International Trade[1]

Lecture Note 2: Ricardo with a Continuum of Goods and Two Countries

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[1]The notes are still preliminary and in beta. Please, if you find any typo or mistake, send it to malfaro@ualberta.ca.

Contents

Notation

This is a derivation

This is some comment

This is a comment on advanced topics which are not part of the course (you can ignore it without loss of continuity regarding the text)

- The symbol ":=" means "by definition".
- I denote vectors by bold lowercase letters (for instance, $\mathbf{x}$) and matrices by bold capital letters (for instance, $\mathbf{X}$).
- To differentiate between the verb "maximize" and the operator "maximum", I denote the former with "max" and the latter with "sup" (i.e., supremum). The same caveat applies to "minimize" and "minimum", where I use "min" and "inf", with the latter indicating infimum.
- "iff" means "if and only if"
- $\exp(x)$ is the function e^x.
- Random variables are denoted with a bar below. For instance, $\underline{x}$.

These notes contain hyperlinks in blue and red text. If you are using Adobe Acrobat Reader, you can click on the link and easily navigate back by pressing Alt+Left Arrow.

1 Ricardian Models

These notes introduce one of the cornerstone models in International Trade: the Ricardian model. In particular, we present the version by **Dornbusch, Fischer, and Samuelson (1977)** (hencefort, DFS). Unlike the original work of Ricardo, which considers two goods and two countries, DFS deals with two countries and a continuum of goods. This makes it possible to think each good as a point in an interval, which has some convenient features such as enabling the use of calculus techniques.

The goal of studying DFS is twofold. First, it helps us become familiar with techniques employed in frameworks with a continuum of goods. Second, it unveils the mechanisms and main insights behind the Ricardian model.

DFS is somewhat limited for empirical use, as it only handles two countries. This is why the next lecture discusses a generalization by Eaton and Kortum (2002), which overcomes such a restriction by considering multiple countries and a continuum of goods. Eaton and Kortum (2002) has become one of the most important papers in the field over the last two decades. Nonetheless, understanding DFS thoroughly facilitates the comprehension of Ricardian models, as similar mechanisms are at work in both models.

2 Setup

There is a world economy that comprises two countries. We respectively refer to them as Home and Foreign, and denote them by (H) and (F). To differentiate variables belonging to one country from the other, we use a tilde over any variable belonging to (F).

Country (H) has L agents, while (F) has $\widetilde{L}$. Each agent makes consumption choices and works. In particular, they choose which industry to work in based on where they get paid more. We simplify the characterization of the labor market by assuming that each agent offers one unit of labor inelastically. This means that each agent offers one unit of labor, regardless of the prevailing wage. Furthermore, labor is mobile between the industries of a same country, but immobile between countries.

Regarding goods, there is a continuum of industries with each comprising one good. Formally, the set of industries is given by $J := [0, 1]$, with a typical element denoted $j \in J$. Given that there is only one good, we use the terms ***goods*** and ***industries*** interchangeably.

2.1 Supply

The market structure is *perfect competition*. Furthermore, each good is produced using labor exclusively, with all workers having the same level of productivity. By assuming that there is only one production factor, we abstract from other inputs, such as capital or even heterogeneity among workers of a same country (which, essentially, would be modeled as different inputs). This assumption keeps the model simple, allowing us to focus on differences in productivity between countries.

All firms in a country have access to the same technology for producing goods, but these technologies differ across countries. We also suppose constant returns to scale, implying that marginal costs are constant. The existence of only one input makes it possible to describe the technology by either the unit labor requirements or the marginal productivity of labor. Depending on the specific interpretation we are after, it is more convenient to use one or the other.

The **unit labor requirement** of good j is defined as the necessary amount of labor to produce one unit of good j. We denote it by $a(j)$ and $\widetilde{a}(j)$ for (H) and (F), respectively. Since there is only one factor of production, this information completely determines the **marginal productivity of labor**: it is just the inverse of the unit labor requirements, i.e., $\frac{1}{a(j)}$ for (H) and $\frac{1}{\widetilde{a}(j)}$ for (F).

As is shown below, specialization in the Ricardian model is identified through the concept of **comparative advantages**. To formalize it, it is necessary to have knowledge of the relative efficiency between countries given by $\frac{\widetilde{a}(j)}{a(j)}$ for good j—knowing the absolute efficiency, which is given by $a(j)$ and $\widetilde{a}(j)$, is not necessary. Following DFS, we define the relative efficiency of good j as a new variable, $A(j) := \frac{\widetilde{a}(j)}{a(j)}$.

Without loss of generality, we order the set of goods J according to a decreasing order of home-country comparative advantages. This implies that (H) has comparative advantages in low-indexed goods $j \in J$, while (F) has comparative advantages in high-indexed goods $j \in J$. Formally, this is captured supposing that $A'(j) < 0$.[1]

[1] We are allowed to make this order because we are working with only one input. With more than one input, it is required that the efficiency in production can be collapsed into single composite factor. In that way, we can represent efficiency through a real-valued number.

2.2 Demand

We suppose that the preferences of agents are identical in both countries. Specifically, hey are given by a Cobb-Douglas utility function sharing the same parameters.

For its description, consider country (H), and let $c(j)$ be the consumption of a typical household in the economy. Then,

$$U\left[(c(j))_{j\in J}\right] := \exp\left[\int_0^1 \beta(j)\ln c(j)\ \mathrm{d}j\right], \tag{1}$$

where $\beta(j) > 0$ for any $j \in J$ and $\int_0^1 \beta(j)\ \mathrm{d}j = 1$. Notice that, since we assume identical preferences in each country, the βs are the same in both (H) and (F).

The budget constraint is given by

$$y = \int_0^1 p(j)\, c(j)\ \mathrm{d}j,$$

where y is the household's income. The maximization problem determines an optimal consumption of a good j given by:

$$c(j) = \beta(j)\frac{y}{p(j)}.$$

With a Cobb-Douglas utility function, $\beta(j)$ **represents the fraction of income spent on good** j.[2] It is worth remarking that **all goods are essential under a Cobb Douglas utility function**. This means that each good is consumed in positive quantities, and $c(j)$ only equals zero if $p(j) \to \infty$.

