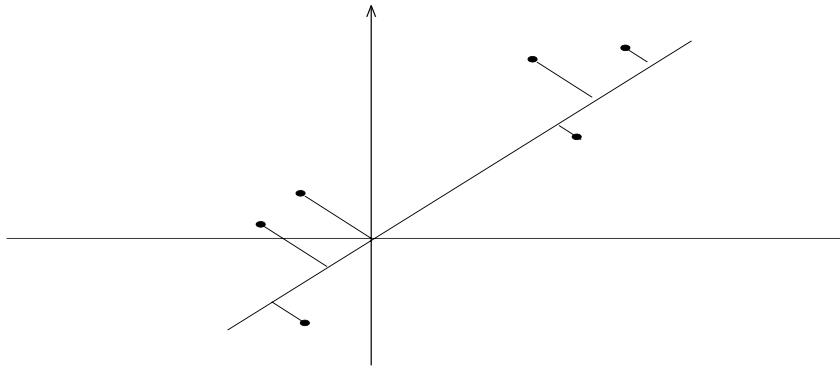
The straight line is found by minimizing the <u>shortest distance</u> between the line and the points squared

Let $\mathbf{C} = [\mathbf{A} \quad \mathbf{B}]$.

Let the <u>singular value decomposition (SVD)</u> of $\mathbf{C}$ be

$$\mathbf{C} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{H},$$

Remarks: • The columns of $\mathbf{U}$ are the eigenvectors of $\mathbf{C}\mathbf{C}^{H}$.

Total Least Squares (TLS) solution to $\mathbf{Ax} = \mathbf{b}$.

- Recall $\mathbf{x}_{LS}$ is obtained by perturbing $\mathbf{b}$ only, i.e,

$$\mathbf{Ax}_{LS} = \mathbf{b} + \mathbf{e}_{LS}. \qquad \mathbf{e}_{LS}^H \mathbf{e}_{LS} = \quad \min.$$

- $\mathbf{x}_{TLS}$ is obtained by perturbing both $\mathbf{A}$ and $\mathbf{b}$, i.e.,

$$(\mathbf{A} + \mathbf{E}_{TLS})\,\mathbf{x}_{TLS} = \mathbf{b} + \mathbf{e}_{TLS},$$

$$\|[\mathbf{E}_{TLS} \quad \mathbf{b}_{TLS}]\|_F = \quad \text{minimum},$$

where $\|.\|_F$ is <u>Frobenius matrix norm</u>,

$$\|\mathbf{G}\|_F = \sum_i \sum_j |g_{ij}|^2,$$

$$g_{ij} = (ij)^{th} \quad \text{element of } \mathbf{G}.$$

Ex.

$$\begin{bmatrix} 3 & 3-\delta \\ 4 & 4-\delta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

$$\mathbf{Q} = \frac{1}{5} \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}.$$

$$\mathbf{QAx} = \mathbf{Qb} \quad \text{gives} \quad \begin{bmatrix} 5 & 5+\frac{\delta}{5} \\ 0 & -\frac{7\delta}{5} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} \\ -\frac{7}{5} \end{bmatrix}.$$

$$\Rightarrow \begin{cases} x_2 = \frac{1}{\delta} \\ x_1 = -\frac{1}{\delta} \quad \text{(same as exact solution)} \end{cases}$$

Remark: For large number of overdetermined equations, QR method needs about <u>twice</u> as much computation as solving Normal equation in (1).

- **QR Method**: (Numerically more robust).

$$\mathbf{A}\mathbf{x} = \mathbf{b}.$$

Using <u>Householder transformation</u>, we can find an orthonormal matrix $\mathbf{Q}$ $\left(i.e, \quad \mathbf{Q}\mathbf{Q}^H = \mathbf{I}\right)$, such that

$$\begin{bmatrix} T \\ .... \\ 0 \end{bmatrix} \mathbf{x} = \mathbf{Q}\mathbf{A}\mathbf{x} = \mathbf{Q}\mathbf{b} = \begin{bmatrix} z_1 \\ \cdots \\ z_2 \end{bmatrix},$$

where $\mathbf{T}$ is a <u>square, upper triangular matrix</u>, and

$$\min \quad \mathbf{e}^H \mathbf{e} = \mathbf{z}_2^H \mathbf{z}_2$$

$$\Rightarrow \mathbf{T}\mathbf{x}_{LS} = \mathbf{z}_1$$

Back Substitution to find $\mathbf{x}_{LS}$

Solution to Normal equation :

$$\mathbf{x} = \left(\mathbf{A}^H\mathbf{A}\right)^{-1}\mathbf{A}^H\mathbf{b} = \begin{bmatrix} \frac{49}{\delta} + 2 \\[2mm] -\frac{49}{\delta} \end{bmatrix}.$$

Big Difference between this and the exact solution !

Ex.

$$\begin{bmatrix} 3 & 3 - \delta \\ 4 & 4 + \delta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \mathbf{A}\mathbf{x} = \mathbf{b}$$

where δ is a small number.

Exact solution:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\delta} \\ \frac{1}{\delta} \end{bmatrix}$$

Assume that due to truncation errors, $\delta^2 = 0$.

$$\mathbf{A}^H \mathbf{A} \doteq \begin{bmatrix} 25 & 25 + \delta \\ 25 + \delta & 25 + 2\delta \end{bmatrix}, \mathbf{A}^H \mathbf{b} = \begin{bmatrix} 1 \\ 1 + 2\delta \end{bmatrix}.$$

$$\begin{bmatrix} 1 & l_{21}^* & \cdots & l_{n1}^* \\ 0 & 1 & \cdots & l_{n2}^* \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \mathbf{L}^H \mathbf{x}_{LS} = \mathbf{D}^{-1}\mathbf{y} = \begin{bmatrix} \frac{y_1}{d_1} \\ \vdots \\ \frac{y_n}{d_n} \end{bmatrix}$$

$$\Rightarrow \begin{cases} x_n = \frac{y_n}{d_n} \\ x_k = \frac{y_k}{d_k} - \sum_{j=k+1}^{n} l_{jk}^* x_j, \quad k = n-1, \cdots \end{cases}$$

Remarks:

- Solving Normal equations may be sensitive to numerical errors.

Back - Substitution to solve:

$$\mathbf{LDL}^H \mathbf{x}_{LS} = \mathbf{g}$$

Let

$$\mathbf{y} = \mathbf{DL}^H \mathbf{x}_{LS}.$$

$$
\begin{bmatrix}
1 & 0 & \cdots & 0 \\
l_{21} & 1 & \cdots & 0 \\
\vdots & & \ddots & \\
l_{n1} & l_{n2} & \cdots & 1
\end{bmatrix}
\begin{bmatrix}
y_1 \\
y_2 \\
\vdots \\
y_n
\end{bmatrix}
=
\begin{bmatrix}
g_1 \\
\vdots \\
g_n
\end{bmatrix}
$$

$$
\begin{cases}
y_1 = g_1 \\
y_2 = g_2 - l_{21} y_1 \\
\vdots \\
y_k = g_k - \sum_{j=1}^{k-1} l_{kj} y_j, \quad k = 3, \cdots, n.
\end{cases}
$$

Cholesky Decomposition:

$$\mathbf{C} = \mathbf{LDL}^H,$$

where $\quad \mathbf{L} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ l_{21} & 1 & \cdots & 0 \\ \vdots & & \ddots & \\ l_{n1} & l_{n2} & \cdots & 1 \end{bmatrix} \quad$ (Lower Triangular Matrix)

$$\mathbf{D} = \begin{bmatrix} d_1 & & & 0 \\ & \ddots & & \\ 0 & & & d_n \end{bmatrix}, \quad d_i > 0.$$

Computational Aspects of LS

- **Solving Normal Equations**

$$\left(\mathbf{A}^H \mathbf{A}\right) \mathbf{x}_{LS} = \mathbf{A}^H \mathbf{b}.$$ (1)

This equation is called Normal equation.

Let

$$\mathbf{A}^H \mathbf{A} = \mathbf{C}, \quad \mathbf{A}^H \mathbf{b} = \mathbf{g}.$$

$$\mathbf{C}\mathbf{x}_{LS} = \mathbf{g}, \quad \text{where } \mathbf{C} \text{ is positive definite.}$$

Ex:

$$\mathbf{A} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{x}_{LS} = ?$$

$$
\begin{aligned}
\mathbf{x}_{LS} &= (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{b} \\
&= \left(\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1
\end{aligned}
$$

$$\mathbf{A}\mathbf{x}_{LS} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} (1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$\mathbf{e}_{LS} = \mathbf{A}\mathbf{x}_{LS} - \mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Illustration of LS solution:

Let

$$\mathbf{A} = [\mathbf{a}_1 \;\; \vdots \;\; \mathbf{a}_2].$$

$$\mathbf{x}_{LS} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
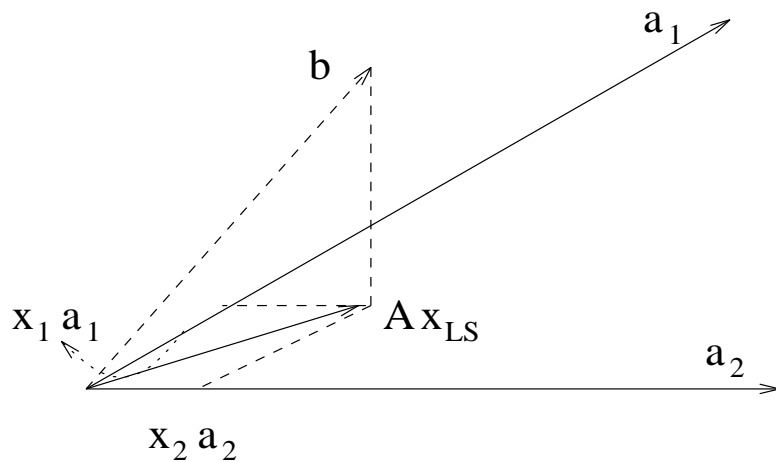
$$
\begin{aligned}
\mathbf{e}^H \mathbf{e} &= (\mathbf{A}\mathbf{x} - \mathbf{b})^H (\mathbf{A}\mathbf{x} - \mathbf{b}) \\
&= \mathbf{x}^H \mathbf{A}^H \mathbf{A}\mathbf{x} - \mathbf{x}^H \mathbf{A}^H \mathbf{b} - \mathbf{b}^H \mathbf{A}\mathbf{x} + \mathbf{b}^H \mathbf{b} \\
&= \left[\mathbf{x} - (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{b} \right]^H (\mathbf{A}^H \mathbf{A}) \left[\mathbf{x} - (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{b} \right] \\
&+ \left[\mathbf{b}^H \mathbf{b} - \mathbf{b}^H \mathbf{A}(\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{b} \right]
\end{aligned}
$$

Remark: • The 2nd term above is independent of $\mathbf{x}$.

• $\mathbf{e}^H \mathbf{e}$ is minimized if

$$
\boxed{\mathbf{x} = (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{b}}
$$
  LS Solution

Ex:
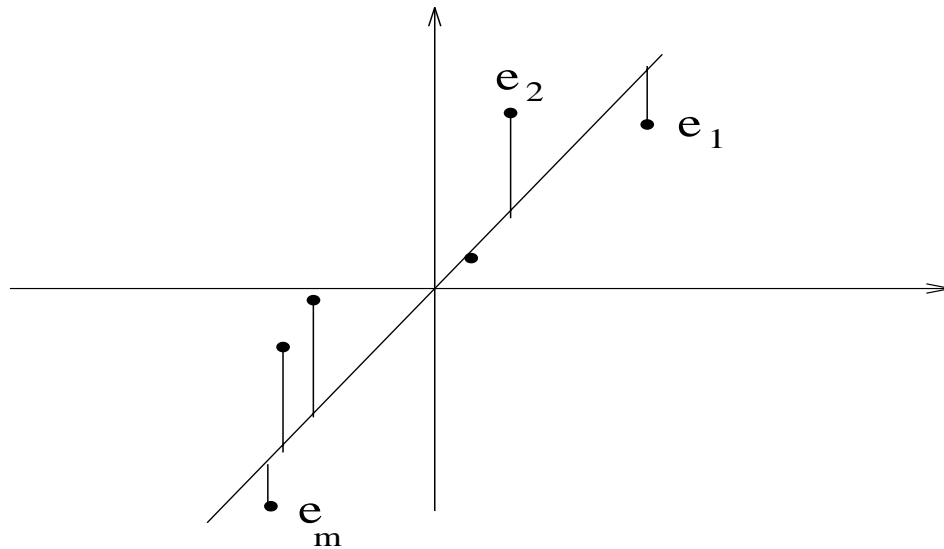
Remarks: • $\mathbf{A}\mathbf{x}_{LS} = \mathbf{b} + \mathbf{e}_{LS}$

• We see that $\mathbf{x}_{LS}$ is found by perturbing $\mathbf{b}$ so that a solution exists.

Least Squares (LS) Solution

for Overdetermined Equations:

- Objective of LS solution:

$$\text{Let} \quad \mathbf{e} = \mathbf{A}\mathbf{x} - \mathbf{b}$$

Find $\quad \mathbf{x}_{LS} \quad$ so that $\quad \mathbf{e}^H \mathbf{e} \quad$ is minimized.

$$\text{Let} \quad \mathbf{e} = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_m \end{bmatrix}$$

Euclidean Norm $= \mathbf{e}^H \mathbf{e} = |e_1|^2 + |e_2|^2 + \cdots + |e_m|^2$

Solving Linear Equations:

Consider $\mathbf{A}^{m \times n} \mathbf{x}^{n \times 1} = \mathbf{b}^{m \times 1}$.

• When $m = n$ and $\mathbf{A}$ is full rank, $\mathbf{x} = \mathbf{A}^{-1} \mathbf{b}$.

• When $m > n$ and $\mathbf{A}$ is full rank n, then the solution exists if $\mathbf{b}$ is in the n-dimensional subspace of the m-dimensional space that is determined by the columns in $\mathbf{A}$.

Ex:

$$\mathbf{A} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\text{If} \quad \mathbf{b} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \mathbf{x} = 3.$$

$$If \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{x} =? \quad \text{does not exist !}$$

Remarks:

- The overdetermined linear equations may be solved with Least Squares or Total Least Squares Methods.

- M should be chosen based on the trade-off between information contained in the large lags of $\hat{r}(k)$ and the accuracy of $\hat{r}(k)$.

- Overdetermined YW -equation may also be obtained for AR signals.

Overdetermined Modified Yule - Walker Equation $(M > p)$

$$\begin{bmatrix} \hat{r}(q) & \cdots & \hat{r}(q-p+1) \\ \vdots & & \vdots \\ \hat{r}(q+p-1) & \cdots & \hat{r}(q) \\ \vdots & & \vdots \\ \hat{r}(q+M-1) & \cdots & \hat{r}(q+M-p) \end{bmatrix} \begin{bmatrix} \hat{a}_1 \\ \vdots \\ \hat{a}_p \end{bmatrix} \approx - \begin{bmatrix} \hat{r}(q+1) \\ \vdots \\ \hat{r}(q+p) \\ \vdots \\ \hat{r}(q+M) \end{bmatrix}$$

Ex: Consider

$$x(n) = \cos(\omega_1 n + \phi_1) + \cos(\omega_2 n + \phi_2),$$

where ϕ_1 and ϕ_2 are independent and uniformly distributed on $[0,2\pi]$.

$$r(k) = \frac{1}{2}\cos(\omega_1 k) + \frac{1}{2}\cos(\omega_2 k).$$
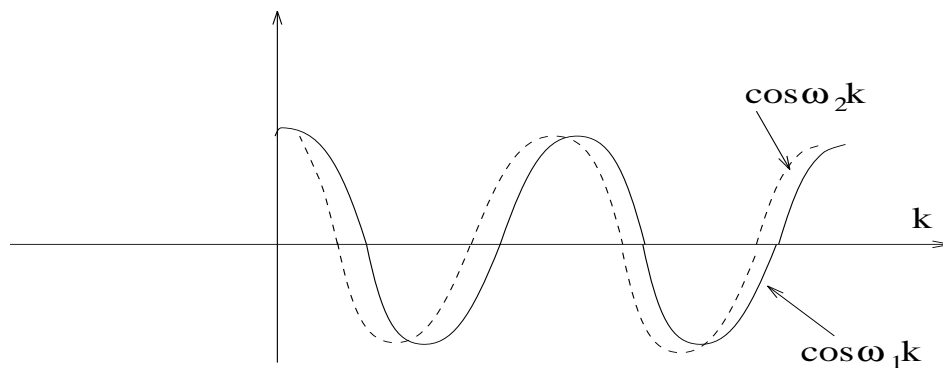
Note that when $\omega_1 \approx \omega_2$, large values of k are needed to distinguish $\cos(\omega_1 k)$ and $\cos(\omega_2 k)$.

Remark: This comment is true for both AR and ARMA models.

ARMA PSD Estimate:

$$\hat{P}(\omega) = \frac{\sum_{k=-q}^{q} \hat{\gamma}_k e^{-j\omega k}}{\left| \hat{A}(\omega) \right|^2}$$

Remarks:

- This method is called <u>modified YW ARMA Spectral Estimator</u>

- $\hat{P}(\omega)$ is <u>not</u> guaranteed to be $\geq 0, \quad \forall \, \omega$, due to the MA part.

- The AR estimates $\hat{a}_1, \cdots, \hat{a}_p$ have <u>reasonable accuracy</u> if the ARMA <u>poles and zeroes are well inside the unit circle</u>.

