International Trade[1]

Lecture Note: Review of the CES Demand

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[1]The notes are still preliminary and in beta. If you find any typo or mistake, please send it to malfaro@ualberta.ca.

Contents

Notation

This is a derivation

This is some comment

This is a comment on advanced topics which are not part of the course (you can ignore it without loss of continuity regarding the text)

- The symbol ":=" means "by definition".
- I denote vectors by bold lowercase letters (for instance, $\mathbf{x}$) and matrices by bold capital letters (for instance, $\mathbf{X}$).
- To differentiate between the verb "maximize" and the operator "maximum", I denote the former with "max" and the latter with "sup" (i.e., supremum). The same caveat applies to "minimize" and "minimum", where I use "min" and "inf", with the latter indicating infimum.
- "iff" means "if and only if"
- $\exp(x)$ is the function e^x.
- Random variables are denoted with a bar below. For instance, $\underline{x}$.

These notes contain hyperlinks in blue and red text. If you are using Adobe Acrobat Reader, you can click on the link and easily navigate back by pressing Alt+Left Arrow.

1 Introduction

This note introduces the **constant elasticity of substitution** (**CES**) demand system. This demand system is widely used not only in International Trade, but also in several other fields such as Macroeconomics.

The popularity of the CES stems from its tractability for theoretical and empirical analysis. Additionally, it is based on homothetic preferences, ensuring several convenient properties.[1]

2 Setup

We consider an industry that consists of a differentiated good. This good comprises a discrete set of varieties $\Omega := \{1, 2, ..., M\}$, where each variety $\omega \in \Omega$ is characterized by its price p_ω and a parameter $z_\omega > 0$ referred to as variety ω's appeal.

We refer to z_ω as appeal, as we define it as broader concept than quality: it captures any non-price aspect of ω (objective or subjective) that induces a consumer to increase its quantity demanded.

The demand is determined by the decisions of a representative consumer. This agent allocates an exogenous expenditure y to the industry. Furthermore, the preferences are represented by a CES utility function:

$$U\left[(q_\omega)_{\omega\in\Omega}\right] := \left[\sum_{\omega\in\Omega} (z_\omega)^{\frac{\delta}{\sigma}} (q_\omega)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}, \tag{1}$$

where $\delta > 0$, $\sigma > 1$, and q_ω is the quantity demanded of ω.

The parameter σ is known as the **elasticity of substitution**, where $\sigma > 1$ ensures that varieties are (imperfect) substitutes, rather than complements.

Depending on the value σ, the CES gives rise different utility functions as special cases:

- If $\sigma \to \infty$, then U is the linear utility function.
- If $\sigma \to 0$, then U is the Leontief utility function (perfect complements).
- If $\sigma \to 1$, then U is the Cobb Douglas utility function.

[1] There are two common demand systems used in Economics for empirical work: the CES and the Logit demand. The latter is primarily used in Industrial Organization and resembles the CES in several respects. This is why a deep understanding the CES will help you if you ever use the Logit.

As any monotone transformation still represents the same preferences, it is common to find alternative representations of the CES. For instance, under the simplifying assumption that $z_\omega = 1$ for all $\omega \in \Omega$,

$$U\left[(q_\omega)_{\omega\in\Omega}\right] := \sum_{\omega\in\Omega} q_j^\rho,$$
$$U\left[(q_\omega)_{\omega\in\Omega}\right] := \left(\sum_{\omega\in\Omega} q_j^\rho\right)^{\frac{1}{\rho}}.$$

The latter functional form is equivalent to (1) when assuming $\rho := \frac{\sigma-1}{\sigma}$. We opted for our specification in terms of σ, as it expresses all our results in terms of the elasticity of substitution.

2.1 Optimal Choices

The consumer's optimization problem is

$$\max_{(q_\omega)_{\omega\in\Omega}} U\left[(q_\omega)_{\omega\in\Omega}\right] = \left[\sum_{\omega\in\Omega} (z_\omega)^{\frac{\delta}{\sigma}} (q_\omega)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} \text{ subject to } y = \sum_{\omega\in\Omega} p_\omega q_\omega. \tag{2}$$

Solving this problem, the optimal quantity demanded of variety ω is

$$q_\omega = y\frac{(z_\omega)^\delta (p_\omega)^{-\sigma}}{\mathbb{P}^{1-\sigma}}, \tag{3}$$

where $\mathbb{P}$ is a price index defined by

$$\mathbb{P} := \left[\sum_{\omega\in\Omega} (z_\omega)^\delta (p_\omega)^{1-\sigma}\right]^{\frac{1}{1-\sigma}}. \tag{4}$$

The CES utility function has no corner solutions and has a unique interior solution. To characterize the solution through the first-order conditions, note that maximizing $\left[\sum_{\omega\in\Omega} (z_\omega)^{\frac{\delta}{\sigma}} (q_\omega)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}$ is equivalent to maximizing $\sum_{\omega\in\Omega} (z_\omega)^{\frac{\delta}{\sigma}} (q_\omega)^{\frac{\sigma-1}{\sigma}}$. Hence, the Lagrangian is

$\mathscr{L} = \sum_{\omega\in\Omega} (z_\omega)^{\frac{\delta}{\sigma}} (q_\omega)^{\frac{\sigma-1}{\sigma}} + \lambda\left(Y - \sum_{\omega\in\Omega} p_\omega q_\omega\right)$,

where λ is the Lagrange multiplier.

