Name: ....
Student Number: ....

EE 504

Middle East Technical University
Electrical – Electronics Engineering Department

Final Exam

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3rd June, 2004
C.Candan

Problem 1: (17 pts)

Part 1: Determine if the following statements are TRUE or FALSE (2 pts each)

a) [..TRUE..] RLS filters converge faster than LMS filters. (at least an order of magnitude)

b) [.FALSE.] RLS filters converge at the fastest rate when the input is white noise. (input independent)

c) [..TRUE.] By design Kalman filter is an unbiased estimator.

d) [..TRUE..] For stationary signals Kalman filters converge to the Wiener solution for all initial conditions.

Part 2: Explain the relation between the following types of filters. Discuss the differences in cost function, the differences in their operation, the conditions at which both filters are equivalent etc. (3 pts each)

e) Wiener and LMS Filters:

LMS is an iterative method for the solution of Wiener equations. LMS is a gradient based approximation technique. Gradients are calculated from instantenous values. LMS filters approaches corresponding Wiener filter as iterations increase.

f) LMS and RLS Filters:

LMS is based on statistical signal properties. RLS is based on the signal (or realization of an ensemble) itself. For stationary signals, RLS → LMS as iterations increase. LMS is preferrable when SNR is less than 1, RLS is preferrable otherwise.

g) Wiener and Kalman Filters:

Kalman filters use the known system model at the estimation of signal values. Both filters operate on statistically defined signals. The cost function of both filters are the same. Kalman Filter → IIR Causal Wiener filter as iterations increase. Kalman F. is applicable to non-stationary signals, while Wiener F. is not.

Problem 2: (Parts a, b, c, d = 4pts, 4pts, 3pts, 2pts ; total : 13pts)
Solve the following short questions.

a) $w(n)$ is a zero mean, unit variance white noise source. $s(n) = w(n) + w(n-1)$. Determine the coefficients ($\{a, b\}$ and $\{c, d\}$) of the following optimal predictors.

$$\hat{s}(n+1) = as(n) + bs(n-1)$$
$$\hat{s}(n+2) = cs(n) + ds(n-1)$$

Show your steps.

$$r_s(k) = r_w(k) * h(k) * h(-k) = \delta(k) * (\delta(n) + \delta(n-1)) * (\delta(n) + \delta(n+1))$$
$$= \delta(n+1) + 2\delta(n) + \delta(n-1)$$

$$r_s(k) = \begin{cases} 1 & |k|=1 \\ 2 & k=0 \\ 0 & \text{other} \end{cases}$$

$\hat{s}(n+1)$: $\underline{R}_s \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} r_s(1) \\ r_s(2) \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2/3 \\ -1/3 \end{bmatrix}$

$\hat{s}(n+2)$: $\underline{R}_s \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} r_s(2) \\ r_s(3) \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

b) Fit a line in the least square error sense to the following points $(x, y) = \{(0, 0), (1, 2), (2, 3)\}$. Show your steps.

$y = mx + n$

$$\underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}}_{\underline{A}} \underbrace{\begin{bmatrix} m \\ n \end{bmatrix}}_{\underline{x}} = \underbrace{\begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}}_{\underline{b}} \rightarrow \hat{\underline{x}}_{LS} = (\underline{A}^T\underline{A})^{-1}\underline{A}^T\underline{b} = \begin{bmatrix} 5 & 3 \\ 3 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 8 \\ 5 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 3 & -3 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} 8 \\ 5 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 1/6 \end{bmatrix}$$

c) Set up the state equations for the Kalman filter to fit a line to the following points $(x, y) = \{(0, 0), (1, 2), (2, 3)\}$. (Do not run any iterations, just set up the state equations).

