First Order Logic

Department of Electrical and Electronics Engineering
Spring 2005
Dr. Afşar Saranlı

References: Artificial Intelligence: A Modern Approach, 2nd Ed., Russel & Norvig

Overview

- Why First-Order Logic (FOL)?
- Syntax and Semantics of FOL,
- Fun with sentences,
- Wumpus world in FOL

Pros and Cons of FOL

- 😊 Propositional logic is **declarative**: pieces of syntax correspond to facts
- 😊 Propositional logic allows partial/disjunctive/negated information (unlike most data structures and databases)
- 😊 Propositional logic is **compositional**: meaning of $B_{1,1} \wedge P_{1,2}$ is derived from meaning of $B_{1,1}$ and of $P_{1,2}$
- 😊 Meaning in propositional logic is **context-independent** (unlike natural language, where meaning depends on context)
- ☹ Propositional logic has very limited expressive power (unlike natural language)
  E.g., cannot say "pits cause breezes in adjacent squares"
  except by writing one sentence for each square

First Order Logic

Whereas propositional logic assumes world contains **facts**,
first-order logic (like natural language) assumes the world contains

- Objects: people, houses, numbers, theories, Ronald McDonald, colors, baseball games, wars, centuries . . .
- Relations: red, round, bogus, prime, multistoried . . .,
  brother of, bigger than, inside, part of, has color, occurred after, owns, comes between, . . .
- Functions: father of, best friend, third inning of, one more than, end of . . .

Logics in General

Language	Ontological Commitment	Epistemological Commitment
Propositional logic	facts	true/false/unknown
First-order logic	facts, objects, relations	true/false/unknown
Temporal logic	facts, objects, relations, times	true/false/unknown
Probability theory	facts	degree of belief
Fuzzy logic	facts + degree of truth	known interval value

Syntax of FOL: Basic Elements

Constants	$KingJohn,\ 2,\ UCB, \ldots$
Predicates	$Brother,\ >, \ldots$
Functions	$Sqrt,\ LeftLegOf, \ldots$
Variables	$x,\ y,\ a,\ b, \ldots$
Connectives	$\wedge\ \vee\ \neg\ \Rightarrow\ \Leftrightarrow$
Equality	$=$
Quantifiers	$\forall\ \exists$

Atomic Sentences

$$\begin{aligned}\text{Atomic sentence} &= predicate(term_1, \ldots, term_n) \\ &\quad \text{or } term_1 = term_2\end{aligned}$$

$$\begin{aligned}\text{Term} &= function(term_1, \ldots, term_n) \\ &\quad \text{or } constant \text{ or } variable\end{aligned}$$

E.g., $Brother(KingJohn, RichardTheLionheart)$
$> (Length(LeftLegOf(Richard)), Length(LeftLegOf(KingJohn)))$

Complex Sentences

Complex sentences are made from atomic sentences using connectives

$$\neg S, \quad S_1 \wedge S_2, \quad S_1 \vee S_2, \quad S_1 \Rightarrow S_2, \quad S_1 \Leftrightarrow S_2$$