A household's indirect utility function is

$$v(\mathbf{p}, y) := \frac{y}{\mathbb{P}},$$

with $\mathbb{P} := \exp\left(\int_0^1 \beta(j)\ln\left(\frac{p(j)}{\beta(j)}\right)\ \mathrm{d}j\right)$. In the model, welfare will be evaluated through v and represents *welfare per capita*. In particular, v can be interpreted as a measure of real income per capita.

[2] Strictly speaking, since there is a continuum of goods, β is the density function of the expenditure's shares. Notice that the particular values of a density function could be greater than 1. This is just a consequence of working with a continuum. Nonetheless, if it is easier for you, stick to an interpretation with a discrete bur large number of goods.

3 Autarky

Autarky refers to a closed economy, which will act as a benchmark for comparing outcomes under free trade. Each economy is in equilibrium under autarky when the goods and labor markets clear.[3] Next, we proceed to describe the equilibrium in each market in (H). A similar characterization applies to country (F) by adding a tilde to each variable.

3.1 Autarky Equilibrium

Let w be wages and $p(j)$ the price of good j. We denote any equilibrium variable with a superscript *aut,* which stands for autarky. To understand how the equilibrium is identified, some remarks are in order.

First, each good is essential from the consumer's point of view: for any finite price, there is a positive consumption of each good. Thus, **we anticipate an equilibrium where all goods have to be produced**. In particular, since we are dealing with a closed economy, all goods have to be produced in the country. This contrasts to what occurs in the case of free trade, where it is possible that one country specializes in the production of some goods and importing those goods not produced in the country.

Regarding this point, it is important to note that we assume free mobility of labor within each country. Additionally, although each agent offers one unit of labor regardless of the wage, they can select the industry in which they work. Therefore, we can deduce that **the wages paid in each industry have to be equal**. This arises because, if one industry paid more than the other, then all workers would move to that industry. This would result in no production in the other industry. Therefore, we require that each agents earns the same, irrespective of the industry in which they work.

We denote the wage prevailing in autarky as w^{aut}. This is taken as the numéraire, and so $w^{aut} := 1$. Moreover, perfect competition determines that the price of good j equals its marginal cost:

$$p^{aut}(j) = w^{aut} a(j) \text{ for almost all } j \in J. \tag{2}$$

Additionally, given $y^{aut} = w^{aut}$, the consumption per capita of good j in autarky is given

[3] In fact, it is the labor market plus a unitary mass of goods markets since the set of goods is $[0, 1]$.

by $c^{aut}(j) = \beta(j) \frac{w^{aut}}{p^{aut}(j)}$, with total demand $C(j) := Lc(j)$.

Notice that all the values on the right-hand side of (2) are known. This implies that the equation fully identifies the price in autarky for each good j. Consequently, **prices and wages are pinned down independently of the quantities consumed**. In other words, **prices and wages are not affected by the demand side**, and entirely determined by the supply side—assumptions made regarding the demand side will only impact the quantities consumed.

3.2 Welfare in Autarky

We measure welfare per capita through the indirect utility function, which can be interpreted as real income. Welfare in aurtaky can be expressed in logarithms by

$$\ln v^{aut}(\mathbf{p}, y) := \int_0^1 \beta(j) \ln\left(\frac{\beta(j)}{a(j)}\right) \mathrm{d}j.$$

$v^{aut}(\mathbf{p}, y)$ is given by the utility function (1) evaluated at the vector of consumptions $\left(c^{aut}(j)\right)_{j\in[0,1]}$. Since $c^{aut}(j) = \beta(j) \frac{w^{aut}}{p^{aut}(j)}$, we have that

$v^{aut}(\mathbf{p}, y) := \exp\left[\int_0^1 \beta(j) \ln\left(\beta(j) \frac{y^{aut}}{p^{aut}(j)}\right) \mathrm{d}j\right]$

Applying logs to both sides and using that $y^{aut} = w^{aut}$ and $p^{aut}(j) = w^{aut} a(j)$:

$\Rightarrow \ln v^{aut}(\mathbf{p}, y) = \int_0^1 \beta(j) \ln\left(\beta(j) \frac{w^{aut}}{w^{aut} a(j)}\right) \mathrm{d}j$

$\Rightarrow \ln v^{aut}(\mathbf{p}, y) = \int_0^1 \beta(j) \ln\left(\frac{\beta(j)}{a(j)}\right) \mathrm{d}j.$

To draw some conclusions about the autarkic equilibrium, let's compare country (H) in two scenarios: one with unit labor requirements a, and another with unit labor requirements $\widetilde{a}$ (i.e., those of (F)). Then, the logarithm of welfare ratios is

$$\ln \frac{v^{aut}(\mathbf{p}, y)}{\widetilde{v}^{aut}(\mathbf{p}, y)} = \int_0^1 \beta(j) \ln[A(j)] \mathrm{d}j,$$

where $\int_0^1 \beta(j) \ln[A(j)] \mathrm{d}j > 0$ indicates that $v^{aut}(\mathbf{p}, y) > \widetilde{v}^{aut}(\mathbf{p}, y)$.

$\ln v^{aut}(\mathbf{p}, y) - \ln \widetilde{v}^{aut}(\mathbf{p}, y) = \int_0^1 \beta(j) \ln\left(\frac{\beta(j)}{a(j)}\right) \mathrm{d}j - \int_0^1 \beta(j) \ln\left(\frac{\beta(j)}{\widetilde{a}(j)}\right) \mathrm{d}j$

$\Rightarrow \ln \frac{v^{aut}(\mathbf{p}, y)}{\widetilde{v}^{aut}(\mathbf{p}, y)} = \int_0^1 \beta(j) [\ln(\widetilde{a}(j)) - \ln(a(j))] \mathrm{d}j$

$\Rightarrow \ln \frac{v^{aut}(\mathbf{p}, y)}{\widetilde{v}^{aut}(\mathbf{p}, y)} = \int_0^1 \beta(j) \ln[A(j)] \mathrm{d}j$

Suppose that $\widetilde{a}(j) < a(j)$. It is easy to see that the agent gets more utility when the country's efficiency is $\widetilde{a}$. The reason is that the country can produce more of each good with the same endowment of workers. Formally, this follows because $A(j) < 1$ for all j.