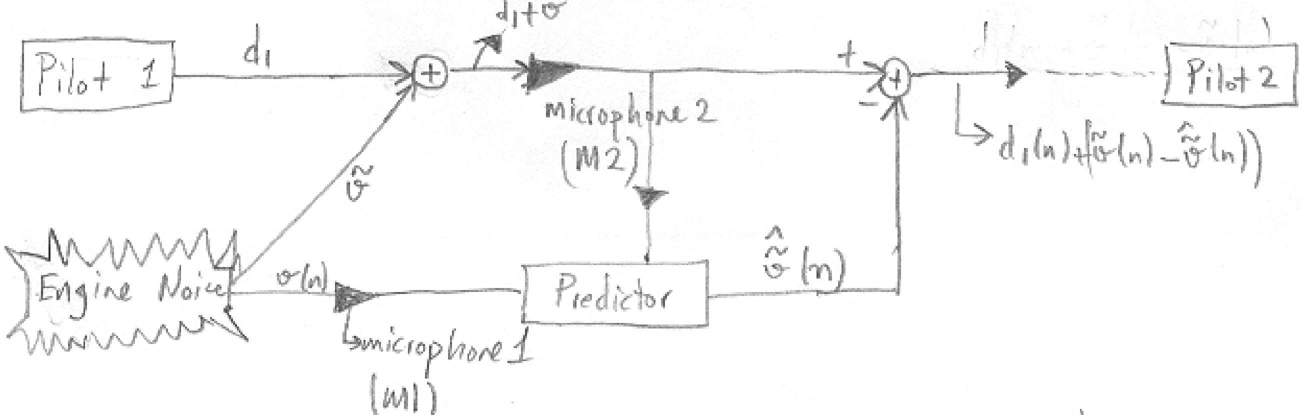
$$\hat{\underline{x}}(n+1) = \hat{\underline{x}}(n) = \begin{bmatrix} \hat{m} \\ \hat{n} \end{bmatrix}$$ for $n = 0, 1, 2$

$$y(n) = [n \quad 1]\hat{x}(n) + v(n)$$

$y(0) = 0$, $y(1) = 2$, $y(2) = 3$

$v(n)$ → measurement noise (approximation error)

g) Noise Cancellation: The pilots in airplanes use headsets with microphones to talk to each other. Engine noise in cockpit may leak into the microphones and disturb the conversation. Design a system to cancel the engine noise leakage into the headsets. State where to put the microphones and show what to do the with the recorded signals with a block diagram.



M1 should be placed close to the engine, d_1 should not leak into M1.
M2 is in cockpit (microphone of headset)

Problem 3: (25pts) Consider a system consisting of two sensors, each making a single measurement of an unknown constant x. Each measurement is noisy and may be modeled as follows

$$y(1) = x + v(1)$$
$$y(2) = x + v(2)$$

where $v(1)$ and $v(2)$ are zero mean and uncorrelated random variables with variance σ_1^2 and σ_2^2 respectively. Note that $\sigma_1^2 \neq \sigma_2^2$.

a) In absence of any other information, we seek the best linear estimate of x of the form

$$\hat{x} = k_1 y(1) + k_2 y(2)$$

Find the values for k_1 and k_2 that yield an **unbiased** estimate of x that minimizes the mean-square error, $E\{(x - \hat{x})^2\}$.

b) Repeat part a) for the case where the measurement errors are correlated,

$$E\{v(1)v(2)\} = \rho\sigma_1\sigma_2$$

c) Repeat part a) within the framework of Kalman filtering, treating the measurements $y(1)$ and $y(2)$ sequentially.

a) Unbiased Estimator: $E\{\hat{x}\} = E\{x\} \longrightarrow k_1 + k_2 = 1$ $\left(\begin{matrix} v(1) \\ v(2) \end{matrix}\right)$ zero mean

Problem is minimize $J(k_1, k_2) = E\{(x-\hat{x})^2\}$ with the constraint $k_1 + k_2 = 1$

Method 1: Lagrange Multipliers $J(k_1, k_2) = E\{(x-\hat{x})^2\} + \lambda(1 - k_1 - k_2)$

$$\frac{\partial J(k_1,k_2)}{\partial k_1} = 0 \quad , \quad \frac{\partial J(k_1,k_2)}{\partial k_2} = 0 \quad , \quad \frac{\partial J}{\partial \lambda} = 0$$

Solve the equation system.

