Julia As Your First Programming Language: A Book for Scientists

(Version September 30, 2025)

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WELCOME TO THE SITE!

The website is still work in progress in terms of writing, content, and subjects covered.

The chapters included so far can be found **here** and a pdf version **here**. I'll continue adding new content as I go.

If you find mistakes, typos, or have any suggestions, please open an issue on the **book's GitHub page**. Your feedback is greatly appreciated!

WHY JULIA?

Scientific computing demands both an intuitive design and execution speed. Julia is built to deliver both.

SIMPLE AND EXPRESSIVE: Julia's syntax is concise and close to mathematical notation, so you can translate models into readable code that's easy to maintain.

INTERACTIVE PROTOTYPING: rapidly build and explore models in real-time, enabling quick iteration and immediate result inspection.

HIGH PERFORMANCE: Julia is engineered for speed. It's capable of achieving speeds comparable to C or Fortran, without forcing you to rewrite performance-critical parts in another language. Moreover, it offers seamless support for parallel computing, including native multithreading and GPU acceleration.

A UNIFIED COMPUTATIONAL PLATFORM: No need to switch between multiple languages. In Julia, you can preprocess data, perform statistical analysis, generate graphical representations, and implement numerical models. Furthermore, Julia interoperates with other languages and tools, allowing you to leverage the strengths of Python, R, or C++. Its ecosystem includes mature packages for data analysis, plotting, and computational modeling.

A QUICK OVERVIEW OF THE BOOK

AUDIENCE: The book is intended **for an audience with little or no background in programming**. This doesn't mean that we solely cover basic topics. Rather, it defines the book's approach of starting from elementary concepts, **gradually introducing more advanced concepts as we progress**.

APPROACH: Throughout the book, **I've made a conscious effort to distinguish between what's essential and what's ancillary**, with the latter clearly labelled as optional. My goal is that you don't become bogged down in particular details, while still having the possibility of exploring topics further if you wish.

TOPICS: The book focuses on the foundational concepts of the language, without pursuing an exhaustive examination of all its features. My philosophy is that **you can easily incorporate additional features if you grasp the logic of the language**.

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1a. Installation and Resources

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INTRODUCTION

We start by covering the essential steps to install Julia and VS Code. The latter is a code editor to write and execute code in multiple languages. We'll conclude by providing some valuable online resources for Julia's users.

INSTALLING JULIA

> **Remark**
> All the links mentioned on the website are included in Links, located in the left navigation bar.

To download Julia and access its official documentation, visit **Julia's official website**. Note that the installation process depends on your computer's operating system.

INSTALLING VS CODE

Once Julia is installed, you'll need an editor to write scripts and visualize outputs. There are numerous alternatives in this respect. **Our website supposes that you use Visual Studio Code (aka VS Code)**, which is free, officially supported by Julia, and runs on any operating system (i.e., Windows, macOS, and Linux). One of the key benefits of VS Code is the possibility of installing plugins to extend the editor's capabilities. In fact, you'll need to add the *Julia Language Support* plugin for running Julia.

> **Privacy-Oriented Version of VS Code**
> VS Code is open-source software created and maintained by Microsoft. If you want a more private alternative that disables telemetry and tracking, **VSCodium** is a rebuild of VS Code.

Links to other popular editors can be found on **Useful Links**, including Vim, Emacs, NotePad, and Sublime. These editors are officially supported by Julia (except Sublime). I strongly recommend getting proficient in either VS Code or one of these alternatives. This will allow you **to master a single tool for coding in multiple programming languages**.

> **Warning!**

> Avoid getting used to specialized editors built for one particular language, such as RStudio for R (or its newer version **Posit**). The editors I mentioned were designed for coding, regardless of the programming language you choose. Mastering a general code editor will enhance your coding skills—you'll be able to apply the same tools and keyboard shortcuts to every language you work with.

JULIA'S RESOURCES FOR HELP

There are two official resources for learning Julia.

1. **Julia's official documentation**. Written by Julia's developers.
2. **Julia Discourse**. Official forum for asking questions.

INSTALLING R AND PYTHON (*OPTIONAL*)

Julia offers a seamless integration with other programming languages like R and Python, allowing you to export data from Julia, perform specific operations, and then import the results back into Julia. This interoperability is particularly useful when a desired function is only available in one of these languages.

For those familiar with R and Python, **this note** outlines some noteworthy differences with respect to Julia. Additionally, this **cheat sheet** provides a quick reference on syntax differences among Matlab, Python, and Julia.

1b. Running Julia

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INTRODUCTION

In the following, we cover the basic steps for getting started with Julia. As we haven't introduced any tools available in Julia (e.g., functions), we'll keep the discussion to a bare minimum. Specifically, we'll limit ourselves to setting up Julia in VS Code and presenting methods to add comments and file paths.

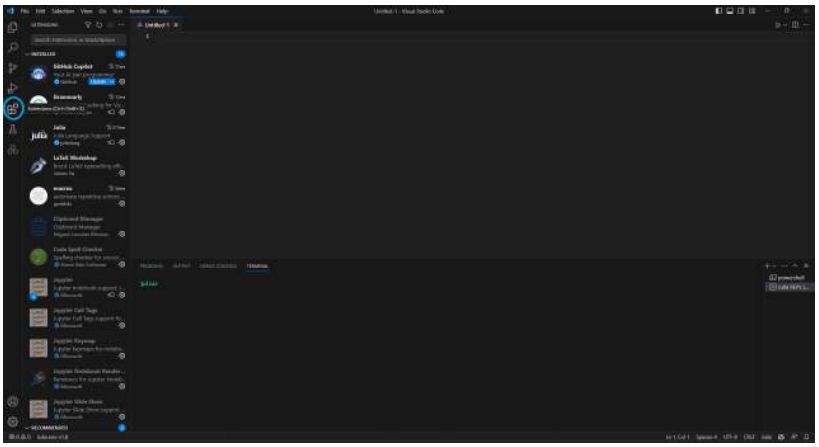
USING JULIA IN VS CODE

The REPL (Read-Eval-Print Loop) is an interactive programming environment. It lets users input commands and immediately obtain outputs through a command-line interface. When you run `julia.exe`, the REPL is automatically activated and displays the `julia>` prompt, where you can enter commands.

▼ Screenshot

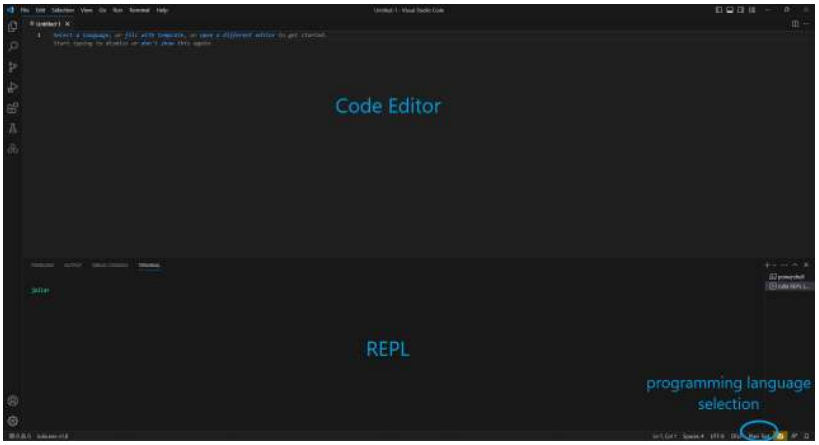
Throughout this website, we'll assume that you're working with a code editor, rather than interacting directly with the REPL. In particular, VS Code will be our code editor of choice, including its privacy-focused alternative VS Codium. To get started, you'll need to install the Julia extension of VS Code. This can be found by navigating to the Extensions tab, as indicated below by the blue circle.

▼ Screenshot



The layout of VS Code displays the REPL at the bottom of the screen, with code written in the area above it. To execute code, you'll also need to specify the programming language you're using. This can be achieved by clicking on the language option located at the bottom corner of the screen, or by using the keyboard shortcut Ctrl+k + M and typing "julia". All this is demonstrated in the screenshot below.

▼ Screenshot



ADDING COMMENTS IN A SCRIPT

Comments are text annotations ignored during execution, serving as a means to document your code. To add a *single-line comment*, simply precede the text with the # symbol. This symbol can be placed anywhere on a line, with any text that follows disregarded by Julia. Alternatively, you can add *multi-line comments* by delimiting the text with #= at the beginning and =# at the end.

```
# This is an example of a comment

x = 2   # `x=2` is run, but anything after `#` won't

#=  This is an example of a longer comment.
    It can be split into several lines, and
    can have any length. =#
```

PATHS OF FILES AND FOLDERS

File management systems vary across operating systems, determining that the syntax for file paths also differs. To accommodate this, Julia provides two approaches. The first one provides an operating system-specific syntax. Below, we illustrate its application for a file `C:\user\file.jl` on Windows and `/user/file.jl` on Linux/macOS. There's also a platform-agnostic alternative to make your code more portable, provided by the `joinpath` function. This is the preferred option, as it can be used with any operating system.

```
# On Windows (note the double \\)
"C:\\user\\file.jl"

# On Unix-based systems (e.g., macOS or Linux)
"/user/file.jl"

# on any operating system
joinpath("/", "user", "file.jl")
```

Two special paths have convenient shortcuts worth mentioning:

- `@__DIR__` identifies the directory where your script is saved.
  For instance, if your script is in `C:\user\julia`, then `joinpath(@__DIR__, "graphs")` refers to `C:/user/julia/graphs`.

- `homedir()` indicates the user's home directory.
  This refers to `C:\Users\username` on Windows (where "username" is your actual user), and is the equivalent of `~` on Linux. For instance, you could access your Google Drive's folder located on either `C:\Users\username\GoogleDrive` or `\home\username\GoogleDrive` by the command `joinpath(homedir(), "GoogleDrive")`.

EXECUTING CODE FROM A FILE

Julia also allows you to work **non-interactively**, by executing code from a script stored in a file. The following example illustrates its implementation, running a file located at `C:\user\julia\graphs.jl` on Windows and at `/users/julia/graphs.jl` on macOS/Linux systems.

```
include(joinpath("/", "user", "julia", "graphs.jl"))
```

1c. VS Code (OPTIONAL)

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FEATURES AND KEYBOARD SHORTCUTS

We present a few keyboard shortcuts and handy features for VS Code. They also apply to its privacy-focused alternative VS Codium. Remarkably, these features are largely language-agnostic, holding regardless of the programming language you're working with.

For visual illustration, the features discussed are accompanied by GIFs. To view these GIFs, simply click "Example", or alternatively press Alt+↑ or Alt+↓ to open and close all of them simultaneously.

TO RUN A SCRIPT

Select the script to be executed and press Ctrl+Enter

▶ Example

TO FORMAT EXPRESSIONS AND MAKE THEM MORE LEGIBLE

Select the script to be formatted and press Ctrl+k + Ctrl+f. Sometimes, activating this tool requires running it twice.

▶ Example

TO ALIGN EQUAL SIGNS

This feature requires the VS Code Extension "Better Align". It aligns consecutive lines by using the equal sign and other symbols as a reference. It's implemented by pressing Alt + a.

▶ Example

See also the extension "Cursor Align", which aligns code by clicking the position on each line.

TO EXTEND THE CURSOR VERTICALLY

Hold down Alt+Ctrl + press ↑ or ↓

▶ Example

TO SEE THE DOCUMENTATION OF A FUNCTION

It requires hovering over the function.

▶ Example

Alternatively, you can go to the REPL, press ?, and then type the function's name you want to search for.

▶ Example

TO AUTOCOMPLETE A WORD

Start typing a word + press Tab when you see the option list.

▶ Example

TO INTRODUCE UNICODE CHARACTERS (TAB COMPLETION)

Type a unicode character/command, press Ctrl + Space to open an option list, and then choose the option and press Tab.

▶ Example

In Julia, Greek letters and math have the same syntax as Latex. To add them, you need to start with \ (e.g., `\eq` for ≠) and use Tab completion.

TO SELECT THE SAME WORD MULTIPLE TIMES

Select the word and then press Ctrl+d for selecting each additional time it appears. This is useful when you want to change part of the expression.

▶ Example

TO HIDE PART OF THE SCRIPT

Given a code block, add `#region` at the beginning and `#endregion` at the end.

▶ Example

When you have several lines indented, VS Code allows you to hide the block automatically. The following example shows this for a function.

▶ Example

TO TURN MULTIPLE LINES INTO A COMMENT

Select all the lines you want to interpret as a comment rather than code. Then, press Ctrl + /.

▶ Example

1d. A Minimal Set of Good Practices

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REMARKS

We conclude this chapter by reviewing various principles to write code. They represent a minimum set of good practices that apply regardless of the programming language used. By adhering to these guidelines, you'll be able to write clear and maintainable code. I suggest incorporating these suggestions into your workflow from the very beginning, as it'll render the learning process smoother.

Several of the suggestions we present might seem inconsequential to you at this point, or give the impression that their importance is exaggerated. For small projects, there's some truth to this—they won't have a substantial impact. However, as projects grow in size and complexity, following these principles becomes crucial. [1] It's not uncommon to revisit your own code after a few months (or even days!) and struggle to understand it. When this occurs, extending the code becomes a daunting task, often resulting in non-reusable code.

As usual, the devil is in the details: the challenge here lies in interpreting and implementing these suggestions effectively. Many of them rely on the reader's judgment, as they require a subjective assessment of when and how to apply them. For example, one suggestion we'll present is to use clear and descriptive names. However, determining what constitutes "clear" or "unclear" is ultimately a matter of personal interpretation. Hopefully, the implementation of the suggestions will become apparent as we move forward and apply these concepts.

WRITE EASY-TO-READ CODE

Code is read more often than it's written. I can't stress enough the importance of this statement. It has a stark implication: write code that is easy to read, even if this requires additional effort or some extra verbosity.

If you end up coding extensively in your future career, you'll likely learn this lesson the hard way. I certainly did. One of the first times I had to reuse an old script, I was completely clueless about my own code. As a consequence, I had to rewrite the entire script from scratch, as making sense of the old code would've taken longer.

> **Remark**
> If you're concerned that more readable code requires excessive typing, remember that you can use Tab Completion to autocomplete names. Additionally, AI tools like GitHub Copilot will suggest code while you type, thereby also mitigating the inconvenience.

To illustrate this point, suppose you're reading a script that cleans some data. Imagine in particular that you come across a line that has two possible expressions: `na.rm=TRUE` and `dropmissing=true`. Even if you're unfamiliar with the language's syntax or the concept of missing data, you could likely infer the meaning of `dropmissing=true`: discard entries with no values provided. On the contrary, `na.rm=TRUE` offers no clue. Although this example may appear somewhat abstract, it actually highlights how to discard missing observations in R and Julia: `na.rm=TRUE` corresponds to R and `dropmissing=true` to Julia. [2]

The example also reveals why typing `na.rm=TRUE` might be tempting: it's short and requires less typing. However, it's essential to weigh the long-term benefits of readable code. Although typing more might seem inconvenient in the short term, it represents a minimal effort compared to the future costs of ambiguous code. Moreover, you may feel confident that you'll remember what you intended to write, but it's common to be puzzled by code you wrote just days before.

The benefits of clear code become apparent when you read a script written in an unfamiliar programming language: if the code is well-written and clearly structured, you might grasp the logic and tasks being performed. [3]

Several tips arise as a consequence of this. We list them below.

USE NAMES WITH A CLEAR MEANING

Clear names don't only refer to variables and functions, but files as well. In particular, you should avoid abbreviating. Code editors can be very helpful in this regard, by offering word auto-completion. This feature requires typing the first letters of each word and then pressing Tab. [4]

Avoiding abbreviations has the additional benefit of making it easier to substitute expressions. For instance, suppose you name a variable `re`, and later decide to replace it with a different name. Then, the substitution process becomes more challenging, as the search will also capture functions like `replace` and `repeat`.

Finally, using descriptive names reduces the need for comments. If the code is self-explanatory, comments become only necessary for exceptionally complex code or clarifications that go beyond what's written.

INDENT AND ALIGN YOUR CODE

The implementation details of this suggestion have already been covered in the previous section. For further details, please refer to that section.

When writing code sequentially, VS Code automatically provides indentation. You can also format a selected portion of code by pressing Ctrl + k, followed by f. Alternatively, to format the entire script, use the shortcut Alt + Shift + L. [5]

To illustrate how this feature improves readability, consider the following (somewhat exaggerated) example.

```
if x>0 display("x is a positive number") else display("x is a non-positive number") end

function example(a,b)
x=a/10#rescaling x
output=2*b+x
return output
end
```

```
if x > 0
    display("x is a positive number")
else
    display("x is a non-positive number")
end

function example(a, b)
    x      = a / 10                # rescaling x
    output = 2 * b +  x

    return output
end
```

To further improve readability, I suggest also aligning code blocks. Several plugins in VS Code can assist with this task, such as "Better Align" and "Cursor Align". Their use is demonstrated below.

```
this_is_a_variable = 1
x = 3
another_var = 2

computations_here = x + another_var
more_calcs = this_is_a_variable * another_var
```

```
this_is_a_variable = 1
x                  = 3
another_var        = 2

computations_here  = x + another_var
more_calcs         = this_is_a_variable * another_var
```

FOOTNOTES

[1.] For real-world examples, read "Brief Story" on this link or the perspective of a former worker from Oracle.

[2.] Python also tends to employ abbreviations that can hinder readability. For instance, to count the number of characters of a variable `x`, Python calls `len(str(x))` while Julia calls `length(string(x))`.

[3.] One way to learn how to write clear code is through AI chatbots, which are pretty good at providing highly readable examples.

[4.] You could eventually use the option "find and replace", whereby you substitute abbreviations for their full name. However, this is error-prone, and you may end up replacing unrelated expressions by substituting all words at once.

[5.] Unlike Python, Julia only uses indentation for readability purposes. It doesn't affect how code is executed.

2a. Overview and Goals

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> **Remark**
> Throughout the book, I made some deliberate choices regarding whether and when to introduce certain subjects. Considering this, I'll include a section called "Overview and Goals" prior to each chapter, which elucidates my rationale for these choices. The goal is to contextualize the book's approach, offering readers some guidance on the best way to engage with the material.

The current chapter introduces the concept of variables and types, covering single-element objects (numbers and characters) and collections (primarily vectors and tuples). At this early stage, **we only scratch the surface of the topics**. In particular, the chapter doesn't cover any object in depth, and even excludes important ones like dictionaries. The reason is pedagogical: I didn't want to overwhelm readers with details about objects or types, considering that core programmatic concepts like functions and for-loops haven't yet been introduced.

In light of this, Chapter 2 should be understood as a minimal background on objects, sufficient for progressing into the basics of working programmatically.

The main skills you should gain from Chapter 2 are:

- familiarizing yourself with Julia's syntax, and
- distinguishing between scalars (single-element objects) and collections.

2b. Variables, Types, and Operators

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INTRODUCTION

This section introduces the concepts of **variables** and **types**. We'll also present the notion of **operators**, focusing on their syntax. To ensure a smooth learning experience, I've minimized the reliance on objects that we haven't covered yet. The only one introduced is vectors, whose elements are enclosed in brackets (e.g., `[1,2,3]`).

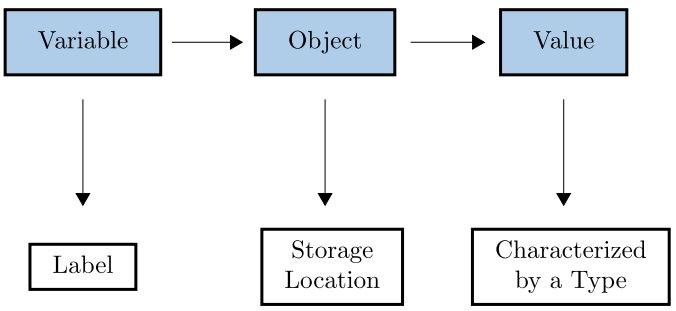
VARIABLES

When a program is executed, the computer stores data in RAM (Random Access Memory). Each piece of data in RAM is referred to as an **object** and is assigned a unique memory address. These addresses are typically represented in hexadecimal format (e.g., `0x00007e0966dc0dd0`).

Furthermore, every object is associated with a value and a type. **Values** represent the actual data contained within the object. In turn, **types** define the nature of the data stored, providing the computer with critical information for handling the object internally.

Since directly referencing memory addresses would be impractical, we instead define **variables**. They act as human-readable **labels** for objects, simplifying our interaction with the data. Linking objects with a variable relies on the so-called **assignment operator** `=`, which creates a binding between the variable name and the object's memory location. [1] This allows developers to interact with data through symbolic identifiers, rather than raw memory locations.

VARIABLES

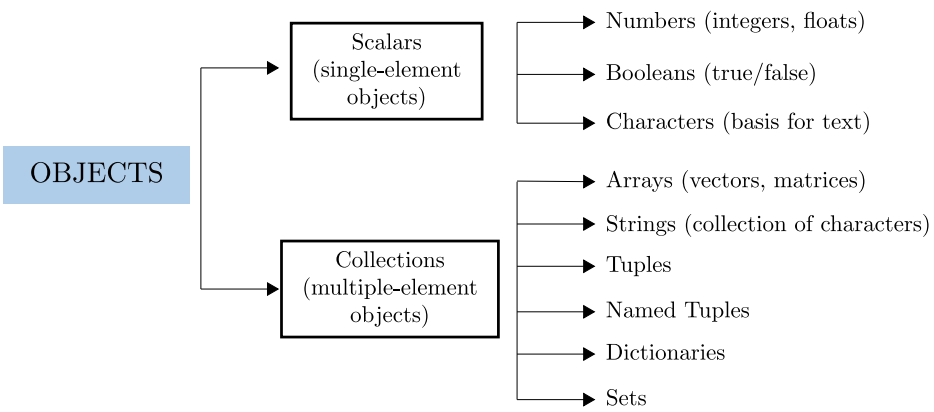


To illustrate this process, let's consider executing the command `x = "Hello"`. When this is run, several actions take place. First, the computer reserves a memory location to store the object (e.g., at address `0x1234`). This object is then assigned the specific value `"Hello"`, which in Julia is an instance of the type `String`.

At the same time, we're assigning the label `x` to this object, so that `x` points to the memory address `0x1234`. This means that, every time we use `x` in our code, we're actually accessing the object stored at memory address `0x1234`. It's important to note that `x` isn't the object or the value itself, but rather a reference to the memory allocation `0x1234`. This explains why we can define multiple labels to reference exactly the same object (i.e., the same memory address).

CLASSIFICATION OF OBJECTS

Objects are typically characterized according to the number of elements they contain: **scalars** refer to single-element objects, and **collections** refer to objects containing multiple elements. Below, we outline some objects encompassed in each category.



NAMES FOR VARIABLES

Variable names in Julia can be defined using Unicode characters, thus offering a wide range of possibilities. This feature enables you to use Greek letters, Chinese characters, symbols, and even emoticons. [2] Underscores `_` are also permitted, which can be helpful for separating words within variable names (e.g., `intermediate_result`). [3] Importantly, names are case-sensitive, so that `bar` and `Bar` are treated as two distinct variables.

```
a          = 2
A          = 2              # variable `A` is different from `a`

new_value = 2               # underscores allowed

β          = 2              # Greek letters allowed

中國        = 2               # Chinese characters allowed

x̄          = 2              # decorations allowed
x₁         = 2
ẋ          = 2

🐒         = 2              # emoticons allowed
```

Warning!

Julia doesn't let you delete variables. Once a variable is created, it remains in memory until the program terminates. If a variable is taking up too much memory, you can free up space by reassigning it to a smaller object.

Notation for Variable Names

Julia's developers adopt the convention of using **snake-case notation for variable names**. This format consists of lowercase letters and numbers, with words separated by underscores. (e.g., `snake_case_var1`). Note that this is only a convention, not a language's requirement.

UPDATING VARIABLES

It's possible to assign a new value to a variable using the variable itself. This approach is referred to as **updating a variable**.

```
x  = 2

x  = x + 3    # 'x' now equals 5
```

Julia offers a concise syntax for updating values, based on the so-called **update operators**. They're implemented by prefixing the assignment operator `=` with the operator to be applied, as demonstrated below.

```
x  = 2

x  = x + 3
x += 3        # equivalent

x  = x * 3
x *= 3        # equivalent

x  = x - 3
x -= 3        # equivalent
```

TYPES

Before diving into the intricacies of Julia, it's essential to familiarize yourself with the basics of its type system. This initial overview will only provide the minimum necessary for the upcoming chapters. A comprehensive treatment of types, including their role in performance optimization, will be deferred to Part II of this website. For now, the focus is on core definition and notation.

> **Notation for Types**
> Julia's developers adopt the convention of using **CamelCase** notation for denoting types, where every first letter is capitalized (e.g., `MyType`). Note that this is only a convention, not a language's requirement.

As previously mentioned, types define the nature of values, specifying all the information the computer needs for their storage and manipulation. To better illustrate types, let's split the discussion in terms of scalars and collections.

Common numeric types for scalars include `Int64` for integers, `Float64` for decimal numbers, and `Bool` for binary values (`true` and `false` values). [4] Likewise, the type `Char` represents individual characters, serving as the building block for the `String` type. `String` is the standard type in Julia for representing text, and its values consist of sequences of characters.

Collections, on the other hand, often require **type parameters** for a full characterization of their types. These parameters can be incorporated into any type, and have the goal of providing additional information about its contents.

Type parameters are denoted using `{}` after the type's name. For instance, the type `Vector{Int64}` indicates that the collection represents a vector exclusively containing elements of type `Int64` (e.g., `[2, 4, 6]`). Here, `Int64` serves as a type parameter. Note that type parameters are optional and therefore can be omitted when not needed. Indeed, this is the case with the types for scalars mentioned above.

> **Type Annotations**
> You can explicitly declare the type of a variable by using **type annotations**, via the `::` operator. For example, `x::String` ensures that `x` can only store string values throughout the program, resulting in an error if you attempt to reassign `x` with a value of a different type.

CONCRETE TYPES AND ABSTRACT TYPES

In Julia, **types are organized hierarchically**, creating relations of supertypes and subtypes. This hierarchy gives rise to the notions of abstract and concrete types.

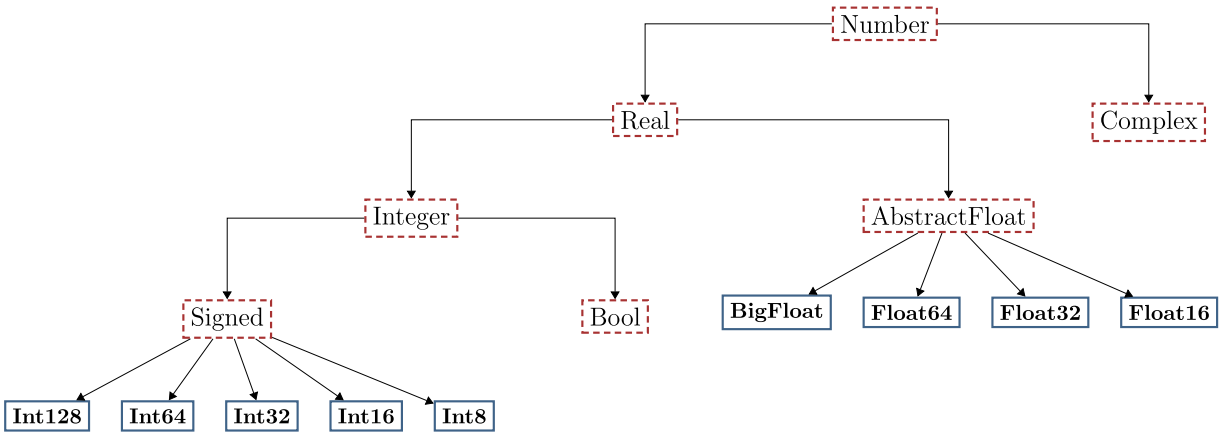
An **abstract type** is a set of types that serve as a parent to other types. The `Any` type in Julia is a prime example of abstract type. It acts as the root of the hierarchy, thus comprising all possible subtypes—by definition, every type in Julia is a subtype of `Any`.

In contrast, a **concrete type** is an irreducible unit, representing a terminal node in the hierarchy and therefore lacking subtypes. Concrete types include in particular primitive types, which represent the most fundamental types that computers use to perform calculations. Examples of primitive types are `Int64` and `Float64`, which directly map to low-level hardware representations.

Abstract types provide great flexibility for writing code. For example, the abstract type `Number` defined in Julia encompasses all possible numeric types (e.g., `Float64`, `Int64`, `Float32`). By declaring a variable as `Number`, programmers avoid unnecessarily constraining their programs to specific numeric representations or precision.

To demonstrate this hierarchy, we consider the concrete types comprised by `Number`. The names included in the table match the exact names in Julia. Note, nonetheless, that the full subtype hierarchy of `Number` is broader than the simplified representation presented. [5]

EXAMPLE OF THE ABSTRACT TYPE "NUMBER"



Note: Dashed red borders indicate abstract types, while solid blue borders indicate concrete types.

OPERATORS

In programming, **operators** are symbols that represent operations performed on objects. They can be thought of as syntactic sugar for functions, as we'll see in the next chapters. In fact, almost all operators in Julia can be employed as functions.

For instance, the symbol `+` in `x + y` is an operator that performs the addition of `x` and `y`. Likewise, the symbols `x` and `y` are referred to as the **operands**, representing the operator's inputs to perform its calculation. *Operators follow specific syntax rules based on the number of operands they require*. Understanding this syntax will prove useful for several topics covered later on the website. Next, we define and illustrate the syntax through several examples. At this point, just focus on how operators are written, even if their specific functions are not yet clear.

- **Unary operators**: They take *one operand*, with the operator written to the left of it. [6] Formally, their syntax is `<operator>x`, such as `√x` or `-x`.

- **Binary operators**: They take *two operands*, and the operator is written between them. [7] Formally, their syntax is `x <operator> y`, such as `x + y` or `x^y` for x^y.

- **Ternary operators**: They take *three operands*. Formally, their syntax is `x <operator1> y <operator2> z`. Ternary operators are rare, which is why the specific operator `x ? y : z` is directly referred to as *the* ternary operator. We'll see that this operator performs a conditional evaluation, returning `y` if `x` is true and `z` returned otherwise.

FOOTNOTES

[1.] While it's common to say that "a variable has a specific type", this is a simplification. Technically, it's the value of the variable that has a specific type, not the variable itself.

[2.] You can insert Unicode characters by copying and pasting them from a list like this one. Alternatively, you can use tab completion with the commands listed in the Julia documentation.

[3.] Not all symbols are allowed. For instance, names with common mathematical symbols like `x^` or `%x` aren't permitted. Additionally, numbers are allowed, but they can't be included as the first character (e.g., `2x` is invalid).

[4.] The suffix `64` in these types represents the precision of the number. This represents the maximum number of significant digits or decimals that a type can accurately represent.

[5.] The subtype `Signed` from `Integers` represents positive or negative integers. Although not included in the graph, there's also a type called `Unsigned`, which only accepts positive integers.

[6.] Operators to the left of the operand are known as **prefix operators**. Conversely, operators written to the right of the operand are known as **postfix operators**, and Julia has a few of them (e.g., `'` to transpose a vector or matrix `x`, which is written as `x'`). Despite this, we won't use postfix operators on this website.

[7.] Operators with this syntax are called **infix operators**.

2c. Numbers

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INTRODUCTION

The previous section introduced the concept of variables, distinguishing between those containing a single element (scalars) and collections. This section expands on scalars, exclusively focusing on those holding numeric values.

NUMBERS

Computers store numbers in various formats, treating integers and decimal numbers as separate entities. Even within each category of numbers, multiple representations emerge depending on the intended level of precision. This precision is determined by the number of bits allocated to store values in memory, which in turn defines the maximum range of values that a data type supports. [1] The representation just described extends well beyond Julia, and is intrinsic to how computers operate at a fundamental level.

In modern computers, numbers typically have a default size of 64 bits, and Julia's default types for numbers are:

- `Int64` for integers.
- `Float64` for decimal numbers. [2]

> **Remark**
> Julia provides the type `Int` as a more versatile option than `Int64`, which adapts to your computer's architecture: `Int` defaults to `Int64` on 64-bit systems and `Int32` on 32-bit systems. Since most modern machines operate on a 64-bit architecture, `Int` typically defaults to `Int64`. Note that there's no equivalent type `Float` for floating-point numbers, with Julia always defaulting to `Float64`.

It's worth emphasizing that `Int64` and `Float64` are two different data types. Thus, while `1` is a value with type `Int64`, the same value becomes `1.0` as a `Float64` type.

NUMBERS

```
x = 1        # `Int64`

y = 1.0      # `Float64`
z = 1.       # alternative notation for `1.0`
```

Remark

To enhance code readability, you can break up long numbers by inserting underscores `_`.

NOTATION FOR NUMBERS

```
x = 1000000
y = 1_000_000              # equivalent to `x` and more readable

x = 1000000.24
y = 1_000_000.24               # '_' can be used with decimal numbers
```

The type `Float64` encompasses not only decimal numbers, but also two special values: `Inf` for infinity and `NaN` for indeterminate expressions such as 0/0 (`NaN` stands for "not a number"). Considering this, all the following variables have type `Float64`.

FLOAT64

```
x = 2.5

y = 10/0

z = 0/0
```

```
julia> x
2.5
julia> y
Inf
julia> z
NaN
```

BOOLEAN VARIABLES

A distinct numeric type is `Bool`, which facilitates the representation of **Boolean variables**. These variables can only take on the values `true` and `false`. Internally, they're implemented as integers, with `true` corresponding to `1` and `false` to `0`. Because of this implementation, Julia accepts `1` and `0` interchangeably with `true` and `false`.

Boolean expressions come into play when evaluating conditions, such as checking whether a number exceeds a certain value or whether a string matches a specific pattern. These conditional evaluations yield Boolean values, and can then be employed to control the flow of the program. Some examples of Boolean values are presented below.

BOOLEAN VARIABLES

```
x = 2
y = 1

z = (x > y)        # is `x` greater than `y` ?
z = x > y          # same operation (don't interpreted it as 'z = x')
```

```
julia> z
true
```

ARITHMETIC OPERATORS

Numbers can be manipulated through a variety of **arithmetic operators**. These operators are represented by symbols akin to those in other programming languages.

Julia's Arithmetic Operator	Meaning
`x + y`	addition
`x - y`	subtraction
`x * y`	product
`x / y`	division
`x^y`	power (x^y)

It's worth noting that all the operators presented above are *binary*. Consequently, they adhere to the syntax `x <symbol> y`, as indicated in our discussion on operators.

FOOTNOTES

[1.] For instance, 8-bit integers can only represent values from -128 to 127. Likewise, 32-bit floating-point numbers, used for decimal numbers, can represent up to 7 significant digits of precision.

[2.] The term "Float" stands for "floating point" and is how computers represent decimal numbers.

2d. Strings

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INTRODUCTION

This section presents types for text representation, distinguishing between characters and strings. The coverage will be concise, as the website won't focus on string analysis. However, a minimal coverage is necessary as string variables are important for tasks like specifying paths, saving files, and other core functionalities.

CHARACTERS

In Julia, the `Char` type is used to represent individual characters. A character `x` is defined by using single quotes, as in `'x'`. Given its support for Unicode characters, `Char` encompasses not only numbers and letters, but also a wide range of symbols. This is shown below.

```
# x equals the character 'a'
x = 'a'

# 'Char' allows for Unicode characters
x = 'β'
y = '🐒'
```

Notice that characters must be enclosed in single quotes `' '`, even for symbols like 🐒. Otherwise, Julia will interpret the expression as a variable.

```
# any character is allowed for defining a variable
🐒 = 2          # 🐒 represents a variable, just like if we had defined x = 2

y  = 🐒         # y equals 2
z  = '🐒'       # z equals the character 🐒
```

STRINGS

We'll rarely use the type `Char` directly. Instead, we'll work with the so-called type `String`. This is an ordered collection of characters, making it possible to represent text.

Strings can be defined through either double quotes `" "` or triple quotes `""" """`. The latter is particularly convenient for handling newlines, such as when the text has to span multiple lines. [1]

```
x = "Hello, beautiful world"

x = """Hello, beautiful world"""
```

STRING INTERPOLATION

String interpolation allows you to embed Julia code within a string, which is then evaluated and replaced in the string with its value.

To interpolate an expression, you must simply prefix the string with the `$` symbol. If the expression contains spaces, you'll need to enclose it in curly braces, like `$()`. Both cases are exemplified below.

```
number_students = 10


output_text     = "There are $(number_students) students in the course"
```

```
julia> x
"There are 10 students in the course"
```

```
number_matches  = 50
goals_per_match = 2

output_text     = "Last year, Messi scored $(number_matches * goals_per_match) goals"
```

```
julia> x
"Last year, Messi scored 100 goals"
```

FOOTNOTES

[1.] For more on the differences between double and triple quotes, see here

2e. Arrays (Vectors and Matrices)

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INTRODUCTION

So far, we've explored variables representing single-element objects. Now, we'll shift our focus to **collections**, defined as **variables comprising multiple elements**. Julia provides several forms of collections, including:

- Arrays (including vectors and matrices)
- Tuples and Named Tuples
- Dictionaries
- Sets

Arrays represent one of the most common data structures for collections. They are formally defined as objects with type `Array{T,d}`, where `d` is the array's dimension and `T` is its elements' type (e.g., `Int64` or `Float64`).

Two special categories of arrays are **vectors** (1-dimensional arrays) and **matrices** (2-dimensional arrays). Vectors are represented by the type `Vector{T}`, which is an alias for `Array{T,1}`. For its part, matrices use the type `Matrix{T}`, which is an alias for `Array{T,2}`. Although we provide a subsection about matrices at the end, this is labeled as optional. The reason is that vectors are sufficient for conveying the topics of this website.

> **Remark**
>
> **Julia uses 1 as an array's first index**. This contrasts with many other languages (e.g., Python), where 0 is used as the first index.

VECTORS

Vectors in Julia are defined as *column-vectors*, and their elements are separated by a comma or a semicolon.

```
x = [1, 2, 3]          #= column-vector (defined using commas or semicolons)
                           Vector{Int64} (alias for Array{Int64, 1}) =#

x = [1; 2; 3]          # equivalent notation to define `x`
```

```
julia> x
3-element Vector{Int64}:
 1
 2
 3
```

> **Remark**
>
> Arrays can hold elements of various types, such as numbers and strings. For example, `[1, 2.5, "Hello"]` is a valid vector in Julia, identifying its elements as having type `Any` (recall that `Any` encompasses all the possible types supported by Julia). While arrays mixing types can be created, they're highly discouraged for several reasons, including performance.

ACCESSING A VECTOR'S ELEMENTS

Given a vector `x`, we can access its *i*-th element with `x[i]` and retrieve all its elements with `x[:]`.

```
x = [4, 5, 6]
```

```
julia> x
3-element Vector{Int64}:
 1
 2
 3
```

```
julia> x[2]
5
julia> x[:]
3-element Vector{Int64}:
 4
 5
 6
```

It's also possible to access a subset of `x`'s elements. There are several approaches to achieve this, and we'll only present two basic ones at this point. The simplest method involves setting the indices **via a vector**, using the syntax `x[<vector>]`.

```
x = [4, 5, 6, 7, 8]
```

```
julia> x
3-element Vector{Int64}:
 1
 2
 3
```

```
julia> x[[1,3]] # elements of 'x' with indices 1 and 3
2-element Vector{Int64}:
 4
 6
julia> x[1,3] # be careful! this is the notation used for matrices, indicating 'x[row 1,
column 3]'
ERROR: BoundsError: attempt to access 5-element Vector{Int64} at index [1, 3]
```

The second approach sets the indices **via ranges**. These are denoted as `<first>:<steps>:<last>`, with Julia assuming increments of one if we omit `<steps>`. To respectively express the first and last index in a range, you can use the keywords `begin` and `end`.

```
x = [4, 5, 6, 7, 8]
```

```
julia> x
3-element Vector{Int64}:
 1
 2
 3
```

```
julia> x[1:2] # steps with unit increments (assumed by default)
2-element Vector{Int64}:
 4
 5
julia> x[1:2:5] # steps with increments of 2 (explicitly indicated)
3-element Vector{Int64}:
 4
 6
 8
julia> x[begin:end] # all elements. Equivalent to 'x[:]' or 'x[1:end]'
3-element Vector{Int64}:
 4
 5
 6
 7
 8
```

MATRICES (*OPTIONAL*)

Matrices can be defined as collections of row- or column-vectors. If they're created through multiple row vectors, each row has to be separated by a semicolon `;`. If we instead adopt multiple column vectors, their elements need to be separated by a space.

Note that row vectors are considered as special cases of matrices, with their elements separated by a space—they're matrices with multiple columns having one element.

```
X = [1 2 ; 3 4]        #= matrix as a collection of row-vectors, separated by semicolons
                          Matrix{Int64} (alias for Array{Int64, 2})=#

X = [ [1,3] [2,4] ]     # identical to `C`, but defined through a collection of column-
vectors

Y = [1 2 3]            #= row-vector (defined without commas)
                          Matrix{Int64} (alias for Array{Int64, 2}) =#
```

```
julia> X
2×2 Matrix{Int64}:
 1  2
 3  4
julia> Y
1×3 Matrix{Int64}:
 1  2  3
```

ACCESSING A MATRIX'S ELEMENTS

Given a matrix `X`, we can access its element at row `r` and column `c` by `X[r,c]`. Likewise, the i-th element of a row vector is accessed with `X[i]`. [1] Moreover, we can select all elements across the row `r` by `X[r,:]`, and all elements of column `c` by `X[:,c]`.

```
X = [5 6 ; 7 8] # matrix

Y = [4 5 6]     # row-vector
```

```
julia> X
2×2 Matrix{Int64}:
 5  6
 7  8
julia> X[2,1]
7
julia> X[1,:]
2-element Vector{Int64}:
 5
 6
julia> X[:,2]
2-element Vector{Int64}:
 6
 8
julia> Y[2]
5
```

To access a subset of elements, you must follow the same approaches as with vectors, but applied to either rows or columns.

```
X = [5 6 ; 7 8]
```

```
julia> X
2×2 Matrix{Int64}:
 5  6
 7  8
julia> X[[1,2],1]
2-element Vector{Int64}:
 5
 7
julia> X[1:2,1]
2-element Vector{Int64}:
 5
 7
julia> X[begin:end,1]
2-element Vector{Int64}:
 5
 7
```

FOOTNOTES

[1.] We could also use this approach for any matrix, as Julia also accepts a linear index for matrices. For instance, a 3x3 matrix accepts indices between 1 and 9. However, unless you want to iterate over all elements of a matrix, the notation `X[r,c]` is easier to interpret.

2f. Tuples

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INTRODUCTION

We continue our exploration of *collections*, defined as objects storing multiple elements. Having previously focused on arrays, we'll now turn our attention to another form of collection known as *tuples*.

The defining characteristic of tuples is their fixed size and immutability. This implies that, once a tuple is created, its elements cannot be added, removed, or modified. While these restrictions might initially seem limiting, they also bring substantial performance gains when working with small collections.

At this point, we'll only touch on the basics. A more in-depth exploration of tuples will follow in Part II, after we've developed the necessary tools to understand the role of tuples in high-performance scenarios.

> **Warning!**
> **Tuples should only be used when the collection comprises a small number of elements**. Large tuples will result in slow computations at best, or directly trigger fatal errors. For large vectors, you should keep relying on vectors.

DEFINITION OF TUPLES

The syntax for accessing the *i*-th element of a tuple `x` is `x[i]`, similar to vectors. Likewise, defining tuples requires enclosing their elements in parentheses `()`, which contrasts with the square brackets `[]` used in vectors.

When tuples comprise more than one element, the use of `()` is optional and its omission is in fact a common practice. On the contrary, single-element tuples have stricter syntax rules: you must use parentheses `()` and a trailing comma `,` after the element. For example, a tuple with the single element `10` is represented as `(10,)`. This notation differentiates a tuple from the expression `(10)`, which would be interpreted simply as the number 10.

```
x = (4,5,6)
x =  4,5,6            #alternative notation
```

```
julia> x
(4, 5, 6)
julia> x[1]
4
```

```
x = (10,)    # not x = (10) (it'd be interpreted as x = 10)
```

```
julia> x
(10,)
julia> x[1]
10
```

TUPLES FOR ASSIGNMENTS

Tuples are particularly useful for simultaneously assigning values to multiple variables. This is achieved by placing **a tuple on the left-hand side of** `=` and **a collection on the right-hand side** (either another tuple or a vector). The following examples demonstrate both options.

```
(x,y) = (4,5)
 x,y  =  4,5        #alternative notation
```

```
julia> x
4
julia> y
5
```

```
(x,y) = [4,5]
 x,y  = [4,5]       #alternative notation
```

```
julia> x
4
julia> y
5
```

As we'll see later, this technique is commonly employed when a function returns multiple values, enabling you to unpack the returned values into individual variables.

3a. Overview and Goals

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The upcoming Chapters 3 and 4 will cover three core tools for programming: functions, conditional statements, and for-loops. Chapter 3 in particular focuses on **functions**, which constitute the backbone of Julia programming. As they're tightly linked to achieving high performance, we'll dedicate considerable time to discussing their usage.

Our coverage of functions will be organized into three categories, based on who defines them:

- ***i)*** built-in functions,
- ***ii)*** third-party functions, and
- ***iii)*** user-defined functions.

The first two types of functions become available in the workspace via packages, which may be loaded implicitly or explicitly. This connection between packages and functions leads us into exploring the concepts together in Section 3b. Instead, user-defined functions are left for Section 3c.

A firm grasp of functions requires understanding variable scope, including the distinction between global and local variables. By establishing this difference, we'll frame functions as self-contained mini-programs designed to perform a specific task. Both subjects are presented together in Section 3d. This perspective on functions will lead to the identification of good practices for using functions, which will have significant implications for the structure of code as we progress. At this point, nonetheless, it suffices if you start becoming familiar with this view.

Finally, we'll introduce the concept of broadcasting in Section 3e. Mastering this technique is crucial, as it lets you seamlessly apply the same function to each element in a collection. Broadcasting is a widely used technique not only in Julia, but also in other programming languages like Python.

3b. Function Calls and Packages

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INTRODUCTION

Broadly speaking, functions can be broken down into three categories:

- ***i)*** built-in functions,
- ***ii)*** third-party functions, and
- ***iii)*** user-defined functions.

This section focuses on i) and ii), relegating iii) to the next section. We consider in particular how to call functions, which in turn leads us to discuss packages.

> **Notation for Functions**
>
> Julia's developers suggest a **snake-case notation for function names**. This consists of lowercase letters, numbers, and possibly underscores to separate words (e.g., `snake_case123`). Note this is only a convention, not a language's requirement.

PACKAGES

When you start a new session in Julia, only a handful of very basic functions are available (e.g., those for sums, products, and subtractions). This is a deliberate choice made by Julia developers, who rely on **packages** to incorporate functions into the workspace. In fact, both built-in and third-party functions are contained in packages—the only difference is that the former are loaded by default.

The approach is not unique to Julia. However, Julia embraces this philosophy more profoundly than other programming languages. Thus, it doesn't even include standard functions such as averages or standard deviations, which are instead relegated to a package called `Statistics`. [1]

This design philosophy is rooted in a programming principle known as **modularity**. The principle promotes the development of small reusable modules, rather than large intertwined code. Its main advantage is to let packages evolve independently, without bugs and deprecations spreading across the entire Julia ecosystem. The practical implication of this feature is that users need to load several packages in each session, even to perform simple tasks.

LOADING PACKAGES AND CALLING FUNCTIONS

The concept of packages is tightly related to modules. Formally, **modules** are independent blocks of code, each acting as a separate workspace, that export a defined set of functions. In fact, when you start Julia, you're implicitly writing your script in a module called `Main`.

Packages are a special type of modules, which additionally include information about their **dependencies**. Dependencies are defined as the necessary packages that must be loaded to run the package itself.

