Solutions

C. Candan

EE 504

Middle East Technical University
Electrical – Electronics Engineering Department

Midterm #1

Problem	
1	
2	
3	
4	
5	
Total:	

16th April, 2004

Problem 1: (25pts)
Determine whether the following statements are TRUE or FALSE. Explain your reasoning. (Correct answer 2pts, correct explanation 3pts)

$$\overbrace{A(z)}^{Causal} \quad B(z) = \overbrace{B^{+}(z)}^{causal} + \overbrace{B^{-}(z)}^{Non-Causal}$$

a) The causal IIR Wiener filter is the causal part of the IIR Wiener filter.

FALSE:
$$H^{Causal}(z) = \frac{1}{\sigma_0^2 Q(z)} \cdot \left[\frac{P_{dx}(z)}{Q(z^{-1})}\right]_+ = A(z)\,B^{+}(z)$$

Also see Hayes Example 7.2.9 for a numerical example (Also covered in class)

$$H^{Non-Causal}(z) = \frac{1}{\sigma_0^2 Q(z)} \cdot \frac{P_{dx}(z)}{Q(z^{-1})} = A(z)B(z) \quad ; \quad \left[H^{NC}(z)\right]_+ = A(z)B^{+}(z) + \left[A(z)B^{-}(z)\right]_+$$

b) For the signal $x[n] = d[n] + v[n]$, where $d[n]$ is the desired signal and $v[n]$ is noise; it is possible to construct a Wiener filter to estimate $d[n]$ in the minimum mean square error sense. The constructed filter minimizes the energy of the $v[n]$ in the estimate. Hence it maximizes the output SNR.

$x[n] \rightarrow \boxed{W(z)} \rightarrow \hat{d}(n)$; $d(n) \rightarrow \boxed{W(z)} \rightarrow \tilde{d}(n)$; $v(n) \rightarrow \boxed{W(z)} \rightarrow \tilde{v}(n)$

FALSE: $E\{(d-\hat{d})^2\}$

$$w_*: \min_{w} E\{(d-\hat{d})^2\}$$

$$SNR_{out} = \frac{E\{\tilde{d}(n)^2\}}{E\{\tilde{v}(n)^2\}} = \frac{w^T R_d w}{w^T R_v w}$$

$w: \max_{SNR\ w}\{SNR_{out}\} \neq w_*$ → If $v[n]$ is white, $R_v = \sigma_v^2 I$

$\rightarrow SNR_{out} = w^T R_d w / \sigma_v^2 w^T w \Rightarrow w_{SNR} \Rightarrow R_d w_{max} = \lambda_{max} w$ maximizes SNR_{out}

c) A FIR Wiener filter is designed to estimate a noise corrupted signal. The input to the filter is decided to be amplified by factor of two. (the desired signal stays the same) The minimum MSE filter with the amplified input has the same output power.

TRUE:

Before: $x[n] \rightarrow \boxed{W_1(z)} \rightarrow \hat{d}(n)$

$$R_x w_1 = r_{dx}$$

After: $2x[n] \rightarrow \boxed{W_2(z)} \rightarrow \hat{d}(n)$

$$R_x^{after} = 4R_x \qquad r_{dx}^{after} = 2 r_{dx}$$

$$\rightarrow (4R_x) w_2 = (2 r_{dx})$$

$$w_2 = w_1/2$$

$$E\{\hat{d}_2(n)^2\} = w_2^T R_x^{after} w_2 = w_1^T R_x w_1 = E\{\hat{d}_1(n)^2\}$$

d) A Wiener filter is designed to reduce the noise level of a signal. The ratio of the signal power at the output to the signal power at the input is ρ ($\rho < 1$). If the output of the Wiener filter is filtered one more time with the same Wiener filter (two Wiener filters in cascade), the signal power ratio of the cascaded filter is better (greater) than ρ^2.