Method 2: minimize $J(k_1, 1-k_1)$ (Easier for this question)

$$\frac{\partial J(k_1, 1-k_1)}{\partial k_1} = 0 \longrightarrow E\{(x-\hat{x})(y(1) - y(2))\} = 0$$

$$E\{(x - k_1 \underbrace{y(1)}_{x+v(1)} - (1-k_1)\underbrace{y(2)}_{x+v(2)})(v(1) - v(2))\} = 0$$

$$E\{(k_1 v(1) + (1-k_1)v(2))(v(1) - v(2))\} = 0$$

$$E\{k_1 v_1^2 - (1-k_1)v_2^2 + v_1 v_2(1 - 2k_1)\} = 0$$

$$k_1 \sigma_1^2 - (1-k_1)\sigma_2^2 = 0 \longrightarrow \boxed{k_1 = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}}$$

$$k_1 = \frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2} \quad ; \quad k_2 = \frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2} \longrightarrow \boxed{\hat{x} = \frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}\, y(1) + \frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}\, y(2)}$$

Observations : ① If $\sigma_1^2 = \sigma_2^2 \longrightarrow \hat{x} = \frac{1}{2}\big(y(1)+y(2)\big) \longrightarrow$ LS estimate.

② If $\left.\begin{array}{l}\sigma_1^2 \text{ is finite}\\ \sigma_2^2 \text{ is } \infty\end{array}\right\} \longrightarrow \hat{x} = y(1)$

ⓑ $E\{v_1 v_2\} = \rho\,\sigma_1\sigma_2 \longrightarrow k_1\sigma_1^2 - (1-k_1)\sigma_2^2 + (1-2k_1)\rho\,\sigma_1\sigma_2 = 0$

$$k_1 = \frac{\sigma_2^2 - \rho\,\sigma_1\sigma_2}{\sigma_1^2 - 2\rho\,\sigma_1\sigma_2 + \sigma_2^2}$$

$$k_2 = 1 - k_1$$

ⓒ $\left.\begin{array}{l} x(n+1) = x(n)\\ y(n) = x(n) + v(n)\end{array}\right\}$ $\underline{A} = \underline{C} = 1$

$Q_v(1) = \sigma_1^2 \quad , \quad Q_v(2) = \sigma_1^2$

$$P(n|n-1) = P(n-1|n-1) = P(n-1)$$

$$K(n) = \frac{P(n-1)}{P(n-1) + Q_v(n)}$$

$$P(n) = (1 - K(n))\,P(n-1) = \frac{P(n-1)\,Q_v(n)}{P(n-1) + Q_v(n)}$$

Initialization:

$P(0) = \infty \longleftarrow$ infinite variance

$\hat{x}(0) = 0$

$\hat{x}(1)$:

$$P(1) = \frac{P(0)\cdot\sigma_1^2}{P(0)+\sigma_1^2} = \sigma_1^2$$

$$K(1) = \frac{P(0)}{P(0)+\sigma_1^2} = 1$$

$$\hat{x}(1) = \hat{x}(0) + K(1)\,[y(1) - 0]$$

$$\boxed{\hat{x}(1) = y(1)}$$

$\hat{x}(2)$: $\longrightarrow$

$$P(2) = \frac{P(1)\,\sigma_2^2}{P(1)+\sigma_2^2} = \frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}$$

$$K(2) = \frac{P(1)}{P(1)+\sigma_2^2} = \frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}$$

$$\hat{x}(2) = \hat{x}(1) + K(2)\,[y(2) - y(1)] = y(1) + \frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}\,[y(2) - y(1)]$$

$$\boxed{\hat{x}(2) = \frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}\, y(1) + \frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}\, y(2)}$$

Problem 4: (20pts) Suppose that we wish to estimate a signal $d(n)$ from the noisy observation

$$x(n) = d(n) + v(n)$$

where $v(n)$ is unit variance white noise that is uncorrelated with $d(n)$. The signal $d(n)$ is an AR(1) process that is generated by the difference equation

$$d(n) = 0.8d(n-1) + w(n)$$

where $w(n)$ is white noise with variance $\rho_w^2 = 0.36$. Therefore $r_d(k) = (0.8)^{|k|}$. Find the optimum causal Wiener Filter for estimating $d(n)$ from $x(n)$.