- Very poor estimates $\hat{a}_1, \cdots, \hat{a}_p$ occur when ARMA poles and zeroes are closely-spaced and nearby unit circle. (This is narrowband signal case).

$$
\begin{aligned}
\gamma_k &= E\left[y(n)y(n-k)\right] \\
&= E\left[A(z)x(n)A(z)x(n-k)\right] \\
&= E\left[\sum_{i=0}^{p} a_i x(n-i) \sum_{j=0}^{p} a_j x(n-j-k)\right] \\
&= \sum_{i=0}^{p}\sum_{j=0}^{p} a_i a_j r(k+j-i).
\end{aligned}
$$

Since $\hat{a}_1, \cdots, \hat{a}_p$ may be computed with the modified YW- Method

$$
\begin{cases}
\hat{\gamma}_k = \sum_{i=0}^{p}\sum_{j=0}^{p} \hat{r}(k+j-i)\hat{a}_i\hat{a}_j, \quad \hat{a}_0 \overset{\triangle}{=} 1, \quad k = 0,1,\cdots,q \\
\hat{\gamma}_{-k} = \gamma_k.
\end{cases}
$$

What about the MA part of the ARMA PSD?

$$\text{Let} \quad y(n) = (1 + b_1 z^{-1} + \cdots + b_q z^{-q}) u(n).$$

The ARMA model becomes

$$(1 + a_1 z^{-1} + \cdots + a_p z^{-p}) x(n) = y(n)$$
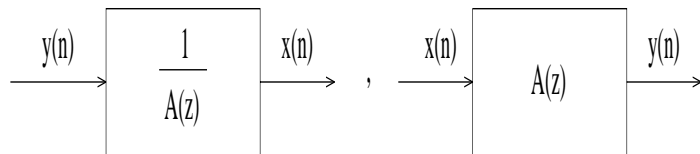
$$P_x(\omega) = \left| \frac{1}{A(\omega)} \right|^2 P_y(\omega).$$

Let γ_k be the autocorrelation function of $y(n)$. Then (see MA signals).

$$P_y(\omega) = \sum_{k=-q}^{q} \gamma_k e^{-j\omega k}$$

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Remarks:

(1) Replacing $\hat{r}(k)$ for $r(k)$ above, we can solve for $\hat{a}_1, \cdots, \hat{a}_p$.

(2) The matrix on the left side

- is nonsingular under mild conditions.

- is Toeplitz.

- is NOT symmetric.

- Levinson - type fast algorithms exist.

To solve for $a_1, \cdots, a_p$ we need p equations. Using $r(k) = r(-k)$ gives

$$
\begin{bmatrix}
r(q+1) & r(q) & \cdots & r(q-p+1) \\
r(q+2) & r(q+1) & \cdots & r(q-p+2) \\
\vdots & & \ddots & \\
r(q+p) & r(q+p-1) & \cdots & r(q)
\end{bmatrix}
\begin{bmatrix}
1 \\
a_1 \\
\vdots \\
a_p
\end{bmatrix}
= 0.
$$

$$
\Rightarrow
\begin{bmatrix}
r(q) & \cdots & r(q-p+1) \\
r(q+1) & \ddots & \\
\vdots & & \\
r(q+p-1) & \cdots & r(q)
\end{bmatrix}
\begin{bmatrix}
a_1 \\
\vdots \\
a_p
\end{bmatrix}
= -
\begin{bmatrix}
r(q+1) \\
r(q+2) \\
\vdots \\
r(q+p)
\end{bmatrix}
$$

<u>k ≥ q+1</u>

$$\begin{bmatrix} 1 & a_1 & \cdots & a_p \end{bmatrix} \begin{bmatrix} r(-k) \\ r(-k+1) \\ \vdots \\ r(-k+p) \end{bmatrix} = \begin{bmatrix} 1 & b_1 & \cdots & b_q \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{0}.$$

$$\Rightarrow \begin{bmatrix} r(-(q+1)) & r(-q) & \cdots & r(-(q+1)+p) \\ r(-(q+2)) & r(-(q+1)) & \cdots & r(-(q+2)+p) \\ \vdots & & \ddots & \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \\ \vdots \\ a_p \end{bmatrix} = \mathbf{0}.$$

This is the modifed YW - Equation

$$
\begin{bmatrix} 1 & a_1 & \cdots & a_p \end{bmatrix}
\begin{bmatrix} r(0) \\ r(1) \\ \vdots \\ r(p) \end{bmatrix}
=
\begin{bmatrix} 1 & b_1 & \cdots & b_q \end{bmatrix}
\begin{bmatrix} \sigma^2 \\ \sigma^2 h_1 \\ \vdots \\ \sigma^2 h_q \end{bmatrix}
$$

k = 1:

$$
\begin{bmatrix} 1 & a_1 & \cdots & a_p \end{bmatrix}
\begin{bmatrix} r(-1) \\ r(0) \\ \vdots \\ r(p-1) \end{bmatrix}
=
\begin{bmatrix} 1 & b_1 & \cdots & b_q \end{bmatrix}
\begin{bmatrix} 0 \\ \sigma^2 \\ \sigma^2 h_1 \\ \vdots \\ \sigma^2 h_{q-1} \end{bmatrix}
$$

$\vdots$

ARMA Signals: (Also called Pole -Zero Model).

$(1 + a_1 z^{-1} + \cdots + a_p z^{-p})x(n) = (1 + b_1 z^{-1} + \cdots + b_q z^{-q})u(n).$

Let us write $x(n)$ as MA(∞):

$x(n) = u(n) + h_1 u(n-1) + h_2 u(n-2) + \cdots$

$$\Rightarrow \begin{cases} E\left[x(n)u(n)\right] = \sigma^2. \\ E\left[u(n)x(n-k)\right] = 0, \quad k \geq 1 \end{cases}$$

ARMA model can be written as

$$\begin{bmatrix} 1 & a_1 & \cdots & a_p \end{bmatrix} \begin{bmatrix} x(n) \\ x(n-1) \\ \vdots \\ x(n-p) \end{bmatrix} = \begin{bmatrix} 1 & b_1 & \cdots & b_q \end{bmatrix} \begin{bmatrix} u(n) \\ u(n-1) \\ \vdots \\ u(n-q) \end{bmatrix}$$

• Next we shall multiply both sides by $x(n-k)$ and take $E\{.\}$.

Remarks: • Estimating $b_1, \cdots, b_q$ is a <u>nonlinear</u> problem.

$$\text{A simple estimator is} \quad \hat{P}(\omega) = \sum_{k=-q}^{q} \hat{r}(k)e^{-j\omega k}.$$

* This is <u>exactly Blackman - Tukey</u> method with rectangular window of length $2q + 1$.

* No matter whether $\hat{r}(k)$ is biased or unbiased estimate, this $\hat{P}(\omega)$ may be < 0 .

* When <u>unbiased</u> $\hat{r}(k)$ is used, $\hat{P}(\omega)$ is unbiased.

* <u>To ensure $\hat{P}(\omega) \geq 0$</u> , $\forall \, \omega$, we may use <u>biased $\hat{r}(k)$ and a window</u> with $W(\omega) \geq 0, \quad \forall \, \omega$. For this case, $\hat{P}(\omega)$ is biased.

This is again exactly BT-method.

• A <u>most used</u> MA spectral estimator is based on a <u>Two-Stage Least Squares Method.</u> See the discussions on ARMA later.

MA Signals:

$$
\begin{aligned}
x(n) &= B(z)u(n) \\
&= u(n) + b_1 u(n-1) + \cdots + b_q u(n-q) \\
r(k) &= E\left[x(n)x(n-k)\right] \\
&= E\left\{[u(n) + \cdots + b_q u(n-q)]\right. \\
&\qquad \left.[u(n-k) + \cdots + b_q u(n-q-k)]\right\}
\end{aligned}
$$

$$
\boxed{|k| > q: \qquad r(k) = 0}
$$

$$
\boxed{
\begin{aligned}
|k| < q: \quad r(k) &= \sigma^2 \sum_{l=0}^{q-k} b_l b_{l+k}, \quad q > k \geq 0 \\
r(k) &= r(-k). \qquad -q < k < 0 \\
b_0 = 1, b_1, &\cdots, b_q = \quad \text{real.}
\end{aligned}
}
$$

$$
\Rightarrow \qquad \boxed{P(\omega) = \sum_{k=-q}^{q} r(k) e^{-j\omega k}.}
$$

Recall

$$\delta_{n+1} = \delta_n(1 - k_{n+1}^2).$$

If $\mathbf{R}_{n+1} > 0$,

$$\Rightarrow r(0) > 0, \quad \delta_n > 0, \quad n = 1, 2, \cdots, p,$$

$$k_{n+1}^2 = \frac{\delta_n - \delta_{n+1}}{\delta_n}$$

$$\text{Since} \quad \delta_n - \delta_{n+1} < \delta_n,$$

$$k_{n+1}^2 < 1 \quad \Rightarrow \quad |k_{n+1}| < 1.$$

If $\quad |k_n| < 1, \quad r(0) > 0$,

$$\Rightarrow \quad k_{n+1}^2 < 1.$$

$$\Rightarrow \begin{cases} \delta_0 = r(0) > 0, \\ \delta_{n+1} = \delta_n(1 - k_{n+1}^2) > 0, \quad n = 1, 2, \cdots, p - 1 \end{cases}$$

$$\Rightarrow \quad \mathbf{U}_{n+1}^T \mathbf{R}_{n+1} \mathbf{U}_{n+1} = \mathbf{D}_{n+1}.$$

$$\Rightarrow (*) \quad \text{proven !}$$

$$\text{Since} \quad \mathbf{U}_{n+1}^{-1} \mathbf{R}_{n+1}^{-1} \left(\mathbf{U}_{n+1}^T \right)^{-1} = \mathbf{D}_{n+1}^{-1},$$

$\Rightarrow$

$$\boxed{\mathbf{R}_{n+1}^{-1} = \mathbf{U}_{n+1} \mathbf{D}_{n+1}^{-1} \mathbf{U}_{n+1}^T.}$$

$\mathbf{U}_{n+1} \mathbf{D}_{n+1}^{-\frac{1}{2}}$ is called $\underline{\text{Cholesky Factor}}$ of $\mathbf{R}_{n+1}^{-1}$

- Consider the determinant of $\mathbf{R}_{n+1}$:

$$\det(\mathbf{R}_{n+1}) = \det(\mathbf{D}_{n+1}) = r(0)\Pi_{k=1}^n \delta_k$$

$$\Rightarrow \quad \det(\mathbf{R}_{n+1}) = \delta_n \det(\mathbf{R}_n)$$

$$\Rightarrow \quad \mathbf{R}_{n+1} > 0, \quad n = 1, 2, \cdots, p, \quad \text{iff} \quad r(0) > 0$$

$$\text{and} \quad \delta_k > 0, \quad k = 1, 2, \cdots, p.$$

$$\Rightarrow \begin{bmatrix} r(0) & \mathbf{r}_k^T \\ \mathbf{r}_k & \mathbf{R}_k \end{bmatrix} \begin{bmatrix} 1 \\ \boldsymbol{\theta}_k \end{bmatrix} = \begin{bmatrix} \delta_k \\ \mathbf{0} \end{bmatrix}$$

$$\Rightarrow r(0) + \mathbf{r}_k^T \boldsymbol{\theta}_k = \delta_k \quad \Rightarrow r(0) + \boldsymbol{\theta}_k^T \mathbf{r}_k \quad = \delta_k$$

$$\underline{\mathbf{r}_k + \mathbf{R}_k \boldsymbol{\theta}_k = 0} \quad \Rightarrow \quad \mathbf{r}_k^T + \boldsymbol{\theta}_k^T \mathbf{R}_k^T$$

$$= \quad \mathbf{r}_k^T + \boldsymbol{\theta}_k^T \mathbf{R}_k \quad = 0$$

$$\Rightarrow \mathbf{U}_{k+1}^T \mathbf{R}_{k+1} \mathbf{U}_{k+1} \quad = \quad \begin{bmatrix} \delta_k & \mathbf{0} \\ \mathbf{U}_k^T \mathbf{r}_k & \mathbf{U}_k^T \mathbf{R}_k \end{bmatrix} \begin{bmatrix} 1 & \mathbf{0} \\ \boldsymbol{\theta}_k & \mathbf{U}_k \end{bmatrix}$$

$$= \quad \begin{bmatrix} \delta_k & \mathbf{0} \\ \mathbf{U}_k^T \mathbf{r}_k + \mathbf{U}_k^T \mathbf{R}_k \boldsymbol{\theta}_k & \mathbf{U}_k^T \mathbf{R}_k \mathbf{U}_k \end{bmatrix}$$

$$= \quad \begin{bmatrix} \delta_k & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_k \end{bmatrix} = \mathbf{D}_{k+1}$$

Suppose $(*)$ is true for $n = k - 1$, i.e.,

$$\mathbf{U}_k^T \mathbf{R}_k \mathbf{U}_k = \mathbf{D}_k.$$

Consider $n = k$:

$$
\mathbf{U}_{k+1}^T \mathbf{R}_{k+1} \mathbf{U}_{k+1} =
\begin{bmatrix} 1 & \boldsymbol{\theta}_k^T \\ \mathbf{0} & \mathbf{U}_k^T \end{bmatrix}
\begin{bmatrix} r(0) & \mathbf{r}_k^T \\ \mathbf{r}_k & \mathbf{R}_k \end{bmatrix}
\begin{bmatrix} 1 & \mathbf{0} \\ \boldsymbol{\theta}_k & \mathbf{U}_k \end{bmatrix}
$$

$$
=
\begin{bmatrix} r(0) + \boldsymbol{\theta}_k^T \mathbf{r}_k & \mathbf{r}_k^T + \boldsymbol{\theta}_k^T \mathbf{R}_k \\ \mathbf{U}_k^T \mathbf{r}_k & \mathbf{U}_k \mathbf{R}_k \end{bmatrix}
\begin{bmatrix} 1 & \mathbf{0} \\ \boldsymbol{\theta}_k & \mathbf{U}_k \end{bmatrix}
$$

Since $\quad \mathbf{R}_{k+1} \begin{bmatrix} 1 \\ \boldsymbol{\theta}_k \end{bmatrix} = \begin{bmatrix} \delta_k \\ \mathbf{0} \end{bmatrix}$

111

<u>n = 1 :</u>

$$\begin{bmatrix} r(0) & r(1) \\ r(1) & r(0) \end{bmatrix} \begin{bmatrix} 1 \\ a_{1,1} \end{bmatrix} = \begin{bmatrix} \delta_1 \\ 0 \end{bmatrix}.$$

$$\Rightarrow \begin{bmatrix} 1 & a_{1,1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r(0) & r(1) \\ r(1) & r(0) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ a_{1,1} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & a_{1,1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \delta_1 & r(1) \\ 0 & r(0) \end{bmatrix}$$

$$= \begin{bmatrix} \delta_1 & 0 \\ 0 & r(0) \end{bmatrix}.$$

Proof (for the second property above only): We first use induction to prove:

$$
\underbrace{\begin{bmatrix} 1 & a_{n,1} & \cdots & a_{n,n} \\ & \ddots & & \vdots \\ 0 & & 1 & a_{1,1} \\ & & & 1 \end{bmatrix}}_{\mathbf{U}^T_{n+1}}
\underbrace{\begin{bmatrix} r(0) & \cdots & r(n) \\ & \ddots & \\ r(n) & & r(0) \end{bmatrix}}_{\mathbf{R}_{n+1}}
\underbrace{\begin{bmatrix} 1 & & & \\ a_{n,1} & 1 & & \\ \vdots & & \ddots & \\ a_{n,n} & \cdots & a_{1,1} & 1 \end{bmatrix}}_{\mathbf{U}_{n+1}}
$$

$$
= \underbrace{\begin{bmatrix} \delta_n & & & \\ & \ddots & & \mathbf{0} \\ \mathbf{0} & & \delta_1 & \\ & & & r(0) \end{bmatrix}}_{\mathbf{D}_{n+1}} \qquad (*)
$$

$$
\boldsymbol{\theta}_2 = \begin{bmatrix} \theta_1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} \tilde{\theta}_1 \\ 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} -\rho \\ 0 \end{bmatrix} + 0 \begin{bmatrix} -\rho \\ 1 \end{bmatrix} = \begin{bmatrix} -\rho \\ 0 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}
$$

Properties of LDA:

- $|k_n| < 1, \quad n = 1, 2, \cdots, p, \quad$ and $\quad r(0) > 0,$ iff

$$
A_n(z) = 1 + a_{n,1} z^{-1} + \cdots + a_{n,n} z^{-n} = 0
$$

has roots inside the unit circle.