TRUE:

$x[n] \rightarrow \boxed{H(z)} \rightarrow \hat{d}(n)$

$x(n) = d(n) + v(n)$

$d(n) \rightarrow \boxed{H(z)} \rightarrow \tilde{d}_1(n)$

$d(n) \rightarrow \boxed{H(z)} \rightarrow \boxed{H(z)} \rightarrow \tilde{d}_2(n)$

$$\rho = \frac{E\{(\tilde{d}_1(n))^2\}}{E\{(d(n))^2\}}$$

$$E\{\tilde{d}_1(n)^2\} = \int_{-\infty}^{\infty} P_d(e^{jw})|H(e^{jw})|^2 dw \leq \sqrt{\int P_d(e^{jw})dw \int P_d(e^{jw})|H(e^{jw})|^4 dw}$$

(Cauchy Schwarz)

$$\rho^2 \leq \frac{\int P_d(e^{jw})|H(e^{jw})|^4 dw}{\int P_d(e^{jw})\,dw} = \frac{E\{(\tilde{d}_2(n))^2\}}{E\{(d(n))^2\}}$$

e) The error of the causal IIR predictor is white. And the filter $E(z) = 1 - H^{min-MSE}_{Causal predictor}(z)$ is the minimum phase whitening filter.

TRUE:

$$\hat{x}(n) = \sum_{k=1}^{\infty} h(k)\,x(n-k) \longrightarrow \text{Causal IIR predictor.}$$

$$e(n) = x(n) - \hat{x}(n) \longrightarrow E(z) = X(z) - H^{min-MSE}_{Causal predictor}(z)X(z)$$

$$= \underbrace{\left(1 - H^{min\ MSE}_{causal\ predictor}(z)\right)}_{\text{Filter}} X(z)$$

$$E\{e(n)x(n-k)\} = 0 \quad \text{for } k \geq 1 \rightarrow \text{Orthogonality principle}$$

$e(n)$ is white

$r_e(k) = E\{e(n)e(n-k)\} = E\{e(n)x(n-k)\} - e(n)\sum h(\ell)x(n-k-\ell)\} = 0 \quad$ for $k \geq 1$

Problem 2: (15pts)
We would like to design a minimum MSE three-step predictor with a first order filter. That is

$$\hat{x}[n+3] = w(0)x[n] + w(1)x[n-1]$$

should produce minimum mean square error estimate of $x[n+3]$.
a) What are the Wiener-Hopf equations for this predictor ?

b) If the values of $r_x(k)$ for lags k=0 to k=4 are

$$\mathbf{r}_x = [100.1 - 0.2 - 0.9] = [\,1 \quad 0 \quad 0{,}1 \quad -0{,}2 \quad -0{,}9\,]$$

solve the Wiener-Hopf equation system.

c) Calculate the minimum mean square error.

a) $J = E\left\{ \left(x(n+3) - \hat{x}(n+3) \right)^2 \right\}$

$$\frac{\partial J}{\partial w(k)} = 0 \longrightarrow E\left\{ \left(x(n+3) - \hat{x}(n+3) \right) x[n-k] \right\} = 0 \quad ; \quad k = \{0, 1\}$$

$$E\left\{ \hat{x}(n+3)\, x(n-k) \right\} = E\left\{ x(n+3)\, x(n-k) \right\} \quad ;$$

$$\hat{x}(n+3) = \sum_{\ell=0}^{1} w(k)\, x(n-\ell)$$

$$\sum_{\ell=0}^{1} w(\ell)\, r_x(k-\ell) = r_x(k+3) \quad ; \quad k = \{0, 1\}$$

$$\begin{bmatrix} r_x(0) & r_x(-1) \\ r_x(1) & r_x(0) \end{bmatrix} \begin{bmatrix} w(0) \\ w(1) \end{bmatrix} = \begin{bmatrix} r_x(3) \\ r_x(4) \end{bmatrix}$$

