Polyphase Decomposition

The Decomposition

- Consider an arbitrary sequence $\{x[n]\}$ with a z-transform $X(z)$ given by

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

- We can rewrite $X(z)$ as

$$X(z) = \sum_{k=0}^{M-1} z^{-k} X_k(z^M)$$

where

$$X_k(z) = \sum_{n=-\infty}^{\infty} x_k[n] z^{-n} = \sum_{n=-\infty}^{\infty} x[Mn+k] z^{-n}$$

$$0 \le k \le M-1$$

Polyphase Decomposition

- The subsequences $\{x_k[n]\}$ are called the ***polyphase components*** of the parent sequence $\{x[n]\}$
- The functions $X_k(z)$, given by the z-transforms of $\{x_k[n]\}$, are called the ***polyphase components*** of $X(z)$

Polyphase Decomposition

- The relation between the subsequences $\{x_k[n]\}$ and the original sequence $\{x[n]\}$ are given by

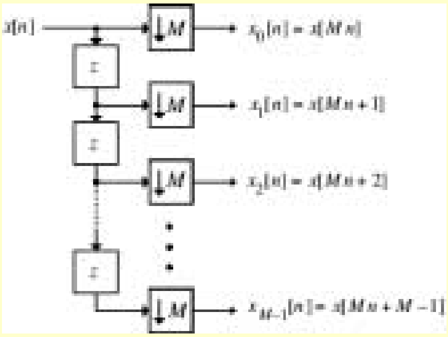
$$x_k[n] = x[Mn+k], \quad 0 \le k \le M-1$$

- In matrix form we can write

$$X(z) = \begin{bmatrix} 1 & z^{-1} & \cdots & z^{-(M-1)} \end{bmatrix} \begin{bmatrix} X_0(z^M) \\ X_1(z^M) \\ \vdots \\ X_{M-1}(z^M) \end{bmatrix}$$

Polyphase Decomposition

- A multirate structural interpretation of the polyphase decomposition is given below



Polyphase Decomposition

- The ***polyphase decomposition of an FIR transfer function*** can be carried out by inspection
- For example, consider a length-9 FIR transfer function:

$$H(z) = \sum_{n=0}^{8} h[n] z^{-n}$$

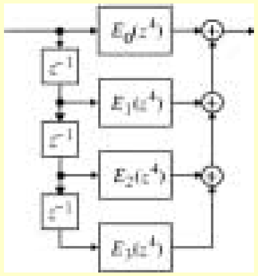
Polyphase Decomposition

- Its 4-branch polyphase decomposition is given by

$$H(z) = E_0(z^4) + z^{-1}E_1(z^4) + z^{-2}E_2(z^4) + z^{-3}E_3(z^4)$$

where

$$E_0(z) = h[0] + h[4]z^{-1} + h[8]z^{-2}$$
$$E_1(z) = h[1] + h[5]z^{-1}$$
$$E_2(z) = h[2] + h[6]z^{-1}$$
$$E_3(z) = h[3] + h[7]z^{-1}$$



Polyphase Decomposition

- The ***polyphase decomposition of an IIR transfer function*** $H(z) = P(z)/D(z)$ is not that straightforward
- One way to arrive at an M-branch polyphase decomposition of $H(z)$ is to express it in the form $P'(z)/D'(z^M)$ by multiplying $P(z)$ and $D(z)$ with an appropriately chosen polynomial and then apply an M-branch polyphase decomposition to $P'(z)$

Polyphase Decomposition

- ***Example*** - Consider $H(z) = \dfrac{1-2z^{-1}}{1+3z^{-1}}$
- To obtain a 2-band polyphase decomposition we rewrite $H(z)$ as

$$H(z) = \frac{(1-2z^{-1})(1-3z^{-1})}{(1+3z^{-1})(1-3z^{-1})} = \frac{1-5z^{-1}+6z^{-2}}{1-9z^{-2}} = \frac{1+6z^{-2}}{1-9z^{-2}} + \frac{-5z^{-1}}{1-9z^{-2}}$$

- Therefore,

$$H(z) = E_0(z^2) + z^{-1}E_1(z^2)$$

where

$$E_0(z) = \frac{1+6z^{-1}}{1-9z^{-1}}, \quad E_1(z) = \frac{-5}{1-9z^{-1}}$$

Polyphase Decomposition

- Note: The above approach increases the overall order and complexity of $H(z)$
- However, when used in certain multirate structures, the approach may result in a more computationally efficient structure
- An alternative more attractive approach is discussed in the following example

Polyphase Decomposition

- ***Example*** - Consider the transfer function of a 5-th order Butterworth lowpass filter with a 3-dB cutoff frequency at 0.5π:

$$H(z) = \frac{0.0527864\,(1+z^{-1})^5}{1+0.633436854z^{-2}+0.0557281z^{-4}}$$

- It is easy to show that $H(z)$ can be expressed as

$$H(z) = \frac{1}{2}\left[\left(\frac{0.105573+z^{-2}}{1+0.105573z^{-2}}\right)+z^{-1}\left(\frac{0.52786+z^{-2}}{1+0.52786z^{-2}}\right)\right]$$