- $|k_n| < 1, \quad n = 1, 2, \cdots, p, \quad$ and $\quad r(0) > 0 \quad$ iff $\quad \mathbf{R}_{n+1} > 0$

LDA: Initialization:

$$
\begin{cases}
\theta_1 = -\frac{r(1)}{r(0)} = -\frac{\rho}{1} = -\rho \\[2mm]
\delta_1 = r(0) - \frac{r^2(1)}{r(0)} = 1 - \rho^2. \\[2mm]
k_1 = \theta_1 = -\rho.
\end{cases}
$$

$$
r_1 = \rho,
$$

$$
\begin{aligned}
k_2 &= -\frac{r(2) + \theta_1^T \tilde{r}_1}{\delta_1} \\[3mm]
&= -\frac{\rho^2 + (-\rho)\rho}{1 - \rho^2} = 0 \\[3mm]
\delta_2 &= \delta_1(1 - k_2^2) = (1 - \rho^2)(1 - 0^2) \\[2mm]
&= 1 - \rho^2 = \sigma^2
\end{aligned}
$$

Ex:

$$\begin{bmatrix} 1 & \rho & \rho^2 \\ \rho & 1 & \rho \\ \rho^2 & \rho & 1 \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \sigma^2 \\ 0 \\ 0 \end{bmatrix}.$$

Straightforward Solution:

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = -\begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}^{-1} \begin{bmatrix} \rho \\ \rho^2 \end{bmatrix}$$

$$= -\frac{1}{(1-\rho^2)} \begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix} \begin{bmatrix} \rho \\ \rho^2 \end{bmatrix}$$

$$= \begin{bmatrix} -\rho \\ 0 \end{bmatrix} \Rightarrow \sigma^2 = 1 - \rho^2.$$

LDA: Initialization:

$$n = 1: \quad \mathbf{R}_2 = \begin{bmatrix} r(0) & r(1) \\ r(1) & r(0) \end{bmatrix} \begin{bmatrix} 1 \\ \theta_1 \end{bmatrix} = \begin{bmatrix} \delta_1 \\ 0 \end{bmatrix}$$

$$\boxed{\theta_1 = -\frac{r(1)}{r(0)}} \qquad O(1) \quad \text{flops}$$

$$\boxed{\delta_1 = r(0) - \frac{r^2(1)}{r(0)}} \qquad O(1) \quad \text{flops}$$

$$\boxed{k_1 = \theta_1}$$

For $n = 1, 2, \cdots, p - 1$, do:

$$k_{n+1} = -\frac{r(n+1) + \boldsymbol{\theta}_n^T \tilde{\mathbf{r}}_n}{\delta_n} \qquad \sim n \quad \text{flops}$$

$$\delta_{n+1} = \delta_n(1 - k_{n+1}^2) \qquad O(1) \quad \text{flops}$$

$$\boldsymbol{\theta}_{n+1} = \begin{bmatrix} \boldsymbol{\theta}_n \\ 0 \end{bmatrix} + k_{n+1} \begin{bmatrix} \tilde{\boldsymbol{\theta}}_n \\ 1 \end{bmatrix}. \qquad \sim n \quad \text{flops}$$

Proof:

$$\mathbf{R}_{n+2} \begin{bmatrix} 1 \\ \boldsymbol{\theta}_{n+1} \end{bmatrix} = \mathbf{R}_{n+2} \left\{ \begin{bmatrix} 1 \\ \boldsymbol{\theta}_n \\ 0 \end{bmatrix} + k_{n+1} \begin{bmatrix} 0 \\ \tilde{\boldsymbol{\theta}}_n \\ 1 \end{bmatrix} \right\}$$

$$= \begin{bmatrix} \delta_n \\ \mathbf{0} \\ \alpha_n \end{bmatrix} + k_{n+1} \begin{bmatrix} \alpha_n \\ \mathbf{0} \\ \delta_n \end{bmatrix}$$

$$= \begin{bmatrix} \delta_n + k_{n+1}\alpha_n \\ \mathbf{0} \\ \alpha_n + k_{n+1}\delta_n \end{bmatrix} = \begin{bmatrix} \delta_{n+1} \\ \mathbf{0} \\ 0 \end{bmatrix}.$$

Result:

$$\text{Let} \quad k_{n+1} = -\frac{\alpha_n}{\delta_n}. \quad \text{Then}$$

$$\boldsymbol{\theta}_{n+1} = \begin{bmatrix} \boldsymbol{\theta}_n \\ 0 \end{bmatrix} + k_{n+1} \begin{bmatrix} \tilde{\boldsymbol{\theta}}_n \\ 1 \end{bmatrix}.$$

$$\delta_{n+1} = \delta_n(1 - k_{n+1}^2)$$

Consider:

$$\mathbf{R}_{n+2} \begin{bmatrix} 1 \\ \boldsymbol{\theta}_n \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{R}_{n+1} & \vdots & r(n+1) \\ & \vdots & r(n) \\ & \vdots & \vdots \\ & \vdots & r(1) \\ r(n+1) & \cdots & \vdots & r(0) \end{bmatrix} \begin{bmatrix} 1 \\ \boldsymbol{\theta}_n \\ \cdots \\ 0 \end{bmatrix} = \begin{bmatrix} \delta_n \\ \mathbf{0} \\ \cdots \\ \alpha_n \end{bmatrix}$$

$$\text{Let} \quad \mathbf{r}_n = \begin{bmatrix} r(1) \\ \vdots \\ r(n) \end{bmatrix}.$$

Then $\alpha_n = r(n+1) + \boldsymbol{\theta}_n{}^T \tilde{\mathbf{r}}_n.$

Proof:

$$\mathbf{A} = \begin{bmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ a_1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_1 \\ a_{n-1} & \cdots & a_1 & a_0 \end{bmatrix}$$

$$\begin{aligned} \Rightarrow \mathbf{A}_{ij} &= a_{|i-j|} \\ \tilde{\mathbf{c}}_i &= \mathbf{c}_{n-i+1} = \sum_{k=1}^{n} \mathbf{A}_{n-i+1,k} b_k \\ &= \sum_{k=1}^{n} a_{|n-i+1-k|} b_k \\ &= \sum_{m=1}^{n} a_{|m-i|} b_{n-m+1} = \sum_{m=1}^{n} \mathbf{A}_{m,i} \tilde{b}_m \quad (m = n - k + 1) \\ &= (\mathbf{A}\tilde{\mathbf{b}})_i \end{aligned}$$

Let $\mathbf{A}$ = Symmetric and Toeplitz.

$$\text{Let} \quad \tilde{\mathbf{b}} = \begin{bmatrix} b_n \\ b_{n-1} \\ \vdots \\ b_1 \end{bmatrix}, \text{with} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

Then if $\mathbf{c} = \mathbf{A}\mathbf{b}$

$$\Rightarrow \tilde{\mathbf{c}} = \mathbf{A}\tilde{\mathbf{b}}$$

LDA solves

$$
\mathbf{R}_{n+1}
\begin{bmatrix} 1 \\ \cdots \\ \boldsymbol{\theta}_n \end{bmatrix}
=
\begin{bmatrix} \delta_n \\ \cdots \\ \mathbf{0} \end{bmatrix}
$$

<u>recursively in n,</u> starting from $n = 1$.

Remark:

For $n = 1, 2, \cdots, p$,

• LDA needs $\approx p^2$ flops

• Regular matrix inverses need $\approx p^4$ flops.

Levinson - Durbin Algorithm (LDA)

Let

$$\mathbf{R}_{n+1} = \begin{bmatrix} r(0) & r(1) & \cdots & r(n) \\ r(1) & r(0) & & \\ \vdots & & \ddots & \\ r(n) & r(n-1) & & r(0) \end{bmatrix}, \quad (\text{ real signal })$$

$$n = 1, 2, \cdots, p$$

$$\text{Let} \quad \boldsymbol{\theta}_n = \begin{bmatrix} a_{n,1} \\ \vdots \\ a_{n,n} \end{bmatrix},$$

<u>Efficient Methods for solving</u>

$$\mathbf{Ra} = -\mathbf{r} \quad \text{or} \quad \hat{\mathbf{R}}\hat{\mathbf{a}} = -\hat{\mathbf{r}}$$

- Levinson - Durbin Algorithm.

- Delsarte - Genin Algorithm.

- Gohberg - Semencul Formula for $\mathbf{R}^{-1}$ or $\hat{\mathbf{R}}^{-1}$

(Sometimes, we may be interested in not only $\mathbf{a}$ but also $\mathbf{R}^{-1}$)

Remarks:

- If $\{\hat{r}(k)\}$ is a <u>positive definite</u> sequence and if $a_1, \cdots, a_p$ are found by solving $\mathbf{R}\mathbf{a} = -\mathbf{r}$, then the roots of polynomial $1 + a_1 z^{-1} + \cdots + a_p z^{-p}$ are <u>inside the unit circle</u>.

- The AR system thus obtained is BIBO <u>stable</u>

- <u>Biased estimate $\{\hat{r}(k)\}$</u> should be used in YW-equation to obtain a stable AR system:

Remarks:

- When we only have N samples, $\{r(k)\}$ is not available. $\{\hat{r}(k)\}$ may be used to replace $\{r(k)\}$ to obtain $\hat{a}_1, \cdots, \hat{a}_p$.

$\Rightarrow$ This is the <u>Yule - Walker Method.</u>

- $\mathbf{R}$ is a positive semidefinite matrix. $\mathbf{R}$ is positive definite unless $x(n)$ is a sum of less than $\lfloor \frac{p}{2} \rfloor$ sinusoids.

- $\mathbf{R}$ is Toeplitz.

- Levinson - Durbin algorithm is used to solve for a efficiently

- AR models are most frequently used in practice.

- Estimation of AR parameters is a well-established topic.

$$\Rightarrow \begin{bmatrix} r(0) & r(-1) & \cdots & r(-p) \\ r(1) & r(0) & \cdots & r(-p+1) \\ \vdots & \ddots & \ddots & \\ r(p) & r(p-1) & \cdots & r(0) \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \\ \vdots \\ a_p \end{bmatrix} = \begin{bmatrix} \sigma^2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

$$\mathbf{Ra} = -\mathbf{r} \Leftrightarrow \begin{bmatrix} r(0) & \cdots & r(-p+1) \\ \vdots & \ddots & \\ r(p-1) & \cdots & r(0) \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_p \end{bmatrix} = - \begin{bmatrix} r(1) \\ \vdots \\ r(p) \end{bmatrix}.$$

$\underline{k \geq 1,}$

$$E\left\{ x(n-k) \begin{bmatrix} x(n) & x(n-1) & \cdots & x(n-p) \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \\ \vdots \\ a_p \end{bmatrix} \right\} = 0.$$

$$\Rightarrow \begin{bmatrix} r(k) & r(k-1) & \cdots & r(k-p) \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \\ \vdots \\ a_p \end{bmatrix} = 0. \quad (**)$$

<u>k = 0</u>,

$$E\left\{x(n)\begin{bmatrix}x(n) & x(n-1) & \cdots & x(n-p)\end{bmatrix}\begin{bmatrix}1\\a_1\\\vdots\\a_p\end{bmatrix}\right\} = \sigma^2.$$

$$\Rightarrow \begin{bmatrix}r(0) & r(-1) & \cdots & r(-p)\end{bmatrix}\begin{bmatrix}1\\a_1\\\vdots\\a_p\end{bmatrix} = \sigma^2. \quad (*)$$

Computing Coefficients from $r(k)$.

AR signals.

Let $\quad \frac{1}{A(z)} = 1 + \alpha_1 z^{-1} + \alpha_2 z^{-2} + \cdots$

$x(n) = \frac{1}{A(z)} u(n) = u(n) + \alpha_1 u(n-1) + \cdots$

$$\begin{cases} E\left[x(n)u(n)\right] = \sigma^2 \\ E\left[x(n-k)u(n)\right] = 0, \ \text{k} \geq 1 \end{cases}$$

Since $\quad A(z)x(n) = u(n)$

$x(n) + a_1 x(n-1) + \cdots + a_p x(n-p) = u(n)$

$$\begin{bmatrix} x(n) & x(n-1) & \cdots & x(n-p) \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \\ \cdots \\ a_p \end{bmatrix} = u(n)$$

$$\begin{bmatrix} c_{p+1} & c_p & \cdots & c_{p-q+1} \\ \vdots & \ddots & \ddots & \vdots \\ & & \ddots & \\ c_{p+q} & & & c_p \end{bmatrix} \begin{bmatrix} 1 \\ b_1 \\ \vdots \\ b_q \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} c_p & \cdots & c_{p-q+1} \\ \vdots & \ddots & \\ c_{p+q-1} & \cdots & c_p \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_q \end{bmatrix} = - \begin{bmatrix} c_{p+1} \\ \vdots \\ c_{p+q} \end{bmatrix} . (*)$$

Remark: Once $b_1, \cdots, b_q$ are computed with $(*)$ $a_1, \cdots, a_p$ can be computed with $(\diamond)$.

Since $\dfrac{B(z)}{A(z)} = \dfrac{1}{C(z)} \Rightarrow B(z)C(z) = A(z)$

$$\Rightarrow \left[1 + b_1 z^{-1} + \cdots + b_q z^{-q}\right]\left[1 + c_1 z^{-1} + \cdots\right]$$

$$= \left[1 + a_1 z^{-1} + \cdots + a_p z^{-p}\right]$$

$$
\begin{bmatrix}
1 & 0 & \cdots & \cdots & 0 \\
c_1 & 1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & & \vdots \\
c_p & \ddots & \ddots & \ddots & \\
c_{p+1} & \ddots & \ddots & \ddots & 1 \\
\vdots & \ddots & \ddots & & c_1 \\
\vdots & & & & c_p \\
\vdots & & & &
\end{bmatrix}
\begin{bmatrix}
1 \\
b_1 \\
b_2 \\
\vdots \\
b_q
\end{bmatrix}
=
\begin{bmatrix}
1 \\
a_1 \\
\vdots \\
a_p \\
0 \\
\vdots
\end{bmatrix}
\qquad (\diamond)
$$

Relationships Among Models

- An MA(q) or ARMA(p,q) model is equivalent to an AR(∞).

- An AR(p) or ARMA(p,q) model is equivalent to an MA(∞) model

Ex:

$$H(z) = \frac{1 + 0.9z^{-1}}{1 + 0.8z^{-1}} = \quad \text{ARMA(1,1)}$$

$$
\begin{aligned}
H(z) \quad &= \quad \frac{1}{(1 + 0.8z^{-1})\frac{1}{(1+0.9z^{-1})}} \\[2ex]
&= \quad \frac{1}{(1 + 0.8z^{-1})(1 - 0.9z^{-1} + 0.81z^{-2} + \cdots)} \\[2ex]
&= \quad \text{AR}(\infty).
\end{aligned}
$$

Remark:Let $\text{ARMA}(p,q) = \frac{B(z)}{A(z)} = \frac{1}{C(z)} = \text{AR}(\infty)$.

From $a_1, \cdots, a_p, b_1, \cdots, b_q$, we can find $c_1, c_2, \cdots$ and vice versa.

Remark:

- If poles of $\frac{1}{A(z)}$ $\underline{\text{inside unit circle}}$, $\quad H(z) = \frac{B(z)}{A(z)}$ is BIBO $\underline{\text{stable}}$.

- If zeroes of $B(z)$ inside unit circle, $H(z) = \frac{B(z)}{A(z)}$ is minimum phase.

- We chose $H(z)$ so that both its zeroes and poles are inside unit circle.

$$\text{u(n)} \longrightarrow \boxed{H(z) = \frac{B(z)}{A(z)}} \longrightarrow \text{x(n)}$$
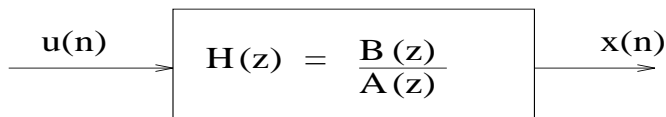
Stable and
Minimum Phase system

Consider

$$P(z) = \sigma^2 \frac{B(z)B(\frac{1}{z})}{A(z)A(\frac{1}{z})}.$$

Remark: • If α is zero for $P(z)$, so is $\frac{1}{\alpha}$.

• If β is a pole for $P(z)$, so is $\frac{1}{\beta}$.

• Since the $a_1, \cdots, a_p, b_1, \cdots, b_q$ are real, the poles and zeroes of $P(z)$ occur in complex conjugate pairs.
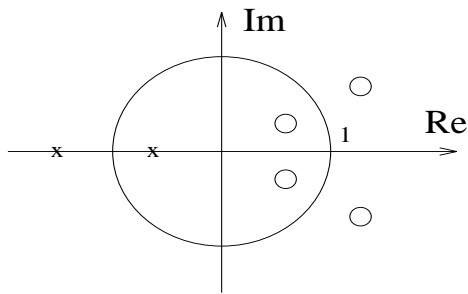
Spectral Factorization:

$$H(\omega) = \frac{B(\omega)}{A(\omega)}$$

$$P(\omega) = \sigma^2 \left| \frac{B(\omega)}{A(\omega)} \right|^2 = \frac{\sigma^2 B(\omega) B^*(\omega)}{A(\omega) A^*(\omega)}.$$

$$A(\omega) = 1 + a_1 e^{-j\omega} + \cdots + a_p e^{-jp\omega}$$

$b_1, \cdots, b_q, a_1, \cdots, a_p$ are real coefficients.

$$
\begin{aligned}
A^*(\omega) &= 1 + a_1 e^{j\omega} + \cdots + a_p e^{jp\omega} \\
&= 1 + a_1 \frac{1}{z} + \cdots + a_p \frac{1}{z^p} = A(\frac{1}{z})
\end{aligned}
$$

$$\boxed{P(z) = \sigma^2 \frac{B(z) B(\frac{1}{z})}{A(z) A(\frac{1}{z})}.}$$

Remark: If $a_1, \cdots, a_p, b_1, \cdots, b_q$ are complex,

$$P(z) = \sigma^2 \frac{B(z) B^*\left(\frac{1}{z^*}\right)}{A(z) A^*\left(\frac{1}{z^*}\right)}$$

ARMA Model: ARMA(p,q)

$$P(\omega) = \sigma^2 \left| \frac{B(\omega)}{A(\omega)} \right|^2.$$

AR Model: AR(p)

$$P(\omega) = \sigma^2 \left| \frac{1}{A(\omega)} \right|^2.$$

MA Model: MA(q)

$$P(\omega) = \sigma^2 |B(\omega)|^2.$$

Remark: • AR models peaky PSD better .

• MA models valley PSD better.

• ARMA is used for PSD with both peaks and valleys.