E.g. $Sibling(KingJohn, Richard) \Rightarrow Sibling(Richard, KingJohn)$

$>(1,2) \vee \leq(1,2)$

$>(1,2) \wedge \neg{>}(1,2)$

Truth in First Order Logic

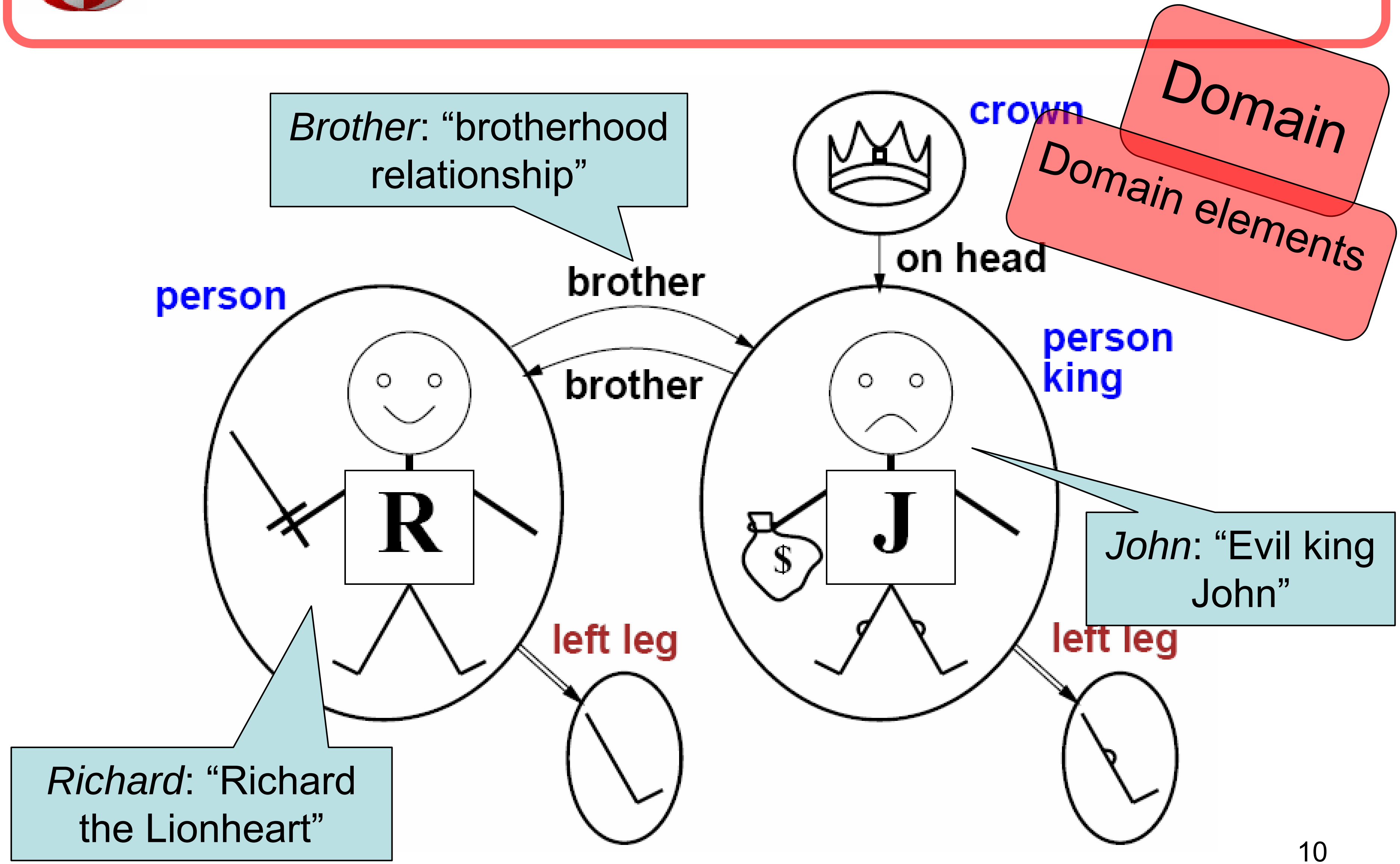
- Sentences are true with respect to a model and an interpretation

- Model contains ≥ 1 objects (domain elements) and relations among them

  Interpretation specifies referents for
    constant symbols $\rightarrow$ objects
    predicate symbols $\rightarrow$ relations
    function symbols $\rightarrow$ functional relations

- An atomic sentence $predicate(term_1, \ldots, term_n)$ is true
  iff the objects referred to by $term_1, \ldots, term_n$
  are in the relation referred to by $predicate$

Models in FOL: An example



Truth Example

- Consider the interpretation in which
  $Richard \rightarrow$ Richard the Lionheart
  $John \rightarrow$ the evil King John
  $Brother \rightarrow$ the brotherhood relation

- Under this interpretation, $Brother(Richard, John)$ is true just in case Richard the Lionheart and the evil King John are in the brotherhood relation in the model

“Possible Models” for FOL: Lots!!

- Entailment in propositional logic can be computed by enumerating models
- We **can** enumerate the FOL models for a given KB vocabulary:
- For each number of domain elements n from 1 to ∞
  - For each k-ary predicate P_k in the vocabulary
    - For each possible k-ary relation on n objects
      - For each constant symbol C in the vocabulary
        - For each choice of referent for C from n objects . . .
- Computing entailment by enumerating FOL models is not easy!

