9f. Reductions

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INTRODUCTION

Reductions are a computational strategy for **operations that take collections as input and return a single element**. Such operations arise naturally in a wide range of contexts, such as when computing summary statistics (e.g., averages, variances, or maxima of collections).

The reduction process works by iteratively applying an operation to pairs of elements, accumulating the results at each step until the final output is obtained. For example, to compute $\boxed{\text{sum}(x)}$ for a vector \boxed{x} , a reduction would start by adding the first two elements, then add the third element to the partial total, and continue this cumulative process until all elements have been summed.

The method is particularly convenient when we need to transform a vector's elements prior to aggregating the result. By operating on scalars, **reductions sidestep the need to materialize intermediate outputs**, thus reducing memory allocations. This means that, if for example you have to compute [sum(log.(x))], a reduction would avoid the creation of the intermediate vector [log.(x)].

AN INTUITION

Reductions are implemented using a for-loop, with an operator applied to pairs of elements and the resulting output updated in each iteration. A classic example is the summation of all numeric elements in a vector. This involves applying the addition operator + to pairs of elements, iteratively updating the accumulated sum. This process is demonstrated below.

```
x = rand(100)
foo(x) = sum(x)
julia> foo(x)
48.447
```

```
x = rand(100)
function foo(x)
  output = 0.

for i in eachindex(x)
    output = output + x[i]
  end
  return output
end

julia> foo(x)
48.447
```

```
x = rand(100)

function foo(x)
    output = 0.

for i in eachindex(x)
        output += x[i]
    end

    return output
end

julia> foo(x)
48.447
```

In reductions, it's common to see implementations like the last tab, which are based on <u>update operators</u>. Recall that they turn an expression like x = x + a into x + a.

IMPLEMENTING REDUCTIONS

The implementation of reductions relies on either a <u>binary operator</u> or a two-argument function during each iteration. An example of a binary operator is +, which we used above. However, we could've also used its two-argument function form, replacing output = output + x[i] with output = +(output, x[i]).

The use of two-argument functions expands the scope of reductions, enabling us to compute more complex operations. For instance, it allows us to compute the maximum value of a vector x. This requires the function x, where x where x returns the maximum of the scalars x and y.

Formally, a reduction requires the binary operation to satisfy **two mathematical properties**:

• **Associativity**: the way in which operations are grouped must not change the result. For example, addition is associative because (a + b) + c = a + (b + c).

• Existence of an identity element: there exists an element that, when combined with any other element through a binary operation, leaves that element unchanged. For example, the identity element of addition is 0 because a + 0 = a.

The following list indicates the identity elements of each operation.

Operation Identity Element

Sum	0
Product	1
Maximum	-Inf
Minimum	Inf

Identity elements constitute the initial values of the iterative process. Based on them, we next implement reductions for several operations. The examples define the function fool to show the desired outcome, while fool provides the same output via a reduction.

```
x = rand(100)

foo1(x) = sum(x)

function foo2(x)
    output = 0.

for i in eachindex(x)
    output += x[i]
    end

    return output
end
```

```
x = rand(100)

fool(x) = prod(x)

function foo2(x)
    output = 1.

for i in eachindex(x)
    output *= x[i]
    end

   return output
end
```

```
x = rand(100)

foo1(x) = maximum(x)

function foo2(x)
    output = -Inf

for i in eachindex(x)
    output = max(output, x[i])
  end

return output
end
```

```
x = rand(100)

fool(x) = minimum(x)

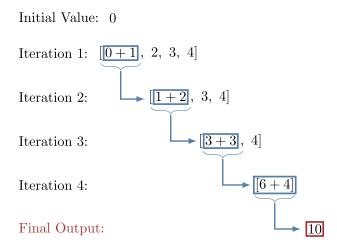
function foo2(x)
    output = Inf

for i in eachindex(x)
    output = min(output, x[i])
  end

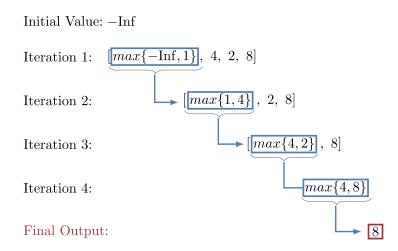
return output
end
```

Below, we visually illustrate how reductions operate in these examples.

REDUCTION 1: sum of [1,2,3,4]



REDUCTION 2: maximum of [1,4,2,8]



AVOIDING MEMORY ALLOCATIONS VIA REDUCTIONS

One of the primary advantages of reductions is that they avoid the memory allocation of intermediate results.