Polyphase Decomposition

- Therefore $H(z)$ can be expressed as

$$H(z) = E_0(z^2) + z^{-1}E_1(z^2)$$

where

$$E_0(z) = \frac{1}{2}\left(\frac{0.105573+z^{-1}}{1+0.105573z^{-1}}\right)$$

$$E_1(z) = \frac{1}{2}\left(\frac{0.52786+z^{-1}}{1+0.52786z^{-1}}\right)$$

Polyphase Decomposition

- Note: In the above polyphase decomposition, branch transfer functions $E_i(z)$ are ***stable allpass functions***
- Moreover, the decomposition has not increased the order of the overall transfer function $H(z)$

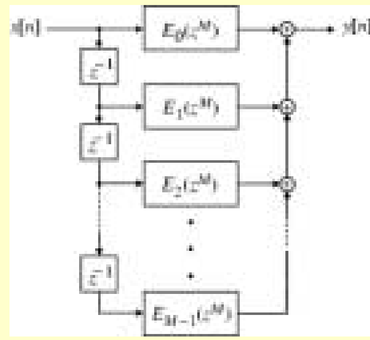
FIR Filter Structures Based on Polyphase Decomposition

- We shall demonstrate later that a parallel realization of an FIR transfer function $H(z)$ based on the polyphase decomposition can often result in computationally efficient multirate structures
- Consider the M-branch ***Type I polyphase decomposition*** of $H(z)$:

$$H(z) = \sum_{k=0}^{M-1} z^{-k} E_k(z^M)$$

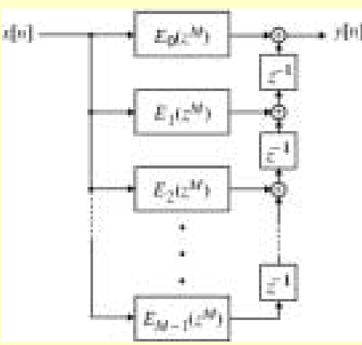
FIR Filter Structures Based on Polyphase Decomposition

- A direct realization of $H(z)$ based on the ***Type I polyphase decomposition*** is shown below



FIR Filter Structures Based on Polyphase Decomposition

- The transpose of the Type I polyphase FIR filter structure is indicated below



FIR Filter Structures Based on Polyphase Decomposition

- An alternative representation of the transpose structure shown on the previous slide is obtained using the notation

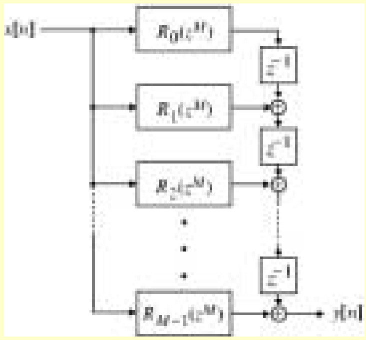
$$R_\ell(z^M) = E_{M-1-\ell}(z^M), \qquad 0 \le \ell \le M-1$$

- Substituting the above notation in the Type I polyphase decomposition we arrive at the ***Type II polyphase decomposition***:

$$H(z) = \sum_{\ell=0}^{M-1} z^{-(M-1-\ell)} R_\ell(z^M)$$

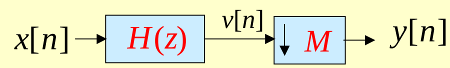
FIR Filter Structures Based on Polyphase Decomposition

- A direct realization of $H(z)$ based on the ***Type II polyphase decomposition*** is shown below



Computationally Efficient Decimators

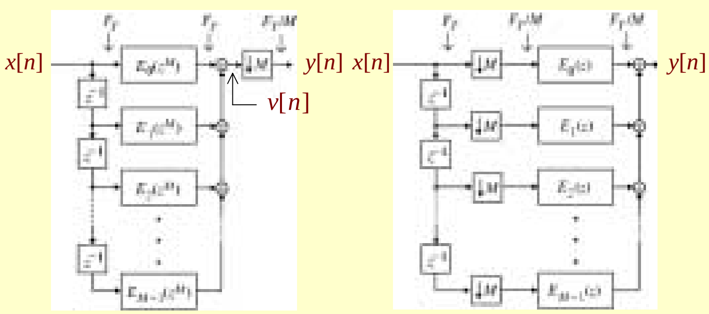
- Consider first the single-stage factor-of-M decimator structure shown below



- We realize the lowpass filter $H(z)$ using the Type I polyphase structure as shown on the next slide

Computationally Efficient Decimators

- Using the ***cascade equivalence #1*** we arrive at the computationally efficient decimator structure shown below on the right