In Difference Equation Form,

$$x(n) = -\sum_{k=1}^{p} a_k x(n-k) + \sum_{k=0}^{q} b_k u(n-k).$$

In Z-transform Form, $z = e^{j\omega}$

$$H(z) = \frac{B(z)}{A(z)},$$

$$A(z) = 1 + a_1 z^{-1} + \cdots + a_p z^{-p}$$

$$B(z) = 1 + b_1 z^{-1} + \cdots + b_q z^{-q}$$

Unit Delay line
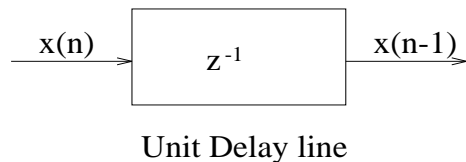
Notation sometimes used : $\quad z^{-1} x(n) = x(n-1)$

$$\text{Then:} \quad x(n) = \frac{B(z)}{A(z)} u(n)$$
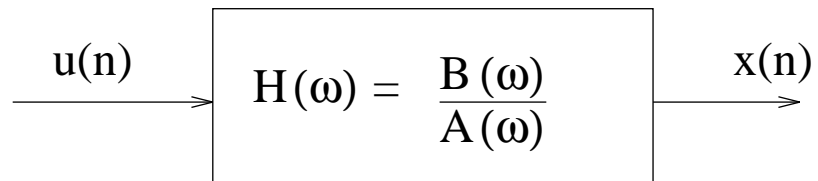
u(n) = zero-mean white noise of variance σ^2.

$$P_{xx}(\omega) = \sigma^2 \left| \frac{B(\omega)}{A(\omega)} \right|^2.$$

Remark:

The rational spectra can be associated with a signal obtained by filtering white noise of power σ^2 through a rational filter with

$$H(\omega) = \frac{B(\omega)}{A(\omega)}.$$

Rational Spectra:

$$P(\omega) = \sigma^2 \left| \frac{B(\omega)}{A(\omega)} \right|^2$$

$$A(\omega) = 1 + a_1 e^{-j\omega} + \cdots + a_p e^{-jp\omega}$$

$$B(\omega) = 1 + b_1 e^{-j\omega} + \cdots + b_q e^{-jq\omega}.$$

Remark: • We mostly consider real valued signals here.

• $a_1, \cdots, a_p, b_1, \cdots, b_q$ are real coefficients.

• Any <u>continuous PSD</u> can be approximated <u>arbitrarily close</u> by a <u>rational PSD</u>.

Parsimony Principle:

Better estimates may be obtained by using an appropriate data model with fewer unknowns.

Appropriate Data Model.

• If data model wrong, $\hat{P}(f)$ will always be biased.
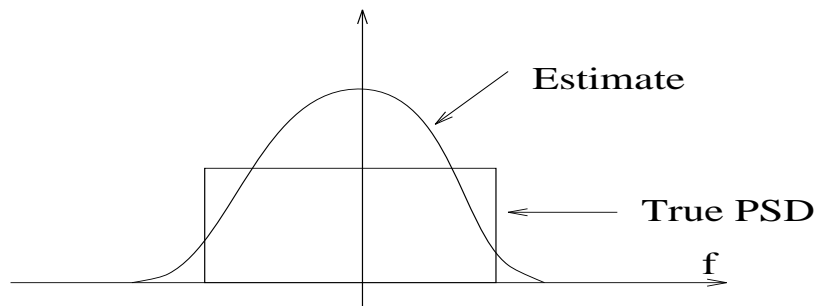
• To use parametric methods, reasonably correct *'a priori'* knowledge on data model is necessary.

PARAMETRIC METHODS

Parametric Modeling

Ex.

$$P(f) = \frac{r(0)}{\sqrt{2\pi}\sigma_f} e^{-\frac{1}{2}\left(\frac{f}{\sigma_f}\right)^2}, |f| \leq \frac{1}{2}$$
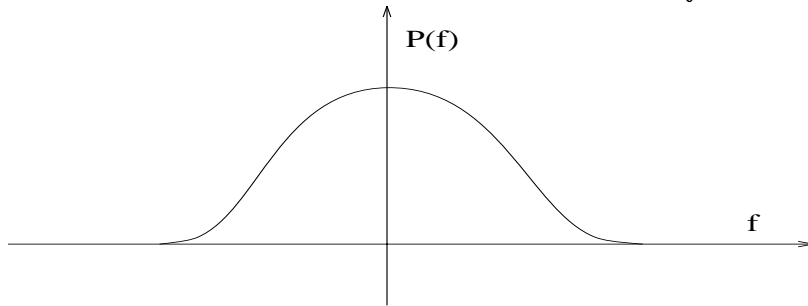
Remark: • $P(f)$ is described by 2 unknowns: $r(0)$ and σ_f.

• Once we know $r(0)$ and σ_f, we know $P(f)$, the PSD.

• Nonparametric methods assume no knowledge on $P(f)$ – too many unknowns.

• Parametric Methods attempt to estimate $r(0)$ and σ_f.

Implementation of $\hat{P}_D(\omega)$

- Zero pad $x(n)$ so that $x(n)$ has N' points, $N' >> N$.

- Calculate $\hat{P}_p(\omega_k)$ with FFT.

$$\omega_k = \frac{2\pi}{N'}k, \quad k = 0, \cdots, N' - 1.$$

-

$$\hat{P}_D(\omega_k) = \frac{1}{2J + 1} \sum_{j=k-J}^{k+J} \hat{P}_p(\omega_j).$$

$\hat{P}_p(\omega)$ $\bullet \bullet \bullet$ $\underbrace{\cdots \ \bullet \bullet \ \cdots}_{\text{2J+1 points averaging}}$ $\bullet$

$$\Downarrow$$

$$\hat{P}_D(\omega_k)$$

Daniell Method:

$$\hat{P}_D(\omega) = \frac{1}{2\pi\beta} \int_{\omega-\beta\pi}^{\omega+\beta\pi} \hat{P}_p(\psi)d\psi.$$
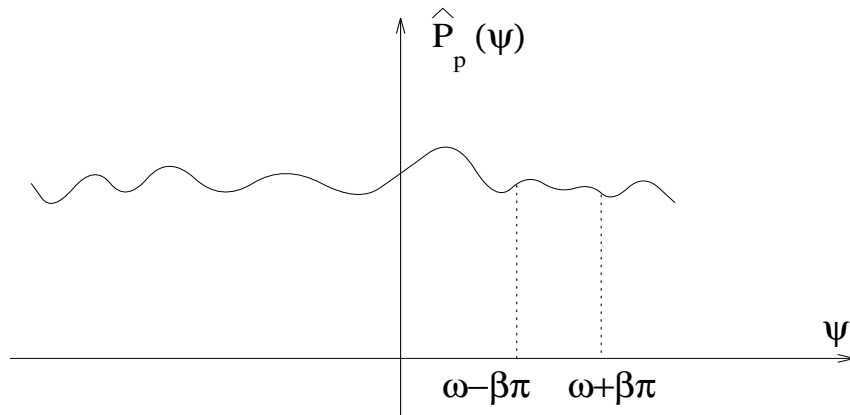
Remark: • $\hat{P}_D(\omega)$ is a special case of $\hat{P}_{BT}(\omega)$ with

$$w(n) \text{ in } \hat{P}_{BT}(\omega) \overset{DTFT}{\longleftrightarrow} W(\omega) = \begin{cases} \frac{1}{\beta}, \omega \in [-\beta\pi, \beta\pi] \\ 0, \text{else} . \end{cases}$$

• The larger the β, the lower the variance, but the poorer the resolution.

$$\hat{P}_l(\omega) = \frac{1}{MP} \left| \sum_{n=0}^{M-1} w(n)x_l(n)e^{-j\omega n} \right|^2$$

$$\hat{P}_W(\omega) = \frac{1}{S} \sum_{l=1}^{S} \hat{P}_l(\omega)$$

Remarks: • By allowing $x_l(n)$ to overlap, we hope to have a larger S, the number of $\hat{P}_j(\omega)$ we average .

 Practical examples show that $\hat{P}_W(\omega)$ may offer lower variance than $\hat{P}_B(\omega)$, but not significantly.

• $\hat{P}_W(\omega)$ may be shown to be $\hat{P}_{BT}(\omega)$ -type estimator, under reasonable approximation.

• $\hat{P}_W(\omega)$ can be easily computed with FFT -favored in practice

• $\hat{P}_{BT}(\omega)$ is theoretically favored.

• 50% overlap in general

Welch Method:

- $x_l(n)$ may overlap in the Welch method.

- $x_l(n)$ may be windowed before computing Periodogram.

$x_1(n)$

$x_2(n)$

$\cdots$

$x_S(n)$
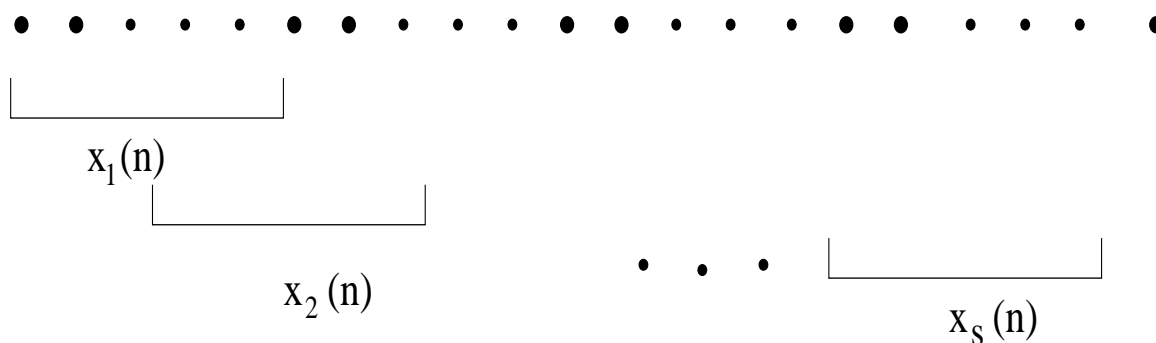
Let $w(n)$ be the window applied to $\quad x_l(n), l = 1, .., S, n = 0, .., M-1$

Let

$$\text{P} = \text{power of} \quad w(n) = \frac{1}{M} \sum_{n=0}^{M-1} |w(n)|^2$$

Bartlett Method

x(n):

$$\underbrace{\bullet\,\bullet\,\bullet\,\cdots}_{x_1(n)}\ \underbrace{\bullet\,\bullet\,\bullet\,\cdots}_{x_2(n)}\ \bullet\,\bullet\,\bullet\,\cdots\ \underbrace{\bullet\,\bullet\,\cdots}_{x_L(n)}$$

- $x(n)$ is an N point sequence.

- $x_l(n), l = 1, \cdots, L$, are M point sequences.

- $x_l(n)$ are non-overlapping. $L = \frac{N}{M}$.

$$\hat{P}_l(\omega) = \frac{1}{M}\left|\sum_{n=0}^{M-1} x_l(n)e^{-j\omega n}\right|^2$$

$$\hat{P}_B(\omega) = \frac{1}{L}\sum_{l=1}^{L}\hat{P}_l(\omega).$$

Remark:

- $\hat{P}_B(\omega) \geq 0, \forall\, \omega$.

- For large M and L, $\hat{P}_B(\omega) \approx [\ \hat{P}_{BT}(\omega)$ using $w_R(n)\ ]$

Window design for $\hat{P}_{BT}(\omega)$

Let $\beta_m = 3$dB <u>main</u> lobe width.

Resolution of $\hat{P}_{BT}(\omega) \sim \beta_m$ Variance of $\hat{P}_{BT}(\omega) \sim \frac{1}{\beta_m}$.

> • Choice of β_m is based on the trade-off between resolution
>   and variance, and N
> • Choice of window shape is based on leakage, and N.

• **Practical rule of thumb:**

1. $M \leq \frac{N}{10}$.

2. Window shape based on trade-off between smearing and leakage.

3. Window shape for $\hat{P}_{BT}(\omega) \geq 0, \quad \forall \omega$

Remark: • <u>Other methods</u> for Non-parametric Spectral
Estimation include: <u>Bartlett</u>, <u>Welch</u>, <u>Daniell</u> Methods.

• All try to reduce variance at the expense of poorer resolution.

Ex:

$$x(2n) \stackrel{DTFT}{\longleftrightarrow} \frac{1}{2} X\left(\frac{\omega}{2}\right).$$
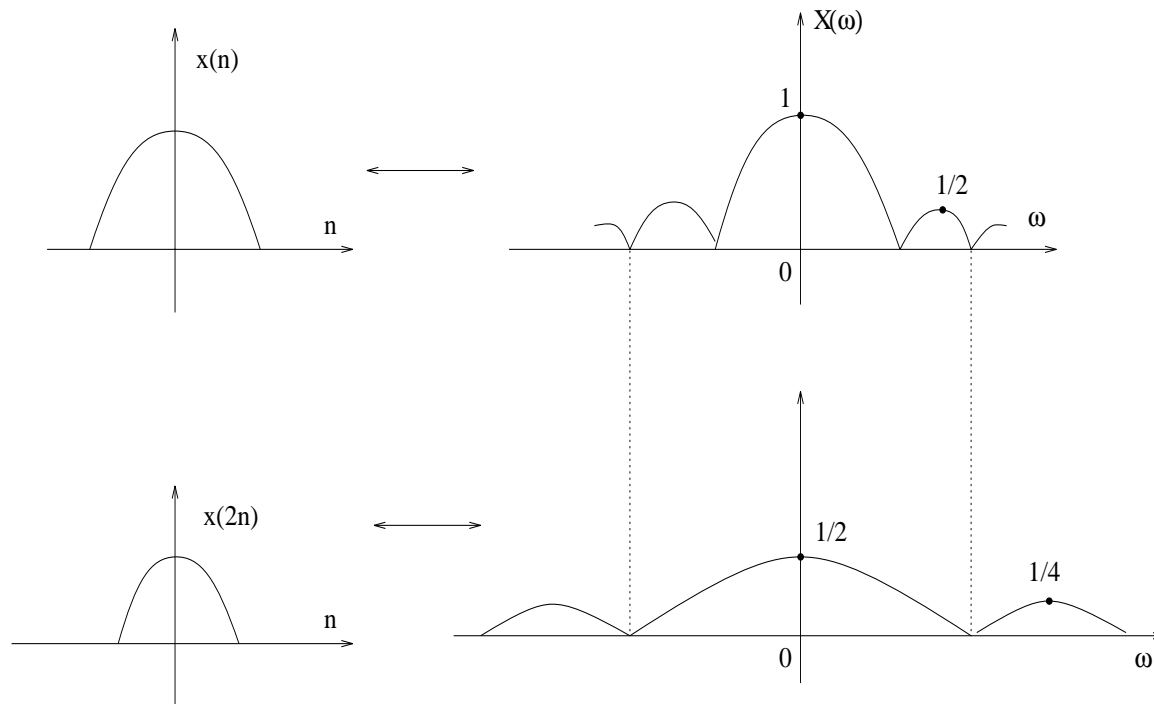
Remark: • Once the window shape is fixed, M $\uparrow \to N_e \uparrow \to \beta_e \downarrow$.

$\Rightarrow$ M $\uparrow \to$ main lobe width $\downarrow$.

$$N_e \beta_e = \frac{\sum_{n=-(M-1)}^{M-1} w(n)}{\frac{1}{2\pi} \int_{-\pi}^{\pi} W(\omega) d\omega} \frac{\int_{-\pi}^{\pi} W(\omega) d\omega}{2\pi \sum_{n=-(M-1)}^{M-1} w(n)} = 1$$

$\Rightarrow$ $\boxed{N_e \beta_e = 1}$ (Time Bandwidth product.)

Remark:

• If a signal decays slowly in one domain, it is more concentrated in the other domain.

• Window shape determines the side lobe level relative to $W(0)$.

- **Equivalent Bandwidth β_e:**

$$2\pi\beta_e = \frac{\int_{-\pi}^{\pi} W(\omega)d\omega}{W(0)}$$

Since $\quad w(n)\overset{DTFT}{\longleftrightarrow}W(\omega).$

$$w(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} W(\omega)e^{j\omega n}d\omega.$$

$$\Rightarrow w(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} W(\omega)d\omega.$$

$$W(\omega) = \sum_{n=-(M-1)}^{M-1} w(n)e^{-j\omega n}.$$

$$\Rightarrow W(0) = \sum_{n=-(M-1)}^{M-1} w(n)$$

Ex.

$$w_B(n) = \begin{cases} 1 - \frac{|n|}{M}, & -(M-1) \le n \le (M-1) \\ 0, & \text{else} \end{cases}$$

$N_e = M$
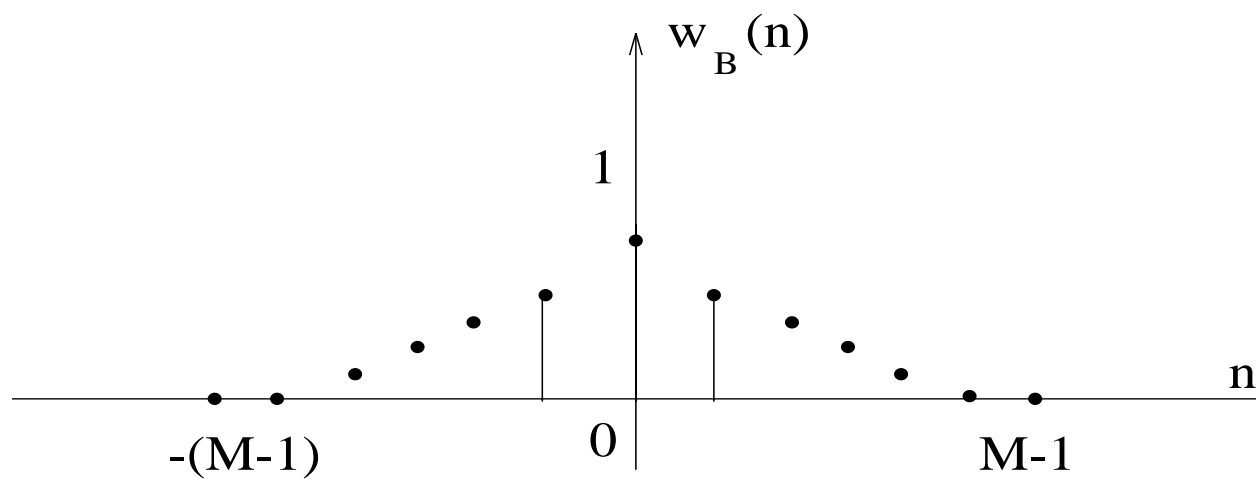
Time-Bandwidth Product

- **Equivalent Time Width N_e:**

$$N_e = \frac{\sum_{n=-(M-1)}^{M-1} w(n)}{w(0)}$$

Ex.

$$N_e = \frac{\sum_{k=-(M-1)}^{M-1}(1)}{1} = 2M - 1.$$
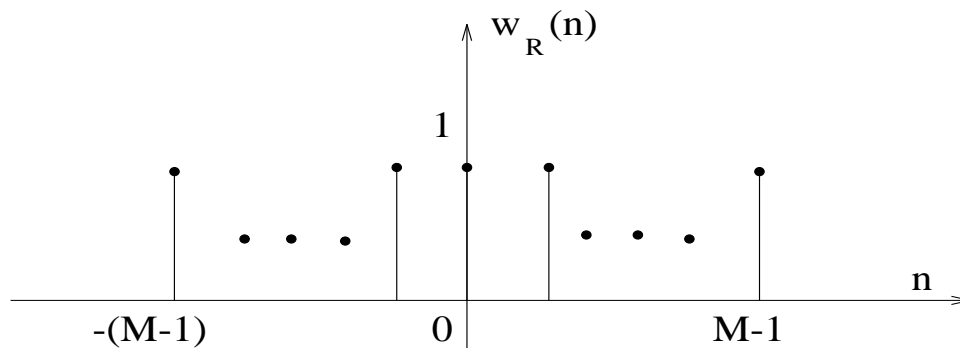
Theorem:

If $\mathbf{A} \geq 0$ (positive semidefinite) $\mathbf{B} \geq 0$ then $\mathbf{A} \odot \mathbf{B} \geq 0$.