Properties of Sets of Objects

- Now we can represent objects…
- What if we want to express properties of entire sets of objects?
- Or… some of them?
- Does FOL allow that?

Yes!

Universal Quantification

Existential Quantification

Universal Quantification

- $\forall \langle variables \rangle \; \langle sentence \rangle$
- Everyone at Berkeley is smart:
  $\forall x \;\; At(x, Berkeley) \Rightarrow Smart(x)$

Universal Quantification

- $\forall \langle variables \rangle \ \langle sentence \rangle$
- Everyone at Berkeley is smart:
  $\forall x \ \ At(x, Berkeley) \Rightarrow Smart(x)$
- $\forall x \ \ P$ is true in a model m iff P is true with x being **each** possible object in the model
- **Roughly** speaking, equivalent to the conjunction of instantiations of P

$$
\begin{aligned}
& (At(KingJohn, Berkeley) \Rightarrow Smart(KingJohn)) \\
\wedge\ & (At(Richard, Berkeley) \Rightarrow Smart(Richard)) \\
\wedge\ & (At(Berkeley, Berkeley) \Rightarrow Smart(Berkeley)) \\
\wedge\ & \ldots
\end{aligned}
$$

A Common Mistake to Avoid

- Typically, $\Rightarrow$ is the main connective with $\forall$
- Common mistake: using $\land$ as the main connective with $\forall$:

  $$\forall x \;\; At(x, Berkeley) \land Smart(x)$$

- What does it mean?

A Common Mistake to Avoid

- Typically, $\Rightarrow$ is the main connective with $\forall$
- Common mistake: using $\wedge$ as the main connective with $\forall$:

$$\forall x \;\; At(x, Berkeley) \wedge Smart(x)$$

- What does it mean?
- Means: “*Everyone is at Berkeley and everyone is smart*”.
- Is NOT what we wanted to say!

Existential Quantification

- $\exists \langle variables \rangle \; \langle sentence \rangle$
- Someone at Stanford is smart:
  $\exists x \;\; At(x, Stanford) \wedge Smart(x)$

Existential Quantification

- $\exists \langle variables \rangle \ \ \langle sentence \rangle$
- Someone at Stanford is smart:
  $\exists x \ \ At(x, Stanford) \wedge Smart(x)$
- $\exists x \ \ P$ is true in a model m iff P is true with x being **some** possible object in the model
- **Roughly** speaking, equivalent to the disjunction of instantiations of P

$$
\begin{aligned}
& (At(KingJohn, Stanford) \wedge Smart(KingJohn)) \\
\vee \ & (At(Richard, Stanford) \wedge Smart(Richard)) \\
\vee \ & (At(Stanford, Stanford) \wedge Smart(Stanford)) \\
\vee \ & \ldots
\end{aligned}
$$

Another Common Mistake to Avoid

- Typically, $\wedge$ is the main connective with $\exists$
- Common mistake: using $\Rightarrow$ as the main connective with $\exists$:

$$\exists x \;\; At(x, Stanford) \Rightarrow Smart(x)$$

x	*y*	*x*→*y*
0	0	1
0	1	1
1	0	0
1	1	1

- is true if there is anyone who is not at Stanford!

Properties of Quantifiers

- $\forall x \ \forall y$ is the same as $\forall y \ \forall x$ (why??)
- $\exists x \ \exists y$ is the same as $\exists y \ \exists x$ (why??)
- $\exists x \ \forall y$ is **not** the same as $\forall y \ \exists x$
- $\exists x \ \forall y \ Loves(x, y)$
  "There is a person who loves everyone in the world"
- $\forall y \ \exists x \ Loves(x, y)$
  "Everyone in the world is loved by at least one person"

Properties of Quantifiers

- Quantifier duality: each can be expressed using the other

$$\forall x \;\; Likes(x, IceCream) \qquad \neg\exists x \;\; \neg Likes(x, IceCream)$$

$$\exists x \;\; Likes(x, Broccoli) \qquad \neg\forall x \;\; \neg Likes(x, Broccoli)$$