b) $$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} w(0) \\ w(1) \end{bmatrix} = \begin{bmatrix} -0{,}2 \\ -0{,}9 \end{bmatrix}$$

$$w(0) = -0{,}2$$
$$w(1) = -0{,}9$$

c) $J^{*} = E\{(\underbrace{x(n+3) - \hat{x}(n+3)}_{e(n)})^2\}$ for $w(0) = -0{,}2$, $w(1) = -0{,}9$

$$= E\{e(n)\,x(n+3)\} - E\{e(n)\,\hat{x}(n+3)\}$$

$\hat{x}(n+3) \to \sum_{\ell=0}^{1} w(0)\,x(n-\ell)$

$$= E\{e(n)\,x(n+3)\} - \sum_{\ell=0}^{1} w(0)\,\cancel{E\{e(n)\,x(n-\ell)\}}^{\,0}$$ for $w(0) = -0{,}2$, $w(1) = -0{,}9$

$e(n) \to x(n+3) - \hat{x}(n+3)$

$$= r_x(0) - \sum_{\ell=0}^{1} w(\ell)\, r_x(\ell+3)$$

$$= 1 - \left((-0{,}2)^2 + (-0{,}9)^2\right)$$

$$= 0{,}15$$

Problem 3) cont.

e) $$J(\underline{h}) = 1 - 2[h_1 \;—\; h_N]\begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix} + [h_1\; h_2 \;—\; h_N]\begin{bmatrix}3 & 1 & 1 & \cdots & 1\\1 & 3 & 1 & \cdots & 1\\\vdots & & & & \\1 & \cdots & & & 3\end{bmatrix}\begin{bmatrix}h_1\\h_2\\\vdots\\h_N\end{bmatrix}$$

$$J(h, h, —, h) = 1 - 2.Nh + 2\begin{pmatrix} h_1h_2 + h_1h_3 + — + h_1h_N + \\ + h_2h_3 + — + h_2h_N \\ + h_3h_4 \cdots + h_3h_N \\ + h_{N-1}h_N \end{pmatrix} + 3.Nh^2$$

$$= 1 - 2Nh + 2.\frac{(N-1)N}{2}.h^2 + 3Nh^2$$

$$= 1 - 2Nh + N(N+2)h^2$$

Optimal estimator is biased $E\{\hat{d}\} \neq E\{d\}$

but consistent

$1 - N/N+2 \to 0$ as $N \to \infty$

$$\frac{\partial J(h, h, h, h)}{\partial h} = 0 \longrightarrow -2N + 2N(N+2)h = 0 \to h_* = \frac{1}{N+2}$$

, $J(h_*, h_*, -, h_*) = 1 - N/N+2$

3

Problem 3: (20pts)

A random variable d is to be estimated from its noisy observations. The measurement system introduces noise uncorrelated with d.

$$m[n] = d + v[n]$$

The noise $v[n]$ is white with zero mean and has the variance of 2. The variance of the r.v. d is 1.

a) Calculate the mean square error of the estimate $\hat{d} = \frac{1}{3}\sum_{n=1}^{3} m[n]$.

b) Find the minimum MSE estimator with three observations and calculate the minimum MSE.

c) Find the general MSE relation for the estimator $\hat{d} = \sum_{n=1}^{3} h[n]m[n]$.

d) Assume $h[1] = h[2] = h[3] = h$ in part c) and plot the MSE vs. h curve. Find the minimum point.

e) Using the results of part d) find the Nth order estimator. Find the minimum error corresponding to the Nth order estimator. What can you say about the bias and the consistency of the optimal estimators ?