Remark: If $\hat{r}(k)$ is a biased estimate, $\hat{P}_p(\omega) \geq 0 \; \forall \; \omega$. Then if $W(\omega) \geq 0 \; \forall \; \omega$, we have $\hat{P}_{BT}(\omega) \geq 0 \; \forall \; \omega$.

Remark: Nonnegative definite (positive semidefinite) window sequences: <u>Bartlett, Parzen.</u>

$$\hat{\mathbf{R}}_{BT} = \begin{bmatrix} w(0) & \cdots & w(M-1) & 0 & \cdots \\ & \ddots & & & \\ w[-(M-1)] & \cdots & w(0) & & \\ 0 & \ddots & & \ddots & \\ \vdots & & & & \end{bmatrix}$$

$$\odot \begin{bmatrix} \hat{r}(0) & \cdots & \hat{r}(N-1) & 0 & \cdots \\ & \ddots & & & \\ \hat{r}[-(N-1)] & \cdots & \hat{r}(0) & & \\ 0 & \ddots & & \ddots & \\ \vdots & & & & \end{bmatrix}$$

$\odot$ = Hadamard matrix product:

$(ij)^{th}$ element: $(\mathbf{A} \odot \mathbf{B})_{ij} = \mathbf{A}_{ij}\mathbf{B}_{ij}$

Remark: • $\hat{P}_{BT}(\omega) \geq 0 \ \forall \ \omega$ iff $\{w(k)\hat{r}(k)\}$ is a positive semidefinite sequence.

• $\hat{P}_{BT}(\omega) \geq 0 \ \forall \ \omega$ iff

$\hat{\mathbf{R}}_{BT} =$

$$
\begin{bmatrix}
w(0)\hat{r}(0) & \cdots & w(M-1)\hat{r}(M-1) & 0 & \cdots \\
 & \ddots & & \ddots & \\
w[-(M-1)]\hat{r}[-(M-1)] & \cdots & w(0)\hat{r}(0) & & \\
0 & & \ddots & & \ddots \\
\vdots & & & &
\end{bmatrix}
$$

is positive semidefinite, i.e, $\hat{\mathbf{R}}_{BT} \geq 0$.

Theorem: Let $Y(\omega) \overset{DTFT}{\longleftrightarrow} y(n), \quad -(N-1) \leq n \leq N-1$

Then $Y(\omega) \geq 0 \ \forall \ \omega$ iff

$$
\begin{bmatrix}
y(0) & y(1) & \cdots & y(N-1) & 0 & \cdots \\
y(-1) & y(0) & \cdots & y(N-2) & y(N-1) & \cdots \\
\vdots & & \ddots & & & \\
y[-(N-1)] & \cdots & & y(0) & y(1) & \cdots \\
0 & \ddots & \ddots & \ddots & \ddots & \\
\vdots & & & & &
\end{bmatrix}
$$

is positive semidefinite.

In other words, $Y(\omega) \geq 0 \ \forall \ \omega$ iff

$\cdots, 0, \cdots, 0, y[-(N-1)], \cdots, y(0), y(1), \cdots, y(N-1), 0, \cdots$ is a
positive semidefinite sequence.

Remark: • BT spectral estimator is "locally" weighted average of periodogram $\hat{P}_p(\omega)$.

• The smaller the M, the poorer the resolution of $\hat{P}_{BT}(\omega)$ but the lower the variance.

• Resolution of $\hat{P}_{BT}(\omega) \propto \frac{1}{M}$.

• Variance of $\hat{P}_{BT}(\omega) \propto \frac{M}{N} \quad \underset{N \to \infty}{\overset{M \text{ fixed}}{\longrightarrow}} 0$.

• For fixed M, $\hat{P}_{BT}(\omega)$ is asymptotically biased but variance $\longrightarrow 0$.

Question: When is $\hat{P}_{BT}(\omega) \geq 0 \ \forall \ \omega$?

REFINED METHODS

 Decrease variance of $\hat{P}(\omega)$ by increasing bias or
 decreasing resolution .

Blackman - Tukey (BT) Method

Remark: The $\hat{r}(k)$ used in $\hat{P}_c(\omega)$ is poor estimate for large lags k.

$$M < N: \quad \hat{P}_{BT}(\omega) = \sum_{k=-(M-1)}^{M-1} w(k)\hat{r}(k)e^{-j\omega k},$$

where $w(k)$ is called lag window.

Remark: If $w(k)$ is rectangular, $w(k)\hat{r}(k)$ is a truncated version of $\hat{r}(k)$.

If $\hat{r}(k)$ is a biased estimate, and $w(k)\overset{DTFT}{\longleftrightarrow}W(\omega)$

$$\boxed{\hat{P}_{BT}(\omega) = \tfrac{1}{2\pi} \int_{-\pi}^{\pi} W(\omega - \psi)\hat{P}_p(\psi)d\psi \ .}$$

$$\lim_{N \to \infty} E\left[\hat{P}_p(\omega_1)\hat{P}_p(\omega_2)\right] = P(\omega_1)P(\omega_2) + P^2(\omega_1)\delta(\omega_1 - \omega_2).$$

$$\lim_{N \to \infty} E\left\{\left[\hat{P}_p(\omega_1) - P(\omega_1)\right]\left[\hat{P}_p(\omega_2) - P(\omega_2)\right]\right\}$$

$$= \begin{cases} P^2(\omega_1), & \omega_1 = \omega_2 \\ 0, & \omega_1 \neq \omega_2 \text{ (uncorrelated if } \omega_1 \neq \omega_2) \end{cases}$$

Remark: • $\hat{P}_p(\omega)$ is not a consistent estimate.

• If $\omega_1 \neq \omega_2, \hat{P}_p(\omega_1)$ and $\hat{P}_p(\omega_2)$ are uncorrelated with each other.

• This variance result is also true for

$$y(n) = \sum_{k=0}^{\infty} h(k)x(n-k),$$

where $x(n)$ is zero-mean complex Gaussian white noise.

x(n) | h (n) | y(n)
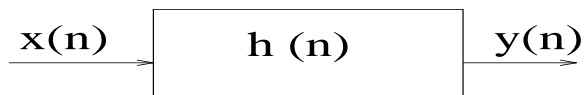
For Gaussian complex white noise,

$$E\left[x(k)x^*(l)x(m)x^*(n)\right] = \sigma^4\left[\delta(k-l)\delta(m-n) + \delta(k-n)\delta(l-m)\right].$$

$$
\begin{aligned}
E\left[\hat{P}_p(\omega_1)\hat{P}_p(\omega_2)\right] &= \frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1}\sum_{m=0}^{N-1}\sum_{n=0}^{N-1} E\left[x(k)x^*(l)x(m)x^*(n)\right] \\
&\quad e^{-j\omega_1(k-l)}e^{-j\omega_2(m-n)} \\
&= \sigma^4 + \frac{\sigma^4}{N^2}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} e^{-j(\omega_1-\omega_2)(k-l)} \\
&= \sigma^4 + \frac{\sigma^4}{N^2}\left|\sum_{k=0}^{N-1} e^{j(\omega_1-\omega_2)k}\right|^2 \\
&= \sigma^4 + \frac{\sigma^4}{N^2}\left\{\frac{\sin\left[(\omega_1-\omega_2)\frac{N}{2}\right]}{\sin\frac{(\omega_1-\omega_2)}{2}}\right\}^2
\end{aligned}
$$

Remark: If $x(n)$ is zero-mean complex Gaussian white noise, $\hat{P}_p(\omega)$ is an <u>unbiased</u> estimate.

$$
\bullet \quad r(k) \;=\; \sigma^2 \delta(k).
$$

$$
E\left[\hat{P}_p(\omega)\right] \;=\; \sum_{k=-(N-1)}^{N-1} \left(1 - \frac{|k|}{N}\right) r(k) e^{-j\omega k} = \sigma^2
$$

$$
\bullet \quad P_p(\omega) \;=\; \sum_{k=-\infty}^{\infty} r(k) e^{-j\omega k} = \sigma^2
$$

$$
\;=\; E\left[\hat{P}_p(\omega)\right].
$$

Variance Analysis

We shall consider the case $x(n)$ is zero-mean circularly symmetric complex <u>Gaussian white noise.</u>

$$\odot \begin{cases} E[x(n)x^*(k)] = \sigma^2 \delta(n-k). \\ E[x(n)x(k)] = 0 \quad \text{for all} \quad n,k. \end{cases}$$

$\odot$ is equivalent to:

$$\begin{cases} E[\ \text{Re}(x(n))\text{Re}(x(k))] = \frac{\sigma^2}{2}\delta(n-k). \\ E[\text{Im}(x(n))\text{Im}(x(k))] = \frac{\sigma^2}{2}\delta(n-k). \\ E[\text{Re}(x(n))\text{Im}(x(k))] = 0. \end{cases}$$

Remark: The real and imaginary parts of $x(n)$ are $N(0, \frac{\sigma^2}{2})$ and independent of each other.

Remark:

• The side lobes of $W_B(\omega)$ transfer power from high power frequency bins to low power frequency bins —— leakage.

• Smearing and leakage cause more problems to peaky $P(\omega)$ than to flat $P(\omega)$.

If $P(\omega) = \sigma^2$, for all ω, $E[\hat{P}_p(\omega)] = P(\omega)$.

• Bias of $\hat{P}_p(\omega)$ decreases as $N \to \infty$. (asymptotically unbiased.)

$W_B(\omega)$, Main lobe, Side lobes, ω, 0
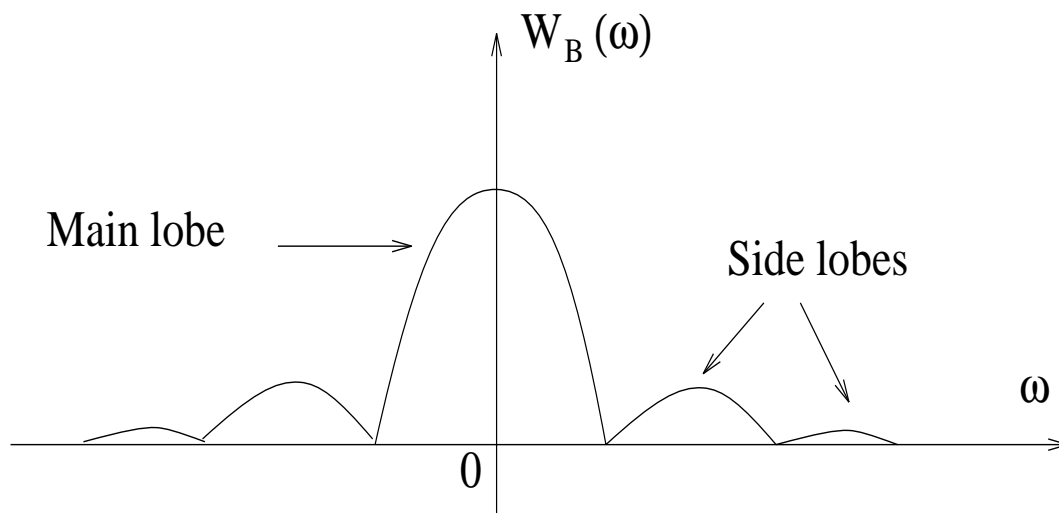
$$\boxed{3 \text{ dB power width of main lobe} \approx \tfrac{2\pi}{N} \text{ (or } \tfrac{1}{N} \text{ in Hz)} .}$$

Remark:• The main lobe of $W_B(\omega)$ smears or smooths $P(\omega)$.

• Two peaks in $P(\omega)$ that are separated less than $\frac{2\pi}{N}$ cannot be resolved in $\hat{P}_p(\omega)$.

• $\frac{1}{N}$ in Hz is called spectral resolution limit of periodogram methods.

- When $\hat{r}(k)$ is <u>unbiased</u> estimate,

$$E\left[\hat{P}_p(\omega)\right] = \frac{1}{2\pi}\int_{-\pi}^{\pi} P(\psi)W_R(\omega - \psi)d\psi \ .$$

$$w_R(k)\overset{DTFT}{\longleftrightarrow}W_R(\omega)$$
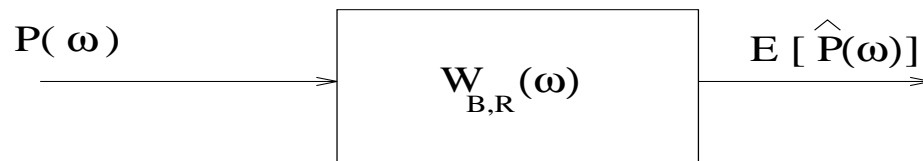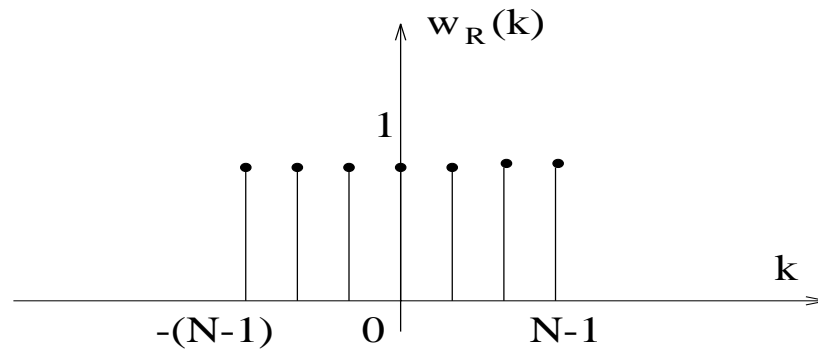
Bartlett or Triangular Window.
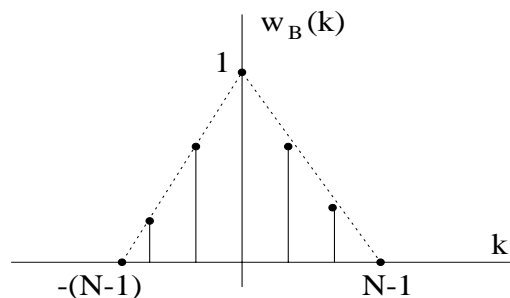
$$E\left[\hat{P}_p(\omega)\right] = \sum_{k=-\infty}^{\infty} \left[w_B(k)r(k)\right] e^{-j\omega k}$$

$$\text{Let} \quad w_B(k) \overset{DTFT}{\longleftrightarrow} W_B(\omega)$$

$$\boxed{E\left[\hat{P}_p(\omega)\right] = \frac{1}{2\pi} \int_{-\pi}^{\pi} P(\psi)W_B(\omega - \psi)d\psi.}$$

PROPERTIES OF PERIODOGRAM

Bias Analysis

- When $\hat{r}(k)$ is a <u>biased</u> estimate,

$$E\left[\hat{P}_p(\omega)\right] = E\left[\hat{P}_c(\omega)\right] = E\left\{\sum_{k=-(N-1)}^{N-1} \hat{r}(k)e^{-j\omega k}\right\}$$

$$k \geq 0, \quad E\left[\hat{r}(k)\right] = \frac{N-k}{N}r(k),$$

$$k < 0, \quad E\left[\hat{r}(k)\right] = E\left[r^*(-k)\right] = \frac{N+k}{N}r^*(-k) = \frac{N-|k|}{N}r(k),$$

$$E\left[\hat{P}_p(\omega)\right] = \sum_{k=-(N-1)}^{N-1} \left(1 - \frac{|k|}{N}\right)r(k)e^{-j\omega k}.$$

Asymptotic Properties of $\hat{a}_{ML}(r_1, ..., r_N)$

(a) $\hat{a}_{ML}(r_1, ..., r_N) \xrightarrow[N \to \infty]{} a$ ($\hat{a}_{ML}$ is a consistent estimate.)

(b) $\hat{a}_{ML}$ is asymptotically efficient.

(c) $\hat{a}_{ML}$ is aymptotically Gaussian.

Ex. $r = g^{-1}(a) + e, \quad e \sim N(0, \sigma^2). \quad \hat{a}_{ML} = ?$ efficient ?

$$\text{Let} \quad b = g^{-1}(a). \quad \text{Then} \quad a = g(b)$$

$$\frac{\partial}{\partial a} \ln f(r|a) = \frac{1}{\sigma^2} \left(r - g^{-1}(a) \right) \frac{dg^{-1}(a)}{da} \Big|_{a = \hat{a}_{ML}} = 0$$

$$\hat{a}_{ML} = g(r) = g(\hat{b}_{ML}).$$

Invariance property of ML estimator

- If $a = g(b)$ then $\hat{a}_{ML} = g(\hat{b}_{ML})$.