Decimator structure based on Type I polyphase decomposition

Computationally Efficient Decimators

- To illustrate the computational efficiency of the modified decimator structure, assume $H(z)$ to be a length-N structure and the input sampling period to be $T = 1$
- Now the decimator output $y[n]$ in the original structure is obtained by down-sampling the filter output $v[n]$ by a factor of M

Computationally Efficient Decimators

- It is thus necessary to compute $v[n]$ at

$$n = \ldots, -2M, -M, 0, M, 2M, \ldots$$

- Computational requirements are therefore N multiplications and $(N-1)$ additions per output sample being computed
- However, as n increases, stored signals in the delay registers change

Computationally Efficient Decimators

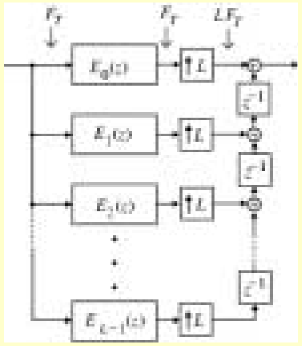
- Hence, all computations need to be completed in one sampling period, and for the following $(M-1)$ sampling periods the arithmetic units remain idle
- The modified decimator structure also requires N multiplications and $(N-1)$ additions per output sample being computed

Computationally Efficient Decimators and Interpolators

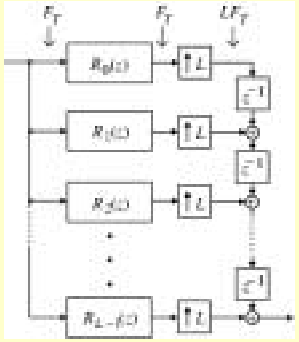
- However, here the arithmetic units are operative at all instants of the output sampling period which is M times that of the input sampling period
- Similar savings are also obtained in the case of the interpolator structure developed using the polyphase decomposition

Computationally Efficient Interpolators

- Figures below show the computationally efficient interpolator structures



Interpolator based on
Type I polyphase decomposition



Interpolator based on
Type II polyphase decomposition

Computationally Efficient Decimators and Interpolators

- More efficient interpolator and decimator structures can be realized by exploiting the symmetry of filter coefficients in the case of linear-phase filters $H(z)$
- Consider for example the realization of a factor-of-3 ($M = 3$) decimator using a length-12 Type 1 linear-phase FIR lowpass filter

Computationally Efficient Decimators and Interpolators

- The corresponding transfer function is

$$H(z) = h[0] + h[1]z^{-1} + h[2]z^{-2} + h[3]z^{-3} + h[4]z^{-4} + h[5]z^{-5}$$
$$+ h[5]z^{-6} + h[4]z^{-7} + h[3]z^{-8} + h[2]z^{-9} + h[1]z^{-10} + h[0]z^{-11}$$

- A conventional polyphase decomposition of $H(z)$ yields the following subfilters:

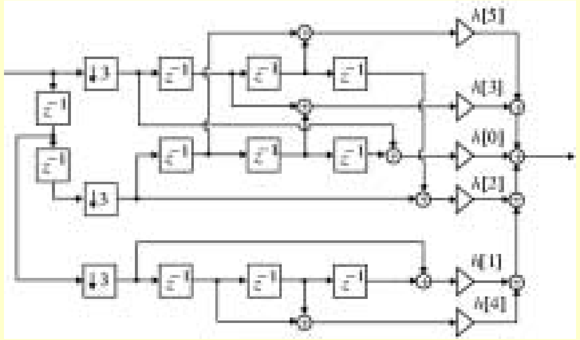
$$E_0(z) = h[0] + h[3]z^{-1} + h[5]z^{-2} + h[2]z^{-3}$$
$$E_1(z) = h[1] + h[4]z^{-1} + h[4]z^{-2} + h[1]z^{-3}$$
$$E_2(z) = h[2] + h[5]z^{-1} + h[3]z^{-2} + h[0]z^{-3}$$

Computationally Efficient Decimators and Interpolators

- Note that $E_1(z)$ still has a symmetric impulse response, whereas $E_0(z)$ is the mirror image of $E_2(z)$
- These relations can be made use of in developing a computationally efficient realization using only 6 multipliers and 11 two-input adders as shown on the next slide

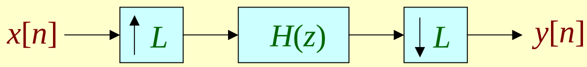
Computationally Efficient Decimators and Interpolators

- Factor-of-3 decimator with a linear-phase decimation filter



A Useful Identity

- The cascade multirate structure shown below appears in a number of applications



- Equivalent time-invariant digital filter obtained by expressing $H(z)$ in its L-term Type I polyphase form $\sum_{k=0}^{L-1} z^{-k} E_k(z^L)$ is shown below