EE 504

Homework #1
Due: March 11, 2005

P.1: We have studied the estimation of r.v. x from r.v. y in the last lectures. In one of the examples, the estimator had the form $\widehat{x} = ay + b$. The minimum MSE for this estimator is derived as $J_* = \rho_x^2(1 - \rho_{xy}^2)$. We have noted in class that it is possible to reach zero error for this estimator (estimation matches the true value) if the correlation coefficient of x and y is ± 1. In this problem we examine the reverse argument.

Show that if x and y have the correlation coefficient of ± 1, x has to be in the form $x = ay + b$, matching the structure of the estimator.

P.2: We have derived the optimal MSE estimator as $\varphi(x) = E\{y|x\}$, where y is the r.v. to be estimated, x is the observation.

Assume that the observation x on unknown r.v. y can take four different values, $x \in \{1, 2, 3, 4\}$. We denote MSE estimator which minimizes the error $E\{(y - c(x))^2\}$ by $c(x) = E\{y|x\}$.

The minimum error estimator for this problem is a lookup table, mapping observations $x = \{1, 2, 3, 4\}$ to $\{c(1), c(2), c(3), c(4)\}$. Find c(1),c(2),c(3),c(4) by minimizing the following:

$$\begin{aligned} E\{(y - c(x))^2\} &= p\{x = 1\}E\{(y - c(1))^2|x = 1\} + \cdots \\ &\cdots + p\{x = 4\}E\{(y - c(4))^2|x = 4\} \end{aligned}$$

P.3: Show that the optimal estimator for the Gaussian r.v.'s with Gaussian distributed observations is the linear estimator.

The pdf of the jointly distributed Gaussian r.v. (zero mean):

$$f(x, y) = \frac{1}{2\pi\rho_1\rho_2\sqrt{1 - r^2}} \exp\left[-\frac{1}{2(1 - r^2)}\left(\frac{x^2}{\rho_1^2} - 2r\frac{xy}{\rho_1\rho_2} + \frac{y^2}{\rho_2^2}\right)\right]$$

Show that $E\{y|x\}$ is a straight line. Assume non-zero mean. (Hint: See Papoulis page 176)

P.4: It is possible to use different cost functions for the estimation problem. Modify the usual cost function of $E\{(y - \hat{y})^2\}$ to $E\{|y - \hat{y}|\}$ and show that the minimum absolute error estimator $\hat{y} = c$ is the median of the r.v. y. (Hint: See Papoulis page 178)

P.5: We have derived the linear minimum MSE estimator of a scalar as

$$\hat{y}_N = \frac{1}{N+c}\sum_{k=1}^{N} x_k$$

where $c = \frac{\rho_v^2}{E\{y^2\}} = \frac{1}{SNR}$ (ρ_v^2: variance of noise on observations, SNR: Signal to Noise Ratio)

Implement the following MATLAB experiment comparing the MSE optimal linear estimator with the mean estimator.

Setup:
i. Randomly generate a single scalar y from zero mean, unit variance Gaussian distribution.
ii. Generate a 1xN noise vector from zero mean, ρ_v variance Gaussian distribution.
iii. Form the observations by adding the noise vector to y
iv. Generate the min. MSE estimate, and the mean estimate from the observation vector. Calculate the square of the error for each estimate.

a) Take N=8, $\rho_v^2 = 1$, repeat i-iv for 1000 times. Calculate the average of the errors found in step iv. Compare the resulting average errors of two estimators.

b) Repeat part a for the following range of noise variances $\rho_v^2 = 0.1 : 0.1 : 10$. Plot the average error vs. SNR curve for each estimator. You can use logarithmic scale for x and y axes.

c) Implement the recursive estimator for both estimators. For N=300, $\rho_v^2 = 1$ repeat the steps i-iii. Instead of running batch estimator (step iv), run the recursive estimators on the observation vector. Plot the estimation output vs. (number of observations) curve for both estimators. Comment on the bias of the min. MSE estimator.

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