Getting access to a package's functions requires loading the package via either the keyword `import` or `using`. The primary difference between the two is how functions are eventually called in your code. If the package is loaded via `import`, the function's name must include a prefix with the package's name. On the contrary, no prefix is needed when the package is loaded with `using`.

Below, we demonstrate each approach by calling the function `mean` from the package `Statistics`. This package isn't loaded by default, but it comes pre-installed with Julia.

```
x = [1,2,3]

import Statistics   #getting access to its functions will require the prefix `Statistics.`
Statistics.mean(x)
```

```
x = [1,2,3]

using Statistics      #no need to add the prefix `Statistics.` to call its functions (although it's possible to do so)
mean(x)
```

BUILT-IN FUNCTIONS

Formally, Julia's built-in functions are contained in two packages known as `Core` and `Base`. Both are automatically loaded in every Julia session, with their functions accessible as if we had executed `using Core` and `using Base`. This determines that their functions don't require adding a prefix to be called. [2]

Among mathematical functions, the syntax of their most common ones is the following.

Function in Julia	Meaning
`log(x)`	$\ln(x)$
`exp(x)`	e^x
`sqrt(x)`	$\sqrt{x}$
`abs(x)`	$\lvert x \rvert$
`sin(x)`	$\sin(x)$

Function in Julia	Meaning
`cos(x)`	$\cos(x)$
`tan(x)`	$\tan(x)$

Operators as Functions

Most of the symbols employed as operators are also available as functions. This is illustrated below for several arithmetic operators:

```
+(2,3)      # same as 2 + 3
-(2,3)      # same as 2 - 3
*(2,3)      # same as 2 * 3
/(2,3)      # same as 2 / 3
^(2,3)      # same as 2 ^ 3
```

WHY USING "IMPORT" IF IT'S MORE VERBOSE?

When a function's name is shared across multiple packages, at least one of the packages must be loaded via `import` to prevent naming conflicts. For instance, given the package `Statistics` and another one called `MyPackage` containing a function called `mean`, Julia will throw an error if you don't load one of them with `import`. [3]

Using `import` not only avoids naming conflicts, but may also reduce ambiguity in the meaning of a function. For instance, consider a function called `rank`. This name could reference a wide range of concepts, depending on the context (e.g., the rank of a matrix, the order in a list). However, explicitly identifying the package when the function is called could shed some light on its intended meaning.

Remark

`import` may also be useful if you have custom functions that are widely applied across your projects. For example, consider a function called `table_in_pdf`, which exports Julia tables to a PDF with some predefined format. While the name of the function makes it clear what it's doing, a user could wonder if this function comes from a standard package. You could hint that this isn't the case, by placing the function in a package called `userDefined`. In this way, you can load the package using `import UserDefined`, and then calling the function via `UserDefined.table_in_pdf`.

APPROACHES TO LOADING PACKAGES AND CALLING FUNCTIONS

The concepts discussed so far will probably be all you need to use packages in Julia. However, there are a few additional features worth mentioning.

First, users can load only a subset of functions from a package. This possibility is particularly relevant for heavy packages, which may take a significant time to fully load. For instance, if we only need the function `mean` from `Statistics`, the following two approaches achieve the same result.

```
x = [1,2,3]

import Statistics: mean
mean(x)                    # no prefix needed
```

```
x = [1,2,3]

using Statistics: mean
mean(x)
```

Note that this approach deems it unnecessary to add the package's name as a prefix, even when the package is loaded via `import`.

Another handy feature is the possibility of assigning custom names to either packages or functions. This becomes particularly useful when names are lengthy.

```
x = [1,2,3]

import Statistics as st
st.mean(x)
```

```
x = [1,2,3]

import Statistics: mean as average
average(x)                    # no prefix needed

using Statistics: mean as average
average(x)
```

Again, notice that the function's name doesn't require any prefix when it's called, even with `import`.

MACROS

Macros are ubiquitous in Julia. They enable the automation of tasks that otherwise would be tedious and time-consuming to perform. On this website, we'll only cover how to apply macros, without exploring how to define them. The reason is that creating macros requires knowledge of Julia's metaprogramming capabilities, which is beyond the scope of this website.

While the utility of macros may not be immediately obvious at this point, this will become clearer once we start applying them in subsequent sections.

APPLYING MACROS

Macros and functions share similarities, with both performing operations on inputs and producing outputs. Their key distinction lies in their handling of inputs and outputs: macros manipulate code syntax (statements or expressions), whereas functions process data values (variables or evaluated expressions).

Formally, macros are denoted by prefixing the symbol `@` to their name. They take an entire code expression as their argument and transform it. For example, a macro might take `x = some_function(y)` as input, potentially modifying each individual component (`x`, `=`, or `some_function(y)`), inserting new code, or reorganizing the code structure. The final output is a modified version of the original expression, which is then integrated into the program during execution.

A key purpose of macros is to automate code transformations. For example, consider the `@.` macro in Julia, which appends a dot `.` to every operator and function call in a statement. For now, ignore the impact of adding dots to your code, which will be explained in an upcoming section. Instead, focus on how macros operate at the syntactic level to rewrite entire code blocks.

```
# both are equivalent
   z .= foo.(x .+ y)
@. z  = foo(x  + y)          # @. adds . to '=', 'foo', and '+'
```

> **Warning!** - Caution with Macro Usage
>
> Applying macros requires extreme caution: they could act as black boxes and hence lead to unexpected behaviors. In fact, macros tend to be a common source of bugs. Make sure you understand which part of the expression a macro modifies and how.

FOOTNOTES

[1.] The extent to which Julia advocates for this principle is evident in `Statistics` itself, where functions for computing distributions are included in another package called `Distributions`.

[2.] Some built-in functions may require a prefix. For instance, this is what occurs with the function called `isgreater`, which must be called via `Base.isgreater`. Furthermore, some submodules are also loaded by default in each session. For instance, the function `Base.Iterators.accumulate` is part of the submodule `Iterators` from `Base`, and can be directly called using `Iterators.accumulate`.

[3.] Defining a function that shares the name of another package's function isn't necessarily an oversight by developers. For instance, we could implement our own `mean` function in a package called `MyPackage`, which aims at computing averages more efficiently in certain applications.

3c. Defining Your Own Functions

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INTRODUCTION

Recall that functions can be classified into *i)* built-in functions, *ii)* third-party functions, and *iii)* user-defined functions. The previous section has covered the first two, and **we now focus on iii)**.

USER-DEFINED FUNCTIONS

The first step to define your own functions is giving names. Function names follow similar rules to variable names. In particular, they accept Unicode characters, enabling the user to define functions such as `∑(x)`. Once you create them, functions can be called without invoking any prefix. This means that a function `foo` can be called by simply executing `foo(x)`. [1]

There are two approaches to defining functions. We'll refer to each as the **standard form** and the **compact form**. The standard form is the most general and allows you to write both short and long functions. On the other hand, the compact form is employed for single-line functions and is reminiscent of mathematical definitions. To illustrate each form, consider a function `foo` that sums two variables `x` and `y`.

STANDARD FORM

```
function foo(x,y)
    x + y
end
```

COMPACT FORM

```
foo(x,y) = x + y
```

The output of the compact form is given by the only operation contained. For its part, the standard form defaults to returning the last line as its output, allowing you to specify the output via the keyword `return`. To return multiple outputs, you can use a collection, with tuples being the most common choice. [2]

These approaches to specifying an output are illustrated below.

EXPLICIT OUTPUT

```
function foo(x,y)
    term1 = x + y
    term2 = x * y

    return term2
end
```

```
julia> foo(10,2)
20
```

IMPLICIT OUTPUT

```
function foo(x,y)
    term1 = x + y
    term2 = x * y            # output returned
end
```

```
julia> foo(10,2)
20
```

MULTIPLE OUTPUTS

```
function foo(x,y)
    term1 = x + y
    term2 = x * y

    return term1, term2      # a tuple, using the notation that omits the parentheses
end
```

```
julia> foo(10,2)
2-element Vector{Int64}:
 12
 20
```

AN EXPRESSION AS OUTPUT

```
function foo(x,y)
    term1 = x + y
    term2 = x * y

    return term1 + term2
end
```

```
julia> foo(10,2)
32
```

> **Functions without Arguments**
>
> It's possible to define functions that don't require arguments, as we show below.

FUNCTIONS WITHOUT ARGUMENTS

```
function foo()
    a = 1
    b = 1
    return a + b
end
```

An example of functions without arguments is `Pkg.update()`, which was introduced when we studied packages.

The Order In Which Functions Are Defined is Irrelevant

A function can be defined anywhere in the code. In fact, you can define a function that calls another function, even if the latter hasn't been defined yet. To illustrate this, consider the following two code snippets, which are functionally equivalent.

CODE SNIPPET 1

```
foo1(x) = 2 + foo2(x)

foo2(x) = 1 + x
```

```
julia> foo1(2)
5
```

CODE SNIPPET 2

```
foo2(x) = 1 + x

foo1(x) = 2 + foo2(x)
```

```
julia> foo1(2)
5
```

POSITIONAL AND KEYWORD ARGUMENTS

Up to this point, we've been defining and calling functions using the notation `foo(x,y)`. A key characteristic of this syntax is that arguments are passed in a specific order, so that `foo(2,4)` assigns the first argument to `x` and the second to `y`. This approach is known as **positional arguments**.

However, a major drawback of positional arguments is their susceptibility to silent errors: if we accidentally swap the positions of the arguments, the function may still provide an output. As the number of arguments grows, the likelihood of introducing such bugs increases, making it more

challenging to identify and resolve errors.

To circumvent this issue, we can rely on **keyword arguments**. This approach requires function calls to explicitly specify their arguments, making their order irrelevant. For example, `foo(x=2,y=4)` and `foo(y=4,x=2)` would then be valid and equivalent.

The following examples illustrate how to define and call functions using both positional and keyword arguments. Additionally, we'll establish that the approaches can be combined. Note that positional arguments necessarily require a semicolon during function definitions, but accept either a semicolon or a comma during function calls.

POSITIONAL ARGUMENTS

```
foo(x, y) = x + y
```

```
julia> foo(1,2)
2
```

KEYWORD ARGUMENTS

```
foo(; x, y) = x + y
```

```
julia> foo(x=1,y=1)
2
julia> foo(; x=1, y=1) # alternative notation (only for calling 'foo')
2
```

POSITIONAL AND KEYWORD ARGUMENTS

```
foo(x; y) = x + y
```

```
julia> foo(1 ; y=1)
2
julia> foo(1 , y=1) # alternative notation
2
```

KEYWORD ARGUMENTS WITH DEFAULT VALUES

Keyword arguments accept default values, allowing users to omit certain arguments when the function is called. The following examples illustrate how this feature works in practice, where the omitted arguments take on their default values.

POSITIONAL AND KEYWORD ARGUMENTS

```
foo(x; y=1) = x + y
```

```
julia> foo(1) # equivalent to foo(1,y=1)
2
```

OMITTING POSITIONAL ARGUMENTS

```
foo(; x=1, y=1) = x + y
```

```
julia> foo() # equivalent to foo(x=1,y=1)
2
julia> foo(x=2) # equivalent to foo(x=2,y=1)
3
```

USING ARGUMENTS AS INPUTS OF OTHER ARGUMENTS

When a function is called, its arguments are evaluated sequentially from left to right. This property enables users to define subsequent arguments in terms of previous ones. For example, given `foo(;x,y)`, the default value of `y` could be set based on the value of `x`.

PRIOR ARGUMENTS AS DEFAULT VALUES

```
foo(; x, y = x+1) = x + y
```

```
julia> foo(x=2) #function run with implicit value 'y=3'
5
```

SPLATTING

Given a function `foo(x,y)`, you can set the values of `x` and `y` through a tuple or vector `z`. The implementation relies on the splat operator `...`, which unpacks the individual elements of a collection and passes them as separate arguments.

TUPLE SPLATTING

```
foo(x,y) = x + y

z = (2,3)
```

```
julia> foo(z...)
5
```

VECTOR SPLATTING

```
foo(x,y) = x + y

z = [2,3]
```

```
julia> foo(z...)
5
```

ANONYMOUS FUNCTIONS

Anonymous functions offer a third way to define functions. Unlike the previous methods, they're commonly introduced with a different purpose: to serve as inputs to other functions. [3]

As the name suggests, anonymous functions aren't referenced by a name. Their syntax resembles the arrow notation from mathematics (e.g. $x \mapsto \sqrt{x}$). Specifically, single-argument functions are expressed as `x -> <body of the function>`. Likewise, functions with two or more arguments are expressed by `(x,y) -> <body of the function>`.

To demonstrate the role of anonymous functions, let's consider the built-in function `map(<function>, <collection>)`. This applies `<function>` element-wise to each element of `<collection>`. For example, `map(add_two, x)` applies the function `add_two(a) = a + 2` to each element of `x = [1,2,3]`, thus returning `[3,4,5]`. Applying `map` in this way requires defining `add_two` beforehand, which unnecessarily pollutes the namespace if `add_two` won't be reused. Anonymous functions provide an elegant solution, by directly embedding the operation within `map`. In this way, an anonymous function effectively eliminates the need of creating a temporary function like `add_two`.

VIA COMPACT FUNCTION

```
x          = [1, 2, 3]
add_two(a) = a + 2

result     = map(add_two, x)
```

```
julia> result
3-element Vector{Int64}:
 3
 4
 5
```

VIA ANONYMOUS FUNCTION

```
x          = [1, 2, 3]



result     = map(a -> a + 2, x)
```

```
julia> result
3-element Vector{Int64}:
 3
 4
 5
```

The function `map` can also demonstrate the syntax of anonymous functions with multiple arguments. In those cases, the syntax becomes `map(<function>, <array1>, <array2>)`. For instance, `map(+, [1,2], [2,4])` provides the sum of each pair of numbers, yielding `[3,6]`.

VIA COMPACT FUNCTION

```
x        = [1,2,3]
y        = [4,5,6]

add(a,b) = a + b
result   = map(add_two, x, y)
```

```
julia> result
3-element Vector{Int64}:
 3
 4
 5
```

VIA ANONYMOUS FUNCTION

```
x        = [1,2,3]
y        = [4,5,6]



result   = map((a,b) -> a + b, x, y)
```

```
julia> result
3-element Vector{Int64}:
 5
 7
 9
```

THE "DO-BLOCK" SYNTAX

Anonymous functions can help keep our code tidy, but they may not be practical for functions that span multiple lines. This inconvenience can be addressed by what's known as **do-blocks**. They allow us to insert the anonymous function separately, and then pass it as the first argument to a function call. Given a function `foo(<inner function>, <vector>)`, its generic implementation is as follows.

PRIOR ARGUMENTS TO DEFINE DEFAULT VALUES

```
foo(<vector>) do <arguments of inner function>
    # body of inner function
    end
```

To illustrate the notation with a concrete scenario, let's revisit the `map` function example and rewrite it using a do-block.

WITH COMPACT FUNCTION

```
x          = [1, 2, 3]
add_two(a) = a + 2

result     = map(add_two, x)
```

```
julia> result
3-element Vector{Int64}:
 3
 4
 5
```

WITH ANONYMOUS FUNCTION

```
x          = [1, 2, 3]


result     = map(a -> a + 2, x)
```

```
julia> result
3-element Vector{Int64}:
 3
 4
 5
```

WITH DO-BLOCK SYNTAX

```
x          = [1, 2, 3]

result     = map(x) do a
                a + 2
                end
```

```
julia> result
3-element Vector{Int64}:
 3
 4
 5
```

Do-blocks also accept anonymous functions with multiple arguments, as shown below.

WITH COMPACT FUNCTION

```
x = [1,2,3]
y = [4,5,6]

add(a,b) = a + b
result   = map(add_two, x, y)
```

```
julia> result
3-element Vector{Int64}:
 5
 7
 9
```

WITH ANONYMOUS FUNCTION

```
x        = [1,2,3]
y        = [4,5,6]


result   = map((a,b) -> a + b, x, y)
```

```
julia> result
3-element Vector{Int64}:
 5
 7
 9
```

WITH DO-BLOCK SYNTAX

```
x        = [1,2,3]
y        = [4,5,6]

result   = map(x,y) do a,b       # not (a,b)
                a + b
                end
```

```
julia> result
3-element Vector{Int64}:
 5
 7
 9
```

A FUNCTION'S DOCUMENTATION (*OPTIONAL*)

To conclude this section, we cover how to document functions. This can be done by adding a string expression immediately before the function definition. Once this is done, the documentation can be accessed in the same manner as with built-in functions: you can type the function's name in the REPL after pressing `?`, or directly hover over the function's name in VS Code. [4]

STANDARD FORM

```
"This function is written in a standard way. It takes a number and adds two to it."
function add_two(a)
   a + 2
end
```

COMPACT FORM

```
"This function is written in a compact form. It takes a number and adds three to it."
add_three(a) = a + 3
```

For further details, see the official documentation.

FOOTNOTES

[1.] The method to call a function actually depends on the **module** in which it's defined, and whether this module has been "imported" or "used". We won't cover modules on this website. However, they're essential when working for large projects, as each module operates as an independent workspace with its own variables. When initiating a new session in Julia, you're actually working within a module called `Main`.

[2.] The reason for this is that tuples are more performant than vectors when the number of elements is small.

[3.] Anonymous functions are also known as *lambda functions* in other languages.

[4.] Here, we explained how to access a function's documentation, under the subtitle "To See The Documentation of a Function".

3d. Variable Scope & Relevance of Functions

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INTRODUCTION

Variable scope refers to the code block in which a variable is accessible. The concept allows us to distinguish between **global variables**, which are accessible in any part of the code, and **local variables**, which are confined to specific blocks like functions or loops. The existence of scopes determines that the same variable `x` could refer to different objects, depending on where it's called.

When it comes to functions, Julia adheres to specific rules for variable scope. Specifically, given a variable `x` defined outside a function:

- if a new variable `x` is defined inside a function or is passed to a function as an argument, then `x` is considered *local* to that function. This means that any reference to `x` within the function refers to the local variable, without any relation to the variable `x` defined outside the function,
- if a function doesn't define a new `x` nor `x` is a function argument, then `x` refers to the variable defined outside the function (i.e., the global variable).

In this section, we'll show how these rules work in practice.

GLOBAL AND LOCAL VARIABLES

A variable that is local to a function exists solely within that function's scope. This means that these variables cease to exist once the function finishes executing. Consequently, any attempt to reference local variables outside the function will result in an error.

Variables local to a function encompass:

1. the function arguments,
2. the variables defined in the function body.

Any other variable included in a function that's not *i)* or *ii)* necessarily refers to a global variable.

Understanding which variables are local or global is essential for predicting a program's behavior. This is because a local variable may share the same name as a global one, without them being related. The following examples help clarify the differences between global and local variables.

```
x = "hello"

function foo(x)                 # 'x' is local, unrelated to 'x = hello' above
    y = x + 2                   # 'y' is local, 'x' refers to the function argument

    return x,y
end
```

```
julia> foo(1)
1                           # local x
3                           # local y
julia> x
"hello"
julia> y
ERROR: UndefVarError: y not defined
```

```
z = 2

function foo(x)
    y = x + z                       # 'x' refers to the function argument, 'z' refers to the
global

    return x,y,z
end
```

```
julia> foo(1)
1                           # local x
3                           # local y
2                           # global z
julia> x
ERROR: UndefVarError: x not defined
julia> z
2
```

THE ROLE OF FUNCTIONS

In programming, **functions** can be understood **as self-contained mini-programs to represent specific tasks**. Under this interpretation, local variables simply act as labels that help articulate the mechanics of the task. Consequently, their inaccessibility outside the function emerges naturally. [1]

To explain this view of functions, consider a variable `x`, along with another variable `y` computed by transforming `x` through a function `f`. In particular, assume a transformation that doubles `x`, so that `y = 2 * x`. The following are two approaches to calculating `y`.

```
x = 3

double() = 2 * x
y        = double()
```

```
x = 3

double(x) = 2 * x
y         = double(x)
```

```
x = 3

double(🐒) = 2 * 🐒
y          = double(x)
```

The function in Approach 1 relies on the global variable `x`. This practice is highly discouraged for several reasons. Firstly, it prevents the reusability of the function, as it's specifically designed to double the global variable `x`, rather than acting as a mini-program that doubles *any* variable.

Second, the inclusion of the global variable `x` compromises the function's self-containment, as the function's output depends on the value of `x` at the moment of execution. If you work on a long project, this will turn the code prone to bugs.

Lastly, global variables have a detrimental impact on performance, a topic we'll study later on the website. In fact, global variables in Julia are directly a performance killer.

In contrast, Approach 2 refers to `x` as a local variable. This `x` is unrelated to the global variable `x`—it simply serves as a label to identify the variable to be doubled. Indeed, we could've replaced `x` with any other label, as demonstrated in Approach 3 through the monkey emoji, `🐒`.

By avoiding referencing any variable outside its scope, Approach 2 makes the function self-contained. This allows users to easily anticipate the consequence of executing `double` through a simply inspection of the function, eliminating the need to review the entire codebase. Thus, Approach 2 aligns with the interpretation of a function as a self-contained mini-program: the function embodies the task of doubling a variable, turning the function reusable and applicable to any variable. In this context, applying `double` to the global variable `x` becomes just one possible application.

RECOMMENDATIONS FOR THE USE OF FUNCTIONS

Structuring code around functions offers numerous advantages. However, to fully realize these benefits, users must adhere to certain principles when writing code. This section outlines a few of them and should be considered as a mere introduction to the subject. The topic will be investigated further, when we explore high performance.

AVOID GLOBAL VARIABLES IN FUNCTIONS

Global variables are strongly discouraged. This is not only due to the reasons mentioned previously, but also because they can have a devastating impact on performance. The easiest solution to this issue is to pass global variables as function arguments. This practice will actually become second nature once you start viewing functions as self-contained mini-programs. Specifically, by adopting this

perspective, you'll conceive local variables as labels to describe a task, rather than references to global variables. This shift in mindset can help you write more efficient and maintainable code.

AVOID REDEFINING VARIABLES WITHIN FUNCTIONS

The suggestion applies to both local variables and function arguments. Redefining these variables can have several disadvantages, including reduced code readability and potential performance degradation. Therefore, it's recommended that you define new variables instead of redefining existing ones. This approach is demonstrated in the following example.

```
function foo(x)
   x      = 2 + x           # redefines the argument

   y      = 2 * x
   y      = x + y           # redefines a local variable
end
```

```
function foo(x)
   z      = 2 + x           # new variable

   y      = 2 * x
   output = z + y           # new variable
end
```

(OPTIONAL) - Another Issue of Redefining Variables

MODULARITY

We've emphasized the importance of viewing functions as self-contained mini-programs, designed to perform specific tasks. This perspective leads us to highlight the importance of **modularity**: the practice of breaking down a program into multiple small functions, each with its own distinct purpose, inputs, and outputs.

The primary benefit of modularity is the ability to work with independent code blocks. By keeping these blocks separate, we can decompose complex problems into multiple manageable tasks, making it easier to test and debug code. Additionally, modularity makes it possible to eventually improve or substitute parts of the code, without breaking the entire program.

A helpful way to understand this principle is by considering the analogy of building a Lego minifigure. In the first step, multiple blocks are created independently, each representing a specific part of the figure, such as the legs, torso, arms, and head. Then, in the second stage, these individual blocks are brought together and assembled into an integrated minifigure.

This two-step approach offers several advantages. By focusing on each block individually, we can concentrate and refine each part without worrying about the entire structure. Additionally, it provides great flexibility: since each block is created independently, we can modify specific blocks without having to rebuild the entire figure. For instance, if we want to change the figure's head, we can simply swap out the corresponding block, without starting from scratch.

The principle of modularity is closely tied to the suggestion of writing short functions. Some proponents even argue that functions should be limited to fewer than five lines of code Indeed, entire books have been written based on this principle. Although this viewpoint may be considered rather extreme, it clearly emphasizes the advantages of avoiding lengthy functions.

(OPTIONAL) - Example of Modularity

FOOTNOTES

[1.] Local variables play a similar role to integration variables in math. Formally, in $\int f(t)\ \mathrm{d}t$ for some function f, t just represents a symbol indicating over which variable we're integrating. The integral could be equivalently expressed using any other integration variable, such as x in $\int f(x)\ \mathrm{d}x$.

3e. Map and Broadcasting

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INTRODUCTION

This section explores element-wise operations on **iterable collections**. A collection is characterized as iterable when its elements can be accessed sequentially, thus including examples like vectors, tuples, and ranges.

The first approach covered is the `map` function, which applies a given function to each element of a collection. This function is particularly convenient for avoiding for-loops when transforming collections.

After this, we'll shift our focus to a fundamental technique in Julia known as **broadcasting**. This enables the application of functions and operators element-wise, while maintaining concise and expressive code. Broadcasting is quite versatile, supporting operations on collections of equal size or combinations of scalars and same-size collections. Its distinctive syntax, which involves appending a dot `.` to the function/operator, makes it easily identifiable throughout the code.

> **Remark**
>
> The terms **broadcasting** and **vectorization** will be used interchangeably throughout the website, although strictly speaking they're not equivalent. [1] Furthermore, vectorization has multiple meanings, depending on the context in which the definition is applied.

> **Warning!**
>
> Later on the website, we'll explore **for-loops** as an alternative approach to transforming arrays. Several languages strongly recommend vectorizing operations to improve speed, instead highly discouraging for-loops. **Such advice does not apply to Julia**. In fact, when it comes to optimizing code in Julia, for-loops are often the key to achieving faster performance.
>
> Considering this, the main advantage of vectorization in Julia is to streamline code without sacrificing speed.

THE "MAP" FUNCTION

The `map` function is available in most programming languages, allowing you to take a collection and generate a new one with transformed elements. It can be applied in two ways, depending on the number of inputs passed.

In its simplest form, `map` takes a single-argument function `foo` and a collection `x`. Its syntax is `map(foo,x)`, returning a new collection with `foo(x[i])` as *i*-th element. `map` is commonly applied with an anonymous function playing the role of `foo`, as illustrated below.

```
x = [1, 2, 3]


z = map(log,x)
```

```
julia> z
3-element Vector{Float64}:
 0.0
 0.69315
 1.09861
julia> [log(x[1]), log(x[2]), log(x[3])]
3-element Vector{Float64}:
 0.0
 0.69315
 1.09861
```

```
x = [1, 2, 3]


z = map(a -> 2 * a, x)
```

```
julia> z
3-element Vector{Int64}:
 2
 4
 6
julia> [2*x[1], 2*x[2], 2*x[3]]
3-element Vector{Int64}:
 2
 4
 6
```

The second way to apply `map` arises when the function `foo` takes multiple arguments. In case `foo` is a two-argument function, the syntax is `map(foo, x, y)`, returning a new collection whose *i*-th element is `foo(x[i], y[i])`. When the collections `x` and `y` have different sizes, `foo` **is applied element-wise until the shortest collection is exhausted**. This rule applies even when either `x` or `y` is a scalar, in which case `map` would return a single element.

For demonstrating its use, let's consider the addition operation. As you may recall, `+` denotes both an operator (e.g., `2 + 3`) and a function (e.g., `+(2, 3)`). By using `+` in particular as a function, `map` can perform element-wise additions across multiple collections.

```
x = [ 1, 2, 3]
y = [-1,-2,-3]

z = map(+, x, y)        # recall that `+` exists as both operator and function
```

```
julia> z
3-element Vector{Int64}:
 0
 0
 0
julia> [+(x[1],y[1]), +(x[2],y[2]), +(x[3],y[3])]
3-element Vector{Int64}:
 0
 0
 0
```

```
x = [ 1, 2, 3]
y = [-1,-2,-3]

z = map((a,b) -> a+b, x, y)
```

```
julia> z
3-element Vector{Int64}:
 0
 0
 0
julia> [x[1]+y[1], x[2]+y[2], x[3]+y[3]]
3-element Vector{Int64}:
 0
 0
 0
```

```
x = [ 1, 2, 3]
y = [-1,-2]

z = map(+, x, y)        # recall that `+` is both an operator and a function
```

```
julia> z
2-element Vector{Int64}:
 0
 0
julia> [+(x[1],y[1]), +(x[2],y[2])]
2-element Vector{Int64}:
 0
 0
```

```
x = [ 1, 2, 3]
y =  -1

z = map(+, x, y)        # recall that `+` is both an operator and a function
```

```
julia> z
1-element Vector{Int64}:
 0
julia> [+(x[1],y[1])]
1-element Vector{Int64}:
 0
```

BROADCASTING

The function `map` can rapidly become unwieldy when dealing with complex functions or multiple arguments. This is where broadcasting comes into play, offering a more streamlined syntax.

Next, we'll explore the concept of broadcasting in a step-by-step manner. First, we'll show how it applies to collections of equal size, covering both functions and operators. After this, we'll demonstrate that broadcasting accepts combinations of scalars and collections, even though it typically doesn't support operations with collections of different sizes. In such instances, the scalar is treated as a vector that matches the size of the corresponding collections.

Unlike other programming languages, **broadcasting is an intrinsic feature of Julia** and thereby applicable to *any* function or operator, including user-defined ones.

BROADCASTING FUNCTIONS

Broadcasting expands the versatility of functions, allowing them to be applied element-wise to a collection. This feature is implemented by appending a dot ***after*** the name of the function, as in `foo.(x)`.

Remarkably, **any function** `foo` **has a broadcasting counterpart** `foo.`. This entails that broadcasting is automatically available for user-defined functions. Furthermore, it determines that broadcasting isn't restricted to numeric collections, but to any type of collection.

Similarly to `map`, broadcasting can be applied to both single- and multiple-argument functions. Each case warrants separate consideration.

As for single-argument functions, broadcasting `foo` over a collection `x` returns a new collection with `foo(x[i])` as its *i*-th element. The following examples demonstrate this.

```
# `log(a)` is a function appying to scalars `a`

x         = [1,2,3]
```

```
julia> log.(x)
3-element Vector{Float64}:
 0.0
 0.69315
 1.09861
julia> [log(x[1]), log(x[2]), log(x[3])] # identical to log.(x)
3-element Vector{Float64}:
 0.0
 0.69315
 1.09861
```

```
square(a) = a^2     #user-defined function for a single element 'a'

x         = [1,2,3]
```

```
julia> square.(x)
3-element Vector{Int64}:
 1
 4
 9
julia> [square(x[1]), square(x[2]), square(x[3])] # identical to square.(x)
3-element Vector{Int64}:
 1
 4
 9
```

As for multiple-argument functions, suppose a function `foo` and collections `x` and `y`. Then, `foo.(x,y)` returns a new collection with `foo(x[i],y[i])` as its *i*-th element.

Importantly, **collections with different sizes aren't allowed**, establishing a clear contrast between broadcasting and `map`. The sole exception to this rule is when one of the objects is a scalar, as we'll see later.

Below, we provide several examples. The first example in particular makes use of the built-in function `max`, which provides the maximum value among its scalar arguments.

```
# 'max(a,b)' returns 'a' if 'a>b', and 'b' otherwise

x        = [0, 4, 0]
y        = [2, 0, 8]
```

```
julia> max.(x,y)
3-element Vector{Float64}:
 2
 4
 8
julia> [max(x[1],y[1]), max(x[2]),y[2]), max(x[3]),y[3])] # identical to max.(x,y)
3-element Vector{Float64}:
 2
 4
 8
```

```
foo(a,b) = a + b        # user-defined function for single elements 'a' and 'b'

x        = [-2, -4, -10]
y        = [ 2,  4,  10]
```

```
julia> foo.(x)
3-element Vector{Int64}:
 0
 0
 0
julia> [foo(x[1],y[1]), foo(x[2]),y[2]), foo(x[3]),y[3])] # identical to foo.(x,y)
3-element Vector{Float64}:
 0
 0
 0
```

Remark

Broadcasting applies not only to numeric functions, but to any function. For instance, consider the built-in function `string`, which concatenates its arguments to form a sentence (e.g., `string("hello ","world")` returns `"hello world"`).

```
country = ["France", "Canada"]
is_in   = [" is in "  , " is in "]
region  = ["Europe", "North America"]
```

```
julia> string.(country, is_in, region)
2-element Vector{String}:
 "France is in Europe"
 "Canada is in North America"
```

BROADCASTING OPERATORS

It's also possible to **broadcast operators**, making them apply element-wise. Its use requires prepending a dot ***before*** the operator.

For its application, it's helpful to recall the classification of operators by the number of operands, as this determines their syntax. Specifically, the syntax of *unary operators* is `<symbol>x`, so that `.√x` broadcasts `√`. Likewise, the syntax for *binary operators* is `x <symbol> y`, such that `x .+ y` computes the element-wise sum of vectors `x` and `y`, resulting in `[x[1]+y[1], x[2]+y[2], ...]`.

```
x = [ 1,  2,  3]
y = [-1, -2, -3]
```

```
julia> x .+ y
3-element Vector{Int64}:
 0
 0
 0
```

```
x = [1, 2, 3]
```

```
julia> .√x
3-element Vector{Float64}:
 1.0
 1.41421
 1.73205
```

BROADCASTING OPERATORS WITH SINGLE-ELEMENT OBJECTS

In all the cases covered so far, broadcasting was applied with inputs of the same size. In general, collections of dissimilar size, such as `x = [1,2]` and `y=[3,4,5]`, aren't allowed.

One exception to this rule occurs when broadcasting applies to vectors of equal size combined with scalars. In these cases, scalars are treated as objects having the same size as the vectors, with all entries equal to the scalar. For example, given `x = [1,2,3]` and `y = 2`, the expression `x .+ y` produces the same result as defining `y = [2,2,2]` and then executing `x .+ y`. This is demonstrated below.

```
x = [0,10,20]
y = 5
```

```
julia> x .+ y
3-element Vector{Int64}:
 5
 15
 25
```

```
x = [0,10,20]
y = [5, 5, 5]
```

```
julia> x .+ y
3-element Vector{Int64}:
 5
 15
 25
```

Remark

We emphasize that broadcasting can be applied to any iterable collection. Thus, the example based on strings presented above can be rewritten as follows.

```
country = ["France", "Canada"]
is_in   = " is in "
region  = ["Europe", "North America"]
```

```
julia> string.(country, is_in, region)
2-element Vector{String}:
 "France is in Europe"
 "Canada is in North America"
```

ITERABLE OBJECTS

So far, our examples have focused on broadcasting using vectors as collections. Furthermore, we've explored the technique by treating functions and operators separately, which sheds light on the underlying mechanics of broadcasting. Next, we'll take a more comprehensive perspective, applying broadcasting to other types of collections and to expressions combining functions and operators.

We first show that broadcasting can be applied to any iterable object, including tuples and ranges.

```
x = (1, 2, 3)     # or simply x = 1, 2, 3
```

```
julia> log.(x)
(0.0, 0.69315, 1.09861)
julia> x .+ x
(2, 4, 6)
```

```
x = 1:3
```

```
julia> log.(x)
(0.0, 0.69315, 1.09861)
julia> x .+ x
(2, 4, 6)
```

```
x = (1, 2, 3)    # or simply x = 1, 2, 3
y = 1:3
```

```
julia> x .+ y
(2, 4, 6)
```

Furthermore, it's possible to simultaneously broadcast operators and functions. Given the pervasiveness of such operations, Julia provides the macro `@.` for an effortless application. The macro should be added at the beginning of the statement, and has the effect of automatically adding a "dot" to each operator and function found.

To demonstrate its use, consider adding two vectors element-wise, which we then transform by squaring the elements of the resulting vector.

```
x = [1, 0, 2]
y = [1, 2, 0]

temp = x .+ y
z    = temp .^ 2
```

```
julia> temp
3-element Vector{Int64}:
 2
 2
 2
julia> z
3-element Vector{Int64}:
 4
 4
 4
```

```
x = [1, 0, 2]
y = [1, 2, 0]

square(x) = x^2
```

```
julia> square.(x .+ y)
3-element Vector{Int64}:
 4
 4
 4
```

```
x = [1, 0, 2]
y = [1, 2, 0]

square(x) = x^2
```

```
julia> @. square(x + y)
3-element Vector{Int64}:
 4
 4
 4
```

BROADCASTING FUNCTIONS VS BROADCASTING OPERATORS

We've demonstrated that both functions and operators can be broadcasted. This lets us implement operations in two distinct ways: either broadcast a function that operates on a single element or define a function that directly performs the broadcasted operation.

The examples below demonstrate that the same output is obtained using either approach. For the illustration, we suppose that the goal is to square each element of `x`.

```
x                = [1, 2, 3]

number_squared(a) = a ^ 2          # function for a single element 'a'
```

```
julia> number_squared.(x)
3-element Vector{Int64}:
 1
 4
 9
```

```
x                = [1, 2, 3]

vector_squared(x) = x .^ 2          # function for a vector 'x'
```

```
julia> vector_squared(x) # '.' not needed (it'd be redundant)
3-element Vector{Int64}:
 1
 4
 9
```

While both approaches yield the same output, **defining a function that operates on a scalar is the more advisable choice**. This is due to a couple of reasons. Firstly, a function like `number_squared(a)` enables users to seamlessly perform computations on both scalars and collections. This is achieved by simply choosing between executing the function or its broadcasted version. A corollary of this is that scalar functions avoid committing to a specific application. Secondly, the notation `number_square.(x)` explicitly conveys that the operation is element-wise, an aspect that would remain hidden in `vector_squared(x)`.

BROADCASTING OVER ONLY ONE ARGUMENT

When we broadcast a function or operator over some vectors `x` and `y`, both objects are simultaneously iterated. However, there are instances where we only want to iterate over one argument, keeping the other argument fixed.

A typical scenario is when we need to check whether elements from `x` match any values in a predefined list `y`. To illustrate this, we first introduce the function `in(a, list)`. This assesses whether the scalar `a` equals some element in the vector `list`. For instance, executing `in(2, [1,2,3])` returns `true`, because `2` belongs to `[1,2,3]`.

Suppose now that, instead of a scalar `a`, we have a vector `x`. The goal then is to verify whether *each* of the elements in `x` is present in `list = [1,2,3]`. Below, we show that this operation can't be directly implemented by broadcasting `in`.

```
x    = [1, 2]
list = [1, 2, 3]
```

```
julia> in.(x, list)
ERROR: DimensionMismatch: arrays could not be broadcast to a common size; got a dimension
with lengths 2 and 3
```

```
x    = [1, 2, 4]
list = [1, 2, 3]
```

```
julia> in.(x, list)
3-element BitVector:
 1
 1
 0
```

In the first example, `in.(x, list)` errors because `x` and `list` should either have the same size or one of them be a scalar. The second example does produce an output, but not the one we're looking for: it checks whether `1==1`, `2==2`, and `4==3`. Instead, our goal is to determine if `1` is in `[1,2,3]`, if `2` is in `[1,2,3]`, and if `3` is in `[1,2,3]`.

Intuitively, we need a mechanism to inform Julia that `list` should be treated as a single element while iterating over `x`. This can be accomplished in two different ways: either by enclosing `list` in a collection (e.g., a vector or tuple) or by using the `Ref` function.

As for the first approach, let's consider a tuple as the wrapping collection. Then, the implementation would require `(list,)`, which converts the variable into a tuple whose only element is the tuple itself. [2] Explaining the specifics of `Ref` is beyond our current scope. What matters for practical purposes is that `Ref(list)` makes `list` be treated as a singe element. Below, we demonstrate each approach.

```
x    = [2, 4, 6]
list = [1, 2, 3]        # 'x[1]' equals the element 2 in 'list'
```

```
julia> in.(x, [list])
3-element BitVector:
 1
 0
 0
```

```
x    = [2, 4, 6]
list = [1, 2, 3]        # 'x[1]' equals the element 2 in 'list'
```

```
julia> in.(x, (list,))
3-element BitVector:
 1
 0
 0
```

```
x    = [2, 4, 6]
list = [1, 2, 3]        # 'x[1]' equals the element 2 in 'list'
```

```
julia> in.(x, Ref(list))
3-element BitVector:
 1
 0
 0
```

The output vector we obtain in each case is what's known as a `BitVector`, where `1` corresponds to `true` and `0` to `false`. Therefore, the result is `[true, false, false]`, reflecting that `x[1]` is `2` and `2` belongs to `list`, whereas `x[2]` and `x[3]` don't equal any element in `list`.

> **Warning!**
> It's possible to use any collection to wrap `list`. However, we'll see in Part II of the book that there's some performance penalty involved when vectors are created. Consequently, **you should stick to `(list,)` rather than `[list]` when implementing this approach**.

While the previous example focused on the broadcasting of functions, the same principle applies to operators. This can be illustrated through the `∈` operator, which serves a similar purpose to the `in` function. Just like `in`, the `∈` operator determines whether a particular element exists within a collection. [3]

```
x    = [2, 4, 6]
list = [1, 2, 3]
```

```
julia> x .∈ (list,) # only 'x[1]' equals an element in 'list'
3-element BitVector:
 1
 0
 0
```

```
x    = [2, 4, 6]
list = [1, 2, 3]
```

```
julia> x .∈ Ref(list) # only 'x[1]' equals an element in 'list'
3-element BitVector:
 1
 0
 0
```

CURRYING AND FIXING ARGUMENTS (*OPTIONAL*)

Currying is a technique that transforms the evaluation of a function with multiple arguments into evaluating a sequence of functions, each with a single argument. [4] For instance, the curried version of `f(x,y)` would be written `f(x)(y)` and provide an identical output.

Our interest in currying lies in its ability to simplify broadcasting: it enables the treatment of an argument as a single object, without the need to use `Ref` or encapsulate objects as vectors/tuples. The technique could seem confusing for new users. In particular, it requires a good understanding of functions as first-class objects, entailing that functions can be treated as variables themselves. My primary goal is that you can at least recognize the syntax of currying, and thus be able to read code that applies the technique.

We start by illustrating how currying can be applied in general.

```
addition(x,y) = 2 * x + y
```

```
julia> addition(2,1)
5
```

```
addition(x,y) = 2 * x + y

# the following are equivalent
curried(x)    = (y -> addition(x,y))
curried       = x -> (y -> addition(x,y))
```

```
julia> curried(2)(1)
5
```

```
addition(x,y) = 2 * x + y
curried(x)    = (y -> addition(x,y))

# the following are equivalent
f             = curried(2)          # function of 'y', with 'x' fixed to 2
g(y)          = addition(2,y)
```

```
julia> f(1)
5
julia> g(1)
5
```

The key to understanding the syntax is that `curried(x)` is a function itself, with `y` as its argument. The second tab illustrates this clearly through the equivalence between `f = curried(2)` and `addition(2,y)`. These functions help us understand the logic behind curry, but are only useful for the specific case of `x=2`. Instead, `curried(x)` allows the user to call the function through `curried(x)(y)`, and so be used for any `x`.

As for broadcasting, any function `foo` in Julia can be broadcasted through `f.`. And we've determined that `curried(x)` is a function just like any other. Therefore, `curried(x)` plays the same role as `foo`, and so we can broadcast over `y` for a fixed `x` through `curried(x).(y)`.

```
a             = 2
b             = [1,2,3]

addition(x,y) = 2 * x + y
curried(x)    = (y -> addition(x,y))   # 'curried(x)' is a function, and 'y' its argument
```

```
julia> curried(a).(b)
3-element Vector{Int64}:
 5
 6
 7
```

```
a              = 2
b              = [1,2,3]

addition(x,y) = 2 * x + y
curried(x)    = (y -> addition(x,y))

#the following are equivalent
f    = curried(a)              # 'foo1' is a function, and 'y' its argument
g(y) = addition(2,y)
```

```
julia> f.(b)
3-element Vector{Int64}:
 5
 6
 7
julia> g.(b)
3-element Vector{Int64}:
 5
 6
 7
```

Let's now explore how the currying technique can help treat a vector as a single element in broadcasting. To illustrate this, consider the function `in` used previously. This function has a built-in curried version, which can be applied through `in(list).(x)` for vectors `list` and `x`. To better demonstrate its usage, the following example compares an implementation with `Ref`, the built-in curried `in`, and our own curry implementation.

```
x    = [2, 4, 6]
list = [1, 2, 3]
```

```
julia> in.(x,Ref(list))
3-element BitVector:
 1
 0
 0
```

```
x    = [2, 4, 6]
list = [1, 2, 3]

our_in(list_elements)  =  (x  ->  in(x,list_elements))      #  'our_in(list_elements)'  is  a
function
```

```
julia> our_in(list).(x) # it broadcasts only over 'x'
3-element BitVector:
 1
 0
 0
```

```
x    = [2, 4, 6]
list = [1, 2, 3]
```

```
julia> in(list).(x) # similar to 'our_in'
3-element BitVector:
 1
 0
 0
```

FOOTNOTES

[1.] Vectorization refers to applications restricted to arrays of the same size, with broadcasting being an extension of it that allows for scalars.

[2.] Recall that tuples with a single element must be written with a trailing comma, as in `(list,)`. Instead, `(list)` is interpreted as `list`, and hence treated as a vector.

[3.] `∈` can also be applied as a function, with its syntax mirroring that of `in`. Thus, `∈(a, list)` for a scalar `a` yields the same results as `in(a, list)`.

[4.] The name comes from the mathematician Haskell Curry, not the spice!

4a. Overview and Goals

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After studying functions, Chapter 4 covers another two core tools for programming: **conditions** and **for-loops**.

At this point, we'll simply define the concepts, without emphasizing much on the most effective ways to apply them. Basically, you should focus on the approaches and syntax to express conditions and for-loops.

We also relegate the analysis of techniques that combine functions, conditions, and for-loops. The following chapters will show that their simultaneous use gives rise to important concepts of Julia's language, such as in-place functions.

4b. Conditions

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INTRODUCTION

This section lays the basics for incorporating conditions into our programs. Formally, conditions are defined as functions and operators that return true or false as their output. A common example of a condition is `x > y`.

To get the most out of this section, you should keep in mind the classification of operators discussed here. This establishes that operators can be categorized according to their number of operands. Specifically, **unary operators** act on a single operand and precede it (i.e. `<operator>x`), whereas **binary operators** take two operands and are placed between them (i.e. `x <operator> y`).

CONDITIONS

Conditions are represented as values with type `Bool`, evaluating to either `true` or `false`. These values are internally represented as integers restricted to `1` and `0`.

The representation of Boolean values in the REPL varies depending on their dimension: scalar `Bool` values are displayed as `true` and `false`, while `Bool` vectors use `1` and `0`. This is illustrated below.

```
x = 2

#`y` provides `true` or `false` as its output
y = (x > 0)
```

```
julia> y
true
```

```
x = 2

#'z' provides 'true' and 'false' as its output, represented by 1s and 0s
z = [x > 0, x < 0]
```

```
julia> z
2-element Vector{Bool}:
 1
 0
```

> **Warning!**
> Parentheses are optional when writing single conditions, allowing us to write `y = x > 0` rather than `y = (x > 0)`. Nonetheless, the former

syntax is somewhat ambiguous, with the risk of being potentially misinterpreted as `(y = x) > 0`. To avoid confusion, it's a good practice to always include parentheses. This is especially true when working with multiple conditions, where outcomes can be drastically altered.

The condition in the previous example was defined via the operator `>`. More generally, conditions accept **comparison operators**, which are *binary operators* that compare values of various types (e.g., numbers and strings). The next list defines the most common ones.

Comparison Operator	Meaning
`x == y`	equal
`x ≠ y` or `x != y`	not equal
`x < y`	lower than
`x ≤ y` or `x <= y`	lower or equal than
`x > y`	greater than
`x ≥ y` or `x >= y`	greater or equal than

Remark

The non-standard characters appearing in the table can be written using tab completion:

- `≠` via `\ne`, which stands for "not equal",
- `≥` via `\ge`, which stands for "greater or equal",
- `≤` via `\le`, which stands for "lower or equal".

Remark

Comparison operators are also available as functions. For instance, the following expressions are all valid:

```
==(1,2)      # same as 1 == 2
 ≠(1,2)      # same as 1 ≠ 2
 ≥(1,2)      # same as 1 ≥ 2
>=(1,2)      # same as 1 ≥ 2
 >(1,2)      # same as 1 > 2
```

LOGICAL OPERATORS

Logical operators allow us to combine multiple conditions into a single one. Formally, they take `Bool` expressions as their operands, and return another `Bool` as their output. The following are the main logical operators used in Julia.