Also, note that $\int_0^1 \beta(j) \ln[A(j)] \,\mathrm{d}j$ is a weighted average. Consequently, since $A(j)$ is decreasing, the consumer is better off under the technology $\widetilde{a}$ when the βs give more weight to low-j goods.

The comparison we have made does not imply anything about trade. We have simply compared two closed economies with different levels of productivity, with the goal of determining which would have greater welfare under autarky. This comparison also highlights that, conditional on the levels of consumption, aspects of production such as profits or exports can be disregarded for determining a country's welfare.

4 Free Trade

We consider the case of free trade. This refers to a scenario with zero trade costs and absence of any other trade barrier. The assumption is equivalent to supposing that all goods are perfectly tradable.

The equilibrium of the economy now requires not only that the goods and labor markets clear in each country. We also need to add a condition of how each country's imports are financed. As usual in the literature, we will ask for balanced trade: in each country, the values of its exports have to be equal to the value of its imports[4] In other words, the total income generated by each country's production equals its expenditures on goods.

4.1 Wages and Goods Prices

We begin by describing how prices are set in an open economy, with wages w and $\widetilde{w}$ given. First, notice that, since there are no trade costs and goods are completely homogeneous, the price of good j in each country must be the same. In other terms, **the law of one price holds**.

Furthermore, goods are homogeneous in every single dimension except possibly prices. Thus, consumers will buy the good j from the country that sells it cheaper. Since prices are equal to marginal cost in each country,

$$p^{trade}(j) = \inf\left\{wa(j), \widetilde{w}\widetilde{a}(j)\right\}.$$

[4] In fact, it is enough to ask for the impossibility of running trade deficits. The absence of trade surplus would follow because, cince consumers satisfy nonsatiability, then leaving goods not consumed is incompatible with utility maximization.

The equation implies that any good $j \in J$ such that $wa(j) < \widetilde{w}\widetilde{a}(j)$ is produced in (H), while $j \in J$ satisfying $wa(j) > \widetilde{w}\widetilde{a}(j)$ is produced in (F). Equivalently, we can reexpress the inequalities by saying that any $j \in J$ is produced at (H) if $\frac{w}{\widetilde{w}} < A(j)$, and in (F) if $\frac{w}{\widetilde{w}} < A(j)$. Which country does produce the good j^{*} that satisfies $\frac{w}{\widetilde{w}} = A(j^{*})$? We do not make any assumption on this regard. Good j^{*} could be produced in (H), (F), or both. Since there is a continuum of goods, any specific good is negligible for the economy (technically, it is of zero measure) and thus does not affect the equilibrium values. This is in contrast to what happens when there is a discrete number of goods, where the marginal good affects the equilibrium.

The identification of the threshold good j^{*} allows us to define the set of goods produced by each country. Since $A(j)$ is strictly decreasing, there exists a unique j^{*} such that

$$\frac{w}{\widetilde{w}} = A(j^{*}). \tag{3}$$

Given this, ountry (H) will produce all goods $j \in [0, j^{*}]$, while (F) will produce goods $j \in [j^{*}, 1]$.[5] Recall that which country produces in particular j^{*} is irrelevant, since there is a continuum of goods. I have assumed in particular that both countries produce it.

From this relation, we can infer that $w > \widetilde{w}$ **if** (H) **has absolute advantages relative to** (F) **for all** $j \in J$. The result follows by noticing that (H) has absolute advantages in all goods when $a(j) < \widetilde{a}(j)$ for all $j \in J$. This implies that $A(j^{*}) > 1$ for any j^{*}, and so $w > \widetilde{w}$.

Given a value of j^{*}, relative wages $\frac{w}{\widetilde{w}}$ are completely determined by $A(j^{*})$. So, in principle, $w > \widetilde{w}$ would arise if (H) has absolute advantages for the set of goods around the value j^{*}. However, j^{*} is determined by the whole model and could vary depending after some shock to the economy. Therefore, it is necessary to have absolute advantages of (H) for all goods to ensure that the result is global and not local (i.e., that always holds).

4.2 Demand side

We have already shown that, given prices and wages, there is a constant fraction of income spent on each good j due to the Cobb Douglas assumption. Formally, this

[5] Another relation to pin down the value of j^{*} might be used. Notice that, since $A(j)$ is decreasing, we know that the function is invertible. Formally, $\exists A^{-1}$ such that $A^{-1} : \mathbb{R}_{+} \to Z$. The function A^{-1} provides the index of good that has a specific relative efficiency. Thus, since $\frac{w}{\widetilde{w}} = A(j^{*})$, then $j^{*} = A^{-1}\left(\frac{w}{\widetilde{w}}\right)$.

means that $c(j) = \beta(j)\frac{y}{p(j)}$ and $\widetilde{c}(j) = \beta(j)\frac{\widetilde{y}}{p(j)}$ for any $j \in J$. Notice that, even though $\beta(j)$ is the same for both countries (identical preferences) and the law of one price holds (because there are no trade costs), differences in wages across countries due to different technologies could determine different levels of consumption. However, it is worth keeping in mind that the expenditure share of each good j in terms of income would remain the same, even if the incomes of each country are different. This follows because good j's expenditure shares is given by $\beta(j)$.

Let $\upsilon(j^*)$ be the fraction of total income spent on goods produced by (H), and $1-\upsilon(j^*)$ the fraction of total income spent on goods produced by (F). Each term is given by

$$\upsilon(j^*) := \int_0^{j^*} \beta(j)\,\mathrm{d}j,$$
$$1-\upsilon(j^*) = \int_{j^*}^{1} \beta(j)\,\mathrm{d}j.$$

4.3 Equilibrium

In equilibrium, the labor and goods markets in each country clear and there is balanced trade. Given that the total income of (H) and (F) is respectively wL and $\widetilde{w}\widetilde{L}$, trade is balanced when

$$\underbrace{[1-\upsilon(j^*)]\,wL}_{(H)\text{'s imports}} = \underbrace{\upsilon(j^*)\,\widetilde{w}\widetilde{L}}_{(H)\text{'s exports}}.$$

This equation can be reexpressed by

$$\frac{w}{\widetilde{w}} = \frac{\upsilon(j^*)}{1-\upsilon(j^*)}\frac{\widetilde{L}}{L}, \tag{4}$$

which gives the relative wages that are consistent with balanced trade.