- $\hat{a}_{ML}$ may not be efficient. $\hat{a}_{ML}$ is not efficient if $g(\cdot)$ is a nonlinear function.

Independent measurements $r_1, ..., r_N$ available, where r_i may or may not be Gaussian.

Assume

$$\hat{a}_{ML} = \frac{1}{N} \sum_{i=1}^{N} r_i.$$

Law of large numbers: $\hat{a}_{ML} \xrightarrow[N \to \infty]{} a$

Central Limit Theorem:

$\hat{a}_{ML}$ has Gaussian distribution as $N \to \infty$.

Remarks:

(1) If $\hat{a}(r)$ is unbiased, $Var[\hat{a}(r)\,] \geq$ CRB.

(2) If an efficient estimate $\hat{a}(r)$ exists, i.e,

$$\frac{\partial}{\partial a} \ln f(r|a) = c[\hat{a}(r) - a]. \quad (c \quad \text{is independent of} \quad r.)$$

then

$$0 = \frac{\partial}{\partial a} \ln f(r|a)|_{a=\hat{a}_{ML}(r)} \text{ results in } \hat{a}_{ML}(r) = \hat{a}(r).$$

$\Rightarrow$

$$\boxed{\text{If an efficient estimate exists, it is } \hat{a}_{ML}.}$$

(3) If an efficient estimate <u>does not</u> exist, how good $\hat{a}_{ML}(r)$ is depends on each specific problem.

<u>No estimator can achieve the CR-bound</u>. Bounds (for example, Bhattacharya, Barankin) larger than the CR-bound may be found.

$$\frac{\partial}{\partial a} \ln f(r|a) \Big|_{a=\hat{a}_{ML}} = 0 \quad \Rightarrow \underline{\hat{a}_{ML} = r}$$

$$\frac{\partial}{\partial a} \ln f(r|a) = \frac{1}{\sigma^2}(a - \hat{a}_{ML})$$

$$\Rightarrow -\sigma^2 \frac{\partial}{\partial a} \ln f(r|a) = \hat{a}_{ML} - a$$

$$\Rightarrow \quad \hat{a}_{ML} \text{ efficient} \begin{cases} E\left[(\hat{a}_{ML} - a)^2 \Big| a\right] = CRB \\ E\left[\hat{a}_{ML}\right] = E\left[r\right] = a, \quad \text{unbiased} \end{cases}$$

Remark: $\bullet$ MSE $= Var[\hat{a}_{ML}] = Var[r] = \sigma^2$.

$$\bullet I(a) = E\left\{\left[\frac{\partial}{\partial a} \ln f(r|a)\right]^2 \Big| a\right\} = E\left\{\left[\frac{1}{\sigma^2}(a - r)\right]^2\right\} = \frac{1}{\sigma^4}\sigma^2 = \frac{1}{\sigma^2}$$

$$\Rightarrow \quad \text{CRB} = \frac{1}{I(a)} = \sigma^2 = Var[\hat{a}_{ML}].$$

Efficient Estimate:

An estimate is efficient if

(a.) It is unbiased

(b.) It achieves the CR - bound, i.e, $E\left\{[\hat{a}(r) - a]^2 | a\right\} = \text{CRB}$.

Ex. $r = a + e$

where a is unknown constant, $e \sim N(0, \sigma^2)$. $\hat{a}_{ML} = ?$ efficient ?

$$f(r|a) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(r-a)^2}$$

$$\ln f(r|a) = \ln \frac{1}{\sqrt{2\pi}\sigma} - \frac{1}{2\sigma^2}(r-a)^2.$$

$$\frac{\partial}{\partial a} \ln f(r|a) = -\frac{1}{2\sigma^2} 2(r-a)$$

$$= \frac{1}{\sigma^2}(a-r).$$

Schwarz Inequality:

$$\int_{-\infty}^{\infty} g_1(x)g_2(x)dx \leq \left[\int_{-\infty}^{\infty} g_1{}^2(x)dx\right]^{\frac{1}{2}} \left[\int_{-\infty}^{\infty} g_2{}^2(x)dx\right]^{\frac{1}{2}},$$

where " $=$ " holds iff $g_1(x) = cg_2(x)$ for some constant c (c is independent of x).

$$\Rightarrow \left[1 + \frac{\partial}{\partial a}B(a)\right]^2 \leq \left\{\int_{-\infty}^{\infty} [\hat{a}(r) - a]^2 f(r|a)dr\right\}$$

$$\cdot \underbrace{\left\{\int_{-\infty}^{\infty} \left[\frac{\partial}{\partial a}\ln f(r|a)\right]^2 f(r|a)dr\right\}}_{I(a)}$$

where "$=$" holds iff

$$\boxed{\hat{a}(r) - a = c\frac{\partial}{\partial a}\ln f(r|a).}$$

(where c is a constant independent of r).

Proof:
$$\text{Proof:} \quad B(a) \;=\; E\left[\hat{a}(r) - a | a\right]$$

$$=\; \int_{-\infty}^{\infty} \left[\hat{a}(r) - a\right] f(r|a) dr$$

$$\frac{\partial}{\partial a} B(a) \;=\; \int_{-\infty}^{\infty} \left[\hat{a}(r) - a\right] \frac{\partial}{\partial a} f(r|a) dr - \underbrace{\int_{-\infty}^{\infty} f(r|a) dr}_{=1}$$

$$1 + \frac{\partial}{\partial a} B(a) \;=\; \int_{-\infty}^{\infty} \left[\hat{a}(r) - a\right] f(r|a) \frac{\partial}{\partial a} f(r|a) \frac{1}{f(r|a)} dr$$

$$\text{But} \quad \frac{\partial}{\partial a} \ln f(r|a) \;=\; \frac{\frac{\partial}{\partial a} f(r|a)}{f(r|a)}$$

$$1 + \frac{\partial}{\partial a} B(a) \;=\; \int_{-\infty}^{\infty} \left[\hat{a}(r) - a\right] f(r|a) \frac{\partial}{\partial a} \ln f(r|a) dr$$

$$\Rightarrow \left\{ \int_{-\infty}^{\infty} \left[\hat{a}(r) - a\right] \sqrt{f(r|a)} \left[\left(\frac{\partial}{\partial a} \ln f(r|a) \right) \sqrt{f(r|a)} \right] dr \right\}^2$$

$$= \left[1 + \frac{\partial}{\partial a} B(a) \right]^2.$$

Cramér - Rao Bound.

Let $B(a) = E[\hat{a}(r)|a] - a$ denote the bias of $\hat{a}(r)$, where r is the measurement.

Then

$$MSE = E\left[(\hat{a}(r) - a)^2 | a\right] \geq \frac{\left[1 + \frac{\partial}{\partial a} B(a)\right]^2}{E\left\{\left[\frac{\partial}{\partial a} \ln f(r|a)\right]^2 | a\right\}} \ .$$

* The denominator of the CRB is known as Fisher's Information, $I(a)$.

* If $B(a) = 0$, the numerator of CRB is 1.

Ex. x is a measurement of an uniformly distributed random variable on $[0, \theta]$, where θ is an unknown constant. $\hat{\theta}_{ML} = ?$

$\hat{\theta}_{ML} = x$
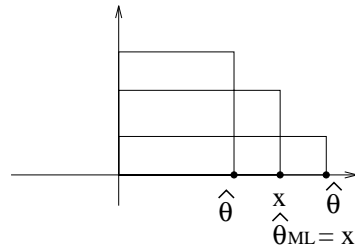
Question: What if two independent measurements x_1 and x_2 are taken ?

$\hat{\theta}_{ML} = \max (x_1, x_2)$.
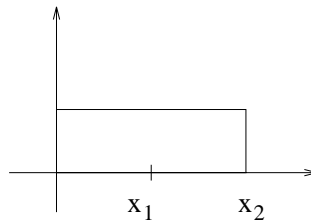
Ex. $y = a + e$

Three independent measurements y_1, y_2, y_3 are taken.

$\hat{a}_{ML} = ?$ Bias $= ?$ Variance $= ?$

$$f(y_i|a) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(y_i - a)^2}{2\sigma^2}}.$$

$$f(y_1, y_2, y_3|a) = \Pi_{i=1}^3 \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(y_i - a)^2}{2\sigma^2}}.$$

$$\frac{\partial f(y_1, y_2, y_3|a)}{\partial a}\Big|_{a = \hat{a}_{ML}} = 0$$

$$\Rightarrow \hat{a}_{ML} = \frac{1}{3}(y_1 + y_2 + y_3).$$

$$E[\hat{a}_{ML}] = E\left[\frac{1}{3}(y_1 + y_2 + y_3)\right] = a.$$

$$Var[\hat{a}_{ML}] = \frac{1}{9}Var(y_1 + y_2 + y_3)$$

$$= \frac{1}{9}(\sigma^2 + \sigma^2 + \sigma^2) = \frac{\sigma^2}{3}.$$

Maximum Likelihood (ML) Estimate of a:

Say $y = 5$, we want to find $\hat{a}$ so that it is most likely that the measurement is 5

$$\frac{\partial f(y|a)}{\partial a}\Big|_{a = \hat{a}_{ML}} = 0.$$
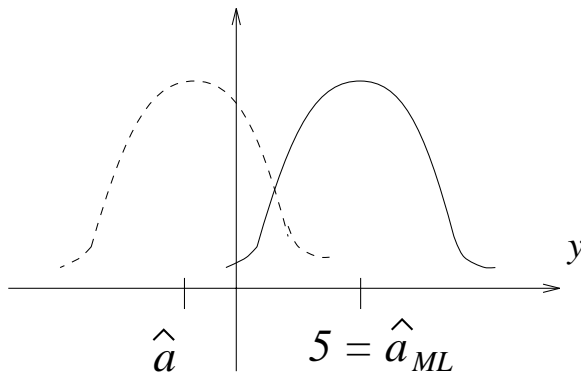
- $\Rightarrow \underline{\hat{a}_{ML} = y}$
- $E[\hat{a}_{ML}] = E[y] = E[a + n] = a$
- $Var[\hat{a}_{ML}] = Var[y] = \sigma^2$

Ex. Measurement

$$y = a + e,$$

Where a is an unknown constant and e is $N(0,\sigma^2)$.

Find $\hat{a}$ from y ?

Pdf of y:
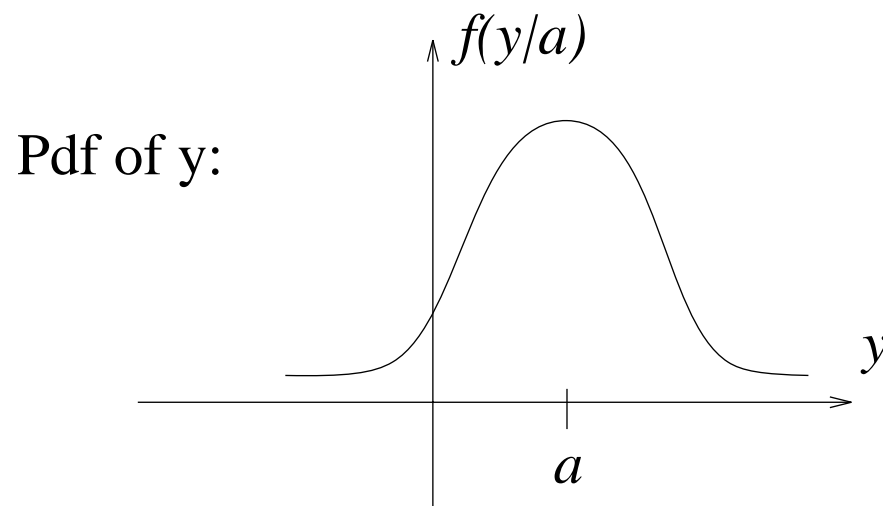
FUNDAMENTALS OF ESTIMATION THEORY

Properties of a Good Estimator for a constant scalar a

- <u>Small Bias:</u>

$$\text{Bias} = E[\hat{a}] - a$$

- <u>Small Variance:</u>

$$\text{Variance} = E\left\{ (\hat{a} - E[\hat{a}])^2 \right\}$$

- <u>Consistent:</u>

$$\hat{a} \rightarrow a \text{ as Number of measurements} \rightarrow \infty.$$

$$\begin{cases} X(2p) = \sum_{n=0}^{N-1} \left[x(n) + x(n + \frac{N}{2}) \right] W^{kn}, & k = 2p = 0, 2, ... \\ X(2p+1) = \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) - x(n + \frac{N}{2}) \right] W^{kn}, & k = 2p + 1, \end{cases}$$

which requires $2 \left(\frac{N}{2} \right)^2$ complex multiplication

This process is continued till 2 points.

- Remark: An $N = 2^m$ -pt FFT requires $O(N \log_2 N)$ complex multiplications.

- Zero padding may be used so that $N = 2^m$.

- Zero padding will not change resolution of $\hat{P}_p(\omega)$.

Let

$$
\begin{aligned}
W &= e^{-j\frac{2\pi}{N}}, N = 2^m \\
X(k) &= \sum_{n=0}^{N-1} x(n) W^{kn} \\
&= \sum_{n=0}^{\frac{N}{2}-1} x(n) W^{kn} + \sum_{n=\frac{N}{2}}^{N-1} x(n) W^{kn} \\
&= \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) + x\left(n + \frac{N}{2}\right) W^{\frac{Nk}{2}} \right] W^{kn}
\end{aligned}
$$

Note:

$$
W^{\frac{Nk}{2}} = e^{-j\frac{2\pi}{N}\frac{Nk}{2}} = e^{-j\pi k}
$$

$$
= \begin{cases} 1, & \text{even k} \\ -1, & \text{odd k} \end{cases}
$$

General Comments on $\hat{P}_p(\omega)$ and $\hat{P}_c(\omega)$.

- $\hat{P}_p(\omega)$ and $\hat{P}_c(\omega)$ provide POOR estimate of $P(\omega)$. (The variances of $\hat{P}_p(\omega)$ and $\hat{P}_c(\omega)$ are high.)

Reason: $\hat{P}_p(\omega)$ and $\hat{P}_c(\omega)$ are from a <u>single</u> realization of a random process.

- Compute $\hat{P}_p(\omega)$ via FFT.

Recall DFT: (N^2 complex multiplication)

$$X(k) = \sum_{i=0}^{N-1} x(i) e^{-j\frac{2\pi}{N}ki}$$

$$\hat{P}_p(k) = \frac{1}{N}|X(k)|^2.$$

Remark:

- With biased $\hat{r}(k)$, $\hat{P}_c(\omega) = \hat{P}_p(\omega) \geq 0$, for all ω

- $E[\hat{r}(k)] \neq r(k)$

$E[\hat{r}(k)] \longrightarrow r(k)$, as $N \to \infty \Rightarrow$ Asymptotically unbiased.

- $\hat{\mathbf{R}} = \begin{bmatrix} \hat{r}(0) & \hat{r}(1) & \cdots & \hat{r}(N-1) \\ \hat{r}^*(1) & \hat{r}(0) & \cdots & \hat{r}(N-2) \\ \vdots & \vdots & \vdots & \vdots \\ \hat{r}^*(N-1) & \hat{r}^*(N-2) & \cdots & \hat{r}(0) \end{bmatrix}$,

with $\hat{r}(k)$ biased estimate. Then $\hat{\mathbf{R}}$ is positive semidefinite.

Ex.
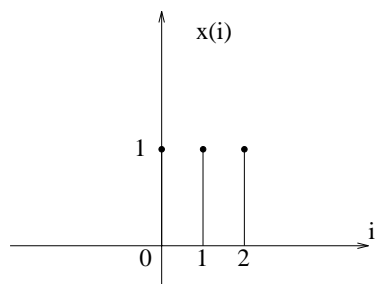
$\hat{r}(0) = \frac{1}{3}\sum_0^2 (1)(1) = 1.$

$\hat{r}(-1) = \hat{r}(1) = \frac{1}{3}\sum_1^2 (1)(1) = \frac{2}{3}.$

$\hat{r}(-2) = \hat{r}(2) = \frac{1}{3}\sum_2^2 (1)(1) = \frac{1}{3}.$
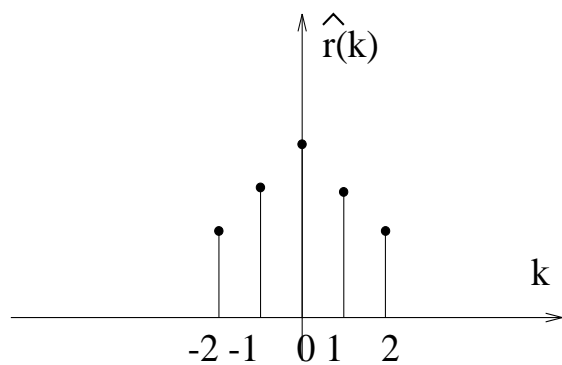
DTFT
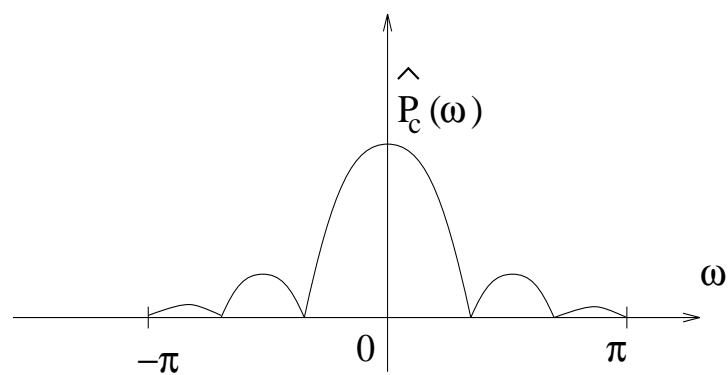
<u>Biased Estimate of $r(k)$</u> (used more often!)

$$\begin{cases} k \geq 0, & \hat{r}(k) = \frac{1}{N} \sum_{i=k}^{N-1} x(i)x^*(i-k), \\ k < 0, & \hat{r}(k) = \hat{r}^*(-k), \end{cases}$$

Remark:

$$E[\hat{r}(k)] = \frac{1}{N} \sum_{i=k}^{N-1} E[x(i)x^*(i-k)]$$

$$= \frac{1}{N} \sum_{i=k}^{N-1} r(k) = \frac{N-k}{N} r(k)$$

$$\longrightarrow r(k), \text{ as } N \to \infty$$

(Asymptotically unbiased)

Remark:

- $\hat{r}(k)$ is a bad estimate of $r(k)$ for large k.