The first-order condition for variety ω is

$(z_\omega)^{\frac{\delta}{\sigma}} \frac{\sigma-1}{\sigma} (q_\omega)^{\frac{-1}{\sigma}} - \lambda p_\omega = 0$

$\Rightarrow \frac{\sigma-1}{\sigma} (q_\omega)^{\frac{-1}{\sigma}} = -\lambda p_\omega (z_\omega)^{-\frac{\delta}{\sigma}}$

$\Rightarrow \left(\frac{\sigma-1}{\sigma}\right)^{-\sigma} q_\omega = -(\lambda)^{-\sigma} (p_\omega)^{-\sigma} (z_\omega)^\delta$

where we have simplified the notation by directly referring to the optimal solution as q_ω.

There are M first-order conditions, one for each $\omega \in \Omega$. Take $\omega', \omega'' \in \Omega$, and divide their corresponding first-order conditions:

$$\frac{q_{\omega'}}{q_{\omega''}} = \left(\frac{z_{\omega'}}{z_{\omega''}}\right)^\delta \left(\frac{p_{\omega'}}{p_{\omega''}}\right)^{-\sigma} \Rightarrow \frac{p_{\omega'} q_{\omega'}}{p_{\omega''} q_{\omega''}} = \left(\frac{z_{\omega'}}{z_{\omega''}}\right)^\delta \left(\frac{p_{\omega'}}{p_{\omega''}}\right)^{1-\sigma}$$

Now, fix ω'' and sum over ω',
$\sum_{\omega'\in\Omega} \frac{p_{\omega'}q_{\omega'}}{p_{\omega''}q_{\omega''}} = \sum_{\omega'\in\Omega} \left(\frac{p_{\omega'}}{p_{\omega''}}\right)^{1-\sigma}$
$\Rightarrow \frac{1}{p_{\omega''}q_{\omega''}} \sum_{\omega'\in\Omega} p_{\omega'}q_{\omega'} = \left(\frac{1}{z_{\omega''}}\right)^{\delta} \left(\frac{1}{p_{\omega''}}\right)^{1-\sigma} \sum_{\omega'\in\Omega} (z_{\omega'})^{\delta} (p_{\omega'})^{1-\sigma}$.
Using that $y = \sum_{\omega'\in\Omega} p_{\omega'}q_{\omega'}$ and defining $\mathbb{P} := \left[\sum_{\omega'\in\Omega} (z_{\omega'})^{\delta} (p_{\omega'})^{1-\sigma}\right]^{\frac{1}{1-\sigma}}$, we get $q_\omega = y\,(z_\omega)^{\delta} (p_\omega)^{-\sigma} \mathbb{P}^{\sigma-1}$.

Likewise, the expenditure on variety ω is defined by $r_\omega := p_\omega q_\omega$, where r stands for revenue. It is given by

$$r_\omega = y\frac{(z_\omega)^{\delta} (p_\omega)^{1-\sigma}}{\mathbb{P}^{1-\sigma}}.$$

This expression allows us to identify ω's market share, which is formally defined by $s_\omega := \frac{y_\omega}{y}$ and given by

$$\begin{aligned} s_\omega &= \frac{(z_\omega)^{\delta} (p_\omega)^{1-\sigma}}{\mathbb{P}^{1-\sigma}}, \\ &= \frac{(z_\omega)^{\delta} (p_\omega)^{1-\sigma}}{\sum_{\omega\in\Omega} (z_\omega)^{\delta} (p_\omega)^{1-\sigma}}. \end{aligned} \tag{5}$$

Market shares play a crucial role for models based on the CES demand, as several key expressions can be rewritten in terms of them. Indeed, market shares create a direct link between these empirical models and the data.

3 CES as Representation of Differentiated Goods

The CES captures the demand for a good that is both horizontally and vertically differentiated. Next, we analyze each form of differentiation separately.

3.1 Horizontal Differentiation

To simplify the explanation of horizontal differentiation, let's assume that all varieties have the same unitary appeal. Formally, $z_\omega = 1$ for each $\omega \in \Omega$. When a good is horizontally differentiated, the agent perceives **each variety as unique**. This implies that there is no notion of one variety being superior or inferior to another—varieties are simply different from one another.

Furthermore, CES preferences exhibit strict convexity, a property that is known as **love for variety** when the number of varieties is endogenous. This feature describes the agent's attitude towards varieties that are distinct: the consumer prefers diversifying consumption across varieties, rather than only consuming a strict subset of varieties. Love for variety ensures that any new variety that becomes available will be consumed.

Typical scenarios conceived by this approach involve a preference for different meals daily or the diversification o clothing colors. This characterization contrasts with an agent having strong preferences for a few meals or colors, who additionally may not consume new varieties introduced to the market.

