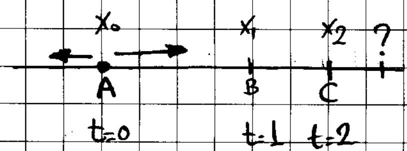
EE557 ESTIMATION THEORY:

Example: Object Tracking



noise term

① $x_{k+1} = x_k + T.v \; (+u_k) + w_k$

assume constant vel.

② $x_{k+1} = x_k + Tv_k + \frac{T^2}{2} \cdot w_k$

$v_{k+1} = v_k + T. w_k$

$y_k = x_k + v_k$ observation noise

Deterministic Systems

General Case
$$\begin{cases} x_{k+1} = f_k(x_k, u_k) \\ y_k = h_k(x_k) \end{cases}$$

x_{k+1}: vector, u_k: vector

$x_k \in \mathbb{R}^n$

$u_k \in \mathbb{R}^m$

$y_k \in \mathbb{R}^r$

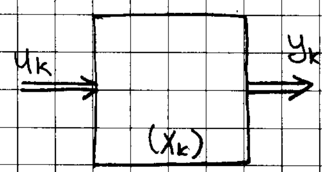
Linear Sys
$$\begin{cases} x_{k+1} = A_k x_k + B_k u_k \\ y_k = C_k x_k \end{cases}$$

LTI
$$\begin{cases} x_{k+1} = A x_k + B u_k \\ y_k = C x_k \end{cases}$$

OR

$y_{k+1} = a_0 y_k + a_1 y_{k-1} + \cdots + a_n y_{k-n} + b_0 u_k + \cdots$

Stochastic Systems



$x_{k+1} = f_k(x_k, u_k, w_k)$

random quantity

$y_k = h_k(x_k, v_k)$ ($y_k \equiv x_k \rightarrow$ complete obsr. case)

①
$x_{k+1} = Ax_k + Bu_k + w_k$

$y_k = Cx_k + v_k$

Basic Random Variables # (Vectors)

② $x_0, w_0, w_1, \ldots, v_0, v_1, \ldots$

③ Joint probability distribution functions of $\{x_0, w_0, w_1, \ldots, v_0, v_1, \ldots\}$

①, ② and ③ defines a stochastic system

① or $y_{k+1} = a_0 y_k + a_1 y_{k-1} + \cdots + a_n y_{k-n}$
$+ b_0 u_k + b_1 u_{k-1} + \cdots + b_m u_{k-m}$
$+ c_0 w_k + c_1 w_{k-1} + \cdots$

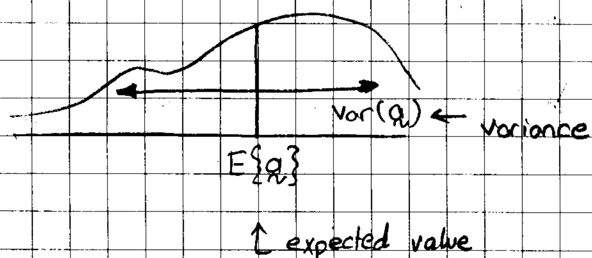
Definition: A stochastic system or model is given by specifying;

1) f_k and h_k for $k \geq 0$

2) The joint probability distribution $\{x_0, w_0, w_1, \ldots, v_0, v_1, \ldots\}$ where x_0 is the initial state, $\{w_0, w_1, \ldots\}$ are input disturbance and $\{v_0, v_1, \ldots\}$ are the

measurement noise.

$\underset{\sim}{a} \longrightarrow F_{\underset{\sim}{a}}(a) \rightarrow$ probability distribution func.



- Solving a stochastic model means obtaining the probability distribution functions of the states.

__Example:__ Coin example, (2 coins)

$\Omega = \{HH, HT, TH, TT\}$ the set of possible outcomes

events: subsets of Ω

$$\left.\begin{array}{l} p(HH) = 0.3 \\ p(\{HH, HT\}) = 0.5 \\ \vdots \end{array}\right\} \text{Probabilities}$$

p: event $\longrightarrow [0,1]$

__Probability Space:__

$\{\Omega, \mathcal{F}, p\}$ $\quad p: \mathcal{F} \to [0,1]$

$\mathcal{F}$: events which are subsets of Ω

$p(\Omega) = 1$, $p(\emptyset) = 0$

__Random Variable:__

$\underset{\sim}{x}: \Omega \to \mathbb{R}^n$ (it is a function)

$$\underset{\sim}{a}(\omega) = \begin{cases} 1 & \text{if } \omega = HH \\ 2 & \text{if } \omega = HT \\ 3 & \text{if } \omega = TH \\ 4 & \text{if } \omega = TT \end{cases}$$

Probability Distribution Function: (of $\underset{\sim}{x}$)

$$F_{\underset{\sim}{x}}(\alpha) = \text{Prob}\{\underset{\sim}{x} \le \alpha\}$$
$$= \text{Prob}\{\underset{\sim}{x}(\omega) \le \alpha\}$$
$$= \text{Prob}\{\omega \mid \underset{\sim}{x}(\omega) \le \alpha\}$$

$$P_{\underset{\sim}{x}}(X) \triangleq \text{Prob}(\{\omega \mid \underset{\sim}{x}(\omega) \in X\})$$

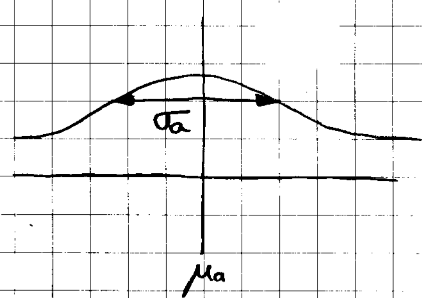
- Density function of $\underset{\sim}{x}$:

$$f_{\underset{\sim}{x}}(\alpha) = \frac{d}{d\alpha} F_{\underset{\sim}{x}}(\alpha)$$

__Example:__

$\underset{\sim}{a}$ gaussian (normal)

$$f_{\underset{\sim}{a}}(a) = \frac{1}{\sqrt{2\pi \sigma_a^2}} e^{-\left(\frac{(a-\mu_a)^2}{2\sigma_a^2}\right)}$$



Expected Value (mean):

$$E\{\underset{\sim}{a}\} = E_{\underset{\sim}{a}} = \int_{-\infty}^{\infty} \underset{\sim}{a} \cdot f_{\underset{\sim}{a}}(a)\, da$$

$$E_{\underset{\sim}{a}} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} f_{\underset{\sim}{a}}(a_1, a_2)\, da_1\, da_2 = \begin{pmatrix} \bar{a}_1 \\ \bar{a}_2 \end{pmatrix} = \mu_{\underset{\sim}{a}}$$

Variance:

$$Var(\underset{\sim}{a}) = E\{a-\bar{a}\}^2 = E(a-\bar{a})^2$$

$$Var(\underset{\sim}{a}) = E\{(a-\bar{a})^T(a-\bar{a})\} = \underbrace{E\{\|a-\bar{a}\|^2\}}_{\text{scalar}}$$

$$Cov(\underset{\sim}{a}) = E\{(a-\bar{a})(a-\bar{a})^T\}_{n\times n} \quad \text{matrix}$$

Example:

$$\underset{\sim}{x} = \begin{pmatrix} \underset{\sim}{x_1} \\ \underset{\sim}{x_2} \end{pmatrix}$$

$$E(\underset{\sim}{x}) = \begin{pmatrix} \iint_{-\infty}^{\infty} x_1 f(x_1,x_2)\,dx_1\,dx_2 \\ \iint_{-\infty}^{\infty} x_2 \underbrace{f(x_1,x_2)}_{P_{\underset{\sim}{x}}(x_1,x_2)}\,dx_1\,dx_2 \end{pmatrix} = \begin{pmatrix} E(x_1) \\ E(x_2) \end{pmatrix}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 f(x_1,x_2)\,dx_1\,dx_2 = \int_{-\infty}^{\infty} x_1 \underbrace{\int_{-\infty}^{\infty} f(x_1,x_2)\,dx_2}_{f_{\underset{\sim}{x_1}}(x_1)}\,dx_1$$

$$= \int_{-\infty}^{\infty} x_1 f_{\underset{\sim}{x_1}}(x_1)\,dx_1 = E(x_1)$$

Let g be a function of $(\underset{\sim}{x})$

$$E(g(\underset{\sim}{x})) = \int_{R^n} g(\underset{\sim}{x}) \underbrace{P_{\underset{\sim}{x}}(x)}_{f_{\underset{\sim}{x}}(x)}\,dx$$

Let $\underset{\sim}{x}$ is an n-dimensional random vector

$$Var(\underset{\sim}{x}) = E\Big(\underbrace{(\underset{\sim}{x}-\bar{x})^T(\underset{\sim}{x}-\bar{x})}_{g(\underset{\sim}{x})}\Big) \rightarrow \text{a scalar}$$

$$\bar{x} = E(\underset{\sim}{x})$$

Covariance Matrix:

$$Cov(\underset{\sim}{x}) = E\{(\underset{\sim}{x}-\bar{x})\cdot(\underset{\sim}{x}-\bar{x})^T\} \rightarrow \text{a matrix}$$

Conditional Distribution:

$$P(A|B) = \frac{P(AB)}{P(B)}$$

A given B

$\underset{\sim}{x}$ $\quad$ $\underset{\sim}{y}$ $\quad$ $P_{\underset{\sim}{x}\underset{\sim}{y}}(x,y)$

$\downarrow$ $\quad$ $\downarrow$

$p_{\underset{\sim}{x}}(x)$ $\quad$ $p_{\underset{\sim}{y}}(x)$

$$P_{\underset{\sim}{x}|\underset{\sim}{y}}(x|y) = \frac{P_{\underset{\sim}{x}\underset{\sim}{y}}(x,y)}{P_{\underset{\sim}{y}}(y)}$$

Example:

$$\underset{\sim}{x} = \begin{cases} 1 & P(\underset{\sim}{x}=1)=0.4 \\ 0 & P(\underset{\sim}{x}=0)=0.6 \end{cases}$$

$$\underset{\sim}{y} = \begin{cases} 1 & P(\underset{\sim}{y}=1)=0.7 \\ 0 & P(\underset{\sim}{y}=0)=0.3 \end{cases}$$

$$P_{\underset{\sim}{x},\underset{\sim}{y}}(x,y) = \begin{cases} x=0,\ y=0 & P(\underset{\sim}{x}=0,\underset{\sim}{y}=0)=0.1 \\ x=1,\ y=0 & P(\underset{\sim}{x}=1,\underset{\sim}{y}=0)=0.2 \\ x=0,\ y=1 & P(\underset{\sim}{x}=0,\underset{\sim}{y}=1)=0.5 \\ x=1,\ y=1 & P(\underset{\sim}{x}=1,\underset{\sim}{y}=1)=0.2 \end{cases}$$

$+ = 0.3$

$$P_{\underset{\sim}{x}|\underset{\sim}{y}}(\underset{\sim}{x}=0|\underset{\sim}{y}=0) = \frac{P_{\underset{\sim}{x},\underset{\sim}{y}}(\underset{\sim}{x}=0,\underset{\sim}{y}=0)}{P_{\underset{\sim}{y}}(\underset{\sim}{y}=0)} = \frac{0.1}{0.3} = \frac{1}{3}$$

$$P_{\underset{\sim}{x}|\underset{\sim}{y}}(\underset{\sim}{x}=0|\underset{\sim}{y}=1) = \frac{P_{\underset{\sim}{x}\underset{\sim}{y}}(\underset{\sim}{x}=0,\underset{\sim}{y}=1)}{P_{\underset{\sim}{y}}(\underset{\sim}{y}=1)} = \frac{0.5}{0.7} = \frac{5}{7}$$

- $E(g(x,y)|z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, p_{x,y/z}(x,y/z)\,dx\,dy$

  $= f(z)$ (a random variable)

- $E\{\underbrace{E\{x/y\}}_{f(y)}\} = E\{x\}$ (a constant value)

Example:

$E(x) = 1*0.4 + 0*0.6 = 0.4$

$E(y) = 1*0.7 + 0*0.3 = 0.7$

$$E\{x/y\} = f(y) = \begin{cases} E\{x \mid y=0\} \\ E\{x \mid y=1\} \end{cases}$$

$E\{x \mid y=0\} = 1\,P(x=1/y=0) + 0.P(x=0/y=0)$

$= 2/3$

$E\{x / y=1\} = 1.P(x=1 / y=1) + 0.P(x=0/y=1)$

$= 2/7$

$$\Rightarrow E\{x/y\} = f(y) = \begin{cases} 2/3 & \text{if } y=0 \\ 2/7 & \text{if } y=1 \end{cases}$$

$E\{f(y)\} = E\{E\{x/y\}\} = \frac{2}{3}\cdot p(y=0) + \frac{2}{7} p(y=1)$

$= \frac{2}{\cancel{3}}\cdot\frac{\cancel{3}}{10} + \frac{2}{\cancel{7}}\cdot\frac{\cancel{7}}{10} = \frac{4}{10}$

—o—

$x_{k+1} = f_k(x_k, u_k, w_k)$

$y_k = h_k(x_k, v_k)$

x_0

Basic Random Variables:

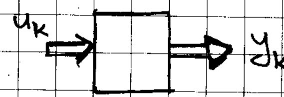
$\{x_0, w_0, w_1, \dots, v_0, v_1, \dots\}$

$\rightarrow p_{x_0,w_0,w_1,\dots,v_0,v_1,\dots}(x_0, w_0, w_1, \dots, v_0, v_1, \dots)$

needed

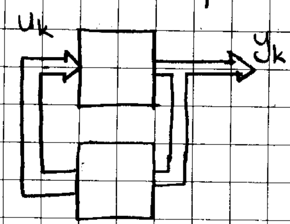
$x_{k+1} = f_k(x_k, u_k)$

$y_k = h_k(x_k)$



open loop

closed loop



$u_k = g_k(y_0, y_1, y_2, \dots)$

Example:

$x_{k+1} = x_k + u_k + w_k$

$y_k = x_k$

assume $\{x_0, w_0, w_1, \dots\}$ independent, zero mean

variance σ^2

feedback control

$g_k(y^k) = g_k(y_0, y_1, \dots, y_k) \triangleq -y_k = u_k = -x_k$

$x_{k+1} = \cancel{x_k} - \cancel{x_k} + w_k = w_k$

$E\{x_k\} = E\{w_k\} = 0$

$Var\{x_k\} = Var\{w_k\} = \sigma^2$

→ pre-determined inputs

$x_1 = x_0 + u_0 + w_0$

$x_2 = x_0 + u_0 + w_0 + u_1 + w_1$

$x_k = x_0 + \sum_{i=0}^{k-1} u_i + \sum_{i=0}^{k-1} w_i$

$$E\{x_k\} = E\{x_0 + \sum_{i=0}^{k-1} u_i + \sum_{i=0}^{k-1} w_i\}$$

$$E\{x_k\} = 0 + \sum_{i=0}^{k-1} u_i + 0 \Rightarrow 0$$

$$Var\{x_k\} = Var\{x_0 + \cancel{\sum_{i=0}^{k-1} u_i} + \sum_{i=0}^{k-1} w_i - \cancel{\sum_{i=0}^{k-1} u_i}\}$$

$$= Var\{x_0 + \sum_{i=0}^{k-1} w_i\} = \sigma^2 + k\sigma^2 = (k+1)\sigma^2$$

- In this course, our aim is to minimize

$$J = Min\ E\{\sum_{i=0}^{k} x_i^2\}$$

$$J^g = E\{\sum_{i=0}^{k} w_i^k\}^2 = (k+1)\sigma^2 \leftarrow \text{closed-loop}$$

$$J^u = E\{\sum_{i=0}^{k} (x_0 + \sum_{j=0}^{i-1} u_j + \sum_{j=0}^{i-1} w_j)^2\} \leftarrow \text{open-loop}$$

$$= \sum_{i=0}^{k} [\sigma^2 + (\sum_{j=0}^{i-1} u_j)^2 + i\sigma^2]$$

$$= \sum_{i=0}^{k} (1+i)\sigma^2 + \underbrace{\sum_{i=0}^{k} (\sum_{j=0}^{i-1} u_j)^2}_{\geq 0} \geq \sigma^2 \sum_{i=0}^{k} (i+1) = \sigma^2 \frac{k(k+1)}{2}$$

—o—

Deterministic Case

$$x_{k+1} = f_k(x_k, u_k)$$

$$y_k = h_k(x_k) \qquad x_k, u_k, u_{k+1}$$

$$x_{k+2} = f_{k+1}(x_{k+1}, u_{k+1}) = f_{k+1}(f_k(x_k, u_k), u_{k+1})$$

Stochastic Case

$$x_{k+1} = f_k(x_k, u_k, w_k) \qquad x_0$$

$$y_k = h_k(x_k, v_k)$$

$$u_k = g_k(y^k) = g_k(y_0, y_1, y_2, \dots, y_k)$$

$$P^g_{x^g_{k+1}|u^g_k, x^g_k}(x^g_{k+1} | u^g_k, x^g_k)$$

$$\Updownarrow ?? =$$

$$P^g_{x^g_{k+1}|u^g_0, u^g_1, \dots, x^g_0, x^g_1, \dots, x^g_k}(x^g_{k+1} | u^g_0, \dots, x^g_0, \dots)$$

$$P^g_{x^g_{k+1}|x^g_k, u^g_k}(x^g_{k+1} \in X_{k+1} | x^g_k, u^g_k)$$

$$x^g_{k+1} = f_k(x^g_k, u^g_k, w_k)$$

$$x^g_{k+1} \in X_{k+1} \Leftrightarrow w_k \in W_k$$

$$\Rightarrow P^g_{w_k}(w_k \in W_k | x^g_k, u^g_k)$$

Assumption:

Assume that $\{x_0, w_0, w_1, \dots, v_0, v_1, \dots\}$ are all independent.

$$x_1 = f_0(x_0, \overbrace{g_0(y_0)}^{u_0}, w_0) \Rightarrow \text{function of } x_0, v_0, w_0$$

which are independent of $w_1 \Rightarrow x_1 \leftrightarrow w_1$ independent

$$u_1 = g_1(y_1, y_0) = g_1(h_1(x_1, v_1), h_0(x_0, v_0))$$

$$= g_1(h_1(f_1(x_0, u_0, w_0), v_1), h_0(x_0, v_0))$$

$$= g_1(h_1(f_1(x_0, h_0(x_0, v_0), w_0), v_0), v_1), h_0(x_0, v_0))$$

$$\Rightarrow w_1 \leftrightarrow u_1 \text{ independent}$$

$$\Rightarrow P^g_{w_1}(w_1 \in W_1 / x^g_1, u^g_1) = P^g_{w_1}(W_1)$$

Lemma: Suppose that the assumption holds. Then for any feedback law g,

$$P^g(X_{k+1} | x^g_k, \dots, x_0, u^g_k, \dots, u^g_0) = P^g(X_{k+1} | u_k, x_k) \quad \star$$

Moreover these conditional densities do not depend on "g".