a) $$E\{(d-\hat{d})^2\} = E\left\{\left(d - \frac{m_1+m_2+m_3}{3}\right)^2\right\} = E\left\{\left(d - \frac{v_1+v_2+v_3+3d}{3}\right)^2\right\}$$

$$= \frac{E\{(v_1+v_2+v_3)^2\}}{9} = \frac{3.\sigma_v^2}{9} = \frac{2}{3}$$

b) $$J = E\{(d - \underline{h}^T\underline{m})^2\}$$

$$\nabla_h J = 0 \longrightarrow -2E\{(d-\underline{h}^T\underline{m}).\underline{m}\} = 0 \longrightarrow E\{\underline{m}\underline{m}^T\}.\underline{h} = E\{d\underline{m}\}$$

$$E\{m_k . m_\ell\} = E\{(d+v_k)(d+v_\ell)\} = \sigma_d^2 = 1 \quad , \quad k \neq \ell$$

$$E\{m_k^2\} = E\{(d+v_k)(d+v_k)\} = \sigma_d^2 + \sigma_v^2 = 3 \quad , \quad k = \ell$$

$$E\{d m_k\} = E\{d(d+v_k)\} = \sigma_d^2 = 1 \quad , \forall k$$

$$\begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}\begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Longrightarrow h_1 = h_2 = h_3 = 1/5$$

$$J_* = E\{(d - \underline{h}_*^T \underline{m})^2\} = \overbrace{E\{d^2\} - E\{d\,\underline{h}_*^T \underline{m}\}}^{E\{(d-\underline{h}_*^T\underline{m})d\}} - E\{(d - \underline{h}_*^T \underline{m})\,\underline{h}_*^T \underline{m}\}$$

$$= E\{d^2\} - h_*^T E\{d\,\underline{m}\} - \underline{h}_*^T \cancel{E\{(d - h_*^T m)\underline{m}\}}$$

$$= 1 - [\,1/5 \;\; 1/5 \;\; 1/5\,]\begin{bmatrix}1\\1\\1\end{bmatrix} - [\,1/5 \;\; 1/5 \;\; 1/5\,]\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$= 2/5$$

---

c) $$J(\underline{h}) = E\{(d - \underline{h}^T \underline{m})^2\} = E\{(d - \underline{h}^T \underline{m})d\} - E\{(d - \underline{h}^T \underline{m})\underline{h}^T \underline{m}\}$$

$$= E\{d^2\} - 2\,\underline{h}^T E\{d\,\underline{m}\} + \underline{h}^T E\{\underline{m}\,\underline{m}^T\}\underline{h}$$

$$= 1 - 2\,[\,h_1 \;\; h_2 \;\; h_3\,]\begin{bmatrix}1\\1\\1\end{bmatrix} + [\,h_1\, h_2\, h_3\,]\begin{bmatrix}3 & 1 & 1\\1 & 3 & 1\\1 & 1 & 3\end{bmatrix}[\,h$$

$$= 1 - 2(h_1 + h_2 + h_3) + 2(h_1 h_2 + h_1 h_3 + h_2 h_3) + 3(h_1^2 + h_2^2 +$$

---

d) $J(h_1, h_2, h_3) =$ $\longrightarrow$ $J(h,h,h) = 1 - 6h + 6h^2 + 9h^2$

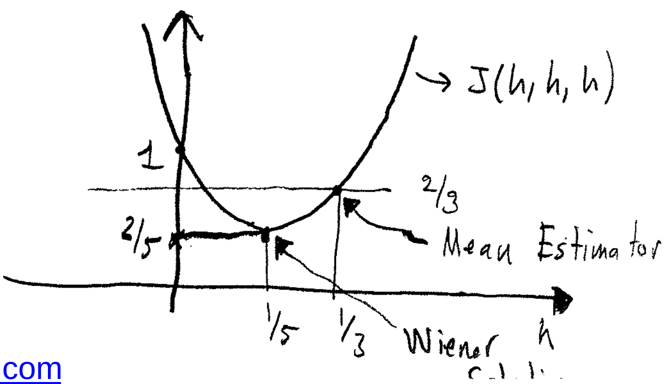
$$= 1 - 6h + 15h^2$$

$$J(1/5, 1/5, 1/5) = 1 - 6/5 + 15/25 = 1 - 3/5 = 2/5 \qquad (\text{part } b)$$

$$J(1/3, 1/3, 1/3) = 1 - 2 + 15/9 = 2/3 \qquad (\text{part } a)$$

$$\frac{\partial J(h,h,h)}{\partial h} = -6 + 30h\,,$$

$$\frac{\partial J(h,h,h)}{\partial h} = 0 \rightarrow h = 1/5$$



Problem 4: (20pts)