To show that the CES represents strictly convex preferences, we need to show that indifference curves are strictly convex.
Indifference curves are the combinations of goods that provide the same utility. To derive them, we use that

$$\mathrm{d}U = \sum_{\omega \in \Omega} \frac{\partial U}{\partial q_\omega} \mathrm{d}q_\omega$$

where $\frac{\partial U}{\partial q_\omega} = \left(\sum_{\omega \in \Omega} q_\omega^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}-1} q_\omega^{\frac{\sigma-1}{\sigma}-1}$.
Fix the utility, so that $\mathrm{d}U = 0$. Moreover, suppose that $\mathrm{d}q_{\omega'} \neq 0$ and $\mathrm{d}q_{\omega''} \neq 0$, with $\mathrm{d}q_\omega = 0$ for any $\omega \in \Omega \setminus \{\omega', \omega''\}$.
Then,

$$0 = \left[\left(\sum_{\omega \in \Omega} q_\omega^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}-1} q_{\omega'} \right] \mathrm{d}q_{\omega'} + \left[\left(\sum_{\omega \in \Omega} q_\omega^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}-1} q_{\omega''} \right] \mathrm{d}q_{\omega''}.$$

From this we get,

$$\frac{\mathrm{d}q_{\omega'}}{\mathrm{d}q_{\omega''}} = -\left(\frac{q_{\omega''}}{q_{\omega'}} \right)^{-\frac{1}{\sigma}},$$

and therefore $\frac{\mathrm{d}q_{\omega'}}{\mathrm{d}q_{\omega''}} < 0$.
Finally, indifference curves are convex since

$$\frac{\mathrm{d}^2 q_{\omega'}}{(\mathrm{d}q_{\omega''})^2} = \frac{1}{\sigma} \frac{1}{q_{\omega''}} \left(\frac{q_{\omega''}}{q_{\omega'}} \right)^{-\frac{1}{\sigma}} > 0.$$

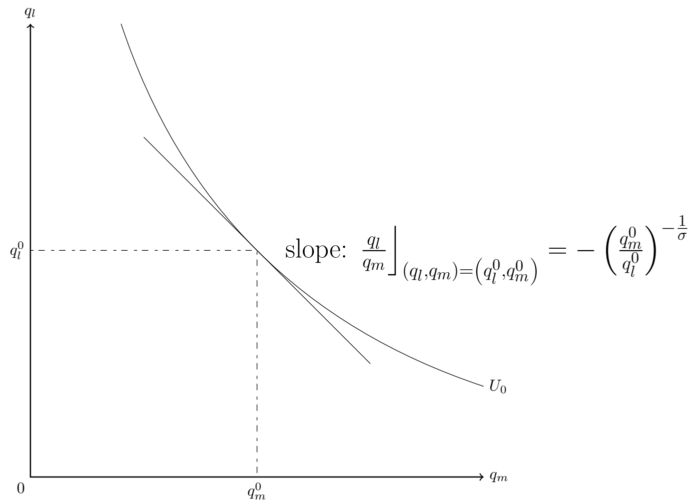
The intensity in which a consumer perceives varieties as different is governed by the elasticity of substitution, σ. This parameter measures the degree of substitution of a variety relative to the rest of varieties.

Since σ is constant for all varieties, the degree of substitution is the same for every good, and in particular not affected by the number of varieties available. This means that there are no crowding effects: when a firm enters an industry and introduces a new variety, the degree of substitution between varieties is unchanged.

To formally show the role of σ, consider two varieties l and m. The indifference curves are then given by

$$\frac{\mathrm{d}q_l}{\mathrm{d}q_m} = -\left(\frac{q_m}{q_l} \right)^{-\frac{1}{\sigma}}.$$

Graphically:



In particular, the indifference curve when an agent does not consume the variety m (i.e., $q_m \to 0$) is

$$\lim_{q_m \to 0} \frac{\partial q_l}{\partial q_m} = -\infty.$$

The equation indicates that the consumer is willing to give up an infinitely large amount of variety l to consume m. This formalizes that the agent finds any variety valuable, reflecting a preference for diversification. In particular, the consumer will buy a new variety regardless of the price.

We can also show that the introduction of new varieties is valuable in an alternative way. Suppose that the agent is consuming an identical quantity $\widetilde{q}$ of each variety:

$$\begin{aligned} U_1 &= \left(\sum_{\omega \in \Omega} \widetilde{q}^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}, \\ &= M^{\frac{\sigma}{\sigma-1}} \widetilde{q}. \end{aligned}$$

Let's compare this value to a scenario where we halve the consumption per variety, but double the varieties available. Formally, the number of varieties becomes $2M$ and the quantities per variety $\frac{\widetilde{q}}{2}$, so that

$$U_2 = (2M)^{\frac{\sigma}{\sigma-1}} \frac{\widetilde{q}}{2}.$$

Using that $\sigma > 1$, it is easy to see that

$$(2M)^{\frac{\sigma}{\sigma-1}} \frac{\widetilde{q}}{2} > M^{\frac{\sigma}{\sigma-1}} \widetilde{q},$$

and therefore $U_2 > U_1$. The result captures intuitively the idea behind *love of variety*:

the agent prefers a basket with a greater variety of items, rather than a basket containing more of the same original items.

3.2 Vertical Differentiation

When a good is vertically differentiated, the agent perceives some varieties as superior to others. A typical example is computers where, other things equal, faster process are always preferred by consumers.