Logical Operator	Meaning
`x && y`	`x` and `y`
`x || y`	`x` or `y`
`!x`	negation of `x`

Notice that `&&` and `||` follow the syntax rules of *binary* operators.

```
x = 2
y = 3

# are both variables positive?
z1 = (x > 0) && (y > 0)

# is either `x` or `y` (or both) positive?
z2 = (x > 0) || (y > 0)
```

```
julia> z1
true
julia> z2
true
```

Another operator taking conditions as their operands is the "not" operator, represented by `!`. This is a unary operator that inverts a condition's value, changing `true` to `false` and vice versa. To use it, you simply place `!` at the start of the condition (i.e., before the parentheses).

As an illustration, the variables `y1` and `y2` below become equivalent via `!`.

```
x = 2

# is `x` positive?
y1 = (x > 0)

# is `x` not less than zero nor equal to zero? (equivalent)
y2 = !(x ≤ 0)
```

```
julia> y1 #identical output as 'y2'
true
```

LOGICAL OPERATORS AS SHORT-CIRCUIT OPERATORS

A key feature of `&&` and `||` is that they're **short-circuit operators**. This means that, once an operand is evaluated, the remaining operands are evaluated only if the previous operands didn't establish the truth or falseness of the expression. Specifically:

- `(x > 0) || (y > 0)`
  This expression is true when *at least one condition* is satisfied. Thus, Julia begins by analyzing `x > 0`. If this expression is true, it immediately returns `true`, without evaluating any subsequent expression. Only when `x > 0` is false will Julia evaluate `y > 0`.

- `(x > 0) && (y > 0)`
  This expression is true if *both conditions* are satisfied. Thus, Julia begins by analyzing `x > 0`. If this expression is false, it immediately returns `false`, without evaluating any subsequent expression. Only when `x > 0` is true will Julia evaluate `y > 0`.

Since not all operands are always evaluated, it's possible to get a result even if some operands contain invalid expressions. This is shown in the next example, where we include invalid Julia code as a condition.

```
x = 10
```

```
julia> (x < 0) && (this-is-not-even-legitimate-code)
false
julia> (x > 0) && (this-is-not-even-legitimate-code)
ERROR: UndefVarError: `this` not defined
```

```
x = 10
```

```
julia> (x > 0) || (this-is-not-even-legitimate-code)
true
julia> (x < 0) || (this-is-not-even-legitimate-code)
ERROR: UndefVarError: `this` not defined
```

PARENTHESIS IN MULTIPLE CONDITIONS

The inclusion of parentheses isn't crucial when working with only two conditions. This is because expressions like `(x > 0) && (y > 0)` can be safely written as `x > 0 && y > 0`, without much risk of confusion.

On the contrary, when dealing with three or more conditions, the lack of parentheses can drastically impact the expected behavior of an expression. The following example illustrates this point.

```
x = 5
y = 0
```

```
julia> x < 0 && y > 4 || y < 2
true
```

```
x = 5
y = 0
```

```
julia> (x < 0) && (y > 4 || y < 2)
false
```

```
x = 5
y = 0
```

```
julia> (x < 0 && y > 4) || (y < 2)
true
```

In the example, the expression without parenthesis is equivalent to the last tab's, since `&&` **has higher precedence than** `||` in Julia: when both `&&` and `||` are used, `&&` will be evaluated first.

To avoid confusion when more than two conditions are incorporated, **we'll always add parentheses**. This improves readability and spares us the need to memorize specific rules. The next optional subsection covers Julia's precedence rules in more detail. However, if you'll consistently enclose conditions in parentheses, you can safely skip it.

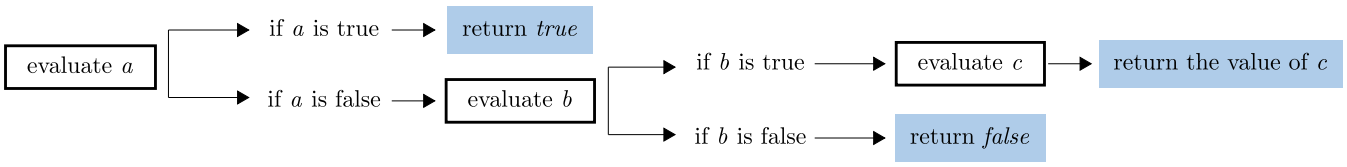
MULTIPLE CONDITIONS WITHOUT PARENTHESES (*OPTIONAL*)

To simplify the explanation, let's focus on cases with three conditions. These conditions will be represented through `Bool` variables `a`, `b`, and `c`, with each variable possibly representing expressions like `x > 0`.

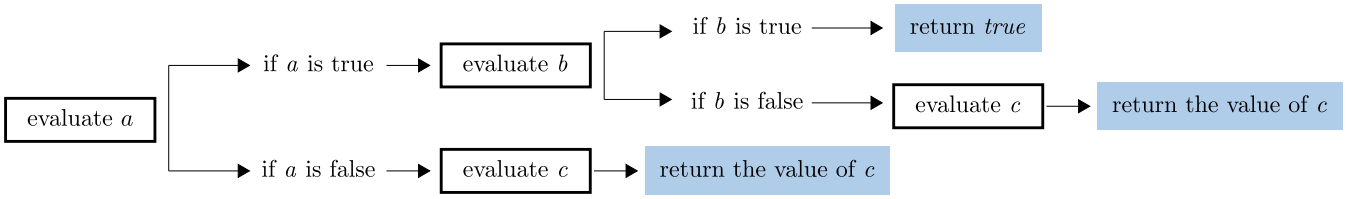
To understand how Julia groups three conditions without parentheses, there are two rules you need to know. First, `&&` has higher precedence than `||`. This means that `a && b || c` is equivalent to `(a && b) || c`, whereas `a || b && c` is equivalent to `a || (b && c)`. Second, `&&` and `||` are short-circuit operators. Thus, `a && b` immediately returns `false` if its first operand `a` is false, without evaluating the second operand `b`. Likewise, `a || b` returns `true` if the first operand `a` is true, without evaluating the second operand `b`.

The following diagrams describe the process for evaluating `a && b || c` and `a || b && c`, based on these two rules.

CASE 1: `a || b && c` is equivalent to `a || (b && c)`



CASE 2: `a && b || c` is equivalent to `(a && b) || c`



To illustrate the rules in practice, let's go through several examples that combine true/false values for `a`, `b`, and `c`. In these examples, we'll use the invalid expression `does-not-matter`. This is to emphasize that some conditions aren't necessarily evaluated thanks to the short-circuit behavior of `&&` and `||`.

```
julia> false || true && true
true
julia> false || true && false
false
julia> true || does-not-matter
true
```

```
julia> true && false || true
true
julia> true && false || false
false
julia> false && does-not-matter || true
true
```

FUNCTIONS TO CHECK CONDITIONS ON VECTORS: "ALL" AND "ANY"

Julia provides two built-in functions called `all` and `any` to evaluate multiple conditions in a collection. The function `all` returns `true` if *every condition* is true, whereas `any` returns `true` if *at least one condition* is true. The functions **require either directly specifying the conditions through a Boolean vector or defining the condition to check through a function**. Next, we cover each case separately.

VECTORS FOR REPRESENTING MULTIPLE CONDITIONS

In the following, we demonstrate the syntax of `all` and `any` when they take a Boolean vector as their argument.

```
a                 = 1
b                 = -1

# function indicating whether all elements satisfy the condition
are_all_positive = all([a > 0, b > 0])

# function indicating whether at least one element satisfies the condition
is_one_positive  = any([a > 0, b > 0])
```

```
julia> are_all_positive
false
julia> is_one_positive
true
```

The function `all` returns `true` only when all the conditions are satisfied, thus requiring that each vector's entry is positive. This doesn't hold in the example, since `b = -1`. Conversely, `any` returns `true` when at least one of the conditions holds, thus requiring at least one element in the vector to be positive. This is satisfied in the example, since `a = 1`.

As we indicated, `all` and `any` do not support passing multiple conditions as separate arguments. This entails that expressions like `all(a > 0, b > 0)` aren't allowed. Nevertheless, this restriction actually makes the functions more flexible, as they **enable the use of broadcasting operations for checking multiple conditions**. For example, the following code snippet implements the same operations as above, but through a vector `x`.

```
x                 = [1, -1]

are_all_positive = all(x .> 0)
is_one_positive  = any(x .> 0)
```

```
julia> are_all_positive
false
julia> is_one_positive
true
```

FUNCTIONS FOR REPRESENTING MULTIPLE CONDITIONS

In addition to expressing conditions through vectors, `all` and `any` allow **passing a function to represent the condition to check**. The syntax for this is `all(<function>, <array>)` and `any(<function>, <array>)`, where `<function>` can be an anonymous function. The following examples demonstrate how to implement `all(x .> 0)` and `any(x .> 0)` using this approach.

```
x                = [1, -1]

are_all_positive = all(i -> i > 0, x)
is_one_positive  = any(i -> i > 0, x)
```

```
julia> are_all_positive
false
julia> is_one_positive
true
```

By passing a function as an argument, `all` and `any` can additionally be employed **to evaluate the same condition across multiple vectors**. This is achieved by broadcasting `all` and `any`.

```
x                = [1, -1]
y                = [1,  1]

are_all_positive = all.(i -> i > 0, [x,y])
is_one_positive  = any.(i -> i > 0, [x,y])
```

```
julia> are_all_positive # all elements in 'y' are positive, but not in 'x'
2-element BitVector:
 0
 1
julia> is_one_positive # at least one element of 'x' or 'y' is positive
2-element BitVector:
 1
 1
```

4c. Conditional Statements

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INTRODUCTION

Programs routinely perform alternative operations depending on their execution flow. To handle these possibilities, programs rely on conditional statements, which enable the execution of specific code blocks only when certain conditions are met.

Each code block of a conditional statement is referred to as a **branch**. Based on the number of branches, there are three types of conditional statements:

- **if-then statements**, which consist of a **single branch**. They run a specific operation only if a condition is met, with no operation performed otherwise.

- **if-else statements**, which consist of **two branches**. They run a specific operation if a condition is met, and another if the condition isn't satisfied.

- **if-else-if statements**, which consist of **three or more branches**. They comprise a series of conditions, with each branch executing a different code block.

Next, we cover each in depth. The presentation builds heavily on the logical operators introduced in the previous section. If you haven't read it, I highly recommend doing so before continuing.

IF-THEN STATEMENTS

If-then statements execute an operation only when a condition is met, doing nothing instead when the condition isn't satisfied. These statements can be constructed via:

- the `if` keyword,
- the logical operator `&&`,
- the logical operator `||`.

The approach via `if` keyword is self-explanatory. As for the logical operators, `&&` executes an operation if the condition is true, whereas `||` does it when the condition is *not* satisfied. In fact, `||` is equivalent to `&&` with its condition negated.

Below, we illustrate the syntax for each form. The examples rely on the `println` function, which displays the text passed as argument in the REPL.

```
x = 5

if x > 0
    println("x is positive")
end
```

```
"x is positive"
```

```
x = 5

(x > 0) && (println("x is positive"))
```

```
"x is positive"
```

```
x = 5

(x ≤ 0) || (println("x is positive"))
```

```
"x is positive"
```

Note that if-then statements imply that no action would've been taken if, for instance, we had used `x = -1` as a condition—it's only when `x > 0` that `println` is executed.

The `if` approach offers the most flexibility, making it ideal for complex conditional statements. However, it's somewhat verbose for simple conditional statements. For these cases, `&&` and `||` are preferred, as they help us keep the code streamlined.

A common application of `||` is in conjunction with the function `error` to handle errors. This construct immediately interrupts the script's execution when the condition isn't satisfied, displaying the provided message as the argument of `error`. For instance, consider a function `foo(x)` that requires non-negative values for `x`. To enforce this, you could include `x > 0 || error("x must be positive")` at the beginning of the function. If `foo(x)` is then called with a non-positive `x`, it'll immediately halt its execution and print the error message "x must be positive" in the REPL.

> **Remark**
>
> Note that `&&` and `||` behave like if-then statements when they combine a condition with an operation. This is different from using them exclusively with conditions, where all operands would be `Bool` values.

IF-ELSE STATEMENTS

If-else statements execute an operation when a condition is true and another operation when the condition is false. There are two forms to write these statements.

The first one is the most flexible and uses the `if` keyword in combination with `else`. The second method relies on the so-called **ternary operator**, which requires the keywords `?` and `:` via the syntax `<condition> ? <operation if true> : <operation if false>`. This is referred to as *the* ternary operator because it's the only operator in most programming languages that takes three arguments.

We illustrate the syntax of both approaches below.

```
x = 5

if x > 0
    println("x is positive")
else
    println("x is not positive")
end
```

```
"x is positive"
```

```
x = 5

x > 0 ? println("x is positive") : println("x is not positive")
```

```
"x is positive"
```

The function `ifelse` offers an alternative for constructing if-else expressions. This function takes three arguments: a condition, an expression to be evaluated if the condition is true, and another one if false. [1]

One advantage of using a function for an if-else statement is that it supports broadcasting. This is particularly helpful when creating vectors whose elements vary according to a condition, as demonstrated below.

```
x                     = [4, 2, -6]

are_elements_positive = ifelse.(x .> 0, true, false)
```

```
julia> are_elements_positive
3-element BitVector:
 1
 1
 0
```

```
x                    = [4, 2, -6]

x_absolute_value     = ifelse.(x .≥ 0, x, -x)
```

```
julia> x_absolute_value
3-element Vector{Int64}:
 4
 2
 6
```

> **Remark**
>
> Broadcasting `ifelse` **requires broadcasting both** `ifelse` **and the condition**. The first example, for instance, would throw an error if we execute `ifelse.(x>0, true, false)`. This is because `x > 0` would attempt to check if the *entire vector* is positive, which is an operation undefined in Julia.

IF-ELSE-IF STATEMENTS

So far, we've analyzed conditional statements that handle only two possibilities: one when the condition is met, and another if it isn't. This binary structure can be limiting when multiple alternatives need to be considered. Basically, it forces you to nest several `if` and `else` statements to manage additional conditions.

To simplify this process, we can use the `elseif` keyword to extend the `if` and `else` approach. This is illustrated below.

```
x = -10

if x > 0
    println("x is positive")
elseif x == 0
    println("x is zero")
end
```

```
x = -10

if x > 0
    println("x is positive")
elseif x == 0
    println("x is zero")
else
    println("x is negative")
end
```

```
"x is negative"
```

The first examples showcase the benefits provided by the approach. Specifically, `elseif` eliminates the need to explicitly specify actions for every possible scenario. Instead, it performs an action if `x` is positive, another action if `x` is zero, but it does nothing otherwise. In contrast, using `if` and `else` would require an exhaustive approach, where all possible contingents must be accounted for.

Likewise, the second example demonstrates that combinations of the `if`, `else`, and `elseif` keywords are possible.

FOOTNOTES

[1.] The function `ifelse` does *not* behave as a **short-circuit operator**. This means that all the operations are computed, despite that only one of them will ultimately be returned as output.

4d. For-Loops

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INTRODUCTION

A key feature of programming is its ability to automate repetitive tasks, making **for-loops** play a crucial role in coding. They let you execute the same block of code repeatedly, treating each element in a list as a different input.

While for-loops are fundamental in every programming language, their importance is especially pronounced in Julia: unlike other languages (e.g., Matlab, Python, and R), which often discourage for-loops in performance-critical code, Julia relies on them to achieve high performance.

The role of for-loops in optimizing performance will be explored in Part II. Here, we'll primarily introduce the tool itself, focusing on its syntax, constructions, and common iteration techniques.

SYNTAX

For-loops delimit their scope via the keywords `for` and `end`. To illustrate their syntax, consider the function `println(a)`, which evaluates `a` and displays its output in the REPL. In case `a` is a string, `println(a)` simply displays the word stored in `a`. The following script repeatedly applies `println` to display each word contained in a collection.

FOR-LOOPS SYNTAX

```julia
for x in ["hello","beautiful","world"]
    println(x)
end
```

```
"hello"
"beautiful"
"world"
```

Remark

The keyword `in` can be replaced by `∈` or `=`. [1] Consequently, the following constructions are all equivalent.

IN

```julia
for x in ["hello","beautiful","world"]
    println(x)
end
```

∈

```
for x ∈ ["hello","beautiful","world"]
    println(x)
end
```

=

```
for x = ["hello","beautiful","world"]
    println(x)
end
```

Furthermore, we can employ any character or term to describe the iteration variable. For instance, we iterate below using `word`.

ALTERNATIVE NAME FOR ITERATION VARIABLE

```
for word in ["hello","beautiful","world"]
    println(word)
end
```

```
"hello"
"beautiful"
"world"
```

Based on this example, we can identify three components that characterize a for-loop:

- **A code block to be executed**: represented in the example by `println(x)`.
- **A list of elements**: represented in the example by `["hello","beautiful","world"]`. This specifies the elements over which we'll apply the code block. The list can contain elements with any data type (e.g., strings, numbers, and even functions). The only requirement is that the list must be an **iterable object**, defined as a collection whose elements can be accessed individually. An example of iterable object is vectors, as in the example. However, we'll also introduce others most commonly used, such as ranges.
- **An iteration variable**: represented in the example by `x`. This serves as a label that takes on the value of each element in the list, one at a time, during each iteration. The iteration variable is a local variable, with no significance outside the for-loop. Its sole purpose is to provide a convenient way to access and manipulate the elements of the list within the loop.

In the following sections, we'll explore different objects that can serve as lists. Furthermore, we'll show that these lists can comprise elements not immediately obvious. A typical example is functions, making it possible to apply different functions to the same object.

Always Wrap For-Loops in Functions

At this stage of the website, we're still introducing fundamental concepts. Thus, we're presenting subjects in their simplest form for

> learning purposes. In particular, this explains why for-loops will be written in the global scope.
>
> However, **you should always wrap for-loops in functions**. Executing for-loops outside a function severely degrades performance, and is additionally subject to different rules regarding variable scoping. [2]

ITERATING OVER INDICES

So far, we've considered a simple list like `["hello", "beautiful", "world"]` to demonstrate how for-loops work. In real applications, however, manually specifying each element in a list is impractical. Fortunately, when a list follows a predictable pattern (e.g., a sequence of numbers), we can simply describe the pattern that generates those elements.

Building on this insight, we'll next explore how to define ranges. They let users define a sequence of numbers, which is particularly useful to access elements of a collection through their indices.

RANGES

Ranges in Julia are defined via the syntax `<begin>:<steps>:<end>`, where `<begin>` represents the starting index and `<end>` the ending index. Likewise, `<steps>` sets the increment between values, defaulting to one when the term is omitted. We can also reverse the order of the sequence, by providing a negative value for `<steps>`. All this is demonstrated below.

RANGE WITH STEPS GIVEN

```julia
for i in 1:2:5
    println(i)
end
```

```
1
3
5
```

RANGE WITH REVERSE ORDER

```julia
for i in 3:-1:1
    println(i)
end
```

```
3
2
1
```

> **Remark**
>
> The application of ranges isn't limited to for-loops. They can also define vectors when used in combination with the `collect` function.

CREATING A VECTOR FROM A RANGE

```
x = collect(4:6)
```

```
julia> x
3-element Vector{Int64}:
 4
 5
 6
```

ITERATING OVER INDICES OF AN ARRAY

Ranges can be employed to access elements of a collection. When combined with a for-loop, it makes it possible to apply the same code block to each element of a vector.

Specifically, the expression `1:length(x)`, where `length(x)` returns the number of elements in `x`, allows iteration over all indices of a vector `x`. The same functionality can be achieved with the function `eachindex(x)`. In fact, this is the recommended approach for iterating over all elements, as it returns an iterator optimized for each iterable object.

1:LENGTH(X)

```
x = [4, 6, 8]

for i in 1:length(x)
    println(x[i])
end
```

```
4
6
8
```

EACHINDEX

```
x = [4, 6, 8]

for i in eachindex(x)
    println(x[i])
end
```

```
4
6
8
```

Remark

There are other approaches to iterating over all indices of a vector `x`. For instance, you can use `LinearIndices(x)`, or `firstindex(x):lastindex(x)` to specify a range from the first to the last index of `x`.

This multiplicity of methods exists to handle non-standard indices, such as those provided by the `OffsetArrays.jl` package. This package sets the first index of arrays to 0, a common convention in many programming languages. Nevertheless, unless you're developing a package for other users, you don't need to worry about which approach to implement. Indeed, they can all be used interchangeably, as shown below.

EACHINDEX

```
x = [4, 6, 8]

for i in eachindex(x)
    println(x[i])
end
```

```
4
6
8
```

1:LENGTH(X)

```
x = [4, 6, 8]

for i in 1:length(x)
    println(x[i])
end
```

```
4
6
8
```

LINEARINDICES

```
x = [4, 6, 8]

for i in LinearIndices(x)
    println(x[i])
end
```

```
4
6
8
```

FIRSTINDEX(X):LASTINDEX(X)

```
x = [4, 6, 8]

for i in firstindex(x):lastindex(x)
    println(x[i])
end
```

```
4
6
8
```

> Among the available alternatives, `eachindex` is preferable because it automatically selects the most efficient method for each type of collection. Additionally, the syntax is consistent across all indexing conventions.

RULES FOR VARIABLE SCOPE IN FOR-LOOPS

Similar to functions, for-loops create a new variable scope. In fact, the scoping rules for both are similar, with one key difference: **for-loops can modify global variables, whereas functions cannot**.

> **Warning!**
>
> The general scoping rules presented here apply universally, except in rare edge cases that result from poor coding practices. Since this scenario is uncommon, we only outline it next.
>
> Basically, the issue occurs when *i)* the for-loop is *not* wrapped in a function, *ii)* a local variable shares the same name as a global variable, and *iii)* the script is run non-interactively (i.e., using the function `include` and a script file). [3]
>
> Unless the three conditions hold simultaneously, you don't have to worry about this scenario. And even if this occurs, Julia will display a warning in the REPL indicating that there's a problem with your code.

To formalize the variable scope of for-loops, we'll refer to a variable `x`. The rules governing its scope are:

- the variable of iteration `x` is always local, regardless of whether there's a variable `x` defined outside the for-loop.
- if there's no variable named `x` outside the for-loop, `x` is a new local variable. Moreover, this variable won't be accessible outside the for-loop.
- if there's a variable named `x` outside the for-loop, `x` refers to this variable.

The following code snippets illustrate the first two rules, which exclusively refer to local variables. The second example is particularly noteworthy, as it highlights **a common mistake made by beginners**: running a for-loop that defines a local variable, and then trying to access it outside the for-loop.

ITERATION VARIABLE IS LOCAL

```
x = 2

for x in ["hello"]          # this 'x' is local, not related to 'x = 2'
    println(x)
end
```

```
"hello"
```

DEFINING LOCAL VARIABLE

```
#no `x` outside the for-loop

for word in ["hello"]
    x = word                # `x` is local to the for-loop, not available outside it
end
```

```
julia> x
ERROR: UndefVarError: x not defined
```

Likewise, the following example demonstrates the consequences of the last rule we mentioned. This refers to the consequences of variable scope for global variables.

REFERRING TO THE GLOBAL X

```
x = [2, 4, 6]

for i in eachindex(x)
    x[i] * 10        # it refers to the `x` outside of the for-loop
end
```

```
julia> x
3-element Vector{Int64}:
 20
 40
 60
```

REASSIGNING THE GLOBAL X

```
x = [2, 4, 6]

for word in ["hello"]
    x = word                        # it reassigns the `x` defined outside the for-loop
end
```

```
julia> x
"hello"
```

ARRAY COMPREHENSIONS

To seamlessly create arrays via for-loops, you can use **array comprehensions**. Their syntax is `[<expression> for... if...]`, where `<expression>` denotes either an operation or a function.

For illustration purposes, consider a vector `x`. Suppose that the goal is to create a vector `y` with elements equal to the square of the corresponding element in `x`. The following code snippets show two approaches to creating `y` via array comprehensions.

COMPREHENSION USING AN OPERATION

```
x      = [1,2,3]


y      = [a^2 for a in x]              # or y = [x[i]^2 for i in eachindex(x)]
```

```
julia> y
3-element Vector{Int64}:
  1
  4
  9
```

COMPREHENSION USING A FUNCTION

```
x      = [1,2,3]

foo(a) = a^2
y      = [foo(a) for a in x]           # or y = [foo(x[i]) for i in eachindex(x)]
```

```
julia> y
3-element Vector{Int64}:
  1
  4
  9
```

Array comprehensions also allow for creating vectors based on conditions. In such instances, the condition must be placed at the end of the expression.

COMPREHENSION WITH CONDITION

```
x = [i for i in 1:4 if i ≤ 3]
```

```
julia> x
3-element Vector{Int64}:
 1
 2
 3
```

> **Remark**
>
> Array comprehensions can also create matrices. Its syntax demands a comma to separate the description of each dimension.

COMPREHENSION FOR MATRICES

```
y = [i * j for i in 1:2, j in 1:2]
```

```
julia> y
2×2 Matrix{Int64}:
 1  2
 2  4
```

ITERATING OVER MULTIPLE OBJECTS

Thus far, we've considered for-loops that iterate over single values. We now extend their application to **simultaneous iterations over multiple values**. Specifically, we'll examine two scenarios: simultaneous iterations over two lists and over both the indices and values of a vector.

ITERATING OVER TWO LISTS

Depending on how elements should be combined, we can define two approaches to simultaneously iterating over two lists `x` and `y`. First, the function `Iterators.product(x,y)` makes it possible to iterate over all the possible combinations of elements. This function is part of the package `Iterators`, imported by default in each Julia session.

Alternatively, you can iterate over all the ordered pairs of `x` and `y`. This is implemented through the function `zip(x,y)`, which provides the pair of *i*-th elements from `x` and `y` in the *i*-th iteration.

MULTIPLE ITERATORS (ALL COMBINATIONS)

```
list1 = [1, 2]
list2 = [3, 4]

for (a,b) in Iterators.product(list1,list2) #it takes all possible combinations
    println([a,b])
end
```

```
[1,3]
[2,3]
[1,4]
[2,4]
```

MULTIPLE ITERATORS (PAIRS)

```
list1 = [1, 2]
list2 = [3, 4]

for (a,b) in zip(list1,list2)              #it takes pairs of elements with the same index
    println([a,b])
end
```

```
[1,3]
[2,4]
```

Using `zip`, we can also iterate over multiple values via array comprehensions.

MULTIPLE ITERATORS (ALL COMBINATIONS)

```
x = [i * j for i in 1:2 for j in 1:2]
```

```
julia> x
4-element Vector{Int64}:
 1
 2
 2
 4
```

MULTIPLE ITERATORS (PAIRS)

```
x = [i * j for (i,j) in zip(1:2, 1:2)]
```

```
julia> x
2-element Vector{Int64}:
 1
 4
```

SIMULTANEOUSLY ITERATING OVER INDICES AND VALUES

To iterate over each pair of index-value of a vector, we can employ the `enumerate` function.

FOR-LOOPS

```
x = ["hello", "world"]

for (index,value) in enumerate(x)
    println("$index $value")
end
```

```
"1 hello"
"2 world"
```

ARRAY COMPREHENSION

```
x = [10, 20]


y = [index * value for (index,value) in enumerate(x)]
```

```
julia> y
2-element Vector{Int64}:
 10
 40
```

ITERATING OVER FUNCTIONS

Functions in Julia are **first-class objects**, also referred to as **first-class citizens**. This means that functions can be manipulated just like any other data type, such as strings and numbers. In particular, this property makes it possible to store functions in a vector and apply them sequentially to an object. The following example illustrates this by computing descriptive statistics of a vector `x`.

ITERATION OVER FUNCTIONS

```
x                       = [10, 50, 100]
list_functions          = [maximum, minimum]

descriptive(vector,list) = [foo(vector) for foo in list]
```

```
julia> descriptive(x, list_functions)
4-element Vector{Real}:
 100
  10
```

FOOTNOTES

1. Recall that `∈` can be written through tab completion using the command `\in`.

2. In fact, older versions of Julia were restricting the use of for-loops in the global scope.

3. There are two methods to execute a script. The first method is what we've been using so far, where you work interactively with Julia. This includes running commands in the REPL's prompt `julia>` and the execution of a script through a code editor. The second method consists of executing files that store scripts through the function `include`.

5a. Overview and Goals

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Thus far, we've laid the groundwork by introducing the fundamentals of Julia. We've covered in particular variables (single-element and collections) and core programming tools (functions, conditions, and for-loops). At this initial stage, our emphasis was primarily on helping you familiarize with the core approaches and their syntax. However, we didn't delve into any of these concepts, nor did we explore how the tools can be applied and combined.

Equipped now with a foundational knowledge of the concepts, we're ready to explore each in greater depth. **Chapter 5 in particular focuses on mutable collections**, using vectors as their primary example. As we begin to integrate these tools, it may take some time to get fully comfortable with their usage. In fact, you may occasionally need to revisit the sections on functions, conditions, and for-loops.

Despite that our focus is on vectors, many of the lessons we'll learn are applicable across all mutable collections. For instance, this is the case for concepts such as indexing and in-place operations. Other techniques presented extend even further, making their application universal across programming languages. Examples of this include the notion of mutability, along with the distinction between assignments and mutations.

5b. Mutable and Immutable Objects

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INTRODUCTION

Objects in programming can be broadly classified into two categories: mutable and immutable. **Mutable objects** allow their elements to be modified, appended, or removed at will. They're designed for flexibility, with **vectors** constituting a prime example.

In contrast, **immutable objects** are inherently unchangeable: they prevent additions, removals, or modifications of their elements. A common example of immutable object is **tuples**. Immutability effectively locks variables into a read-only state, safeguarding against unintended changes. Moreover, it can result in potential performance gains, as we'll show in Part II of this website.

This section will be relatively brief, focusing solely on the distinctions between mutable and immutable objects. Subsequent sections will expand on their uses and properties.

> **Remark**
> A popular package called `StaticArrays` provides an implementation of **immutable vectors**. We'll explore this package in the context of high performance, as it greatly speeds up computations that involve small vectors.

EXAMPLES OF MUTABILITY AND IMMUTABILITY

To illustrate the consequences of immutability, the following examples attempt to modify existing elements of a collection. The examples rely on vectors as an example of a mutable object and tuples for immutable ones. Additionally, we present the case of strings as another example of immutable object. Recall that strings are essentially sequences of characters, usually employed to represent text.

```
x = [3,4,5]
```

```
julia> x[1] = 0
julia> x
3-element Vector{Int64}:
 0
 4
 5
```

```
x = (3,4,5)
```

```
julia> x[1] = 0
ERROR: MethodError: no method matching setindex!(::Tuple{Int64, Int64, Int64}, ::Int64,
::Int64)
```

```
x = "hello"
```

```
julia> x[1]
'h': ASCII/Unicode U+0068 (category Ll: Letter, lowercase)
julia> x[1] = "a"
ERROR: MethodError: no method matching setindex!(::String, ::Int64, ::Int64)
```

The key characteristic of mutable objects is their ability to modify existing elements. Moreover, mutability also commonly allows for the dynamic addition and removal of elements. In a subsequent section, we'll present various methods for implementing this functionality. For now, we simply demonstrate the concept by using the functions `push!` and `pop!`, which respectively add and remove an element at the end of a collection.

```
x = [3,4]

push!(x, 5)       # add element 5 at the end
```

```
julia> x
3-element Vector{Int64}:
 3
 4
 5
```

```
x = [3,4,5]

pop!(x)           # delete last element
```

```
julia> x
2-element Vector{Int64}:
 3
 4
```

```
x = (3,4,5)

pop!(x)           # error, just like push!(x, <some element>)
```

```
ERROR: MethodError: no method matching pop!(::Tuple{Int64, Int64, Int64})
```

5c. Assignments vs Mutations

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INTRODUCTION

The upcoming sections will be entirely devoted to **vector mutation**. However, to properly cover this subject, we first need to introduce some preliminary concepts, including:

- the distinction between assignments and mutations
- methods for initializing arrays to eventually mutate them
- techniques for vector indexing to select elements

The current section in particular focuses on **distinguishing between assignments and mutations of variables**. The difference between both operations can easily go unnoticed by new users, as both operations use the operator `=`, despite being fundamentally different. Clearly delineating these operations is important not only for Julia, but also other programming languages.

SOME BACKGROUND

Recall that **variables** serve as labels for **objects**, with objects in turn holding **values**. Moreover, objects can be classified according to the number of **elements** contained, ranging from single-element objects (e.g. integers and floating-point numbers) to collections (e.g. vectors).

NUMBER

$$\underbrace{2}_{\text{object}}$$

VECTOR

$$\underbrace{\overset{\text{elements}}{[4, 5, 6]}}_{\text{object}}$$

The distinction between objects and their elements is crucial for the remainder of the section. This is because *assignments apply to objects, whereas mutations apply to elements*. More specifically, assignments rebind variables to new objects, while mutations simply modify existing elements of an existing object.

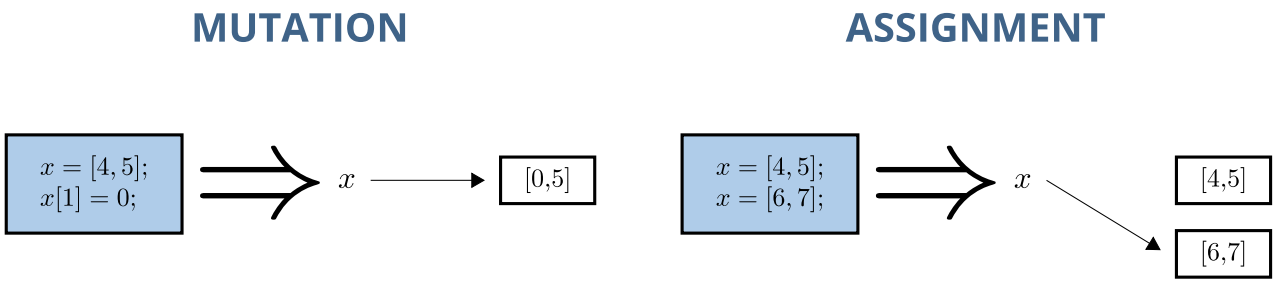
ASSIGNMENTS VS MUTATIONS

Assignments bind variables to objects, a process implemented via the `=` operator. For instance, `x = 3` and `x = [1,2,3]` are examples of assignments, where `3` and `[1,2,3]` represent the objects being assigned to `x`.

Mutations, by contrast, **modify the elements of an object, without creating a new one**. These operations also rely on the `=` operator, with `x[1] = 0` being an example of mutation.

Despite sharing the same operator `=`, assignments and mutations are conceptually distinct. This difference can be better appreciated by **visualizing objects as specific memory addresses**. This implies that assignments like `x = [4,5]` involve two steps: *i*) finding a memory location to store the object with value `[4,5]`, and *ii*) labeling the memory address as `x` for easy access. In contrast, a mutation modifies the contents of the object, but without changing its memory address.

To illustrate this, consider the example `x = [4,5]`, where the object `[4,5]` is stored in a particular memory location. If you then run `x = [6,7]`, `x` becomes associated with a new object containing `[6,7]`, thus constituting an *assignment*. However, if you execute `x[1] = 0` afterwards, the operation modifies the original object `[6,7]` to `[6,0]`. This operation constitutes a *mutation* because `x` continues to reference the same memory address, even though its content has changed.



Remark

You can mutate all the elements of `x`, without this necessarily entailing a new assignment. For example, this occurs when we modify the values of `x` by mutating `x[:]`.

```
x    = [4,5]

x[:] = [0,0]
```

```
julia> x
2-element Vector{Int64}:
 0
 0
```

> The distinction is particularly important since mutations tend to be faster than creating new objects. This will become relevant in Part II, where we explore strategies for speeding up operations.

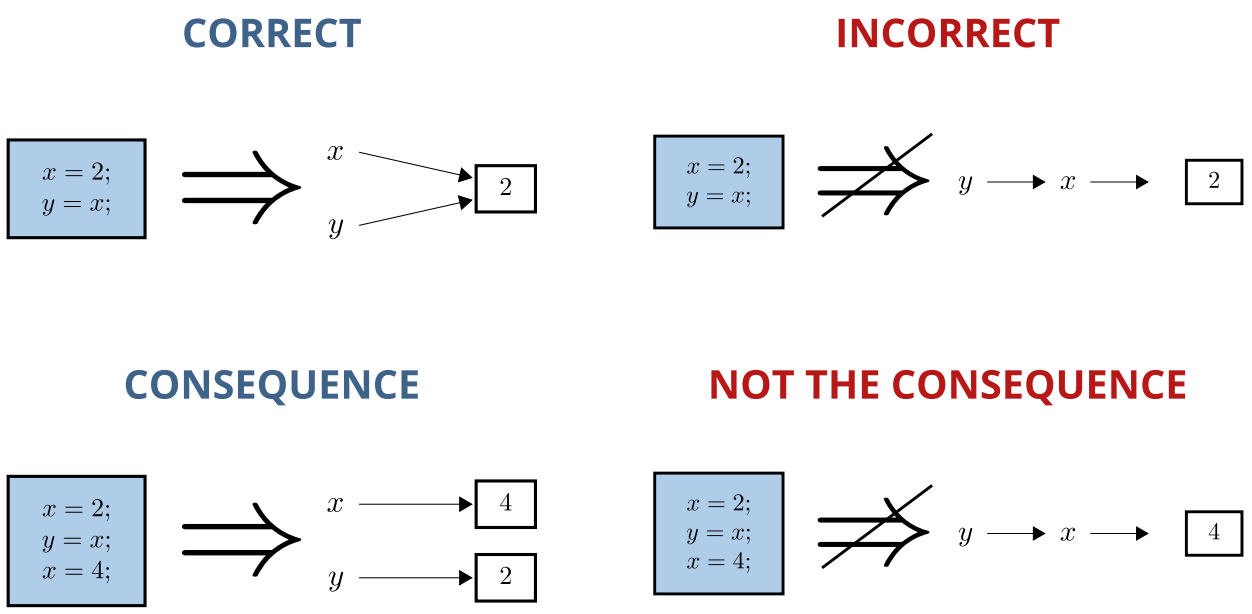
ALIAS VS COPY

So far, we've emphasized the critical distinction between assignments and mutations. Since both operations rely on the `=` operator, we must then inquire when `=` will entail one or the other operation. Next, we explore in particular cases like `y = x`, characterized by entire objects on each side of `=`. Other cases are left for the upcoming sections, after we introduce the concept of slices (i.e, subsets of elements from a vector).

In Julia, executing `y = x` makes `y` another name for the object referenced by `x`. This means that `x` and `y` become different labels for the same underlying object. Formally, it's said that `y` constitutes an **alias** of `x`.

Note that `y = x` shouldn't be understood as binding `y` to `x` itself. Rather, it means that `y` becomes another label for the object that `x` references. This subtle distinction carries a significant practical implication: reassigning `x` to a new object won't affect `y`'s reference.

To clarify this further, let's consider an example where we first execute `x = 2` and then `y = x`. At this point, both `x` and `y` reference the same object, which holds the value `2`. If we eventually execute `x = 4`, the variable `x` will start pointing to a new object that holds the value `4`. However, this won't affect the original object that `x` was referencing. As a result, `y` will still point to the original object with value `2`. This behavior is illustrated below.



```
x = 2    #'x' points to an object with value 2
y = x    #'y' points to the same object as 'x' (do not interpret it as 'y' pointing to 'x')

x = 4    #'x' now points to another object (but 'y' still points to the object holding 2)
```

```
julia> x
4
julia> y
2
```

Remark

Two variables could comprise identical elements and yet refer to different objects.

This can be demonstrated using the operators `==` and `===`, which assess two different types of equality. Specifically, `x == y` checks whether `x` and `y` have equal values, regardless of whether they refer to the same object. In contrast, `x === y` checks whether both `x` and `y` point to the same object, thus verifying if they share the same memory address. By applying these operators, the following example illustrates that objects with identical elements aren't necessarily referencing the same object.

```
x = [4,5]

y = x
```

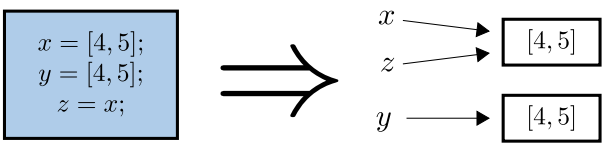
```
julia> x == y
true                    #`x` and `y` have identical elements
julia> x === y
true                        #`x` and `y` DO point to the same
object
```

```
x = [4,5]

y = [4,5]
```

```
julia> x == y
true                    #`x` and `y` have identical elements
julia> x === y
false                    #`x` and `y` DO NOT point to the same
object
```

GRAPHICAL REPRESENTATION



We've indicated that the operation `y = x` creates an alias of `x`, making `y` and `x` two different labels for the same object. This implies that **modifying the elements of either `x` or `y` will necessarily change the elements held by both**. The following diagram and code snippet illustrate this.

GRAPHICAL REPRESENTATION

```
x    = [4,5]
y    = x

x[1] = 0
```

```
julia> x
2-element Vector{Int64}:
 0
 5
julia> y
2-element Vector{Int64}:
 0
 5
```

If you instead want to treat `x` and `y` as separate objects, you must apply the function `copy`. This creates a *new object* with identical elements as the original. In this way, any modification to the new object won't affect the original one, allowing you to work with `x` and `y` independently.

```
x    = [4,5]
y    = copy(x)

x[1] = 0
```

```
julia> x
2-element Vector{Int64}:
 0
 5
julia> y
2-element Vector{Int64}:
 4
 5
```

5d. Initializing Vectors

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INTRODUCTION

We continue introducing preliminary concepts for mutations. The previous section distinguished between the use of `=` for assignments and mutations. Now, we'll deal with approaches to creating vectors.

Our presentation starts by outlining the process of initializing vectors, where memory is reserved without assigning initial values. We'll then discuss how to create vectors filled with predefined values such as zeros or ones. Finally, we show how to concatenate multiple vectors into new ones.

INITIALIZING VECTORS

Creating an array involves two steps: reserving memory for holding its content and assigning initial values to its elements. However, if you don't intend to populate the array with values right away, it's more efficient to only initialize the array. This means reserving memory space, but without setting any initial values.

Technically, initializing an array entails creating an array filled with `undef` values. These values represent arbitrary content in memory at the moment of allocation. Importantly, while `undef` displays concrete numbers when you output the array's content, they're meaningless and vary every time you initialize a new array.

There are two methods for creating vectors with `undef` values. The first one requires you to explicitly specify the type and length of the array, which is accomplished via `Vector{<elements' type>}(undef, <length>)`. The second approach is based on the function `similar(x)`, which creates a vector with the same type and dimensions as an existing vector `x`.

```
x_length = 3

x = Vector{Int64}(undef, x_length)  # `x` can hold `Int64` values, and is initialized with
3 undefined elements
```

```
julia> x
3-element Vector{Int64}:
 140724480121488
   2497084710592
   2497285012816
```

```
y = [3,4,5]

x = similar(y)                        # `x` has the same type as `y`, which is Vector{Int64}
(undef, 3)
```

```
julia> x
3-element Vector{Int64}:
 2497063587648
 2497063587664
 2497355825296
```

The example demonstrates that `undef` values don't follow any particular pattern. Moreover, these values vary in each execution, as they reflect any content held in RAM at the moment of allocation. In fact, a more descriptive way to call `undef` values would be **uninitialized values**.

CREATING VECTORS WITH GIVEN VALUES

In the following, we present several approaches to creating arrays filled with predefined values.

VECTORS WITH RANGE

If our goal is to generate a sequence of values, we can employ the function `collect(<range>)`. Recall that the syntax for defining ranges is `<start>: <steps>: <stop>`, where `<steps>` establishes the gap between elements.

```
some_range = 2:5

x          = collect(some_range)
```

```
julia> x
4-element Vector{Int64}:
 2
 3
 4
 5
```

Notice that when a range is created, `<steps>` implicitly dictates the number of elements to be generated. Alternatively, you could specify the number of elements to be stored, letting `<steps>` be implicitly determined. This is achieved by the function `range`, whose syntax is `range(<start>, <end>, <number of elements>)`. [1]

The following code snippet demonstrates the use of `range`, by generating five evenly spaced elements between 0 and 1.

```
x = range(0, 1, 5)
```

```
julia> x
0.0:0.25:1.0
```

```
x = range(start=0, stop=1, length=5)
```

```
julia> x
0.0:0.25:1.0
```

```
x = range(start=0, length=5, stop=1)    # any order for keyword arguments
```

```
julia> x
0.0:0.25:1.0
```

VECTORS WITH SPECIFIC VALUES

We can also create vectors of some given length filled with the same repeated value. In particular, the functions `zeros` and `ones` respectively create vectors with zeros and ones. By default, these functions define `Float64` elements, although you can specify a different type in the first argument of the function.

```
length_vector = 3

x             = zeros(length_vector)
```

```
julia> x
3-element Vector{Float64}:
 0.0
 0.0
 0.0
```

```
length_vector = 3

x             = zeros(Int, length_vector)
```

```
julia> x
3-element Vector{Int64}:
 0
 0
 0
```

```
length_vector = 3

x             = ones(length_vector)
```

```
julia> x
3-element Vector{Float64}:
 1.0
 1.0
 1.0
```

```
length_vector = 3

x             = ones(Int, length_vector)
```

```
julia> x
3-element Vector{Int64}:
 1
 1
 1
```

For creating Boolean vectors, Julia provides two convenient functions called `trues` and `falses`.

```
length_vector = 3

x             = trues(length_vector)
```

```
julia> x
3-element BitVector:
 1
 1
 1
```

```
length_vector = 3

x             = falses(length_vector)
```

```
julia> x
3-element BitVector:
 0
 0
 0
```

VECTORS FILLED WITH A REPEATED OBJECT

To define vectors comprising elements different from zeros or ones, Julia provides the `fill` function. Unlike the previous functions, this accepts any arbitrary scalar to be repeated.

```
length_vector    = 3
filling_object   = 1

x                = fill(filling_object, length_vector)
```

```
julia> x
3-element Vector{Int64}:
 1
 1
 1
```

```
length_vector    = 3
filling_object   = [1,2]

x                = fill(filling_object, length_vector)
```

```
julia> x
3-element Vector{Vector{Int64}}:
 [1, 2]
 [1, 2]
 [1, 2]
```

```
length_vector    = 3
filling_object   = [1]

x                = fill(filling_object, length_vector)
```

```
julia> x
3-element Vector{Vector{Int64}}:
 [1]
 [1]
 [1]
```

CONCATENATING VECTORS

Finally, we can create a vector `z` that merges all the elements of two vectors `x` and `y`. One simple approach for doing this is through `z = [x ; y]`. While this method is suitable for concatenating a few vectors, it becomes impractical with a large number of vectors, and directly infeasible when the number of vectors to concatenate is unknown.

For these scenarios, we can instead employ the function `vcat`, which merges all its arguments into one vector. By use of the splat operator `...`., the function can also be applied with a single argument that comprises a list of vectors. [2]

```
x = [3,4,5]
y = [6,7,8]


z = vcat(x,y)
```

```
julia> x
6-element Vector{Int64}:
 3
 4
 ⋮
 7
 8
```

```
x = [3,4,5]
y = [6,7,8]

A = [x, y]
z = vcat(A...)
```

```
julia> x
6-element Vector{Int64}:
 3
 4
 ⋮
 7
 8
```

Closely related to vector concatenation is the `repeat` function, which defines a vector containing the same object multiple times. Importantly, unlike `fill`, `repeat` **requires an array as its input**, throwing an error if a scalar is passed in particular.

```
nr_repetitions   = 3
vector_to_repeat = [1,2]

x                = repeat(vector_to_repeat, nr_repetitions)
```

```
julia> x
6-element Vector{Int64}:
 1
 2
 ⋮
 1
 2
```

```
nr_repetitions   = 3
vector_to_repeat = [1]

x                = repeat(vector_to_repeat, nr_repetitions)
```

```
julia> x
3-element Vector{Int64}:
 1
 1
 1
```

```
nr_repetitions   = 3
vector_to_repeat = 1

x                = repeat(vector_to_repeat, nr_repetitions)
```

```
ERROR: MethodError: no method matching repeat(::Int64, ::Int64)
```

ADDING, REMOVING, AND REPLACING ELEMENTS (*OPTIONAL*)

> **Warning!**
>
> This subsection requires knowledge of a few **concepts that we haven't discussed yet**. As such, it's marked as optional.
>
> One such concept is **in-place functions**, identified by the symbol `!` appended to the function's name. The symbol is simply a notation added by developers, hinting that the function modifies the value of at least one of its arguments. In-place functions will be explored thoroughly later).
>
> Another concept introduced is **pairs**, which will also be examined comprehensively in a future section. For the purposes of this subsection, it's sufficient to know that pairs are denoted by `a => b`, where `a` in our application refers to some value and `b` represents its corresponding replacement value.