A filter is called zero phase if it satisfies $w(n) = w(-n)$. The zero phase filters do not introduce any phase distortions to the input signals. The system function of zero-phase filters can be written as:

$$W(z) = w(0) + \sum_{k=1}^{P} w(k)[z^{-k} + z^{k}]$$

a) Derive Wiener-Hopf equations for the optimum zero-phase smoothing filter.

b) Determine the first order zero phase filter $W(z) = w(0) + w(1)[z + z^{-1}]$ for $r_d(k) = 4(0.5)^{|k|}$ and $r_v(k) = \delta(k)$. (As usual $d[n]$ and $v[n]$ uncorrelated, $v[n]$ zero mean, $x[n] = d[n] + v[n]$ is the input to the filter)

c) Imaginary phase filters are defined by $w(n) = -w(n)$ and $w(0) = 0$. Determine the Wiener-Hopf equations for imaginary phase filters.

d) Show the relation between unconstrainted Wiener filter (arbitrary phase) and zero and imaginary phase Wiener filters.

a) $J = E\{(d(n) - \hat{d}(n))^2\}$; $\hat{d}(n) = w(0)x(n) + \sum_{k=1}^{P} w(k)[x(n-k) + x(n+k)]$

$\frac{\partial J}{\partial w(0)} = 0 \rightarrow E\{(d(n) - \hat{d}(n))x(n)\} = 0 \rightarrow \boxed{r_x(0)w(0) + \sum_{\ell=1}^{P} w(\ell)[r_x(\ell) + r_x(-\ell)] = r_{dx}(0)}$

$\frac{\partial J}{\partial w(k)} = 0 \rightarrow E\{(d(n) - \hat{d}(n))[x(n-k) + x(n+k)]\} = 0$; $k = \{1, \ldots$

$$\boxed{w(0)[r_x(k) + r_x(-k)] + 2\sum_{\ell=1}^{P} w(\ell)[r_x(k-\ell) + r_x(k+\ell)] = r_{dx}(k) + \ldots + r_{dx}(-k)}$$

$k = \{1, P\}$

$$\begin{bmatrix} r_x(0) & 2r_x(1) & 2r_x(2) & \ldots & 2r_x(P) \\ r_x(1) & r_x(0)+r_x(2) & r_x(-1)+r_x(3) & \ldots & r_x(1-P)+r_x(1+P) \\ r_x(2) & r_x(1)+r_x(3) & r_x(0)+r_x(4) & \ldots & r_x(2-P)+r_x(2+P) \\ \vdots & & & & \\ r_x(P) & r_x(P+1)+r_x(P-1) & \ldots & & r_x(0)+r_x(2P) \end{bmatrix} \begin{bmatrix} w_0 \\ \vdots \\ w_P \end{bmatrix} = \begin{bmatrix} r_{dx}(0) \\ \frac{r_{dx}(-1)+r_{dx}(1)}{2} \\ \frac{r_{dx}(2)+\ldots+r_{dx}(-2)}{2} \\ \vdots \\ \frac{r_{dx}(P)+\ldots+r_{dx}(-P)}{2} \end{bmatrix}$$

b) $r_{dx}(k) = E\{ d(n)\ x(n-k)\} = E\{ d(n)[d(n-k)+v(n-k)]\}$

$$= r_d(k)$$

$r_x(k) = E\{ x(n)x(n-k)\} = E\{(d(n)+v(n))(d(n-k)+v(n-k))\}$

$$= r_d(k) + r_v(k)$$

$r_{dx}(k) = 4 \cdot 0.5^{|k|}$

$r_x(k) = 4 \cdot 0.5^{|k|} + \delta(k)$

$$\begin{bmatrix} 5 & 4 \\ 2 & (5+1) \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$w_0 = 8/11 \quad ; \quad w_1 = 1/11$$