The intensity in which a good is preferred relative to others is captured by the parameter z_ω. Specifically, (1) indicates that a higher z_ω yields more utility from consuming variety ω.

The fact z_ω enters directly into the utility function explains why this variable is referred to appeal rather than quality: it encompasses all aspects, beyond prices that impact the consumer's decision. As such, appeal could represent not only objective features overhauling a variety, but also psychological factors. Overall, the consumer derives more utility from consuming ω when z_ω is higher, and we remain agnostic about why this occurs.

The terms $(z_\omega)_{\omega\in\Omega}$ capture vertical differentiation in relative terms, rather than absolute terms. This means that what matters for variety ω is z_ω in comparison to $(z_\omega)_{\Omega\backslash\omega}$. This can be easily observed by noting that any monotone transformation of the utility function still represents the same preferences. Hence, we could work instead with the following utility function

$$U\left[(q_\omega)_{\omega\in\Omega}\right] = \frac{\left[\sum_{\omega\in\Omega}(z_\omega)^{\frac{\delta}{\sigma}}(q_\omega)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}}{\left(\sum_{\omega'\in\Omega} z_{\omega'}\right)^{\frac{\delta}{\sigma}}}.$$

This means we can express the utility function through a normalized parameter $\widetilde{z}_\omega := \frac{z_\omega}{\sum_{\omega'\in\Omega} z_{\omega'}} \in (0,1)$, such that

$$U\left[(q_\omega)_{\omega\in\Omega}\right] := \left[\sum_{\omega\in\Omega}(\widetilde{z}_\omega)^{\frac{\delta}{\sigma}}(q_\omega)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}.$$

The degree to which vertical aspects influence consumption is governed by the parameter δ. This can be appreciated in the appeal elasticity of demand:

$$\frac{\mathrm{d}\ln q_\omega}{\mathrm{d}\ln z_\omega} = \delta\left(1 - s_\omega\right),$$

with $\frac{\mathrm{d}\ln q_\omega}{\mathrm{d}\ln z_\omega} = \delta$ when firms have a negligible market share.

We express q_ω as a function $q_\omega\left[p_\omega, z_\omega, \mathbb{P}\left((p_{\omega'}, z_{\omega'})_{\omega'\in\Omega}\right)\right]$. Taking logs of (3),

$$\ln q_\omega = \ln y + \delta \ln z_\omega - \sigma \ln p_\omega - (1-\sigma)\ln \mathbb{P},$$

and so

$$\frac{\mathrm{d}\ln q_\omega}{\mathrm{d}\ln z_\omega} = \delta - (1-\sigma)\frac{\partial \ln \mathbb{P}}{\partial \ln z_\omega}.$$

Taking logs of (4),

$$\ln \mathbb{P} = \frac{1}{1-\sigma}\ln\left[\sum_{\omega\in\Omega}(z_\omega)^\delta (p_\omega)^{1-\sigma}\right],$$

and taking the derivative with respect to z_ω,

$$\frac{\partial \ln \mathbb{P}}{\partial \ln z_\omega} = \frac{\delta}{1-\sigma}\frac{(z_\omega)^\delta (p_\omega)^{1-\sigma}}{\sum_{\omega\in\Omega}(z_\omega)^\delta (p_\omega)^{1-\sigma}}.$$

Finally, using (5), we determine that

$$\frac{\partial \ln \mathbb{P}}{\partial \ln z_\omega} = \frac{\delta}{1-\sigma}s_\omega,$$

and so $\frac{\mathrm{d}\ln q_\omega}{\mathrm{d}\ln z_\omega} = \delta(1-s_\omega)$.

In the literature, there are more specific versions of the CES that account for vertical differentiation. They can be understood as special cases by defining δ accordingly, implicitly assuming a given importance of appeal aspects for consumers.

One of these ways is by defining (1) with $\delta = \sigma - 1$, so that utility becomes

$$U\left[(q_\omega)_{\omega\in\Omega}\right] := \left[\sum_{\omega\in\Omega}(z_\omega q_\omega)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}.$$

Variety ω's optimal demand implies an expenditure on ω given by

$$r_\omega = y\frac{(p_\omega/z_\omega)^{1-\sigma}}{\mathbb{P}^{1-\sigma}},$$

where $\mathbb{P} := \left[\sum_{\omega\in\Omega}(p_\omega/z_\omega)^{1-\sigma}\right]^{\frac{1}{1-\sigma}}$.

This variant of the CES utility captures scenarios where consumers decide according to **the price per unit of quality provided by the variety**.