Next, we show how to add, remove, and replace elements of a vector. To add a single element in particular, the methods are as follows.

```
x                 = [3,4,5]
element_to_insert = 0


push!(x, element_to_insert)                # add 0 at the end - faster
```

```
julia> x
4-element Vector{Int64}:
 3
 4
 5
 0
```

```
x                 = [3,4,5]
element_to_insert = 0


pushfirst!(x, element_to_insert)           # add 0 at the beginning - slower
```

```
julia> x
4-element Vector{Int64}:
 0
 3
 4
 5
```

```
x                 = [3,4,5]
element_to_insert = 0
at_index          = 2

insert!(x, at_index, element_to_insert)     # add 0 at index 2
```

```
julia> x
4-element Vector{Int64}:
 3
 0
 4
 5
```

```
x                 = [3,4,5]
vector_to_insert  = [6,7]


append!(x, vector_to_insert)                # add 6 and 7 at the end
```

```
julia> x
5-element Vector{Int64}:
 3
 4
 5
 6
 7
```

The function `push!` is particularly helpful to collect results in a vector. This is because, as it doesn't require any prior knowledge about the number of elements to be stored, we can dynamically grow the vector by adding more results. Notice that adding elements at the end via `push!` is faster than doing so at the beginning via `pushfirst!`.

Analogous functions exist to remove elements, as shown below.

```
x                 = [5,6,7]


pop!(x)                           # delete last element
```

```
julia> x
2-element Vector{Int64}:
 5
 6
```

```
x                 = [5,6,7]


popfirst!(x)                      # delete first element
```

```
julia> x
2-element Vector{Int64}:
 6
 7
```

```
x                  = [5,6,7]
index_of_removal   = 2

deleteat!(x, index_of_removal)      # delete element at index 2
```

```
julia> x
2-element Vector{Int64}:
 5
 7
```

```
x                  = [5,6,7]
indices_of_removal = [1,3]

deleteat!(x, indices_of_removal)    # delete elements at indices 1 and 3
```

```
julia> x
1-element Vector{Int64}:
 6
```

Emulating the behavior of `deleteat!`, we can also indicate which elements should be retained.

```
x               = [5,6,7]
index_to_keep   = 2

keepat!(x, index_to_keep)
```

```
julia> x
1-element Vector{Int64}:
 6
```

```
x               = [5,6,7]
indices_to_keep = [2,3]

keepat!(x, index_to_keep)
```

```
julia> x
1-element Vector{Int64}:
 6
```

Finally, we can replace specific values with new ones. This can be done by creating a new copy via `replace` or by updating the original vector with `replace!`.

Both functions make use of pairs `a => b`, where `a` is some value and `b` its corresponding replacement value. Note that these functions perform substitutions based on values, rather than indices.

```
x = [3,3,5]

replace!(x, 3 => 0)              # in-place (it updates x)
```

```
julia> x
3-element Vector{Int64}:
 0
 0
 5
```

```
x = [3,3,5]

replace!(x, 3 => 0, 5 => 1)      # in-place (it updates x)
```

```
julia> x
3-element Vector{Int64}:
 0
 0
 1
```

```
x = [3,3,5]

y = replace(x, 3 => 0)           # new copy
```

```
julia> y
3-element Vector{Int64}:
 0
 0
 5
```

```
x = [3,3,5]

y = replace(x, 3 => 0, 5 => 1)   # new copy
```

```
julia> y
3-element Vector{Int64}:
 0
 0
 1
```

FOOTNOTES

[1.] Note that `range` represents a convenient syntax for `<start> : 1 / <number of elements> : <end>`.

[2.] Recall that the operator `...` splits a collection into multiple arguments. This enables the use of a vector or tuple to denote multiple function arguments. For further details, see here under the subsection "Splatting".

5e. Slices: Copies vs Views

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INTRODUCTION

This section concludes the coverage of preliminary concepts for studying mutations by focusing on the behavior of a vector's **slice**. This is defined as a subset of a vector's elements, formally represented as `x[<indices>]`.

Importantly, **slices act differently depending on how they're included in a statement**, functioning as either:

- **copies** of the original vector, thus creating a new object at a new memory address.
- **views** of the original vector, where the original object and the slice share the same memory address.

In the following, we explain the distinction between copies and views in detail. Understanding it is crucial for mutating slices, as mutations can only occur when the slice references the original object. In contrast, if a slice acts as a copy, the parent object and the slice are unrelated, with changes to the slice having no impact on the original object.

SLICES AND THE ASSIGNMENT OPERATOR

Vector mutation involves modifying slices through the operator `=`. For this to be possible, a prerequisite is that the slice references the original object. Nonetheless, the behavior of slices in assignments varies depending on their position within the expression.

Specifically, **slices on the left-hand (LHS) side of `=` act as views**. In this case, slices reference the original elements, thus allowing for the mutation of its parent object. In contrast, **slices on the right-hand side (RHS) of `=` create a copy**. Since copies point to a new object, any modification to the slice won't affect the original object.

The following code snippet demonstrates both behaviors.

```
x    = [4,5]


x[1] = 0        # 'x[1]' is a view and mutates 'x'
```

```
julia> x
2-element Vector{Int64}:
 0
 5
```

```
x    = [4,5]
y    = x[1]      # 'y' is unrelated to 'x' because 'x[1]' is a copy

x[1] = 0         # it mutates 'x' but does NOT modify 'y'
```

```
julia> y
4
```

Aliasing vs Copy

Objects on the RHS of `=` are only treated as copies when it comes to **slices**, such as in statements `y = x[<indices>]`. Instead, if we insert the whole object `x` on the RHS of `=`, as in `y = x`, the operation creates an alias. In this case, `y` and `x` will reference the same object, and so any modification made to `y` will also be reflected in `x`.

```
x    = [4,5]
y    = x         # the whole object (a view)

x[1] = 0         # it DOES modify 'y'
```

```
julia> y
2-element Vector{Int64}:
 4
 5
```

```
x    = [4,5]
y    = x[:]      # a slice of the whole object (a copy)

x[1] = 0         # it does NOT modify 'y'
```

```
julia> y
2-element Vector{Int64}:
 0
 5
```

THE FUNCTION 'VIEW'

Identifying when slices act as copies or views is relevant for high performance, as views eliminate the overhead associated with memory allocations. Although this topic will be explored in Part II of the website, such considerations underscore the importance of distinguishing between copies and slices, beyond their use in assignments.

As a rule of thumb, **slices typically default to creating copies**. This is the case when, for instance, a slice is passed as a function argument or when used within an expression not involving an assignment. These scenarios are illustrated below.

```
x = [3,4,5]

#the following slices are all copies
log.(x[1:2])

x[1:2] .+ 2

[sum(x[:]) * a for a in 1:3]

(sum(x[1:2]) > 0) && true
```

In all these cases, transforming slices into views requires an explicit indication. To achieve this, you need to employ the function `view`. Its syntax is `view(x, <indices>)`, where `<indices>` represent the subset of indices defining the slice. To demonstrate its usage, we revisit and compare the previous code snippet.

```
x = [3,4,5]

#we make explicit that we want views
log.(view(x,1:2))

view(x,1:2) .+ 2

[sum(view(x,:)) * a for a in 1:3]

(sum(view(x,:)) > 0) && true
```

```
x = [3,4,5]

#the following slices are all copies
log.(x[1:2])

x[1:2] .+ 2

[sum(x[:]) * a for a in 1:3]

(sum(x[1:2]) > 0) && true
```

These examples reveal the potential verbosity involved when `view` isn't used sparingly. To address this issue, Julia provides the macros `@view` and `@views`.

The `@view` macro is equivalent to `view`, allowing you to write `@view x[1:2]` instead of `view(x, 1:2)`. However, its advantages are somewhat limited: it saves only a few characters, and additionally necessarily requires parentheses when multiple slices are used (e.g., `@view(x[1:2]) .+ @view(x[2:3])`). In contrast, the `@views` macro significantly streamlines notation, by automatically converting *every* slice within an expression into a view.

```
x = [4,5,6]

# the following are all equivalent
       y = view(x, 1:2)  .+ view(x, 2:3)
       y = @view(x[1:2]) .+ @view(x[2:3])
@views y = x[1:2] .+ x[2:3]
```

One of the most notable applications of `@views` is in functions. By placing `@views` at the beginning of a function, you automatically convert every slice within the function body and its arguments into a view.

```
@views function foo(x)
  y = x[1:2] .+ x[2:3]
  z = sum(x[:]) .+ sum(y)

  return z
end
```

```
function foo(x)
  y = @view(x[1:2]) .+ @view(x[2:3])
  z = sum(@view x[:]) .+ sum(y)

  return z
end
```

5f. Array Indexing

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INTRODUCTION

In order to mutate vectors, you first need to identify the elements you wish to modify. This process is known as **vector indexing**. We've already covered several basic methods for indexing, including vectors itself and ranges (e.g., `x[[1,2,3]]` or `x[1:3]`). While these approaches are effective for simple selections, they fall short for more complex scenarios, for example precluding selections based on conditions.

This section expands our toolkit by introducing some additional forms of indexing. The techniques presented primarily build on broadcasting Boolean operations.

LOGICAL INDEXING

Logical indexing (also known as *Boolean indexing* or *masking*) allows you to select elements based on conditions. Considering a vector `x`, this is achieved using a Boolean vector `y` of the same length as `x`, which acts as a filter: `x[y]` retains elements where `y` is `true` and excludes those where `y` is `false`.

LOGICAL INDEXING

```
x = [1,2,3]
y = [true, false, true]
```

```
julia> x[y]
2-element Vector{Int64}:
 1
 3
```

OPERATORS AND FUNCTIONS FOR LOGICAL INDEXING

Logical indexing becomes a powerful tool when we leverage broadcasting operations, allowing you to easily specify conditions via Boolean vectors. For instance, to select all the elements of `x` lower than 10, you can broadcast a comparison operator or a custom function.

INDEXING VIA BROADCASTING OPERATOR

```
x            = [1, 2, 3, 100, 200]

y            = x[x .< 10]
```

```
julia> y
3-element Vector{Int64}:
 1
 2
 3
```

INDEXING VIA BROADCASTING FUNCTION

```
x            = [1, 2, 3, 100, 200]

condition(a) = (a < 10)              #function to eventually broadcast
y            = x[condition.(x)]
```

```
julia> y
3-element Vector{Int64}:
 1
 2
 3
```

When dealing with multiple conditions, the conditions must be combined using the logical operators `&&` and `||`. [1] The following example illustrates the syntax for doing this. Note that *all* operators must be broadcasted, since logical operators only work with scalar values.

INDEXING VIA BROADCASTING OPERATOR

```
x            = [3, 6, 8, 100]

# numbers greater than 5, lower than 10, but not including 8
y            = x[(x .> 5) .&& (x .< 10) .&& (x .≠ 8)]
```

```
julia> y
1-element Vector{Int64}:
 6
```

INDEXING VIA @.

```
x            = [3, 6, 8, 100]

# numbers greater than 5, lower than 10, but not including 8
y            = x[@. (x > 5) && (x < 10) && (x ≠ 8)]
```

```
julia> y
1-element Vector{Int64}:
 6
```

INDEXING VIA BROADCASTING FUNCTION

```
x            = [3, 6, 7, 8, 100]

# numbers greater than 5, lower than 10, but not including 8
condition(a) = (a > 5) && (a < 10) && (a ≠ 8)          #function to eventually broadcast
y            = x[condition.(x)]
```

```
julia> y
1-element Vector{Int64}:
 6
```

The example reveals that directly broadcasting *operators* may result in verbose code, due to the repeated use of dots in the expression. In contrast, approaches based on functions or the macro `@.` keep the syntax simple, reducing **boilerplate code**.

LOGICAL INDEXING VIA `IN` AND `∈`

> **Remark**
>
> The symbols `∈` and `∉` used in this section can be inserted via tab completion:
>
> - `∈` by `\in`
> - `∉` by `\notin`

Another approach to selecting elements through logical indexing involves `in` and `∈`. Each of these symbols is available as a function and an operator, and they check whether a *scalar* `a` belongs to a given collection `list`. For simplicity, next we'll refer to `in` as a function and `∈` as an operator.

The function `in(a, list)` evaluates whether the scalar `a` matches any element in the vector `list`, yielding the same result as `a ∈ list`. For example, both `in(2, [1, 2, 3])` and `2 ∈ [1, 2, 3]` return `true`, as `2` is an element of `[1,2,3]`.

By replacing the scalar `a` with a collection `x`, `in` and `∈` can define Boolean vectors via broadcasting. Recall, though, that broadcasting defaults to iterating over pairs of elements. This means that executing `in.(x, list)` or `x .∈ list` will result in a simultaneous iteration over each pair of `x` and `list`. However, this isn't the desired operation. Rather, our goal is to check whether each element of `x` belongs to `list`, which requires treating `list` as a single object. This can be accomplished in several ways, as it was shown **here**, including wrapping `list` in `Ref`.

As an illustration, below we create a vector `y` that contains the minimum and maximum of the vector `x`.

FUNCTIONS 'IN' AND '∈'

```
x           = [-100, 2, 4, 100]
list = [minimum(x), maximum(x)]

# logical indexing (both versions are equivalent)
bool_indices = in.(x, Ref(list))    #`Ref(list)` can be replaced by `(list,)`
bool_indices = (∈).(x,Ref(list))

y            = x[bool_indices]
```

```
julia> bool_indices
4-element BitVector:
 1
 0
 0
 1
julia> y
2-element Vector{Int64}:
 -100
  100
```

OPERATORS 'IN' AND '∈'

```
x           = [-100, 2, 4, 100]
list = [minimum(x), maximum(x)]

# logical indexing
bool_indices = x .∈ Ref(list)       #only option, not possible to broadcast `in`


y            = x[bool_indices]
```

```
julia> bool_indices
4-element BitVector:
 1
 0
 0
 1
julia> y
2-element Vector{Int64}:
 -100
  100
```

Remark

The `in` function has an alternative curried version, allowing the user to directly broadcast `in` while treating `list` as a single element. The syntax for doing this is `in(list).(x)`, as shown in the example below.

CURRIED 'IN'

```
x           = [2, 4, 100]
list = [minimum(x), maximum(x)]

#logical indexing
bool_indices = x[in(list).(x)]   #no need to use `Ref(list)`
y            = x[bool_indices]
```

```
julia> bool_indices
4-element BitVector:
 1
 0
 0
 1
julia> y
2-element Vector{Int64}:
 -100
  100
```

Remark

The functions and operators `in` and `∈` allow for negated versions `!in` and `∉` (equivalent to `!∈`), which select elements *not* belonging to a set.

Below, we apply them to retain the elements of `x` that are not its minimum or its maximum.

FUNCTIONS '!IN' AND '∉'

```
x           = [-100, 2, 4, 100]
list = [minimum(x), maximum(x)]

#identical vectors for logical indexing
bool_indices = (!in).(x, Ref(list))
bool_indices  =  (∉).(x,  Ref(list))                #or  `(!∈).(x,
Ref(list))`
```

```
julia> bool_indices
4-element BitVector:
 0
 1
 1
 0
julia> x[bool_indices]
2-element Vector{Int64}:
 2
 4
```

OPERATORS '!IN' AND '∉'

```
x           = [-100, 2, 4, 100]
list = [minimum(x), maximum(x)]

#vector for logical indexing
bool_indices = x .∉ Ref(list)
```

```
julia> bool_indices
4-element BitVector:
 0
 1
 1
 0
julia> x[bool_indices]
2-element Vector{Int64}:
 2
 4
```

THE FUNCTIONS 'FINDALL' AND 'FILTER'

We close this section by presenting two additional methods for element selection. They're provided by the functions `filter` and `findall`.

The function `filter` returns the *elements* of a vector `x` satisfying a given condition. Despite what the name may suggest, `filter` retains elements rather than discard them. The condition is specified by a function that returns a Boolean scalar.

'FILTER'

```
x = [5, 6, 7, 8, 9]

y = filter(a -> a < 7, x)
```

```
julia> y
2-element Vector{Int64}:
 5
 6
```

The function `findall` does the same as `filter`, but returns the *indices* of `x`. With `findall`, the condition can be stated in two ways: either via a Boolean scalar function or a Boolean vector.

'FINDALL' - VIA FUNCTION

```
x = [5, 6, 7, 8, 9]

y = findall(a -> a < 7, x)
z = x[findall(a -> a < 7, x)]
```

```
julia> y
2-element Vector{Int64}:
 1
 2
julia> z
2-element Vector{Int64}:
 5
 6
```

'FINDALL' - VIA BOOLEAN VECTOR

```
x = [5, 6, 7, 8, 9]

y = findall(x .< 7)
z = x[findall(x .< 7)]
```

```
julia> y
2-element Vector{Int64}:
 1
 2
julia> z
2-element Vector{Int64}:
 5
 6
```

FOOTNOTES

[1.] The logical operators `&&` and `||` were introduced in the section about conditional statements.

5g. In-Place Operations

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INTRODUCTION

This section focuses on **in-place operations**, a term that encompasses any method that mutates collections. These operations are characterized by the reuse of existing objects, rather than generating new ones, giving rise to the expression " *modifying values in place*."

Understanding whether an operation mutates an object or generates a new one is crucial, as the outcomes may differ depending on the application. Furthermore, even if results were identical, in-place modifications commonly entail performance benefits relative to the creation of new objects. This aspect will be explored in Part II of the website, when we discuss high performance.

> **Remark**
>
> Before proceeding, I recommend reviewing the definitions of slices introduced in the previous section. Recall that, given a vector `x`, a **slice** refers to a subset of `x`'s elements selected via `x[<indices>]`.
>
> Moreover, a slice can act as a **copy**, in which case we're creating a new object with its own memory address. Alternatively, the slice can behave as a **view**, thereby referencing the original memory address of `x`. The distinction is crucial, as it'll determine whether modifying the slice will affect the original data or not.

MUTATIONS VIA COLLECTIONS

A simple way to mutate a vector is to replace an entire slice with another collection. This is achieved using statements of the form `x[<indices>] = <expression>`, where `<expression>` must match the length of `x[<indices>]`. The approach effectively mutates `x` because a slice on the left-hand side of `=` behaves as a view, thus referencing the original object. Below, we provide a few examples of this approach.

```
x          = [1, 2, 3]

x[2:end]   = [20, 30]
```

```
julia> x
3-element Vector{Int64}:
  1
 20
 30
```

```
x          = [1, 2, 3]

x[x .≥ 2] = [2, 3] .* 10
```

```
julia> x
3-element Vector{Int64}:
  1
 20
 30
```

A common application of this method involves defining `<expression>` through elements from either the original vector or the slice being modified. This allows for self-referencing updates of the variable.

```
x          = [1, 2, 3]

x[2:end]   = [x[i] * 10 for i in 2:length(x)]
```

```
julia> x
3-element Vector{Int64}:
  1
 20
 30
```

```
x          = [1, 2, 3]

x[x .≥ 2] = x[x .≥ 2] .* 10
```

```
julia> x
3-element Vector{Int64}:
  1
 20
 30
```

Importantly for the mutations via for-loops, a scalar can be used on the right-hand side of `=` for single-element slices.

```
x          = [1, 2, 3]

x[3]       = 30
```

```
julia> x
3-element Vector{Int64}:
  1
  2
 30
```

Warning! - Vectors can only be mutated by objects of the same type

When a vector is defined, the type of elements that the vector can hold is implicitly defined. Consequently, attempting to replace elements with a different type will result in an error. For instance, the following examples only admit mutations with values of type `Int64`.

```
x          = [1, 2, 3]     # Vector{Int64}

x[2:3]     = [3.5, 4]      # 3.5 is Float64
```

```
ERROR: InexactError: Int64(3.5)
```

```
x          = [1, 2, 3]     # Vector{Int64}

x[2:3]       =  [3.0,  4]         #  3.0  is  Float64  but  accepts
conversion
```

```
julia> x
3-element Vector{Int64}:
 1
 3
 4
```

MUTATIONS VIA FOR-LOOPS

Replacing a single-element slice with a scalar value enables mutations through for-loops. This is implemented by substituting the value of a single element during each iteration.

To illustrate the procedure, let's consider a typical application of this approach: populating vectors with values. The process involves initializing a vector, and then iterating over its elements to assign desired values.

```julia
x    = Vector{Int64}(undef, 3)  # `x` is initialized with 3 undefined elements

x[1] = 0
x[2] = 0
x[3] = 0
```

```
julia> x
3-element Vector{Int64}:
 0
 0
 0
```

```julia
x    = Vector{Int64}(undef, 3)  # `x` is initialized with 3 undefined elements

for i in eachindex(x)
    x[i] = 0
end
```

```
julia> x
3-element Vector{Int64}:
 0
 0
 0
```

The approach relies on `x[i]` on the left-hand side of `=` acting as a view. Alternatively, we could leverage the function `view` to create a variable that contains all the elements to be modified. This allows us to work with for-loops that mutate entire objects, rather than a subset of the original object.

The following example demonstrates this by mutating a vector of zeros. Note that the function `zeros` defaults to zeros with type `Float64`, explaining why `1` is automatically converted to `1.0`.

```julia
x     = zeros(3)


for i in 2:3
    x[i] = 1
end
```

```
julia> x
3-element Vector{Float64}:
 0.0
 1.0
 1.0
```

```
x     = zeros(3)
slice = view(x, 2:3)

for i in eachindex(slice)
    slice[i] = 1
end
```

```
julia> x
3-element Vector{Float64}:
 0.0
 1.0
 1.0
```

> **Warning!** - For-Loops Should be Wrapped in Functions
>
> Recall that for-loops should always be wrapped in functions, as failing to do so severely affects performance. In the next section, where we'll cover mutating functions, mutations via for-loops will be revisited.

MUTATIONS VIA .=

In terms of syntax, broadcasting serves as a streamlined alternative to for-loops. This principle even extends to mutations, which can be implemented by broadcasting the assignment operator `=`. The syntax for this is `x[<indices>] .= <expression>`, where `<expression>` can be either a *vector* or a *scalar*.

In the specific case of `x[<indices>]` on the left-hand side and a vector for `<expression>`, the `.=` operator produces the same outcome as using `=` with a corresponding collection. In fact, using `=` rather than `.=` in these cases tends to be more performant.

```
x       = [3, 4, 5]

x[1:2]  = x[1:2] .* 10
```

```
julia> x
3-element Vector{Int64}:
 30
 40
  5
```

```
x       = [3, 4, 5]

x[1:2] .= x[1:2] .* 10    # identical output (less performant)
```

```
julia> x
3-element Vector{Int64}:
 30
 40
  5
```

Considering this, the primary use cases of `.=` for mutating `x` involve expressions like:

- `x[<indices>] .= <scalar>`, and
- `y .= <expression>` where `y` is either a view of `x` or `x` itself.

Next, we analyze each case separately.

SCALARS ON THE RIGHT-HAND SIDE OF =

Applying `=` to replace multiple elements with the *same* scalar value requires a collection matching the number of elements being substituted. However, with the `.=` operator, you can streamline the process by simply writing `x[<indices>] .= <scalar>`.

For instance, the following code snippet uses this approach to replace every negative value in `x` with zero.

```
x          = [-2, -1, 1]

x[x .< 0] .= 0
```

```
julia> x
3-element Vector{Int64}:
 0
 0
 1
```

VARIABLES ON THE LEFT-HAND SIDE OF =

Since both mutations and assignments rely on `=`, it's essential to **distinguish between in-place operations and reassignments**. In particular, the latter doesn't modify the original object, but actually creates a new one.

We've already shown that placing slices on the left-hand side of `=` results in mutations. Now, let's consider cases where an entire object like `x` appears on the left-hand side. In these cases, we need to be careful, as only `.=` will result in a mutation, whereas `=` will perform a reassignment.

For example, suppose our goal is to modify *all* the elements of a vector `x`. All the following approaches determine that `x` holds the same values, but only the last two achieve this by mutating `x`.

```
x    = [1, 2, 3]

x    = x .* 10
```

```
julia> x
3-element Vector{Int64}:
 10
 20
 30
```

```
x    = [1, 2, 3]

x   .= x .* 10
```

```
julia> x
3-element Vector{Int64}:
 10
 20
 30
```

```
x    = [1, 2, 3]

x[:] = x .* 10
```

```
julia> x
3-element Vector{Int64}:
 10
 20
 30
```

Notice that the difference between approaches could go unnoticed, as they all yield the same outcome. However, we'll see in Part II of the website that these approaches can entail big differences in performance. In particular, reusing the original memory address of `x` tends to be more performant than creating a new memory address for `x`.

This difference will also manifest when using the macro `@.` for a seamless broadcasting. Depending on where `@.` is placed relative to `=`, we could end up with an assignment or a mutation.

```
x    = [1, 2, 3]

x   .= x .* 10
```

```
julia> x
3-element Vector{Int64}:
 10
 20
 30
```

```
x    = [1, 2, 3]

@. x = x  * 10
```

```
julia> x
3-element Vector{Int64}:
 10
 20
 30
```

```
x    = [1, 2, 3]

x    = @. x * 10
```

```
julia> x
3-element Vector{Int64}:
 10
 20
 30
```

VIEW ALIASES ON THE LEFT-HAND SIDE OF =

The previous case exemplified a scenario where the ultimate outcome is the same, regardless of whether we employ `=` or `.=`. Instead, the results will differ when view aliases are on the left-hand side. View aliases allow us to work with `slice = view(x, <indices>)`, rather than `x[<indices>]`. Defining view aliases is convenient when we need to perform various operations over the same slice. This avoids repeatedly referencing the slice via `x[<indices>]`, which would be inefficient, error-prone, and tedious.

In these cases, it's only when we use `.=` that we'll perform a mutation.

```
x      = [1, 2, 3]

slice  = view(x, x .≥ 2)
slice .= slice .* 10        # same as 'x[x .≥ 2] = x[x .≥ 2] .* 10'
```

```
julia> x
3-element Vector{Int64}:
  1
 20
 30
```

```
x      = [1, 2, 3]

slice  = view(x, x .≥ 2)
slice  = slice .* 10        # this does NOT modify `x`
```

```
julia> x
3-element Vector{Int64}:
 1
 2
 3
```

There are a few additional incorrect uses that can emerge with view aliases. To demonstrate them, let's consider replacing all negative values in `x` with zero. Below, only the first approach achieves the desired outcome.

```
x      = [-2, -1, 1]

slice  = view(x, x .< 0)
slice .= 0
```

```
julia> x
3-element Vector{Int64}:
 0
 0
 1
```

```
x      = [-2, -1, 1]

slice  = x[x .< 0]          # 'slice' is a copy
slice .= 0                  # this does NOT modify `x`
```

```
julia> x
3-element Vector{Int64}:
 -2
 -1
  1
```

```
x      = [-2, -1, 1]

slice  = view(x, x .< 0)
slice  = 0                  # this does NOT modify `x`
```

```
julia> x
3-element Vector{Int64}:
 -2
 -1
  1
```

5h. In-Place Functions

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INTRODUCTION

This section continues exploring **approaches to mutating vectors**. The emphasis is in particular on **in-place functions**, defined as functions that mutate at least one of their arguments.

Many built-in functions in Julia have a corresponding in-place counterpart. These versions can be easily identified by the symbol `!` appended to their names. In-place functions enable users to store the output in one of the function's arguments, thereby avoiding the creation of a new object. They can also be used to update the values of a variable directly. For example, given a vector `x`, `sort(x)` returns a new vector with ordered elements, but without altering the original `x`. In contrast, the in-place version `sort!(x)` directly stores the result within `x` itself.

The benefits of in-place functions will become evident in Part II, when discussing high-performance computing. Essentially, by reusing existing objects, in-place functions eliminate the overhead associated with creating new objects.

IN-PLACE FUNCTIONS

In-place functions, also known as **mutating functions**, are characterized by their ability to modify at least one of their arguments. For example, given a vector `x`, the following function `foo(x)` constitutes an example of in-place function, as it modifies the content of `x`.

```
y = [0,0]

function foo(x)
    x[1] = 1
end
```

```
julia> y
2-element Vector{Int64}:
 0
 0
julia> foo(y) #it mutates 'y'
julia> y
2-element Vector{Int64}:
 1
 0
```

> **Functions Can't Reassign Variables**

While functions are capable of mutating values, they **can't reassign variables defined outside their scope**. Any attempt to redefine such variable will be interpreted as the creation of a new local variable. [1]

The following code illustrates this behavior by redefining a function argument and a global variable. The output reflects that `foo` in each example treats the redefined `x` as a new local variable, only existing within `foo`'s scope.

```
x = 2

function foo(x)
    x = 3
end
```

```
julia> x
2
julia> foo(x)
julia> x #functions can't redefine variables globally, only mutate them
2
```

```
x = [1,2]

function foo()
    x = [0,0]
end
```

```
julia> x
2-element Vector{Int64}:
 1
 2
julia> foo()
julia> x #functions can't redefine variables globally, only mutate them
2-element Vector{Int64}:
 1
 2
```

BUILT-IN IN-PLACE FUNCTIONS

In Julia, many built-in functions that take vectors as arguments are available in two forms: a "standard" version and an in-place version. To distinguish between them, Julia's developers follow a convention that **any function ending with `!` corresponds to an in-place function**.

Appending ! To A Function Has No Impact at All

> **Appending `!` to a function doesn't change the function's behavior**. It's simply a convention adopted by Julia's developers to emphasize the mutable nature of the operation. Its purpose is to alert users about the potential side effects of the function, thus preventing unintended modifications of objects.

To illustrate these forms, let's start considering single-argument functions. In particular, we'll focus on `sort`. This arranges the elements of a vector in ascending order, with the option `rev=true` implementing a descending order. In its standard form, `sort(x)` creates a new vector containing the sorted elements, leaving the original vector `x` unchanged. In contrast, the in-place version `sort!(x)` updates the original vector `x` directly, overwriting its values with the sorted result.

```
x = [2, 1, 3]

y = sort(x)
```

```
julia> x
3-element Vector{Int64}:
 2
 1
 3
julia> y
3-element Vector{Int64}:
 1
 2
 3
```

```
x = [2, 1, 3]

sort!(x)
```

```
julia> x
3-element Vector{Int64}:
 1
 2
 3
```

Regarding multiple-argument functions, it's common to include an argument whose sole purpose is to store outputs. For instance, given a function `foo` and a vector `x`, the built-in function `map(foo, x)` has an in-place version `map!(foo, <output vector>, x)`.

```julia
x      = [1, 2, 3]


output = map(a -> a^2, x)
```

```
julia> x
3-element Vector{Int64}:
 1
 2
 3
julia> output
3-element Vector{Int64}:
 1
 4
 9
```

```julia
x      = [1, 2, 3]
output = similar(x)             # we initialize `output`

map!(a -> a^2, output, x)       # we update `output`
```

```
julia> x
3-element Vector{Int64}:
 1
 2
 3
julia> output
3-element Vector{Int64}:
 1
 4
 9
```

MUTATIONS VIA FOR-LOOPS

Recall that for-loops in Julia should always be wrapped in functions. This not only prevents issues with variable scope, but is also key for performance, as we'll discuss in Part II.

In this context, the ability of functions to mutate their arguments is crucial. It determines that we can initialize vectors with `undef` values, pass them to a function, and fill them through a function via for-loops. The examples below illustrate this application.

```julia
x = [3,4,5]

function foo!(x)
    for i in 1:2
        x[i] = 0
    end
end
```

```
julia> foo!(x)

julia> x
3-element Vector{Int64}:
 0
 0
 5
```

```julia
x = Vector{Float64}(undef, 3)           # initialize a vector with 3 elements

function foo!(x)
    for i in eachindex(x)
        x[i] = 0
    end
end
```

```
julia> foo!(x)

julia> x
3-element Vector{Int64}:
 0
 0
 0
```

FOOTNOTES

[1.] Strictly speaking, it's possible to reassign a variable by using the `global` keyword. However, its use is typically discouraged, explaining why we won't cover it.

6a. Overview and Goals

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The previous chapter equipped us with techniques for indexing and modifying vectors, expanding our toolkit for working with data collections. This section builds on this knowledge to achieve several goals.

Firstly, we'll **introduce additional types for collections**, including dictionaries and named tuples. Building on our grasp of tuples and vectors, we're now well-positioned to understand the unique features of these alternatives and understand when they're more suitable.

Secondly, we'll **expand on tools for streamlining code**, which will become indispensable in your daily use of Julia. These tools will make your coding experience smoother, by reducing boilerplate code and improving syntax readability. One notable example is the use of pipes.

Thirdly, we'll introduce several standard functions for manipulating vectors, enabling you to perform operations such as removing duplicates and sorting elements.

To conclude the chapter, we'll **put into practice all the tools we've covered**. This will be done through a hypothetical scenario involving a YouTuber's earnings. This hands-on approach will demonstrate how to apply the tools learned, helping you bridge the gap between theory and practice. Furthermore, it'll lay the foundation for more advanced data analysis tools: by mastering the application of fundamentals such as vector indexing, you'll be well-equipped to seamlessly transition to typical data-analysis tools (e.g., the `DataFrames` package).

6b. Named Tuples and Dictionaries

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INTRODUCTION

Our previous discussions on collections have centered around vectors and tuples. The current section expands on the subject by offering a more comprehensive analysis of tuples. Additionally, we introduce two new types of collections: **named tuples** and **dictionaries**.

We'll also cover how to characterize collections through keys and values, methods for the manipulation of collections, and approaches to transforming one collection into another.

KEYS AND VALUES

Most collections in Julia are characterized by **keys**. They serve as unique identifiers for their elements and are paired with a corresponding **value**.. [1] For instance, the vector `x = [2, 4, 6]` has the indices `1, 2, 3` as its keys, and `2, 4, 6` as their respective values.

Keys are more general than indices: while indices are limited to integer identifiers, keys can be any valid Julia object (e.g., strings, numbers, or other objects).

Julia provides the functions `keys` and `values` to extract the keys and values of a collection. The following code snippets demonstrate their usage with vectors and tuples, whose keys are represented by indices. Note that neither `keys` nor `values` return a vector, requiring the `collect` function to obtain a vector representation of the keys or values.

```
x = [4, 5, 6]
```

```
julia> collect(keys(x))
3-element Vector{Int64}:
 1
 2
 3
julia> collect(values(x))
3-element Vector{Int64}:
 4
 5
 6
```

```
x = (4, 5, 6)
```

```
julia> collect(keys(x))
3-element Vector{Int64}:
 1
 2
 3
julia> collect(values(x))
3-element Vector{Int64}:
 4
 5
 6
```

THE TYPE `PAIR`

Collections of key-value pairs in Julia are represented by the type `Pair{<key type>, <value type>}`. Although we won't directly work with objects of this type, they form the basis for constructing other collections such as dictionaries and named tuples.

A key-value pair can be created by using the operator `=>`, as in `<key> => <value>`. For instance, `"a" => 1` represents a pair, where `a` is the key and `1` the corresponding value. Alternatively, pairs can be created using the function `Pair(<key>, <value>)`, where `Pair("a",1)` is equivalent to the previous example.

Given a pair `x`, its key can be accessed via either `x[1]` or `x.first`. Likewise, its value is retrieved using either `x[2]` or `x.second`. All this is demonstrated below.

```
some_pair = ("a" => 1)       # or simply 'some_pair = "a" => 1'

some_pair = Pair("a", 1)     # equivalent
```

```
julia> some_pair
"a" => 1
julia> some_pair[1]
"a"
julia> some_pair.first
"a"
```

```
some_pair = ("a" => 1)       # or simply 'some_pair = "a" => 1'

some_pair = Pair("a", 1)     # equivalent
```

```
julia> some_pair
"a" => 1
julia> some_pair[2]
1
julia> some_pair.second
1
```

THE TYPE `SYMBOL`

The type used to represent keys may vary across collections. A commonly one used for keys is `Symbol`, which offers an efficient way to represent string-based identifiers. A symbol named `x` is written as `:x` and can be constructed from a string using the function `Symbol(<string>)`. [2]

```
vector_symbols = [:x, :y]
```

```
julia> vector_symbols
2-element Vector{Symbol}:
 :x
 :y
```

```
vector_symbols = [Symbol("x"), Symbol("y")]
```

```
julia> vector_symbols
2-element Vector{Symbol}:
 :x
 :y
```

NAMED TUPLES

> **Warning!**
>
> **Tuples and named tuples are only suitable for small collections**. Using them with large collections can result in poor performance or even fatal errors (such as stack overflows). For large collections, arrays remain the preferred choice.
>
> Defining what qualifies as *small* is challenging, and unfortunately there's no definitive answer. We can only indicate that collections with fewer than 10 elements are certainly small, while those exceeding 100 elements clearly exceed the intended use.

Named tuples share several properties with regular tuples, including their **immutability**. However, they also exhibit some notable differences. One important distinction is that **the keys of named tuples are objects of type** `Symbol`, in contrast to the numerical indices used for regular tuples.

Named tuples also differ syntactically, requiring being enclosed in parentheses `()`. Omitting them is not possible, unlike with regular tuples. Furthermore, when creating a single-element named tuple, the syntax requires either a trailing comma `,` after the element (similar to regular tuples) or a leading semicolon `;` before the element. [3]

To construct a named tuple, each element must be specified in the format `<key> = <value>`, such as `a = 10`. Alternatively, a pair `<key with Symbol type> => <value>` can be used, as in `:a => 10`. Once a named tuple `nt` is created, the element `a` can be accessed either by key lookup `nt[:a]` or by dot syntax `nt.a`.

The following code snippets illustrate these concepts.

```
# all 'nt' are equivalent
nt = (  a=10, b=20)
nt = (; a=10, b=20)

nt = (  :a => 10, :b => 10)
nt = (; :a => 10, :b => 10)
```

```
julia> nt
(a = 10, b = 20)
julia> nt.a
10
julia> nt[:a] #alternative way to access 'a'
10
```

```
# all 'nt' are equivalent
nt = (  a=10,)
nt = (; a=10 )

nt = (  :a => 10,)
nt = (; :a => 10 )

#not 'nt =  (a = 10)'  -> this is interpreted as 'nt = a = 10'
#not 'nt = (:a => 10)' -> this is interpreted as a pair
```

```
julia> nt
(a = 10, )
julia> nt.a
10
julia> nt[:a] #alternative way to access 'a'
10
```

Remark

To see the list of keys and values, we can employ the functions `keys` and `values`.

```
nt = (a=10, b=20)
```

```
julia> collect(keys(nt))
2-element Vector{Symbol}:
 :a
 :b
julia> values(nt)
(10, 20)
```

DISTINCTION BETWEEN THE CREATION OF TUPLES AND NAMED TUPLES

It's possible to create named tuples from existing variables. For instance, given variables `x = 10` and `y = 20`, one can define `nt = (; x, y)`. This creates a named tuple with keys `x` and `y`, and corresponding values `10` and `20`.

The semicolon `;` plays a crucial role in this construction, as it distinguishes named tuples from regular tuples. Omitting it, as in `nt = (x, y)`, would result in a regular tuple instead.

```
x   = 10
y   = 20

nt  = (; x, y)
tup = (x, y)
```

```
julia> nt
(x = 10, y = 20)
julia> tup
(10, 20)
```

```
x   = 10



nt  = (; x)
tup = (x, )
```

```
julia> nt
(x = 10,)
julia> tup
(10,)
```

DICTIONARIES

Dictionaries are collections of key-value pairs, exhibiting three distinctive features:

- **Dictionary keys can be any object**: strings, numbers, and other objects are possible.
- **Dictionaries are mutable**: elements can be modified, added, and removed after creation.
- **Dictionaries are unordered**: keys have no inherent order.

The function `Dict` can be used to create dictionaries, where each argument is a key-value pair written in the form `<key> => <value>`.

```
some_dict = Dict(3 => 10, 4 => 20)
```

```
julia> some_dict
Dict{Int64, Int64} with 2 entries:
  4 => 20
  3 => 10
julia> some_dict[1]
10
```

```
some_dict = Dict("a" => 10, "b" => 20)
```

```
julia> some_dict
Dict{String, Int64} with 2 entries:
  "b" => 20
  "a" => 10
julia> some_dict["a"]
10
```

```
some_dict = Dict(:a => 10, :b => 20)
```

```
julia> some_dict
Dict{Symbol, Int64} with 2 entries:
  :a => 10
  :b => 20
julia> some_dict[:a]
10
```

```
some_dict = Dict((1,1) => 10, (1,2) => 20)
```

```
julia> some_dict
Dict{Tuple{Int64, Int64}, Int64} with 2 entries:
  (1, 2) => 20
  (1, 1) => 10
julia> some_dict[(1,1)]
10
```

Note that regular dictionaries are inherently unordered, meaning that the access to their elements doesn't follow any pattern. The following example illustrates this, by collecting the dictionary keys into a vector. [4]

```
some_dict      = Dict(3 => 10, 4 => 20)

keys_from_dict = collect(keys(some_dict))
```

```
julia> keys_from_dict
2-element Vector{Int64}:
 4
 3
```

```
some_dict      = Dict("a" => 10, "b" => 20)

keys_from_dict = collect(keys(some_dict))
```

```
julia> keys_from_dict
2-element Vector{String}:
 "b"
 "a"
```

```
some_dict      = Dict(:a => 10, :b => 20)

keys_from_dict = collect(keys(some_dict))
```

```
julia> keys_from_dict
2-element Vector{Symbol}:
 :a
 :b
```

```
some_dict      = Dict((1,1) => 10, (1,2) => 20)

keys_from_dict = collect(keys(some_dict))
```

```
julia> keys_from_dict
2-element Vector{Tuple{Int64, Int64}}:
 (1, 2)
 (1, 1)
```

CREATING TUPLES, NAMED TUPLES, AND DICTIONARIES

Tuples, named tuples, and dictionaries can be constructed from other collections. The only requirement is that the source collection possesses a key-value structure.

To demonstrate this possibility, we begin by creating **dictionaries** from a variety of collections.

```
vector = [10, 20] # or tupl = (10,20)



dict = Dict(pairs(vector))
```

```
julia> dict
Dict{Int64, Int64} with 2 entries:
  2 => 20
  1 => 10
```

```
keys_for_dict   = [:a, :b]
values_for_dict = [10, 20]


dict = Dict(zip(keys_for_dict, values_for_dict))
```

```
julia> dict
Dict{Symbol, Int64} with 2 entries:
  :a => 10
  :b => 20
```

```
keys_for_dict   = (:a, :b)
values_for_dict = (10, 20)


dict = Dict(zip(keys_for_dict, values_for_dict))
```

```
julia> dict
Dict{Symbol, Int64} with 2 entries:
  :a => 10
  :b => 20
```

```
nt_for_dict = (a = 10, b = 20)



dict = Dict(pairs(nt_for_dict))
```

```
julia> dict
Dict{Symbol, Int64} with 2 entries:
  :a => 10
  :b => 20
```

```
keys_for_dict      = (:a, :b)
values_for_dict    = (10, 20)
vector_keys_values     =     [(keys_for_dict[i],     values_for_dict[i])     for     i     in
eachindex(keys_for_dict)]

dict = Dict(vector_keys_values)
```

```
julia> dict
Dict{Symbol, Int64} with 2 entries:
  :a => 10
  :b => 20
```

Likewise, we can define a **tuple** from other collections, as shown below.

```
a               = 10
b               = 20

tup             = (a, b)
```

```
julia> tup
(10, 20)
```

```
values_for_tup = [10, 20]

tup             = (values_for_tup... ,)
```

```
julia> tup
(10, 20)
```

```
values_for_tup = [10, 20]

tup             = Tuple(values_for_tup)
```

```
julia> tup
(10, 20)
```

Finally, **named tuples** can also be constructed from other collections.

```
a = 10
b = 20


nt = (; a, b)
```

```
julia> nt
(a = 10, b = 20)
```

```
keys_for_nt   = [:a, :b]
values_for_nt = [10, 20]


nt = (; zip(keys_for_nt, values_for_nt)...)
```

```
julia> nt
(a = 10, b = 20)
```

```
keys_for_nt   = [:a, :b]
values_for_nt = [10, 20]


nt = NamedTuple(zip(keys_for_nt, values_for_nt))
```

```
julia> nt
(a = 10, b = 20)
```

```
keys_for_nt   = (:a, :b)
values_for_nt = (10, 20)


nt = NamedTuple(zip(keys_for_nt, values_for_nt))
```

```
julia> nt
(a = 10, b = 20)
```

```
keys_for_nt        = [:a, :b]
values_for_nt      = [10, 20]
vector_keys_values = [(keys_for_nt[i], values_for_nt[i]) for i in eachindex(keys_for_nt)]

nt = NamedTuple(vector_keys_values)
```

```
julia> nt
(a = 10, b = 20)
```

```
dict = Dict(:a => 10, :b => 20)



nt = NamedTuple(vector_keys_values)
```

```
julia> nt
(a = 10, b = 20)
```

DESTRUCTURING TUPLES AND NAMED TUPLES

Previously, we demonstrated how to create tuples and named tuples from variables. Next, we show that the reverse operation is also possible, where **values are extracted from a tuple or named tuple and assigned to separate variables**. This process is known as **destructuring**, enabbling users to "unpack" the values of a collection into distinct variables.

Destructuring involves the assignment operator `=` with either a tuple or named tuple on the left-hand side. The choice between one or the other determines what objects can be used on the right-hand side. Tuples on the left-hand side are quite flexible, allowing values to be unpacked from a variety of collections. Named tuples on the left-hand side, instead, necessarily require a named tuple on the right-hand side. Next, we develop each case separately.

DESTRUCTURING COLLECTIONS THROUGH TUPLES

Given a collection `list` with two elements, destructuring via tuples allows us to unpack its values into the variables `x` and `y`. The syntax for this is `<tuple> = <collection>`, as in `x,y = list`. In the following, we illustrate the process by considering different objects as `list`.

```
list = [3,4]

x,y  = list
```

```
julia> x
3
julia> y
4
```

```
list = 3:4

x,y  = list
```

```
julia> x
3
julia> y
4
```

```
list = (3,4)

x,y  = list
```

```
julia> x
3
julia> y
4
```

```
list = (a = 3, b = 4)

x,y  = list
```

```
julia> x
3
julia> y
4
```

In addition to unpacking all elements, destructuring can also be applied to only a subset of elements. The assignment is then performed in sequential order, following the collection's inherent order.

Importantly, this method excludes the possibility of skipping any specific value. When a value must be disregarded, the conventional approach is to bind this value to the special variable name `_`. This symbol serves purely as a placeholder to indicate that the value is unimportant and has no impact on execution.