$$H_{NC}(z) = \frac{P_{dx}(z)}{P_x(z)} \quad ; \quad P_x(z) = \sigma_0^2\, Q(z)\, Q(1/z)$$

$$H_{causal}(z) = \frac{1}{\sigma_0^2 Q(z)} \left[\frac{P_{dx}(z)}{Q(1/z)} \right]_+$$

$$P_x(z) = P_d(z) + P_v(z)$$
$$= \mathcal{Z}\{0.8^{|k|}\} + 1$$
$$= \frac{1-(0.8)^2}{(1-0.8z^{-1})(1+0.8z^{-1})} + 1$$
$$= 1.6\,\frac{(1-0.5z^{-1})(1-0.5z)}{(1-0.8z^{-1})(1-0.8z)}$$

$$P_{dx}(z) = P_d(z)$$
$$= \frac{0.36}{(1-0.8z^{-1})(1-0.8z)}$$

Factorize $P_x(z) = \sigma_0^2 \cdot Q(z)\, Q(1/z) \longrightarrow \sigma_0^2 = 1.6$

$$Q(z) = \frac{1-0.5z^{-1}}{1-0.8z^{-1}}$$ min-phase (causal and invertible)

$$H_{causal}(z) = \frac{1}{\sigma_0^2 Q(z)} \left[\frac{P_{dx}(z)}{Q(1/z)} \right]_+ = \frac{1}{\sigma_0^2 Q(z)} \left[\frac{0.36}{(1-0.8z^{-1})\cancel{(1-0.8z)}} \cdot \frac{\cancel{(1-0.8z)}}{(1-0.5z)} \right]_+$$

$$= \frac{1}{\sigma_0^2 Q(z)} \left[\frac{0.36}{(1-0.8z^{-1})(1-0.5z)} \right]_+$$

$$\frac{0.36}{(1-0.8z^{-1})(1-0.5z)} = \frac{0.36z^{-1}}{(1-0.8z^{-1})(z^{-1}-0.5)} = \underbrace{\frac{0.6}{(1-0.8z^{-1})}}_{\text{Causal.}} + \underbrace{\frac{0.3}{z^{-1}-0.5}}_{\downarrow}$$

$$\underbrace{\frac{0.3z}{1-0.5z}}_{\text{non-causal}}$$

$$H_{causal}(z) = \frac{1}{\sigma_0^2\,\phi(z)} \cdot \frac{0.6}{(1-0.8z^{-1})}$$

$$= \frac{1}{1.6} \cdot \frac{\cancel{(1-0.8z^{-1})}}{(1-0.5z^{-1})} \cdot \frac{0.6}{\cancel{(1-0.8z^{-1})}}$$

$$= \frac{0.375}{1-0.5z^{-1}}$$

$$\boxed{h_{causal}(n) = 0.375\left(\tfrac{1}{2}\right)^n u(n)}$$

Problem 5: (15pts) In some applications, it is necessary to delay the update of the filter coefficients for a short period of time. The delayed LMS algorithm which has a filter coefficient update equation is given by

$$\mathbf{w}_{n+1} = \mathbf{w}_n + \mu e(n - n_0)\mathbf{x}(n - n_0)$$

where

$$e(n - n_0) = d(n - n_0) - \hat{d}(n - n_0)$$

Note that if the delay, n_0, equals to zero then we have the conventional LMS algorithm.

For $n_0 = 1$, determine the values of μ for which the delayed LMS algorithm converges in the mean.

$$\hat{d}(n) = \underline{w}_n^T \cdot \underline{x}(n) \longrightarrow \hat{d}(n-1) = \underline{w}_{n-1}^T \cdot \underline{x}(n-1)$$

$$\underline{w}_{n+1} = \underline{w}_n + \mu\left(d(n-1) - \underline{w}_{n-1}^T \underline{x}(n-1)\right)\underline{x}(n-1)$$

$$E\{\underline{w}_{n+1}\} = E\{\underline{w}_n\} + \mu E\{d(n-1)\underline{x}(n-1)\} - \mu E\{\underline{x}(n-1)\underline{x}(n-1)^T \underline{w}_{n-1}\}$$

Independence Assumption

$$E\{\underline{w}_{n+1}\} = E\{\underline{w}_n\} - \mu E\{\underline{x}(n-1)\underline{x}(n-1)^T\} E\{\underline{w}_{n-1}\} + \mu E\{d(n-1)\underline{x}(n-1)\}$$

$$\boxed{E\{\underline{w}_{n+1}\} = E\{\underline{w}_n\} - \mu \underline{\underline{R}}_x E\{\underline{w}_{n-1}\} + \mu \underline{r}_{dx}} \quad (1)$$