4 Price Elasticity of Demand

The price elasticity of variety ω is defined by

$$\varepsilon_\omega := -\frac{\mathrm{d}q_\omega/q_\omega}{\mathrm{d}p_\omega/p_\omega} := -\frac{\mathrm{d}\ln q_\omega}{\mathrm{d}\ln p_\omega},$$

where $q_\omega(p_\omega, \mathbb{P})$ is given by (3). Since $\mathbb{P}$ depends on p_ω due to (4), we get

$$\varepsilon_\omega := -\left[\frac{\partial \ln q_\omega}{\partial \ln p_\omega} + \frac{\partial \ln q_\omega}{\partial \ln \mathbb{P}}\frac{\partial \ln \mathbb{P}}{\partial \ln p_\omega}\right],$$

which equals

$$\varepsilon_\omega = \sigma - (\sigma - 1)\, s_\omega. \tag{6}$$

Taking logs of (3),

$$\ln q_\omega = \ln y + \delta \ln z_\omega - \sigma \ln p_\omega - (1-\sigma)\ln \mathbb{P},$$

and so

$$\frac{\mathrm{d}\ln q_\omega}{\mathrm{d}\ln p_\omega} = -\sigma - (1-\sigma)\frac{\partial \ln \mathbb{P}}{\partial \ln p_\omega}.$$

Taking logs of (4),

$$\ln \mathbb{P} = \frac{1}{1-\sigma}\ln\left[\sum_{\omega\in\Omega}(z_\omega)^\delta (p_\omega)^{1-\sigma}\right],$$

and derivating with respect to p_ω,

$$\frac{\partial \ln \mathbb{P}}{\partial p_\omega} = \frac{1}{1-\sigma}\frac{(1-\sigma)(z_\omega)^\delta (p_\omega)^{-\sigma}}{\sum_{\omega\in\Omega}(z_\omega)^\delta (p_\omega)^{1-\sigma}},$$

$$\Rightarrow \frac{\partial \ln \mathbb{P}}{\partial \ln p_\omega} = \frac{(z_\omega)^\delta (p_\omega)^{1-\sigma}}{\sum_{\omega\in\Omega}(z_\omega)^\delta (p_\omega)^{1-\sigma}}.$$

Finally, using (5), we determine that

$$\frac{\partial \ln \mathbb{P}}{\partial \ln p_\omega} = s_\omega,$$

and the result follows.

Equation (6) implies that a firm with higher market share faces a less elastic demand. Thus, the CES parsimoniously captures market power through a firm's market share. Likewise, (5) determines that a higher market share is explained by a lower price or a higher appeal.

Furthermore, note that s_ω affects the price elasticity through the term $\frac{\partial \ln \mathbb{P}}{\partial \ln p_\omega} = s_\omega$. Moreover, we know that the price index represents the aggregate conditions of the market. Both aspects entail that the strength of firm ω's impact on the industry conditions depends on its market share.

This fact has important implications for models that assume a continuum of varieties. In that case, the effect of ω's price on the price index is negligible, implying that

$$\varepsilon_\omega = \sigma$$

Consequently, *when the number of varieties is infinite, the price elasticity of demand equals the elasticity of substitution.*

5 Welfare

The CES utility function can be shown to represent homothetic preferences, with various properties derived from this assumption. One of these properties is that the indirect utility function can be expressed as real income, $\frac{y}{\mathbb{P}}$.

This means formally that we can always define real numbers $\mathbb{Q}$ and $\mathbb{P}$, such that $\mathbb{Q}\mathbb{P} = y$ and with $\mathbb{Q}$ being the indirect utility function V. In the case of the CES, $\mathbb{Q}$ and $\mathbb{P}$ are given by

$$\mathbb{Q} := \left[\sum_{\omega\in\Omega} (z_\omega)^{\frac{\delta}{\sigma}} q_\omega^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}},$$

$$\mathbb{P} := \left[\sum_{\omega\in\Omega} (z_\omega)^{\delta} (p_\omega)^{1-\sigma}\right]^{\frac{1}{1-\sigma}},$$

where $\mathbb{Q}$ uses the optimal demands given by (3).

The fact that $\mathbb{Q}\mathbb{P} = y$ justifies why $\mathbb{Q}$ is referred to as a **quantity index** and $\mathbb{P}$ as a **price index**: we can interpret $\mathbb{Q}$ as a representative basket of varieties, and $\mathbb{P}$ as the necessary income to buy one basket.

Moreover, $\mathbb{Q}\mathbb{P} = y$ implies that

$$\mathbb{Q} = \frac{y}{\mathbb{P}},$$

with $\mathbb{Q}$ being equal to the consumer's indirect utility function. This means that one unit of utility corresponds to one unit of $\mathbb{Q}$. Formally, $V = 1$ is equivalent to $\mathbb{Q} = 1$. Based on this result, $\mathbb{P}$ also reflects the consumer's valuation of the basket, **as it is the minimum expenditure function to achieve one unit of utility**. In formal terms, $\mathbb{Q} = 1$ when income is given by $y = \mathbb{P}$, implying that $V = 1$.[2]

We can prove directly the indirect utility function equals real income. The indirect utility function V corresponds to the utility function evaluated at the optimal quantities:

$V = \left[\sum_{\omega\in\Omega} (z_\omega)^{\frac{\delta}{\sigma}} (q_\omega)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}$

$\Rightarrow V = \left[\sum_{\omega\in\Omega} (z_\omega)^{\frac{\delta}{\sigma}} \left(\frac{(z_\omega)^{\delta}(p_\omega)^{-\sigma}}{\mathbb{P}^{1-\sigma}} y\right)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}$

$\Rightarrow V = \left[\sum_{\omega\in\Omega} (z_\omega)^{\frac{\delta}{\sigma}} \left(\frac{(z_\omega)^{\delta}(p_\omega)^{-\sigma}}{\mathbb{P}^{1-\sigma}} y\right)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}$

[2]The price index also plays another role. To see this, denote the Lagrange multiplier of the optimization problem (2) by λ^*. Remember that $\lambda^* := \frac{\partial V}{\partial y}$ due to the Envelope Theorem, and so λ^* is the marginal utility of income: the impact on optimal utility of one additional unit of income. The relation between λ^* arises since $\lambda^* = \frac{1}{\mathbb{P}}$, so that $\frac{1}{\mathbb{P}}$ is the marginal utility of income.