For illustration, we'll use a vector as an example of `list`. Nonetheless, the same principle applies to any collection.

```
list  = [3,4,5]

(x,)  = list
```

```
julia> x
3
```

```
list  = [3,4,5]

x,y   = list
```

```
julia> x
3
julia> y
4
```

```
list  = [3,4,5]

_,_,z = list                    # _ skips the assignment of that value
```

```
julia> z
5
```

```
list  = [3,4,5]

x,_,z = list                    # _ skips the assignment of that value
```

```
julia> x
3
julia> z
5
```

DESTRUCTURING WITH NAMED TUPLES ON BOTH SIDES

Destructuring can also be applied with named tuples on the left-hand side. In this case, values are extracted by directly referencing field names, rather than relying on their positional order. The main advantage of this approach is that variables can be assigned in any order, provided their names correspond to some field in the named tuple.

```
nt             = (; key1 = 10, key2 = 20, key3 = 30)

(; key2, key1) = nt            # keys in any order
```

```
julia> key1
10
julia> key3
30
```

```
nt            = (; key1 = 10, key2 = 20, key3 = 30)

(; key3)      = nt            # only one key
```

```
julia> key3
30
```

Remark

When destructuring with a tuple on the left-hand side and a named tuple on the right-hand side, keep in mind that the assignment is carried out strictly by position. This means that variable names on the left don't influence the assignment operation. In other words, the keys of the named tuple are completely ignored during the process.

```
nt = (; key1 = 10, key2 = 20, key3 = 30)

 key2, key1    = nt            # variables defined according to
POSITION
(key2, key1)   = nt            # alternative notation
```

```
julia> key2
10
julia> key1
20
```

```
nt = (; key1 = 10, key2 = 20, key3 = 30)

(; key2, key1) = nt            # variables defined according to
KEY
 ; key2, key1  = nt            # alternative notation
```

```
julia> key1
10
julia> key2
20
```

The same caveat applies to assignments of single variables.

```
nt       = (; key1 = 10, key2 = 20)

(key2,)   = nt                 # variable defined according to
POSITION
```

```
julia> key2
10
```

```
nt       = (; key1 = 10, key2 = 20)

(; key2) = nt            # variable defined according to KEY
```

```
julia> key2
20
```

APPLICATIONS OF DESTRUCTURING

Destructuring named tuples is particularly valuable in scientific modelling, where numerous parameters are referenced repeatedly. By grouping all these parameters into a single named tuple, they can be passed to a function as *one* consolidated argument. When functions are defined following this procedure, the named tuple is then destructed at the beginning of the function body to extract the needed parameters.

```
β = 3
δ = 4
ϵ = 5

# function 'foo' only uses 'β' and 'δ'
function foo(x, δ, β)
    x * δ + exp(β) / β
end
```

```
julia> foo(2, δ, β)
14.695
```

```
parameters_list = (; β = 3, δ = 4, ϵ = 5)



# function 'foo' only uses 'β' and 'δ'
function foo(x, parameters_list)
    x * parameters_list.δ + exp(parameters_list.β) / parameters_list.β
end
```

```
julia> foo(2, parameters_list.β, parameters_list.δ)
14.695
```

```
parameters_list = (; β = 3, δ = 4, ϵ = 5)



# function 'foo' only uses 'β' and 'δ'
function foo(x, parameters_list)
    (; β, δ) = parameters_list

    x * δ + exp(β) / β
end
```

```
julia> foo(2, parameters_list)
14.695
```

Destructuring also provides an elegant solution for retrieving multiple outputs from a function. This makes it possible to unpack the returned outputs into separate variables. In the example below, the function `foo` returns tuple, which is then unpacked into variables `x`, `y`, and `z`.

```
function foo()
    out1 = 2
    out2 = 3
    out3 = 4

    out1, out2, out3
end

x, y, z = foo()
```

```
function foo()
    out1 = 2
    out2 = 3
    out3 = 4

    [out1, out2, out3]
end

x, y, z = foo()
```

Another common use of destructuring arises when only a subset of a function's outputs is needed. While both tuples and named tuples can be applied for this purpose, tuples offer greater flexibility as they can be combined with various types of collections. In contrast, named tuples are restricted to returning another named tuple as the function's output, thus requiring prior knowledge of the field names.

The following example illusates this functionality by extracting only the first and third output of `foo`.

```
function foo()
    out1 = 2
    out2 = 3
    out3 = 4

    out1, out2, out3
end

x, _, z = foo()
```

```
function foo()
    out1 = 2
    out2 = 3
    out3 = 4

    [out1, out2, out3]
end

x, _, z = foo()
```

```
function foo()
    out1 = 2
    out2 = 3
    out3 = 4

    (; out1, out2, out3)
end

(; out1, out3) = foo()
```

FOOTNOTES

[1.] Not all collections map keys to values. For example, the type `Set`, which represents a group of unique unordered elements, doesn't have a key-value structure.

[2.] `Symbol` also enables the programmatic creation of variables. A typical use case arises in data analysis, where symbols are employed to generate new columns in tabular data structures.

[3.] The semicolon notation `;` may seem odd, but it actually comes from the syntax for keyword arguments in functions.

[4.] The package `OrderedCollections` addresses this, by offering a special dictionary called `OrderedDict`. It behaves similarly to regular dictionaries, including their syntax, but endows the dictionary with an order.

6c. Chaining Operations

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INTRODUCTION

This section introduces two approaches to computing outputs involving multiple intermediate steps. First, we introduce the so-called **let blocks**, which create a new scope that returns the last line as its output. Let blocks offer a concise way to wrap a sequence of operations, making them similar to functions but with less syntactic clutter. They also help maintain a tidy namespace, as all intermediate variables will be local and therefore inaccessible outside the block.

The second approach is based on **pipes**, which chain a series of operations and return the final output. As the built-in pipe can become unwieldy beyond single-argument functions, we also present an alternative based on the `Pipe` package.

LET BLOCKS

Let blocks are particularly convenient when performing a series of operations, but only the final result is of interest. To illustrate their utility, consider the task of computing the rounded logarithm of `a`'s absolute value, formally expressed as $\mathrm{round}\left(\ln\left(|a|\right)\right)$.

In Julia, this operation can be implemented using the expression `round(log(abs(a)))`, where `round(a)` returns the nearest integer to `a`. However, the expression is dense and hard to read because of the nested parentheses, with the issue potentially exacerbated if variables or functions had long names.

One straightforward way to improve clarity is to break the whole operation into multiple steps: *i)* compute the absolute value of `a`, *ii)* compute the logarithm of the result, and *iii)* round the resulting output. While this can be implemented through three intermediate variables that store the output in each step, this approach would clutter our namespace and potentially obscure the nested nature of the operations.

A more refined solution is to introduce a **let-block**. This construct resembles functions in several respects. It delineates a new scope by the `let` and `end` keywords, enabling multiple calculations to be performed locally. The result of the last calculation is then returned as the output of the let-block. Like functions, let-blocks also allow arguments to be passed by specifying them after the `let` keyword.

To highlight the benefits of let-blocks, the following examples compare various approaches to computing `round(log(abs(a)))`.

```
a      = -2

output = round(log(abs(a)))
```

```
julia> output
1.0
julia> temp1
julia> temp2
4
```

```
a      = -2

temp1  = abs(a)
temp2  = log(temp1)
output = round(temp2)
```

```
julia> output
1.0
```

```
a      = -2

output = let b = a          # 'b' is a local variable having the value of 'a'
   temp1 = abs(b)
   temp2 = log(temp1)
   round(temp2)
end
```

```
julia> output
1.0
julia> temp1 #local to let-block
ERROR: UndefVarError: `temp1` not defined
julia> temp2 #local to let-block
ERROR: UndefVarError: `temp2` not defined
```

```
a      = -2

output = let a = a          # the 'a' on the left of `=` defines a local variable
   temp1 = abs(a)
   temp2 = log(temp1)
   round(temp2)
end
```

```
julia> output
1.0
julia> temp1 #local to let-block
ERROR: UndefVarError: `temp1` not defined
julia> temp2 #local to let-block
ERROR: UndefVarError: `temp2` not defined
```

Let Blocks Can Mutate Variables

Let blocks behave like functions regarding assignments and mutation. This means that you can mutate their arguments, but can't reassign variables.

```
x = [2,2,2]

output = let x = x
   x[1] = 0
end
```

```
julia> x
3-element Vector{Int64}:
 0
 2
 2
```

```
x = [2,2,2]

output = let x = x
   x = 0
end
```

```
julia> x
3-element Vector{Int64}:
 2
 2
 2
```

Because mutations can occur within let-blocks, you should exercise caution to prevent unintended consequences in the global scope.

PIPES

For operations consisting of multiple intermediate step, pipes constitutes an alternative to let-blocks. Unlike let-blocks, they're specifically designed to chain operations together, with each step taking the output of the previous step as its input. These steps are separated with the `|>` keyword.

Pipes are particularly well-suited for sequential applications of single-argument functions. To illustrate, let's revisit the example presented above for let blocks.

```
a      = -2

output = round(log(abs(a)))
```

```
julia> output
1.0
```

```
a      = -2

output = a |> abs |> log |> round
```

```
julia> output
1.0
```

Let Blocks and Pipes For Long Names

Both approaches are useful to create temporary aliases for variables with lengthy names. This allows users to assign meaningful names to variables, while preserving code readability.

```
variable_with_a_long_name = 2

output           =           variable_with_a_long_name           -
log(variable_with_a_long_name) / abs(variable_with_a_long_name)
```

```
julia> output
1.6534264097200273
```

```
variable_with_a_long_name = 2

temp   = variable_with_a_long_name
output = temp - log(temp) / abs(temp)
```

```
julia> output
1.6534264097200273
```

```
variable_with_a_long_name = 2

output = variable_with_a_long_name  |>
         a -> a - log(a) / abs(a)
```

```
julia> output
1.6534264097200273
```

```
variable_with_a_long_name = 2

output = let x = variable_with_a_long_name
    x - log(x) / abs(x)
end
```

```
julia> output
1.6534264097200273
```

BROADCASTING PIPES

Just like any other operator, pipes can be broadcasted by prefixing them with a dot `.`. Thus, `.|>` indicates that the subsequent operation must be applied element-wise to the preceding output. For example, the expression `x .|> abs` is equivalent to `abs.(x)`.

To demonstrate this use, suppose we want to transform each element of `x` with the logarithm of its absolute values, and then sum the results.

```
x      = [-1,2,3]

output = sum(log.(abs.(x)))
```

```
julia> output
1.791759469228055
```

```
x      = [-1,2,3]

temp1  = abs.(x)
temp2  = log.(temp1)
output = sum(temp2)
```

```
julia> output
1.791759469228055
```

```
x      = [-1,2,3]

output = x .|> abs .|> log |> sum
```

```
julia> output
1.791759469228055
```

PIPES WITH MORE COMPLEX OPERATIONS

So far, our examples of pipes have followed a simple pattern, with each step consisting of a single-argument function. However, this form precludes the application of pipes to multiple-argument functions or even operations. For example, it prevents the inclusion of expressions like `foo(x,y)` or `2 * x`.

To address this limitation, we can **combine pipes with anonymous functions**. This enables users to specify how the output of the previous step is integrated into the subsequent operation. By doing so, the utility of pipes is significantly expanded, as demonstrated below.

```
a      = -2

output = round(2 * abs(a))
```

```
a      = -2

temp1  = abs(a)
temp2  = 2 * temp1
output = round(temp2)
```

```
a      = -2

output = a |> abs |> (x -> 2 * x) |> round

#equivalent and more readable
output = a              |>
         abs            |>
         x -> 2 * x     |>
         round
```

PACKAGE PIPE

Combining pipes and anonymous functions can result in cumbersome code, defeating the very own purpose of using pipes in the first place.

The `Pipe` package provides a convenient solution, eliminating the need for anonymous functions. By prefixing the operation chain with the `@pipe` macro, you can reference the output of the previous step by the symbol `_`. Additionally, for single-argument operations that don't require anonymous functions, `@pipe` maintains the same syntax as built-in pipes.

To illustrate its convenience, below we reimplement the last code snippet.

```
#
a      = -2

output = a |> abs |> (x -> 2 * x) |> round

#equivalent and more readable
output =       a            |>
               abs          |>
               x -> 2 * x   |>
               round
```

```
using Pipe
a = -2

output = @pipe a |> abs |> 2 * _ |> round

#equivalent and more readable
output = @pipe a            |>
               abs          |>
               2 * _        |>
               round
```

FUNCTION COMPOSITION (*OPTIONAL*)

An alternative approach to nesting functions is through the composition operator `∘`. This symbol can be inserted by tab completion through `\circ`, and its functionality is the same as in Mathematics. Specifically, for some functions `f` and `g`, `(f ∘ g)(x)` is equivalent to `f(g(x))`.

The operator `∘` can be considered as an alternative to piping, as it provides the same output as `x |> f |> g`. Moreover, `∘` is also available as a function, where `∘(f,g)(x)` is equivalent to `(f ∘ g)(x)`. The following examples demonstrate its use.

```
a        = -1



# all `output` are equivalent
output   = log(abs(a))
output   = a |> abs |> log
output   = (log ∘ abs)(a)
output   = ∘(log, abs)(a)
```

```
julia> output
0.0
```

```
a        = 2
outer(a) = a + 2
inner(a) = a / 2

# all `output` are equivalent
output   = (a / 2) + 2
output   = outer(inner(a))
output   = a |> inner |> outer
output   = (outer ∘ inner)(a)
output   = ∘(outer, inner)(a)
```

```
julia> output
3.0
```

Importantly, the resulting function from function composition can be broadcasted. To understand this notation more clearly, you should think of compositions as defining a new function: $h := f \circ g$. This entails that $h(x) := (f \circ g)(x)$, and therefore $h(x) := (f \circ g)(x) = f[g(x)]$. Given this, broadcasting `h` requires `h.(x)`, which is equivalent to `(f ∘ g).(x)` or `∘(f,g).(x)`.

```julia
x        = [1, 2, 3]



# all `output` are equivalent
output   = log.(abs.(x))
output   = x .|> abs .|> log
output   = (log ∘ abs).(x)
output   = ∘(log, abs).(x)
```

```
julia> output
3-element Vector{Float64}:
 0.0
 0.6931471805599453
 1.0986122886681098
```

```julia
x        = [1, 2, 3]
outer(a) = a + 2
inner(a) = a / 2

# all `output` are equivalent
output   = (x ./ 2) .+ 2
output   = outer.(inner.(x))
output   = x .|> inner .|> outer
output   = (outer ∘ inner).(x)
output   = ∘(outer, inner).(x)
```

```
julia> output
3-element Vector{Float64}:
 2.5
 3.0
 3.5
```

We can also broadcast the composition operator `∘` itself, enabling the simultaneous application of multiple functions to the same object. For instance, the following implementation ensures that each function takes the absolute value of its argument.

```julia
a            = -1

inners       = abs
outers       = [log, sqrt]
compositions = outers .∘ inners

# all `output` are equivalent
output       = [log(abs(a)), sqrt(abs(a))]
output       = [foo(a) for foo in compositions]
```

```
julia> compositions
2-element Vector{ComposedFunction{O, typeof(abs)} where O}:
 log ∘ abs
 sqrt ∘ abs
julia> output
2-element Vector{Float64}:
 0.0
 1.0
```

6d. Useful Functions for Vectors

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INTRODUCTION

This section provides an overview of essential functions for manipulating vectors. We cover in particular common operations like sorting, finding unique elements, counting occurrences, and ranking data. Our ultimate goal is to illustrate the utility of these functions in a practical context, as we'll do in the next section.

SORTING VECTORS

The `sort` function allows the user to arrange elements in a specific order. Tt sorts elements in ascending order by default, with the possibility of a descending order by setting the keyword argument `rev = true`.

The function comes in two variants: `sort`, which returns a new sorted copy, and `sort!`, the in-place version that directly updates the vector.

SORT (ASCENDING)

```
x = [4, 5, 3, 2]

y = sort(x)
```

```
julia> y
4-element Vector{Int64}:
 2
 3
 4
 5
```

SORT (DESCENDING)

```
x = [4, 5, 3, 2]

y = sort(x, rev=true)
```

```
julia> y
4-element Vector{Int64}:
 5
 4
 3
 2
```

SORT!

```
x = [4, 5, 3, 2]

sort!(x)
```

```
julia> x
4-element Vector{Int64}:
 2
 3
 4
 5
```

Both `sort(x)` and `sort!(x)` have the option of defining the sorting order based on transformations of `x`. Specifically, given a function `foo`, the sorted order can be determined by the values of `foo(x)`. We demonstrate this below through the function `sort`, whose implementation requires the keyword argument `by`.

SORT - ABSOLUTE

```
x      = [4, -5, 3]


y      = sort(x, by = abs)      # 'abs' computes the absolute value
```

```
julia> abs.(x)
3-element Vector{Int64}:
 4
 5
 3
julia> y
3-element Vector{Int64}:
  3
  4
 -5
```

SORT - QUADRATIC

```
x      = [4, -5, 3]

foo(a) = a^2
y      = sort(x, by = foo)      # same as sort(x, by = x -> x^2)
```

```
julia> foo.(x)
3-element Vector{Int64}:
 16
 25
  9
julia> y
3-element Vector{Int64}:
  3
  4
 -5
```

SORT - NEGATIVE

```
x      = [4, -5, 3]

foo(a) = -a
y      = sort(x, by = foo)      # same as sort(x, by = x -> -x)
```

```
julia> foo.(x)
3-element Vector{Int64}:
 -4
  5
 -3
julia> y
3-element Vector{Int64}:
  4
  3
 -5
```

RETRIEVING INDICES OF SORTED ELEMENTS

While `sort` returns the ordered *values* of the vectors, you may also be interested in the *indices* of the sorted elements. This functionality is provided by the function `sortperm`, which returns the indices of `x` that would result in `sort(x)`. In other words, `x[sortperm(x)] == sort(x)` returns `true`. [1]

EXAMPLE 1

```
x          = [1, 2, 3, 4]

sort_index = sortperm(x)
```

```
julia> sort_index
4-element Vector{Int64}:
 1
 2
 3
 4
```

EXAMPLE 2

```
x          = [3, 4, 5, 6]

sort_index = sortperm(x)
```

```
julia> sort_index
4-element Vector{Int64}:
 1
 2
 3
 4
```

EXAMPLE 3

```
x          = [1, 3, 4, 2]

sort_index = sortperm(x)
```

```
julia> sort_index
4-element Vector{Int64}:
 1
 4
 2
 3
```

Note that the elements in the first two examples are already in ascending order. As a result, `sortperm` returns the trivial permutation `[1, 2, 3, 4]`. In contrast, the last example features an unordered vector `x = [1, 3, 4, 2]`. Thus, the resulting vector `[1, 4, 2, 3]` indicates that the smallest element is at index 1, the second smallest is at index 4, the third smallest is at index 2, and the largest at index 3.

Like `sort`, `sortperm` also supports retrieving indices in descending order. This requires including the keyword argument `rev = true`.

EXAMPLE 1

```
x          = [9, 3, 2, 1]

sort_index = sortperm(x, rev=true)
```

```
julia> sort_index
4-element Vector{Int64}:
 1
 2
 3
 4
```

EXAMPLE 2

```
x          = [9, 5, 3, 1]

sort_index = sortperm(x, rev=true)
```

```
julia> sort_index
4-element Vector{Int64}:
 1
 2
 3
 4
```

EXAMPLE 3

```
x          = [9, 3, 5, 1]

sort_index = sortperm(x, rev=true)
```

```
julia> sort_index
4-element Vector{Int64}:
 1
 3
 2
 4
```

Finally, `sortperm` also accepts the keyword argument `by` to define a custom transformation.

SORT - ABSOLUTE

```
x      = [4, -5, 3]


value  = sort(x, by = abs)      # 'abs' computes the absolute value
index  = sortperm(x, by = abs)
```

```
julia> abs.(x)
3-element Vector{Int64}:
 4
 5
 3
julia> value
3-element Vector{Int64}:
  3
  4
 -5
julia> index
3-element Vector{Int64}:
 3
 1
 2
```

SORT - QUADRATIC

```
x      = [4, -5, 3]

foo(a) = a^2
value  = sort(x, by = foo)      # same as sort(x, by = x -> x^2)
index  = sortperm(x, by = foo)
```

```
julia> foo.(x)
3-element Vector{Int64}:
 16
 25
  9
julia> value
3-element Vector{Int64}:
  3
  4
 -5
julia> index
3-element Vector{Int64}:
 3
 1
 2
```

SORT - NEGATIVE

```
x      = [4, -5, 3]

foo(a) = -a
value  = sort(x, by = foo)      # same as sort(x, by = x -> -x)
index  = sortperm(x, by = foo)
```

```
julia> foo.(x)
3-element Vector{Int64}:
 -4
  5
 -3
julia> value
3-element Vector{Int64}:
  4
  3
 -5
julia> index
3-element Vector{Int64}:
 1
 3
 2
```

AN EXAMPLE

One common application of `sortperm` is to reorder one variable based on the values of another. For example, suppose we want to assess the daily failures of a machine. Focusing on the first three days of the month, the following code snippet ranks these days by their corresponding failure counts.

DAYS SORTED BY LOWEST NUMBER OF FAILURES

```
days     = ["one", "two", "three"]
failures = [8, 2, 4]

index            = sortperm(failures)
days_by_failures = days[index]                    # days sorted by lowest failures
```

```
julia> index
3-element Vector{Int64}:
 2
 3
 1
julia> days_by_earnings
3-element Vector{String}:
 "two"
 "three"
 "one"
```

REMOVING DUPLICATES

The function `unique` removes duplicates from a vector, returning a vector that contains each element only once. The function comes in two variants `unique` provides a new copy, and `unique!`, the in-place version that directly updates the original vector.

UNIQUE

```
x = [2, 2, 3, 4]

y = unique(x)       # returns a new vector
```

```
julia> x
4-element Vector{Int64}:
 2
 2
 3
 4
julia> y
3-element Vector{Int64}:
 2
 3
 4
```

UNIQUE!

```
x = [2, 2, 3, 4]

unique!(x)          # mutates 'x'
```

```
julia> x
3-element Vector{Int64}:
 2
 3
 4
```

The `StatsBase` package provides a related function called `countmap`, which counts the occurrences of each element in a vector. It returns a dictionary where the unique elements act as keys, and their corresponding values represent the number of times each element appears.

Note that the keys in the resulting dictionary are unsorted by default. If you prefer sorted keys, you must apply the `sort` function to the result. This will automatically transform an ordinary dictionary into an object with type `OrderedDict`.

UNSORTED COUNT

```
using StatsBase
x           = [6, 6, 0, 5]

y           = countmap(x)              # Dict with `element => occurrences`

elements    = collect(keys(y))
occurrences = collect(values(y))
```

```
julia> y
Dict{Int64, Int64} with 3 entries:
  0 => 1
  5 => 1
  6 => 2
julia> elements
3-element Vector{Int64}:
 0
 5
 6
julia> occurrences
3-element Vector{Int64}:
 1
 1
 2
```

SORTED COUNT

```julia
using StatsBase
x           = [6, 6, 0, 5]

y           = sort(countmap(x))        # OrderedDict with `element => occurrences`

elements    = collect(keys(y))
occurrences = collect(values(y))
```

```
julia> y
OrderedCollections.OrderedDict{Int64, Int64} with 3 entries:
  0 => 1
  5 => 1
  6 => 2
julia> elements
3-element Vector{Int64}:
 0
 5
 6
julia> occurrences
3-element Vector{Int64}:
 1
 1
 2
```

ROUNDING NUMBERS

Julia provides standard functions to approximate numerical values to a specific precision:

- `round` approximates the number to its nearest integer.
- `floor` approximates the number down to its nearest integer.
- `ceil` approximates the number up to its nearest integer.

Below, we show that these functions are quite flexible, allowing users to specify the output's type (e.g., `Int64` or `Float64`), the number of decimals places via the keyword argument `digits`, and the significant digits.

ROUND

```julia
x = 456.175

round(x)                        # 456.0

round(x, digits=1)              # 456.2
round(x, digits=2)              # 456.18

round(Int, x)                   # 456

round(x, sigdigits=1)           # 500.0
round(x, sigdigits=2)           # 460.0
```

FLOOR

```
x = 456.175

floor(x)                        # 456.0

floor(x, digits=1)              # 456.1
floor(x, digits=2)              # 456.17

floor(Int, x)                   # 456

floor(x, sigdigits=1)           # 400.0
floor(x, sigdigits=2)           # 450.0
```

CEIL

```
x = 456.175

ceil(x)                         # 457.0

ceil(x, digits=1)               # 456.2
ceil(x, digits=2)               # 456.18

ceil(Int, x)                    # 457

ceil(x, sigdigits=1)            # 500.0
ceil(x, sigdigits=2)            # 460.0
```

RANKINGS

Instead of sorting a vector, you may be interested in determining the rank position of each element. The `StatsBase` package offers two functions for this purpose: `competerank` and `ordinalrank`. The main difference between them lies in how they handle tied elements: `competerank` assigns the same rank to tied elements, while `ordinalrank` assigns consecutive ranks. Both functions return ranks where 1 corresponds to the lowest value. The keyword argument `rev = true` allows you to invert the ranking, so that the highest value corresponds to a rank of 1.

RANK (SAME RANK FOR TIES)

```
using StatsBase
x = [6, 6, 0, 5]

y = competerank(x)
```

```
julia> y
4-element Vector{Int64}:
 3
 3
 1
 2
```

DESCENDING RANK (SAME RANK FOR TIES)

```julia
using StatsBase
x = [6, 6, 0, 5]

y = competerank(x, rev=true)
```

```
julia> y
4-element Vector{Int64}:
 1
 1
 4
 3
```

RANK (UNIQUE POSITIONS)

```julia
using StatsBase
x = [6, 6, 0, 5]

y = ordinalrank(x)
```

```
julia> y
4-element Vector{Int64}:
 3
 4
 1
 2
```

DESCENDING RANK (UNIQUE POSITIONS)

```julia
using StatsBase
x = [6, 6, 0, 5]

y = ordinalrank(x, rev=true)
```

```
julia> y
4-element Vector{Int64}:
 1
 2
 4
 3
```

Do not confuse `ordinalrank` and `sortperm`

The function `ordinalrank` indicates the position of each value in the *sorted* vector, while `sortperm` indicates the position of each value in the *unsorted* vector.

'ORDINALRANK'

```
using StatsBase
x = [3, 1, 2]

y = ordinalrank(x)
```

```
julia> y
3-element Vector{Int64}:
 3
 1
 2
```

'SORTPERM'

```
using StatsBase
x = [3, 1, 2]

y = sortperm(x)
```

```
julia> y
3-element Vector{Int64}:
 2
 3
 1
```

EXTREMA (MAXIMUM AND MINIMUM)

We conclude by presenting a method for finding the indices and values of extrema in collections. The following examples are based on the maximum, with similar functions available for the minimum.

VALUE

```
x = [6, 6, 0, 5]

y = maximum(x)
```

```
julia> y
6
```

INDEX

```
x = [6, 6, 0, 5]

y = argmax(x)
```

```
julia> y
1
```

VALUE AND INDEX

```
x = [6, 6, 0, 5]

y = findmax(x)
```

```
julia> y
(6, 1)
```

Julia additionally provides the function `max` and `min`, which respectively return the maximum and minimum of their *arguments*. These functions will become particularly useful for procedures based on binary operations (e.g., the so-called reductions).

'MAX' FUNCTION

```
x = 3
y = 4

z = max(x,y)
```

```
julia> z
4
```

FOOTNOTES

[1.] The name `sortperm` originates from "sorting permutation". Although the name might seem somewhat opaque, it arises because the operation returns the permutation of indices that would sort the original vector.

6e. Illustration - Johnny, the YouTuber

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INTRODUCTION

Through a practical example, this section will demonstrate the convenience of the following features:

1. Boolean indexing for working with subsets of the data
2. organizing code around functions
3. pipes to enhance code readability
4. use of views to modify subsets of the data

DESCRIBING THE SCENARIO

We'll explore the stats of Johnny's YouTube channel during a month. He has a median of 50,000 visits per video, with a few viral videos exceeding 100,000 visits. The information at our disposal is:

- `nr_videos`: 30 (one per day).
- `visits`: viewers per video (in thousands).
- `payrates`: Dollars paid per video for 1,000 visits. They range from $2 to $6. The fluctuation is consistent with YouTube's payment model, which depends on a video's feature (e.g., content, duration, retention).

The scenario is modeled by some mock data. The details of how data are generated are unimportant, but were added below for the sake of completeness. Ultimately, what matters is that the mock data creates the variables `visits` and `payrates`.

```julia
using StatsBase, Distributions
using Random; Random.seed!(1234)

function audience(nr_videos; median_target)
    shape = log(4,5)
    scale = median_target / 2^(1/shape)

    visits = rand(Pareto(shape,scale),  nr_videos)

    return visits
end

nr_videos = 30

visits   = audience(nr_videos, median_target = 50)      # in thousands of visits
payrates = rand(2:6, nr_videos)                         # per thousands of visits
```

```
julia> visits # in thousands
30-element Vector{Float64}:
 38.8086
 70.8113
   ⋮
 72.3673
 30.2565
julia> payrates # per thousand visits
30-element Vector{Int64}:
 5
 3
 ⋮
 2
 4
```

The variables `visits` and `payrates` enable us to calculate the total payment per video.

```julia
earnings = visits .* payrates
```

```
julia> earnings
30-element Vector{Float64}:
 194.043
 212.434
    ⋮
 144.735
 121.026
```

SOME GENERAL INFORMATION

We begin by examining the per-view payments made by YouTube. We first show that Johnny's payments range from $2 to $6. Moreover, using the `countmaps` function from the `StatsBase` package, we conclude that Johnny has eight videos reaching the maximum payment of $6.

```
range_payrates  = unique(payrates) |> sort
```

```
julia> range_payrates
5-element Vector{Int64}:
 2
 3
 4
 5
 6
```

```
using StatsBase
occurrences_payrates = countmap(payrates) |> sort
```

```
julia> occurrences_payrates
OrderedDict{Int64, Int64} with 5 entries:
  2 => 5
  3 => 6
  4 => 8
  5 => 5
  6 => 6
```

We can also provide some insights into Johnny's most profitable videos. By applying the `sort` function, we can isolate his top 3 highest-earning videos. Moreover, we can apply the `sortperm` function to identify their indices, allowing us to extract the payment per view and total visits associated with each video.

```
top_earnings    = sort(earnings, rev=true)[1:3]
```

```
julia> top_earnings
3-element Vector{Float64}:
 7757.81
  693.813
  672.802
```

```
indices         = sortperm(earnings, rev=true)[1:3]

sorted_payrates = payrates[indices]
```

```
julia> sorted_payrates
3-element Vector{Int64}:
 6
 6
 6
```

```
indices         = sortperm(earnings, rev=true)[1:3]

sorted_visits   = visits[indices]
```

```
julia> sorted_visits
3-element Vector{Float64}:
 1292.97
  115.636
  112.134
```

BOOLEAN VARIABLES

In the following, we demonstrate how to use Boolean indexing to extract and characterize subsets of data. Our focus will be on characterizing Johnny's viral videos, defined as those that have surpassed a threshold of 100k visits. In particular, we'll determine the number of visits and revenue generated by them.

To identify the viral videos, we'll create a `Bool` vector, where `true` identifies a viral video. This vector can then be employed as a logical index, allowing us to selectively extract data points from other variables. In the example below, we apply it to compute the total visits and earnings attributable to the viral videos.

```
# characterization of viral videos
viral_threshold = 100
is_viral        = (visits .≥ viral_threshold)

# stats
viral_nrvideos  = sum(is_viral)
viral_visits    = sum(visits[is_viral])
viral_revenue   = sum(earnings[is_viral])
```

```
julia> viral_nrvideos
4
julia> viral_visits
1625.05
julia> viral_revenue
9750.3
```

Boolean indexing also enables subsetting data satisfying multiple conditions. For instance, we can apply this technique to calculate the proportion of viral videos for which YouTube paid more than $3 per thousand visits.

```
# characterization
viral_threshold    = 100
payrates_above_avg = 3

is_viral           = (visits .≥ viral_threshold)
is_viral_lucrative = (visits .≥ viral_threshold) .&& (payrates .> payrates_above_avg)

# stat
proportion_viral_lucrative = sum(is_viral_lucrative) / sum(is_viral) * 100
```

```
julia> proportion_viral_lucrative
100.0
```

Rounding Outputs

You can express results with rounded numbers via the function `round`. By default, this returns the nearest integer expressed as a `Float64` number.

The function also offers additional specifications. For instance, the number of decimal places in the approximation can be controlled via the `digits` keyword argument. Furthermore, it's possible to represent the number as an integer using either `Int` or `Int64` as an argument. [1]

```
rounded_proportion = round(proportion_viral_lucrative)
```

```
julia> rounded_proportion
100.0
```

```
rounded_proportion = round(proportion_viral_lucrative, digits=1)
```

```
julia> rounded_proportion
100.0
```

```
rounded_proportion = round(Int64, proportion_viral_lucrative)
```

```
julia> rounded_proportion
100
```

FUNCTIONS TO REPRESENT TASKS

The approach employed so far allows for a quick exploration of Johnny's viral videos. However, it lacks the structure needed for a systematic analysis across different subsets of the data. To address this limitation, we can capture the characterization of videos through a function.

Recall that a well-designed function should embody a single clearly defined task. In our case, the goal is to subset data and extract key statistics, including the number of videos, visits, and revenue generated. Furthermore, the function should remain independent of any specific application, so it can be reused to analyze different groups of videos without rewriting code each time.

The function below implements this task taking three arguments: the raw data (`visits` and `payrates`) and a condition that defines the subset of data (`condition`). By keeping the condition generic, the function is flexible enough to target any subset of videos. The example also showcases the convenience of pipes to compute intermediate temporary steps.

```
#
function stats_subset(visits, payrates, condition)
    nrvideos = sum(condition)
    audience = sum(visits[condition])

    earnings = visits .* payrates
    revenue  = sum(earnings[condition])

    return (; nrvideos, audience, revenue)
end
```

```
using Pipe
function stats_subset(visits, payrates, condition)
    nrvideos = sum(condition)
    audience = sum(visits[condition])


    revenue  = @pipe (visits .* payrates) |> x -> sum(x[condition])

    return (; nrvideos, audience, revenue)
end
```

```
using Pipe
function stats_subset(visits, payrates, condition)
    nrvideos = sum(condition)
    audience = sum(visits[condition])


    revenue  = @pipe (visits .* payrates) |> sum(_[condition])

    return (; nrvideos, audience, revenue)
end
```

Below, we demonstrate the reusability of the function by characterizing various subsets of data.

```
viral_threshold  = 100
is_viral         = (visits .≥ viral_threshold)
viral            = stats_subset(visits, payrates, is_viral)
```

```
julia> viral
(nrvideos = 4, audience = 1625.05, revenue = 9750.3)
```

```
viral_threshold  = 100
is_notviral      = .!(is_viral)      # '!' is negating a boolean value and we broadcast it
notviral         = stats_subset(visits, payrates, is_notviral)
```

```
julia> notviral
(nrvideos = 26, audience = 1497.02, revenue = 5687.67)
```

```
days_to_consider = (1, 10, 25)      # subset of days to be characterized
is_day           = in.(eachindex(visits), Ref(days_to_consider))
specific_days    = stats_subset(visits, payrates, is_day)
```

```
julia> specific_days
(nrvideos = 3, audience = 182.939, revenue = 1030.33)
```

VARIABLE MUTATION

Suppose Johnny is exploring ways to increase viewership through targeted advertising. His projections suggest that ads will boost viewership per video by 20%. However, due to budget constraints, Johnny must choose between promoting either his non-viral or viral ones. To make an informed decision, Johnny decides to leverage the data at his disposal to crunch some rough estimates. In particular, he'll base his decision on the earnings he would've earned if he had run targeted ads.

The first step in this process involves creating a modified copy of `visits`. This should now reflect the anticipated increase in viewership from running ads on the targeted videos (either viral or non-viral). With this updated audience data, Johnny can then apply the previously defined `stats_subset` function to estimate the potential earnings. By comparing the estimations for each group of targeted video, Johnny can determine which strategy offers the higher return on investment.

```
# 'temp' modifies 'new_visits'
new_visits      = copy(visits)
temp            = @view new_visits[new_visits .≥ viral_threshold]
temp           .= 1.2 .* temp

allvideos       = trues(length(new_visits))
targetViral     = stats_subset(new_visits, payrates, allvideos)
```

```
julia> targetViral
(nrvideos = 30, audience = 3447.08, revenue = 17388.0)
```

```
# 'temp' modifies 'new_visits'
new_visits      = copy(visits)
temp            = @view new_visits[new_visits .< viral_threshold]
temp           .= 1.2 .* temp

allvideos       = trues(length(new_visits))
targetNonViral  = stats_subset(new_visits, payrates, allvideos)
```

```
julia> targetNonViral
(nrvideos = 30, audience = 3421.47, revenue = 16575.5)
```

Given the results in each tab, promoting viral videos appears to be the more profitable option.

Be Careful with Misusing 'view'

Updating `temp` requires an in-place operation to mutate the parent object. In our case, this was achieved via the broadcasted operator `.=`. Below, we present some implementations that fail to produce the intended result.

```
new_visits = copy(visits)


temp  = @view new_visits[new_visits .≥ viral_threshold]
temp .= temp .* 1.2
```

```
new_visits = visits      # it creates an alias, it's a view of
the original object!!!

# 'temp' modifies 'visits' -> you lose the original info
temp  = @view new_visits[new_visits .≥ viral_threshold]
temp .= temp .* 1.2
```

```
new_visits = copy(visits)

# wrong -> not using `temp .= temp .* 1.2`
temp  = @view new_visits[new_visits .≥ viral_threshold]
temp  = temp .* 1.2      # it creates a new variable 'temp', it
does not modify 'new_visits'
```

Use of "Let Blocks" To Avoid Bugs

In the code above, "Target Viral" and "Target Non-Viral" reference variables with identical names. This creates the risk of accidentally referring to a variable from the wrong scenario.

A practical way to mitigate this risk is by employing "let blocks". Since each let block introduces its own scope, this helps maintain a clean namespace and prevents variable collisions.

```
targetViral      = let visits = visits, payrates = payrates,
threshold = viral_threshold
    new_visits = copy(visits)
    temp       = @view new_visits[new_visits .≥ threshold]
    temp      .= 1.2 .* temp

    allvideos  = trues(length(new_visits))
    stats_subset(new_visits, payrates, allvideos)
end
```

```
julia> targetViral
(nrvideos = 30, audience = 3447.08, revenue = 17388.0)
```

```
targetNonViral  = let visits = visits, payrates = payrates,
threshold = viral_threshold
    new_visits = copy(visits)
    temp       = @view new_visits[new_visits .< threshold]
    temp      .= 1.2 .* temp

    allvideos  = trues(length(new_visits))
    stats_subset(new_visits, payrates, allvideos)
end
```

```
julia> targetNonViral
(nrvideos = 30, audience = 3421.47, revenue = 16575.5)
```

BROADCASTING OVER A LIST OF FUNCTIONS

A function like `stats_subset` is useful for computing a fixed set of summary statistics. However, since the choice of statistics is hard-coded into the function's definition, the output can't be changed without rewriting the code. This rigidity makes the function less reusable across different analytical contexts.

A more flexible approach consists of specifying which statistics to compute at the time of use. Julia makes this possible because functions are *first-class objects*, entailing that functions behave just like any other variable. This feature lets us define a list of statistical functions, eventually applying them element-wise to the variables we want to characterize.

Below, we apply this methodology to characterize the variable `visits`.

```
list_functions = [sum, median, mean, maximum, minimum]

stats_visits   = [fun(visits) for fun in list_functions]
```

```
julia> stats_visits
5-element Vector{Float64}:
 3447.08
   64.8765
  114.903
 1551.56
   28.2954
```

The same methodology can also be employed for characterizing multiple variables at once. In fact, broadcasting makes this straightforward to implement. For instance, below we simultaneously characterize `visits` and `earnings`.

```
list_functions = [sum, median, mean, maximum, minimum]

stats_various  = [fun.([visits, payrates]) for fun in list_functions]
```

```
julia> stats_various
5-element Vector{Vector{Float64}}:
 [3447.08, 121.0]
 [64.8765, 4.0]
 [114.903, 4.03333]
 [1551.56, 6.0]
 [28.2954, 2.0]
```

One major limitation of the current method is its inability to reflect each statistic's name. To address this, we can collect all statistics in a named tuple, enabling the access of each through its name. For instance, given a named tuple `stats_visits`, it'll become possible to retrieve the average value of `visits` by `stats_visits.mean` or `stats_visits[:mean]`.

To assign names to the statistics within the named tuple, we'll use the `Symbol` type. This translates strings into identifiers that can act as keys of a named tuple, enabling programmatic access to each statistic.

```
vector_of_tuples = [(Symbol(fun), fun(visits)) for fun in list_functions]
stats_visits     = NamedTuple(vector_of_tuples)
```

```
julia> stats_visits
(sum = 3447.08, median = 64.8765, mean = 114.903, maximum = 1551.56, minimum = 28.2954)
julia> stats_visits.mean
114.903
julia> stats_visits[:median]
64.8765
```

FOOTNOTES

[1.] Recall that the type `Int` defaults to `Int64` on 64-bit systems and to `Int32` for 32-bit systems. Most modern computers fall into the former category, explaining why we usually employ `Int64`.

7a. Overview and Goals

Martin Alfaro
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The first part of the website has laid the groundwork for working with Julia. This demanded introducing fundamental data types, such as scalars, vectors, and tuples. Alongside these, we've covered essential programming constructs, including functions, conditionals, and for-loops. While these concepts may vary in syntax and usage across different programming languages, their underlying principles remain universal.

In the second part of the website, we'll shift our attention to one of Julia's most distinctive strengths: **high-performance computing**. When paired with its intuitive syntax and interactive nature, this feature makes Julia an ideal choice for scientific applications.

The domain of high-performance computing is vast and complex. Moreover, each subject has idiosyncratic features that make certain optimizations more or less relevant. Given this breadth, I've made deliberate choices about what to include and exclude. The challenge lay in striking the right balance between providing sufficient background knowledge for explaining a technique, while avoiding unnecessary specificity.

Considering this inherent trade-off, I've chosen the subjects with the goal of equipping readers with practical knowledge for optimizing code, without overwhelming them with excessive detail. In particular, the primary focus will be on what I consider to be the essentials for performance in Julia: **type stability** and **reductions in memory allocations**. The former in particular constitutes a prerequisite for achieving high performance in Julia, making it necessary for any further optimization.

The discussion of high performance in Julia will lead us to consider its type system. Nonetheless, some valuable concepts related to it have been left out. In particular, the concept of `struct`, which allows users to create their own custom objects, won't be covered. There are two reasons for this omission. First, while important for project development, the subject can be bypassed when analyzing high performance, without compromising its understanding. Second, the section included on types is already long enough—adding more subjects could divert the reader's attention away from the primary focus, which is learning high-performance techniques.

7b. When To Optimize Code?

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INTRODUCTION

Julia has been praised as solving the "two-language problem". This refers to the difficulty of finding a language that's fast, but still easy to read and write. Although it's true that Julia has some advantages relative to other languages, claims like this can be quite misleading for someone new to programming—it wrongly suggests that Julia is the only language you'll need to learn, regardless of your specific coding domain.

In reality, each programming language is designed with certain purposes in mind. Consequently, it's quite likely that you'll need to learn multiple programming languages, even if your focus is narrow. This is particularly true in data analysis, where a package implementing a specific task may only be available in one language. I, for one, tend to use Julia as my main language for data analysis, but complement it with libraries from R and Python when the task requires it. [1]

Getting the best performance in any language is also not immediate. It requires you to write code appropriately, with implementations that tend to be software-specific and involve several trade-offs. [2] Overall, the claim that "Julia is fast" should be replaced by "Julia *can* be fast." Considering this, the upcoming chapters aim to equip you with the essential tools to unlock Julia's performance capabilities.

WHEN SHOULD WE CARE ABOUT SPEED?

Achieving high performance often comes with trade-offs, and thus should never be the sole consideration when writing code. Optimizing performance frequently means rewriting parts of your script, which can reduce readability and make the code harder to maintain in the long run. Additionally, implementing these improvements requires significant time and effort, including tasks such as testing, identifying bottlenecks, and integrating third-party packages.

Considering this, you should assess your goals before embarking on any optimization efforts. Keep in mind that **most of YOUR time will be spent on writing, reading, and debugging code**—reducing the computer's execution time by a millisecond may not be worth the trade-off if it demands investing hours. Moreover, even if speed is crucial for your project, you should prioritize which parts of the code to optimize. Typically, only a few operations impact runtime critically, with the rest having a negligible effect.

With these caveats in mind, the suggestions we'll present in the upcoming chapters serve a dual purpose. Firstly, they represent essential rules for speed—not adhering to them would severely undermine performance, thereby negating any advantages of using Julia. Secondly, several tips we'll

consider have a minimal impact on code's readability, if any. In summary, the procedures to be presented will help you unlock Julia's speed, without sacrificing code readability or entailing excessive additional work.

FOOTNOTES

[1.] Julia has the capacity of calling programs from other software such as R or Python. Python and R also have this feature.

[2.] This explains the disparate results often seen in online benchmarks, where code can be written inefficiently in one language and highly optimized in another. Moreover, since languages tend to excel at certain tasks, it's possible to cherry-pick examples that make a particular language appear faster.

7c. Benchmarking Execution Time

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INTRODUCTION

This section introduces standard tools for benchmarking code performance. Our website reports results based on the `BenchmarkTools` package, which is currently the most mature and reliable option in the Julia ecosystem. That said, the newer `Chairmarks` package has demonstrated notable improvements in execution speed compared with `BenchmarkTools`. I recommend adopting `Chairmarks` once it's achieved sufficient stability and adoption within the community.

To set the stage, we'll start by addressing some key points for interpreting benchmark results. We'll also look at Julia's built-in `@time` macro, whose limitations explain why `BenchmarkTools` and `Chairmarks` should be used instead.

TIME METRICS

Julia uses the same time metrics described below, regardless of whether you use `BenchmarkTools` or `Chairmarks`. For quick reference, these metrics can be accessed at any point **in the left bar** under "**Notation & Hotkeys**".

Unit	Acronym	Measure in Seconds
Seconds	`s`	1
Milliseconds	`ms`	10^{-3}
Microseconds	`µs`	10^{-6}
Nanoseconds	`ns`	10^{-9}

Alongside execution times, each package also reports the amount of **memory allocated on the heap**, typically referred to simply as **allocations**. These allocations can play a major role in overall performance, and usually indicate suboptimal coding practices. As we'll explore in later sections, monitoring allocations tends to be crucial for achieving high performance.

"TIME TO FIRST PLOT"

The expression "Time to First Plot" refers to a side effect of how Julia operates, where the first execution in any new session takes longer than subsequent ones. This latency isn't a bug. Rather, it's a direct consequence of the language's design, which relies on a just-in-time (JIT) compiler: Julia compiles the code for executing functions in their first run, translating them into highly optimized machine code on the fly. This compilation process will be thoroughly covered in upcoming sections.

The first time you run any function, Julia generates low-level machine instructions to carry out the function's operations. This process of translating human-readable code into machine-executable instructions is called **compilation**. Unlike other programming languages, Julia relies on a just-in-time (JIT) compiler, where this code is compiled on-the-fly when a function is first run. This compilation process will be thoroughly covered in upcoming sections.

In each new session, this compilation penalty is incurred only once per function and set of argument types. Once a function is compiled, its machine code is cached, making all subsequent calls faster. The consequence is that the resulting overhead isn't a major hindrance for large projects, where startup costs are quickly amortized. However, it does mean that Julia may not be the best option for quick one-off analyses, such as running a simple regression or producing a quick exploratory plot.

The latency caused by this feature varies significantly across functions, making it difficult to generalize its impact. While it may be imperceptible for simple functions like `sum(x)`, it can be noticeable for rendering a high-quality plot. Indeed, drawing a first plot during a session can take several seconds, explaining the origin of the term "Time to First Plot".

> **Warning!**
> The Time-to-First-Plot issue has been significantly mitigated since `Julia 1.9`, thanks to improvements in precompilation. Each subsequent release is reducing this overhead even further.

@TIME

Julia comes with a built-in macro called `@time`, allowing you to get a quick sense of an operation's execution time. The results provided by this macro, nonetheless, suffer from several limitations that make it unsuitable for rigorous benchmarking.