To illustrate, consider the operation $\boxed{\text{sum}(\log_{\cdot}(x))}$ for a vector \boxed{x} . Its computation involves transforming \boxed{x} into $\boxed{\log_{\cdot}(x)}$, and then summing the transformed elements. By default, broadcasting internally creates a new vector to store the result of $\boxed{\log_{\cdot}(x)}$, thereby allocating memory. However, in many cases like this one, we're only interested in the final sum, not the intermediate result. Therefore, an approach that bypasses the allocation of $\boxed{\log_{\cdot}(x)}$ is beneficial. Reductions accomplish this by defining a scalar $\boxed{\text{output}}$, which is iteratively updated by summing the transformed values of \boxed{x} .

```
x = rand(100)

foo1(x) = sum(log.(x))

function foo2(x)
    output = 0.

    for i in eachindex(x)
        output += log(x[i])
    end

    return output
end

julia> @btime foo1($x)
    315.584 ns (1 allocation: 896 bytes)

julia> @btime foo2($x)
    296.119 ns (0 allocations: 0 bytes)
```

```
x = rand(100)

foo1(x) = prod(log.(x))

function foo2(x)
    output = 1.

    for i in eachindex(x)
        output *= log(x[i])
    end

    return output
end

julia> @btime foo1($x)
    311.840 ns (1 allocation: 896 bytes)

julia> @btime foo2($x)
    296.061 ns (0 allocations: 0 bytes)
```

```
x = rand(100)

foo1(x) = maximum(log.(x))

function foo2(x)
    output = -Inf

for i in eachindex(x)
    output = max(output, log(x[i]))
    end

    return output
end

julia> @btime foo1($x)
    482.602 ns (1 allocation: 896 bytes)
julia> @btime foo2($x)
    374.961 ns (0 allocations: 0 bytes)
```

```
x = rand(100)

foo1(x) = minimum(log.(x))

function foo2(x)
    output = Inf

for i in eachindex(x)
    output = min(output, log(x[i]))
    end

    return output
end

julia> @btime foo1($x)
    487.156 ns (1 allocation: 896 bytes)

julia> @btime foo2($x)
    368.502 ns (0 allocations: 0 bytes)
```

REDUCTIONS VIA BUILT-IN FUNCTIONS

In the previous examples, reductions were implemented manually through explicit for-loops. While this approach makes the underlying mechanics transparent, it also introduces considerable verbosity. In real-world scenarios, for-loops tend to compromise code readability. To address this issue, Julia provides several streamlined alternatives for expressing reductions in a more concise and idiomatic way.

In particular, several functions come with an additional method for applying reductions to a transformed version of the input vector. Such methods are present for functions like $\boxed{\text{sum}}$, $\boxed{\text{prod}}$, $\boxed{\text{maximum}}$, and $\boxed{\text{minimum}}$. The syntax is given by $\boxed{\text{foo}(<\text{transforming function}>, \times)}$, where $\boxed{\text{foo}}$ is one of the functions mentioned and $\boxed{\textbf{x}}$ is the vector to be transformed. For instance, the following examples consider reductions for a log transformation.

```
x = rand(100)
foo(x) = maximum(log, x) #same output as maximum(log.(x))
julia> @btime foo($x)
579.940 ns (0 allocations: 0 bytes)
```

```
x = rand(100)
foo(x) = minimum(log, x) #same output as minimum(log.(x))
julia> @btime foo($x)
577.516 ns (0 allocations: 0 bytes)
```

To keep matters simple, we've used the built-in function \log for transforming x. However, the approach supports any form of function, including anonymous ones.

```
x = rand(100)
foo(x) = sum(a -> 2 * a, x)  #same output as sum(2 .* x)
julia> @btime foo($x)
  6.493 ns (0 allocations: 0 bytes)
```

```
x = rand(100)

foo(x) = prod(a -> 2 * a, x)  #same output as prod(2 .* x)

julia> @btime foo($x)
  6.741 ns (0 allocations: 0 bytes)
```

All these functions additionally accept transforming functions that require multiple arguments. To incorporate this possibility, it's necessary to call the multiple variables using $\boxed{\mathtt{zip}}$, and referring to each variable through indices. We illustrate this below, where the transforming function is $\boxed{\mathtt{x} \cdot \mathtt{*} \cdot \mathtt{y}}$.

```
x = rand(100); y = rand(100)

foo(x,y) = maximum(a -> a[1] * a[2], zip(x,y)) #same output as maximum(x .* y)

julia> @btime foo($x)

172.580 ns (0 allocations: 0 bytes)
```

```
x = rand(100); y = rand(100)

foo(x,y) = minimum(a -> a[1] * a[2], zip(x,y)) #same output as minimum(x .* y)

julia> @btime foo($x)
    166.969 ns (0 allocations: 0 bytes)
```

THE "REDUCE" AND "MAPREDUCE" FUNCTIONS

Beyond the specific functions discussed, reductions can be applied to any operation that meets the requirements of a reduction. In Julia, this generality is provided through the functions reduce and mapreduce. The function reduce applies a binary operation directly to the elements of a collection, combining them into a single result. By contrast, mapreduce first transforms each element of the collection and then applies the reduction, thereby unifying the roles of map and reduce in a single step.