---

c) $J = E\{(d(n) - \hat{d}(n))^2\} \Rightarrow \quad \hat{d}(n) = \sum_{k=1}^{P} w_o(k)(x(n-k) - x(n+k))$

$\frac{\partial J}{\partial w(k)} = 0 \longrightarrow E\{(d(n) - \hat{d}(n))(x(n-k) - x(n+k))\} = 0 \quad k = \{1, \ldots, P\}$

$$\sum_{\ell=1}^{P} w_o(\ell)\left[r_x(k-\ell) - r_x(k+\ell) - r_x(-k-\ell) + r_x(-k+\ell)\right] = r_{dx}(k \ldots - r_{dx}(- \ldots$$

$$\boxed{\sum_{\ell=1}^{P} w_o(\ell)\left[r_x(k-\ell) - r_x(k+\ell)\right] = \frac{r_{dx}(k) - r_{dx}(- \ldots}{2}}$$

---

d) See Back Side of Problem 1

Problem 4 cont.

e) Zero Phase equations:

$$w_e(0) \longrightarrow r_x(0)\,w_e(0) + \sum_{\ell=1}^{p} w_e(\ell)\left[r_x(\ell) + r_x(-\ell)\right] = r_{dx}(0)$$

$$r_x(0)\,w_e(0) + \sum_{\ell=1}^{p} w_e(\ell)\,r_x(-\ell) + \underbrace{\sum_{\ell=1}^{p} w_e(\ell)\,r_x(\ell)}_{\sum_{\ell=-1}^{-p} w_e(-\ell)\,r_x(-\ell)} = r_{dx}(0)$$

$w_e(-\ell) \rightarrow w_e(\ell)$

$$r_x(0)\,w_e(0) + \sum_{\ell=1}^{p} w_e(\ell)\,r_x(-\ell) + \sum_{\ell=-p}^{-1} w_e(\ell)\,r_x(-\ell) = r_{dx}(0)$$

$$\boxed{\sum_{\ell=-p}^{p} w_e(\ell)\,r_x(-\ell) = r_{dx}(0)} \quad (1)$$

$$w_e(k) \longrightarrow w(0)\left[r_x(k) + r_x(-k)\right] + 2\sum_{\ell=1}^{p} w_e(\ell)\left[r_x(k-\ell) + r_x(k+\ell)\right] = r_{dx}(k) + r_{dx}(-k)$$

$k = \{1, \dots, p\}$

$$w(0)\left[r_x(k) + r_x(+k)\right] + \sum_{\ell=1}^{p}(\quad) + \underbrace{\sum_{\ell=1}^{p}(\quad)}_{\sum_{\ell=-p}^{-1} w_e(-\ell)\left[r_x(k+\ell) + r_x(k-\ell)\right]} = r_{dx}(k) + r_{dx}(-k)$$

$$(2) \quad \boxed{\sum_{\ell=-p}^{p} w_e(\ell)\left[r_x(k+\ell) + r_x(k-\ell)\right] = r_{dx}(k) + r_{dx}(-k)}$$

Imaginary Phase Equations:

See next page

Imaginary Phase Equations:

$$w_o(k) \longrightarrow 2\sum_{\ell=1}^{P} w_o(\ell)\left[r_x(k-\ell) - r_x(k+\ell)\right] = r_{dx}(k) - r_{dx}(-k)$$