First, a measurement based on just a single execution is often unreliable, as runtimes can fluctuate significantly due to background processes on your computer. Additionally, if that run is a function's first call, the measurement will include compilation overhead. The extra time Julia spends generating machine code inflates the reported runtime, making it unrepresentative of subsequent calls.

While running `@time` multiple times can address these issues, its most significant flaw arises when benchmarking functions. This is because `@time` mischaracterizes function arguments as global variables. We'll show in upcoming sections that global variables have a marked detrimental effect on performance. Consequently, the time reported doesn't accurately reflect how the function would perform in practice.

The following example illustrates the use of `@time`, highlighting the difference in execution time between the first and subsequent runs.

```
x = 1:100

@time sum(x)          # first run                      -> it incorporates compilation time
@time sum(x)          # time without compilation time -> relevant for each subsequent run
```

```
 0.002747 seconds (3.56 k allocations: 157.859 KiB, 99.36% compilation time)
 0.000003 seconds (1 allocation: 16 bytes)
```

PACKAGE "BENCHMARKTOOLS"

A more reliable alternative for measuring execution time is provided by `BenchmarkTools`, which addresses the shortcomings of `@time` in several ways.

First, it reduces result variability by running operations multiple times and then computing summary statistics. It also measures the execution time of functions without compilation latency, since the package discards the first run for the reported timing. Additionally, the package allows users to explicitly control variable scope: by prefixing function arguments with the `$` symbol, they're treated as local variables during a function call.

The package offers two macros, depending on the level of detail required: `@btime`, which only reports the minimum time, and `@benchmark`, which provides detailed statistics. Below, we demonstrate their use.

```
using BenchmarkTools

x = 1:100
@btime sum($x)         # provides minimum time only
```

```
 2.314 ns (0 allocations: 0 bytes)
```

```
using BenchmarkTools

x = 1:100
@benchmark sum($x)     # provides more statistics than `@btime`
```

In later sections, we'll exclusively benchmark functions. This means that you should always prefix the function arguments with `$`. **Omitting `$` will lead to inaccurate results**, including incorrect reports of memory allocations.

The following example demonstrates the consequence of excluding `$`, where the runtimes reported are higher than the actual runtime.

```
using BenchmarkTools
x = rand(100)

@btime sum(x)
```

```
 14.465 ns (1 allocation: 16 bytes)
```

```
using BenchmarkTools
x = rand(100)

@btime sum($x)
```

```
 6.546 ns (0 allocations: 0 bytes)
```

PACKAGE "CHAIRMARKS"

A new alternative for benchmarking code is the `Chairmarks` package. Its notation closely resembles that of `BenchmarkTools`, with the macros `@b` and `@be` providing a similar functionality to `@btime` and `@benchmark` respectively. The main benefit of `Chairmarks` is its speed, as it can be orders of magnitude faster than `BenchmarkTools`.

As with `BenchmarkTools`, measuring the execution time of functions requires prepending function arguments with `$`.

```
using Chairmarks
x = rand(100)

display(@b sum($x))        # provides minimum time only
```

```
6.550 ns
```

```
using Chairmarks
x = rand(100)

display(@be sum($x))       # analogous to `@benchmark` in BenchmarkTools
```

```
Benchmark: 3856 samples with 3661 evaluations
 min    6.679 ns
 median 6.815 ns
 mean   6.785 ns
 max    14.539 ns
```

REMARK ON RANDOM NUMBERS FOR BENCHMARKING

When comparing the performance of different methods, we must ensure that our measurements aren't skewed by variations in the input data. This implies each approach must be tested using *the exact same set of values*. This guarantees that differences in execution time can be attributed solely to the efficiency of the method itself, rather than to a change in the inputs.

Such consistency can be achieved by using random number generators. They rely on a **random seed**, which is an arbitrary starting point that dictates the entire sequence of values they produce. By setting the same seed before each test, we can generate identical deterministic sequences of random numbers across multiple runs. Importantly, **any arbitrary number can be used for the seed**. The only requirement is that the same number is employed, so that you replicate the exact same set of random numbers.

Random number generation is provided by the package `Random`. Below, we demonstrate its use by setting the seed `1234` before executing each operation. Note, though, that any other number could be used.

```
using Random

Random.seed!(1234)      # 1234 is an arbitrary number, use any number you want
x = rand(100)

Random.seed!(1234)
y = rand(100)           # identical to `x`
```

```
using Random

Random.seed!(1234)      # 1234 is an arbitrary number, use any number you want
x = rand(100)


y = rand(100)           # different from `x`
```

For presentation purposes, code snippets on this website will omit the lines dedicated to setting the random seed. While adding these code lines is essential for ensuring reproducibility, their inclusion in every example would create unnecessary clutter. Below, we illustrate the code that will be displayed throughout the website, along with the actual code executed.

```
using Random
Random.seed!(123)

x = rand(100)

y = sum(x)
```

```
# We omit the lines that seet the seed


x = rand(100)

y = sum(x)
```

BENCHMARKS IN PERSPECTIVE

When evaluating approaches for performing a task, execution times are often negligible, typically on the order of nanoseconds. Yet, this doesn't mean that the choice of method is without practical consequence.

While it's true that operations in isolation may have an insignificant impact on a program's overall runtime, **the relevance of benchmarks emerges when these operations are performed repeatedly**. This includes cases where the operation is called in a for-loop or in iterative procedures (e.g., solving systems of equations or maximizing functions). In these situations, small differences in timing are amplified as they are replicated hundreds, thousands, or even millions of times.

AN EXAMPLE

To illustrate this matter, let's consider a concrete example. Suppose we want to double each element of a vector `x`, and then calculate their sum. In the following, we'll compare two different approaches to accomplish this task.

The first method will be based on `sum(2 .* x)`, with `x` treated as a global variable. As we'll discuss in later sections, this approach is relatively inefficient. A more performant alternative is given by `sum(a -> 2 * a, x)`, where `x` is passed as a function argument. While we haven't explained why this implementation is better, it's sufficient to note that both methods produce the same result. The measured runtimes of each approach are as follows.

```
x      = rand(100_000)

foo()  = sum(2 .* x)
```

```
 35.519 μs (5 allocations: 781.37 KiB)
```

```
x      = rand(100_000)

foo(x) = sum(a -> 2 * a, x)
```

```
 6.393 μs (0 allocations: 0 bytes)
```

The results reveal that the second approach achieves a significant speedup, requiring less than 15% of the time taken by the slower method. However, even the "slow" approach is remarkably fast, taking less than 0.0001 seconds to execute.

This pattern will be recurring in our benchmarks, where absolute execution times are often negligible. In such cases, the relevance of our conclusions heavily depends on the context. If the operation is only performed once in isolation, readability should be the primary consideration for choosing a method. On the other hand, if the operation is executed repeatedly, small differences in performance might accumulate and become meaningful, making the faster approach a more suitable choice.

To make this point concrete, let's revisit the functions from the previous example and call them inside a for-loop that runs 100,000 times. Since our sole goal is to repeat the operation, the iteration variable itself plays no role. In such cases, it's common practice to employ a **throwaway variable**: placeholder that exists only to satisfy the loop's syntax, without ever being referenced. This convention signals to other programmers that the variable's value can be safely ignored. In our example, `_` serves this purpose, simply reflecting that each iteration performs exactly the same operation.

```
x      = rand(100_000)
foo()  = sum(2 .* x)

function replicate()
   for _ in 1:100_000
      foo()
   end
end
```

```
 5.697 s (500000 allocations: 74.52 GiB)
```

```
x      = rand(100_000)
foo(x) = sum(a -> 2 * a, x)

function replicate(x)
   for _ in 1:100_000
      foo(x)
   end
end
```

```
 677.130 ms (0 allocations: 0 bytes)
```

The example starkly reveals the consequences of calling these functions within a for-loop. The execution time of the slow version now jumps to more than 20 seconds, while the fast version finishes in under one second. Such a stark contrast underscores the importance of optimizing functions that are executed repeatedly: even seemingly minor improvements can accumulate into pronounced performance gains.

7d. Preliminaries on Types

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INTRODUCTION

High performance in Julia depends critically on the notion of type stability. The definition of this concept is relatively straightforward: a function is type-stable when the types of its expressions can be inferred from the types of its arguments. When this property holds, Julia can specialize its computation method, resulting in significant performance gains.

Despite its simplicity, type stability is subject to various nuances. In fact, a careful consideration of the property requires a solid foundation in two key areas: **Julia's type system and the inner workings of functions**. The current section equips you with the necessary knowledge to grasp the former, deferring the internals of functions to the next section. **The explanations will focus on the case of scalars and vectors**, leaving more complex objects for subsequent sections.

Before you continue, I recommend reviewing the basics of types introduced here.

> **Warning!**
> **The subject is covered only to the extent necessary for understanding type stability**. Julia's type system is indeed quite vast, and a comprehensive exploration would warrant a dedicated chapter.

BASICS OF TYPES

Variables in Julia serve as mere labels for objects, where objects in turn hold values with certain types. The most common types for scalars are `Float64` and `Int64`, whose vector counterparts are `Vector{Float64}` and `Vector{Int64}`. Recall that `Vector` is an alias for a one-dimensional array, so that a type like `Vector{Float64}` is equivalent to `Array{Float,1}`.

> **`Int` As an Alternative to `Int64`**
> You'll notice that packages tend to use `Int` as the default type for integers. The type `Int` is an alias that adapts to your CPU's architecture. Since most modern computers are 64-bit systems, `Int` is equivalent to `Int64`. Nonetheless, `Int` becomes `Int32` on 32-bit systems.

Julia's type system is organized in a hierarchical way. This feature allows for the definition of subsets and supersets of types, which in the context of types are referred to as **subtypes** and **supertypes**. [1] For instance, the type `Any` is a supertype that includes all possible types in Julia, thus occupying the highest position in any type hierarchy. Another example of supertype is `Number`, which encompasses all numeric types (`Float64`, `Float32`, `Int64`, etc.).

Supertypes provide great flexibility for writing code. They enable the grouping of values to define operations in common. For instance, defining `+` for the abstract type `Number` ensures its applicability to all numeric types, regardless of whether they are integers, floats, or their numerical precision.

A special supertype known as `Union` will be instrumental for our examples. The construction is useful for variables that can potentially hold values with different types. It's syntax is `Union{<type1>, <type2>, ...}`, so that a variable with type `Union{Int64, Float64}` could be either an `Int64` or `Float64`. Note that, by definition, union types are always supertypes of their arguments.

> **Union of Types to Account for Missing Values**
>
> Unions of types emerge naturally in data analysis workflows, especially when handling missing observations. In Julia, these values are represented by the type `Missing`. Thus, if we load a column that contains both integers and empty entries, this is usually stored with type `Vector{Union{Missing,Int64}}`.

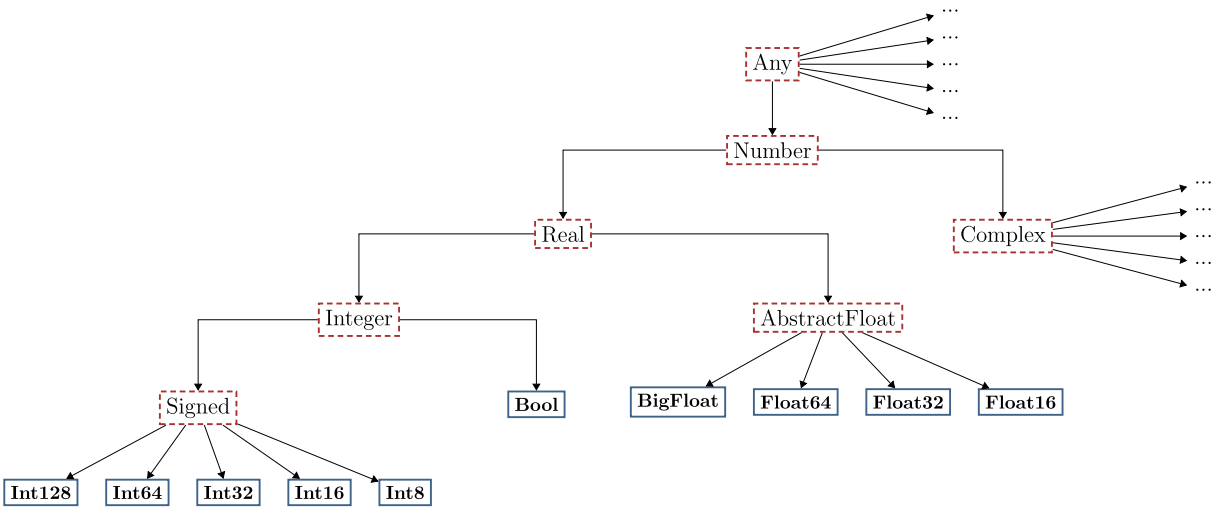
ABSTRACT AND CONCRETE TYPES

The hierarchical nature of types makes it possible to represent subtypes and supertypes as trees. Such structure gives rise to the notions of abstract and concrete types.

An **abstract type** acts as a parent category, necessarily breaking down into subtypes. The type `Any` in Julia is a prime example. In contrast, a **concrete type** represents an irreducible unit that therefore lacks subtypes. Concrete types are considered final, in the sense that they can't be further specialized within the hierarchy.

The diagram below illustrates the difference between abstract and concrete types for scalars. In particular, we present the hierarchy of the type `Number`, where the labels included match the corresponding type name in Julia. [2]

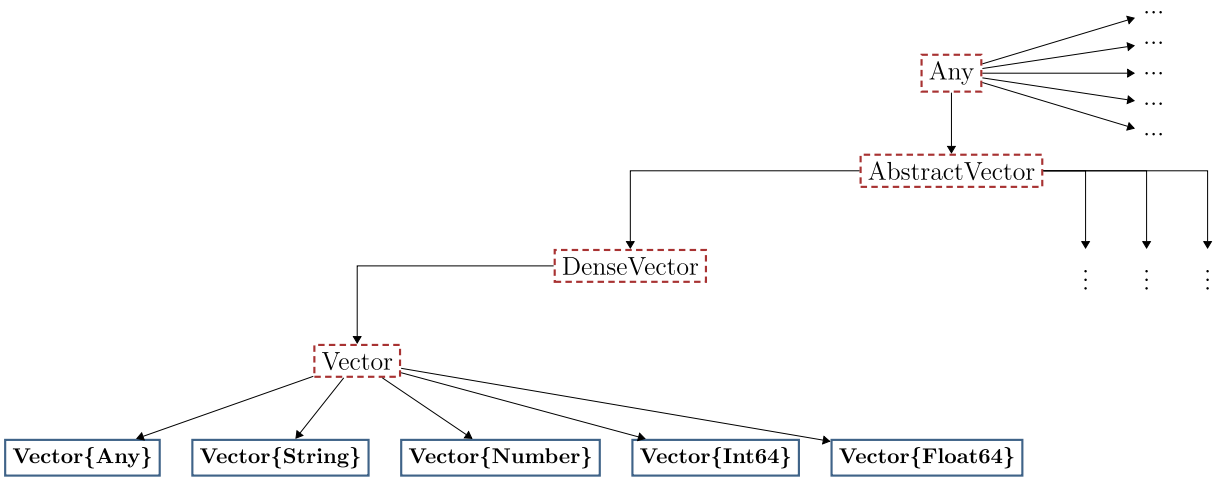
HIERARCHY OF TYPE NUMBER



Note: Dashed red borders indicate abstract types, while solid blue borders indicate concrete types.

The distinction between abstract and concrete types for scalars is relatively straightforward. Instead, the same distinction becomes more nuanced when vectors are considered, as shown in the diagram below.

HIERARCHY OF TYPE VECTOR



Note: Dashed red borders indicate abstract types, while solid blue borders indicate concrete types.

The tree reveals that `Vector{T}` **for a given type** `T` **is a concrete type**. This means that variables can be instances of `Vector{T}`, where `Vector{T}` doesn't have subtypes. The consequence is that a vector like `Vector{Int64}` isn't a subtype of `Vector{Any}`, even though `Int64` is a subtype of `Any`.

This behavior stands in stark contrast to scalars, where `Any` is an abstract type. However, it aligns perfectly with the concept of vectors as collections of *homogeneous elements*, meaning that all elements share the same type.

ONLY CONCRETE TYPES CAN BE INSTANTIATED, ABSTRACT TYPES CAN'T

In Julia, **instantiation** refers to the process of creating an object with a specific type. A key principle of Julia's type system is that **only concrete types can be instantiated**, implying that values can never be represented by abstract types. This distinction helps clarify the meaning of some widespread expressions used in Julia. For example, stating that a variable has type `Any` shouldn't be interpreted literally. Rather, it means the variable can hold values of any concrete type, considering that all concrete types are subtypes of `Any`.

This distinction will become crucial for what follows, particularly for type-annotating variables. It implies that declaring a variable with an abstract type restricts the set of possible concrete types it can hold, even though the variable will ultimately adopt a concrete type.

RELEVANCE FOR TYPE STABILITY

At this point, you may be wondering how all these concepts relate to type stability. The connection becomes clear when you consider how Julia performs computations.

High performance in Julia relies heavily on specializing the computation method. We'll see that this specialization is unattainable in the global scope, as Julia treats global variables as potentially holding values of any type. In contrast, when code is wrapped in a function, the execution process begins by determining the concrete types of each function argument. This information is then used to infer the concrete types of all the expressions within the function body.

When this inference succeeds and all expressions have unambiguous concrete types, the function is considered **type stable**. Type stability enables Julia to specialize its computation method and generate optimized machine code. If, instead, expressions could potentially take on multiple concrete types, performance is substantially degraded, as Julia must consider a separate implementation for each possible type.

For scalars and vectors, type stability essentially requires that expressions ultimately operate on **primitive types**. Examples of numeric primitive types include integers and floating-point numbers, such as `Int64`, `Float64`, and `Bool`. Thus, applying functions like `sum` to a `Vector{Int64}` or `Vector{Float64}` allows for full specialization, whereas applying them to a `Vector{Any}` prevents it.

> **String Objects**
>
> For text representation, the character type `Char` serves as a primitive type. Since a `String` is internally represented as a collection of `Char` elements, operations on `String` objects can also achieve type stability.

THE OPERATOR <: TO IDENTIFY SUPERTYPES

The rest of this section is dedicated to operators and functions for working with types. Specifically, we'll introduce the operator `<:`, which checks whether one type is a subtype of another. Then, we'll examine strategies for constraining variables to specific types.

It's possible that you won't need to apply any of the techniques we present, as Julia automatically attempts to infer types when functions are called. Nonetheless, understanding these operators is essential for grasping upcoming material.

USE OF `<:`

The symbol `:<` tests whether a type `T` is a subtype of another type `S`. It can be used as an operator `T <: S` or as a function `<:(T,S)`. For example, `Int64 <: Number` and `<:(Int64, Number)` verifiy whether `Int64` is a subtype of `Number`, thus returning `true`. Below, we provide further examples.

```
# all the statements below are `true`
Float64 <: Any
Int64   <: Number
Int64   <: Int64
```

```
# all the statements below are `false`
Float64 <: Vector{Any}
Int64   <: Vector{Number}
Int64   <: Vector{Int64}
```

The fact that `Int64 <: Int64` evaluates to `true` illustrates a fundamental principle: **every type is a subtype of itself**. Moreover, in the case of concrete types, this is the only subtype.

THE KEYWORD `WHERE`

By combining `<:` with `Union`, you can also check whether a type belongs to a given set of types. For example, `Int64 <: Union{Int64, Float64}` assesses whether `Int64` equals `Int64` or `Float64`, thus returning `true`.

The approach can be made more widely applicable by using the `where` keyword with a type parameter `T`. [3] The syntax is `<type depending on T> where T <: <set of types>`. This entails that `T` cover multiple possible types.

```
# all the statements below are `true`
Float64 <: Any
Int64   <: Union{Int64, Float64}
Int64   <: Union{T, String} where T <: Number       # `String` represents text
```

```
# all the statements below are `true`
Vector{Float64} <: Vector{T} where T <: Any
Vector{Int64}   <: Vector{T} where T <: Union{Int64, Float64}
Vector{Number}  <: Vector{T} where T <: Any

# all the statements below are `false`
Vector{Float64} <: Vector{Any}
Vector{Int64}   <: Vector{Union{Int64, Float64}}
Vector{Number}  <: Vector{Any}
```

```
# all the statements below are `true`
Vector{Float64} <: Vector{<:Any}
Vector{Int64}   <: Vector{<:Union{Int64, Float64}}
Vector{Number}  <: Vector{<:Any}

# all the statements below are `false`
Vector{Float64} <: Vector{Any}
Vector{Int64}   <: Vector{Union{Int64, Float64}}
Vector{Number}  <: Vector{Any}
```

Types relying on parameters like `T` are called **parametric types**. In the example above, these types allow us to distinguish between a concrete type like `Vector{Any}` and a set of concrete types `Vector{T} where T <: Any`, where the latter encompasses `Vector{Int64}`, `Vector{Float64}`, `Vector{String}`, etc.

Warning! - The Type `Any`

When we omit `<:` and simply write `where T`, Julia implicitly interprets the statement as `where T <: Any`. This is why the following equivalences hold.

```
# all the statements below are `true`
Float64       <: Any
Float64        <: T where T <: Any              # identical to
the line above
Vector{Int64} <: Vector{T} where T <: Any
```

```
# all the statements below are `true`
Float64       <: Any
Float64        <: T where T                     # identical to
the line above
Vector{Int64} <: Vector{T} where T
```

TYPE-ANNOTATING VARIABLES

In the following, we present methods for **type-annotating variables**. The techniques introduced can be used either to assert a variable's type **during an assignment** or to restrict the types of **function arguments**.

Specifically, there are two approaches to type-annotating variables. The first one relies on the binary operator `::`, and its syntax is `x::<type>`. The second approach leverages the Boolean binary operator `<:`, combined with `::` and the keyword `where`. Its syntax is `x::T where T <: <type>` (note that `T` accepts any other letter).

Next, we illustrate both methods, separately considering type-annotations for both assignments and function arguments.

ASSIGNMENTS

Let's start illustrating the approaches based on scalar assignments. Each tab below declares an identical type for `x` and for `y`.

```
x::Int64                = 2      # only reassignments to `Int64` are possible

y::Number               = 2       # only reassignments to `Float64`, `Float32`, `Int64`, etc
are possible
```

```
julia> x = 2.5
ERROR: InexactError: Int64(2.5)
julia> y = 2.5
2.5
julia> y = "hello"
ERROR: MethodError: Cannot convert an object of type String to an object of type Number
```

```
x::T where T <: Int64   = 2      # only reassignments to `Int64` are possible

y::T where T <: Number = 2       # only reassignments to `Float64`, `Float32`, `Int64`, etc
are possible
```

```
julia> x = 2.5
ERROR: InexactError: Int64(2.5)
julia> y = 2.5
2.5
julia> y = "hello"
ERROR: MethodError: Cannot convert an object of type String to an object of type Number
```

> **Warning!** - Modifying Types
>
> Once a type for `x` has been assigned, the type can't be changed. The only way to fix this is by starting a new session.

The fact that `x` holds the same type across all tabs follows because `T <: Float64` can only represent `Float64`. More specifically, `Float64` is a concrete type, which by definition has no subtypes other than itself. Considering this, scalar types are usually asserted using `::` rather than `<:`.

While this behavior holds for scalars, it doesn't apply to vectors. Specifically, using `::` in combination with `Vector{Number}` establishes that the object will have `Vector{Number}` as its concrete type. Instead, `Vector{T} where T <: Number` indicates that the elements of the vector will adopt a concrete subtype of `Number`.

```
# `x` will always be `Vector{Any}`
x::Vector{Any}                  = [1,2,3]

# `y` will always be `Vector{Number}`
y::Vector{Number}               = [1,2,3]
```

```
julia> typeof(x)
Vector{Any} (alias for Array{Any, 1})
julia> typeof(y)
Vector{Number} (alias for Array{Number, 1})
```

```
# `x` is Vector{Int64} and could eventually become `Vector{Float64}`, `Vector{String}`, etc
x::Vector{T} where T <:Any      = [1,2,3]

# `x` is Vector{Int64} and could eventually become `Vector{Float64}`, `Vector{Int32}`, etc
y::Vector{T} where T <: Number = [1,2,3]
```

```
julia> typeof(x)
Vector{Int64} (alias for Array{Int64, 1})
julia> typeof(y)
Vector{Int64} (alias for Array{Int64, 1})
```

The principles outlined apply even when a variable isn't explicitly type-annotated. The reason is that **an assignment without `::` implicitly assigns the type `Any` to the variable**, where `Any` is the supertype encompassing all possible types. For example, the statements `x = 2` and `x::Any = 2` are equivalent.

The same occurs when omitting `<:` from the expression `where T`, which implicitly takes `T <: Any`. Thus, for instance, `x = 2` is equivalent to `x::T where T = 2` or `x::T where T <: Any = 2`. Considering this, all variables listed below have their types constrained in a similar manner.

```
# all are equivalent
a      = 2
b::Any = 2
```

```
# all are equivalent
a                    = 2
b::T where T         = 2
c::T where T <: Any = 2
```

Once we recognize that variables default to the type `Any`, it becomes clear why they can be reassigned with values of different types. For instance, given `a = 1`, executing `a = "hello"` afterwards is valid, since `a` is implicitly type-annotated with `Any`.

> **Warning!** - One-liner Statements Using `where`
>
> Be careful with one-liner statements using `where`, especially when `where T` is shorthand for `where T <: Any`. These concise statements can easily lead to confusion, as demonstrated below.

```
a::T where T = 2                    # this is not `T = 2`, it's
`a = 2`

a::T where {T}          = 2           # slightly less confusing
notation
a::T where {T <: Any} = 2             # slightly less confusing
notation
```

```
foo(x::T) where T = 2               # this is not `T = 2`, it's
`foo(x) = 2`

foo(x::T) where {T}           = 2     # slightly less confusing
notation
foo(x::T) where {T <: Any} = 2      # slightly less confusing
notation
```

FUNCTIONS

Function arguments can also be type-annotated. This is illustrated below, where functions are restricted to integer inputs exclusively.

```
function foo1(x::Int64, y::Int64)
    x + y
end
```

```
julia> foo1(1, 2)
3
julia> foo1(1.5, 2)
ERROR: MethodError: no method matching foo1(::Float64, ::Int64)
```

```julia
function foo2(x::Vector{T}, y::Vector{T}) where T <: Int64
    x .+ y
end
```

```
julia> foo2([1,2], [3,4])
2-element Vector{Int64}:
 4
 6
julia> foo2([1,2], [3.0, 4.0])
ERROR: MethodError: no method matching foo2(::Vector{Int64}, ::Vector{Float64})
```

Note that when both function arguments are annotated with the same type parameter `T`, they're constrained to have exactly the same type. Also notice that types like `Int64` preclude the use of `Float64`, even for numbers like `3.0`. To allow greater flexibility, you should introduce separate type parameters and annotate them with a common abstract type like `Number`.

```julia
function foo2(x::T, y::T) where T <: Number
    x + y
end
```

```
julia> foo2(1.5, 2.0)
3.5
julia> foo2(1.5, 2)
ERROR: MethodError: no method matching foo2(::Float64, ::Int64)
```

```julia
function foo3(x::T, y::S) where {T <: Number, S <: Number}
    x + y
end
```

```
julia> foo3(1.5, 2.0)
3.5
julia> foo3(1.5, 2)
3.5
```

The greatest flexibility is achieved when we don't type-annotate function arguments at all, as they will implicitly default to `Any`. This can be observed below, where all tabs define identical functions. Ultimately, type-annotating function arguments is only needed to prevent invalid usage (e.g., to ensure that `log` isn't applied to a negative value).

```julia
function foo(x, y)
    x + y
end
```

```julia
function foo(x::Any, y::Any)
    x + y
end
```

```
function foo(x::T, y::S) where {T <: Any, S <: Any}
    x + y
end
```

```
function foo(x::T, y::S) where {T, S}
    x + y
end
```

CREATING VARIABLES WITH SOME TYPE

To conclude this section, we present an approach for converting values into a specific type. The approach makes use of the so-called **constructors**, which are functions that create new instances of a concrete type. They're useful for transforming a variable `x` into another type.

Constructors are implemented by functions of the form `Type(x)`, where `Type` should be replaced with the literal name of the type (e.g., `Vector{Float64}`). Like any other function, `Type` also supports broadcasting.

```
x = 1

y = Float64(x)
z = Bool(x)
```

```
julia> y
1.0
julia> z
true
```

```
x = [1, 2, 3]

y = Vector{Any}(x)
```

```
julia> y
3-element Vector{Any}:
 1
 2
 3
```

```
x = [1, 2, 3]

y = Float64.(x)
```

```
julia> y
3-element Vector{Float64}:
 1.0
 2.0
 3.0
```

Remark

Parametric types can be used as constructors. Moreover, although abstract types can't be instantiated, they may still serve as constructors. In such cases, Julia will attempt to convert the object to a specific concrete type, although not all abstract types can be used for this purpose.

```
x = 1

y = Number(x)
```

```
julia> typeof(y)
Int64
```

```
x = [1, 2]

y = (Vector{T} where T)(x)
```

```
julia> typeof(y)
Vector{Int64}
```

```
x = 1

z = Any(x)
```

```
ERROR: MethodError: no constructors have been defined for Any
```

An alternative to transform `x`'s type into `T` is given by the function `convert(T,x)`. Note this only works when a valid conversion exists, as otherwise it'll fail.

```
x = 1

y = convert(Float64, x)
z = convert(Bool, x)
```

```
julia> y
1.0
julia> z
true
```

```
x = [1, 2, 3]

y = convert(Vector{Any}, x)
```

```
julia> y
3-element Vector{Any}:
 1
 2
 3
```

```
x = [1, 2, 3]

y = convert.(Float64, x)
```

```
julia> y
3-element Vector{Float64}:
 1.0
 2.0
 3.0
```

FOOTNOTES

1. Types don't necessarily follow a subtype-supertype hierarchy. For example, `Float64` and `Vector{String}` exist independently, without a hierarchical relationship. This fact will become clearer when the concepts of abstract and concrete types are defined.

2. The `Signed` subtype of `Integers` allows for the representation of negative and positive integers. Julia also offers the type `Unsigned`, which only accepts positive integers and comprises subtypes such as `UInt64` and `UInt32`.

3. `T` can be replaced by any other letter

7e. Functions: Type Inference and Multiple Dispatch

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PhD in Economics

INTRODUCTION

In Julia, functions are key for achieving high performance. This is by design, as functions have been engineered from the outset to generate efficient machine code.

However, to fully unlock their potential, we must first understand the underlying process of function calls. Essentially, when a function is called, Julia attempts to identify concrete types for its variables, eventually selecting the most suitable method for the function. At the heart of the process are three interconnected mechanisms: **dispatch**, **compilation**, and **type inference**. This section will provide a detailed explanation of each concept.

FUNCTION'S VARIABLE SCOPE

To fully appreciate why functions in Julia are central to performance, we must first revisit the notion of variable scope. A function's **local variables** encompass all variables defined within a function's scope, including its arguments and any variable created inside the function body. These variables exist only during the function's execution and are inaccessible from outside. **Global variables**, on the other hand, refer to any variable defined outside a function's scope and remain accessible throughout the program's execution.

From a practical perspective, a key takeaway from this section is that **wrapping code in functions is crucial for high performance in Julia**. Instead, relying on global variables imposes a significant performance penalty.

Note that the use of global variables isn't confined to operations in the global scope. It also takes place when a function references variables that aren't passed as arguments. This is why the detrimental effect of global variables will appear in all the following cases.

GLOBAL SCOPE

```
x     = 2

y     = 3 * x
```

```
julia> y
6
```

FUNCTION USING A GLOBAL VARIABLE

```
x     = 2

foo() = 3 * x
```

```
julia> foo()
6
```

Recall that an expression like `x = 2` is shorthand for `x::Any = 2`, reflecting that global variables default to `Any` if they aren't explicitly type-annotated. Also remember that only concrete types can be **instantiated**, meaning that values can only adopt a concrete type. Consequently, `x::Any` shouldn't be understood as `x` having type `Any`, but rather that `x` can take on any concrete type that's a subtype of `Any`. Since `Any` sits at the top of Julia's type hierarchy, this simply means that `x`'s types are unrestricted.

Working with a global variable like `x` prevents specialization of `*`. The reason is that Julia treats global variables as potentially embodying any value and therefore any type. The underlying logic for this behavior is that, even if a variable holds some value at a specific moment, the user may reassign it at any point in the program. Because of this possibility, Julia must consider multiple possible methods for its computation, one for each possible concrete type of `x`. In practice, this results in Julia generating code with multiple branches, potentially involving type checks, conversions, and object creations. The consequence is degraded performance.

Even if we had type-annotated `x` with a concrete type like `x::Int64 = 2`, the performance limitations wouldn't completely go away. Certain optimizations can only be implemented when both the scope of variables is clearly delimited and values are known. When this is the case, Julia can gain a comprehensive view of all the operations to be performed, creating opportunities for further optimizations.

Functions in Julia were designed to address all these considerations. They achieve this following a series of steps, which we cover next.

FUNCTIONS AND METHODS

A **function** is a name that can be associated with multiple implementations known as methods. A **method** defines a specific function body for a given combination of argument types and number of arguments. The list of methods associated with a function `foo` can be retrieved by running the command `methods(foo)`.

To make this concrete, let's define several methods for some function `foo`. Creating methods requires type-annotating `foo`'s arguments with the operator `::` during its definition. By using this strategy, we can set a distinct body function for each unique combination of types.

To keep matters simple, let's start considering a scenario where all the methods of `foo` have the same number of arguments.

METHODS

```
foo(a,b)                  = a + b
foo(a::String, b::String) = "This is $a and this is $b"
```

```
julia> methods(foo)
2 methods for generic function "foo" from Main
julia> foo(1,2)
3
julia> foo("some text", "more text")
"This is some text and this is more text"
```

Since `foo(a,b)` is equivalent to `foo(a::Any,b::Any)`, the first method sets the behavior of `foo` for any combination of input types. Such behavior is subsequently overridden by the creation of the method `foo(a::String, b::String)`. By doing this, we provide an alternative function body for `a` and `b` with type `String`. The existence of multiple methods explains the differences in outputs obtained: the first method of `foo` is called with `foo(1, 2)`, whereas `foo("some text", "more text")` triggers the second method.

The example also reveals that **methods don't need to comprise similar operations**. Although mixing drastically different operations under a single function name isn't recommended, allowing function bodies to differ is relevant for performance. In particular, it's commonly used for tailoring the computation algorithms to each specific type combination, thus optimizing the overall performance of a function.

Also note that **methods don't need to have the same number of arguments**. For instance, it's possible to define the following methods for a function `bar`.

METHODS WITH DIFFERENT NUMBERS OF ARGUMENTS

```
bar(x)       = x
bar(x, y)    = x + y
bar(x, y, z) = x + y + z
```

```
julia> methods(bar)
3 methods for generic function "bar" from Main
julia> bar(1)
1
julia> bar(1, 2)
3
julia> bar(1, 2, 3)
6
```

Defining methods with different number of arguments is particularly useful for extending a function's behavior. A prime application is given by the function `sum`. So far, we've only used its simplest form `sum(x)`, which adds all the elements of a collection `x`. However, `sum` also supports additional methods. One of them is `sum(<function>, x)`, where the elements of `x` are transformed via `<function>` before being summed.

METHODS FOR 'SUM'

```
x = [2, 3, 4]

y = sum(x)          # 2 + 3 + 4
z = sum(log, x)     # log(2) + log(3) + log(4)
```

FUNCTION CALLS

Building upon our understanding of function definition and methods, let's now analyze the process triggered when a function is called. In the following, all our explanations will be based on the following function `foo`:

EXAMPLE FUNCTION 'FOO'

```
foo(a, b) = 2 + a * b
```

```
julia> foo(1, 2)
4
julia> foo(3, 2)
8
julia> foo(3.0, 2)
8.0
```

In Julia, defining a function like `foo(a, b)` is shorthand for creating a **method** with the signature `foo(a::Any, b::Any)`. Thus, the function body `foo(a,b)` holds for all possible type combinations of `a` and `b`.

When `foo(1, 2)` is called, Julia evaluates the expression `2 + a * b` by following a series of steps.

The process begins with what's known as **multiple dispatch**, where Julia selects which method of a function to execute. Importantly, this decision is based solely on the types of the arguments, not their values. Specifically, Julia begins by identifying the concrete types of the function arguments. In our example where `a = 1` and `b = 2`, both are identified as `Int64`. The information on types is then used to select a *method*, which defines the function body and hence the operations to be performed. This process involves searching through the available methods of `foo` to find the most specific one for the concrete types of `a` and `b`. In our example, `foo` has only one method `foo(a,b) = 2 + a * b`, which is defined for all argument types, including `a::Int64` and `b::Int64`. Therefore, the corresponding function body is `2 + a * b`.

The specific version of this method for the signature `foo(a::Int64, b::Int64)` is known as a **method instance**. If the code for the method instance `foo(a::Int64, b::Int64)` already exists, Julia will directly employ it to compute `foo(1,2)`. Otherwise, the compiler generates optimized code for that method instance, stores it in memory for future use, and Julia executes it.

The following diagram depicts the process unfolded when `foo(1,2)` is executed.

MULTIPLE DISPATCH

The process outlined has implications for how the language works. When a function is called with a particular combination of argument types for the first time, Julia incurs an additional cost because it must generate specialized code for those types. This initial delay is often referred to as **Time To First Plot**, a phrase that highlights how the compilation overhead becomes noticeable in interactive workflows such as plotting. Once the code has been compiled, however, Julia stores the resulting method instance, so that subsequent calls with the same argument types can reuse it. This eliminates the compilation step and leads to much faster execution.

The example with `foo` illustrates this process clearly. After evaluating `foo(1, 2)`, Julia has already compiled a method instance for the signature `foo(a::Int64, b::Int64)`. This is why the subsequent call `foo(3, 2)` can be executed immediately by invoking the cached method instance, without any need for recompilation. Instead, the execution of `foo(3.0, 2)` introduces a new combination of argument types, where `a::Float64` and `b::Int64`. Because no compiled method instance yet exists for this signature, Julia must generate one before executing the function.

TYPE INFERENCE

Most considerations for achieving high performance are related to the compilation process. In particular, Julia employs **Just-In-Time Compilation (JIT)**, a term reflecting that compilation happens on the fly during the function call.

A key mechanism in this process is **type inference**, whereby the compiler attempts to identify concrete types for *all* variables and expressions. If the compiler succeeds in identifying concrete types, it can specialize instructions for each operation and yield fast code.

This is the essence behind **type stability**, which we'll cover extensively in the next chapter. For instance, type inference in our example involves determining concrete types for `2`, `a = 1`, and `b = 2`. Since all values have type `Int64`, the compiler can specialize the computation of `2 + a * b` for variables with type `Int64`.

On the contrary, if the compiler is unable to identify concrete types for some expressions, it must create generic code to accommodate multiple combinations of types. This forces Julia to perform type checks and conversions during runtime, significantly degrading performance.

REMARKS ON TYPE INFERENCE

Below, we provide various remarks about type inference that are worth keeping in mind for next sections.

FUNCTIONS DO NOT GUARANTEE IDENTIFICATION OF CONCRETE TYPES

Merely wrapping code in a function doesn't guarantee that the compiler will identify concrete types for all operations. The following example illustrates this.

TYPE-UNSTABLE FUNCTION

```
x      = [1, 2, "hello"]     # Vector{Any}

foo(x) = x[1] + x[2]         # type unstable
```

```
julia> foo(x)
3
```

The issue in the example arises because the compiler assigns the type `Any` to both `x[1]` and `x[2]`, as they come from an object with type `Vector{Any}`. As a consequece, the compiler can't specialize the computation of the operation `+`. The example also highlights that compilation is exclusively based on types, not values. Thus, the generated code ignores the actual values `x[1] = 1` and `x[2] = 2`, which would otherwise indicate a type `Int64`.

GLOBAL VARIABLES INHERIT THEIR GLOBAL TYPE

Type inference is restricted to local variables, with any global variable having its type inherited from the global scope. For instance, consider the following example.

UNANNOTATED GLOBAL VARIABLE

```
a         = 2
b         = 1

foo(a)    = a * b
```

```
julia> foo(a)
2
```

TYPE-ANNOTATED GLOBAL VARIABLE

```
a         = 2
b::Number = 1

foo(a)    = a * b
```

```
julia> foo(a)
2
```

In both examples `b` is a global variable. Consequently, the compiler infers `b`'s type as `Any` in the first tab and as `Number` in the second tab.

TYPE-ANNOTATING FUNCTION ARGUMENTS DOES NOT IMPROVE PERFORMANCE

Identifying concrete types is crucial for achieving high performance. At first glance, this might suggest that explicitly annotating function arguments is necessary for performance, or at least beneficial. However, thanks to type inference, such annotations are redundant. In fact, adding them can be

counterproductive, as they unnecessarily restrict the types accepted by a function, thereby limiting its flexibility and potential applications.

To better appreciate this loss of flexibility, compare the following scripts.

UNANNOTATED FUNCTION

```
foo2(a, b) = a * b
```

```
julia> foo2(0.5, 2.0)
1.0
julia> foo2(1, 2)
2
```

TYPE-ANNOTATED FUNCTION

```
foo1(a::Float64, b::Float64) = a * b
```

```
julia> foo1(0.5, 2.0)
1.0
julia> foo1(1, 2)
ERROR: MethodError: no method matching foo1(::Int64, ::Int64)
```

The function on the first tab only accepts arguments of type `Float64`, implying that even integer variables are disallowed. By contrast, the function on the second tab mirrors the behavior with `Float64` inputs, but additionally allows for other types. This flexibility stems from the implicit type annotation `Any` on the function arguments.

Packages Commonly Type-Annotate Function Arguments

When inspecting the code of packages, you may notice that function arguments are often type-annotated. The reason for this isn't related to performance, but rather to ensure the function's intended usage, safeguarding against inadvertent type mismatches.

For instance, suppose a function that computes the revenue of a theater via `nr_tickets * price`. Importantly, the operator `*` in Julia not only implements product of numbers, but also concatenates words when applied to expressions with type `String`. Without type-annotations, the function could potentially be misused. This is exemplified in the first tab below, with the second tab precluding this possibility by asserting types.

UNANNOTATED FUNCTION

```
revenue1(nr_tickets, price) = nr_tickets * price
```

```
julia> revenue1(3, 2)
6
julia> revenue1("this is ", "allowed")
"this is allowed"
```

TYPE-ANNOTATED FUNCTION

```
revenue2(nr_tickets::Int64, price::Number) = nr_tickets * price
```

```
julia> revenue2(3, 2)
6
julia> revenue2("this is ", "not allowed")
ERROR: MethodError: no method matching revenue2(::String,
::String)
```

8a. Overview and Goals

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In the upcoming chapters, we'll focus on two essential aspects for performance: type stability and reductions in memory allocation. These core principles represent the most basic procedures to achieve high performance, thus acting as the starting point for further optimizations.

This chapter in particular focuses on type stability, whose importance for Julia can't be overstated —**any attempt to generate fast code without ensuring type stability is destined to fail**.

At its core, type stability is rooted in how computers execute operations at a fundamental level. Specifically, regardless of the programming language used, the approach to computing operations differs depending on the inputs' types. This means, for instance, that the internal process for integer operations differs from computations based on floating-point numbers.

The consequence of this feature for performance is that speed demands the identification of concrete types for each variable. With this information available, the computation method can be specialized. Instead, if concrete types can't be identified, the code generated must accommodate multiple potential approaches, one for each possible combination of input types. This introduces additional runtime checks and type conversions, significantly degrading execution speed.

The discussion of type stability will be intertwined with functions, as *type stability requires wrapping code in function as a prerequisite*. The reason for this is that Julia only attempts to infer the types of variables within a function. Wrapping code in a function is only a necessary condition for type stability, and the chapter will provide additional conditions to guarantee the property.

8b. Defining Type Stability

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INTRODUCTION

This section formally defines type stability and reviews the tools employed for its verification. In the next section, we'll begin examining how type stability applies in specific scenarios.

AN INTUITION

In a previous section, we described the process that unfolds when a function is called. To briefly review, let's consider a function `foo(x) = x + 2` and executing `foo(a)` for some variable `a`. We assume that `a` has a specific value assigned and therefore a concrete type, although we omit explicitly stating a value for `a`. In this way, we highlight that the process depends on types, rather than values.

Calling `foo(a)` prompts Julia to identify the concrete type of `a`, which we'll denote as `T`. If a compiled method instance for `foo` with an argument of type `T` already exists, then `foo(a)` is executed immediately. Otherwise, Julia compiles a method instance for evaluating `a + 2`. This code generation leverages type inference, wherein the compiler attempts to deduce concrete types for terms in the function body. The resulting machine code is then stored, making it readily available for subsequent calls of `foo(b)` when `b` has type `T`.

TYPE STABILITY AND PERFORMANCE

The key to generating fast code lies in the information available to the compiler during the compilation stage. This information is primarily gathered through type inference, where Julia identifies the specific type of each variable and expression involved. When the compiler can **accurately predict a single concrete type for the function's output**, the function call is said to be **type stable**.

While this constitutes the formal definition of type stability, a more stringent definition is usually applied in practice: the compiler must be able to **infer unique concrete types for *each* expression within the function**, not only for the final output. This definition aligns with `@code_warntype`, the built-in macro to detect type instabilities.

If the condition is satisfied, the compiler can specialize the computational approach for each operation, resulting in fast execution. Essentially, type stability dictates that there's sufficient information to determine a straight execution path, thus avoiding unnecessary type checks and dispatches at runtime.

In contrast, type-unstable functions generate generic code that accommodates each possible combination of unique concrete types. This results in additional overhead during runtime, where Julia is forced to dynamically gather type information and perform extra calculations based on it. The consequence is a pronounced deterioration in performance.

> **Type Stability Characterizes Function *Calls***
>
> It's common to describe a function as "type stable". Strictly speaking, however, type stability isn't a property of the function. Rather, it's a property of how the function behaves when called with arguments of particular concrete types. The distinction is crucial in practice, since a function may exhibit type stability for certain input types, but not for others.

AN EXAMPLE

To identify type stability in practice, let's consider the following example.

```
x = [1, 2, 3]                    # `x` has type `Vector{Int64}`

@btime sum($x[1:2])              # type stable
```

```
 22.406 ns (1 allocation: 80 bytes)
```

```
x = [1, 2, "hello"]              # `x` has type `Vector{Any}`

@btime sum($x[1:2])              # type UNSTABLE
```

```
 31.938 ns (1 allocation: 64 bytes)
```

The two operations may seem equivalent, as they both ultimately compute `1 + 2`. However, the implementation strategy used in each case differ, with the first approach being faster because the function call is type stable.

Specifically, the output `x[1] + x[2]` in the first tab can be deduced to be `Int64`, thus satisfying the definition of type stability. This occurs because `x[1]` and `x[2]` can be identified as `Int64`, allowing the compiler to generate code specialized for this type. Note that the efficiency of the generated code isn't limited to the given operation: it applies to any call `sum(y)` such that `y` is a `Vector{Float64}`.

In contrast, **the second tab defines a type-unstable function call**. Since `x` has type `Vector{Any}`, it becomes impossible to predict a unique concrete type for `x[1] + x[2]` solely based on `x`'s type. This is because `x[1]` and `x[2]` may embody any concrete type that is a subtype of `Any`. Consequently, the compiler is forced to create code with multiple conditional statements, with each branch handling how to compute `x[1] + x[2]` for a possible type (`Int64`, `Float64`, `Float32`, etc.). This results in slow compiled code, as it'll require extra work during runtime. Furthermore, the degraded performance will be incurred for every call `sum(y)` such that `y` has type `Vector{Any}`.

> **Remark**
> Julia's developers are continually refining the compiler, addressing and mitigating the effects of certain type instabilities. As a result, **many operations that were once type unstable are now type stable**. This means that type stability should be considered a dynamic property of the language, subject to change as the compiler evolves.