It's worth remarking that reductions involving sum, prod, max, and min should still be done via their dedicated functions. These specialized functions have been carefully optimized for their respective tasks and typically outperform the more general alternatives. Consequently, our primary use case of reduce and mapreduce is for other types of reductions not covered or when packages provide their own implementations for reductions. ²

FUNCTION "REDUCE"

The function reduce uses the syntax reduce(<function>, x), where <function> is a two-argument function. The following example demonstrates its use.

```
x = rand(100)
foo(x) = reduce(+, x)  #same output as sum(x)

julia> @btime foo($x)
   6.168 ns (0 allocations: 0 bytes)
```

```
x = rand(100)

foo(x) = reduce(*, x)  #same output as prod(x)

julia> @btime foo($x)
  6.176 ns (0 allocations: 0 bytes)
```

```
x = rand(100)
foo(x) = reduce(max, x) #same output as maximum(x)

julia> @btime foo($x)
    167.905 ns (0 allocations: 0 bytes)
```

Note all the examples presented could've been implemented as <u>we did previously</u>, where we directly applied <code>sum</code>, <code>prod</code>, <code>maximum</code> and <code>minimum</code>.

FUNCTION "MAPREDUCE"

The function [mapreduce] combines the functions [map] and [reduce]: before applying the reduction, [mapreduce] transforms vectors [via] the function [map]. Recall that [map(foo,x)] transforms each element of the collection [x] by applying [foo] element-wise. Thus, [mapreduce(<transformation>, <reduction>, x)] first transforms [x]'s elements through [map], and then applies a reduction to the resulting output.

To illustrate its use, we make use of a log transformation.

```
x = rand(100)

foo(x) = mapreduce(log, *, x) #same output as prod(log.(x))

julia> @btime foo($x)

294.618 ns (0 allocations: 0 bytes)
```

```
x = rand(100)

foo(x) = mapreduce(log, max, x) #same output as maximum(log.(x))

julia> @btime foo($x)

579.808 ns (0 allocations: 0 bytes)
```

Just like with reduce, note that the examples could've been implemented directly as we did previously, through the functions sum, prod, maximum, and minimum.

mapreduce can also be used with anonymous functions and transformations requiring multiple arguments. Below, we illustrate both possibilities. Note that the implementations are the same as with sum, prod, maximum, and minimum.

```
x = rand(100); y = rand(100)

foo(x,y) = mapreduce(a -> a[1] * a[2], +, zip(x,y)) #same output as sum(x .* y)

julia> @btime foo($x)

29.165 ns (0 allocations: 0 bytes)
```

```
x = rand(100); y = rand(100)

foo(x,y) = mapreduce(a -> a[1] * a[2], *, zip(x,y)) #same output as prod(x .* y)

julia> @btime foo($x)

48.221 ns (0 allocations: 0 bytes)
```

```
x = rand(100); y = rand(100)

foo(x,y) = mapreduce(a -> a[1] * a[2], max, zip(x,y)) #same output as maximum(x .* y)

julia> @btime foo($x)

175.634 ns (0 allocations: 0 bytes)
```

REDUCE OR MAPREDUCE?

In principle, reduce could be considered as a special case of mapreduce, where the latter transforms x through the identity function identity(x). Likewise, mapreduce(<transformation>, <operator>, x)
produces the same result as reduce(<operator>, map(<transformation>, x))

Nonetheless, mapreduce should be employed in those cases where the vector input must be transformed. The reason is that mapreduce avoids the internal memory allocations of the transformed vector. This is demonstrated below, where we compute sum(2 .* x) through a reduction.

```
x = rand(100)

foo(x) = mapreduce(a -> 2 * a, +, x)

julia> @btime foo($x)
6.372 ns (0 allocations: 0 bytes)
```

FOOTNOTES

^{1.} In the section <u>Lazy Operations</u>, we'll explore an alternative based on broadcasting that doesn't materialize intermediate results either.

^{2.} For instance, the package Folds provides a parallelized version of both map and mapreduce, enabling the utilization of all available CPU cores. Its syntax is identical to Julia's built-in functions.