There are different ways to arrive at the same equilibrium condition. For instance, we could have used that (H)'s income equals production's value, and that the latter is equal to the world demand of (H)'s goods:

$wL = \underbrace{\upsilon(j^*)\,wL}_{\text{domestic-demand value}} + \underbrace{\upsilon(j^*)\,\widetilde{w}\widetilde{L}}_{\text{foreign-demand value}}$

or, what is same,

$wL = \underbrace{\upsilon(j^*)}_{\text{income's share on goods produced at } (H)} \underbrace{\left(wL + \widetilde{w}\widetilde{L}\right)}_{\text{global income}}$

$\frac{w}{\widetilde{w}} = \frac{\upsilon(j^*)}{1-\upsilon(j^*)}\frac{\widetilde{L}}{L}$

Alternatively, market-clearing of the labor market in each country provides the same equilibrium condition:

$\int_0^{j^*} a(j)\left[Lc(j)+\widetilde{L}\widetilde{c}(j)\right]\mathrm{d}j = L$
$\Rightarrow \int_0^{j^*} a(j)\left[L\beta(j)\frac{w}{p(j)}+\widetilde{L}\widetilde{\beta}(j)\frac{\widetilde{w}}{p(j)}\right]\mathrm{d}j = L,$
and substituting in $p(j) = wa(j)$,
$\Rightarrow \int_0^{j^*}\left[L\beta(j)+\widetilde{L}\beta(j)\frac{\widetilde{w}}{w}\right]\mathrm{d}j = L \Rightarrow \left[L+\widetilde{L}\frac{\widetilde{w}}{w}\right]\int_0^{j^*}\beta(j)\,\mathrm{d}j = L$
$\Rightarrow \left[L+\widetilde{L}\frac{\widetilde{w}}{w}\right]\upsilon(j^*) = L$
$\Rightarrow \frac{w}{\widetilde{w}} = \frac{\upsilon(j^*)}{1-\upsilon(j^*)}\frac{\widetilde{L}}{L}$

We now have two expressions of $\frac{w}{\widetilde{w}}$ that summarize the supply and demand side: $A(j^*) = \frac{w}{\widetilde{w}} = \frac{\upsilon(j^*)}{1-\upsilon(j^*)}\frac{\widetilde{L}}{L}$. Thus, the equilibrium can be obtained by a value j^*, which is implicitly determined by

$$A(j^*) = \frac{\upsilon(j^*)}{1-\upsilon(j^*)}\frac{\widetilde{L}}{L},$$

with relative wages given by $\frac{w}{\widetilde{w}} = A(j^*)$. Once we identify j^*, the equilibrium quantities of each good j are given by:

$$\begin{aligned} Q^{trade}(j) &= Lc^{trade}(j) + \widetilde{L}\widetilde{c}^{trade}(j), \\ &= L\beta(j)\frac{w}{p(j)} + \widetilde{L}\beta(j)\frac{\widetilde{w}}{p(j)}, \end{aligned}$$

and so,

$$Q^{trade}(j) = \frac{\beta(j)}{a(j)}\left(L + \widetilde{L}\frac{\widetilde{w}}{w}\right).$$

4.4 Gains of Trade

Next, we prove the existence of gains of trade. This requires comparing welfare per capita under trade, relative to autarky. Welfare in each respective case is given by

$$\begin{aligned} \ln v^{aut}(\mathbf{p}, y) &:= \int_0^1 \beta(j)\ln\left(\frac{\beta(j)}{a(j)}\right)\mathrm{d}j, \\ \ln v^{trade}(\mathbf{p}, y) &:= \int_0^{j^*} \beta(j)\ln\left(\frac{\beta(j)}{a(j)}\right)\mathrm{d}j + \int_{j^*}^1 \beta(j)\ln\left(\frac{\beta(j)}{\widetilde{a}(j)}A(j^*)\right)\mathrm{d}j. \end{aligned}$$

implying that gains of trade can be identified through

$$\ln\left(\frac{v^{trade}(\mathbf{p}, y)}{v^{aut}(\mathbf{p}, y)}\right) = \int_{j^*}^1 \beta(j)\ln\left(\frac{A(j^*)}{A(j)}\right)\mathrm{d}j.$$

Since $A(j)$ is decreasing, $\frac{A(j^*)}{A(j)} > 1$ when $j > j^*$. Hence, the left-hand side is positive, implying that $v^{trade}(\mathbf{p}, y) > v^{aut}(\mathbf{p}, y)$. Consequently, there are always gains of trade in this model.