If $\underline{\underline{R}}_x = \underline{\underline{Q}}\,\underline{\underline{\Lambda}}\,\underline{\underline{Q}}^T$, $\underline{\underline{\Lambda}} = \mathrm{diag}(\lambda_1, \lambda_2, \text{—}, \lambda_p)$; multiply both sides by $\underline{\underline{Q}}^T$ and call $\underline{\underline{Q}}^T E\{\underline{w}_{n+1}\} = E\{\underline{\underline{Q}}^T \underline{w}_{n+1}\} = E\{\underline{\tilde{w}}_{n+1}\}$

$$E\{\underline{\underline{Q}}^T \underline{w}_{n+1}\} = E\{\underline{\underline{Q}}^T \underline{w}_n\} - \mu \underline{\underline{Q}}^T(\underline{\underline{Q}}\,\underline{\underline{\Lambda}}\,\underline{\underline{Q}}^T) E\{\underline{w}_{n-1}\} + \mu \underbrace{\underline{\underline{Q}}^T \underline{r}_{dx}}_{\tilde{r}_{dx}}$$

$$\boxed{E\{\underline{\tilde{w}}_{n+1}\} = E\{\underline{\tilde{w}}_n\} - \mu \underline{\underline{\Lambda}} E\{\underline{\tilde{w}}_{n-1}\} + \mu \underline{\tilde{r}}_{dx}} \quad (2)$$

Notes: Eq(1) is a coupled system of recursive equations
Eq(2) is the decoupled representation of the same system.

k^{th} row of the equation system of the equation (2):

$$\underbrace{E\{w_{n+1}(k)\}}_{x(n+1)} = \underbrace{E\{w_n(k)\}}_{x(n)} - \mu\lambda_k \underbrace{E\{w_{n-1}(k)\}}_{x(n-1)} + \underbrace{\mu\tilde{r}_{dx}(k)}_{c}$$

$$x(n+1) = x(n) - \mu\lambda_k x(n-1) + c$$

The poles of the recursion above should be in the unit circle.

$$z X(z) = X(z) - \mu\lambda_k z^{-1} X(z) + \ldots$$

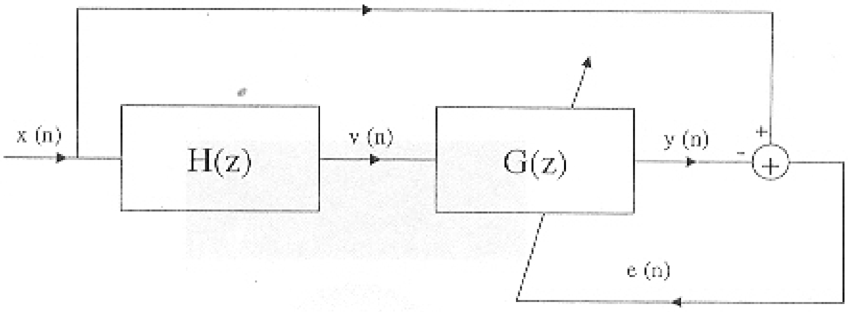
$$(z^2 + z + \mu\lambda) \longrightarrow \text{roots} = \left\{ \frac{1 \mp \sqrt{1 - 4\mu\lambda_k}}{2} \right\}$$

roots inside the unit circle if $|\mu\lambda_k| < 1$
(can also be found from the stability triangle)

Conclusion: For convergence in mean $|\mu\lambda_k| < 1$ for all k.

$$\boxed{0 < \mu < \frac{1}{\lambda_{max}}}$$

Problem 6: (15pts) Consider the second order adaptive recursive filter that is used as a channel equalizer as shown in the figure below.



where

$$H(z) = 1 - z^{-1} + 0.8z^{-2}$$

and

$$G(z) = \frac{b_n(0)}{1 + a_n(1)z^{-1} + a_n(2)z^{-2}}$$

a) The following equations are given to you by a friend.

$$\begin{aligned}
b_{n+1}(0) &= b_n(0) + \mu e(n) f(n) \\
&\left.\begin{aligned}
\cancel{a_{n+1}(1)} &= \cancel{a_n(1) + \mu e(n) g(n)} \\
\cancel{a_{n+1}(2)} &= \cancel{a_n(2) + \mu e(n) h(n)} \\
\cancel{f(n)} &= \cancel{v(n) + b_n(0) f(n-1)} \\
\cancel{g(n)} &= \cancel{v(n) + a_n(1) g(n-1) + a_n(2) g(n-2)}
\end{aligned}\right\} \text{modified.} \\
\cancel{h(n)} &= \cancel{v(n-1) + a_n(1) g(n-1) + a_n(2) g(n-2)} \quad \text{deleted}
\end{aligned}$$