- $E[\hat{r}(k)] = r(k)$ (unbiased)

Proof:

$$E[\hat{r}(k)] = E\left[\frac{1}{N-k}\sum_{i=k}^{N-1} x(i)x^*(i-k)\right]$$

$$= \frac{1}{N-k}\sum_{i=k}^{N-1} r(k) = r(k)$$

- $\hat{P}_c(\omega)$ based on unbiased $\hat{r}(k)$ may be ≤ 0.

Ex.
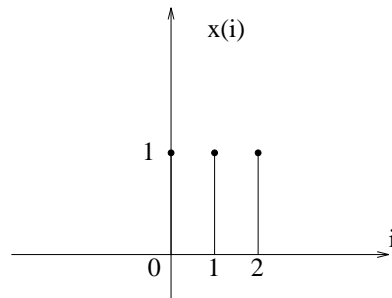
$\hat{r}(0) = \frac{1}{3} \sum_0^2 (1)(1) = 1$, (average of 3 points)

$\hat{r}(-1) = \hat{r}(1) = \frac{1}{2} \sum_1^2 (1)(1) = 1$, (average of 2 points)

$\hat{r}(-2) = \hat{r}(2) = \frac{1}{1} \sum_2^2 (1)(1) = 1$, (average of 1 point)

$\hat{r}(-3) = \hat{r}(3) = 0$.
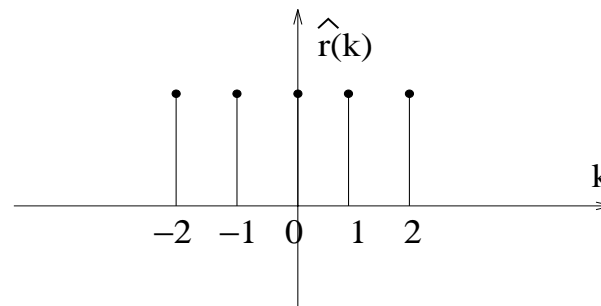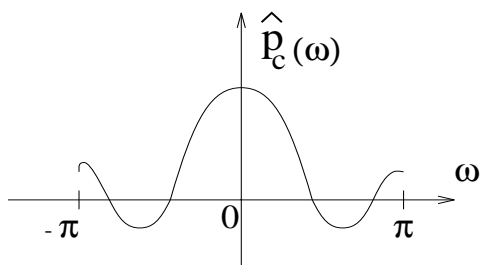
Correlogram (See first PSD definition)

$$\text{Correlogram} = \hat{P}_c(\omega) = \sum_{k=-(N-1)}^{N-1} \hat{r}(k) e^{-j\omega k}.$$

Unbiased Estimate of r(k):

$$\begin{cases} k \geq 0, & \hat{r}(k) = \frac{1}{N-k} \sum_{i=k}^{N-1} x(i) x^*(i-k) \\ k < 0, & \hat{r}(k) = \hat{r}^*(-k) \end{cases}$$

Spectral Estimation Problem

From a finite-length record $\{x(0), ..., x(N-1)\}$, determine an estimate $\hat{P}(\omega)$ of the PSD, $P(\omega)$, for $\omega \in [\text{-}\pi, \pi]$.

NonParametric Methods:

Periodogram:

Recall the second definition of PSD:

$$P(\omega) = \lim_{N \to \infty} E \left\{ \frac{1}{N} \left| \sum_{n=0}^{N-1} x(n) e^{-j\omega n} \right|^2 \right\}.$$

$$\text{Periodogram} = \hat{P}_p(\omega) = \frac{1}{N} \left| \sum_{n=0}^{N-1} x(n) e^{-j\omega n} \right|^2.$$

Remark: • $\hat{P}_p(\omega) \geq 0$ for all ω.

• If $x(n)$ is real, $\hat{P}_p(\omega)$ is <u>even.</u>

• $E[\hat{P}_p(\omega)] = ?$ $\text{Var}[\hat{P}_p(\omega)] = ?$ (to be discussed later on)

PSD for LTI Systems.
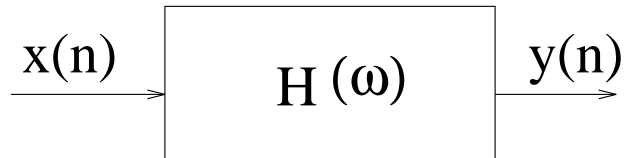
$$P_y(\omega) = P_x(\omega)|H(\omega)|^2.$$

Complex (DE) Modulation.

$$y(n) = x(n)e^{j\omega_0 n}.$$

It is easy to show that

$$r_y(k) = r_x(k)e^{j\omega_0 k}.$$

$$P_y(\omega) = P_x(\omega - \omega_0).$$

Second Definition of PSD.

$$P(\omega) = \lim_{N \to \infty} E \left\{ \frac{1}{N} \left| \sum_{n=0}^{N-1} x(n) e^{-j\omega n} \right|^2 \right\}.$$

This definition is equivalent to the first one under

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=-N+1}^{N-1} |k| \, |r(k)| = 0$$

(which means that $\{r(k)\}$ decays sufficiently fast).

Properties of PSD.

- $P(\omega) \geq 0$ for all ω.

- For real $x(n), r(k) = r(-k), \Rightarrow P(\omega) = P(-\omega), \omega \in [-\pi, \pi]$.

- For complex $x(n), r(k) = r^*(-k)$.

- $r(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P(\omega) d\omega = $ Average power for all frequency.

PSD

Average power between $\omega 1$ and $\omega 2$

ω

$\omega 1$ $\omega 2$
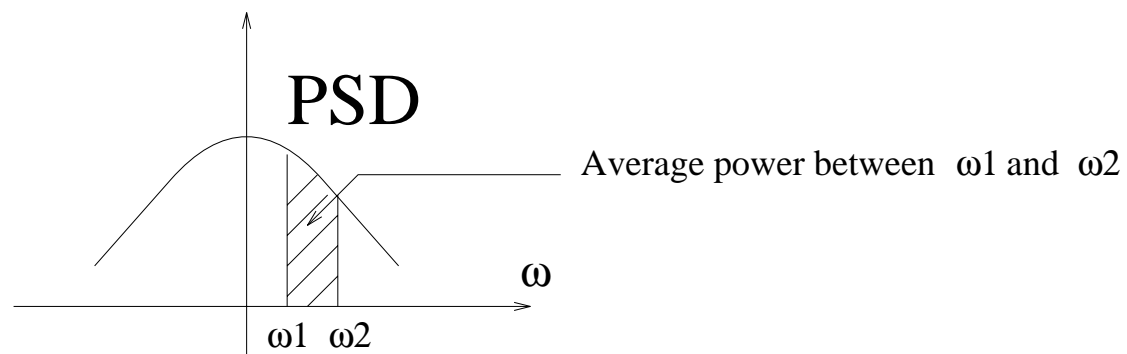
First Definition of PSD:

$$P(\omega) = \sum_{k=-\infty}^{\infty} r(k)e^{-j\omega k}$$

$$r(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P(\omega)e^{j\omega k}\, d\omega$$

Or

$$P(f) = \sum_{k=-\infty}^{\infty} r(k)e^{-j2\pi f k}$$

$$r(k) = \int_{-\frac{1}{2}}^{\frac{1}{2}} P(f)e^{j2\pi f k}\, df$$

Remark: • Since $r(k)$ is discrete, $P(\omega)$ and $P(f)$ are periodic, with period 2π (ω) and 1 (f), respectively.

• We usually consider $\omega \in [-\pi, \pi]$ or $f \in [-\frac{1}{2}, \frac{1}{2}]$.

- Eigendecomposition of $\mathbf{R}$

$\mathbf{R} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{U}^H$,

where $\mathbf{U}^H\mathbf{U} = \mathbf{U}\mathbf{U}^H = \mathbf{I}$

($\mathbf{U}$ is unitary matrix whose columns are eigenvectors of $\mathbf{R}$)

$\boldsymbol{\Sigma} = \mathrm{diag}(\lambda_1, \ldots, \lambda_m)$,

(λ_i are the eigenvalues of $\mathbf{R}$, real, and ≥ 0).

Covariance matrix:

$$
\mathbf{R} = \begin{bmatrix}
r(0) & r(1) & \cdots & r(m-2) & r(m-1) \\
r^*(1) & r(0) & \ddots & \ddots & r(m-2) \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
r^*(m-1) & r^*(m-2) & \cdots & r^*(1) & r(0)
\end{bmatrix}
$$

- It is easy to show that $\mathbf{R}$ is <u>positive semidefinite.</u>

- $\mathbf{R}$ is also <u>Toeplitz</u>.

- Since $\mathbf{R} = \mathbf{R}^H$, $\mathbf{R}$ is <u>Hermitian.</u>

Def: $\underline{\mathbf{A} \text{ is positive semidefinite if } \mathbf{z}^H \mathbf{A} \mathbf{z} \geq 0 \text{ for any } \mathbf{z}.}$

$$(\mathbf{z}^H = (\mathbf{z}^T)^* \text{ Hermitian transpose }).$$

Let

$$\mathbf{A} = \begin{bmatrix} r(0) & r(k) \\ r^*(k) & r(0) \end{bmatrix}$$

$$= E \left\{ \begin{bmatrix} x(n) \\ x(n-k) \end{bmatrix} \begin{bmatrix} x^*(n) & x^*(n-k) \end{bmatrix} \right\}$$

Obviously, $\mathbf{A}$ is positive semidefinite.

Then all eigenvalues of $\mathbf{A}$ are ≥ 0.

$\Rightarrow$ determinant of $\mathbf{A} \geq 0$.

$\Rightarrow r^2(0) - |r(k)|^2 \geq 0.$

- **Power Spectral Density (PSD) of Random Signals.**

Let $\{x(n)\}$ be wide-sense stationary (WSS) sequence with

$$E[x(n)] = 0.$$

$$r(k) = E[x(n)x^*(n-k)].$$

Properties of autocorrelation function $r(k)$.

- $r(k) = r^*(-k)$.

- $r(0) \geq |r(k)|$, for all k

- $0 \leq r(0) =$ average power of $x(n)$.

Let $\rho(k) = \sum_{n=-\infty}^{\infty} x(n)x^*(n-k)$.

$$\sum_{k=-\infty}^{\infty} \rho(k)e^{-j\omega k} = \sum_{k=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} x(n)x^*(n-k)e^{-j\omega n}e^{j\omega(n-k)}$$

$$= \left[\sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}\right]\left[\sum_{s=-\infty}^{\infty} x(s)e^{-j\omega s}\right]^*$$

$$= |X(\omega)|^2 = S(\omega).$$

Remark: $S(\omega)$ is the DTFT of the "autocorrelation" of finite energy sequence $\{x(n)\}$.

ENERGY AND POWER SPECTRAL DENSITIES

• **Energy Spectral Density of Deterministic Signals.**

Finite Energy Signal if

$$0 < \sum_{n=-\infty}^{\infty} |x(n)|^2 < \infty$$

Let $X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$

Parseval's Energy Theorem:

$$\begin{cases} \sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} S(\omega) d\omega, \\ S(\omega) = |X(\omega)|^2 \end{cases}$$

Remark: $|X(\omega)|^2$ "measures" the length of orthogonal projection of $\{x(n)\}$ onto basis sequence $\{e^{-j\omega n}\}$, $\omega \in [-\pi, \pi]$.

- Bounded-Input Bounded-Output (BIBO) Stability:

All poles of $H(z)$ are inside the unit circle for a causal system

(where h(n)=0, n< 0).

- FIR Filter: N=0.

- IIR Filter: N>0.

- Minimum Phase: All poles and zeroes of H(z) are inside the unit circle.

Linear Time-Invariant (LTI) Systems.

- N^{th} order difference equation:

$$\sum_{k=0}^{N-1} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k)$$

- Impulse Response:
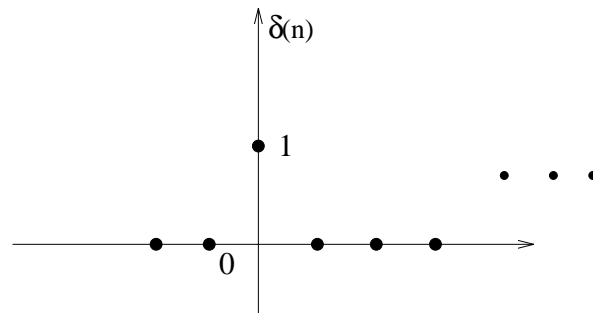
$$h(n) = y(n)\big|_{x(n)=\delta(n)}$$

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}.$$

Z-Transform

$$
\begin{cases}
X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \\
x(n) = \frac{1}{2\pi j} \int_c X(z) z^{n-1} dz
\end{cases}
$$

For finite duration $x(n)$,

$$
X(z) \quad = \quad \sum_{n=0}^{N-1} x(n) z^{-n}
$$

The DFT X(k) is related to X(z) as follows:

$$
X(k) = X(z)\big|_{z=e^{j\frac{2\pi}{N}k}}.
$$
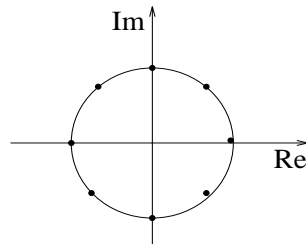
(X(k) evenly sampled on the unit circle of the z-plane)

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Effects of Zero-Padding:
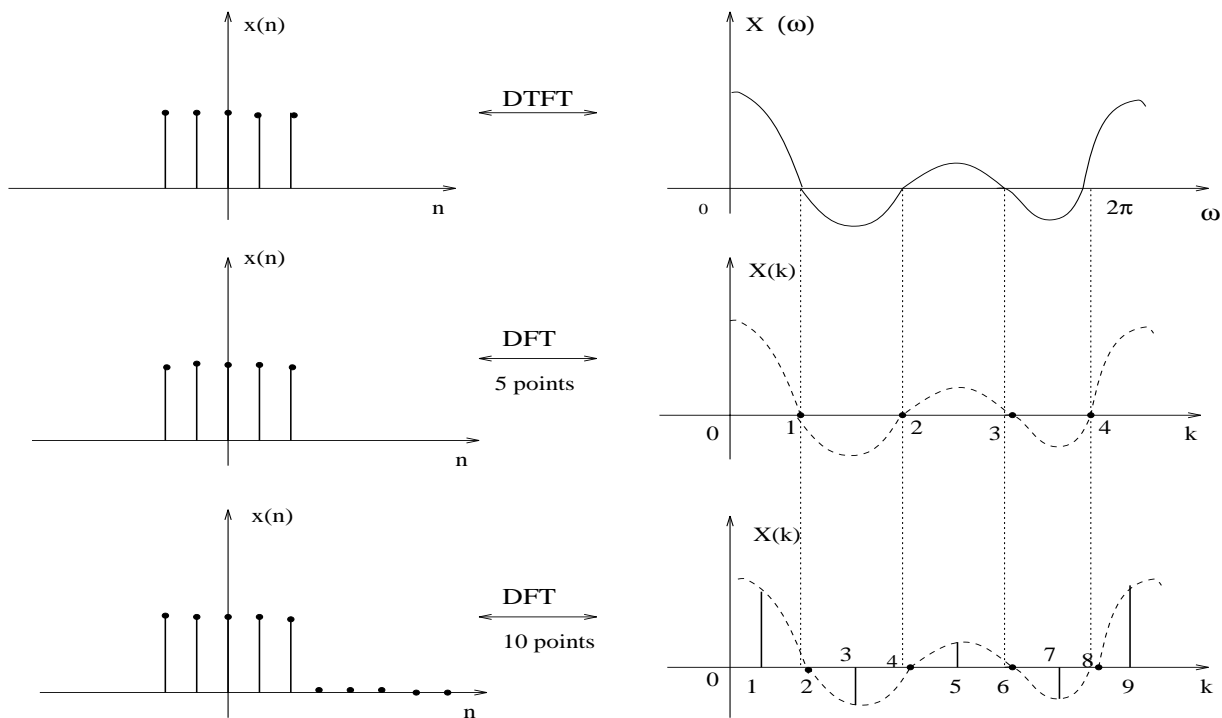
Remark:• The more zeroes padded, the closer X(k) is to X(ω).

• X(k) is a sampled version of X(ω) for finite duration sequences.