- De Morgan rule applies to quantifiers

Fun with Sentences

- Brothers are siblings

  $\forall x, y \;\; Brother(x, y) \;\Rightarrow\; Sibling(x, y).$

- "Sibling" is symmetric

  $\forall x, y \;\; Sibling(x, y) \;\Leftrightarrow\; Sibling(y, x).$

- One's mother is one's female parent

  $\forall x, y \;\; Mother(x, y) \;\Leftrightarrow\; (Female(x) \wedge Parent(x, y)).$

- A first cousin is a child of a parent's sibling

- $\forall x, y \;\; FirstCousin(x, y) \;\Leftrightarrow\; \exists p, ps \;\; Parent(p, x) \wedge Sibling(ps, p) \wedge Parent(ps, y)$

Equality

- Equality sign means two terms refer to the same object
- E.g., *Father*(*John*) = *Henry*
  says that…
- The object referred to by *Father*(*John*) and the object referred to by *Henry* are the same.
- It can also be used with negation:
- E.g., Richard has at least two brothers

$$\exists\, x,y\ Brother(x, Richard) \land Brother(y, Richard) \land \neg(x=y)$$

Interacting with FOL KBs

- *Assertions* and *Queries* in FOL.
- Suppose a wumpus-world agent is using an FOL KB and perceives a smell and a breeze (but no glitter) at $t = 5$:

  $Tell(KB, Percept([Smell, Breeze, None], 5))$
  $Ask(KB, \exists\, a \;\; Action(a, 5))$

- I.e., does KB entail any particular actions at $t = 5$?
- Answer: $Yes,\ \{a/Shoot\}$ $\leftarrow$ substitution (binding list)

Interacting with FOL KBs

- Given a sentence S and a substitution σ,
  $S\sigma$ denotes the result of plugging σ into S; e.g.,
  $S = Smarter(x, y)$
  $\sigma = \{x/Hillary, y/Bill\}$
  $S\sigma = Smarter(Hillary, Bill)$

- $Ask(KB, S)$ returns some/all σ such that $KB \models S\sigma$

KB for the Wumpus World

- "Perception"
  $\forall\, b, g, t \;\; Percept([Smell, b, g], t) \;\Rightarrow\; Smelt(t)$
  $\forall\, s, b, t \;\; Percept([s, b, Glitter], t) \;\Rightarrow\; AtGold(t)$
- Reflex: $\forall\, t \;\; AtGold(t) \;\Rightarrow\; Action(Grab, t)$
- Reflex with internal state: do we have the gold already?
  $\forall\, t \;\; AtGold(t) \wedge \neg Holding(Gold, t) \;\Rightarrow\; Action(Grab, t)$

  $Holding(Gold, t)$ cannot be observed
  $\Rightarrow$ keeping track of change is essential

Deducing Hidden Properties

- Properties of locations:
  $\forall x, t \;\; At(Agent, x, t) \land Smelt(t) \;\Rightarrow\; Smelly(x)$
  $\forall x, t \;\; At(Agent, x, t) \land Breeze(t) \;\Rightarrow\; Breezy(x)$

- Squares are breezy near a pit:

  Diagnostic rule—infer cause from effect
  $$\forall y \;\; Breezy(y) \;\Rightarrow\; \exists x \;\; Pit(x) \land Adjacent(x, y)$$

  Causal rule—infer effect from cause
  $$\forall x, y \;\; Pit(x) \land Adjacent(x, y) \;\Rightarrow\; Breezy(y)$$

  Neither of these is complete—e.g., the causal rule doesn't say whether squares far away from pits can be breezy