$\Rightarrow V = \left[\left(\frac{y}{\mathbb{P}^{1-\sigma}} \right)^{\frac{\sigma-1}{\sigma}} \sum_{\omega\in\Omega} (z_\omega)^\delta (p_\omega)^{1-\sigma} \right]^{\frac{\sigma}{\sigma-1}}$

$\Rightarrow V = \left(\frac{y}{\mathbb{P}^{1-\sigma}} \right) \left[\sum_{\omega\in\Omega} (z_\omega)^\delta (p_\omega)^{1-\sigma} \right]^{\frac{\sigma}{\sigma-1}}$

Given that $\mathbb{P} := \left[\sum_{\omega\in\Omega} (z_\omega)^\delta (p_\omega)^{1-\sigma} \right]^{\frac{1}{\sigma-1}}$ and hence $\left[\sum_{\omega\in\Omega} (z_\omega)^\delta (p_\omega)^{1-\sigma} \right]^{\frac{\sigma}{\sigma-1}} = \mathbb{P}^{-\sigma}$, then $V = \left(\frac{y}{\mathbb{P}^{1-\sigma}} \right) \mathbb{P}^{-\sigma} = \frac{y}{\mathbb{P}}$ and the result follows.

5.1 Welfare Determinants

To keep matters simple, suppose that the prices and quality of each variety are the same. Formally, let $p_\omega = \overline{p}$ and $z_\omega = \overline{z}$ for each ω. Then, the price index given by (4) is

$$\begin{aligned} \mathbb{P} &= \left[\sum_{\omega\in\Omega} (z_\omega)^\delta (p_\omega)^{1-\sigma} \right]^{\frac{1}{1-\sigma}}, \\ &= M^{\frac{1}{1-\sigma}} (\overline{z}_\omega)^{\frac{\delta}{1-\sigma}} \overline{p}_\omega. \end{aligned}$$

Therefore, welfare is

$$\frac{y}{\mathbb{P}} = \mathbb{Q} = \frac{y}{M^{\frac{1}{1-\sigma}} (\overline{z}_\omega)^{\frac{\delta}{1-\sigma}} \overline{p}_\omega}, \tag{7}$$

where the powers of M and $\overline{z}_\omega$ are negative since we suppose $\sigma > 1$.

One unit of the quantity index (i.e., one basket of goods) represents one unit of real income, and hence one unit of utility. Likewise, the price index is the necessary income to obtain one unit of utility, thereby representing the consumer's valuation for one unit of the basket. An implication of this is that **the lower the price index, the lower the income you need to obtain one unit of utility**. Consequently, **a lower price index reflects a higher consumer's valuation of the basket.**

Due to this result, the determinants of the price index completely identify the factors determining the valuation of the basket. Applying logarithms to the definition of the price index,

$$\mathbb{P} = \frac{1}{1-\sigma} \ln M + \frac{\delta}{1-\sigma} \ln \overline{z}_\omega + \ln \overline{p}_\omega,$$

implying that

- $\frac{\partial \ln \mathbb{P}}{\partial \ln M} < 0$: a higher number of varieties decreases the price index, and is hence welfare improving. It reflects that a basket with more varieties is more valuable, as it represents a more diversified basket.

- $\frac{\partial \ln \mathbb{P}}{\partial \ln \bar{z}_\omega} < 0$: a higher appeal of varieties decreases the price index, which increases welfare.
- $\frac{\partial \ln \mathbb{P}}{\partial \ln \bar{z}_\omega} > 0$: a higher price of varieties increases the price index, which reduces welfare.

Note that σ affects the impact of the number of varieties on welfare, as σ captures the intensity in which the consumer loves variety. Specifically, applying logs to $\mathbb{Q}$ and taking the derivative,

$$\frac{\partial \ln \mathbb{Q}}{\partial \ln M} = \frac{1}{\sigma - 1}.$$

This establishes that the impact of M on $\mathbb{Q}$ is lower when σ is higher. The outcome reflects that consumer perceives varieties as less differentiated when σ is higher. In the limit, where $\sigma \to \infty$, the utility function becomes linear, representing a scenario where varieties are seen as perfect substitutes. This explains why $\left.\frac{\partial \ln \mathbb{Q}}{\partial \ln M}\right\rfloor_{\sigma \to \infty} = 0$, as consumption diversification has no value—the consumer's sole concern is total consumption, regardless of whether one or multiple varieties are consumed.

6 A Continuum of Goods

While we have assumed a discrete number M of varieties, it is standard to work with a continuum of varieties. This entails that the number of varieties is infinite, with every point in the interval $[0, M]$ representing a different variety.[3]

The assumption is in particularly adopted in models of monopolistic competition, implying that every variety is negligible for an industry's aggregate conditions. In terms of the CES, this is captured by saying that no firm in isolation is capable of affecting the price index.