$k=\{1,\ldots,P\}$

$$\sum_{\ell=1}^{P}(\;) + \underbrace{\sum_{\ell=1}^{P}(\;)}_{\displaystyle \sum_{\ell=-1}^{-P} \underbrace{w_o(-\ell)}_{-w_o(\ell)}\left[r_x(k+\ell) - r_x(k-\ell)\right]} = r_{dx}(k) - r_{dx}(-k)$$

$$\underbrace{\sum_{\ell=-1}^{-P} w_o(-\ell)\left[r_x(k+\ell) - r_x(k-\ell)\right]}_{\displaystyle \sum_{\ell=-P}^{-1} w_o(\ell)\left[r_x(k-\ell) - r_x(k+\ell)\right]}$$

$$(3) \qquad \boxed{\sum_{\ell=-P}^{P} w_o(\ell)\left[r_x(k-\ell) - r_x(k+\ell)\right] = r_{dx}(k) - r_{dx}(-k)}$$

$$(2)+(3) \Longrightarrow \sum_{\ell=-P}^{P}\left(w_e(\ell)+w_o(\ell)\right) r_x(k-\ell) + \underbrace{\sum_{\ell=-P}^{P}\left(w_e(\ell)-w_o(\ell)\right) r_x(k+\ell)}_{\displaystyle \sum_{\ell=-P}^{P}\left(w_e(-\ell)-w_o(-\ell)\right) r_x(k-\ell)} = 2r_{dx}($$

$$\cancel{2}\sum_{\ell=-P}^{P}\left(w_e(\ell)+w_o(\ell)\right) r_x(k-\ell) = \cancel{2}\, r_{dx}(k)$$

Wiener-Hopf Equation for Arbitrary Phase Filter

$$\longrightarrow \boxed{\sum_{\ell=-P}^{P} w_a(\ell)\, r_x(k-\ell) = r_{dx}(k)} \quad k\in[-P, \quad k\neq 0$$

$$\longrightarrow \boxed{\sum w_a(\ell)\, r_x(-\ell) = r_{dx}(0)} \quad k=0$$

$$\boxed{w_e(k) = \mathrm{Ev}\{w_a(k)\} = w_a(k) + w_a(-k)}$$

So

Arbitrary Phase Wiener Filter for $\sum_{\ell=-1}^{1} w(k)\,x(n-k) = d(n)$

$$\sum_{\ell=-1}^{1} r_x(k-\ell)\, w_a(\ell) = r_{dx}(k) \qquad k = \{-1, 0, 1\}$$

$$\begin{bmatrix} 5 & 2 & 1 \\ 2 & 5 & 2 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} w_a(-1) \\ w_a(0) \\ w_a(1) \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$$

$$w_a(-1) = 1/11$$
$$w_a(0) = 8/11$$
$$w_a(1) = 1/11$$

Optimum Linear Phase Filter Coef: $w_e(k) = \dfrac{w_a(k) + w_a(-k)}{2}$

$$= w_a(k)$$

Optimum Imaginary Phase Filter Coef.: $w_o(k) = 0$

Problem 5: (20pts)
The Pth order single step minimum MSE predictor is defined as

$$\hat{x}[n] = \sum_{k=1}^{P} w_k x[n-k]$$

The prediction error is $\epsilon[n] = x[n] - \hat{x}[n]$.

a) Assume that x[n] is an AR(1) process: $x[n] = \alpha x[n-1] + u[n]$ ($u[n]$ is white with ρ_u variance). Write the Yule-Walker equations for $x[n]$.

b) Determine the auto-correlation sequence of prediction error for first order minimum MSE predictor ($P = 1$).

c) Find the auto-correlation and the variance of error for P=1 when the input is AR(1).

d) Assume that the input is an AR(2) process. Determine the error auto-correlation sequence for P=1, P=2 and P=3. Comment on your results.