The expression for $\ln v^{aut}(\mathbf{p}, y)$ was already derived above. Regarding $v^{trade}(\mathbf{p}, y)$, it is given by the utility function (1) evaluated at the vector of consumptions $\left(c^{trade}(j)\right)_{j\in[0,1]}$. Since $c^{trade}(j) = \beta(j)\frac{w}{p^{trade}(j)}$, we have that
$\ln v^{trade}(\mathbf{p}, y) = \int_0^1 \beta(j)\ln\left[\beta(j)\frac{w}{p^{trade}(j)}\right]\mathrm{d}j$
Moreover, we know that $p^{trade}(j) = wa(j)$ if $j \in [0, j^*]$ and $p^{trade}(j) = \widetilde{w}\widetilde{a}(j)$ if $j \in [j^*, 1]$. Therefore,
$\ln v^{trade}(\mathbf{p}, y) = \int_0^{j^*} \beta(j)\ln\left[\beta(j)\frac{w}{wa(j)}\right]\mathrm{d}j + \int_{j^*}^1 \beta(j)\ln\left[\beta(j)\frac{w}{\widetilde{w}\widetilde{a}(j)}\right]\mathrm{d}j$
and using that $A(j) := \frac{\widetilde{a}(j)}{a(j)}$ and $\frac{w}{\widetilde{w}} = A(j^*)$, then
$\ln v^{trade}(\mathbf{p}, y) = \int_0^{j^*} \beta(j)\ln\left[\frac{\beta(j)}{a(j)}\right]\mathrm{d}j + \int_{j^*}^1 \beta(j)\ln\left[\frac{\beta(j)}{\widetilde{a}(j)}A(j^*)\right]\mathrm{d}j$
Regarding the gains o trade,
$\ln\left(\frac{v^{trade}(\mathbf{p},y)}{v^{aut}(\mathbf{p},y)}\right) = \int_0^{j^*} \beta(j)\ln\left[\frac{\beta(j)}{a(j)}\right]\mathrm{d}j + \int_{j^*}^1 \beta(j)\ln\left[\frac{\beta(j)}{\widetilde{a}(j)}A(j^*)\right]\mathrm{d}j - \int_0^1 \beta(j)\ln\left(\frac{\beta(j)}{a(j)}\right)\mathrm{d}j$
$\Rightarrow \ln\left(\frac{v^{trade}(\mathbf{p},y)}{v^{aut}(\mathbf{p},y)}\right) = \int_{j^*}^1 \beta(j)\ln\left[\frac{\beta(j)}{\widetilde{a}(j)}A(j^*)\right]\mathrm{d}j - \int_{j^*}^1 \beta(j)\ln\left(\frac{\beta(j)}{a(j)}\right)\mathrm{d}j$
$\Rightarrow \ln\left(\frac{v^{trade}(\mathbf{p},y)}{v^{aut}(\mathbf{p},y)}\right) = \int_{j^*}^1 \beta(j)\ln\left[\frac{\beta(j)}{\widetilde{a}(j)}A(j^*)\frac{a(j)}{\beta(j)}\right]\mathrm{d}j$
$\Rightarrow \ln\left(\frac{v^{trade}(\mathbf{p},y)}{v^{aut}(\mathbf{p},y)}\right) = \int_{j^*}^1 \beta(j)\ln\left[\frac{A(j^*)}{A(j)}\right]\mathrm{d}j$

4.5 Differences in Market Size

Once we have determined that trade makes every country better off, we can also obtain other conclusions from the model by performing a **comparative-static analysis**: this requires varying one parameter of the model, and see how the endogenous variables react to this shock.

To illustrate the methodology, let's consider an increase in the population of (F) relative to (H) (or, equivalently, a relative decrease in the population of (H)). Denoting $\widetilde{l} := \frac{\widetilde{L}}{L}$, this consists of an infinitesimal variation of $\widetilde{l}$, which is formally denoted by $\mathrm{d}\widetilde{l} > 0$. Our interest lies in the sign of $\mathrm{d}(w/\widetilde{w})$ and $\mathrm{d}j^*$. Equivalently, we want to obtain a sign for $\frac{\partial(w/\widetilde{w})}{\partial\widetilde{l}}$ and $\frac{\partial j^*}{\partial\widetilde{l}}$.

To analyze the effects of changes in market size, we proceed by differentiating the system of equilibrium conditions, given by equations (3), i.e. $\frac{w}{\widetilde{w}} = A(j^*)$, and (4), i.e. $\frac{w}{\widetilde{w}} = \frac{v(j^*)}{1-v(j^*)}\widetilde{l}$, and then expressing the result in a matrix way:

$$\begin{pmatrix} 1 & -A'(j^*) \\ 1 & \frac{-v'(j^*)}{[1-v(j^*)]^2}\widetilde{l} \end{pmatrix}\begin{pmatrix} \mathrm{d}(w/\widetilde{w}) \\ \mathrm{d}j^* \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{v(j^*)}{1-v(j^*)} \end{pmatrix}\mathrm{d}\widetilde{l}.$$

Dividing both sides by $\mathrm{d}\widetilde{l}$, we obtain our expression of interest:

$$\begin{pmatrix} 1 & -A'(j^*) \\ 1 & \frac{-v'(j^*)}{[1-v(j^*)]^2}\frac{\widetilde{L}}{L} \end{pmatrix}\begin{pmatrix} \frac{\partial(w/\widetilde{w})}{\partial\widetilde{l}} \\ \frac{\partial j^*}{\partial\widetilde{l}} \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{v(j^*)}{1-v(j^*)} \end{pmatrix}.$$

By applying Cramer's rule, we can establish that an increase in the population of (F) relative to (H) results in:

- $\frac{\partial(w/\widetilde{w})}{\partial\widetilde{l}} > 0$: the relative wage of (H) increases, and
- $\frac{\partial j^*}{\partial\widetilde{l}} < 0$: (H) produces and exports less goods, while (F) produces and exports more goods.

The equilibrium conditions are $\frac{w}{\widetilde{w}} = A(j^*)$ and $\frac{w}{\widetilde{w}} = \frac{v(j^*)}{1-v(j^*)}\widetilde{l}$. We differentiate each of them, which means that we let $\mathrm{d}\widetilde{l} \neq 0$ regarding exogenous variables, and determine the impact on $\mathrm{d}(w/\widetilde{w})$ and $\mathrm{d}j$, which are the endogenous variables of the model.

The differentiation gives, respectively, for each equation:

$\mathrm{d}(w/\widetilde{w}) = A'(j^*)\,\mathrm{d}j + 0\,\mathrm{d}\widetilde{l}$

$\mathrm{d}(w/\widetilde{w}) = \left(\frac{v'(j^*)}{[1-v(j^*)]^2}\frac{\widetilde{L}}{L}\right)\mathrm{d}j + \left(\frac{v(j^*)}{1-v(j^*)}\right)\mathrm{d}\widetilde{l}$

In a matrix way, we can express it as it is in the main text, so that

$$\begin{pmatrix} 1 & -A'(j^*) \\ 1 & \frac{-v'(j^*)}{[1-v(j^*)]^2}\frac{\widetilde{L}}{L} \end{pmatrix} \begin{pmatrix} \frac{\partial(w/\widetilde{w})}{\partial\widetilde{l}} \\ \frac{\partial j^*}{\partial\widetilde{l}} \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{v(j^*)}{1-v(j^*)} \end{pmatrix}$$

The matrix of the left-hand side is denominated the Jacobian matrix and it consists of the first derivatives of the endogenous variables. Formally, let $J := \begin{pmatrix} 1 & -A'(j^*) \\ 1 & \frac{-v'(j^*)}{[1-v(j^*)]^2} \end{pmatrix}$ with $\det J = \frac{[1-v(j^*)]^2 A'(j^*) - v'(j^*)}{[1-v(j^*)]^2}$ that satisfies $\det J < 0$.