CHECKING FOR TYPE STABILITY

There are several mechanisms to determine whether a function call is type stable. One of them is based on the `@code_warntype` macro, which reports all the types inferred during a function call. To illustrate its use, consider a function that defines `y` as a transformation of `x`, and then uses `y` to perform some operation.

```
function foo(x)
    y = (x < 0) ?  0  :  x

    return [y * i for i in 1:100]
end
```

```
julia> @code_warntype foo(1.)
```

```
function foo(x)
    y = (x < 0) ?  0  :  x

    return [y * i for i in 1:100]
end
```

```
julia> @code_warntype foo(1)
```

The output of `@code_warntype` can be difficult to interpret. Nonetheless, the addition of colors facilitates its understanding:

- If all lines are **blue**, the function is **type stable**. This means that Julia can identify a unique concrete type for each variable.
- If at least one line is **red**, the function is **type unstable**. It reflects that one variable or more could potentially adopt multiple possible types.
- **Yellow** lines indicate type instabilities that the compiler can handle effectively (in the sense that they have a reduced impact on performance). As a rule of thumb, **you can safely ignore them**.

> **Warning!**

> Throughout the website, we'll refer to **type instabilities** as those indicated by a red warning exclusively. Yellow warnings will be mostly ignored.

In the provided example, the compiler attempts to infer concrete types. This is done by identifying two pieces of information, given `x`'s concrete type:

i) the type of `y`,

ii) the type of `y * i` where `i` has type `Int64`, implicitly defining the type of `[y * i for i in 1:100]`.

The example clearly demonstrates that **the same function can be type stable or unstable depending on the types of its inputs**: `foo` is type stable when `x` has type `Int64`, but type unstable when `x` is `Float64`.

Specifically, in the scenario where `x = 1`, the compiler infers for *i)* that `y` can be equal to either `0` or `x`. Since both `0` and `1` are `Int64`, the compiler identifies a unique type for `y`, given by `Int64`. Regarding *ii)*, `y * i` also yields an `Int64`, as both `i` and `y` have type `Int64`. This determines that `[y * i for i in 1:100]` has type `Vector{Int64}`. Consequently, `foo(1)` is type stable, enabling Julia to invoke a method specialized for integers.

As for `x = 1.0`, the information for *i)* is that `y` could be either `0` or `1.0`. As a result, the compiler can't infer a unique type for `y`, which could be either `Int64` or `Float64`. The `@code_warntype` macro reflects this, identifying `y` as having type `Union{Float64, Int64}`. This ambiguity affects *ii)*, forcing the compiler to consider approaches that handle both `Float64` and `Int64`, and hence preventing specialization. Overall, `foo(1.0)` is type unstable, which has a detrimental impact on performance.

> **Remark**
>
> The conclusions regarding type stability wouldn't have changed if we had considered, for instance, `foo(-2)` or `foo(-2.0)`. This is because the compilation process relies on information about types, not values. More specifically, this means that type stability depends on whether `x` has type `Int64` or `Float64`, regardless of its actual value.

YELLOW WARNINGS MAY TURN RED

Not all instances of type instabilities have the same impact on performance. Their severity is ultimately indicated through a yellow or red warning. Yellow warnings denote a relatively minor impact on performance, typically resulting from isolated computations that Julia can handle effectively. However, repeated execution of these operations may escalate into more serious performance issues, triggering a red warning. The following example demonstrates a scenario like this.

```
function foo(x)
    y = (x < 0) ? 0 : x

    y * 2
end
```

```
julia> @code_warntype foo(1.)
```

```
function foo(x)
    y = (x < 0) ? 0 : x

    [y * i for i in 1:100]
end
```

```
julia> @code_warntype foo(1.)
```

```
function foo(x)
    y = (x < 0) ? 0 : x

    for i in 1:100
      y = y + i
    end

    return y
end
```

```
julia> @code_warntype foo(1.)
```

The yellow warning reflects that `y * 2` could return either a `Float64` or `Int64` value. However, this operation is computed only once and based on two types that the compiler can handle efficiently. Instead, the second tab involves multiple computations `y * i` without knowledge of a unique concrete type for `y`, resulting in a red warning.

Despite this, note that not all yellow warnings will necessarily escalate to a red warning when incorporated into a for-loop. The third tab illustrates this point, reinforcing that not all type instabilities are equally detrimental.

> **For-Loops and Yellow Warnings**
>
> When running for-loops, a yellow warning will always be displayed, even if the operation is type stable. The warning can safely be disregarded in such cases, as it simply reflects the inherent behavior of iterators: they return either the next element to iterate over or `nothing` (a value with type `Nothing`) when the sequence is exhausted.

```julia
function foo()
    for i in 1:100
        i
    end
end
```

```
julia> @code_warntype foo()
```

8c. Type Stability with Scalars and Vectors

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INTRODUCTION

The previous section has defined type stability, along with approaches to checking whether the property holds. In this section, we start the analysis of type stability for specific objects. We cover in particular the case of scalars and vectors, providing practical guidance for achieving type stability with them.

TYPES OF SCALARS AND VECTORS

Recall that the formal definition of a type-stable function is that the function's output type can be inferred from its argument types. In practice, however, we often rely on a more stringent definition, which requires that the compiler can infer a single concrete type for each expression within the function body. This property guarantees that every operation is specialized, resulting in optimal performance. Nevertheless, simply demanding that the output's type can be inferred already offers benefits, as it ensures that type instability won't be propagated when the function is called in other operations.

The principle applied to scalars is straightforward, demanding operations be performed on variables with the same concrete type (e.g., `Float64`, `Int64`, `Bool`). In contrast, type stability for vectors rather requires that the *elements* have a concrete type. The following table identifies scalars and vectors satisfying this property.

Objects Whose Elements Have Concrete Types

Scalars	Vectors
`Int`	`Vector{Int}`
`Int64`	`Vector{Int64}`
`Float64`	`Vector{Float64}`
`Bool`	`BitVector`

Note: `Int` defaults to `Int64` or `Int32`, depending on your CPU's architecture.

Next, we'll delve into type stability in scalars and vectors, considering each case separately.

TYPE STABILITY WITH SCALARS

To make the definition of type stability for scalars operational, let's revisit some concepts about types. Recall that only concrete types like `Int64` or `Float64` can be instantiated, while abstract types like `Any` or `Number` can't.

Instantiation simply means that all values ultimately adopt a unique concrete type. For instance, a variable `x::Number = 2` shouldn't be interpreted as `x` having the type `Number`. Instead, it means that `x` can only be reassigned to values whose concrete type is a subtype of `Number`. Ultimately, `x` must have a concrete type, which in this case is `Int64`.

In this context, type instability may arise when operations mix `Int64` and `Float64`, although this isn't always the case. To illustrate this, we'll start showing some scenarios where mixing these types doesn't cause issues.

TYPE PROMOTION AND CONVERSION

Julia employs various mechanisms to handle cases combining `Int64` and `Float64`. The first one is part of a concept known as **type promotion**, which converts dissimilar types to a common one whenever possible. The second one emerges when variables are type-annotated, in which case Julia engages in **type conversion**. By transforming values to the respective type declared, this feature also prevents the mix of types.

Both mechanisms are illustrated below.

```
foo(x,y)   = x * y

x          = 2
y          = 0.5

z             = foo(x,y)          # type stable: mixing `Int64` and `Float64` results in
`Float64`
```

```
julia> z
1.0
```

```
foo(x,y)   = x * y

x::Float64 = 2               # this is converted to `2.0`
y          = 0.5

z           = foo(x,y)         # type stable: `x` and `y` are `Float64`, so predictable type
of output
```

```
julia> z
1.0
```

In the first tab, the output's type depends on the argument's types. However, in all cases the output's type can be predicted, since mixing `Int64` and `Float64` results in `Float64` due to automatic type promotion. As for the second tab, Julia transforms the value of `x` to make it consistent with the type-annotation declared. Consequently, `x * y` is computed as the product of two values with type `Float64`.

TYPE INSTABILITY WITH SCALARS

While type promotion and conversion can handle certain situations, they certainly don't cover all cases. One such scenario is when a scalar's value depends on a conditional statement and each branch returns a value of a different type. In this situation, since the compiler only considers the types and not values, it can't determine which branch is relevant for the function call. As a result, it'll generate code that accommodates both possibilities, as it happens in the following example.

```
function foo(x,y)
    a = (x > y) ?  x  :  y

    [a * i for i in 1:100_000]
end

foo(1, 2)           # type stable   -> `a * i` is always `Int64`
```

```
julia> @btime foo(1,2)
  23.800 µs (2 allocations: 781.30 KiB)
```

```
function foo(x,y)
    a = (x > y) ?  x  :  y

    [a * i for i in 1:100_000]
end

foo(1, 2.5)         # type UNSTABLE -> `a * i` is either `Int64` or `Float64`
```

```
julia> @btime foo(1,2.5)
  43.200 µs (2 allocations: 781.30 KiB)
```

In the example, type instability will inevitably arise if `x` and `y` have different types. Note that type promotion is of no help here. The reason is that this mechanism only ensures that `a * i` will be converted to `Float64` if `a` is `Float64`, considering that `i` is `Int64`. However, the compiler also needs to consider the possibility that `a` could be `Int64`, in which case `a * i` would be `Int64`.

Given this ambiguity, the method instance created must be capable of handling both scenarios. Then, during runtime, Julia will gather more information to disambiguate the situation, and select the relevant computation implementation.

TYPE STABILITY WITH VECTORS

Vectors in Julia are formally defined as collections of elements sharing a homogeneous type. Since operations based on vectors ultimately handle individual elements, type stability is contingent on whether the type of their elements is concrete.

In this context, it's important to distinguish between the type of the object and of its elements. This is because vectors having elements with a concrete type are themselves concrete, but elements with abstract types will still give rise to vectors with concrete types. This is clearly observed with `Vector{Any}`, a concrete type comprising elements with the abstract type `Any`.

Before the analysis of specific scenarios, we start by considering type conversion applied to vectors. This mechanism prevents the mix of types when vectors are defined.

TYPE PROMOTION AND CONVERSION

By definition, vectors require all their elements to share the same type. This means that if you mix elements with disparate types, such as `String` and `Int64`, Julia will infer the vector's type as `Vector{Any}`. Despite this, there are cases where elements can be converted to a common type, such as when mixing `Float64` and `Int64`.

The following example shows this mechanism in an assignment, where the vector is not type annotated. In this case, all elements are converted to the most general type among the values included.

```
x = [1, 2, 2.5]      # automatic conversion to `Vector{Float64}`
```

```
julia> x
3-element Vector{Float64}:
 1.0
 2.0
 2.5
```

```
y = [1, 2.0, 3.0]    # automatic conversion to `Vector{Float64}`
```

```
julia> y
3-element Vector{Float64}:
 1.0
 2.0
 3.0
```

When assignments are instead declared with type-annotations and values are of different types, Julia will attempt to perform a conversion. If possible, this ensures that the assigned values conform to the declared type.

```
x1                 = [1, 2.0, 3.0]     # automatic conversion to `Vector{Float64}`

x2::Vector{Int64}  = y1                # conversion to `Vector{Int64}`
```

```
julia> z2
3-element Vector{Number}:
 1.0
 2.0
 2.5
```

```
y1                 = [1, 2, 2.5]       # automatic conversion to `Vector{Float64}`

y2::Vector{Number} = y1                # `y2` is still `Vector{Number}`
```

```
julia> z2
3-element Vector{Number}:
 1.0
 2.0
 2.5
```

```
nr_elements  = 3
z            = Vector{Any}(undef, nr_elements)     # `Vector{Any}` always

z           .= 1
```

```
julia> v
3-element Vector{Any}:
 1
 1
 1
```

TYPE INSTABILITY

When evaluating type stability with vectors, two forms of operations must be considered. The first one involves operations that manipulate individual elements, such as `x[i]`. This scenario is analogous to the case of scalars, and therefore type stability follows the same rules.

The second scenario involves functions operating on the entire vector. In this case, type stability requires that vectors have elements with a concrete type. Note that this condition isn't sufficient to guarantee type stability, which ultimately depends on how the function implements the operation executed.

Nevertheless, packages tend to provide optimized versions of functions. Consequently, functions are typically type stable when users provide vectors with elements of a concrete type. For instance, this is illustrated below by the function `sum`, which adds all elements in a vector.

```
x1::Vector{Int}     = [1, 2, 3]

sum(x1)             # type stable
```

```
x2::Vector{Int64}   = [1, 2, 3]

sum(x2)             # type stable
```

```
x3::Vector{Float64} = [1, 2, 3]

sum(x3)             # type stable
```

```
x4::BitVector       = [true, false, true]

sum(x4)             # type stable
```

In contrast, the following vectors have elements with abstract types, which result in type instability.

```
x5::Vector{Number} = [1, 2, 3]

sum(x5)                 # type UNSTABLE -> `sum` must consider all possible subtypes of
`Number`
```

```
x6::Vector{Any}    = [1, 2, 3]

sum(x6)             # type UNSTABLE -> `sum` must consider all possible subtypes of `Any`
```

8d. Type Stability with Global Variables

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INTRODUCTION

Variables can be categorized as local or global according to the code block in which they live: **global variables** can be accessed and modified anywhere in the code, while **local variables** are only accessible within a specific scope. For this section, the scope of interest is a function, so local variables will exclusively refer to function arguments and variables defined within the function body.

The distinction is especially relevant for this chapter, since **global variables are a common source of type instability**. The reason is that Julia doesn't assign concrete types to global variables. As a result, the compiler is forced to consider multiple possibilities during any computation that involves these variables. Such behavior prevents specialization, leading to reduced performance.

The current section explores two approaches to working with global variables: type-annotations and constants. Defining global variables as constants is a natural choice when values are truly fixed, such as in the case of $\pi = 3.14159$. More broadly, constants can be used in any scenario where they remain unmodified throughout the script. Compared to type annotations, constants offer better performance, as the compiler gains knowledge of *both* the type and value, rather than just the type. This feature allows for further optimizations, effectively making **the behavior of constants within a function indistinguishable from that of a literal value**. [1]

> **Warning!** - You Should Still Wrap Code in a Function
>
> Even if you implement the fixes proposed for global variables, optimal performance still calls for wrapping tasks in functions. The reason is that **functions implement additional optimizations** that aren't possible in the global scope.

WHEN ARE WE USING GLOBAL VARIABLES?

Before exploring approaches for handling global variables, let's first identify scenarios in which global variables arise. To this end, we present two cases, each represented in a different tab below. The first one considers the simplest scenario possible, where operations are performed directly in the global scope. For its part, the second one illustrates a more nuanced case, where a function accesses and operates on a global variable.

The third tab serves as a counterpoint, implementing the same operations but within a self-contained function. By definition, self-contained functions exclusively operate with locally defined variables. Thus, the comparison of the last two tabs highlights the performance lost by relying on global variables.

```
# all operations are type UNSTABLE (they're defined in the global scope)
x = 2

y = 2 * x
z = log(y)
```

```
x = 2

function foo()
    y = 2 * x
    z = log(y)

    return z
end

@code_warntype foo() # type UNSTABLE
```

```
x = 2

function foo(x)
    y = 2 * x
    z = log(y)

    return z
end

@code_warntype foo(x)  # type stable
```

Self-contained functions offer advantages that extend beyond performance gains: they **enhance readability, predictability, testability, and reusability**. These benefits were briefly considered in a previous section, and come from an interpretation of functions as embodying a specific task.

Among other benefits, self-contained functions are easier to reason about, as understanding their logic doesn't require tracking variables across the entire script. Moreover, a function's output depends solely on its input parameters, without any dependence on the script's state regarding global variables. This makes self-contained functions more predictable, additionally simplifying the code debugging process. Finally, by acting as a standalone program with a clear well-defined purpose, self-contained functions can be reapplied for similar tasks, reducing code duplication and facilitating code maintainability.

ACHIEVING TYPE STABILITY WITH GLOBAL VARIABLES

The previous subsection emphasized the benefits of self-contained functions, providing compelling reasons to avoid global variables. Nonetheless, global variables can still be highly convenient in certain scenarios. For instance, this is the case when we work with true constants. Considering this, next we present two approaches that let us work with global variables, while addressing their performance penalty.

CONSTANT GLOBAL VARIABLES

Declaring global variables as constants requires adding the `const` keyword before the variable's name, such as in `const x = 3`. This approach can be applied to variables of any type, including collections.

```
const a = 5
foo()   = 2 * a

@code_warntype foo()        # type stable
```

```
const b = [1, 2, 3]
foo()   = sum(b)

@code_warntype foo()        # type stable
```

Warning! - Avoid Reassignments to Global Variables

Global variables should be declared constants only if their values will remain unchanged throughout the session. Although it's possible to redefine constants, doing so is discouraged. The feature was only introduced to facilitate testing in interactive sessions, eliminating the need to restart Julia after each modification of a constant's value.

Importantly, if you use this option, all functions depending on the constant changed must be re-declared afterwards: any function that isn't redefined will still rely on the constant's original value. Given this requirement, the safest approach is to rerun the entire script after any change.

To illustrate the potential consequences of overlooking this practice, let's compare the following code snippets that execute the function `foo`. Both define a constant value of `x=1`, which is subsequently redefined as `x=2`. The first example runs the script without re-executing the definition of `foo`, in which case the value returned by `foo` is still based on `x = 1`. In contrast, the second example emulates the re-execution of the entire script. This is achieved by rerunning `foo`'s definition, thus ensuring that `foo` relies on the updated value of `x`.

```
const x = 1
foo()   = x
foo()             # it gives 1

x       = 2
foo()             # it still gives 1
```

```
const x = 1
foo()   = x
foo()             # it gives 1

x       = 2
foo()   = x
foo()             # it gives 2
```

TYPE-ANNOTATING A GLOBAL VARIABLE

The second approach to address type instability involves declaring *concrete* types for global variables. This is done by including the operator `::` after the variable's name (e.g., `x::Int64`). For vectors, note that the elements must also have a concrete type. Otherwise, type stability won't be achieved.

```
x::Int64            = 5
foo()               = 2 * x

@code_warntype foo()     # type stable
```

```
y::Vector{Float64} = [1, 2, 3]
foo()               = sum(y)

@code_warntype foo()     # type stable
```

```
z::Vector{Number}  = [1, 2, 3]
foo()               = sum(z)

@code_warntype foo()     # type UNSTABLE
```

DIFFERENCES BETWEEN APPROACHES

The two approaches presented for handling global variables have different implications for both code behavior and performance. The key lies in that **type-annotations assert a variable's type, while constants additionally declare its value**. Next, we analyze the main differences between both approaches.

DIFFERENCES IN CODE

Unlike the case of constants, type-annotations allow you to reassign a global variable without unexpected consequences. This means you don't need to re-run the entire script when redefining a variable.

```
x::Int64 = 5
foo()    = 2 * x
foo()               # output is 10

x        = 2
foo()    = 2 * x
foo()               # output is 4
```

DIFFERENCES IN PERFORMANCE

Type-annotated global variables are more flexible than constants: they require only a declaration of types, without binding to a specific value. This flexibility, however, comes at a performance cost. The reason is that constants not only convey type information, but also act as a promise of immutability throughout the program. As a result, constants behave like literal values embedded directly in code. This allows the compiler to optimize more aggressively. For example, by replacing certain expressions with their precomputed results.

The following code demonstrates this behavior. It performs an operation that can be precomputed if the value of the global variable is known at compile time. Declaring the global variable as a constant allows the compiler to replace the operation with its result, effectively treating it as a hard-coded value. In contrast, merely type-annotating the global variable constrains only its type, without fixing its value. To make the performance difference more evident, we call this operation repeatedly inside a for-loop.

```
const k1  = 2

function foo()
    for _ in 1:100_000
       2^k1
    end
end
```

```
julia> @btime foo()
  0.800 ns (0 allocations: 0 bytes)
```

```
k2::Int64 = 2

function foo()
    for _ in 1:100_000
       2^k2
    end
end
```

```
julia> @btime foo()
  115.600 µs (0 allocations: 0 bytes)
```

Invariance of Operations

Even without declaring variables as constants, the compiler could still recognize the invariance of some operations across repeated calculations. In such cases, it computes the operation once and reuses the result whenever needed.

To illustrate, consider reexpressing each element of `x` as a proportion relative to the sum of elements. A naive implementation would involve a for-loop with `sum(x)` inside the for-loop body, causing `sum(x)` to be recomputed on every iteration. By contrast, when shares are computed through `x ./ sum(x)`, the compiler is smart enough to recognize the invariance of `sum(x)` across iterations. Therefore, it proceeds to its pre-computation, eliminating redundant work.

```
x           = rand(100_000)


foo(x) = x ./ sum(x)
```

```
julia> @btime foo($x)
  49.400 µs (2 allocations: 781.30 KiB)
```

```
x           = rand(100_000)
const sum_x = sum(x)

foo(x) = x ./ sum_x
```

```
julia> @btime foo($x)
  41.500 µs (2 allocations: 781.30 KiB)
```

```
x           = rand(100_000)


function foo(x)
    y    = similar(x)

    for i in eachindex(x,y)
        y[i] = x[i] / sum(x)
    end

    return y
end
```

```
julia> @btime foo($x)
  633.245 ms (2 allocations: 781.30 KiB)
```

FOOTNOTES

[1.] Literal values refer to values expressed directly in the code (e.g., `1`, `"hello"`, or `true`), in contrast to values coming from a variable input.

8e. Barrier Functions

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INTRODUCTION

This section presents an approach to mitigating type instability based on the so-called **barrier functions**. These are defined as type-stable functions embedded within a type-unstable function, where variables of uncertain type are passed as arguments. This design prompts the compiler to infer concrete types for those variables, effectively creating a "barrier" that prevents the propagation of type instability to subsequent operations.

A key benefit of this technique is that **barrier functions are agnostic to the underlying source of type instability**, making them widely applicable across scenarios.

> **Warning!** - Barrier Functions Should Be Considered as a Second Option
>
> Barrier functions are preferred for situations where type instability is either difficult to fix or inherent to the operations performed. Keep in mind that the original function will remain type unstable, entailing different consequences depending on the instability nature. For this reason, it's best to aim for type-stable code from the outset, whenever possible.

APPLYING BARRIER FUNCTIONS

To illustrate the technique, let's revisit a type-unstable function from a previous section. This defines a variable `y` based on `x`, and subsequently performs an operation involving `y`.

```
function foo(x)
    y = (x < 0) ?  0  :  x

    [y * i for i in 1:100]
end

@code_warntype foo(1)        # type stable
@code_warntype foo(1.)       # type UNSTABLE
```

In the example, `0` is an `Int64`, whereas `x` could be either an `Int64` or `Float64`. This leads to two possibilities:

- `x` is an `Int64`: then, `y` will also be an `Int64`, making `foo(1)` type stable.

- `x` is a `Float64`: the compiler then can't determine whether `y` will be an `Int64` or a `Float64`, rendering `foo(1.)` type unstable.

A barrier function can address the type instability of the second case. It requires embedding a type-stable function into `foo`, passing `y` as an argument. The function will then attempt to deduce `y`'s type, allowing the compiler to use this information for subsequent operations. The example below defines `operation` as a barrier function. [1]

```
operation(y) = [y * i for i in 1:100]

function foo(x)
    y = (x < 0) ?  0  :  x

    operation(y)
end

@code_warntype operation(1)     # barrier function is type stable
@code_warntype operation(1.)    # barrier function is type stable

@code_warntype foo(1)           # type stable
@code_warntype foo(1.)          # barrier-function solution
```

With the introduction of `operation`, the variable `y` in `foo(1.)` can still be either an `Int64` or a `Float64`. Nevertheless, this ambiguity no longer matters, as `operation(y)` will determine the type of `y` before the array comprehension is executed. As a result, the expression `[y * i for i in 1:100]` will be computed with a method specialized for the specific type of `y`, ensuring type stability.

Warning!

Barrier Functions should address the type instability *before* the type-unstable operation is executed. Otherwise, we're back to the original issue, where the compiler has to check `y`'s type at each iteration and select a method accordingly.

For example, `foo` in the example below doesn't apply the technique correctly: `y` can be either `Float64` or `Int64`, but `operation(y,i)` only identifies the type inside the for-loop. Thus, the compiler is forced to check `y`'s type at each iteration, which is the original problem we intended to solve.

```
operation(y,i) = y * i

function foo(x)
    y = (x < 0) ?  0  :  x

    [operation(y,i) for i in 1:100]
end

@code_warntype foo(1)          # type stable
@code_warntype foo(1.)         # type UNSTABLE
```

REMARKS ON @CODE_WARNTYPE

Functions introducing barrier functions hinder the interpretation of `@code_warntype`. This is because barrier functions typically mitigate type instability, rather than completely eliminating it. And even if the barrier function successfully eliminates the type instability, a red warning may still be triggered.

To illustrate this, let's start presenting a scenario where the barrier function completely eliminates the type instability. Yet, a red warning shows up.

```
x = ["a", 1]                    # variable with type 'Any'



function foo(x)
    y = x[2]

    [y * i for i in 1:100]
end
```

```
julia> @code_warntype foo(x)
```

```
x = ["a", 1]                    # variable with type 'Any'

operation(y) = [y * i for i in 1:100]

function foo(x)
    y = x[2]

    operation(y)
end
```

```
julia> @code_warntype foo(x)
```

In this example, `y` is defined from an object with type `Vector{Any}`. This leads to a red warning, as `x[2]` has type `Any` and therefore the compiler can't infer a concrete type for `y`. However, no operation is involved at that point, as we're only performing an assignment. Since the only operation performed uses a barrier function, the lack of type information is inconsequential. Overall, type instability is never impacting performance after introducing a barrier function.

In contrast, the example below demonstrates that a barrier function may only alleviate type instability, rather than eliminate it entirely. In this scenario, the operation `2 * x[2]` is type unstable, forcing the compiler to generate code for each possible concrete type of `x[2]`. Nonetheless, this operation has a negligible performance impact on `foo`, justifying why the barrier function only targets the more demanding operation.

```
x = ["a", 1]                    # variable with type 'Any'



function foo(x)
    y = 2 * x[2]

    [y * i for i in 1:100]
end
```

```
julia> @code_warntype foo(x)
```

```
x = ["a", 1]                    # variable with type 'Any'

operation(y) = [y * i for i in 1:100]

function foo(x)
    y = 2 * x[2]

    operation(y)
end
```

```
julia> @code_warntype foo(x)
```

```
x = ["a", 1]                    # variable with type 'Any'

operation(y) = [y * i for i in 1:100]

function foo(z)
    y = 2 * z

    operation(y)
end
```

```
julia> @code_warntype foo(x[2])
```

The effectiveness of a barrier function ultimately hinges on how the function `foo` will be applied. In the given example, the barrier-function solution would be sufficient if `foo` is called only once. Instead, if `foo` is eventually called in a tight loop, the type instability of `2 * x[2]` would be incurred multiple times. In such cases, simultaneously addressing the type instability in `2 * x[2]` could lead to substantial performance benefits.

FOOTNOTES

[1.] In this particular example, there's an easier solution, where `0` is substituted with `zero(x)`. The function `zero(x)` has been designed to return the additive identity (i.e., the null element) of `x`'s type.

8f. Type Stability with Tuples

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INTRODUCTION

A function is considered type stable when, given the types of its arguments, the compiler can accurately predict single concrete types for its expressions. This definition, while universal, takes on different forms when applied to specific objects. So far, we've exclusively concentrated on scalars and vectors, whose conditions for type stability are relatively straightforward.

In this section, we begin our analysis of type stability for other data structures. In particular, we consider tuples. Note that the coverage of tuples automatically encompasses the case of **named tuples**, which are merely tuples having symbols as keys.

Guaranteeing type stability with tuples is more nuanced compared to vectors, as their type characterization demands more information. Its exploration will challenge our understanding of type stability, demanding a clear grasp of its definition and subtleties.

> **Warning! -** Tuples Are Only Suitable For Small Collections
> Remember that tuples should only be used for collections that comprise a few elements. Using them for large collections will result in significant performance degradation or directly trigger fatal errors.

COMPARING TUPLES AND VECTORS

Tuples and vectors are the most common forms of collections in Julia. While both fulfill a similar purpose, they differ significantly in their underlying implementation. In particular, tuples tend to outperform vectors when working with small objects, by avoiding the memory-allocation overhead incurred by vectors. This advantage will be explained in more depth when discussing static vectors, which are essentially tuples that can be manipulated as vectors.

Another key distinction is that tuples possess a more intricate type system in comparison to vectors. To see this, let's compare the information needed to describe each type.

Vectors represent collections of elements sharing a *homogeneous* type and exhibiting a variable size. Thus, the information needed to describe the types of vectors is relatively minor. For instance, a type like `Vector{Float64}` establishes that *all* elements must have type `Float64`, without any restriction on the number of elements to be contained.

For their part, tuples are fixed-size collections that can accommodate *heterogeneous* types. This makes the characterization of a tuple's type more demanding, requiring both the number of elements and the type of *each* element. For instance, the variable `tup = ("hello", 1)` has type `Tuple{String, Int64}`, indicating that the first element has type `String` and the second one `Int64`. Furthermore, it implicitly sets the number of elements to two, as there's no possibility of appending or removing elements.

The fact that the number of elements is part of the type becomes clear when tuples contain `N` elements of the same type `T`. For this case, Julia provides the convenient alias `Ntuple{N, Float64}`, which is just syntactic sugar for `Tuple{T, T,...,T}` where `T` appears `N` times. [1]

In the following, we show that the choice between tuples and vectors may have different implications for type stability.

TUPLE SLICES WITH MIXED TYPES CAN STILL BE TYPE STABLE

One key difference between tuples and vectors in Julia lies in how they handle type information. While tuples explicitly define the type of each individual element, vectors require all elements to be of a uniform type.

Because vectors must maintain a consistent type throughout, attempting to store mixed concrete types within a single vector compels Julia to determine a common type that accommodates them all. For example, if you create a vector containing both `Int64` and `Float64`, Julia will infer the type of the vector as `Vector{Float64}`, the most general type encompassing both integer and float types.

However, when dealing with highly diverse element types within a vector, this process can lead to less efficient behavior. In extreme cases, Julia might resort to using the abstract type `Any`, resulting in a `Vector{Any}`. Working with vectors like this is extremely undesirable from a performance point of view.

This issue particularly affects vector slices, as they inherit the type information from their parent vector. Thus, if the parent vector has been widened to a more general type like `Vector{Any}`, operations performed on those slices will also be subject to that same type instability. The behavior contrasts sharply with **slices of tuples, where each element within the slice retains its concrete type.**

TUPLE

```julia
tup    = (1, 2, "hello")        # type is `Tuple{Int64, Int64, String}`

foo(x) = sum(x[1:2])

@code_warntype foo(tup)         # type stable (output is `Int64`)
```

VECTOR

```
vector = [1, 2, "hello"]        # type is `Vector{Any}`

foo(x) = sum(x[1:2])

@code_warntype foo(vector)      # type UNSTABLE
```

TUPLES CONTAIN MORE INFORMATION THAN VECTORS

Given the differences in type information, conversions between tuples and vectors can pose several challenges for type stability.

To see this, let's start with the simplest case, where a tuple is converted into a vector. The outcome of this conversion is predictable, stemming directly from our previous analysis: type stability will be preserved when the tuple contains all elements having the same type or when heterogeneous types can be promoted to a common concrete type.

For the examples, recall that **each type automatically generates a function** that transforms variables into the corresponding type. For instance, the function `Vector` converts variables to this type.

TYPE-HOMOGENEOUS TUPLES

```
tup = (1, 2, 3)                 # `Tuple{Int64, Int64, Int64}` or just `NTuple{3, Int64}`


function foo(tup)
    x = Vector(tup)             # 'x' has type `Vector(Int64)}`
    sum(x)
end

@code_warntype foo(tup)         # type stable
```

TYPE PROMOTION

```
tup = (1, 2, 3.5)               # `Tuple{Int64, Int64, Float64}`


function foo(tup)
    x = Vector(tup)             # 'x' has type `Vector(Float64)}`
    sum(x)
end

@code_warntype foo(tup)         # type stable
```

TYPE-HETEROGENEOUS TUPLES

```
tup = (1, 2, "hello")          # `Tuple{Int64, Int64, String}`


function foo(tup)
    x = Vector(tup)            # 'x' has type `Vector(Any)}`
    sum(x)
end

@code_warntype foo(tup)        # type UNSTABLE
```

Likewise, **creating a tuple from a vector will inevitably cause type instability**, regardless of the vector's characteristics. The reason is that vectors don't store information about the number of elements they contain. Consequently, when attempting to construct a tuple from a vector, the compiler must account for the possibility of varying numbers of arguments. The result is that each potential number of elements corresponds to a distinct concrete type for the tuple.

VECTOR WITH NON-PRIMITIVE TYPES

```
x   = [1, 2, "hello"]            # 'Vector{Any}' has no info on each individual type


function foo(x)
    tup = Tuple(x)               # 'tup' has type `Tuple`

    sum(tup[1:2])
end

@code_warntype foo(x)            # type UNSTABLE
```

VECTOR WITH PRIMITIVE TYPES

```
x   = [1, 2, 3]                  # 'Vector{Int64}' has no info on the number of elements


function foo(x)
     tup = Tuple(x)                 # 'tup' has type `Tuple{Vararg(Int64)}` (`Vararg` means
"variable arguments")

    sum(tup[1:2])
end

@code_warntype foo(x)            # type UNSTABLE
```

ADDRESSING VARIABLE ARGUMENTS: DISPATCH BY VALUE

A key takeaway from the previous subsection is that defining tuples from vectors invariably introduces type instability. A simple remedy for this is to convert tuples outside the function, which we then pass as function arguments. This is demonstrated in the code snippet below.

TUPLE AS A FUNCTION ARGUMENT

```
x   = [1, 2, 3]
tup = Tuple(x)

foo(tup) = sum(tup[1:2])

@code_warntype foo(tup)         # type stable
```

The approach presented should be your first option when transforming vectors to tuples. Nonetheless, there may be scenarios where defining the tuple inside the function is unavoidable. In such cases, there are a few alternatives.

Note first that simply passing the vector's number of elements as a function argument doesn't solve the issue. The reason is that the compiler generates method instances based on information about types, not values. This means that a function argument like `length(x)` merely informs the compiler that the number of elements can be described as an object with type `Int64`, without providing any additional insight.

Instead, one effective solution is to define the tuple's length using a literal value, as demonstrated below.

NOT A SOLUTION

```
x   = [1, 2, 3]

function foo(x)
    tup = NTuple{length(x), eltype(x)}(x)

    sum(tup)
end

@code_warntype foo(x)        # type UNSTABLE
```

INFLEXIBLE SOLUTION

```
x   = [1, 2, 3]

function foo(x)
    tup = NTuple{3, eltype(x)}(x)

    sum(tup)
end

@code_warntype foo(tup)       # type stable
```

The downside of this solution is that it defeats the purpose of having generic code, as it restricts the function to tuples of a single predetermined size. To eliminate the type instability without constraining functionality, we need to introduce a more advanced solution. This is based on a technique known as **dispatch by value**. Since this approach is more complex to implement, *I recommend using it only when passing the tuple as a function argument is unfeasible*.

Next, we lay out the principles of dispatch by value, and then apply the technique to the specific case of tuples.

DEFINING DISPATCH BY VALUE

Dispatch by value enables passing information about values to the compiler. Nonetheless, implementing this feature requires a workaround, since the compiler only gathers information about types. The hack consists of creating a type that stores values as type parameters. In the case of tuples, this type parameter is simply the vector's number of elements.

The functionality is implemented via the built-in type `Val`, whose use is best explained through an example. Suppose a function `foo` and a value `a` that you wish the compiler to know. The technique requires defining `foo` with a type-annotated argument having no name, `::Val{a}`. After this, you must call `foo` passing an argument `Val(a)`, which instantiates a type with parameter `a`.

To illustrate the use of `Val`, we revisit an example included in previous sections. This considers a variable `y` that could be an `Int64` or `Float64`, contingent upon a condition. The ambiguity of `y`'s type is then transmitted to any subsequent operation, leading to type instability.

Dispatch by value is implemented by defining the condition as a type parameter of `Val`. In this way, the compiler will receive information about whether the condition is `true` or `false`, and therefore have knowledge about `y`'s type. This makes it possible to specialize its operations.

TYPE UNSTABLE

```julia
function foo(condition)
    y = condition ? 1 : 0.5      # either `Int64` or `Float64`

    [y * i for i in 1:100]
end

@code_warntype foo(true)         # type UNSTABLE
@code_warntype foo(false)        # type UNSTABLE
```

SOLUTION VIA "VAL"

```julia
function foo(::Val{condition}) where condition
    y = condition ? 1 : 0.5      # either `Int64` or `Float64`

    [y * i for i in 1:100]
end

@code_warntype foo(Val(true))    # type stable
@code_warntype foo(Val(false))   # type stable
```

> **Warning!**
> The function argument `Val` must be defined with `{}`, as types define their parameters with `{}`. Instead, `Val` must be called with `()` as it has to be called as any other function.

DISPATCHING BY VALUE WITH TUPLES

Let's now revisit the conversion of vectors to tuples. As we previously discussed, type instability arises because vectors don't store the size as part of their type information, leaving the compiler without sufficient information to determine the tuple's type.

Dispatch by value provides a solution to this issue: by passing the vector's length as a type parameter, the function call becomes type stable.

TYPE UNSTABLE

```
x = [1, 2, 3]

function foo(x, N)
    tuple_x = NTuple{N, eltype(x)}(x)

    2 .+ tuple_x
end

@code_warntype foo(x, length(x))        # type UNSTABLE
```

SOLUTION VIA "VAL"

```
x = [1, 2, 3]

function foo(x, ::Val{N}) where N
    tuple_x = NTuple{N, eltype(x)}(x)

    2 .+ tuple_x
end

@code_warntype foo(x, Val(length(x)))   # type stable
```

FOOTNOTES

[1.] Don't confuse `NTuple` as an abbreviation for the type `NamedTuples`. The "N" in the former case refers to a number "N" of elements.

8g. Type Stability with Higher-Order Functions

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INTRODUCTION

Functions in Julia are **first-class objects**, a concept also referred to as **first-class citizens**. This means that functions can be handled just like any other variable: we can define vectors of functions, have functions whose outputs are other functions, and do many more sophisticated things that would be impossible if functions were treated as a different entity.

In particular, the property makes it possible to define **higher-order functions**, which are functions that take another function as an argument. We've already worked with several higher-order functions, often in the form of anonymous functions passed as function arguments. A familiar example is `map(<function>, <collection>)`, which applies `<function>` to every element of `<collection>`.

In this section, the focus will be on conditions under which higher-order functions are type-stable. As we'll discover, these functions present some challenges in this regard.

> **Remark**
> Throughout the explanations, we'll often refer to the function passed as an argument as the *callback function*.

THE ISSUE

In Julia, *each function defines its own unique concrete type*. In turn, this concrete type is a subtype of an abstract type called `Function`. The type `Function` encompasses all possible functions defined in Julia. The design of the type system creates challenges when specializing the computation method of higher-order functions. Specifically, it can potentially lead to a combinatorial explosion of methods, where a unique method is generated for each callback function.

To address this issue, Julia takes a conservative stance, **often choosing not to specialize the methods of high-order functions**. In particular, we'll see that Julia avoids specialization if the callback function isn't explicitly called. The performance in those cases can drop sharply, as the execution runtime would become similar to performing operations in the global scope.

Given this, it's important to pinpoint the scenarios where specialization is inhibited and monitor its consequences. If you notice that performance is severely impaired, there are still ways to enforce specialization. In this section, we'll explore these strategies.

AN EXAMPLE OF NO SPECIALIZATION

Let's illustrate the conditions under which higher-order functions fail to specialize. Consider a scenario where the goal is to sum the transformed elements of a vector `x`. The only requirement imposed is that the transforming function should be generic, allowing us to possibly apply different functions for the transformation.

We implement this construction via a higher-order function `foo`. The function applies `map` to transform `x` through some function `f`, and then applies `sum` to add the transformed elements. To demonstrate how `foo` works, we call it with the function `abs` as the transformation function, which provides absolute values.

```
x = rand(100)

function foo(f, x)
    y = map(f, x)

    sum(y)
end
```

```
julia> @code_warntype foo(abs,x)
```

Even when `foo(abs,x)` isn't specialized, `@code_warntype` **fails to detect any type-stability issues**. This is a consequence of `@code_warntype` evaluating type stability *under the assumption that specialization is attempted*. In our example, this assumption doesn't hold and therefore `@code_warntype` is of no use.

Type instability arises because Julia **avoids specialization when a callback function isn't explicitly called within the function**. In the example, the function `f` only enters `foo` as an argument of `map`, but there's no explicit line calling `f`.

To obtain indirect evidence about the lack of specialization, we can compare the execution times of the original `foo` function with a version that explicitly calls `f`.

```
x = rand(100)

function foo(f, x)
    y = map(f, x)


    sum(y)
end
```

```
julia> foo(abs, x)
48.447
julia> @btime foo(abs, $x)
  195.579 ns (3 allocations: 928 bytes)
```

```julia
x = rand(100)

function foo(f, x)
    y = map(f, x)
    f(1)                # irrelevant computation to force specialization

    sum(y)
end
```

```
julia> foo(abs, x)
48.447

julia> @btime foo(abs, $x)
  45.745 ns (1 allocation: 896 bytes)
```

The comparison reveals a significant reduction in execution time when `f(1)` is added, along with a notable decrease in memory allocations. As we'll demonstrate in future sections, excessive allocations are often indicative of type instability.

FORCING SPECIALIZATION

> **Warning!**
> Exercise caution when forcing specialization. Overly aggressive specialization can degrade performance severely, explaining why Julia's default approach is deliberately conservative. In particular, you should avoid specialization when your script repeatedly calls a high-order function with many unique functions. [1]

Explicitly calling the callback function to circumvent the no-specialization issue isn't ideal, as it introduces an unnecessary computation. Fortunately, alternative solutions exist. One of them is to type-annotate `f`, which provides Julia with a hint to specialize the code for that type of function.

Another solution involves wrapping the function in a tuple before passing it as an argument. This ensures the identification of the function's type, as tuples define a concrete type for each of their elements.

Below, we outline both approaches.

```julia
x = rand(100)


function foo(f::F, x) where F
    y = map(f, x)


    sum(y)
end
```

```
julia> foo(abs, x)
48.447
julia> @btime foo(abs, $x)
  46.686 ns (1 allocation: 896 bytes)
```

```julia
x = rand(100)
f_tup = (abs,)

function foo(f_tup, x)
    y = map(f_tup[1], x)


    sum(y)
end
```

```
julia> foo(f_tup, x)
48.447
julia> @btime foo($f_tup, $x)
  45.101 ns (1 allocation: 896 bytes)
```

FOOTNOTES

[1.] For discussions about the issue of excessive specialization, see here and here.

8h. Gotchas for Type Stability

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INTRODUCTION

This section presents subtle scenarios where type instabilities arise. Since the root cause of the type instability isn't immediately obvious, we refer to these cases as "gotchas" and offer guidance on how to address them. To ensure self-containment, we revisit some examples previously discussed, providing additional recommendations for their mitigation.

GOTCHA 1: INTEGERS AND FLOATS

Remember that `Int64` and `Float64` are distinct types. Even though Julia promotes integers to floating-point numbers in many contexts, mixing them can still inadvertently introduce type instability.

To illustrate this, consider a function `foo` that takes a numeric variable `x` as its argument and performs two tasks. First, it defines a variable `y` that replaces `x`'s negative values with zero. Second, it executes an operation based on the resulting `y`.

In the following, we implement `foo` with an approach that suffers from type instability, and another one that addresses the issue.

```
function foo(x)
    y = (x < 0) ?  0  :  x

    return [y * i for i in 1:100]
end

@code_warntype foo(1)      # type stable
@code_warntype foo(1.)     # type UNSTABLE
```

```
function foo(x)
    y = (x < 0) ?  zero(x)  :  x

    return [y * i for i in 1:100]
end

@code_warntype foo(1)      # type stable
@code_warntype foo(1.)     # type stable
```

The first implementation uses the literal `0`, whose type is `Int64`. If `x` is also `Int64`, no type instability arises. However, if `x` is `Float64`, the compiler treats `y` as potentially `Int64` or `Float64`, thus causing type instability. [1]

Note, though, that Julia can generally handle combinations of `Int64` and `Float64` quite effectively. Thus, this type instability wouldn't be a significant problem if the operation calls `y` only once. Indeed, `@code_warntype` in this case would simply issue a yellow warning, indicating potential for optimization but not necessarily a severe performance bottleneck. However, `foo` in our example repeatedly performs an operation involving `y`, incurring the cost of type instability multiple times. As a result, `@code_warntype` issues a red warning, indicating a more serious performance issue.

The second tab proposes a **solution** for this scenario. It introduces a function that returns the zero element of `x`'s type, instead of `0`. In this way, `y` is created ensuring that types won't be mixed.

This approach to solving type instability can be extended to values different from zero, by use of the function `convert(typeof(x), <value>)` or `oftype(x, <value>)`. Both convert `<value>` to the same type as `x`. For instance, below we reimplement `foo` using the value `5` instead of `0`.

```
function foo(x)
    y = (x < 0) ?  5  :  x

    return [y * i for i in 1:100]
end

@code_warntype foo(1)      # type stable
@code_warntype foo(1.)     # type UNSTABLE
```

```
function foo(x)
    y = (x < 0) ?  convert(typeof(x), 5)  :  x

    return [y * i for i in 1:100]
end

@code_warntype foo(1)      # type stable
@code_warntype foo(1.)     # type stable
```

```
function foo(x)
    y = (x < 0) ?  oftype(x, 5)  :  x

    return [y * i for i in 1:100]
end

@code_warntype foo(1)      # type stable
@code_warntype foo(1.)     # type stable
```

GOTCHA 2: COLLECTIONS OF COLLECTIONS

In data analysis, it's common to work with collections of collections, where one structure contains others nested inside. A familiar example is the `DataFrames` package in Julia, which organizes data into columns representing different variables. Since we haven't introduced this package, we'll consider a simpler, but analogous case: a vector of vectors, whose type is `Vector{Vector}`.

The appeal of `Vector{Vector}` lies in its flexibility. Because the type doesn't constrain the contents of its inner vectors, it can represent heterogeneous data. Thus, we can create datasets that mix strings, floating-point numbers, and integers across columns.

However, this flexibility introduces a drawback. The type system only knows that each element is a vector, without knowing the concrete type of its contents. As a result, the compiler can't infer types when operating on these inner vectors, leading to type instability.

To make the issue more concrete, consider a vector `data` that contains several inner vectors. Suppose we define a function `foo` that takes `data` as its argument and performs some operation on one of its inner vectors, say `vec2`. The first tab below shows the compiler only knows that `vec2` is a vector, but it can't determine the concrete type of its elements. As a result, calls to `foo` suffer from type instability.

A straightforward way to **address** this problem is presented in the second tab. The solution consists of introducing a barrier function that takes the inner vector `vec2` as its argument. The barrier function rectifies the type instability by attempting to identify a concrete type for `vec2`.

```julia
vec1 = ["a", "b", "c"] ; vec2 = [1, 2, 3]
data = [vec1, vec2]

function foo(data)
    for i in eachindex(data[2])
        data[2][i] = 2 * i
    end
end

@code_warntype foo(data)            # type UNSTABLE
```

```julia
vec1 = ["a", "b", "c"] ; vec2 = [1, 2, 3]
data = [vec1, vec2]

foo(data) = operation!(data[2])

function operation!(x)
    for i in eachindex(x)
        x[i] = 2 * i
    end
end

@code_warntype foo(data)            # barrier-function solution
```

Note that the second tab defines the barrier function in-place. This means that the function directly modifies the contents of the inner vector `vec2`, rather than creating a new copy. Consequently, the outer structure `data` is updated as well. This in-place strategy is common in data analysis, where the goal is often to transform a dataset, instead of generating a new one each time its values are modified.

GOTCHA 3: BARRIER FUNCTIONS

Barrier functions are an effective technique to mitigate type instabilities. However, keep in mind that **the parent function may remain type unstable**. When this occurs and instability isn't resolved before executing a repeated operation, the associated performance penalty will be incurred multiple times.