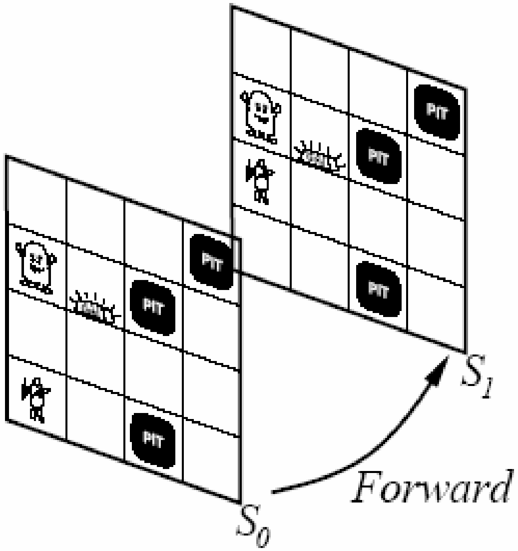
  Definition for the $Breezy$ predicate:
  $$\forall y \;\; Breezy(y) \;\Leftrightarrow\; [\exists x \;\; Pit(x) \land Adjacent(x, y)]$$

Overview of other Knowledge Representation Issues

- We need to deal with time, change.
- Actions and their effects
- Sequence of actions? → Making plans
- *Should you panic? No.*

Keeping track of Change

- Facts hold in situations, rather than eternally
  E.g., $Holding(Gold, Now)$ rather than just $Holding(Gold)$
- Situation calculus is one way to represent change in FOL:
  Adds a situation argument to each non-eternal predicate
  E.g., Now in $Holding(Gold, Now)$ denotes a situation
- Situations are connected by the $Result$ function
  $Result(a, s)$ is the situation that results from doing a in s



Describing Actions I

- "Effect" axiom—describe changes due to action
  $\forall s\ \ AtGold(s) \Rightarrow Holding(Gold, Result(Grab, s))$
- "Frame" axiom—describe **non-changes** due to action
  $\forall s\ \ HaveArrow(s) \Rightarrow HaveArrow(Result(Grab, s))$
- Frame problem: find an elegant way to handle non-change
  (a) representation—avoid frame axioms
  (b) inference—avoid repeated "copy-overs" to keep track of state
- Qualification problem: true descriptions of real actions require endless caveats—what if gold is slippery or nailed down or ...
- Ramification problem: real actions have many secondary consequences—what about the dust on the gold, wear and tear on gloves, ...

Describing Actions II

- Successor-state axioms solve the representational frame problem
- Each axiom is "about" a **predicate** (not an action per se):

$$\begin{array}{lll} \text{P true afterwards} & \Leftrightarrow & [\text{an action made P true} \\ & \vee & \text{P true already and no action made P false}] \end{array}$$

- For holding the gold:

$$\begin{array}{l} \forall\, a, s \;\; Holding(Gold, Result(a,s)) \;\Leftrightarrow \\ \qquad [(a = Grab \wedge AtGold(s)) \\ \qquad \vee (Holding(Gold, s) \wedge a \neq Release)] \end{array}$$

Making Plans

- Initial condition in KB:

  $At(Agent, [1,1], S_0)$
  $At(Gold, [1,2], S_0)$

- Query: $Ask(KB, \exists\, s \;\; Holding(Gold, s))$
  i.e., in what situation will I be holding the gold?

- Answer: $\{s/Result(Grab, Result(Forward, S_0))\}$
  i.e., go forward and then grab the gold

- This assumes that the agent is interested in plans starting at S_0 and that S_0 is the only situation described in the KB

Making Plans: A better way

- Represent plans as action sequences $[a_1, a_2, \ldots, a_n]$

  $PlanResult(p, s)$ is the result of executing p in s

- Then the query $Ask(KB, \exists p \;\; Holding(Gold, PlanResult(p, S_0)))$ has the solution $\{p/[Forward, Grab]\}$

- Definition of $PlanResult$ in terms of $Result$:

  $\forall s \;\; PlanResult([\,], s) = s$

  $\forall a, p, s \;\; PlanResult([a|p], s) = PlanResult(p, Result(a, s))$

- Planning systems are special-purpose reasoners designed to do this type of inference more efficiently than a general-purpose reasoner

Summary

- First-order logic:
  - objects and relations are semantic primitives
  - syntax: constants, functions, predicates, equality, quantifiers
- Increased expressive power: sufficient to define wumpus world
- Situation calculus:
  - conventions for describing actions and change in FOL
  - can formulate planning as inference on a situation calculus KB

Reading Assignment

- Study Russel & Norvig Chapter10: “Knowledge Representation”
- Take notes and bring them to class!!