$x_{k+1} = ax_k + bu_k$: deterministic case

$y_k = cx_k$, $u_k = g(y_0, \dots, y_k)$

x_k^g : known ; u^g : known

$x_{k+1}^g = ax_k^g + bu_k^g$

stochastic case :

$x_{k+1} = ax_k + bu_k + w_k$

$y_k = cx_k + v_k$, $u_k = g(y_0, \dots, y_k)$

Proof:

$x_{k+1}^g = f_k(x_k^g, u_k^g, w_k)$

$y_k^g = h_k(x_k^g, v_k)$, $u_k^g = g_k(y^{g,k})$

$p^g(x_{k+1}^g \in X_{k+1} \mid x_k^g, \dots, x_0^g, u_k^g, \dots, u_0^g)$

$= p^g(f_k(x_k^g, u_k^g, w_k) \in X_{k+1} \mid x_k^g, \dots, x_0^g, u_k^g, \dots, u_0^g)$

Let $W_k = \{ w_k \mid f_k(x_k^g, u_k^g, w_k^g) \in X_{k+1} \}$

$p^g(w_k \in W_k \mid x_k^g, \dots, x_0^g, u_k^g, \dots, u_0^g)^{(*)}$

Claim:

w_k is independent of $x_0, x_1^g, \dots, x_k^g, \dots u_k^g$

Note that if the claim is correct then $(*)$ is equal to $p^g(w_k \in W_k) = p(w_k \in W_k)$ (it is not related with the feedback function "g").

* Note that

$(**)$ $p^g(x_{k+1} \in X_{k+1} \mid x_k^g, u_k^g) = p^g(W_k \mid x_k^g, u_k^g)$

* If the claim is correct

$(**) = p^g(w_k \in W_k) = p(w_k \in W_k)$

Proof of the Claim:

$x_1 = f_0(x_0, u_0^g, w_0)$

$= f_0(x_0, g_0(y_0), w_0)$

$= f_0(x_0, g_0(h(x_0, v_0)), w_0) \rightarrow$ not a function of w_1

Because of the basic assumption w_1 is independent of (x_0, w_0, v_0)

$\therefore$ w_1 is independent of x_1 (1)

w_1 is independent of x_0 (2)

$u_0 = g_0(h_0(x_0, v_0))$: funct. of x_0, v_0

$\therefore$ u_0 is independent of w_1

$u_1 = g_1(h_1(x_1, v_1)) \Rightarrow$ funct. of x_0, v_0, w_0, x_1

(x_1:) function of x_0, w_0, v_0

$\therefore$ w_1 is independent of u_1

$\Rightarrow w_1 \xleftrightarrow[\text{of}]{\text{independent}} x_0, x_1, u_0, u_1$

$\Rightarrow$ If we continue ; at the k^{th} step x_k will be a function of $(x_0, w_0, \dots, w_{k-1}, v_0, \dots, v_{k-1})$

and w_k is independent of x_k

$u_k = f(x_0, w_0, \dots, w_{k-1}, v_0, \dots, v_{k-1})$

and w_k is independent of u_k

$\therefore$ w_k is independent of all past x_i $i \leq k$, u_i $i \leq k$ $\Rightarrow$ claim is correct.

In the two step case

$$p((w_k,w_{k+1})\in W_{k,k+1} \mid x_k^g, u_k^g, u_{k+1}^g) \overset{?}{=}$$
$$p((w_k,w_{k+1})\in W_{k,k+1} \mid x_0^g,\dots,x_k^g, u_0^g,\dots,u_{k+1}^g)$$

$$u_{k+1}^g = g_{k+1}(y^{k+1}) = g_{k+1}(h_{k+1}(x_{k+1},v_{k+1}), h_k(x_k,v_k)\dots)$$
$$= g_{k+1}(h_{k+1}(f(x_k,u_k,\underline{\underline{w_k}}),v_k),\dots)$$

dependent $\Rightarrow$ we cannot drop the given parts at the 2^{nd} step.

Case: u_{k+1} is deterministic

independent

$$p((w_k,w_{k+1})\in W_{k,k+1} \mid u_k^g, x_k^g, \bar{u}_{k+1})$$

independent independent

$$x_k = f_{k-1}(x_{k-1}, u_{k-1}, w_{k-1})$$

$$u_k^g = g_k(y^{k,g}) = g_k(y_1^g, y_2^g, \dots, y_k^g)$$

$h_1(x_1,v_1) \;\dots\; h_k(x_k,v_k)$

func. of $x_0, w_0,\dots,w_{k-1}$, $v_0,\dots,v_{k-1}$

$\Rightarrow p((w_k,w_{k+1})\in W_{k,k+1})$ $\Leftarrow$ left hand side

$p((w_k,w_{k+1})\in W_{k,k+1})$ $\Leftarrow$ right hand side

if u_{k+1} is deterministic $\Rightarrow$ we can drop given parts

Lemma: Suppose that the basic assumption holds, then for any feedback law g $g=(g_0,g_1,g_2,\dots)$ for which

$$g_{k+1} \equiv \bar{u}_{k+1},\; g_{k+2} = \bar{u}_{k+2},\; \dots,\; g_{k+m} = \bar{u}_{k+m}$$

are constant functions

$$p^g(x_{k+m+1}\in X_{k+m+1} \mid x_k,\dots,x_0, \bar{u}_{k+m},\dots,u_k^g,u_0^g)$$
$$= p^g(x_{k+m+1}\in X_{k+m+1} \mid x_k, \bar{u}_{k+m},\dots,\bar{u}_{k+1},u_k^g)$$
$$= P_{w_k,\dots,w_{k+m}}(W_{k,k+m})$$

—o—

$\underset{\sim}{x}_0, \underset{\sim}{x}_1, \dots, \underset{\sim}{x}_k, \dots$

This is a Markov Chain if the pdf of $\underset{\sim}{x}_k$ depends only on $\underset{\sim}{x}_{k-1}$

$$p(x_{k+1}=\alpha \mid x_k, x_{k-1},\dots,x_0) = p(\underset{\sim}{x}_{k+1}=\alpha \mid \underset{\sim}{x}_k)$$

LINEAR SYSTEMS

$$x_{k+1} = Ax_k + Bu_k + Gw_k$$
$$y_k = Cx_k + Hv_k$$

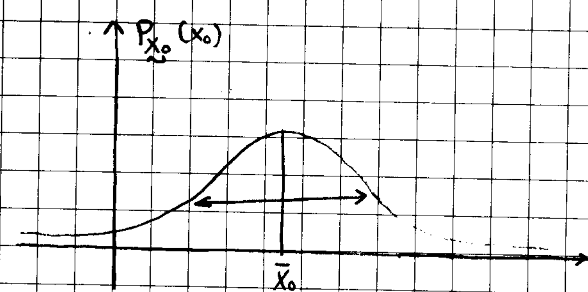
$(x_0, w_0, \dots, v_0, \dots)$ are all independent

- $\underset{\sim}{x}_0$: normal , mean $= \bar{x}_0$

  Covariance matrix : Σ_0

$$P_{\underset{\sim}{x}_0}(x_0) = \frac{1}{\sqrt{(2\pi)^n|\Sigma_0|}} e^{-\frac{1}{2}(x_0-\bar{x}_0)^T\Sigma_0^{-1}(x_0-\bar{x}_0)}$$

if x is one-dimensional



$$P_{\underset{\sim}{x}_0}(x_0) = \frac{1}{\sqrt{2\pi\sigma_0^2}} e^{-\frac{1}{2}(x_0-\bar{x}_0)^2/\sigma_0^2}$$

$w_k \sim N(0, Q)$: normal

covariance matrix (of Q); zero mean (of 0)

$v_k \sim N(0, R)$

Aim #1: Obtain one step transition probability of this system.

$$p(x_{k+1} \mid x_k, u_k) = ?$$

$$x_{k+1} = A x_k + B u_k + G w_k$$

$$\Rightarrow p(x_{k+1} = A x_k + B u_k + G w_k \mid x_k, u_k)$$

- Step I:

$G w_k$ is Gaussian (= Normal) since it is a linear function of a random vector.

$$E\{G w_k\} = G . E\{w_k\} = 0$$

$$Cov(G . w_k) = E\{G w_k . w_k^T G^T\} = G \underbrace{E\{w_k w_k^T\}}_{Q} G^T$$
$$= G Q G^T$$

$p(x_{k+1} = A x_k + B u_k + G w_k \mid x_k, u_k)$: Gaussian

$$E\{A x_k + B u_k + G w_k \mid x_k, u_k\} = \overbrace{A x_k + B u_k}^{\bar{x}_{k+1}} + \underbrace{E\{G w_k\}}_{0}$$

since w_k is independent of x_k, u_k

$$\therefore E\{(x_{k+1} - \bar{x}_{k+1})(x_{k+1} - \bar{x}_{k+1})^T \mid x_k, u_k\}$$

$$= E\{(G w_k)(G w_k)^T \mid x_k, u_k\} = Cov\{G w_k\}$$

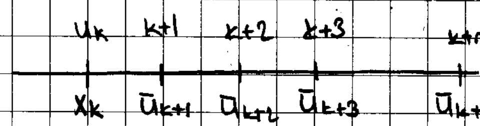
independent

$$= G Q G^T$$

$$p(x_{k+1}^{g} \mid x_k^{g}, u_k^{g}) \sim N(\underbrace{A x_k + B u_k}_{\text{mean } \bar{x}_{k+1}}, \underbrace{G Q G^T}_{\text{covariance}})$$

$$p_{k+1/k}(x_{k+1}) = \frac{1}{\sqrt{(2\pi)^n |G Q G^T|}} e^{-\frac{1}{2}(x_{k+1} - \bar{x}_{k+1})^T (G Q G^T)^{-1} (x_{k+1} - \bar{x}_{k+1})}$$

Aim #2: Obtain the m step conditional pdf of the state.



$$p(x_{k+m+1} \mid x_k^{g}, u_k^{g}, \bar{u}_{k+1}, \ldots, \bar{u}_{k+m})$$

$$x_{k+1} = A x_k + B u_k + G w_k$$

$$x_{k+2} = A^2 x_k + A B u_k + A G w_k + B u_{k+1} + G w_{k+1}$$

$$\vdots$$

$$x_{k+m+1} = A^{m+1} x_k + A^m B u_k + A^{m-1} B u_{k-1} \cdots B u_{k+m}$$
$$+ A^m G w_k + \cdots + G w_{k+m}$$

$$\Rightarrow x_{k+m+1} = A^{m+1} x_k + \sum_{i=0}^{m} A^i B u_{k+m-i} + \overbrace{\sum_{i=0}^{m} A^i G w_{k+m-i}}^{\text{Gaussian}}$$

$$p(x_{k+m+1} \mid x_k, u_k, \bar{u}_{k+1}, \ldots, \bar{u}_{k+m}) \sim N(\bar{x}_{k+m+1}, \Sigma_{k+m+1|k})$$

$$E\{A^{m+1} x_k + \sum_{i=0}^{m} A^i B u_{k+m-i} + \sum_{i=0}^{m} A^i G w_{k+m-i}\}$$

$$= A^{m+1} x_k + \sum_{i=0}^{m} A^i B u_{k+m-i} + \underbrace{\sum_{i=0}^{m} A^i G E\{w_{k+m-i} \mid x_k, u_k, \ldots\}}$$

independent

$$\sum_{i=0}^{m} A^i G \underbrace{E\{w_{k+m-i}\}}_{0}$$

$$= A^{m+1} x_k + \sum_{i=0}^{m} A^i B u_{k+m-i} = E\{x_{k+m+1}\}$$

$Cov(x_{k+1+m} | x_k, u_k, \dots) = \Sigma_{k+m+1|k}$

$= E\left\{ \left(\sum_{i=0}^{m} A^i G w_{k+m-i} \right) \left(\sum_{j=0}^{m} A^j G w_{k+m-j} \right)^T \right\}$

$= E\left\{ \sum_{i=0}^{m} \sum_{j=0}^{m} A^i G w_{k+m-i} \cdot w_{k+m-j}^T G^T A^{j^T} \right\}$

for $i \neq j$ w_{k+m-i} and w_{k+m-j} are independent

$= E\left\{ \sum_{i=0}^{m} A^i G w_{k+m-i} \cdot w_{k+m-i}^T G^T A^{i^T} \right\}$

$= \sum_{i=0}^{m} A^i G Q G^T A^{i^T}$

$= GQG^T + \sum_{i=1}^{m} A^i G Q G^T A^{i^T}$

$= GQG^T + A\left(\sum_{i=1}^{m} A^{i-1} G Q G^T A^{i-1^T} \right) A^T \quad \ell = i-1$

$= GQG^T + A \underbrace{\left(\sum_{\ell=0}^{m-1} A^\ell G Q G^T A^{\ell^T} \right)}_{\Sigma_{k+m|k}} A^T$

$\Rightarrow \Sigma_{k+m+1|k} = A \Sigma_{k+m|k} A^T + GQG^T$

$\Rightarrow \Sigma_{k+1|k} = GQG^T$

$\Rightarrow \Sigma_{k+2|k} = AGQG^T A^T + GQG^T$

$\vdots$ find step m recursively

— o —

$x_{k+1} = A x_k + B u_k + G w_k$

$y_k = C x_k + H v_k$

$Cov(w_k) = Q$, $Cov(v_k) = R$, $Cov(x_0) = \Sigma_0$

$p_{k+1|k}(x_{k+1} | x_k, u_k) \sim N(Ax_k + Bu_k, GQG^T)$

$p_{k+m|k}(x_{k+m} | x_k, u_k, \bar{u}_{k+1}, \dots, \bar{u}_{k+m-1}) \sim N(\bar{x}_{k+m|k}, \Sigma_{k+m|k})$

$\bar{x}_{k+m|k} = A^m x_k + \sum_{i=0}^{m-1} A^i B u_{k+i}$

$\Sigma_{k+m|k} = A \Sigma_{k+m-1|k} A^T + GQG^T$

where $\Sigma_{k|k} = 0$

<u>Covariance Function:</u>

$\Sigma_{k+m|k} = E\left\{ (x_{k+m} - \bar{x}_{k+m})(x_{k+m} - \bar{x}_{k+m})^T \right\}$

Assume that $u_k \equiv \bar{u}_k$ for all k

(deterministic u_k)

<u>Fact I:</u> x_k is normal

$x_0 \sim N(\bar{x}_0, \Sigma_0)$

$x_1 = Ax_0 + Bu_0 + Gw_0$

Note that $Ax_0 + Gw_0$ is gaussian since $Ax_0 + Gw_0 = [A \;\; G] \begin{bmatrix} x_0 \\ w_0 \end{bmatrix}$ and $\begin{bmatrix} x_0 \\ w_0 \end{bmatrix}$ is a gaussian since x_0 and w_0 are gaussian and independent.