Formally, the utility function is

$$U := \left[\int_0^M (z_\omega)^{\frac{\delta}{\sigma}} (q_\omega)^{\frac{\sigma-1}{\sigma}} \, \mathrm{d}\omega \right]^{\frac{\sigma}{\sigma-1}}.$$

[3] It is common to say that there is as a continuum of goods with measure M. This is analogous to say that in the interval $[0, 1]$ there are an infinite number of goods, but the length of the line is equal to 1 and so the measure of goods equals one.

The optimization problem is now

$$\max_{(q_\omega)_{\omega\in[0,M]}} U := \left(\int_0^M (z_\omega)^{\frac{\delta}{\sigma}} (q_\omega)^{\frac{\sigma-1}{\sigma}} \, \mathrm{d}\omega\right)^{\frac{\sigma}{\sigma-1}} \text{ subject to } y = \int_0^M p_\omega q_\omega \, \mathrm{d}\omega,$$

with the same solution as before

$$q_\omega = y \frac{(z_\omega)^\delta (p_\omega)^{-\sigma}}{\mathbb{P}^{1-\sigma}},$$

but with the difference that the price index in the continuum version is

$$\mathbb{P} := \left[\int_0^M (z_\omega)^\delta (p_\omega)^{1-\sigma} \, \mathrm{d}\omega\right]^{\frac{1}{1-\sigma}}.$$

Likewise, the indirect utility function is the same as in the discrete case:

$$V(\mathbb{P}) = \frac{y}{\mathbb{P}}.$$

The optimization problem can be used the solution for the discrete case, and then extending it for a continuum of goods. Alternatively, we can use either calculus of variation or optimal control.

Regarding the former, consider the solution q_ω and an additive perturbation δq_ω. This implies that the Lagrangean can be expressed in terms of $q_\omega + \delta q_\omega$ by,

$\mathscr{L} := \int_0^M (z_\omega)^{\frac{\delta}{\sigma}} [q_\omega + \delta q_\omega]^{\frac{\sigma-1}{\sigma}} \, \mathrm{d}\omega + \lambda \left(Y - \int_0^M p_\omega [q_\omega + \delta q_\omega] \, \mathrm{d}\omega\right)$

$\frac{\partial \mathscr{L}}{\partial \delta} = \frac{\sigma-1}{\sigma} (z_\omega)^{\frac{\delta}{\sigma}} [q_\omega + \delta q_\omega]^{\frac{\sigma-1}{\sigma}-1} q_\omega - \lambda q_\omega p_\omega = 0$

Evaluating the solution at $\delta = 0$,

$\Rightarrow \frac{\partial \mathscr{L}}{\partial \delta}\Big|_{\delta=0} = \frac{\sigma-1}{\sigma} (z_\omega)^{\frac{\delta}{\sigma}} [q_\omega]^{\frac{\sigma-1}{\sigma}-1} q_\omega - \lambda q_\omega p_\omega = 0$

$\Rightarrow \frac{\partial \mathscr{L}}{\partial \delta}\Big|_{\delta=0} = \frac{\sigma-1}{\sigma} (z_\omega)^{\frac{\delta}{\sigma}} [q_\omega]^{\frac{\sigma-1}{\sigma}-1} - \lambda p_\omega = 0$ for all $\omega \in [0, M]$

which gives the same expression as the first-order condition we derived.

Consider the optimal-control problem. Suppose we define an auxiliary variable $Y^R(\omega)$, which is the residual income after consuming good ω. The variable Y^R is constrained to $Y^R(0) = Y$ and $Y^R(M) = 0$. Hence, we can express the control variable as $\frac{\mathrm{d}Y^R}{\mathrm{d}\omega} = -p_\omega q_\omega$, since each good reduces the residual demand. The Hamiltonian is

$\mathscr{H} := (z_\omega)^{\frac{\delta}{\sigma}} (q_\omega)^{\frac{\sigma-1}{\sigma}} + \lambda(-p_\omega q_\omega)$

So taking the first order conditions

$\frac{\partial \mathscr{H}}{\partial q_\omega} = \frac{\sigma-1}{\sigma} (z_\omega)^{\frac{\delta}{\sigma}} (q_\omega)^{\frac{\sigma-1}{\sigma}-1} - \lambda p_\omega = 0$ for each $\omega \in [0, M]$,

which provides the same solution.

One convenient feature of the continuum case is that no firm in isolation can influence aggregate variables. For instance, the price elasticity of demand for ω is

$$\varepsilon_\omega := -\left[\frac{\partial \ln q_\omega}{\partial \ln p_\omega} + \frac{\partial \ln q_\omega}{\partial \ln \mathbb{P}} \frac{\partial \ln \mathbb{P}}{\partial \ln p_\omega}\right].$$

But, since each firm is negligible for the aggregate conditions, $\frac{\partial \ln \mathbb{P}}{\partial \ln p_\omega} = 0$ and therefore

$$\varepsilon_\omega = \sigma.$$

7 CES from a Random Utility Model (OPTIONAL)

We have derived the CES demand by considering a representative consumer exhibiting love for variety. However, this is not the only way to derive a CES demand. Moreover, each approach to deriving the CES demand gives rise to alternative interpretations for the demand parameters.