Ex. Note the Aliasing

Showing One Period
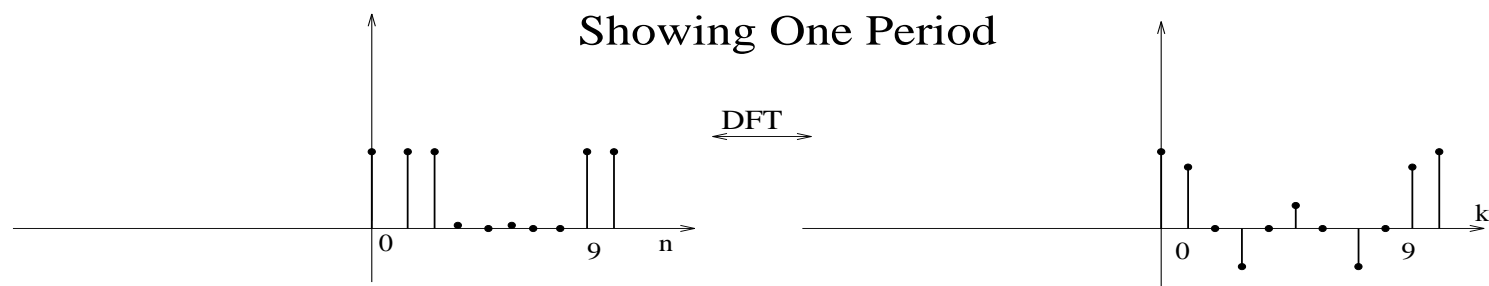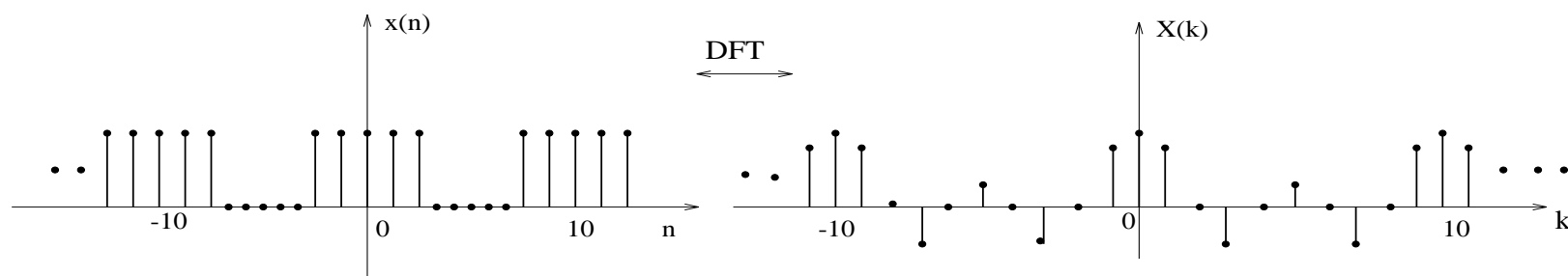
Remarks: For periodic sequences, DFT and DTFT yield similar spectra. IDFT (Inverse DFT) is the same as IDTFT (inverse DTFT) with $X\left(\frac{2\pi k}{N}\right)\delta\left(\omega - \frac{2\pi k}{N}\right)$ replaced by $X(k)$ and $\int$ replaced by $\sum$ (easy for computers).

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Remark: For DTFT, we also have:

Discrete Periodic Signals $\overset{DTFT}{\longleftrightarrow}$ Periodic Discrete Spectra.

Ex.
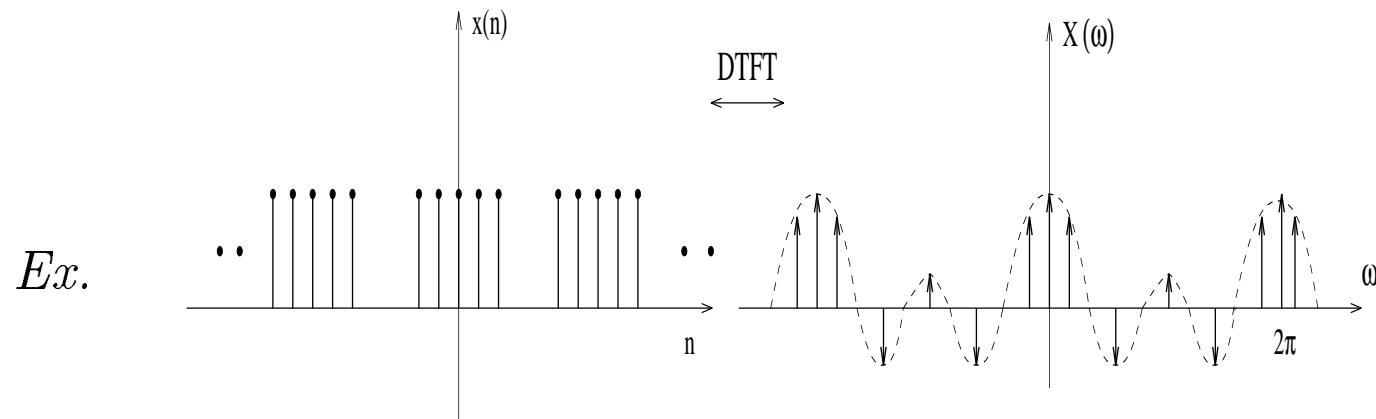
Note The Aliasing

When $x(n + N) = x(n)$,

$$DFT \quad Pair : \begin{cases} x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j2\pi k \frac{n}{N}}, \\ X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi k \frac{n}{N}} \end{cases}$$

For simplicity, we drop T.
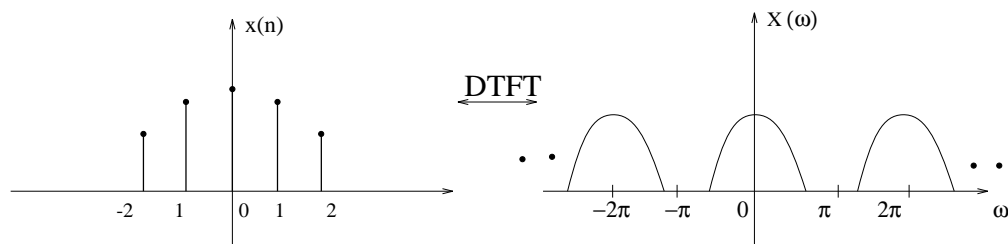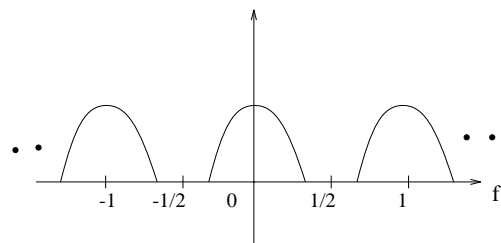
$$DTFT \quad Pair : \begin{cases} X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \\ x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega)e^{j\omega n} d\omega \end{cases}$$

Discrete-Time Signal $x(nT)$ — DTFT — $y(\omega)$
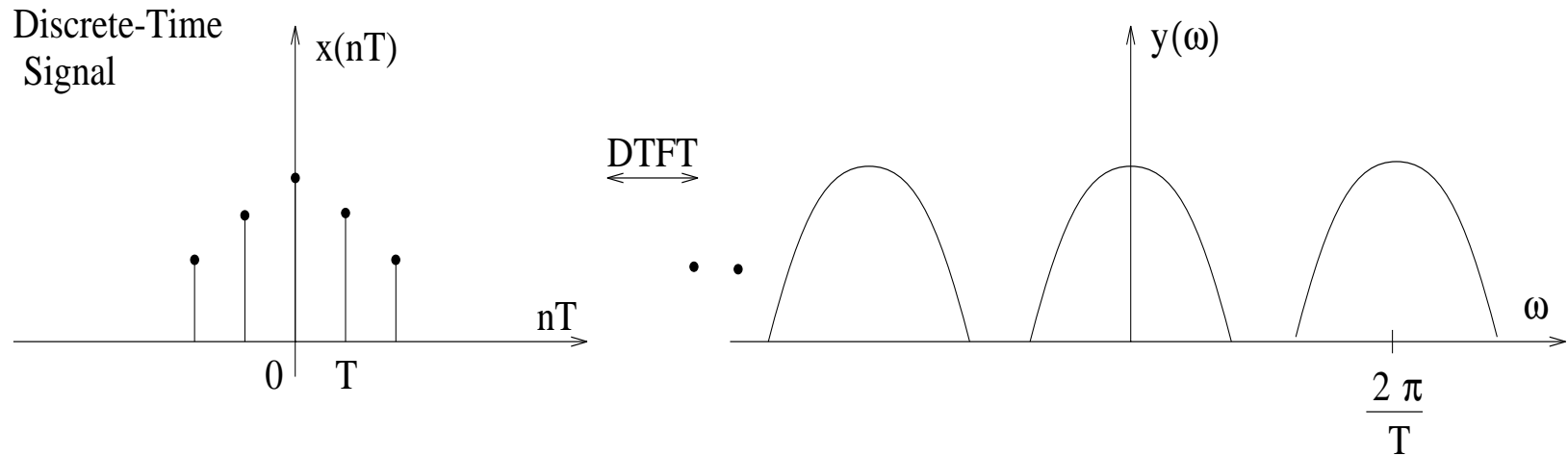
Remarks: Discrete-Time Fourier Transform (DTFT) is the same as Continuous-Time Fourier Transform (CTFT) with $x(nT)\,\delta(t-nT)$ replaced by $x(nT)$ and $\int$ replaced by $\sum$ (easy for computers).

* Fourier Transform (Continuous - Time vs. Discrete-Time)

$$Let \quad y(t) = x(t)s(t) \quad = \quad \sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT)$$

$$CTFT: \quad Y(\omega) \quad = \quad \int_{-\infty}^{\infty} y(t)e^{-j\omega t}\,dt$$

$$= \quad \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT)e^{-j\omega t}\,dt$$

$$= \quad \sum_{n=-\infty}^{\infty} x(nT)e^{-j\omega nT}$$

$$DTFT: \quad Y(\omega) \quad = \quad \sum_{n=-\infty}^{\infty} x(nT)e^{-j\omega nT}$$

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Aliasing Problem:
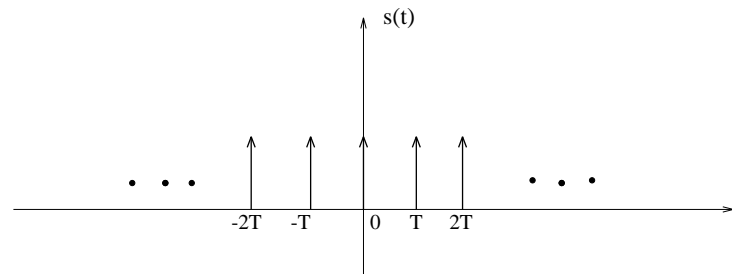
Ex.
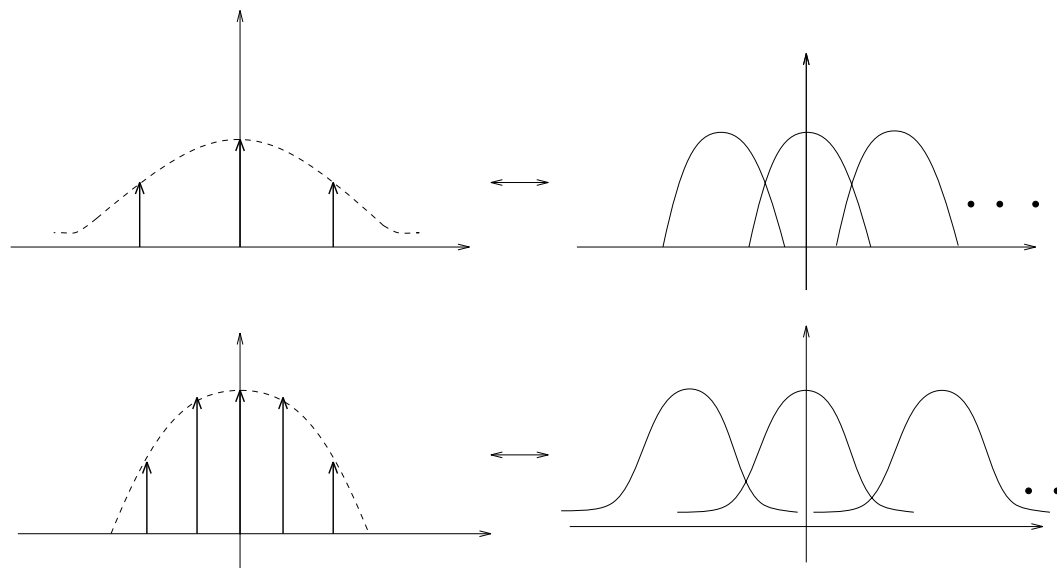
- Discrete signals
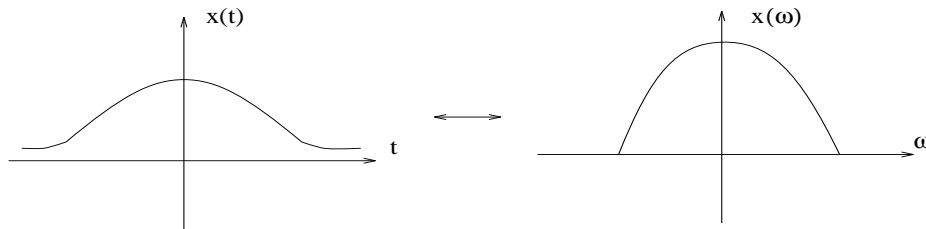
Ex:
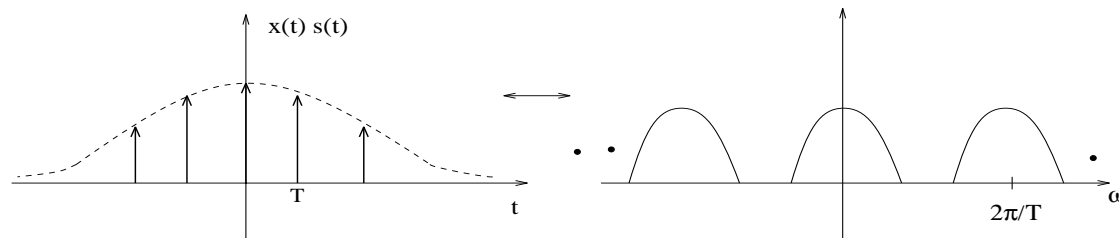
Remark: | **Discrete Signals $\longleftrightarrow$ Periodic Spectra.** |
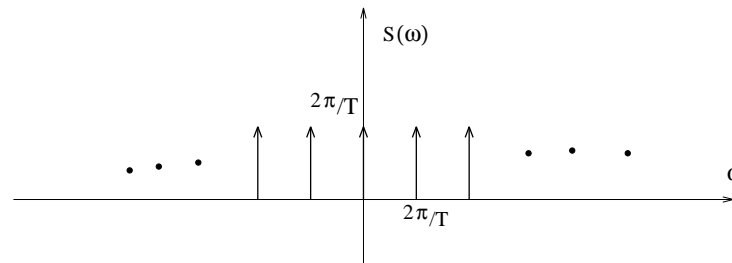
| **Discrete Periodic Signals $\longleftrightarrow$ Periodic Discrete Spectra.** |

Ex.

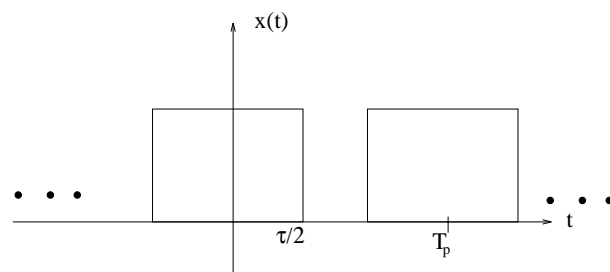$$s(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$
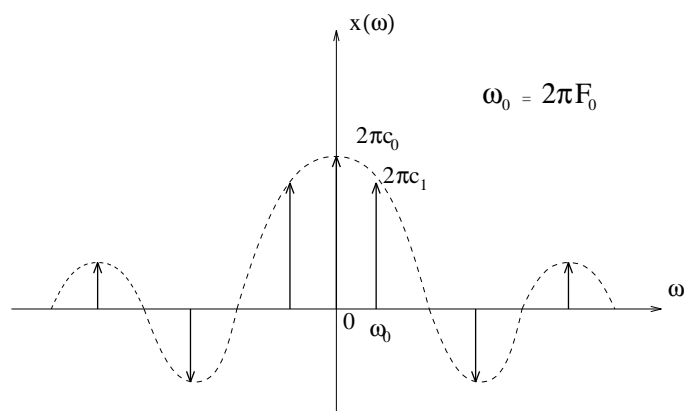

$C_k = \frac{1}{T}$ for all k

Remark:

Periodic Signals $\longleftrightarrow$ Discrete Spectra.

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Ex.

$$e^{j\omega_o t} \xrightarrow{\;FT\;} 2\pi\delta(\omega - \omega_o)$$

REVIEW OF DSP FUNDAMENTALS

Continuous-Time Signals

- Periodic signals

$$x(t) = x(t + T_p)$$

Fourier Series:

$$
\begin{aligned}
x(t) &= \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_o t} \\
c_k &= \frac{1}{T_p} \int_{T_p} x(t) e^{-j2\pi k F_o t}\, dt, \\
F_o &= \frac{1}{T_p}.
\end{aligned}
$$

Some Applications of Spectral Estimation

- Speech

    - Formant estimation (for speech recognition)

    - Speech coding or compression

- Radar and Sonar

    - Source localization with sensor arrays

    - Synthetic aperture radar imaging and feature extraction

- Electromagnetics

    - Resonant frequencies of a cavity

- Communications

    - Code-timing estimation in DS-CDMA systems

Spectral Estimation Methods:

- Classical (Nonparametric) Methods

Ex. Pass the data through a set of band-pass filters and measure the filter output powers.

- Parametric (Modern) Approaches

Ex. Model the data as a sum of a few damped sinusoids and estimate their parameters.

Trade-Offs: (Robustness vs. Accuracy)

- Parametric Methods may offer better estimates if data closely agrees with assumed model.

- Otherwise, Nonparametric Methods may be better.

What is Spectral Estimation?

From a <u>finite record</u> of a <u>stationary</u> data sequence, estimate how the <u>total power</u> is <u>distributed over frequencies</u>, or more practically, over narrow spectral bands (frequency bins).

" Spectral Estimation is ··· an Art "

Petre Stoica

" I hear, I forget;

I see, I remember;

I do, I understand."

A Chinese Philosopher.

EEL 6537 – Spectral Estimation

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