$\therefore$ $x_1 =$ (Normal RV) + (Constant) $\rightarrow$ Normal RV

$\vdots$

$x_k = \underbrace{\underbrace{Ax_{k-1} + Gw_{k-1}}_{\text{Gaussian}} + \underbrace{Bu_{k-1}}_{\text{Constant}}}_{\text{Gaussian}}$

— o —

$x_1 = Ax_0 + Bu_0 + Gw_0$

$x_2 = A^2x_0 + ABu_0 + AGw_0 + Bu_1 + Gw_1$

$\vdots$

$$x_k = A^k x_0 + \underbrace{\sum_{i=0}^{k-1} A^{k-1-i} Bu_i}_{\text{Constant}} + \sum_{i=0}^{k-1} A^{k-1-i} Gw_i$$

Since x_k is a linear combination of $(x_0, w_0, w_1, \ldots, w_{k-1})$ which are independent and gaussian, x_k is also a gaussian.

$$\bar{x}_k = A^k \bar{x}_0 + \sum_{i=0}^{k-1} A^{k-1-i} Bu_i \quad \text{and}$$

$$\Sigma_k = A^k \Sigma_0 (A^k)^T + \sum_{i=0}^{k-1} \mathrm{Cov}(A^{k-1-i} Gw_i)$$

$$= A^k \Sigma_0 (A^k)^T + \sum_{i=0}^{k-1} A^{k-1-i} GQG^T (A^{k-1-i})^T$$

$$= A(A^{k-1}\Sigma_0 (A^{k-1})^T)A^T + \sum_{i=0}^{k-2} A^{k-i-1} GQG^T (A^{k-i-1})^T$$

$$+ \underbrace{A^{k-1-(k-1)}}_{I} GQG^T \underbrace{(A^{k-1-(k-1)})^T}_{I}$$

$\vdots$

$$\Sigma_k = A\Sigma_{k-1}A^T + GQG^T$$

$\Sigma_0 = \mathrm{Cov}(x_0)$ is given

$$\Sigma_{k+m|k} = E\{(x_{k+m} - \bar{x}_{k+m})(x_{k+m} - \bar{x}_{k+m})^T\}$$

$$= E\{(A^{k+m}(\underset{\sim}{x_0} - \bar{x}_0) + \sum_{i=0}^{k+m-1} A^{k+m-1-i} Gw_i)$$

$$* (A^k(\underset{\sim}{x_0} - \bar{x}_0) + \sum_{j=0}^{k-1} A^{k-1-j} Gw_j)^T\}$$

$$= E\{(A^m(x_k - \bar{x}_k) + \sum_{i=0}^{m-1} A^{m-1-i} Gw_{i+k})(x_k - \bar{x}_k)^T$$

$x_{k+1} = Ax_k + Bu_k + Gw_k$

$y_k = Cx_k + Hv_k$

$\{x_0, w_0, w_1, \dots, v_0, \dots\}$ are all independent

$x_0 \sim N(\bar{x}_0, \Sigma_0)$

$w_i \sim N(0, Q) \quad v_i \sim N(0, R)$

$p(x_{k+1} \mid x_k, u_k) \sim N(x_{k+1|k}, \Sigma_{k+1|k})$

$x_{k+1|k} = Ax_k + Bu_k$

$\Sigma_{k+1|k} = GQG^T$

$$p(y_k \mid x_k, u_k) = p(Cx_k + Hv_k \mid x_k, u_k)$$
$$\sim N(Cx_k, HRH^T)$$

$$\Rightarrow p(x_{k+m} \mid x_k, u_k, \bar{u}_{k+1}, \dots, \bar{u}_{k+m-1})$$
$$\sim N\left(A^m x_k + \sum_{i=0}^{m-1} A^{m-1-i} Bu_{k+i}, \Sigma_{k+m|k}\right)$$

$$\Sigma_{k+m|k} = A\Sigma_{\underbrace{k+m-1|k}_{0}} \cdot A^T + GQG^T$$

$$p(x_k) \sim N\left(A^k \bar{x}_0 + \sum_{i=0}^{k-1} A^{k-1-i} B\bar{u}_i, \Sigma_k\right)$$

if $x_0 \sim N(\bar{x}_0, \Sigma_0)$

where $\Sigma_k = A\Sigma_{k-1}A^T + GQG^T$

$$p(y_k) = p(\overbrace{Cx_k}^{\bar{y}_k} + Hv_k)$$

Gaussian Gaussian $\Rightarrow$ Gaussian

(they are independent)

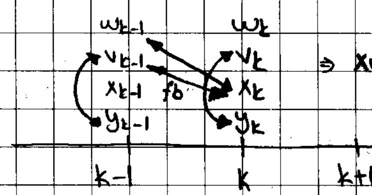
$$\sim N(C\bar{x}_k, \Sigma_k^y)$$

$$\Sigma_k^y = E\{(y_k - \bar{y}_k)(y_k - \bar{y}_k)^T\}$$

$$= E\{(Cx_k + Hv_k - C\bar{x}_k)(Cx_k + Hv_k - C\bar{x}_k)^T\}$$

$$= E\{(C(x_k - \bar{x}_k) + Hv_k)(C(x_k - \bar{x}_k) + Hv_k)^T\}$$

independent



$\Rightarrow x_k$ is independent of w_k and v_k

$$\Rightarrow \Sigma_k^y = C \underbrace{\Sigma_k}_{Cov(x_k)} C^T + \underbrace{HRH^T}_{Cov(v_k)}$$

$$\Sigma_{k+m|k} = Cov(x_{k+m|k}) = E\{(x_{k+m} - \bar{x}_{k+m})(x_{k+m} - \bar{x}_{k+m})\}$$
$$= A^m \Sigma_k$$

Linear, Non-Gaussian Case:

$p(x_{k+1} \mid x_k, u_k) = ?$ we cannot find

$E(x_{k+1} \mid x_k, u_k) = Ax_k + Bu_k$ still holds

$$\Sigma_{k+1|k} = E\{(x_{k+1} - \bar{x}_{k+1})(x_{k+1} - \bar{x}_{k+1})^T \mid x_k, u_k\} = GQG^T$$

—o—

$$\Sigma_k = A^k \Sigma_0 (A^k)^T + \sum_{i=0}^{k-1} A^i GQG^T (A^i)^T$$

if A is a scalar

$$\Sigma_k = \underbrace{a^{2k}}_{positive} \Sigma_0 + \sum_{i=0}^{k-1} a^{2i} \underbrace{(g^2 q)}_{positive}$$

$\Rightarrow$ to have finite Σ_k, $a < 1$

Asymptotic Properties of Σ_k:

$$\Sigma_{k+1} = A\Sigma_k A^T + GQG^T \qquad k \geq 0$$

Assumption: A is a stable matrix

(i.e. all the eigenvalues of A are less than 1 in magnitude)

$A^k \to 0$ as $k \to \infty$

Claim: $(ij)^{th}$ entry of A^k is less than in magnitude $K.\alpha^k$ where $K>0$, $0<\alpha<1$

Theorem 1: Suppose that A is stable, then $\exists$ a positive semidefinite matrix Σ_∞ such that $\lim_{k\to\infty} \Sigma_k = \Sigma_\infty$. Moreover Σ_∞ is the unique solution of

$$\Sigma_\infty = A\Sigma_\infty A^T + GQG^T$$

Proof: Note that $|(A^k)_{ij}| \leq K.\alpha^k$

$$[\Sigma_k]_{ij} = [A^k\Sigma_0(A^k)^T]_{ij} + \left[\sum_{i=0}^{k-1}(A^i)GQG^T(A^i)^T\right]_{ij}$$

$$\left|[\Sigma_k]_{ij}\right| \leq \left|[A^k\Sigma_0(A^k)^T]_{ij}\right| + \left|\left[\sum_{i=0}^{k-1}(A^i)GQG^T(A^i)^T\right]_{ij}\right|$$

$$\leq K^2\alpha^{2k}.M + \sum_{i=0}^{k-1} K^2.\alpha^{2i}.N \to \text{a finite value}$$

$\therefore$ Σ_k will converge a finite matrix

Σ_k is positive semi definite $(\Sigma_k \geq 0)$

$\therefore$ $\Sigma_\infty = \lim_{k\to\infty} \Sigma_k \geq 0$

$$\lim_{k\to\infty} \Sigma_{k+1} = \lim_{k\to\infty}\left(A\Sigma_k A^T + GQG^T\right)$$

$$\Downarrow$$

$$\Sigma_\infty = A\Sigma_\infty A^T + GQG^T \quad \checkmark \quad (*)$$

- Uniqueness of the solution:

Assume there are two solutions of $(*)$ $\underbrace{\Sigma_\infty^1}_{\geq 0}$ and $\underbrace{\Sigma_\infty^2}_{\geq 0}$ $(\Sigma_\infty^1 \neq \Sigma_\infty^2)$

$$\Rightarrow \Sigma_\infty^1 = A\Sigma_\infty^1 A^T + GQG^T$$

$$\Sigma_\infty^2 = A\Sigma_\infty^2 A^T + GQG^T$$

$$\Rightarrow \Sigma_\infty^1 - \Sigma_\infty^2 = \Delta = A\Sigma_\infty^1 A^T - A\Sigma_\infty^2 A^T$$

$$= A(\Sigma_\infty^1 - \Sigma_\infty^2)A^T$$

$$\Rightarrow \Delta = A\Delta A^T$$

$$\Rightarrow A\Delta A^T = A^2\Delta(A^T)^2 = \Delta$$

$$\Rightarrow \Delta = A^m\Delta(A^m)^T \quad m = 0,1,2,\ldots$$

$$\Rightarrow \underbrace{\lim_{m\to\infty}\Delta}_{\Delta} = \underbrace{\lim_{m\to\infty} A^m\Delta(A^m)^T}_{0}$$

$$\Rightarrow \Delta = 0 \Rightarrow \Sigma_\infty^1 - \Sigma_\infty^2 = 0 \Rightarrow \Sigma_\infty^1 = \Sigma_\infty^2$$

contradiction

$\Rightarrow \Sigma_\infty$ is a unique solution of $(*)$

— o —

$$x_{k+1} = Ax_k + Gw_k$$

Example:

$w_k \sim N(0,1)$

$$\begin{bmatrix} x_{k+1}^1 \\ x_{k+1}^2 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix}}_{A}\begin{bmatrix} x_k^1 \\ x_k^2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{G} w_k \quad , \quad x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$x_{k+1}^1 = 2x_k^1 \longrightarrow x_k^1 = 2^k.x_0 \Rightarrow x_k^1 = 2^k$$

$$x_{k+1}^2 = 0.5x_k^2 + w_k$$

$\Sigma_\infty^2 = (0.5)^2 \Sigma_\infty^2 + \underset{G}{1} . \underset{Q}{1} . \underset{G^T}{1} \Rightarrow \Sigma_\infty^2 = 4/3$

$\Rightarrow \Sigma_\infty = \begin{pmatrix} 0 & 0 \\ 0 & 2/3 \end{pmatrix}$ although A is not stable, we obtained a finite Σ_∞.

This is due to selecting G as $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

—o—

Relationship between A and GQG^T

$A \longleftrightarrow GQG^T = \underset{G.\sqrt{Q}}{S} . S^T$

—o— 15.10.2004

$\left.\begin{array}{l} x_{k+1} = Ax_k + Gw_k \\ y_k = Cx_k + Hv_k \end{array}\right\}$ our system

$Cov(w_k) = Q$, $Cov(v_k) = R$, $Cov(x_0) = \Sigma_0$

$GQG^T = \underset{G\sqrt{Q}}{S} . S^T$

<u>Theorem:</u> The following statements are equivalent.

- (A, S) is a reachable pair $\left\{ \text{rank}(S \ AS \ A^2S \ldots) = n \right.$
- The $n \times n$ matrix $\sum_{j=0}^{n-1} A^j S S^T (A^j)^T$ is positive semi-definite
- $\forall x \in R^n$, $\exists$ an input sequence $\{w_0, \ldots, w_{n-1}\}$ which steers the state of the deterministic system $x_{k+1} = Ax_k + Sw_k \quad k \geq 0$ from state $x_0 = 0$ to state $x_n = x$

$\sum_{j=0}^{n-1} A^j S . S^T (A^j)^T = \underbrace{\left[S \ AS \ A^2S \ldots A^{n-1}S \right]}_{\text{rank } n} \begin{bmatrix} S^T \\ S^T A^T \\ \vdots \\ S^T (A^{n-1})^T \end{bmatrix}$.

—o—

<u>Theorem:</u> If the pair (A, S) is reachable and A is stable, then $\Sigma = A \Sigma A^T + S.S^T$ has the unique positive definite solution.

If (A, S) is reachable and $\Sigma = A \Sigma A^T + SS^T$ has a unique positive definite solution then A is stable.

—o—

$x_{k+1} = Ax_k + Gw_k$

$y_k = Cx_k + Hv_k$

$x_0 \sim N(\bar{x}_0, \Sigma_0)$, $w_k \sim N(0, Q)$

$x_k \sim N(0, \Sigma_k)^{(1)}$ $v_k \sim N(0, R)$

$\{x_0, w_0, \ldots, v_0, \ldots\}$ are all independent

$\{x_k\}_{k=0}^\infty$

$p(x_{k+1} | x_k) \sim N(\bar{x}_{k+1|k}, \Sigma_{k+1|k})$

$p(x_{k+1} | x_k, x_{k-1}, \ldots, x_0) = p(x_{k+1} | x_k)^{(2)}$

(2) implies that $\{x_k\}$ is a Markov Chain

(1) and (2) implies that $\{x_k\}$ is Markov and gaussian

$\therefore$ $\{x_k\}$ is GM (Gauss-Markov)

Assume $E\{x_k\} = 0$ and $Cov(x_k) = \Sigma_k$

$Cov(x_{k+1}, x_k) = \Sigma_{k+1,k}$

$p(x_{k+1} | x_k, \ldots, x_0) = p(x_{k+1} | x_k)$

$\Rightarrow E\{x_{k+1} | x_k, \dots, x_0\} = E\{x_{k+1} | x_k\}$

• $$E(x_{k+1} | x_k, \dots, x_0) = E(x_{k+1} | x_k)$$

$$E(x_{k+1} | x_k) = \Sigma_{k+1,k}\Sigma_k^{-1} x_k \quad \textbf{(Claim)} = \hat{x}_{k+1|k}$$

<u>Proof:</u>

Define $\tilde{x}_{k+1|k} \triangleq x_{k+1} - \hat{x}_{k+1|k} \triangleq w_k$

Note that $E\{w_k x_k^T\} = E\{(x_{k+1} - \Sigma_{k+1,k}\Sigma_k^{-1} x_k) x_k^T\}$

$$= E\{x_{k+1}x_k^T - \Sigma_{k+1,k}\Sigma_k^{-1} x_k x_k^T\}$$

$$= \underbrace{E\{x_{k+1} x_k^T\}}_{\Sigma_{k+1,k}} - \Sigma_{k+1,k}\cancel{\Sigma_k^{-1}} \underbrace{E\{x_k x_k^T\}}_{\cancel{\Sigma_k}}$$

$$= 0$$

$\Rightarrow$ w_k and x_k are uncorrelated

$$x_{k+1} - \Sigma_{k+1,k}\Sigma_k^{-1} x_k = w_k$$

$$x_{k+1} = \Sigma_{k+1,k}\Sigma_k^{-1} x_k + w_k$$

Note that, $w_k = x_{k+1} - \Sigma_{k+1}\Sigma_k^{-1} x_k$ is a linear function of x_{k+1} and x_k, therefore w_k is a gaussian.

Note that, x_k is also a gaussian and w_k and x_k are uncorrelated, therefore they are independent.

– Show that $\{x_0, w_0, w_1, \dots\}$ independent

(x_i and w_i are independent : proved)

<u>Controlled Markov Chain Models</u>

$\{x_k\}$ is called a Markov Chain

iff $p(x_{k+1} | x_k, \dots, x_0) = p(x_{k+1} | x_k)$

• Markov Chain Example:

$$\underset{\sim}{x}(k) = \begin{pmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{pmatrix} \qquad \begin{aligned} &\underset{\sim}{x}_1(k) \in \{-1, 1\} \\ &\underset{\sim}{x}_2(k) \in \{-0.3, 0\} \\ &\underset{\sim}{x}_3(k) \in \{-1, 0, 1.5\} \end{aligned}$$

$$\underset{\sim}{x}(k) = \underbrace{\left\{ \overset{x_1}{\begin{pmatrix} -1 \\ -0.3 \\ -1 \end{pmatrix}}, \overset{x_2}{\begin{pmatrix} -1 \\ -0.3 \\ 0 \end{pmatrix}}, \dots \right\}}$$

# of vectors : $2 \times 2 \times 3 = 12$

$p(x_{k+1} = x_1 | x_k = x_1) \triangleq P_{11}$

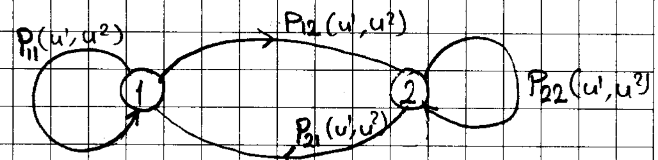
$p(x_{k+1} = x_1 | x_k = x_2) \triangleq P_{21}$

$\vdots$

$$\underset{\sim}{P} = \begin{bmatrix} P_{1,1} & \cdots & P_{1,12} \\ \vdots & & \\ P_{12,1} & \cdots & P_{12,12} \end{bmatrix}_{\substack{12 \times 12 \\ I \times I}}$$

• Controlled Markov Chain Example:

$$\text{Machine} \rightarrow \begin{cases} \text{Operating} & - 1 \\ \text{Failed} & - 2 \end{cases}$$



Maintenance $\rightarrow \left\{ \begin{array}{l} \text{level } 0 \rightarrow \text{low} \\ \text{level } 1 \rightarrow \text{high} \end{array} \right\} u^2$

Operating Condition $\rightarrow \left\{ \begin{array}{l} 0 \rightarrow \text{not used} \\ 1 \rightarrow \text{light use} \\ 2 \rightarrow \text{heavy use} \end{array} \right\} u^1$

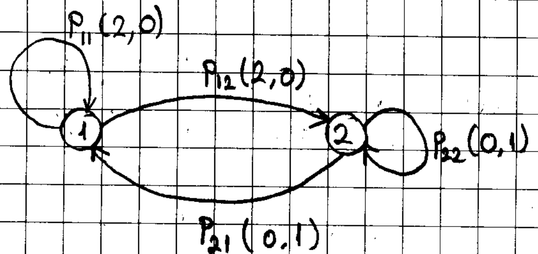
control inputs

$$P_{12}(u^1,u^2) + P_{11}(u^1,u^2) = P_{21}(u^1,u^2) + P_{22}(u^1,u^2) = 1$$

Strategy:

$$u_k = g(x_k) = \begin{pmatrix} u^1_k \\ u^2_k \end{pmatrix}$$

$$g(x_k=1) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \quad g(x_k=2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



$$P^g = \begin{bmatrix} P_{11}(2,0) & P_{12}(2,0) \\ P_{21}(0,1) & P_{22}(0,1) \end{bmatrix} \overset{\text{let}}{=} \begin{bmatrix} 0.7 & 0.3 \\ 0.8 & 0.2 \end{bmatrix}$$

(new machine)

$p(x_0=1)=1$ and $p(x_0=2)=0$

$p(x_1=1 \mid x_0=1) = 0.7$

$p(x_2=1 \mid x_0=1) = ?$

$$P_0 = (1 \quad 0) = [\text{Prob}(x_0=1) \quad \text{Prob}(x_0=2)]$$

$$\begin{aligned} p(x_{k+1}=1) &= p(x_{k+1}=1, x_k=1) + p(x_{k+1}=1, x_k=2) \\ &= p(x_{k+1}=1 \mid x_k=1) \cdot p(x_k=1) \\ &\quad + p(x_{k+1}=1 \mid x_k=2) \cdot p(x_k=2) \\ &= P^g_{11} \cdot p^g(x_k=1) + P^g_{21} \cdot p^g(x_k=2) \\ &= [P^g_{11} \quad P^g_{21}] \begin{bmatrix} p^g(x_k=1) \\ p^g(x_k=2) \end{bmatrix} \\ &= [p^g(x_k=1) \quad p^g(x_k=2)] \begin{bmatrix} P^g_{11} \\ P^g_{21} \end{bmatrix} \quad (1) \end{aligned}$$

$$\begin{aligned} p(x_{k+1}=2) &= p(x_{k+1}=2, x_k=1) + p(x_{k+1}=2, x_k=2) \\ &= p(x_{k+1}=2 \mid x_k=1) \cdot p(x_k=1) \\ &\quad + p(x_{k+1}=2 \mid x_k=2) \cdot p(x_k=2) \end{aligned}$$

$$= [p^g(x_k=1) \quad p^g(x_k=2)] \begin{bmatrix} P^g_{12} \\ P^g_{22} \end{bmatrix} \quad (2)$$

$$\underbrace{[p^g(x_k=1) \quad p^g(x_k=2)]}_{P_k} \underbrace{\begin{bmatrix} P^g_{11} & P^g_{12} \\ P^g_{21} & P^g_{22} \end{bmatrix}}_{P^g} = P_{k+1}$$

$\Rightarrow \boxed{P_{k+1} = P_k P^g}$ an iterative solution

$\Rightarrow P_{k+1} = P_0 (P^g)^{k+1}$

Note that $1 \geq P^g_{ij} \geq 0$ $\quad \sum_{j=1}^{I} P_{ij}(u) = 1$

$$\lim_{k\to\infty} (P)^k \to \begin{cases} \bar{P} = \begin{bmatrix} \bar{p} \\ \bar{p} \\ \bar{p} \end{bmatrix}_{I \times I} & \leftarrow \text{we will mainly deal with} \\ \bar{P} = \begin{bmatrix} 0 & P_{12} & 0 \\ 0 & 0 & P_{13} \\ P_{31} & 0 & 0 \end{bmatrix} \\ \bar{P} = \begin{bmatrix} P_{11} & 0 & \cdots \\ 0 & P_{21} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \end{cases}$$

Example:

goes to 0 — goes to 1

$$P = \begin{bmatrix} 0.1 & 0.9 & 0 \\ 0.2 & 0.3 & 0.5 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow P^2 = \begin{bmatrix} 0.19 & 0.36 & 0.45 \\ 0.08 & 0.27 & 0.65 \\ 0 & 0 & 1 \end{bmatrix}$$

...