Solving the system by applying Cramer's rule:

$\frac{\partial(w/\widetilde{w})}{\partial\widetilde{l}} = \frac{\det\begin{pmatrix} 0 & -A'(j^*) \\ \frac{v(j^*)}{1-v(j^*)} & \frac{v'(j^*)}{[1-v(j^*)]^2} \end{pmatrix}}{\det J} \Rightarrow \frac{\partial(w/\widetilde{w})}{\partial\widetilde{l}} = \frac{A'(j^*)}{\det J}\frac{v(j^*)}{1-v(j^*)} = \frac{A'(j^*)[1-v(j^*)]v(j^*)}{[1-v(j^*)]^2 A'(j^*) - v'(j^*)} > 0$

and

$\frac{\partial j^*}{\partial\widetilde{l}} = \frac{\det\begin{pmatrix} 1 & 0 \\ 1 & \frac{v(j^*)}{1-v(j^*)} \end{pmatrix}}{\det J} \Rightarrow \frac{\partial\partial j^*}{\partial\widetilde{l}} = \frac{1}{\det J}\frac{v(j^*)}{1-v(j^*)}$

$\Rightarrow \frac{\partial j^*}{\partial\widetilde{l}} = \frac{v(j^*)[1-v(j^*)]}{[1-v(j^*)]^2 A'(j^*) - v'(j^*)} < 0$

Once we have identified the impact of an increase in $\bar{l}$, we can also evaluate how this shock affects welfare per capita. This is given by

$$\frac{\mathrm{d}\ln v^{trade}(\mathbf{p}, y)}{\mathrm{d}\widetilde{l}} = \frac{\partial \ln v^{trade}(\mathbf{p}, y)}{\partial j^*}\frac{\partial j^*}{\partial\widetilde{l}} > 0,$$

implying that a greater population of (F) or a lower population of (H) (i.e., increases in $\bar{l}$) makes (H) be better off.

Welfare is given by $\ln v^{trade}(\mathbf{p}, y) = \int_0^{j^*}\beta(j)\ln\left[\frac{\beta(j)}{a(j)}\right]\mathrm{d}j + \int_{j^*}^1 \beta(j)\ln\left[\frac{\beta(j)}{\widetilde{a}(j)}A(j^*)\right]\mathrm{d}j$.

The expression does not depend directly on $\widetilde{l}$ but only through the fact that j^* varies when $\widetilde{l}$ is shocked. Formally, this implies that:

$\frac{\mathrm{d}\ln v^{trade}(\mathbf{p},y)}{\mathrm{d}\widetilde{l}} = \frac{\partial \ln v^{trade}(\mathbf{p},y)}{\partial j^*}\frac{\partial j^*}{\partial\widetilde{l}}$

Through the comparative static analysis, we have already determined that $\frac{\partial j^*}{\partial \widetilde{l}} = \frac{\upsilon(j^*)[1-\upsilon(j^*)]}{[1-\upsilon(j^*)]^2 A'(j^*)-\upsilon'(j^*)} < 0$. So it rests to determine the sign of $\frac{\partial \ln v^{trade}(\mathbf{p},y)}{\partial j^*}$

$\frac{\partial \ln v^{trade}(\mathbf{p},y)}{\partial j^*} = \beta(j^*) \ln\left[\frac{\beta(j^*)}{a(j^*)}\right] - \beta(j^*) \ln\left[\frac{\beta(j^*)}{\widetilde{a}(j^*)} A(j^*)\right] + \int_{j^*}^1 \beta(j) \frac{A'(j^*)}{A(j^*)} \mathrm{d}j$

$\Rightarrow \frac{\partial \ln v^{trade}(\mathbf{p},y)}{\partial j^*} = \beta(j^*) \ln\left[\frac{\beta(j^*)}{a(j^*)}\right] - \beta(j^*) \ln\left[\frac{\beta(j^*)}{a(j^*)}\right] + \frac{A'(j^*)}{A(j^*)} \int_{j^*}^1 \beta(j) \mathrm{d}j$

$\Rightarrow \frac{\partial \ln v^{trade}(\mathbf{p},y)}{\partial j^*} = \frac{A'(j^*)}{A(j^*)} [1-\upsilon(j^*)] < 0.$

Therefore

$$\frac{\mathrm{d} \ln v^{trade}(\mathbf{p},y)}{\mathrm{d}\widetilde{l}} = \underbrace{\frac{\partial \ln v^{trade}(\mathbf{p},y)}{\partial j^*}}_{-} \underbrace{\frac{\partial j^*}{\partial \widetilde{l}}}_{-} > 0$$

Notice that the term $\frac{\mathrm{d} \ln v^{trade}(\mathbf{p},y)}{\mathrm{d}\widetilde{l}}$ not only captures the increase in welfare under trade when $\bar{l}$ is greater, but, also, establishes the change in gains of trade, given by $\ln\left(\frac{v^{trade}(\mathbf{p},y)}{v^{aut}(\mathbf{p},y)}\right)$. To see this, note that

$$\frac{\partial \ln\left(\frac{v^{trade}(\mathbf{p},y)}{v^{aut}(\mathbf{p},y)}\right)}{\partial j^*} = \frac{\partial \ln\left(v^{trade}(\mathbf{p},y)\right)}{\partial j^*} - \underbrace{\frac{\partial \ln\left(v^{aut}(\mathbf{p},y)\right)}{\partial j^*}}_{=0}.$$

so that $\frac{\partial \ln\left(\frac{v^{trade}(\mathbf{p},y)}{v^{aut}(\mathbf{p},y)}\right)}{\partial j^*} = \frac{\partial \ln\left(v^{trade}(\mathbf{p},y)\right)}{\partial j^*}$.

5 Costly Trade and Nontradable goods

While the comparison of autarky and free trade makes the analysis simple, an economy is rarely completely closed or completely open. Next, we consider a more realistic situation, where we introduce the concept of **trade costs**. This reflects that making one unit available in a foreign country entails additional costs, relative to delivering a good domestically. Given the existence of trade costs, the model is able to predict that some goods will be produced only to serve the domestic country. Thus, there is a range of goods that will not be traded.