In particular, we now show that the CES demand can also be derived from a random utility model, where the taste for diversification arises because of an *ideal-variety* interpretation. This approach considers agents having heterogeneous preference that leads them to demand a different variety. For instance, in the context of the soccer jersey industry, we could divide the Spanish population based on their favorite team, such as Barcelona, Real Madrid, Sevilla, etc. By adding up the individual demands, all jerseys would be demanded, determining that aggregate consumption is diversified.

To formalize it, consider a continuum of agents with mass L, and a set of varieties $\Omega := \{1, 2, ..., N\}$. Each consumer makes two choices: which variety to consume and the quantities of it. Unlike the representative-consumer approach, consumers are **constrained to buy only one variety**, although they can buy as much as they want of the variety chosen. This feature explains why the random-utility model is known as discrete-continuous choices.

The population of consumers has heterogeneous tastes for each variety. This is formalized by an indirect utility of variety ω given by

$$V(y, p_\omega) := \ln y - \ln p_\omega + \underline{\varepsilon}_\omega \text{ for each } \omega \in \Omega,$$

where $\underline{\varepsilon}_\omega$ is a random variable. We suppose that each $\underline{\varepsilon}_\omega$ is iid Gumbel distributed, with standard deviation $\mu\frac{\pi}{6}$ where $\mu > 0$.[4] Note that a higher the value for μ increases the variance, and hence represents greater dispersion of tastes. This entails that μ reflects the degree in which preferences are heterogeneous in the population.

The quantity consumed conditional on choosing ω can be obtained by Roy's identity:

[4] The results we show would have been identical if we had defined the indirect utility function by $V(y, p_\omega) := \ln y - \ln p_\omega + \mu\underline{\varepsilon}_\omega$ and assume that $\underline{\varepsilon}_\omega$ is iid and with a Gumbel distribution that has zero mean and unit variance.

$q_{\omega} = -\frac{\partial V(y,p_{\omega})/\partial p_{\omega}}{\partial V(y,p_{\omega})/y}$, implying that

$$q_{\omega}\left(p_{\omega},y\right)=\frac{y}{p_{\omega}}.$$

Once we determine the quantities consumed of each ω, we can inquire on what variety will be chosen. To do this, denote by $\alpha_{\omega}\left(\mathbf{p}\right)$ the proportion of consumers selecting variety ω when prices are $\mathbf{p}$. This term also corresponds to the probability of ω providing the highest indirect utility. Formally,

$$\alpha_{\omega}\left(\mathbf{p}\right):=\Pr\left[\left(\ln y-\ln p_{\omega}+\mu\underline{\varepsilon}_{\omega}\geq\ln y-\ln p_{\omega'}+\mu\underline{\varepsilon}_{\omega'}\right)\left(\forall\omega'\in\Omega\right)\right].$$

By properties of the Gumbel distribution, $\alpha_{\omega}\left(\mathbf{p}\right)$ has the following closed-form solution:

$$\alpha_{\omega}\left(\mathbf{p}\right)=\frac{\exp\left(\frac{\ln p_{\omega}}{\mu}\right)}{\sum_{\omega'\in\Omega}\exp\left(\frac{\ln p_{\omega'}}{\mu}\right)}\Rightarrow\alpha_{\omega}\left(\mathbf{p}\right)=\frac{\left(p_{\omega}\right)^{-\frac{1}{\mu}}}{\sum_{\omega'\in\Omega}\left(p_{\omega'}\right)^{-\frac{1}{\mu}}}.$$

Thus, the aggregate demand is defined by

$$Q_{\omega}\left(\mathbf{p},y\right):=L\alpha_{\omega}\left(\mathbf{p}\right)q_{\omega}\left(p_{\omega},y\right),$$

which equals

$$Q_{\omega}\left(\mathbf{p},y\right)=L\frac{\left(p_{\omega}\right)^{-\frac{1}{\mu}}}{\sum_{\omega'\in\Omega}\left(p_{\omega'}\right)^{-\frac{1}{\mu}}}\frac{y}{p_{\omega}}\Rightarrow Q_{\omega}\left(\mathbf{p},y\right)=Ly\frac{\left(p_{\omega}\right)^{-\frac{1}{\mu}-1}}{\sum_{\omega'\in\Omega}\left(p_{\omega'}\right)^{-\frac{1}{\mu}}}.$$

A direct link to the CES demand is obtained by defining $\sigma:=\frac{1+\mu}{\mu}$, which determines $Q_{\omega}\left(\mathbf{p},y\right)=Ly\frac{\left(p_{\omega}\right)^{-\sigma}}{\sum_{\omega'\in\Omega}\left(p_{\omega'}\right)^{1-\sigma}}$. In this model, σ does not correspond to the elasticity of substitution. Instead, μ measures the degree of taste heterogeneity in the population. In particular, $\frac{\partial\sigma}{\partial\mu}>0$, and so a higher $\uparrow\sigma$ reflects lower heterogeneity in the population.