—o—

$$\lim_{n\to\infty} P^n = \begin{bmatrix} \bar{p} \\ \bar{p} \\ \bar{p} \\ \vdots \end{bmatrix} \qquad \bar{p} = [\bar{p}_1 \quad \bar{p}_2 \cdots \bar{p}_I]$$

$$\bar{p}_i = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} I(x_k = i) \quad \text{w.p. } I$$

indicated function

$$I(x_k = i) = \begin{cases} 1 & \text{if } x_k = i \\ 0 & \text{if } x_k \neq i \end{cases}$$

If $\mathcal{P}^g$ is changing with time

$$P_{k+1}^g = P_k^g \mathcal{P}_k^g$$

$$P_1^g = P_0^g \mathcal{P}_0^g$$

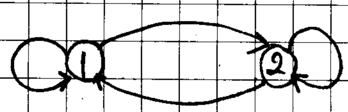
$$P_2^g = P_1^g \mathcal{P}_1^g = P_0^g \mathcal{P}_1^g \mathcal{P}_0^g$$

$$\vdots$$

$$P_{k+1}^g = P_0 \mathcal{P}_0^g . \mathcal{P}_1^g \dots \mathcal{P}_k^g$$

Cost Computation:

Machine Example



$P_{ij}(u^1, u^2)$

- Incremental Cost Function:

$$c(X_k = i, u_k^1 = j, u_k^2 = \ell)$$

$$= c\left(X_k, \begin{pmatrix} u^1 \\ u^2 \end{pmatrix}_k\right) = \begin{cases} c(2, \begin{pmatrix} 0 \\ 0 \end{pmatrix}) = 100 \\ c(2, \begin{pmatrix} 0 \\ 1 \end{pmatrix}) = 150 \\ c(1, \begin{pmatrix} 0 \\ 0 \end{pmatrix}) = 0 \\ c(1, \begin{pmatrix} 1 \\ 0 \end{pmatrix}) = 10 \\ c(1, \begin{pmatrix} 2 \\ 0 \end{pmatrix}) = 10 \\ c(1, \begin{pmatrix} 1 \\ 1 \end{pmatrix}) = 75 \\ \vdots \end{cases}$$

may be time varying

$$E\left\{\sum_{k=0}^{N} c_k(x_k^g, \underset{\sim}{u}_k^g)\right\} = J(g)$$

Since c_k is a random variable, we can only minimize its mean

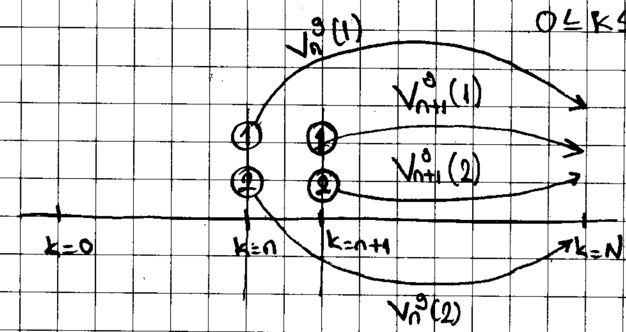
Markov Policy:

$u_k^g = g_k(x_k)$: Markov Policy

Lemma: A controlled Markov Chain with Markov policy generates a state sequence $\{x_k^g\}$ that is Markov.

Computation of $J(g)$:

Incremental Cost : $c_k(i, u)$ $\quad 1 \le i \le I$, $u \in U$, $0 \le k \le N$



find a relation $\begin{pmatrix} V_n^g(1) \\ V_n^g(2) \end{pmatrix}$ and $\begin{pmatrix} V_{n+1}^g(1) \\ V_{n+1}^g(2) \end{pmatrix}$

$$V_N^g = \begin{pmatrix} c_N(1, g_N(1)) \\ c_N(2, g_N(2)) \\ \vdots \\ c_N(I, g_N(I)) \end{pmatrix}$$

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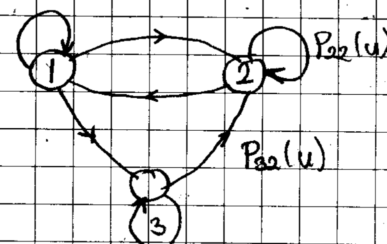
22.10.2004

Controlled Markov Chain $P(u)$



$c_k(x_k, u_k) \leftarrow$ incremental cost

Total cost $\rightarrow E\{\sum_{k=0}^{n} c_k(x_k, u_k)\} = E\{\sum_{k=0}^{n} c_k(x_k, g_k(x_k))\}$

Markov Policy $\rightarrow u_k = g_k(x_k)$

control strategy

$$E^g\left\{\sum_{k=0}^{n} c_k(x_k, g_k(x_k))\right\} = J(g) \quad \text{Total Cost Function}$$

Problem: Given "g", find $J(g)$

$$p_0 \triangleq \left[\text{Prob}\{x_0 = 1\} \ \text{Prob}\{x_0 = 2\} \ \cdots \ \text{Prob}\{x_0 = I\} \right]$$

$$\sum_{i=1}^{I} p_0(i) = 1$$

$$P_{k,ij} \triangleq \text{Prob}\{x_k = j \mid x_{k-1} = i\}$$

$$\underline{P}_k = \begin{bmatrix} P_{k,11} & P_{k,12} & \cdots & P_{k,1I} \\ \vdots & & & \\ P_{k,I1} & \cdots & \cdots & P_{k,II} \end{bmatrix}$$

$$p_k \triangleq \left[\text{Prob}\{x_k = 1\} \ \cdots \ \text{Prob}\{x_k = I\} \right]$$

$$p_{k+1} = \left[\text{Prob}\{x_{k+1} = 1\} \ \cdots \ \text{Prob}\{x_{k+1} = I\} \right]$$

$$\text{Prob}\{x_{k+1} = 1\} = \sum_{l=1}^{I} \text{Prob}\{x_{k+1} = 1, x_k = l\}$$

$$= \sum_{l=1}^{I} \text{Prob}\{x_{k+1} = 1 \mid x_k = l\} \cdot \text{Prob}\{x_k = l\}$$

$$= \sum_{l=1}^{I} P_{k,l1} \cdot p_k(l)$$

$$\Rightarrow p_{k+1} = p_k \underline{P}_k \ \Rightarrow \ p_k^g = p_0^g \cdot \underline{P}_1 \cdot \underline{P}_2 \cdots \underline{P}_{k-1}^g$$

—o—

$$J(g) = E^g\left\{ \sum_{k=0}^{N} c_k(x_k, g_k(x_k)) \right\}$$

$$= \sum_{k=0}^{N} \sum_{i=1}^{I} c_k(i, g_k(i)) \cdot \text{Prob}\{x_k = i\}$$

$$\text{Define } C_k \triangleq \begin{bmatrix} c_k(1, g_k(1)) \\ c_k(2, g_k(2)) \\ \vdots \\ c_k(I, g_k(I)) \end{bmatrix}$$

$$\Rightarrow J(g) = \sum_{k=0}^{N} p_k^g \cdot C_k^g = \sum_{k=0}^{N} p_0^g \cdot \underline{P}_1^g \underline{P}_2^g \cdots \underline{P}_{k-1}^g \cdot C_k^g$$

Recursive Computation of $J(g)$:

$$J(g) = E^g\left\{ \sum_{k=0}^{N} c_k(x_k, g_k(x_k)) \right\}$$

$$\text{Define } V_N^g = \begin{bmatrix} c_N(1, g_N(1)) \\ \vdots \\ c_N(I, g_N(I)) \end{bmatrix}$$

$$V_k^g = \begin{bmatrix} V_k^g(1) \\ \vdots \\ V_k^g(I) \end{bmatrix} \rightarrow V_k^g(i) \triangleq E\left\{ \sum_{l=k}^{N} c_l(x_l, g_l(x_l)) \mid x_k = i \right\}$$

$$J(g) = p_0 V_0^g$$

$$V_k^g(i) = E\left\{ c_k(x_k, g_k(x_k)) + \sum_{l=k+1}^{N} c_l(x_l, g_l(x_l)) \mid x_k = i \right\}$$

$$= c_k(i, g_k(i)) + \underbrace{E^g\left\{ \sum_{l=k+1}^{N} c_l(x_l, g_l(x_l)) \mid x_k = i \right\}}_{A}$$

• Fact: $E\{E\{a \mid bc\} \mid c\} = E\{a \mid c\}$

$E\{E\{a \mid b\}\} = E\{a\}$

$$A: E^g\left\{ \underbrace{E^g\left\{ \sum_{l=k+1}^{N} c_l(x_l, g_l(x_l)) \mid x_{k+1}, x_k = i \right\}}_{V_{k+1}^g(x_{k+1})} \mid x_k = i \right\}$$

$$\Rightarrow A: E^g\left\{ V_{k+1}^g(x_{k+1}) \mid x_k = i \right\}$$

$$= \sum_{j=1}^{I} V_{k+1}^g(j) \cdot \underbrace{\text{Prob}\{x_{k+1} = j \mid x_k = i\}}_{\underline{P}_{k,ij}}$$

$$A: \underbrace{\left[\underline{P}_{k,i1}^g \ \underline{P}_{k,i2}^g \ \cdots \ \underline{P}_{k,iI}^g \right]}_{i^{th} \text{ row of } \underline{P}_k^g} \begin{bmatrix} V_{k+1}^g(1) \\ \vdots \\ V_{k+1}^g(I) \end{bmatrix}$$

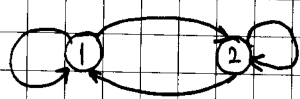
$\therefore V_k^g = P_k \cdot V_{k+1}^g + C_k$

where $C_k = \begin{bmatrix} C_k(1, g_k(1)) \\ \vdots \\ C_k(I, g_k(I)) \end{bmatrix}$, $V_N^g = C_N^g$

and $J(g) = p_0 \cdot V_0^g$

27.10.2004

Example:



$x: \{1, 2\}$ state

$P_{ij}(u)$ $\quad u = g(x)$ $\quad g(1) = 0$

$g(2) = 1$

$u \in U = \{0, 1\}$

$P(u=0) = \begin{bmatrix} 0.5 & 0.5 \\ 0.3 & 0.7 \end{bmatrix}$ $\quad C_k = C(x, u)$

$P(u=1) = \begin{bmatrix} 0.8 & 0.2 \\ 0.9 & 0.1 \end{bmatrix}$

$C(x,u) = \begin{cases} C(1,0) = 1 \\ C(1,1) = 2 \\ C(2,1) = 3 \\ C(2,0) = 4 \end{cases}$

$$\left.\begin{aligned} C_k^g(1) &= c(1, \underbrace{g(1)}_{0}) = 1 \\ C_k^g(2) &= c(2, \underbrace{g(2)}_{1}) = 3 \end{aligned}\right\} C_k^g = \begin{bmatrix} C_k^g(1) \\ C_k^g(2) \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$N = 3$

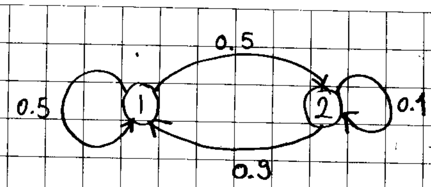
$V_N^g = V_3^g = C_3^g = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$V_2^g = \begin{bmatrix} 1 \\ 3 \end{bmatrix} + P_k^g V_3^g$

$P_k^g = \begin{bmatrix} 0.5 & 0.5 \\ 0.9 & 0.1 \end{bmatrix}$ $\quad$ when $x=1 \Rightarrow g(1) = 0$

$\Rightarrow P(u=0) = [0.5 \ 0.5]$

for $x = 1$



$\Rightarrow V_2^g = \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 0.5 & 0.5 \\ 0.9 & 0.1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4.2 \end{bmatrix}$

$\Rightarrow V_1^g = \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 0.5 & 0.5 \\ 0.9 & 0.1 \end{bmatrix} \begin{bmatrix} 3 \\ 4.2 \end{bmatrix} = \begin{bmatrix} 4.6 \\ 6.12 \end{bmatrix}$

$\Rightarrow V_0^g = \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 0.5 & 0.5 \\ 0.9 & 0.1 \end{bmatrix} \begin{bmatrix} 4.6 \\ 6.12 \end{bmatrix} = \begin{bmatrix} 6.36 \\ 7.752 \end{bmatrix}$

$J(g) = p_0 \cdot \begin{bmatrix} 6.36 \\ 7.752 \end{bmatrix}$ where $p_0 = [1 \ 0]$ (let)

$\Rightarrow J(g) = 6.36$

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$J(g) = E\left\{\sum_{k=0}^{N} C_k(x_k, u_k)\right\}$ as $N \to \infty$, $J(g) \to \infty$

Then define an average objective function.

(1) $J(g) = \lim_{N \to \infty} \frac{1}{N} E\left\{\sum_{k=0}^{N-1} C_k(x_k, g_k(x_k))\right\}$ or

(2) $J(g) = E\left\{\sum \beta^k C_k(x_k, g_k(x_k))\right\}$ where $0 < \beta < 1$

if $C_k(\cdot, \cdot)$ is a bounded function.