Assuming the existence of trade costs makes the model quite general. In particular, it allows us to model autarky and free trade as special cases. As for autarky, it corresponds to a scenario where trade costs are infinite: one unit of the good abroad can be sent only by spending infinite resources. Thus, there is no trade in equilibrium. As for free trade, it emerges as a scenario where trade costs are zero.

There are different ways to model trade costs. We adopt the standard form of the so-called **iceberg transportation costs**: when one unit of good is shipped from the domestic market to the foreign country, a physical portion melts away.

Formally, this means that, if one unit is sent by a firm, then only $\frac{1}{\tau}$ with $\tau > 1$ arrives. This entails that if one unit is consumed abroad, the firm needs to ship τ units. The existence of trade costs creates a **wedge between what the firm produces and what the consumer gets**. Furthermore, **the law of one price does not hold anymore**. Thus, a consumer pays different prices, depending on whether the good is produced by a domestic or foreign firm.

Incorporating this aspect, the price of good j in (H) is still given by the minimum between the domestic or imported alternative. The only difference is that foreign firms need to send τ units of good j if the good is imported. Thus, the marginal cost of the imported good is $\widetilde{w}\widetilde{a}(j)\tau$. The marginal cost to sell domestically is not affected, and hence is the same as in the baseline model.

Formally, the price of good j in (H) is

$$p(j) = \inf\left\{wa(j), \widetilde{w}\widetilde{a}(j)\tau\right\}.$$

Consumers in (H) will opt for local production when $wa(j) < \widetilde{w}\widetilde{a}(j)\tau$ or, what is same,

$$\frac{w}{\widetilde{w}} < A(j)\tau.$$

This allows us to define a cutoff good j^* in country (H) such that a consumer from (H) is indifferent between buying the good locally or importing it. Formally, j^* satisfies that

$$\frac{w}{\widetilde{w}} = A(j^*)\tau. \tag{5}$$

By the same token, the price of good j in (F) is

$$\widetilde{p}(j) = \inf\left\{wa(j)\tau, \widetilde{w}\widetilde{a}(j)\right\}.$$

So, consumers in (F) choose importing from (H) when

$$\frac{w}{\widetilde{w}} < \frac{A(j)}{\tau}.$$

This relation defines the cutoff good $\widetilde{j}^*$ in country (F) such that a consumer from (F) is indifferent between buying the good locally or importing it. Formally,

$$\frac{w}{\widetilde{w}} = \frac{A\left(\widetilde{j}^*\right)}{\tau}. \tag{6}$$

Notice that (5) and (6) make it possible to define when (H) is producing exclusively for its home market or when it additionally exports the good. This occurs since (H) exporting a good means that (F) is importing it. Therefore, (H) produces goods for both the local and export market when both inequalities hold:

$$\frac{w}{\widetilde{w}} < A(j)\,\tau,$$
$$\frac{w}{\widetilde{w}} < \frac{A(j)}{\tau}.$$

Since $\tau > 1$ and both inequalities have to hold, then good j will be consumed locally and exported if $\frac{w}{\widetilde{w}} < \frac{A(j)}{\tau}$.

In addition, (F) will produce for both countries when

$$\frac{w}{\widetilde{w}} > A(j)\,\tau,$$
$$\frac{w}{\widetilde{w}} > \frac{A(j)}{\tau},$$

which implies that (F) will produce for the home country and export good j when $\frac{w}{\widetilde{w}} > A(j)\,\tau$.

The relation between j^* and $\widetilde{j}^*$ becomes important to describe the pattern of trade. In particular, it can be shown that $j^* > \widetilde{j}^*$.

By definition, j^* is such that $\frac{w}{\widetilde{w}} = A(j^*)\,\tau$ and $\widetilde{j}^*$ such that $\frac{w}{\widetilde{w}} = \frac{A(\widetilde{j}^*)}{\tau}$. Therefore,

$$A(j^*)\,\tau = \frac{A\left(\widetilde{j}^*\right)}{\tau}.$$

Consequently, $A(j^*)\,\tau^2 = A\left(\widetilde{j}^*\right)$ and, since $\tau > 1$, then $A(j^*) < A\left(\widetilde{j}^*\right)$. In addition, we have ordered the goods in such a way that $A' < 0$. Thus, $A(j^*) < A\left(\widetilde{j}^*\right)$ can only happen if $j^* > \widetilde{j}^*$.

Since $j^* > \widetilde{j}^*$, we can determine that

$$j \in \begin{cases} \left[0, \widetilde{j}^*\right] & (H) \text{ serves domestic and } (F)\text{'s markets} \\ \left[\widetilde{j}^*, j^*\right] & \text{non-traded goods} \\ \left[j^*, 1\right] & (F) \text{ serves domestic and } (H)\text{'s markets} \end{cases}$$

which completely describes the patterns of trade.

Notice we only have partially described the equilibrium. This is reflected in that the cutoff goods are still a function of the equilibrium relative wages. These cutoffs are endogenous, and we can use the balanced-trade condition to identify them:

$$\underbrace{\left[1-\upsilon\left(j^{*}\right)\right]wL}_{\text{home country's imports}} = \underbrace{\upsilon\left(\widetilde{j}^{*}\right)\widetilde{w}\widetilde{L}}_{\text{home country's exports}},$$

which can be reexpressed as $\frac{w}{\widetilde{w}}=\frac{\upsilon\left(\widetilde{j}^{*}\right)}{1-\upsilon\left(j^{*}\right)}\frac{\widetilde{L}}{L}$.

In sum, all these derivations imply that the system of equilibrium conditions is given by

$$\frac{w}{\widetilde{w}}=\frac{A\left(\widetilde{j}^{*}\right)}{\tau}, \tag{7a}$$

$$\frac{w}{\widetilde{w}}=A\left(j^{*}\right)\tau, \tag{7b}$$

$$\frac{w}{\widetilde{w}}=\frac{\upsilon\left(\widetilde{j}^{*}\right)}{1-\upsilon\left(j^{*}\right)}\widetilde{l}, \tag{7c}$$

where $\widetilde{l}:=\frac{\widetilde{L}}{L}$. As a corollary, the system (7) allows us to pin down $\frac{w}{\widetilde{w}}$, j^{*}, and $\widetilde{j}^{*}$.