Unfortunately not all of these equations are correct. You would like to correct these equations to implement simplified output error method (the filtered signal method). Make necessary corrections to the above equations. You can modify, add or delete the equations so that they correspond to the filtered signal method.

$$f(n) = v(n) + a_n(1) f(n-1) + a_n(2) f(n-2)$$

$$g(n) = y(n) + a_n(1) g(n-1) + a_n(2) g(n-2)$$

$$a_{n+1}(1) = a_n(1) + \mu e(n) g(n-1)$$
$$a_{n+1}(2) = a_n(2) + \mu e(n) g(n-2)$$

b) Modify the equations in part a) to implement the SHARF algorithm using a filter C(z) that is of second order, i.e.

$$C(z) = c(0) + c(1)z^{-1} + c(2)z^{-2}$$

$$e'(n) = c_0 e(n) + c_1 e(n-1) + c_2 e(n-2)$$

$$\left.\begin{aligned} a_{n+1}(k) &= a_n(k) + \mu e'(n) y(n-k) \\ b_{n+1}(k) &= b_n(k) + \mu e'(n) x(n-k) \end{aligned}\right\}$$

filtered Feintuch's method.

Feintuch $\approx$ FIR Adapt Filters.

c) Describe how the filter C(z) should be designed, i.e. what are the design criteria ?

C(z) should be selected such that

$$\mathrm{Re}\left\{ \frac{C(z)}{1 - z^{-1} + 0.8z^{-2}} \right\} > 0 \qquad z = e^{jw}$$

SPR condition

Problem 7: (15pts) An autoregressive process of order 1 is described by the difference equation

$$x(n) = 0.5x(n-1) + w(n)$$

where w(n) is zero-mean white noise with a variance $\rho_w^2 = 0.64$. The observed process $y(n)$ is given by

$$y(n) = x(n) + v(n)$$

where $v(n)$ is zero-mean white noise with a variance $\rho_v^2 = 1$.

a) Write down Kalman filter equations with initial conditions $\hat{x}(0|0) = 0$ and $P(0|0) = 1$.

b) Assuming filter reaches steady state, find the steady state Kalman gain and steady state Kalman Filter.

a) $x(n) = 0.5\,x(n-1) + w(n)$ $\quad \underline{\underline{A}} = 0.5 \quad \underline{\underline{Q_w}} = 0.64$

$y(n) = x(n) + v(n)$ $\quad \underline{\underline{C}} = 1 \quad \underline{\underline{Q_v}} = 1$

$$P(n|n-1) = (0.5)^2\, P(n-1|n-1) + 0.64$$

$$K(n) = \frac{P(n|n-1)}{[P(n|n-1) + 1]}$$

$$P(n|n) = [1 - K(n)]\, P(n|n-1) =$$

b) $$P(n|n) = \left[1 - \frac{P(n|n-1)}{P(n|n-1)+1}\right] P(n|n-1)$$

$$= \frac{P(n|n-1)}{P(n|n-1)+1}$$

$$\boxed{P(n|n) = \frac{0.25\,P(n-1|n-1) + 0.64}{0.25\,P(n-1|n-1) + 1.64}}$$

Call $P_\infty = \lim_{n\to\infty} P(n|n) \longrightarrow P_\infty = \dfrac{0.25\,P_\infty + 0.64}{0.25\,P_\infty + 1.64}$

$$0.25(P_\infty)^2 + 1.39 P_\infty - 0.64 = 0$$

$$P_\infty = \{0.4276, -6\}$$

$P_\infty \Rightarrow$ variance of error as $n \to \infty$ $\therefore P_\infty > 0 \to P_\infty = 0.42$

Steady state ~~filter~~ filter

$$P(n) = \frac{P(n|n-1)}{P(n|n-1)+1} = K(n)$$ (see previous page)

$$K_\infty = \lim_{n\to\infty} K(n) = P_\infty = 0.42$$

$$\boxed{\hat{x}(n) = 0.5\hat{x}(n-1) + 0.4276\left(y(n) - 0.5\hat{x}(n-1)\right)}$$