* We will seek for a solution for the <u>second app.</u>

$J(g) = \sum_{k=0}^{\infty} \beta^k p_0^g P_0^g P_1^g \dots P_{k-1}^g C_k^g$

$V_k^g = \begin{pmatrix} V_k^g(1) \\ \vdots \\ V_k^g(I) \end{pmatrix}$

$V_k^g(i) \triangleq E^g\left\{\sum_{l=k}^{\infty} \beta^l C_l(x_l, g_l(x_l)) \,\middle|\, x_k^g = i\right\}$

$= E^g\left\{\beta^k C_k(x_k, g_k(x_k)) + \sum_{l=k+1}^{\infty} \beta^l C_l(x_l, g_l(x_l)) \,\middle|\, x_k^g = i\right\}$

$= \beta^k C_k(i, g_k(i)) + E\left\{\sum_{l=k+1}^{\infty} \beta^l C_l(x_l, g_l(x_l)) \,\middle|\, x_k^g = i\right\}$

10.Nov.2004 MT1 (Wednesday)

$E\{\underset{\sim}{a}|\underset{\sim}{b}\} = E\{E\{a|bc\}|b\}$

$E\{\underset{\sim}{a}\} = E\{E\{a|b\}\}$

$\Rightarrow V_k^g(i) = \beta^k c_k(i, g_k(i))$

$+E\{\underbrace{E\{\sum_{l=k+1}^{\infty} \beta^l c_l(x_l, g_l(x_l)) \,|\, x_k^g = i, x_{k+1}\}}\,|\, x_k^g = i\}$

$E\{E\{\sum_{l=k+1}^{\infty} \beta^l c_l(x_l, g_l(x_l)) \,|\, x_{k+1}\} \,|\, x_k = i\}$

$\sum_{j=1}^{I} \underbrace{\left(E\{\sum_{l=k+1}^{\infty} \beta^l c_l(x_l, g_l(x_l)) \,|\, x_{k+1} = j\}\right)} \cdot p^g(x_{k+1} = j \,|\, x_k = i)$

$\Rightarrow V_k^g(i) = \beta^k c_k(i, g_k(i)) + \sum_{j=1}^{I} p_{ij}^g \cdot V_{k+1}^g(j)$

$\Rightarrow V_k^g = \beta^k C_k^g + P_k^g V_{k+1}^g$

where $C_k^g = \begin{pmatrix} c_k(1, g_k(1)) \\ \vdots \\ c_k(I, g_k(I)) \end{pmatrix}$

$\Rightarrow V_k^g = \beta^k C_k^g + P_k^g \cdot V_{k+1}^g$ and

$J(g) = p_0 \cdot V_0^g$

- Assume that Stationary Markov Process:

$c_k(.,.) = c(.,.) \quad \forall k$

$P_k^g = P^g \quad \forall k$

$g_k(\cdot) = g(\cdot) \quad \forall k$

$V_k^g(i) = E\{\sum_{l=k}^{\infty} \beta^l c(x_l, g(x_l)) \,|\, x_k^g = i\}$

$= \beta^k E\{\sum_{l=k}^{\infty} \beta^{l-k} c(x_l, g(x_l)) \,|\, x_k^g = i\}$

let $m = l - k$

$\Rightarrow V_k^g(i) = \beta^k E^g\{\sum_{m=0}^{\infty} \beta^m c(x_{m+k}^g, g(x_{m+k}^g)) \,|\, x_k^g = i\}$

Since the system is time invariant

$V_k^g(i) = \beta^k E^g\{\sum_{m=0}^{\infty} \beta^m c(x_m, g(x_m)) \,|\, x_0 = i\}$

$\therefore$

$\Rightarrow V_k^g(i) = \beta^k V_0^g(i) \quad , i = 1, \dots, I$

$\Rightarrow V_k^g = \beta^k V_0^g$

$\Rightarrow \beta^k V_0^g = \beta^k C + P^g \cdot V_{k+1}^g$

$V_{k+1}^g = \beta^{k+1} V_0^g$

$\Rightarrow \cancel{\beta^k} V_0^g = \cancel{\beta^k} C + P^g \cdot \beta^{\cancel{k+1}} V_0^g$

$\Rightarrow V_0^g - \beta P^g V_0^g = C$

$\Rightarrow V_0^g = (I - \beta P^g)^{-1} C$

$\Rightarrow J(g) = p_0 \cdot \underbrace{(I - \beta P^g)^{-1}}_{\text{always exists}} C$

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* Solution to the 1^{st} approach:

$J(g) = \lim_{N \to \infty} \frac{1}{N} E\{\sum_{k=0}^{N-1} c_k(x_k, g_k(x_k))\}$

Properties of P:

- All eigenvalues of P are less than or equal to 1 in magnitude. $\exists$ at least one eigenvalue which is equal to 1.

Note; $P \cdot \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

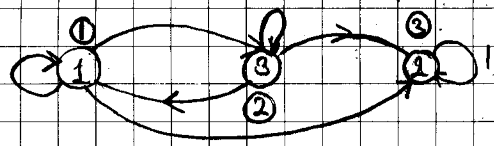
eigenvectors

- P is called reducable or decomposable if $\exists$ a renumbering of the states for which

$P = \begin{bmatrix} P_1 & P_2 \\ 0 & P_3 \end{bmatrix}$ → square matrix, otherwise

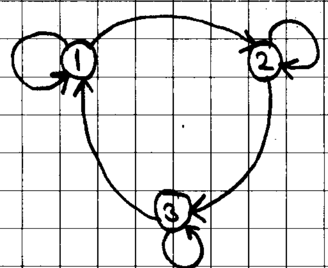
it is called irredusable.

Example:



$$P=\begin{bmatrix} p_{11} & p_{12} & p_{13} \\ 0 & 1 & 0 \\ p_{31} & p_{32} & p_{33}\end{bmatrix} \rightarrow P=\begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ 0 & 0 & 1\end{bmatrix}$$

$\Rightarrow$ P is reducable



$$P=\begin{bmatrix} p_{11} & p_{12} & 0 \\ 0 & p_{22} & p_{23} \\ p_{31} & 0 & p_{33}\end{bmatrix}, \text{ irreducable}$$

Lemma: If P is a transition matrix, then $\lim_{N\to\infty} \frac{1}{N}\sum_{k=0}^{N-1} P^k = \pi$ always exists.

The matrix π is a stochastic matrix and it satisfies the equation $\pi = \pi \cdot P$

Note; π_i is the i^{th} raw of π then,

$\pi_i = \pi_i \cdot P$ i.e. π_i is the left eigenvector of P corresponding to the eigenvalue 1.

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03.11.2004

For stationary "g" and time invariant cost per unit time.

$$J(g) = \lim_{N\to\infty} \frac{1}{N} E\Big\{\sum_{k=0}^{N-1} C_k(x_k^g, g_k(x_k^g))\Big\}$$

$$= \lim_{N\to\infty} \frac{1}{N}\sum_{k=0}^{N-1} p_0 P_0^g P_1^g \dots P_{k-1}^g C_k^g$$

$$J(g) = \lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1} p_0 (P^g)^k \cdot C^g$$

$$= p_0 \underbrace{\Big(\lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}(P^g)^k}_{\pi^g} \cdot C^g\Big)$$

$$= p_0 \cdot \pi^g \cdot C^g$$

Note: $\pi_{I\times I} = \pi_{I\times I} \cdot P_{I\times I}$

$$\pi = \begin{bmatrix}\pi_1 \\ \pi_2 \\ \vdots \\ \pi_I\end{bmatrix} \Rightarrow \begin{aligned} \pi_1 &= \pi_1 \cdot P \\ \pi_2 &= \pi_2 \cdot P \\ &\vdots \\ \pi_I &= \pi_I \cdot P\end{aligned}$$

Lemma: If P is irreducable and aperiodic then

$$\lim_{N\to\infty} P^k = \pi$$

and $\pi = \begin{bmatrix}\pi \\ \pi \\ \vdots \\ \pi\end{bmatrix}$ where $\pi > 0$

—o—

* Assume a controlled Markov chain : $P^k_{I\times I}(u)$

$$p_{ij}^k(u) = Prob(x_{k+1}=j \mid x_k = i)$$

$$J(g) = E\Big\{\sum_{k=0}^{N-1} c_k(x_k, u_k^g) + c_N(x_N)\Big\}$$

aim is to minimize $J(g)$ by an optimum g

• Define;

$$J_k^g = E\Big\{\sum_{\ell=k}^{N-1} c_\ell(x_\ell, u_\ell^g) + c_N(x_N) \;\Big|\; x_k^g \dots x_0^g\Big\}$$

Note that if $\{x_k^g\}$ is a Markov chain

$$J_k^g = E\Big\{\sum_{\ell=k}^{N-1} c_\ell(x_\ell, u_\ell^g) + c_N(x_N) \;\Big|\; x_k^g\Big\}$$

Theorem: Define recursively the functions

$$V_N(i) = C_N(i) \qquad i = 1, 2, \dots, I$$

$$(*)\quad V_k(i) = \inf_{u \in U} \left\{ c_k(i,u) + \sum_{j=1}^{I} V_{k+1}(j) P_{ij}(u) \right\}$$

1) Let $g \in G$, then $V_k(x_k^g) \le J_k^g$ w.p. 1

2) A Markov policy $g = \{g_0, \dots, g_{N-1}\}$ is optimal if for each i, the infimum in $(*)$ is achieved at $g_k(i)$, and then $V_k(x_k^g) = J_k^g$ w.p. 1 and $J^* = J(g) = E\, V_0(x_0)$

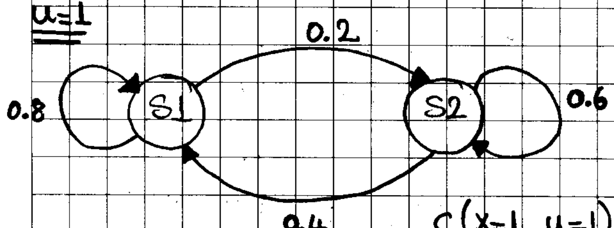
Example: (Machine Example)

State 1: Operating

State 2: Not Operating

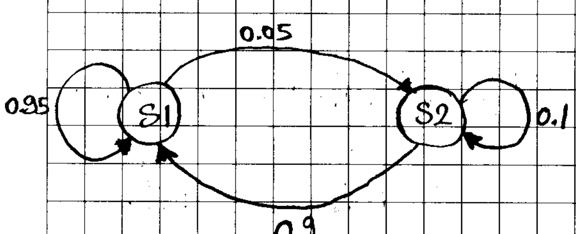
$u \in \{1, 2\}$ ← high maintenance; ↑ low maintenance

<u>u=1</u>



$c(x=1, u=1) = 1$
$c(x=1, u=2) = 4$
$c(x=2, u=1) = 8$
$c(x=2, u=2) = 10$

<u>u=2</u>



$$\begin{bmatrix} C_N(1) \\ C_N(2) \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \end{bmatrix}$$

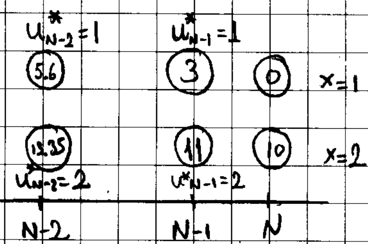
$$\begin{bmatrix} V_N(1) \\ V_N(2) \end{bmatrix} = C_N = \begin{bmatrix} 0 \\ 10 \end{bmatrix}$$

$$\begin{bmatrix} V_{N-1}(1) \\ V_{N-1}(2) \end{bmatrix} = \begin{bmatrix} \inf_{u \in \{1,2\}} \left\{ C_{N-1}(1,u) + \sum_{j=1}^{2} V_N(j) P_{1j}(u) \right\} \\ \inf_{u \in \{1,2\}} \left\{ C_{N-1}(2,u) + \sum_{j=1}^{2} V_N(j) P_{2j}(u) \right\} \end{bmatrix}$$

$$C_{N-1}(1,u) = \begin{cases} 1 & \text{if } u=1 \\ 4 & u=2 \end{cases}$$

$$\Rightarrow V_{N-1}(1) = \inf_{u \in \{1,2\}} \begin{cases} \overbrace{1 + 0.8*0 + 0.2*10}^{3} & u=1 \\ \underbrace{4 + 0.95*0 + 0.05*10}_{4.5} & \text{if } u=2 \end{cases}$$

$\Rightarrow V_{N-1}(1) = 3$ and $u_N^* = 1$



let $N-2 = 0 \Rightarrow N = 2$

$$V_{N-1} = \begin{bmatrix} 3 \\ 11 \end{bmatrix}$$

$$V_{N-2} = \begin{bmatrix} 5.6 \\ 13.35 \end{bmatrix}$$

$$J^* = \text{Prob}(X_0 = 1) * 5.6 + \text{Prob}(X_0 = 2) * 13.35$$

$$\Rightarrow u^* = \begin{cases} 1 & \text{if } x=1 \\ 2 & x=2 \end{cases}$$

DYNAMIC PROGRAMMING

$$x_{k+1} = f_k(x_k, u_k, w_k)$$

$y_k = x_k$ (In general, $y_k = h_k(x_k, v_k)$)

$u_k = g_k(y^k) \in U$

G: set of all possible feedback laws

$C_k(x_k, u_k)$: one-period cost

$C_N(x_N)$

Cost function: $E\left\{\sum_{k=0}^{N-1} C_k(x_k,u_k) + C_N(x_N)\right\} = J(u) = J(g)$

Definition: g^* in G is optimal if

$J(g^*) = J^* = \text{Inf}\{J(g) \mid g \in G\}$

Definition: A feasible law $g = \{g_k\}_{k=0}^{\infty}$ is said to be Markovian or Markov policy if $u_k = g_k(y_k)$ $\quad (y_k \equiv x_k \Rightarrow u_k = g_k(x_k))$

case of complete inf.

05.11.2004

Note: If g is Markovian then $\{x_k^g\}$ is a Markov sequence.

Notation: Let $\tilde{z}$ & $\tilde{w}$ be random variable and let $h(\tilde{z},\tilde{w})$ be a function of them

$$(E_w h)(\tilde{z}) \triangleq \int h(\tilde{z}, w)\underbrace{p_w(w)}_{\text{probability density func}}dw$$

$$(E_w h)(z) \triangleq \int h(z,w)\, p_w(w)\, dw$$

—o—

Given $\{h_k(x)\}_{k=0}^{N}$

* Define a sequence of functions as:

$$H_N(x) \triangleq h_N(x)$$

$$H_k(x) \triangleq h_k(x) + \int H_{k+1}(x')\, p_{x_{k+1}|x_k}(x'|x_k)\, dx'$$

Fact: If $\{h_t\}$ and $\{H_t\}$ are defined as above then,

$$H_t(x_t) = E\left\{\sum_{l=k}^{N} h_l(x_l) \,\Big|\, x_k\right\} = E\left\{\sum_{l=t}^{N} h_l(x_l) \,\Big|\, x_t, \dots, x_0\right\}$$

Lemma: Let $g = (g_0, g_1, \dots)$ be a Markov policy. Define recursively the functions:

$$V_N^g(x) = C_N(x)$$

$$V_k^g(x) = C_k(x, g_k(x)) + E_{w_k} V_{k+1}^g(f_k(x, g_k(x), w_k))$$

Then the random variable $V_k^g(x_k^g)$ satisfies

$$V_k^g(x_k^g) = E\left\{\sum_{l=k}^{N-1} C_l(x_l^g, u_l^g) + C_N(x_N, u_N) \,\Big|\, x_k^g\right\} \qquad k = 0, 1, \dots, N$$

$$J(g) = E\, V_0^g(x_0^g)$$

Example:

$x_{k+1} = a x_k + u_k + w_k \qquad w_k \sim N(0,1)$

$x_0 = 0 \quad , N = 2$

$C_N(x) = 3x^2 \quad , \quad C_k(x) = x^2 + u^2$

$$J(u) = E\{3\tilde{x}_2^2 + \tilde{x}_1^2 + u_1^2 + \cancel{x_0^2}^{\,0} + u_0^2\}$$

let $u_0 = u_1 = 0$

$$V_N^g(x) = V_2^g(x) = C_2(x) = 3x^2$$

$$V_1^g(x) = x^2 + \cancel{u^2}_{\,0} + E_{w_1} V_2^g(ax + \cancel{u}_{\,0} + w_1)$$

$$= x^2 + E_{w_1}\{3(ax + w_1)^2\}$$

$$= x^2 + 3a^2x^2 + 3 = (1+3a^2)x^2 + 3$$

$$V_0^g(x) = C_0(x, 0) + E_{w_0} V_1^g(ax + \cancel{u}^{\,0} + w_0)$$

$$= x^2 + E_{w_0}\{(1+3a^2)(ax + w_0)^2 + 3\}$$

$$= x^2 + (1+3a^2)(a^2x^2 + 1) + 3$$

$$V_0^g(x_0) = 0 + (1+3a^2)(0+1)+3 = 3a^2+4$$
$\overset{\parallel}{0}$

If $x_0 \sim N(1, 0.5)$

$J(g) = E V_0^g(x_0^g)$

$$V_0^g(x_0) = E(x_0^2) + (1+3a^2)E\{a^2x_0^2+1\}+3$$

$E\{x_0\}=1 \quad E\{(x_0-1)^2\}=0.5$

$E\{x_0^2\} - 2E\{x_0\}+1=0.5$

$\Rightarrow E\{x_0^2\}=1.5$

$$\Rightarrow V_0^g(x_0) = 1.5 + (1+3a^2)(1.5a^2+1)+3$$

Proof: Proof is by induction.

$$V_N^g(x_N^g) = C_N(x_N^g)$$

Assume that

$$V_{k+1}^g(x_{k+1}^g) = E\left\{\sum_{l=k+1}^{N-1} c_l(x_l^g, u_l^g) + C_N(x_N^g) \,\middle|\, x_{k+1}^g\right\}$$

$$E\left\{\sum_{l=k}^{N-1} c_l(x_l^g, \underset{\overset{\parallel}{g_l(x_l^g)}}{u_l^g}) + C_N(x_N^g) \,\middle|\, x_k^g\right\}$$

$$= C_k(x_k^g, g_k(x_k^g)) + E\left\{\sum_{l=k+1}^{N-1} c_l(x_l^g, u_l^g) + C_N(x_N^g) \,\middle|\, x_k^g\right\}$$

apply the same trick

$E\{E\{a \mid bc\} \mid c\} = E\{a \mid c\}$

$$= C_k(x_k^g, g_k(x_k^g)) + E\{V_{k+1}(x_{k+1}^g) \mid x_k^g\}$$
$$= C_k(x_k^g, g_k(x_k^g)) + E\{V_{k+1}(f_k(x_k^g, g_k(x_k^g), w_k), w_k \mid x_k^g\}$$
$$V_k(x_k^g) = C_k(x_k^g, g_k(x_k^g)) + E_{w_k} V_{k+1}(f_k(x_k^g, g_k(x_k^g)), w_k)$$

Comparison Principle:

Let $V_k(x)$ $0 \le k \le N$ be functions such that

$$V_N(x) \le C_N(x)$$
$$\vdots$$
$$V_k(x) \le C_k(x,u) + E_{w_k} V_{k+1}(f_k(x,u,w_k))$$

for all x and for all u. Let $g \in G$ be arbitrary then w.p. 1

$V_k(x_k^g) \le J_k^g$ where

$$J_k^g \triangleq E\left\{\sum_{l=k}^{N-1} c_l(x_l^g, u_l^g) + C_N(x_N^g) \,\middle|\, x_0^g, \dots, x_k^g\right\}$$

$$J_0^g = E\left\{\sum_{l=0}^{N-1} c_l(x_l^g, u_l^g) + C_N(x_N^g) \,\middle|\, x_0^g\right\}$$

Theorem: Define recursively the functions

$$V_N(x) \triangleq C_N(x)$$

$$(*)\ V_k(x) = \inf_{u \in U}\left[C_k(x,u) + E_{w_k} V_{k+1}(f_k(x,u,w_k))\right]$$

1) $g \in G$. Then $V_k(x_k^g) \le J_k^g$ w.p. 1

In particular $J(g) \ge E V_0(x_0)$

2) The Markov policy $g = \{g_0, \dots, g_N\}$ in G_M is optimal if the infimum in $(*)$ is achieved at $g_k(x)$, and then $V_k(x_k^g) = J_k^g$ w.p. 1 and