Given this, we can revisit what occurs when there are variations in the relative sizes of countries. By differentiating the system and expressing it in a matrix form:

$$\begin{pmatrix} 1 & -\frac{\upsilon'_{j^*}\upsilon\left(\widetilde{j}^{*}\right)}{\left[1-\upsilon\left(j^{*}\right)\right]^{2}}\widetilde{l} & \frac{-\upsilon'_{\widetilde{j}^*}}{1-\upsilon\left(j^{*}\right)}\widetilde{l} \\ 1 & -A'_{j^*}\tau & 0 \\ 1 & 0 & \frac{-A'_{\widetilde{j}^*}}{\tau} \end{pmatrix} \begin{pmatrix} \mathrm{d}\left(w/\widetilde{w}\right) \\ \mathrm{d}j^{*} \\ \mathrm{d}\widetilde{j}^{*} \end{pmatrix} = \begin{pmatrix} \frac{\upsilon\left(\widetilde{j}^{*}\right)}{1-\upsilon\left(j^{*}\right)} \\ 0 \\ 0 \end{pmatrix} \mathrm{d}\widetilde{l}.$$

By performing comparative statics, we get that

$$\frac{\partial\left(w/\widetilde{w}\right)}{\partial\widetilde{l}}>0,$$

$$\frac{\partial j^{*}}{\partial\widetilde{l}}<0,$$

$$\frac{\partial\widetilde{j}^{*}}{\partial\widetilde{l}}<0.$$

We denote Jb the Jacobian of the matrix on the left-hand side. Formally, $Jb:=\begin{pmatrix} 1 & -\frac{\upsilon'_{j^*}\upsilon\left(\widetilde{j}^{*}\right)}{\left[1-\upsilon\left(j^{*}\right)\right]^{2}}\widetilde{l} & -\frac{\upsilon'_{\widetilde{j}^*}}{1-\upsilon\left(j^{*}\right)}\widetilde{l} \\ 1 & -A'_{j^*}\tau & 0 \\ 1 & 0 & \frac{-A'_{\widetilde{j}^*}}{\tau} \end{pmatrix}$.

First we show that $\det Jb>0$:

$$\det\begin{pmatrix} 1 & -\frac{\upsilon'_{j^*}\upsilon\left(\widetilde{j}^{*}\right)}{\left[1-\upsilon\left(j^{*}\right)\right]^{2}}\widetilde{l} & -\frac{\upsilon'_{\widetilde{j}^*}}{1-\upsilon\left(j^{*}\right)}\widetilde{l} \\ 1 & -A'_{j^*}\tau & 0 \\ 1 & 0 & \frac{-A'_{\widetilde{j}^*}}{\tau} \end{pmatrix} = \underbrace{A'_{j^*}A'_{\widetilde{j}^*}}_{+} - \underbrace{\frac{\upsilon'_{j^*}\upsilon\left(\widetilde{j}^{*}\right)}{\left[1-\upsilon\left(j^{*}\right)\right]^{2}}\widetilde{l}\frac{A'_{\widetilde{j}^*}}{\tau}}_{-} - \underbrace{\frac{\upsilon'_{\widetilde{j}^*}}{1-\upsilon\left(j^{*}\right)}\widetilde{l}A'_{j^*}\tau}_{-} > 0$$

where we have used that $A'_{j^*}, A'_{\widetilde{j}^*} < 0$ and $\upsilon'_{j^*}, \upsilon'_{\widetilde{j}^*} > 0$.

Hence,

$$\frac{\partial(w/\widetilde{w})}{\partial \widetilde{l}} = (\det Jb)^{-1} \det \begin{pmatrix} \frac{\upsilon(\widetilde{j}^*)}{1-\upsilon(j^*)} & -\frac{\upsilon'_{j^*}\upsilon(\widetilde{j}^*)}{[1-\upsilon(j^*)]^2}\widetilde{l} & -\frac{\upsilon'_{\widetilde{j}^*}}{1-\upsilon(j^*)}\widetilde{l} \\ 0 & -A'_{j^*}\tau & 0 \\ 0 & 0 & \frac{-A'_{\widetilde{j}^*}}{\tau} \end{pmatrix}$$

$$\Rightarrow \frac{\partial(w/\widetilde{w})}{\partial \widetilde{l}} = (\det Jb)^{-1} \frac{\upsilon(\widetilde{j}^*)}{1-\upsilon(j^*)} A'_{j^*} A'_{\widetilde{j}^*} > 0$$

$$\frac{\partial j^*}{\partial \widetilde{l}} = (\det Jb)^{-1} \det \begin{pmatrix} 1 & \frac{\upsilon(\widetilde{j}^*)}{1-\upsilon(j^*)} & -\frac{\upsilon'_{\widetilde{j}^*}}{1-\upsilon(j^*)}\widetilde{l} \\ 1 & 0 & 0 \\ 1 & 0 & \frac{-A'_{\widetilde{j}^*}}{\tau} \end{pmatrix}$$

$$\Rightarrow \frac{\partial j^*}{\partial \widetilde{l}} = (\det Jb)^{-1} (-1) \frac{\upsilon(\widetilde{j}^*)}{1-\upsilon(j^*)} \frac{-A'_{\widetilde{j}^*}}{\tau} < 0$$

$$\frac{\partial \widetilde{j}^*}{\partial \widetilde{l}} = (\det Jb)^{-1} \det \begin{pmatrix} 1 & -\frac{\upsilon'_{j^*}\upsilon(\widetilde{j}^*)}{[1-\upsilon(j^*)]^2}\widetilde{l} & \frac{\upsilon(\widetilde{j}^*)}{1-\upsilon(j^*)} \\ 1 & -A'_{j^*}\tau & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \frac{\partial \widetilde{j}^*}{\partial \widetilde{l}} = (\det Jb)^{-1} \frac{\upsilon(\widetilde{j}^*)}{1-\upsilon(j^*)} A'_{j^*}\tau < 0$$