a) $E\{x(n)\,x(n-k)\} = \alpha\, E\{x(n-1)\,x(n-k)\} + E\{u(n)\,x(n-k)\}$

$$r_x(k) = \alpha\, r_x(k-1) + \rho_u^2\,\delta(k)$$

b) $r_\epsilon(k) = E\{\epsilon(n)\,\epsilon(n-k)\}$

$$= E\{\epsilon(n)[x(n-k) - \hat{x}(n-k)]\}$$

For min-MSE predictor (P=1) $\longrightarrow$ $E\{\epsilon(n)\,x(n-1)\} = 0$

$r_\epsilon(0) = E\{\epsilon(n)\,(x(n) - w_1^* x[n-1])\}$

$$= E\{x^2(n) - w_1^* x[n]x[n-1]\} - \overset{0}{\cancel{E\{\epsilon(n)x(n-1)\}}}\, w_1^*$$

$$= r_x(0) - w_1^* r_x(1)$$

$r_\epsilon(1) = E\{\epsilon(n)[x(n-1) - \hat{x}(n-1)]\}$

$$= -w_1^* r_x(2) + w_1^{*2} r_x(1)$$

$r_\epsilon(k) = E\{\epsilon(n)[x(n-k) - w_1^* x(n-k-1)]\} \longrightarrow k \geq 2$

$$= r_x(k) - w_1^* r_x(k-1) - w_1^* r_x(k+1) + w_1^{*2} r_x(k)$$

c) When the process is AR(1),

$$x(n) = \alpha x(n-1) + u(n)$$

Then the first order predictor

$$\hat{x}(n) = w_1 x(n-1)$$

coefficient has to be α to minimize the error.

$$w_{*1} = \alpha.$$

Then $r_\epsilon(0) = r_x(0) - \alpha r_x(1) = \sigma_u^2 \longrightarrow$ (From Yule-Walker)

$r_\epsilon(1) = -\alpha r_x(2) - \alpha^2 r_x(1) = 0 \longrightarrow$ Yule Walker.

$r_\epsilon(k) = 0 \longrightarrow$ Similarly

Then $r_\epsilon(k) = \sigma_u^2 \delta(k) \longrightarrow$ Prediction Error is White

---

d) AR(2): $x(n) = \alpha x(n-1) + \beta x(n-2) + u(n)$

$r_x(k) = \alpha r_x(k-1) + \beta r_x(k-2) + \sigma_u^2 \delta(k) \longleftarrow$ Yule Walker.

P=1

$$w_{*1} = \frac{r_x(1)}{r_x(0)} = \frac{\alpha}{1-\beta}$$

$$r_\epsilon(0) = r_x(0) - \frac{\alpha}{1-\beta} r_x(1)$$

$$= r_x(0)\left[1 - \left(\frac{\alpha}{1-\beta}\right)^2\right]$$

$$r_\epsilon(1) = w_1^2 r_x(2) - w_1 r_x(1)$$

$$= w_1^2(\alpha \underbrace{r_x(1)}_{\alpha/1-\beta\, r_x(0)} + \beta r_x(0)) - \underbrace{w_1 \cdot \frac{\alpha}{1-\beta}}_{w_1^2} r_x(0)$$

$$= w_1^2 r_x(0)\left[\frac{\alpha^2}{1-\beta} + \beta - 1\right]$$

Conc:

$r_\epsilon(k)$ is not white

P=2

$r_\epsilon(0) = \sigma_u^2$

$r_\epsilon(1) = 0$

$r_\epsilon(k) = 0 \quad k \geq 1$

$w_{*1} = \alpha$

$w_{*2} = \beta$

Optimal Coefficients

P=3

$r_\epsilon(0) = \sigma_u^2$

$r_\epsilon(1) = 0$

$r_\epsilon(2) = 0 \quad k \geq 1$

$w_{*1} = \alpha$

$w_{*2} = \beta$

$w_{*3} = 0$