$J^* = J(g) = E V_0(x_0)$

3) A Markov policy $g \in G_M$ is optimal if and only if for each k, the infimum at x_k^g in $(*)$ is achieved by $g_k(x_k^g)$ i.e.

$$V_k(x_k^g) = C(x_k^g, u_k^g) + E_{w_k} V_{k+1}(f_k(x_k^g, g_k(x_k^g), w_k))$$

Example:

$V_3(x) = C_3(x) = 3x^2$ let $a = 0.5$

$$V_2(x) = \operatorname*{Inf}_{u\in U}\left[C_2(x,u) + E_{W_2}\left(3(0.5x+u+w_2)^2\right)\right]$$

$$= \operatorname*{Inf}_{u}\Big[\underbrace{x^2+u^2+3(0.5x+u)^2+3.1}_{h}\Big]$$

$$\frac{\partial h}{\partial u} = 2u + 6(0.5x+u) = 0 \Rightarrow u_2 = \frac{-1.5}{4}x$$

$$\Rightarrow V_2(x) = x^2 + \left(\frac{1.5}{4}\right)^2 x^2 + 3\left(\frac{1}{2}x - \frac{1.5}{4}x\right)^2 + 3$$

$$V_1(x) = \operatorname*{Inf}_{u\in U}\left[x^2+u^2+E_{W_1}\left(A(0.5x+u+w_1)^2+3\right)\right]$$

$$A = \left(1+\left(\frac{1.5}{4}\right)^2+\left(\frac{1}{8}\right)^2\right)$$

$$\Rightarrow u_1 = \alpha x_1$$

(**) $x_{k+1} = f_k(x_k, u_k, w_k)$ $\quad y_k = x_k,\ u_k = g_k(y_k)$

$$E\left\{\sum_{k=0}^{N-1} C_k(x_k,u_k) + C_N(x_N)\right\}$$

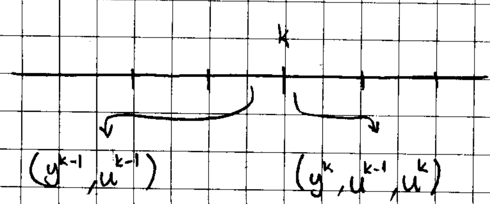
let now $y_k = h_k(x_k, v_k)$ (**)

- Case of partial information

$$P_{x_k|x^{k-1}} = p(x_k|x^{k-1}) = p(x_k|x_{k-1})$$

independent of 'g'

$P_{x_k|\text{known variables at time } k}(x_k \mid \text{known variables at time } k)$ = independent of "g"



Define $z^k \triangleq (y^k, u^{k-1})$

Definition: ξ_k is an information state for the stochastic system (**) if

i) ξ_k is a function of $(y^k, u^{k-1}) = z^k$

ii) ξ_{k+1} can be determined from ξ_k, y_{k+1} and u_k

Conditional Probability as Information State:

Let $\{g\}$ be a feedback policy. Let $\{u_t^g\}, \{u_t^g\}, \{y_t^g\}$ be the corresponding input, state and output sequences.

$u_k = g_k(y^k)$ $\quad z^k = (y^k, u^{k-1})$

$$p^g_{k|k}(x_k|z^k) \triangleq P_{x_k|z^k}(x_k|z^k)$$

$$p^g_{k+1|k}(x_{k+1}|z^k,u_k) \triangleq p_{x_{k+1}|z^k,u_k}(x_k|z^k,u_k)$$

Note 1:

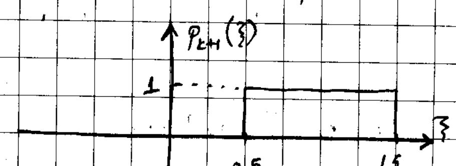
$$p^g(y_{k+1}|x_{k+1},z^k,u_k) = p^g(h_{k+1}(x_{k+1},v_{k+1})|x_{k+1},z^k,u_k)$$
$$= p^g(h_{k+1}(x_{k+1},v_{k+1})|x_{k+1})$$
$$= p(y_{k+1}|x_{k+1})$$

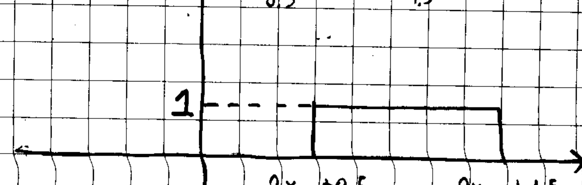
Example:

$x_{k+1} = 0.5x_k + u_k + w_k$

$y_k = 2x_k + v_k$

$$p(y_{k+1}|x_{k+1},z^k,u_k) = p(2x_{k+1}+v_{k+1}|x_{k+1})$$





$$A = p^g_{k+1|k}(x_{k+1} | z^k, u_k) = ?$$

$$B = p_{k+1|k+1}(x_{k+1} | z^{k+1}) = ?$$

$$B: \; p^g_{k+1|k+1}(x_{k+1} | z^{k+1}) = \frac{p^g(x_{k+1}, z^{k+1})}{p^g(z^{k+1})}$$

$$= \frac{p^g(x_{k+1}, y^{k+1}, u^k)}{p^g(z^{k+1})} = \frac{p^g(x_{k+1}, y_{k+1}, y^k, u^k)}{p^g(z^{k+1})}$$

$$= \frac{p^g(y_{k+1} | x_{k+1}, y^k, u^k) \cdot p(x_{k+1}, y^k, u^k)}{p^g(z^{k+1})}$$

$$= \frac{p(y_{k+1} | x_{k+1}) \cdot p(x_{k+1}, z^k, u_k)}{\int p^g(z^{k+1}, x_{k+1})\, dx_{k+1}}$$

$$p(x_{k+1}, z^{k+1}) = p(x_{k+1}, y_{k+1}, z^k, u_k)$$

$$= p(y_{k+1} | x_{k+1}, z^k, u_k) \cdot p(x_{k+1}, z^k, u_k)$$

$$= p(y_{k+1} | x_{k+1}) \cdot p(x_{k+1} | z^k, u_k) \cdot p(z^k, u_k)$$

$$\Rightarrow B = \frac{p(y_{k+1} | x_{k+1}) \cdot p(x_{k+1} | z^k, u_k) \cdot \cancel{p(z^k, u_k)}}{\int p^g(y_{k+1} | x_{k+1}) \cdot p(x_{k+1} | z^k, u_k) \cdot \cancel{p(z^k, u_k)}\, dx_{k+1}}$$

$$\Rightarrow B_{k+1} = \frac{p(y_{k+1} | x_{k+1}) \cdot A_{k+1}}{\int p^g(y_{k+1} | x_{k+1}) A_{k+1}(x_{k+1})\, dx_{k+1}}$$

$$A: \; p(x_{k+1} | z^k, u_k) = \int p(x_{k+1}, x_k | z^k, u_k)\, dx_k$$

$$= \int p(x_{k+1} | x_k, z^k, u_k) \cdot p(x_k | z^k, u_k)\, dx_k$$

$$A_{k+1} = \int p(x_{k+1} | x_k) \underbrace{p(x_k | z^k, u_k)}_{B_k} dx_k$$

$z^k = (y^k, u^{k-1})$

① $$p_{k+1|k+1}(x_{k+1} \mid z^{k+1}) = \frac{p(y_{k+1} \mid x_{k+1}) \cdot p_{k+1|k}(x_{k+1} \mid z^k, u_k)}{\int p(y_{k+1} \mid x_{k+1}) p_{k+1|k}(x_{k+1} \mid z^k, u_k) \, dx_{k+1}}$$

② $$p_{k+1|k}(x_{k+1} \mid z^k, u_k) = \int p(x_{k+1} \mid x_k, u_k) \, p_{k|k}(x_k \mid z^k) \, dx_k$$

$$x_{k+1} = f_k(x_k, u_k, w_k)$$

$$y_{k+1} = h_{k+1}(x_{k+1}, v_{k+1})$$

In the case of Markov Chain:

$x_k \in \{1, 2, 3, \dots, I\}$

π_0 : initial state probability distribution

$\pi_0 = [\text{Prob}(x_0 = 1) \; \cdots \; \text{Prob}(x_0 = I)]$

$p_k(u) = \left[p_{ij}^k(u) \right]_{I \times I}$

$$p_{ij}^k(u) = \text{Prob}(x_k = j \mid x_{k-1} = i)(u)$$

y_k : a function of state

$p(y_k \mid i)$: known density $i = 1, \dots, I$

$$p_{k+1|k+1}(x_{k+1} \mid z^{k+1}) = [\text{Prob}(x_{k+1} = 1 \mid z^{k+1}) \cdots \text{Prob}(x_{k+1} = I \mid z^{k+1})]$$

$$p(y_{k+1} \mid x_{k+1}) = \begin{Bmatrix} p(y_{k+1} \mid x_{k+1} = 1) \\ \vdots \\ p(y_{k+1} \mid x_{k+1} = I) \end{Bmatrix}$$

$$p_{k+1|k}(x_{k+1} \mid z^k, u_k) = [\text{Prob}(x_{k+1} = 1 \mid z^k, u_k) \dots \text{Prob}(x_{k+1} = I \mid z^k, u_k)]$$

Denominator of $p_{k+1|k+1}$:

$$\sum_{j=1}^{I} p(y_{k+1} \mid x_{k+1} = j) \cdot p_{k+1|k}(j \mid z^k, u_k)$$

$$p_{k+1|k+1}(x_{k+1} = i \mid z^{k+1}) = \frac{p(y_{k+1} \mid i) \, p_{k+1|k}(i \mid z^k, u_k)}{\sum_{j=1}^{I} p(y_{k+1} \mid j) \, p_{k+1|k}(j \mid z^k, u_k)}$$

j^{th} element of $P_{k+1|k}(x_{k+1}|z^k,u_k) = \text{Prob}(x_{k+1}=j\,|\,z^k,u_k)$

$$P_{k+1|k}(j|z^k,u_k) = \sum_{l=1}^{I} \underbrace{P_{k+1|k}(j|l,u_k)}_{P^k_{lj}(u_k)}\, P_{k|k}(l|z^k)$$

$$\Rightarrow P_{k+1|k+1}(x_{k+1}|z^{k+1}) = \frac{p(y_{k+1}|i)\cdot \sum_{l=1}^{I} P^k_{li}(u_k)\cdot P_{k|k}(l|z^k)}{\sum_{j=1}^{I}\left(p(y_{k+1}|j)\cdot \sum_{l=1}^{I} P^k_{lj}(u_k)\,P_{k|k}(l|z^k)\right)}$$

$$\underbrace{\begin{bmatrix} P_{k|k}(1|z^k) & P_{k|k}(2|z^k) & \cdots & P_{k|k}(I|z^k)\end{bmatrix}}_{P_{k|k}(x_k|z^k)} \underbrace{\begin{bmatrix} P^k_{1i}(u_k)\\ P^k_{2i}(u_k)\\ \vdots \\ P^k_{Ii}(u_k)\end{bmatrix}}_{i^{th}\text{ column of } P_k(u_k)}$$

Denominator:

$$\sum_{j=1}^{I} p(y_{k+1}|j)\, \underbrace{P_{k|k}(x_k|z^k)}_{\text{row vector}}\ \ j^{th} \text{ column of } P_k(u_k)$$

$$P_{k|k}(x_k|z^k)\cdot P_k(u_k) \underbrace{\begin{bmatrix} p(y_{k+1}|1) & & \\ & \ddots & \\ & & p(y_{k+1}|I)\end{bmatrix}}_{D} \overbrace{\begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix}}^{\underline{1}}$$

$$P_{k+1|k+1}(i|z^{k+1}) = \frac{P_{k|k}(x_k|z^k)\cdot i^{th}\text{ column of } P_k(u_k)\cdot p(y_{k+1}|i)}{P_{k|k}(x_k|z^k)\cdot P_k(u_k)\cdot D\cdot \underline{1}}$$

$$\Rightarrow P_{k+1|k+1}(x_{k+1}|z^{k+1}) = \frac{P_{k|k}(x_k|z^k)\,P_k(u_k)\,D(y_k)}{P_{k|k}(x_k|z^k)\,P_k(u_k)\,D_k(y_k)\cdot \underline{1}}$$

— o —

$x_{k+1} = A_k x_k + B_k u_k + G_k w_k$

$y_k = C_k x_k + H_k v_k$

$\{x_0, w_0, \ldots, v_0, \ldots\}$ independent

$x_0 \sim N(\bar{x}_0, \Sigma_0)$ $\quad w_i \sim N(0,Q)$ $\quad v_i \sim N(0,R)$

$P_{k+1|k+1}(x_{k+1}|z^{k+1}) = ?$

<u>Case 1:</u> $u_k \equiv 0$ (Note $z^k = y^k$)

<u>Note:</u> x_{k+1}, x_k, z^{k+1} jointly normal

$P_{k+1|k+1}(x_{k+1}|y^{k+1}) \sim N(\bar{x}_{k+1|k+1}, \Sigma_{k+1|k+1})$

$P_{k+1|k}(x_{k+1}|y^k) \sim N(\bar{x}_{k+1|k}, \Sigma_{k+1|k})$

Lemma: Let x and y be jointly gaussian random vectors with

$$\left.\begin{aligned} x &\sim N(\bar{x}, \Sigma_x) \\ y &\sim N(\bar{y}, \Sigma_y)\end{aligned}\right\} E\{(x-\bar{x})(y-\bar{y})\} = \Sigma_{xy}$$

Let $\hat{x} \triangleq E(x|y)$, $\tilde{x} \triangleq x - \hat{x}$

then

(i) $\hat{x} = \bar{x} + \Sigma_{xy}\Sigma_y^{-1}(y-\bar{y})$

(ii) $\tilde{x}$ is independent of y and so $\hat{x}$

(iii) $\tilde{x} \sim N(0, \Sigma_{\tilde{x}})$ where $\Sigma_{\tilde{x}} = \Sigma_x - \Sigma_{xy}\Sigma_y^{-1}\Sigma_{xy}^T$

Proof:

$\hat{v} \triangleq \bar{x} + \Sigma_{xy}\Sigma_y^{-1}(y-\bar{y})$

$\tilde{v} \triangleq x - \hat{v}$

Note that;

- $E(\tilde{v}) = E(x-\hat{v}) = E(\underbrace{x-\bar{x}}_{0} - \Sigma_{xy}\Sigma_y^{-1}\underbrace{(y-\bar{y})}_{0})$

$\Rightarrow E(\tilde{v}) = 0$

- $E\{\tilde{v}(y-\bar{y})^T\} = E\{(x-\hat{v})(y-\bar{y})^T\}$

$= E\{x(y-\bar{y})^T\} - E\{\hat{v}(y-\bar{y})^T\}$

$= E\{(x-\bar{x})(y-\bar{y})^T + \bar{x}(y-\bar{y})^T\}$

$- E\{(\bar{x}+\Sigma_{xy}\Sigma_y^{-1}(y-\bar{y}))(y-\bar{y})^T\}$

$= \Sigma_{xy} + 0 - \underbrace{E\{\bar{x}(y-\bar{y})^T\}}_{0} - \Sigma_{xy}\Sigma_y^{-1}E\{(y-\bar{y})(y-\bar{y})^T\}$

$= \Sigma_{xy} - \Sigma_{xy}\Sigma_y^{-1}\Sigma_y = 0$

$\Rightarrow E\{\tilde{v}(y-\bar{y})^T\} = 0 \Rightarrow$ $\tilde{v}$ and y are uncorrelated

$\Rightarrow$ $\tilde{v}$ and y are independent (since Gaussian)

$\Rightarrow$ $\tilde{v}$ and $\hat{v}$ (function of y) are independent

$\hat{x} = E(x|y) = E(\hat{v}+\tilde{v}|y) = E(\hat{v}|y) + \underbrace{E(\tilde{v}|y)}_{0}$

$\Rightarrow \hat{x} = E(x|y) = E(\hat{v}|y)$

$= E(\bar{x}+\Sigma_{xy}\Sigma_y^{-1}(y-\bar{y})|y)$

$\hat{x} = \bar{x} + \Sigma_{xy}\Sigma_y^{-1}(y-\bar{y})$

—o—

$\hat{v} = \bar{x}+\Sigma_{xy}\Sigma_y^{-1}(y-\bar{y})$ $\quad \tilde{v} = x-\hat{v}$

$\Downarrow$

$\hat{x} = \hat{v} = \bar{x}+\Sigma_{xy}\Sigma_y^{-1}(y-\bar{y}) \Rightarrow \tilde{x} = x-\hat{v} = x-\hat{x}$

$\tilde{v}$ and y are independent

$\tilde{x}$ and y are independent

$\tilde{x}$ and $\hat{v}$ are independent

$\therefore$ $\tilde{x}$ and $\hat{x}$ are independent

Show that, $\Sigma_{\tilde{x}} = \Sigma_x - \Sigma_{xy}\Sigma_y^{-1}\Sigma_{xy}^T$

$k \qquad\qquad\qquad k+1$

$\hat{x}_{k|k} = E\{x_k | y^k\} \rightarrow$

$\Sigma_{k|k} = Cov(x_k | y^k) \rightarrow$

$$\left.\begin{aligned}\hat{x}_{k+1|k} &= E\{x_{k+1} | y^k\} \\ \Sigma_{k+1|k} &= Cov(x_{k+1} | y^k)\end{aligned}\right\}\text{No measurement update}$$

the system is $x_{k+1} = A_k x_k + G_k w_k$

$y_k = C_k x_k + H_k v_k$

$x_0 \sim N(0, \Sigma_0)$

$w_k \sim N(0, Q)$

$v_k \sim N(0, R)$

$$E\{x_{k+1} | y^k\} = E\{A_k x_k + G_k w_k | y^k\}$$

$$= A_k \underbrace{E\{x_k | y^k\}}_{\hat{x}_{k|k}} + G_k E\{w_k | y^k\}$$

$$\Rightarrow \hat{x}_{k+1|k} = A_k \hat{x}_{k|k}$$

$$\Sigma_{k+1|k} = A_k \Sigma_{k|k} A_k^T + G_k Q G_k^T$$

Step 2:

$$\hat{y}_{k|k-1} = E\{y_k | y^{k-1}\}$$

$$= E\{C_k x_k + H_k v_k | y^{k-1}\} = C_k \hat{x}_{k|k-1}$$

$$\Sigma^y_{k|k-1} = Cov(y_k | y^{k-1})$$

$$= Cov(C_k x_k + H_k v_k | y^{k-1})$$

$$= C_k \Sigma_{k|k-1} C_k^T + H_k R H_k^T$$

$$E\{x_k \tilde{y}^T_{k|k-1}\} = E\{x_k (C_k \tilde{x}_{k|k-1} + H_k v_k)^T\}$$

$$= E\{x_k \tilde{x}^T_{k|k-1} C_k^T\} + E\{x_k v_k^T H_k^T\}$$

$$= E\{(x_{k|k-1} + \tilde{x}_{k|k-1}) \tilde{x}^T_{k|k-1}\} C_k^T$$

$$\Rightarrow E\{x_k \cdot \tilde{y}^T_{k|k-1}\} = \Sigma_{k|k-1} C_k^T = \Sigma_{xy}$$

Step 3:

$x_{k|k-1}$ is known	$x_{k|k} = E\{x_k | y^k\}$
$\Sigma_{k|k-1}$ is known	
$y_{k|k-1}$ is known	
$\Sigma^y_{k|k-1}$ is known	
$E\{x_k \cdot \tilde{y}^T_{k|k-1}\}$ is known	

$$x_{k|k} = E\{x_k | y^{k-1}, y^k\}$$

$$y_k = \underbrace{y_{k|k-1}}_{\text{estimated part function of } y^{k-1}} + \underbrace{\tilde{y}_{k|k-1}}_{\text{estimation error}}$$

$$\therefore x_{k|k} = E\{x_k | y^{k-1}, \tilde{y}_{k|k-1}\}$$

independent

$$= E\{x_k | y^{k-1}\} + E\{x_k | \tilde{y}_{k|k-1}\} + \cancel{\bar{x}_k}^{\,0}$$

$$x_{k|k} = x_{k|k-1} + \underbrace{E\{x_k | \tilde{y}_{k|k-1}\}}_{\underbrace{E\{x_k \tilde{y}^T_{k|k-1}\}}_{\Sigma_{xy}} \underbrace{Cov(\tilde{y}_{k|k-1})^{-1}}_{\Sigma_y^{-1}} \cdot \underbrace{\tilde{y}_{k|k-1}}_{(y-\bar{y})}} \quad \text{Use Lemma}$$

$$\Rightarrow \boxed{x_{k|k} = x_{k|k-1} + \Sigma_{k|k-1} C_k^T (C_k \Sigma_{k|k-1} C_k^T + H_k R H_k^T)^{-1} (y_k - C_k x_{k|k-1})}$$

$$\Sigma_{k|k} = \Sigma_{k|k-1} - E\{x_k \cdot \tilde{y}^T_{k|k-1}\} Cov(\tilde{y}_{k|k-1})^{-1} E\{x_k \tilde{y}^T_{k|k-1}\}^T$$

$$\Rightarrow \boxed{\Sigma_{k|k} = \Sigma_{k|k-1} - \Sigma_{k|k-1} C_k^T (C_k \Sigma_{k|k-1} C_k^T + H_k R_k H_k^T)^{-1} C_k \Sigma^T_{k|k-1}}$$

Theorem: The conditional density $P_{k|k} \sim N(x_{k|k}, \Sigma_{k|k})$ can be obtained from;

$$x_{k+1|k+1} = A_k x_{k|k} + L_{k+1}(y_{k+1} - C_{k+1}A_k x_{k|k})$$

$$x_{0|0} = L_0 \cdot y_0$$

$$\Sigma_{k+1|k+1} = (I - L_{k+1}C_{k+1})\Sigma_{k+1|k}$$

$$\Sigma_{k+1|k} = A_k \Sigma_{k|k} A_k^T + G_k Q G_k^T$$

$$\Sigma_{0|0} = (I - L_0 C_0)\Sigma_0$$

$$L_k \triangleq \Sigma_{k|k-1} C_k^T (C_k \Sigma_{k|k-1} C_k^T + H_k R H_k^T)^{-1}$$

$$L_0 \triangleq \Sigma_0 C_0^T (C_0 \Sigma_0 C_0^T + H_0 R H_0^T)^{-1}$$

Kalman Gain matrix

—•—

Lemma:

$$Cov(x - E\{x|y\}) \le Cov(x - f(y))$$

$\Rightarrow$ Kalman filter is optimal in this sense

Proof:

$$E\{(x - E\{x|y\}) \cdot g^T(y)\} = 0$$

$$= E\{x g^T(y)\} - \underbrace{E\{E\{x|y\} \cdot g^T(y)\}}_{E\{x \cdot g^T(y)\}} = 0$$

Let $g(y) = E\{x/y\} - f(y)$

$$x - f(y) = \underbrace{\underbrace{x - E(x|y)} + \underbrace{\overbrace{E(x|y) - f(y)}^{g(y)}}}_{\text{uncorrelated}}$$

$$Cov(x - f(y)) = Cov(x - E\{x|y\}) + Cov(E\{x|y\} - f(y))$$

$$\Rightarrow Cov(x - f(y)) \ge Cov(x - E(x|y))$$

Lemma:

Suppose that $\Sigma_{k+1|k}$ and $\bar{\Sigma}_{k+1|k}$ are covariance matrices corresponding to initial conditions $\Sigma_{0|-1}$ and $\bar{\Sigma}_{0|-1}$ respectively.

If $\Sigma_{0|-1} \le \bar{\Sigma}_{0|-1}$ then $\Sigma_{k+1|k} \le \bar{\Sigma}_{k+1|k}$ for all k

Proof:

$$\left.\begin{array}{l} x_0 \sim N(0, \Sigma_0) \\ \xi_0 \sim N(0, M) \end{array}\right\} \{x_0, \xi_0, w_0, \dots, v_0, \dots\} \text{ are independent}$$

two initial conditions : x_0 , $x_0 + \xi_0 = \bar{x}_0$

$\Sigma_{0|-1} = \Sigma_0$ $\quad$ $\bar{\Sigma}_{0|-1} = \Sigma_0 + M$

$$\Rightarrow \Sigma_{0|-1} \le \bar{\Sigma}_{0|-1}$$

$$\bar{x}_0 = x_0 + \xi_0$$

$$\bar{x}_1 = A_0 \bar{x}_0 + G_0 w_0 = A_0 x_0 + A_0 \xi_0 + G_0 w_0 = x_1 + A_0 \xi_0$$

$$\bar{x}_2 = A_1 \bar{x}_1 + G_1 w_1 = A_1 x_1 + A_1 A_0 \xi_0 + G_1 w_1 = x_2 + A_1 A_0 \xi_0$$

$$\Rightarrow \bar{x}_{k+1} = x_{k+1} + A_k A_{k-1} \cdots A_0 \xi_0$$

$$E\{\bar{x}_{k+1} | y^k, \xi_0\} = E\{x_{k+1} + A_k \cdots A_0 \xi_0 | y^k, \xi_0\}$$

ξ_0 is independent of y^k

ξ_0 is independent of x_{k+1}

$$\Rightarrow E\{\bar{x}_{k+1} | y^k, \xi_0\} = E\{x_{k+1} | y^k, \xi_0\} + A_k \cdots A_0 E\{\xi_0 | y^k, \xi_0\}$$

$$= \underbrace{E\{x_{k+1} | y^k\}}_{\hat{x}_{k+1|k}} + A_k \cdots A_0 \xi_0$$

$$\bar{x}_{k+1} = x_{k+1} + A_k A_{k-1} \cdots A_0 \xi_0$$

$$\bar{y}_{k+1} = C_{k+1} \bar{x}_{k+1} + H_{k+1} v_{k+1}$$

$$= C_{k+1} x_{k+1} + C_{k+1} A_k \cdots A_0 \xi_0 + H_{k+1} v_{k+1}$$

$$= y_{k+1} + C_{k+1} A_k \cdots A_0 \xi_0 = f_{k+1}(y_{k+1}, \xi_0)$$

$\Rightarrow \bar{y}^k = f(y^k, \xi_0)$

$$\Sigma_{k+1|k} = Cov(x_{k+1} \mid y^k) = Cov(x_{k+1} - \underbrace{E\{x_{k+1} \mid y^k\}}_{\hat{x}_{k+1|k}})$$

$$= Cov\left(x_{k+1} - E\{\bar{x}_{k+1} \mid y^k, \xi_0\} + A_k \cdots A_0 \xi_0\right)$$

and $\bar{\Sigma}_{k+1|k} = Cov\left(\bar{x}_{k+1} - E\{\bar{x}_{k+1} \mid \bar{y}^k\}\right)$

$\uparrow$

$f(y^k, \xi_0)$

$$\geq Cov\left(\bar{x}_{k+1} - E\{\bar{x}_{k+1} \mid y^k, \xi_0\}\right)$$

Note:

$$Cov(x - E(x|y)) \leq Cov(x - f(y))$$

Theorem:

$$x_{k+1|k} = A_k x_{k|k-1} + A_k L_k (y_k - C_k x_{k|k-1})$$

$$x_{0|-1} = E(x_0) = 0$$

Time invariant case:

$x_{k+1} = A x_k + G w_k$ basic assumption holds

$y_{k+1} = C x_{k+1} + H v_{k+1}$

$x_0 \sim N(0, \Sigma_0)$, $w_k \sim N(0, Q)$, $v_k \sim N(0, R)$

Asymptotic Behaviour of the Kalman Filter:

$$\Sigma_{k+1|k} = A\left(\Sigma_{k|k-1} - \Sigma_{k|k-1} C^T (C \Sigma_{k|k-1} C^T + HRH^T)^{-1} C \Sigma_{k|k-1}^T\right) A^T + GQG^T$$

If A is unstable, then $\Sigma_{k+1|k}$ diverges

If $\Sigma_{k+1|k}$ converges a value (steady-state)

$$\bar{\Sigma} = A(\bar{\Sigma} - \bar{\Sigma} C^T (C \bar{\Sigma} C^T + HRH^T)^{-1} C \bar{\Sigma}) A + GQG^T$$

Algebraic Riccati Equation (ARE)

OPTIMAL CONTROL

(Linear, Gaussian)

$x_{k+1} = A_k x_k + B_k u_k + G_k w_k$

$y_k = C_k x_k + H_k v_k$

$x_0 \sim N(\bar{x}_0, \Sigma_0)$ $w_k \sim (0, Q)$

$v_k \sim N(0, R)$ $\{x_0, w_0, \dots, v_0, \dots\}$ are independent

$$E\left\{\sum_{k=0}^{N-1} C_k(x_k, u_k) + C_N(x_N)\right\} \Leftarrow \text{minimize this}$$

$u_k = g_k(y^k)$ $u_k = g_k(x_{k|k})$

Definition: $g = \{g_0 \cdots g_{n-1}\}$ is seperated if g_k depends on y^k through $x^g_{k|k}$ i.e. $u_k = g_k(x_{k|k})$

Definition: Let $\phi_k : R^n \times U \to R$ be any function. Define;

$$\phi_{k|k}(x,u) \triangleq \int \phi_k(\xi, u) \frac{e^{-\frac{1}{2}(\xi-u)' \Sigma_{k|k}^{-1}(\xi - x)}}{\sqrt{(2\pi)^n |\Sigma_{k|k}|}} d\xi$$

NOTE:

$E\{\phi_k(x_k^g, u_k^g) \mid y^{g,k}\} = \phi_{k|k}(x^g_{k|k}, u^g_k)$

Lemma: Let g be a seperated policy. Define recursively the functions $V_k^g(x)$ where $k = 0, \dots, N$ by

$V_N^g(x) \triangleq C_{N|N}(x)$

$$V_k^g(x) \triangleq C_{k|k}(x, g_k(x)) + \int V_{k+1}(\xi) \frac{e^{-\frac{1}{2}(\xi - A_k x - B_k u)^T \Delta_{k+1}^{-1}(\xi - A_k x - B_k u)}}{\sqrt{(2\pi)^n |\Delta_{k+1}|}} d\xi$$

where $\Delta_{k+1} = \Sigma_{k+1|k} - \Sigma_{k+1|k+1}$

Then the random variable

$$V_k^g(x^g_{k|k}) = J_k^g \triangleq E\left\{\sum_{l=k}^{N-1} C_l(x_l^g, u_l^g) + C_N(x_N^g) \,\Big|\, y^{g,k}\right\}$$

Example:

$x_{k+1} = x_k + u_k + w_k$ $Q = R = 1$

$y_k = x_k + v_k$ $x_0 \sim N(1, 2)$

$\Sigma_{0|-1} = \Sigma_0 = 2$

$L_0 = \Sigma_{0|-1} C_0^T \left[C_0 \Sigma_{0|-1} C_0^T + H_0 R H_0^T\right]^{-1}$

$\Rightarrow L_0 = 2.1.(1.2.1 + 1.1.1)^{-1}$

$\Rightarrow L_0 = 2/3$

$\Sigma_{0|0} = (I - L_0 C_0)\Sigma_0 = \left(1 - \frac{2}{3}\right).2 = 2/3$

$\Sigma_{1|0} = A \Sigma_{0|0} A^T + GQG^T = \frac{2}{3} + 1 = 5/3$

$L_1 = 5/8$, $\Sigma_{1|1} = 5/8$, $\Sigma_{2|1} = 13/8$

$L_2 =$ $\Sigma_{2|2} = 13/21$

$J(g) = E\left\{\sum_{k=0}^{1} C_k(x_k^g, u_k^g) + C_2(x_2^g)\right\}$

$N = 2$

let $C_2(x) = 2x^2$
$C_k(x) = x^2 + u^2$

Let $g(x) = u$ be $u_k = -x_{k|k}$

$$V_2^g(x) = C_{2|2}(x) = \int C_2(\xi) \frac{e^{-\frac{1}{2}(x-\xi)^2 / \Sigma_{2|2}}}{\sqrt{(2\pi)^1 |\Sigma_{2|2}|}} d\xi$$

$$= \int 2\xi^2 \frac{e^{-\frac{1}{2}(x-\xi)^2 / \Sigma_{2|2}}}{\sqrt{2\pi \Sigma_{2|2}}} d\xi$$

$$\int (\xi - x)^2 \frac{e^{-\frac{1}{2}(x-\xi)^2 / \Sigma_{2|2}}}{\sqrt{2\pi \Sigma_{2|2}}} d\xi = \Sigma_{2|2}$$

$E\{x^2\} = E\{[(x - \bar{x}) + \bar{x}]^2\}$

$= E\{(x - \bar{x})^2\} + \bar{x}^2$

$\Rightarrow E\{x^2\} = \sigma_x^2 + \bar{x}^2$

$\Rightarrow V_2^g(x) = 2\left[\Sigma_{2|2} + x^2\right] = 2\left(\frac{13}{21} + x^2\right)$

$\Rightarrow V_2^g(x) = \frac{26}{21} + 2x^2$

$$V_1^g(x) = C_{1|1}(x, g_k(x)) + \int V_2(\xi) \frac{e^{-\frac{1}{2}(\xi - x - u)^2 / \Delta_2}}{\sqrt{2\pi |\Delta_2|}} d\xi$$

$$C_{1|1}(x, g_k(x)) = \int \underbrace{C(\xi, u)}_{(\xi^2 + u^2)} \frac{e^{-\frac{1}{2}(x-\xi)^2 / \Sigma_{1|1}}}{\sqrt{2\pi |\Sigma_{1|1}|}} d\xi$$

$$= (\Sigma_{1|1} + x^2) + \underset{-x}{u^2} = \Sigma_{1|1} + 2x^2$$

$$\Rightarrow C_{1|1}(x, g_k(x)) = 5/8 + 2x^2$$

<u>Proof of the Lemma:</u>

$$J_N^g = E\{C_N(x_N^g) \mid y^{g,N}\}$$

$$= \int C_N(\xi) \frac{e^{-\frac{1}{2}(\xi - x)^T \Sigma_{N|N}^{-1} (\xi - x)}}{\sqrt{(2\pi)^n |\Sigma_{N|N}|}} d\xi$$

$$= C_{N|N}(x_{N|N}^g) = V_N^g(x_{N|N})$$

Proof is continued by induction.

$$E\left\{\sum_{l=k}^{N-1} C_l(x_l^g, u_l^g) + C_N(x_N^g) \mid y^{g,k}\right\} = J_k^g$$

$$= E\left\{C_k(x_k^g, u_k^g) + \sum_{l=k+1}^{N-1} C_l(x_l^g, u_l^g) + C_N(x_N^g) \mid y^{g,k}\right\}$$

$$= \underbrace{E\{C_k(x_k^g, u_k^g) \mid y^{g,k}\}}_{C_{k|k}(x_{k|k}^g, u_k^g)} + E\left\{\sum_{l=k+1}^{N-1} C_l(x_l^g, u_l^g) + C_N(x_N^g) \mid y^{g,k}\right\}$$

$$E\Big\{\underbrace{E\Big\{\sum_{l=k+1}^{N-1} C_l(x_l^g, u_l^g) + C_N(x_N^g) \mid y^{g,k+1}\Big\}}_{V_{k+1}^g(x_{k+1}^g,\, u_{k+1}^g) \text{ by induction hypothesis}} \mid y^{g,k}\Big\}$$

$$\Rightarrow J_k^g = C_{k|k}(x_{k|k}^g, u_{k|k}^g) + E\{V_{k+1}^g(x_{k+1|k+1}^g) \mid y^{g,k}\}$$

$$x_{k+1|k+1}^g = \underbrace{A_k x_{k|k}^g + B_k u_k^g}_{\text{functions of } y^{g,k}} + L_{k+1} \cdot \underbrace{\tilde{y}_{k+1|k}^g}_{\text{independent of } y^{g,k}}$$

zero mean, $\Sigma_{k+1|k}^y$

$$L_{k+1} \cdot \tilde{y}_{k+1|k}^g \xrightarrow{\text{Covariance}} L_{k+1} \operatorname{Cov}(\tilde{y}_{k+1|k}^g) L_{k+1}^T$$

$$= L_{k+1}(C_{k+1} \Sigma_{k+1|k} C_{k+1}^T + H_{k+1} R H_{k+1}^T) L_{k+1}^T$$

$$\Sigma_{k+1|k} C_k^T [\underbrace{C_{k+1} \Sigma_{k+1|k} C_{k+1}^T + H_{k+1} R H_{k+1}^T}_{\Delta}]^{-1}$$

$$= \Sigma_{k+1|k} C_k^T [\Delta]^{-1} C_k \Sigma_{k+1|k} = \underbrace{\Sigma_{k+1|k} - \Sigma_{k+1|k+1}}_{\Delta_{k+1}}$$

$$\Rightarrow J_k^g = C_{k|k}(x_{k|k}^g, u_k^g) + \int V_{k+1}^g(\xi) \frac{e^{-\frac{1}{2}(\xi - A_k x_k^g - B_k u_k^g)^T \Delta_{k+1}^{-1} (\xi - A_k x_k^g - B_k u_k^g)}}{\sqrt{(2\pi)^n |\Delta_{k+1}|}} d\xi$$

$$= \underbrace{\Sigma_{k+1|k} C_k^T [C_{k+1} \Sigma_{k+1|k} C_{k+1}^T + H_{k+1} R H_{k+1}^T]^{-1}}_{\lambda_{k+1}} [A] [A^{-1}] C_k \Sigma_{k+1|k}$$

not system matrix, don't be confused.

$$= \Sigma_{k+1|k} C_k^T [A]^{-1} C_k \Sigma_{k+1|k}$$

$$= \Sigma_{k+1|k} - \Sigma_{k|k+1} = \Delta_{k+1}$$

$$E\{V_{k+1}^g(x_{k+1|k+1}) \mid y^{g,k+1}\}$$

$$\int V_{k+1}(\xi) \frac{1}{\sqrt{(2\pi)|\Delta_{k+1}|}} \cdot e^{-\frac{1}{2}(\xi - A_k x_k - B_k u_k)^T \Delta_{k+1}^{-1} (\xi - A_k x_k - B_k u_k)} d\xi$$

$$= V_k(V_{k|k}^g)$$

$\Rightarrow$ We prooved the lemma.

Lemma: Let $V_k(x)$ of $0 \leq k \leq N$ $x \in \mathbb{R}^n$ be functions such that

$$V_N(x) \leq C_{N|N}(x)$$

$$V_k(x) \leq C_{k|k}(x) + \int V_{k+1}(\xi) \frac{1}{\sqrt{2\pi |\Delta_{k+1}|}} e^{-\frac{1}{2}(\xi - A_k - B_k u) \Delta_{k+1}^{-1} (\xi - A_k - B_k)} d\xi$$

Let $u \in U$ let $g \in G$ be arbitrary, then $V_k(x_{k|k}) \leq J_k^g$

<u>Theorem:</u> Define recursively the functions

$$V_N(x) = C_{N|N}(x)$$

$$V_k(x) = \operatorname{Inf}_u \left\{ c_{k|k}(x, u) + \int V_{k+1}(\xi) N(\xi, A_k + B_k u, \Delta_{k+1}) d\xi \right.$$

1) Let $g \in G$ be arbitrary, then

$$V_k(x_{k|k}^g) \leq J_k^g \quad \text{w.p.} 1$$

2) Suppose that the infimum in $(*)$ is achieved at $g_k(x)$, then the separated policy is optimal. Moreover, $V_k(x_{k|k}^g) = J_k^g$ w.p. 1.

LINEAR QUADRATIC GAUSSIAN PROBLEM

$x_{k+1} = A_k x_k + B_k u_k + G_k w_k$

$y_k = C_k x_k + H_k v_k$

basic assumption is satisfied

- $x_0 \sim N(\bar{x}_0, \Sigma_0)$
- $w_k \sim N(0, Q)$
- $v_k \sim N(0, R)$

- Objective Function

$$J(g) = E\Big\{ \sum_{k=0}^{N-1} \underbrace{x_k^T P_k x_k + u_k^T T_k u_k}_{C_k(x_k, u_k)} \Big) + \underbrace{x_N^T P_N x_N}_{C_N(x_N)}$$

<u>Lemma:</u> $E\{\underset{\sim}{x}' R x\} = \bar{x}' R \bar{x} + tr(R \Sigma)$

where $\bar{x}' = E(x)$

$\Sigma = Cov(x)$

$$V_N(x) = C_{k|k}(x) = x' \overset{S_N}{P_N} x + \overbrace{tr(P \cdot \Sigma_{N|N})}^{S_k}$$

$$V_{k|k}(x) = \inf_u \Big(u T_k u + x P_k x + tr(P_k \Sigma_{k|k}$$

$$+ \int V_{k+1}(\xi) N(\xi, A_k x + B_k u, \Delta_{k+1}) d\xi$$

$$C_{N|N}(x) = \int \xi' P_N \xi N(\xi, x, \Sigma_{k|k}) d\xi$$

<u>Note</u> (Look at the definition $\phi_{k|k}(x, u)$)

$$= x' P_N x + tr(P_N \Sigma_{N|N})$$

$$= \inf_u \Big\{ u' T_k u + x' P_k x + tr\left(P_k \Sigma_{k|k}\right)$$

$$+ \int \underbrace{V_{k+1}(\xi) \cdot N\left(\xi, Ax_k + Bu_k, \Delta_{k+1}\right)}_{???} d\xi \Big\}$$

Lemma: Suppose that $V_{k+1}(x) = x' S_{k+1} x + s_{k+1}$

Then

$V_k(x) = x' S_k x + s_k$ where

$$S_k = P_k + A_k'\left(S_{k+1} - S_{k+1} B_k \left(T_k + B_k' S_{k+1} B_k\right)^{-1} B_k' S_{k+1}\right) A_k$$

$$s_k = s_{k+1} + Tr\left(P_k \Sigma_{k|k} + S_{k+1} \Delta_{k+1}\right)$$

$$u_k^* = -\left[T_k + B_k^T S_{k+1} B_k\right]^{-1} B_k^T S_{k+1} A_k x_k$$

By induction hypothesis

$$V_{k+1} = \xi^T S_{k+1} \xi + s_{k+1}$$

$$V_k(x) = \inf \{ u' T_k u + x' P_k x + tr\left(P_k \Sigma_{k|k}\right) + s_{k+1} + \int \xi' S_{k+1} \xi \, N(\xi, A_k x + B_k u, \Delta_{k+1}) d\xi$$

Note: Suppose that

$u' F u + u' g \Rightarrow$ strictly unique.

$2 F u + g = 0$ $\quad \boxed{u = -\frac{1}{2} F^{-1} g.}$

$u' T_k u \longrightarrow 2 T_k u$

$$\underbrace{x' A_k \Delta_{k+1} A_k x}_{\text{constant}} + 2\left(x' A_k \Delta_{k+1} B_k u_k\right) + u' B_k' \Delta_{k+1} B_k u \longrightarrow$$

$$\longrightarrow 2 B_k' \Delta_{k+1} A_k x + 2 B_k' \Delta_{k+1} B_k u = 0$$

NUMERICAL CONSIDERATIONS ON KALMAN FILTERING

Due to the numerical difficulties in using $\Sigma_{k|k}$, $\sqrt{\Sigma_{k|k}}$ is used to lower the dynamic range.

$x_{k+1} = A_k x_k + G_k w_k$

$y_k = C_k x_k + H_k v_k$ $\quad y_k(w) \in R^m$

$\Sigma_{k+1|k} = A_k \Sigma_{k|k} A_k^T + G_k Q G_k^T$

$L_{k+1} = \Sigma_{k+1|k} C_{k+1}^T \left[C_{k+1} \Sigma_{k+1|k} C_{k+1}^T + H_{k+1} R H_{k+1}^T \right]^{-1}$

$\Sigma_{k+1|k+1} = (I - L_{k+1} C_{k+1}) \Sigma_{k+1|k}$

$x_{k+1|k+1} = A_k x_{k|k} + L_{k+1}(y_{k+1} - C_{k+1} A_k x_{k|k})$

Fact 1: If Q is not diagonal, then define $\bar{w}_k = S_k G_k w_k$, S_k is selected such that $(S_k S_k^T)^{-1} = G_k Q G_k^T$ then $Cov(\bar{w}_k) = I$ and

$x_{k+1} = A_k x_k + S_k^{-1} \bar{w}_k$ (Proof, exercise)

Fact 2: Let $y_k = C_k x_k + H_k v_k$ where $Cov(v_k) = R$. Define $\bar{y}_k = M_k y_k$

$\bar{y}_k = \bar{C}_k x_k + \bar{v}_k \quad Cov(\bar{v}_k) = I$

Proof: $\bar{y}_k = M_k y_k = M_k C_k x_k + \underbrace{M_k H_k v_k}_{\bar{v}_k}$

$\therefore Cov(M_k H_k v_k) = I$

$M_k H_k R H_k^T M_k^T = I$

Choose M_k such that $(M_k^T M_k)^{-1} = H_k R H_k^T$

$\underbrace{M_k^T M_k}_{I} \underbrace{H_k R H_k^T}_{M_k^T M_k} \underbrace{M_k^T M_k}_{I} = M_k^T M_k$ ✓

System Model: $\quad Q$: diagonal, $R = r$

$x_{k+1} = A_k x_k + G_k w_k$

$y_k = C_k x_k + H_k v_k \longleftarrow$ 1 dimensional

Time Update:

$\Sigma_{k+1|k} = A_k \Sigma_{k|k} A_k^T + G_k Q G_k^T$

$\Sigma_{k|k} = U . D . U^T$ where U: unit upper triangular

$$U = \begin{bmatrix} 1 & x & \cdots & x \\ 0 & 1 & \cdots & x \\ \vdots & & & \\ 0 & 0 & \cdots & 1 \end{bmatrix} ; D = \begin{bmatrix} D_{11} & & 0 \\ & \ddots & \\ 0 & & D_{nn} \end{bmatrix}$$

Our aim is to write $\Sigma_{k+1|k} = \tilde{U} \tilde{D} \tilde{U}^T$

$$\tilde{U} = \begin{bmatrix} 1 & & x \\ & \ddots & \\ 0 & & 1 \end{bmatrix}, \tilde{D} = \begin{bmatrix} \tilde{D}_{11} & & 0 \\ & \ddots & \\ 0 & & \tilde{D}_{nn} \end{bmatrix}$$

Replace A_k with A for simplicity.

$\tilde{U} \tilde{D} \tilde{U}^T = A U D U^T A^T + G Q G^T \quad (*)$

Aim is to find $\tilde{U}, \tilde{D}$ such that they have the required form and they satisfy $(*)$

$\tilde{U} \tilde{D} \tilde{U} = A U D U^T A^T + G Q G^T$

$$= \underbrace{\begin{bmatrix} AU & G \end{bmatrix}}_{W} \underbrace{\begin{bmatrix} D & 0 \\ 0 & Q \end{bmatrix}}_{\bar{D}} \begin{bmatrix} (AU)^T \\ G^T \end{bmatrix}$$

$\Rightarrow \tilde{U} \tilde{D} \tilde{U} = W \bar{D} W^T$ where $W = [AU \;\; G]$

$$\bar{D} = \begin{bmatrix} D & 0 \\ 0 & Q \end{bmatrix}$$

Basic Idea: Let $\tilde{U}$ be an arbitrary unit triangular matrix. Then let $W = \tilde{U} V$

$$\tilde{U}\tilde{D}\tilde{U}^T = \tilde{U}V\bar{D}V^T\tilde{U}^T$$

$\tilde{U}$ is the required matrix if $\underbrace{V\bar{D}V^T}_{\tilde{D}}$ is a diagonal matrix

Note that;

1) $\tilde{U}$: upper unit triangular

2) $W = \tilde{U}V$

3) $V\bar{D}V^T$ should be a diagonal matrix

Let $V = \begin{bmatrix} V_1^T \\ V_2^T \\ \vdots \\ V_n^T \end{bmatrix}$ $W = \begin{bmatrix} W_1^T \\ W_2^T \\ \vdots \\ W_n^T \end{bmatrix}$

$$\begin{bmatrix} V_1^T \\ \vdots \\ V_n^T \end{bmatrix} \begin{bmatrix} \bar{D}_{11} & & \\ & \ddots & \\ & & \bar{D}_{nn} \end{bmatrix} \begin{bmatrix} V_1 & \cdots & V_n \end{bmatrix} : \text{diagonal}$$

ij^{th} element

$$\Rightarrow V_i^T D_{ii} V_j = 0 \quad \forall i,j \quad i \neq j$$

$$\begin{bmatrix} W_1^T \\ \vdots \\ W_n^T \end{bmatrix} = \begin{bmatrix} 1 & \tilde{U}_{12} & \cdots & \tilde{U}_{1n} \\ \vdots & \ddots & & \\ 0 & & \ddots & 1 \end{bmatrix} \begin{bmatrix} V_1^T \\ \vdots \\ V_n^T \end{bmatrix}$$

$$\Rightarrow W_n^T = V_n^T$$

$$W_{n-1}^T = V_{n-1}^T + U_{n-1,n} V_n^T$$

$$W_{n-2}^T = V_{n-2}^T + U_{n-2,n-1} V_{n-1}^T + U_{n-2,n} V_n^T$$

$$\text{Span}\{V_n, \ldots, V_j\} = \text{Span}\{w_n, \ldots, w_j\}$$

$$\langle x, y \rangle = x'\bar{D}y$$

$$\langle V_i, V_j \rangle = 0$$

Gram-Shmidth orthogonalization

$$V_n = W_n$$

$$V_{n-1} = W_{n-1} - \frac{\langle W_{n-1}, V_n \rangle}{\langle V_n, V_n \rangle} \cdot V_n$$

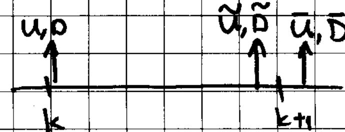
$$V_{n-2} = W_{n-2} - \frac{\langle W_{n-2}, V_{n-1} \rangle}{\langle V_{n-1}, V_{n-1} \rangle} V_{n-1} - \frac{\langle W_{n-2}, V_n \rangle}{\langle V_n, V_n \rangle} V_n$$

Measurement Update:

$$\Sigma_{k+1|k+1} = \Sigma_{k+1|k} - \Sigma_{k+1|k} C_{k+1}^T \left(C_{k+1} \Sigma_{k+1|k} C_{k+1}^T + H_{k+1} R H_{k+1} \right)^{-1} C_{k+1} \Sigma_{k+1|k}$$

$$L_{k+1} = \Sigma_{k+1|k} C_{k+1}^T \frac{1}{C_{k+1} \Sigma_{k+1|k} C_{k+1}^T + H_k H_k^T r}$$

$$\Rightarrow \Sigma_{k+1|k+1} = \Sigma_{k+1|k} - \frac{\Sigma_{k+1|k} C_{k+1}^T C_{k+1} \Sigma_{k+1|k}}{C_{k+1} \Sigma_{k+1|k} C_{k+1}^T + r}$$



$$\Sigma_{k+1|k} = \tilde{U}\tilde{D}\tilde{U}^T$$

$$\Sigma_{k+1|k+1} = \bar{U}\bar{D}\bar{U}^T$$

Problem: Find $\bar{U}, \bar{D}$ such that $\bar{U}$ is unit upper triangular, $\bar{D}$ is diagonal such that

$$\bar{U}\bar{D}\bar{U}^T = \tilde{U}\tilde{D}\tilde{U}^T - \frac{\tilde{U}\tilde{D}\tilde{U}^T C^T C \tilde{U}\tilde{D}\tilde{U}^T}{C\tilde{U}\tilde{D}\tilde{U}^T C^T + r}$$

$$\bar{U}\bar{D}\bar{U}^T = \tilde{U} \underbrace{\left(\tilde{D} - \frac{\tilde{D}\tilde{U}^T C^T C \tilde{U}\tilde{D}}{C\tilde{U}\tilde{D}\tilde{U}^T C^T + r} \right)}_{\hat{U}\hat{D}\hat{U}^T} \tilde{U}^T$$

$$\text{Let } \bar{U} = \tilde{U}\hat{U} = \begin{bmatrix} 1 & \ddots & x \\ 0 & & 1 \end{bmatrix} \begin{bmatrix} 1 & \ddots & x \\ 0 & & 1 \end{bmatrix} = \begin{bmatrix} 1 & & x \\ 0 & \ddots & 1 \end{bmatrix}$$

$$\bar{D} = \hat{D}$$

Solve $\hat{U}\hat{D}\hat{U}^T$

$$F = \tilde{U}^T C^T = \begin{bmatrix} 1 & \cdots & x \\ & \ddots & \\ 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} x \\ \vdots \\ x \end{bmatrix} = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}$$

$$V = \tilde{D}F = \begin{bmatrix} \tilde{D}_{11} & & \\ & \ddots & \\ & & \tilde{D}_{nn} \end{bmatrix} \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} = \begin{bmatrix} \tilde{D}_{11} f_1 \\ \vdots \\ \tilde{D}_{nn} f_n \end{bmatrix}$$

$$\Rightarrow \hat{U}\hat{D}\hat{U}^T = \tilde{D} - \frac{V.V^T}{\alpha} \quad \text{where } \alpha \triangleq C\tilde{U}\tilde{D}\tilde{U}^T C' + r$$