To illustrate this point, let's revisit the last example involving a vector of vectors. Below, we present two incorrect approaches to using a barrier function, followed by a demonstration of its proper application.

```
vec1 = ["a", "b", "c"] ; vec2 = [1, 2, 3]
data = [vec1, vec2]

operation(i) = (2 * i)

function foo(data)
    for i in eachindex(data[2])
        data[2][i] = operation(i)
    end
end

@code_warntype foo(data)            # type UNSTABLE
```

```
vec1 = ["a", "b", "c"] ; vec2 = [1, 2, 3]
data = [vec1, vec2]

operation!(x,i) = (x[i] = 2 * i)

function foo(data)
    for i in eachindex(data[2])
        operation!(data[2], i)
    end
end

@code_warntype foo(data)            # type UNSTABLE
```

```
vec1 = ["a", "b", "c"] ; vec2 = [1, 2, 3]
data = [vec1, vec2]

function operation!(x)
    for i in eachindex(x)
        x[i] = 2 * i
    end
end

foo(data) = operation!(data[2])

@code_warntype foo(data)            # barrier-function solution
```

GOTCHA 4: INFERENCE IS BY TYPE, NOT BY VALUE

Julia's compiler generates method instances solely on the basis of types, without taking the actual values into account. To demonstrate this, consider the following example.

```
function foo(condition)
    y = condition ?  2.5  :  1

    return [y * i for i in 1:100]
end

@code_warntype foo(true)          # type UNSTABLE
@code_warntype foo(false)         # type UNSTABLE
```

At first glance, we might erroneously conclude that `foo(true)` is type stable: the value of `condition` is `true`, so that `y = 2.5` and therefore `y` will have type `Float64`. However, values don't participate in multiple dispatch, meaning that Julia's compiler ignores the value of `condition` when inferring `y`'s type. Ultimately, `y` is treated as potentially being either `Int64` or `Float64`, leading to type instability.

The issue in this case can be easily resolved by replacing `1` by `1.0`, thus ensuring that `y` is always `Float64`. More generally, we could employ similar techniques to the first "gotcha", where values are converted to a specific concrete type.

An alternative solution relies on dispatching by value, a technique we already explored and implemented for tuples. This technique makes it possible to communicate information about values directly to the compiler. It's based on the type `Val` in conjunction with the keyword `where` introduced here.

Specifically, for any function `foo` and value `a` that you seek the compiler to know, you need to include `::Val{a}` as an argument. In this way, `a` is interpreted as a type parameter, which can then be identified using the `where` keyword within the function definition. Finally, when calling `foo`, we need pass `Val(a)` as its input.

Applied to our example, type instability in `foo` is caused because the value of `condition` isn't known by the compiler. Dispatching by value provides a mechanism to explicitly convey this information and hence solve the type instability.

```
function foo(condition)
    y = condition ?  2.5  :  1

    return [y * i for i in 1:100]
end

@code_warntype foo(true)          # type UNSTABLE
@code_warntype foo(false)         # type UNSTABLE
```

```
function foo(::Val{condition}) where condition
    y = condition ?  2.5  :  1

    return [y * i for i in 1:100]
end

@code_warntype foo(Val(true))     # type stable
@code_warntype foo(Val(false))    # type stable
```

GOTCHA 5: VARIABLES AS DEFAULT VALUES OF KEYWORD ARGUMENTS

Functions accept both positional and keyword arguments. The possibility of keyword arguments in particular allows the user to assign default values. If these default values are set through variables rather than literal values, a type instability will be introduced. The reason is that such variables will be treated as global variables.

```
foo(; x) = x

β = 1
@code_warntype foo(x=β)          #type stable
```

```
foo(; x = 1) = x


@code_warntype foo()             #type stable
```

```
foo(; x = β) = x

β = 1
@code_warntype foo()             #type UNSTABLE
```

In case you necessarily need to set a variable as a default value, there are still a few strategies you could follow to restore type stability.

One set of solutions leverages the techniques we introduced for global variables. hese include type-annotating the global variable (*Solution 1a*) or defining it as a constant (*Solution 1b*).

Another strategy involves defining a function that stores the default value. By doing so, you can take advantage of type inference, with the function attempting to infer a concrete type for the default value (*Solution 2*).

You can also solve the type instability by adopting a local approach, where type annotations are added to either the keyword argument (*Solution 3a*) or the default value itself (*Solution 3b*). Note that this isn't necessary when positional arguments are used as default values of keyword arguments (*Solution 4*).

All these scenarios are represented below.

```
foo(; x = β) = x

const β = 1
@code_warntype foo()            #type stable
```

```
foo(; x = β) = x

β::Int64 = 1
@code_warntype foo()            #type stable
```

```
foo(; x = β()) = x

β() = 1
@code_warntype foo()            #type stable
```

```
foo(; x::Int64 = β) = x

β = 1
@code_warntype foo()            #type stable
```

```
foo(; x = β::Int64) = x

β = 1
@code_warntype foo()            #type stable
```

```
foo(β; x = β) = x

β = 1
@code_warntype foo(β)           #type stable
```

GOTCHA 6: CLOSURES CAN EASILY INTRODUCE TYPE INSTABILITIES

Closures are a fundamental concept in programming. They refer to functions that capture and retain access to variables from the scope in which they were defined. In practical terms, a closure arises when **one function is defined inside another**, including the case where anonymous functions are used inside a function.

Although closures provide a convenient way to write modular and self-contained code, they can sometimes introduce type instabilities. While Julia has made progress in mitigating these issues, they have persisted for years and remain a source of potential inefficiency. For this reason, it's essential to understand not only the consequences of using closures carelessly, but also to learn strategies for addressing their performance challenges.

CLOSURES ARE COMMON IN CODING

There are several scenarios where closures emerge naturally. One such scenario is when a task requires multiple steps, but you prefer to keep a single self-contained unit of code. For instance, this approach is particularly useful if a function needs to perform multiple interdependent steps, such as data preparation (e.g., setting parameters or initializing variables) and subsequent computations based on that data. By nesting a function within another, you can keep related code organized and contained within the same logical block, promoting code readability and maintainability.

To illustrate how code implements a task with and without closures, we'll use generic code. This isn't intended to be executed, but rather to demonstrate the underlying structure.

```
function task()
        # <here you define parameters and initialize variables>

    function output()
        # <here you do computations with the parameters and variables>
    end

    return output()
end

task()
```

```
function task()
        # <here, you define parameters and initialize variables>

    return output(<variables>, <parameters>)
end

function output(<variables>, <parameters>)
        # <here, you do some computations with the variables and parameters>
end

task()
```

Although the approach using closures may seem more intuitive, it can easily introduce type instability. This occurs when one of these conditions hold:

- variables or arguments are redefined inside the function (e.g., when updating a variable)
- the order in which functions are defined is altered
- anonymous functions are introduced

Each of these cases is explored below, where we refer to the containing function as the *outer function* and the closure as the *inner function*.

WHEN THE ISSUE ARISES

Let's start examining three examples. They cover all the possible situations where closures could result in type instability.

The first examples reveal that the placement of the inner function could matter for type stability.

```
function foo()
    x           = 1
    bar()       = x

    return bar()
end

@code_warntype foo()      # type stable
```

```
function foo()
    bar(x)      = x
    x           = 1

    return bar(x)
end

@code_warntype foo()      # type stable
```

```
function foo()
    bar()       = x
    x           = 1

    return bar()
end

@code_warntype foo()      # type UNSTABLE
```

```
function foo()
    bar()::Int64 = x::Int64
    x::Int64     = 1

    return bar()
end

@code_warntype foo()      # type UNSTABLE
```

```
function foo()
    x = 1

    return bar(x)
end

bar(x) = x

@code_warntype foo()      # type stable
```

The second example establishes that type instability arises when closures are combined with reassignments of variables or arguments. This issue even emerges when the reassignment involves the same object, including trivial expressions such as `x = x`. The example also reveals that type annotating the redefined variable or the closure doesn't resolve the problem.

```
function foo()
    x            = 1
    x            = 1            # or 'x = x', or 'x = 2'

    return x
end

@code_warntype foo()            # type stable
```

```
function foo()
    x            = 1
    x            = 1            # or 'x = x', or 'x = 2'
    bar(x)       = x

    return bar(x)
end

@code_warntype foo()            # type stable
```

```
function foo()
    x            = 1
    x            = 1            # or 'x = x', or 'x = 2'
    bar()        = x

    return bar()
end

@code_warntype foo()            # type UNSTABLE
```

```julia
function foo()
    x::Int64     = 1
    x            = 1
    bar()::Int64 = x::Int64

    return bar()
end

@code_warntype foo()            # type UNSTABLE
```

```julia
function foo()
    x::Int64     = 1
    bar()::Int64 = x::Int64
    x            = 1

    return bar()
end

@code_warntype foo()            # type UNSTABLE
```

```julia
function foo()
    bar()::Int64 = x::Int64
    x::Int64     = 1
    x            = 1

    return bar()
end

@code_warntype foo()            # type UNSTABLE
```

```julia
function foo()
    x            = 1
    x            = 1            # or 'x = x', or 'x = 2'

    return bar(x)
end

bar(x) = x

@code_warntype foo()            # type stable
```

Finally, the last example deals with situations involving multiple closures. It highlights that the order in which you define them could matter for type stability. The third tab in particular demonstrates that passing closures as function arguments can sidestep the issue. However, such an approach is at odds with how code is generally written in Julia.

```julia
function foo(x)
    closure1(x) = x
    closure2(x) = closure1(x)

    return closure2(x)
end

@code_warntype foo(1)            # type stable
```

```julia
function foo(x)
    closure2(x) = closure1(x)
    closure1(x) = x

    return closure2(x)
end

@code_warntype foo(1)            # type UNSTABLE
```

```julia
function foo(x)
    closure2(x, closure1) = closure1(x)
    closure1(x)           = x

    return closure2(x, closure1)
end

@code_warntype foo(1)            # type stable
```

```julia
function foo(x)
    closure2(x) = closure1(x)

    return closure2(x)
end

closure1(x) = x

@code_warntype foo(1)            # type stable
```

In the following, we'll examine specific scenarios where these patterns emerge. The examples reveal that the issue can occur more frequently than we might expect. For each scenario, we'll also provide a solution that enables the use of a closure approach. Nonetheless, if the function captures a performance critical part of your code, it's probably wise to avoid closures.

"BUT NO ONE WRITES CODE LIKE THAT"

i) Transforming Variables through Conditionals

```
x = [1,2]; β = 1

function foo(x, β)
    (β < 0) && (β = -β)         # transform 'β' to use its absolute value

    bar(x) = x * β

    return bar(x)
end

@code_warntype foo(x, β)        # type UNSTABLE
```

```
x = [1,2]; β = 1

function foo(x, β)
    (β < 0) && (β = -β)         # transform 'β' to use its absolute value

    bar(x,β) = x * β

    return bar(x,β)
end

@code_warntype foo(x, β)        # type stable
```

```
x = [1,2]; β = 1

function foo(x, β)
    δ = (β < 0) ? -β : β        # transform 'β' to use its absolute value

    bar(x) = x * δ

    return bar(x)
end

@code_warntype foo(x, β)        # type stable
```

```
x = [1,2]; β = 1

function foo(x, β)
    β = abs(β)                    # 'δ = abs(β)' is preferable (you should avoid redefining
variables)

    bar(x) = x * δ

    return bar(x)
end

@code_warntype foo(x, β)        # type stable
```

Recall that the compiler doesn't dispatch by value, and so whether the condition holds is irrelevant. For instance, the type instability would still hold if we wrote `1 < 0` instead of `β < 0`. Moreover, the value used to redefine `β` is also unimportant, with the same conclusion holding if you write `β = β`.

ii) Anonymous Functions inside a Function

Using an anonymous function inside a function is another common form of closure. Considering this, type instability also arises in the example above if we replace the inner function `bar` for an anonymous function. To demonstrate this, we apply `filter` with an anonymous function that keeps all the values in `x` that are greater than `β`.

```
x = [1,2]; β = 1

function foo(x, β)
    (β < 0) && (β = -β)          # transform 'β' to use its absolute value

    filter(x -> x > β, x)        # keep elements greater than 'β'
end

@code_warntype foo(x, β)         # type UNSTABLE
```

```
x = [1,2]; β = 1

function foo(x, β)
    δ = (β < 0) ? -β : β         # define 'δ' as the absolute value of 'β'

    filter(x -> x > δ, x)        # keep elements greater than 'δ'
end

@code_warntype foo(x, β)         # type stable
```

```
x = [1,2]; β = 1

function foo(x, β)
    β = abs(β)                     # 'δ = abs(β)' is preferable (you should avoid redefining
variables)

    filter(x -> x > β, x)        # keep elements greater than β
end

@code_warntype foo(x, β)         # type stable
```

iii) Variable Updates

```
function foo(x)
    β = 0                       # or 'β::Int64 = 0'
    for i in 1:10
        β = β + i               # equivalent to 'β += i'
    end

    bar() = x + β               # or 'bar(x) = x + β'

    return bar()
end

@code_warntype foo(1)           # type UNSTABLE
```

```
function foo(x)
    β = 0
    for i in 1:10
        β = β + i
    end

    bar(x,β) = x + β

    return bar(x,β)
end

@code_warntype foo(1)           # type stable
```

```
x = [1,2]; β = 1

function foo(x, β)
    (1 < 0) && (β = β)

    bar(x) = x * β

    return bar(x)
end

@code_warntype foo(x, β)         # type UNSTABLE
```

iv) The Order in Which you Define Functions Could Matter Inside a Function

To illustrate this claim, suppose you want to define a variable `x` that depends on a parameter `β`. However, `β` is measured in one unit (e.g., meters), while `x` requires `β` to be expressed in a different unit (e.g., centimeters). This implies that, before defining `x`, we must rescale `β` to the appropriate unit.

Depending on how we implement the operation, a type instability could emerge.

```julia
function foo(β)
    x(β)                  =  2 * rescale_parameter(β)
    rescale_parameter(β)  =  β / 10

    return x(β)
end

@code_warntype foo(1)      # type UNSTABLE
```

```julia
function foo(β)
    rescale_parameter(β)  =  β / 10
    x(β)                  =  2 * rescale_parameter(β)

    return x(β)
end

@code_warntype foo(1)      # type stable
```

FOOTNOTES

[1.] A similar problem would occur if we replaced `0` by `0.` and `x` is an integer.

9a. Overview and Goals

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INTRODUCTION

In the previous chapter, we began our exploration of high performance in Julia by focusing on type stability. We now shift our attention to memory allocations, a critical aspect of performance optimization.

Memory allocations occur whenever a new object is created, involving the reservation of memory space to store its values. The aspect is crucial for performance, since the approach selected to handle the process can significantly slow down computations. In particular, memory allocations on the heap, simply referred to as *memory allocations*, incur a notable cost due to the additional CPU instructions required for memory management.

Despite this, the interplay between memory allocation and performance is complex. In fact, **reducing memory allocation is neither necessary nor sufficient for speeding up computations**—we'll present instances where the approach allocating more memory turns out to be faster. This apparent paradox arises from a trade-off involved when creating a new object: although allocations can lead to a significant overhead, the resulting objects store their data in contiguous blocks of memory, enabling the CPU to access information more efficiently.

From a practical perspective, it's essential to closely monitor memory usage if performance is critical. **Excessive memory allocation often serves as a red flag**: if two approaches exhibit large differences in memory allocation, their execution speeds are likely to differ significantly as well.

9b. Stack vs Heap

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INTRODUCTION

Memory allocations are a fundamental process in computer programming that occurs whenever a new object is created. It involves setting aside a portion of the computer's Random Access Memory (RAM) to store the object's data. Conceptually, RAM is logically divided into two main areas: the stack and the heap. Importantly, these areas aren't physical locations, but rather conceptual models that govern how memory is managed.

The choice between stack and heap allocation has profound implications for performance. Allocating memory on the heap is a comparatively expensive operation. It requires a systematic search for an available block of memory, bookkeeping to track its status, and an eventual deallocation process to reclaim the space once it's no longer needed. This deallocation process is often handled by a system known as a garbage collector. Stack allocations, by contrast, are simpler and therefore more efficient.

The performance gap between stack and heap allocations can easily become a critical bottleneck if the operation is performed repeatedly. This disparity in performance explains the common convention in programming, including Julia, where **memory allocations will exclusively refer to heap allocations**. While stack allocations are technically a form of memory allocation, their speed and simplicity mean they rarely warrant the same level of scrutiny.

With this distinction in mind, the current section begins our exploration of memory allocations by taking a closer look at the mechanics of the stack and the heap.

STACK ALLOCATIONS

In Julia, typical objects stored on the stack include integers, numbers, characters, and small fixed-size collections like tuples.

Objects on the stack are characterized by having a fixed size, precluding the possibility of dynamically growing or shrinking in size. These characteristics make allocating and deallocating memory on the stack extremely efficient.

The primary limitation of the stack is its limited capacity, making it suitable only for objects with a few elements. Indeed, attempting to allocate more memory than the stack can accommodate will result in a "stack overflow" error, causing program termination. And, even if an object fits on the stack, allocating too many elements can significantly degrade performance. [1]

HEAP ALLOCATIONS

Common objects stored on the heap include arrays (such as vectors and matrices) and strings. Unlike the stack, the heap is designed to allocate and free memory in any order, allowing it to accommodate larger and more complex data structures. Thus, the heap can handle objects as large as the available RAM permits, which additionally could grow or shrink dynamically.

While the heap offers greater flexibility than the stack, its more complex memory management comes at the cost of slower performance. [2] Due to their reduced speed, the current chapter will outline approaches to minimizing heap allocations. Strategies for achieving this includes utilizing the stack whenever possible, and favoring mutation over the creation of new objects.

FOOTNOTES

1. There's no hard and fast rule about how many elements are "too many". Benchmarking is the only reliable way to determine the performance consequences for each particular case. As a rough guideline, objects with more than 100 elements will certainly suffer from poor performance, while those with fewer than 15 elements are likely to benefit from stack allocation.

2. To handle the memory of heap-allocated objects, Julia uses what's known as a *garbage collector*. This mechanism automatically identifies and frees memory no longer in use, which can be computationally expensive.

9c. Objects Allocating Memory

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INTRODUCTION

In the previous section, we introduced the fundamentals of memory allocations, noting that objects can be stored on either the heap or the stack. Furthermore, we introduced typical terminology in programming and Julia, where **allocations exclusively refer to those on the heap**. This convention underlies the common expression that an object "allocates" when it's stored on the heap.

The distinction isn't merely to economize on words. Rather, it reflects a fundamental performance implication: heap allocations are the ones significantly impacting efficiency. They involve a more complex management process than the stack, which can introduce significant computational overhead.

The intimate relationship between performance and heap allocations is even reflected in Julia's built-in benchmarking tools. Macros like `@time` and `@btime` report the total runtime of an operation, along with the heap allocations involved.

Considering the importance of memory allocations on the heap, the current section categorizes objects into those that allocate and those that don't.

NUMBERS, TUPLES, NAMED TUPLES, AND RANGES DON'T ALLOCATE

We start by focusing on objects that don't allocate memory. They include:

- Numbers
- Tuples
- Named Tuples
- Ranges

As these objects don't allocate, neither does their creation, access, or manipulation. This is demonstrated below.

```julia
function foo()
    x = 1; y = 2

    x + y
end
```

```
julia> @btime foo()
  0.800 ns (0 allocations: 0 bytes)
```

```
function foo()
    tup = (1,2,3)

    tup[1] + tup[2] * tup[3]
end
```

```
julia> @btime foo()
  0.800 ns (0 allocations: 0 bytes)
```

```
function foo()
    nt = (a=1, b=2, c=3)

    nt.a + nt.b * nt.c
end
```

```
julia> @btime foo()
  0.800 ns (0 allocations: 0 bytes)
```

```
function foo()
    rang = 1:3

    rang[1] + rang[2] * rang[3]
end
```

```
julia> @btime foo()
  0.800 ns (0 allocations: 0 bytes)
```

ARRAYS AND THEIR SLICES DO ALLOCATE MEMORY

Arrays are among the most common objects allocating memory. Such allocation occurs not only when an array is explicitly created and assigned to a variable. It also occurs whenever a computation produces a new array as its result. The following example illustrates this point.

```
foo()  = [1,2,3]
```

```
julia> @btime foo()
  13.714 ns (1 allocation: 80 bytes)
```

Slicing is another operation that creates arrays and therefore incurs memory allocations. This arises from the default behavior of slicing, which returns a new copy rather than a view of the original object. The sole exception is when a single element is accessed, in which case no new allocation takes place.

```
x      = [1,2,3]

foo(x) = x[1:2]                 # ONE allocation, since ranges don't allocate (but 'x[1:2]'
itself does)
```

```
julia> @btime foo($x)
  16.116 ns (1 allocation: 80 bytes)
```

```
x      = [1,2,3]

foo(x) = x[[1,2]]                    # TWO allocations (one for '[1,2]' and another for
'x[[1,2]]' itself)
```

```
julia> @btime foo($x)
  31.759 ns (2 allocations: 160 bytes)
```

```
x      = [1,2,3]

foo(x) = x[1] * x[2] + x[3]
```

```
julia> @btime foo($x)
  1.400 ns (0 allocations: 0 bytes)
```

Array comprehensions and broadcasting are additional operations that lead to array creation. Notably, broadcasting triggers memory allocation when intermediate results are generated internally, even if they aren't explicitly returned. This behavior is demonstrated in the tab "Broadcasting 2" below.

```
foo()  = [a for a in 1:3]
```

```
julia> @btime foo()
  13.514 ns (1 allocation: 80 bytes)
```

```
x      = [1,2,3]
foo(x) = x .* x
```

```
julia> @btime foo($x)
  15.916 ns (1 allocation: 80 bytes)
```

```
x      = [1,2,3]
foo(x) = sum(x .* x)                # 1 allocation from temporary vector 'x .* x'
```

```
julia> @btime foo($x)
  21.242 ns (1 allocation: 80 bytes)
```

9d. Slice Views to Reduce Allocations

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INTRODUCTION

In a previous discussion on **Slices and Views**, we defined a slice as a subvector derived from a parent vector `x`. Common examples include expressions such as `x[1:2]`, which extracts elements at positions 1 and 2, or `x[x .> 0]`, which selects only those elements that are positive. By default, these operations create a copy of the data and therefore allocate memory. The only exception to this rule occurs when a slice comprises a single object.

In this section, we address the issue of memory allocations associated with slices. To do this, we highlight the role of **views**, which bypass the need for a copy by directly referencing the parent object. The strategy is particularly effective when slices are indexed through ranges. On the contrary, it's not suitable for slices that employ Boolean indexing, like `x[x .> 0]`, where memory allocation will still occur even with views.

Interestingly, we'll show scenarios where **copying data could actually be faster than using views**, despite the additional memory allocation involved. This apparent paradox emerges because copied data is stored in a contiguous block of memory, which facilitates more efficient access patterns. In contrast, views might reference non-contiguous memory locations, potentially leading to slower access times, despite avoiding the initial allocation cost.

VIEWS OF SLICES

We begin by showing that views don't allocate memory *when a slice is indexed by a range*. This behavior can yield performance improvements over regular slices, which create a copy of the data by default.

SLICE AS A COPY

```julia
x = [1, 2, 3]

foo(x) = sum(x[1:2])          # it allocates ONE vector -> the slice 'x[1:2]'
```

```julia
julia> @btime foo($x)
  15.015 ns (1 allocation: 80 bytes)
```

SLICE AS A VIEW

```
x = [1, 2, 3]

foo(x) = sum(@view(x[1:2]))    # it doesn't allocate
```

```
julia> @btime foo($x)
  1.200 ns (0 allocations: 0 bytes)
```

However, **views created through Boolean indexing neither reduce memory allocations nor are more performant**. Therefore, you shouldn't rely on views of these objects to speed up computations. This fact is illustrated below.

BOOLEAN INDEX (COPY)

```
x = rand(1_000)

foo(x) = sum(x[x .> 0.5])
```

```
julia> @btime foo($x)
  662.500 ns (4 allocations: 8.34 KiB)
```

BOOLEAN INDEX (VIEW)

```
x = rand(1_000)

foo(x) = @views sum(x[x .> 0.5])
```

```
julia> @btime foo($x)
  759.770 ns (4 allocations: 8.34 KiB)
```

COPYING DATA MAY BE FASTER

Although views can reduce memory allocations, there are scenarios where copying data can result in faster performance. A detailed comparison of copies versus views will be provided in Part II. Here, we simply remark on this possibility.

Essentially, the choice between copies and views reflects a fundamental trade-off between memory allocation and data access patterns. On the one hand, newly created vectors store data in contiguous blocks of memory, enabling more efficient CPU access and allowing for certain optimizations. On the other hand, views avoid allocation, but may also require accessing data scattered across non-contiguous memory regions.

Below, we illustrate a scenario in which the overhead of creating a copy is outweighed by the benefits of contiguous memory access, making copying the more efficient choice.

COPY

```julia
x = rand(100_000)

foo(x) = max.(x[1:2:length(x)], 0.5)
```

```
julia> @btime foo($x)
  30.100 μs (4 allocations: 781.34 KiB)
```

VIEW

```julia
x = rand(100_000)

foo(x) = max.(@view(x[1:2:length(x)]), 0.5)
```

```
julia> @btime foo($x)
  151.700 μs (2 allocations: 390.67 KiB)
```

9e. Pre-Allocations

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INTRODUCTION

In many scenarios, for-loops entail the creation of new vectors at every iteration, resulting in repeated memory allocation. This dynamic allocation may be unnecessary, particularly if these vectors hold temporary intermediate results that don't need to be preserved for future use. In such situations, performance can be improved through the use of a technique known as pre-allocation.

Pre-allocation involves initializing a vector before the for-loop begins execution, which is then reused during each iteration to store temporary results. By allocating memory upfront and modifying it in place, the overhead associated with repeated vector creation is effectively bypassed.

The performance gains from pre-allocation can be substantial. Remarkably, this technique isn't exclusive to Julia, but rather represents an optimization strategy applicable across programming languages. Its effectiveness ultimately stems from prioritizing the mutation of pre-allocated memory over the creation of new objects, thereby minimizing new assignments on the heap.

Our presentation begins with a review of methods for initializing vectors. This serves as a prerequisite for implementing pre-allocation. We then present two scenarios where pre-allocation proves advantageous, with special emphasis on its advantages within nested for-loops.

> **Remark**
> The review of methods for vector initialization will be relatively brief and centered on performance considerations. For a more detailed review, see the section about vector creation, as well as the sections on in-place assignments and in-place functions.

INITIALIZING VECTORS

Vector initialization refers to the process of creating a vector for subsequently filling it with values. The process typically involves two steps: reserving space in memory and populating that space with some initial values. An efficient approach to initializing a vector involves performing only the first step, keeping whatever content is held in the memory address at the moment of creation. Although Julia will display these contents as numerical values, note that they're essentially arbitrary and meaningless, explaining why they're referred to as `undef` (undefined).

There are two methods for initializing a vector with `undef` values. The first one requires specifying the type and length of the array. Its syntax closely resembles the creation of new vectors. The second one is based on the function `similar(y)`, which creates a vector with the same type and dimension as another existing vector `y`. This approach is particularly useful when your output matches the structure of an input.

Below, we compare the performance of approaches to initializing a vector. In particular, we show that working with `undef` values is faster than populating the vector with specific values. To starkly show these differences, we create a vector with 100 elements and repeat the procedure 100,000 times.

```
x           = collect(1:100)
repetitions = 100_000                       # repetitions in a for-loop

function foo(x, repetitions)
    for _ in 1:repetitions
        Vector{Int64}(undef, length(x))
    end
end
```

```
julia> @btime foo($x, $repetitions)
  1.581 ms (100000 allocations: 85.45 MiB)
```

```
x           = collect(1:100)
repetitions = 100_000                       # repetitions in a for-loop

function foo(x, repetitions)
    for _ in 1:repetitions
        similar(x)
    end
end
```

```
julia> @btime foo($x, $repetitions)
  1.623 ms (100000 allocations: 85.45 MiB)
```

```
x           = collect(1:100)
repetitions = 100_000                       # repetitions in a for-loop

function foo(x, repetitions)
    for _ in 1:repetitions
        zeros(Int64, length(x))
    end
end
```

```
julia> @btime foo($x, $repetitions)
  7.530 ms (100000 allocations: 85.45 MiB)
```

```
x           = collect(1:100)
repetitions = 100_000                       # repetitions in a for-loop

function foo(x, repetitions)
    for _ in 1:repetitions
        ones(Int64, length(x))
    end
end
```

```
julia> @btime foo($x, $repetitions)
  4.674 ms (100000 allocations: 85.45 MiB)
```

```
x           = collect(1:100)
repetitions = 100_000                       # repetitions in a for-loop

function foo(x, repetitions)
    for _ in 1:repetitions
        fill(2, length(x))                  # vector filled with integer 2
    end
end
```

```
julia> @btime foo($x, $repetitions)
  4.877 ms (100000 allocations: 85.45 MiB)
```

> **Remark**
>
> Recall that `_` is a convention adopted for **dummy variables**. These are variables that have a value, but aren't used or referenced anywhere in the code. In the context of for-loops, the sole purpose of `_` is to satisfy the syntax requirements, which expect a variable to iterate over. The symbol `_` is arbitrary and any other could be used in its place.

APPROACHES TO INITIALIZING VECTORS

We can initialize `output` by passing it to the function as a keyword argument. This enables the use of `similar(x)`, where `x` is a previous function's argument. Considering this, the following two implementations turn out to be equivalent.

```
function foo(x)
   output = similar(x)
   # <some calculations using 'output'>
end
```

```
function foo(x; output = similar(x))

   # <some calculations using 'output'>
end
```

When multiple variables of the same type need to be initialized, array comprehension offers a concise solution. A more efficient approach is based on the so-called generators, which will be covered in a subsequent section. At this point, you should only know that the method based on generators doesn't allocate. Furthermore, its syntax is similar to array comprehensions, with the only difference that brackets `[]` are replaced with parentheses `()`.

```
x = [1,2,3]

function foo(x)
    a,b,c = [similar(x) for _ in 1:3]
    # <some calculations using a,b,c>
end
```

```
julia> @btime foo($x)
  49.848 ns (4 allocations: 320 bytes)
```

```
x = [1,2,3]

function foo(x)
    a,b,c = (similar(x) for _ in 1:3)
    # <some calculations using a,b,c>
end
```

```
julia> @btime foo($x)
  35.348 ns (3 allocations: 240 bytes)
```

Notice that, although the example uses `similar(x)`, the same principle applies to other initialization methods such as `Vector{Float64}(undef, length(x))`.

PRE-ALLOCATING VECTORS IN NESTED FOR-LOOPS

When working with vectors in for-loops or through broadcasting, certain operations inevitably create new vectors. This may occur even when the operation ultimately yields a scalar value. The following examples demonstrate this point.

```
x = rand(100)

foo(x) = sum(2 .* x)                    # 2 .* x implicitly creates a temporary vector
```

```
julia> @btime foo($x)
  36.779 ns (1 allocation: 896 bytes)
```

```julia
x = rand(100)

function foo(x)
    output = similar(x)                # you need to create this vector to store the results

    for i in eachindex(x)
        output[i] = 2 * x[i]
    end

    return output
end
```

```
julia> @btime foo($x)
  45.416 ns (1 allocation: 896 bytes)
```

Furthermore, when the outcome of a computation must be preserved, allocating new vectors becomes an inevitable part of the process. However, final outputs could actually serve themselves as intermediate steps in a larger computation that may involve another for-loop. Situations where the output of one for-loop feeds into another fall into the category of **nested for-loops**.

The following example illustrates how each iteration in the second for-loop generates a new vector for the intermediate result.

```julia
x = rand(100)

function foo(x; output = similar(x))
    for i in eachindex(x)
        output[i] = 2 * x[i]
    end

    return output
end

calling_foo_in_a_loop(output,x) = [sum(foo(x)) for _ in 1:100]
```

```
julia> @btime calling_foo_in_a_loop($x)
  6.160 μs (101 allocations: 88.38 KiB)
```

```julia
x = rand(100)

foo(x) = 2 .* x

calling_foo_in_a_loop(x) = [sum(foo(x)) for _ in 1:100]
```

```
julia> @btime calling_foo_in_a_loop($x)
  6.433 μs (101 allocations: 88.38 KiB)
```

Cases like this lead to unnecessary memory allocations, making them strong candidates for pre-allocating intermediate results. By adopting this strategy, we can reuse the same vector across iterations, and hence avoid the repeated overhead of creating new vectors.

To implement this strategy, we require an in-place function that takes the for-loop's output as one of its arguments. This function will eventually be called iteratively, updating its output at each iteration. There are two ways to implement this method, and we analyze each separately in the following.

VIA A FOR-LOOP

The first approach defines an in-place function that updates the values of `output` through a for-loop.

```
x      = rand(100)
output = similar(x)

function foo!(output,x)
    for i in eachindex(x)
        output[i] = 2 * x[i]
    end

    return output
end
```

```
julia> @btime foo!($output, $x)
  5.100 ns (0 allocations: 0 bytes)
```

```
x      = rand(100)
output = similar(x)

function foo!(output,x)
    for i in eachindex(x)
        output[i] = 2 * x[i]
    end

    return output
end

calling_foo_in_a_loop(output,x) = [sum(foo!(output,x)) for _ in 1:100]
```

```
julia> @btime calling_foo_in_a_loop($output, $x)
  1.340 μs (1 allocation: 896 bytes)
```

VIA BROADCASTING

The second option relies on the operator `.=` to update a vector's values. Compared to the previous example, this strategy enables updates with a simpler syntax, allowing `foo!` to be defined in a single line.

```
x      = rand(100)
output = similar(x)

foo!(output,x) = (output .= 2 .* x)
```

```
julia> @btime foo!($output, $x)
  5.800 ns (0 allocations: 0 bytes)
```

```
x      = rand(100)
output = similar(x)

foo!(output,x) = (@. output = 2 * x)
```

```
julia> @btime foo!($output, $x)
  5.400 ns (0 allocations: 0 bytes)
```

```
x      = rand(100)
output = similar(x)

foo!(output,x) = (@. output = 2 * x)

calling_foo_in_a_loop(output,x) = [sum(foo!(output,x)) for _ in 1:100]
```

```
julia> @btime calling_foo_in_a_loop($output,$x)
  1.320 μs (1 allocation: 896 bytes)
```

Warning! - Use of @. to update values

When your goal is to update the values of a vector, recall that `@.` has to be *placed at the beginning* of the statement.

```
# the following are equivalent and define a new variable
   output  = @. 2  * x
   output  =    2 .* x
```

```
# the following are equivalent and update 'output'
@. output  = 2  * x
   output .= 2 .* x
```

PRE-ALLOCATIONS FOR INTERMEDIATE STEPS

So far, our discussion has centered around the benefits of pre-allocating vectors in nested for-loops. However, such benefits extend beyond that specific setting.

Broadly speaking, pre-allocating proves useful when: ***i***) the vector serves an intermediate result that feeds into another operation, and ***ii***) the intermediate result is computed inside a for-loop. When both conditions are met, reusing a pre-allocated vector consistently outperforms creating a new vector at each iteration.

To illustrate, consider a function `foo` that returns `output`, whose computation requires an intermediate variable `temp`. The key feature here is that, even though `foo` itself won't be called inside a nested for-loop, the computation of `output` is complex enough that it's convenient to split it into multiple steps.

If `temp` isn't pre-allocated, a new vector is created at each iteration as shown below.

```
x = rand(100)

function foo(x; output = similar(x))
    for i in eachindex(x)
        temp      = x .> x[i]
        output[i] = sum(temp)
    end

    return output
end
```

```
julia> @btime foo($x)
  14.700 μs (201 allocations: 11.81 KiB)
```

```
x = rand(100)

foo(x) = [sum(x .> x[i]) for i in eachindex(x)]
```

```
julia> @btime foo($x)
  14.600 μs (201 allocations: 11.81 KiB)
```

```
x = rand(100)

function foo(x)
    temp   = [x .> x[i] for i in eachindex(x)]
    output = sum.(temp)
end
```

```
julia> @btime foo($x)
  15.100 μs (202 allocations: 12.69 KiB)
```

Next, we demonstrate different strategies that pre-allocate `temp`. The implementation of the pre-allocation differs in subtle but important ways from the nested-loop case.

First, because `foo` won't be eventually called in a for-loop, the pre-allocation of `temp` can be performed within the function itself. This means we don't need to define `temp` as an argument of `foo`. Second, since all iterations occur within a single for-loop, broadcasting becomes a particularly convenient option for updating `temp`.

VIA A FOR-LOOP OR BROADCASTING

The simplest strategy to avoid the allocations of `temp` is to pre-allocate it and update its values via broadcasting. By contrast, updating `temp` through a for-loop represents a more involved approach. The difference can be observed below.

```julia
x = rand(100)

function foo!(x; output = similar(x), temp = similar(x))
    for i in eachindex(x)
        for j in eachindex(x)
            temp[j] = x[j] > x[i]
        end
        output[i] = sum(temp)
    end

    return output
end
```

```
julia> @btime foo!($x)
  1.850 μs (2 allocations: 1.75 KiB)
```

```julia
x = rand(100)

function foo!(x; output = similar(x), temp = similar(x))
    for i in eachindex(x)
        temp     .= x .> x[i]
        output[i] = sum(temp)
    end

    return output
end
```

```
julia> @btime foo!($x)
  1.660 μs (2 allocations: 1.75 KiB)
```

```julia
x = rand(100)

function foo!(x; output = similar(x), temp = similar(x))
    for i in eachindex(x)
        @. temp      = x > x[i]
           output[i] = sum(temp)
    end

    return output
end
```

```
julia> @btime foo!($x)
  1.660 μs (2 allocations: 1.75 KiB)
```

Note that the two allocations observed are due to the creation of `temp` and `output`, which are unavoidable.

VIA IN-PLACE FUNCTION

Another way to avoid the memory allocations of `temp` is to pre-allocate it via an in-place function. We'll refer to this function as `update_temp!`, which is defined outside the for-loop and executed at each iteration. The advantage of this approach is modularity: by separating the update logic from the

for-loop, we can focus on optimizing `update_temp!` independently if its performance becomes critical. This design also improves code readability and maintainability.

```
x = rand(100)

function update_temp!(x, temp, i)
    for j in eachindex(x)
       temp[j] = x[j] > x[i]
    end
end

function foo!(x; output = similar(x), temp = similar(x))
    for i in eachindex(x)
        update_temp!(x, temp, i)
        output[i] = sum(temp)
    end

    return output
end
```

```
julia> @btime foo!($x)
  1.790 μs (2 allocations: 1.75 KiB)
```

```
x = rand(100)

update_temp!(x, temp, i) = (@. temp = x > x[i])

function foo!(x; output = similar(x), temp = similar(x))
    for i in eachindex(x)
           update_temp!(x, temp, i)
           output[i] = sum(temp)
    end

    return output
end
```

```
julia> @btime foo!($x)
  1.680 μs (2 allocations: 1.75 KiB)
```

ADDING A NESTED FOR-LOOP

Finally, let's study a scenario that combines the two situations discussed earlier: a function `foo` that requires an intermediate variable `temp`, and the repeated invocation of `foo` inside an outer for-loop. In this case, we're layering the complexity of intermediate pre-allocation with the demands of a nested loop structure.

In this combined setting, both `output` and `temp` must be initialized outside the function and passed in as arguments. This ensures that no vector is re-created at each iteration of the inner or outer for-loop. The following example illustrates an implementation that avoids unnecessary allocations by updating `update_temp!` via broadcasting.

```julia
x      = rand(100)



update_temp!(x, temp, i) = (@. temp = x > x[i])

function foo!(x; output = similar(x), temp = similar(x))
    for i in eachindex(x)
            update_temp!(x, temp, i)
            output[i] = sum(temp)
    end
    return output
end

calling_foo_in_a_loop(x) = [foo!(x) for _ in 1:1_000]
```

```
julia> @btime calling_foo_in_a_loop($x)
  1.734 ms (2001 allocations: 1.72 MiB)
```

```julia
x      = rand(100)
output = similar(x)
temp   = similar(x)

update_temp!(x, temp, i) = (@. temp = x > x[i])

function foo!(x, output, temp)
    for i in eachindex(x)
            update_temp!(x, temp, i)
            output[i] = sum(temp)
    end
    return output
end

calling_foo_in_a_loop(x, output, temp) = [foo!(x, output, temp) for _ in 1:1_000]
```

```
julia> @btime calling_foo_in_a_loop($x, $output, $temp)
  1.666 ms (1 allocation: 7.94 KiB)
```

9f. Reductions

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INTRODUCTION

Reductions are a computational strategy for **operations that take collections as input and return a single element**. Such operations arise naturally in a wide range of contexts, such as when computing summary statistics (e.g., averages, variances, or maxima of collections).

The reduction process works by iteratively applying an operation to pairs of elements, accumulating the results at each step until the final output is obtained. For example, to compute `sum(x)` for a vector `x`, a reduction would start by adding the first two elements, then add the third element to the partial total, and continue this cumulative process until all elements have been summed.

The method is particularly convenient when we need to transform a vector's elements prior to aggregating the result. By operating on scalars, **reductions sidestep the need to materialize intermediate outputs**, thus reducing memory allocations. This means that, if for example you have to compute `sum(log.(x))`, a reduction would avoid the creation of the intermediate vector `log.(x)`.

AN INTUITION

Reductions are implemented using a for-loop, with an operator applied to pairs of elements and the resulting output updated in each iteration. A classic example is the summation of all numeric elements in a vector. This involves applying the addition operator `+` to pairs of elements, iteratively updating the accumulated sum. This process is demonstrated below.

```
x = rand(100)

foo(x) = sum(x)
```

```
julia> foo(x)
48.447
```

```julia
x = rand(100)

function foo(x)
    output = 0.

    for i in eachindex(x)
        output = output + x[i]
    end

    return output
end
```

```
julia> foo(x)
48.447
```

```julia
x = rand(100)

function foo(x)
    output = 0.

    for i in eachindex(x)
        output += x[i]
    end

    return output
end
```

```
julia> foo(x)
48.447
```

In reductions, it's common to see implementations like the last tab, which are based on update operators. Recall that they turn an expression like `x = x + a` into `x += a`.

IMPLEMENTING REDUCTIONS

The implementation of reductions relies on either a binary operator or a two-argument function during each iteration. An example of a binary operator is `+`, which we used above. However, we could've also used its two-argument function form, replacing `output = output + x[i]` with `output = +(output, x[i])`.

The use of two-argument functions expands the scope of reductions, enabling us to compute more complex operations. For instance, it allows us to compute the maximum value of a vector `x`. This requires the function `max`, where `max(a,b)` returns the maximum of the scalars `a` and `b`.

Formally, a reduction requires the binary operation to satisfy **two mathematical properties**:

- **Associativity**: the way in which operations are grouped must not change the result. For example, addition is associative because `(a + b) + c = a + (b + c)`.

- **Existence of an identity element**: there exists an element that, when combined with any other element through a binary operation, leaves that element unchanged. For example, the identity element of addition is `0` because `a + 0 = a`.

The following list indicates the identity elements of each operation.

Operation	Identity Element
Sum	0
Product	1
Maximum	-Inf
Minimum	Inf

Identity elements constitute the initial values of the iterative process. Based on them, we next implement reductions for several operations. The examples define the function `foo1` to show the desired outcome, while `foo2` provides the same output via a reduction.

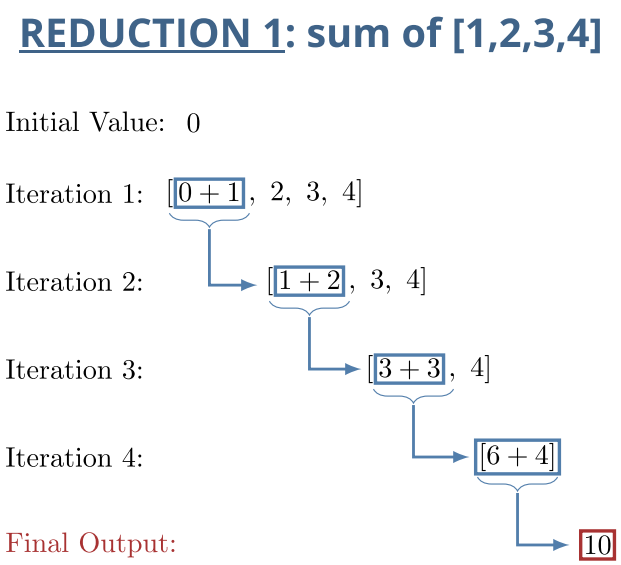
```
x = rand(100)

foo1(x) = sum(x)

function foo2(x)
    output = 0.

    for i in eachindex(x)
        output += x[i]
    end

    return output
end
```

```
x = rand(100)

foo1(x) = prod(x)

function foo2(x)
    output = 1.

    for i in eachindex(x)
        output *= x[i]
    end

    return output
end
```

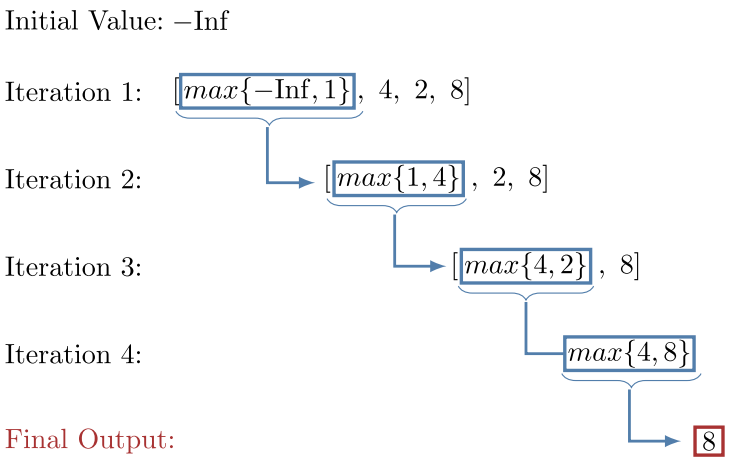
```
x = rand(100)

foo1(x) = maximum(x)

function foo2(x)
    output = -Inf

    for i in eachindex(x)
        output = max(output, x[i])
    end

    return output
end
```

```
x = rand(100)

foo1(x) = minimum(x)

function foo2(x)
    output = Inf

    for i in eachindex(x)
        output = min(output, x[i])
    end

    return output
end
```

Below, we visually illustrate how reductions operate in these examples.



REDUCTION 2: maximum of [1,4,2,8]



AVOIDING MEMORY ALLOCATIONS VIA REDUCTIONS

One of the primary advantages of reductions is that they avoid the memory allocation of intermediate results.

To illustrate, consider the operation `sum(log.(x))` for a vector `x`. Its computation involves transforming `x` into `log.(x)`, and then summing the transformed elements. By default, broadcasting internally creates a new vector to store the result of `log.(x)`, thereby allocating memory. However, in many cases like this one, we're only interested in the final sum, not the intermediate result. Therefore, an approach that bypasses the allocation of `log.(x)` is beneficial. Reductions accomplish this by defining a scalar `output`, which is iteratively updated by summing the transformed values of `x`. [1]

```
x = rand(100)

foo1(x) = sum(log.(x))

function foo2(x)
    output = 0.

    for i in eachindex(x)
        output += log(x[i])
    end

    return output
end
```

```
julia> @btime foo1($x)
  315.584 ns (1 allocation: 896 bytes)
julia> @btime foo2($x)
  296.119 ns (0 allocations: 0 bytes)
```

```julia
x = rand(100)

foo1(x) = prod(log.(x))

function foo2(x)
    output = 1.

    for i in eachindex(x)
        output *= log(x[i])
    end

    return output
end
```

```
julia> @btime foo1($x)
  311.840 ns (1 allocation: 896 bytes)
julia> @btime foo2($x)
  296.061 ns (0 allocations: 0 bytes)
```

```julia
x = rand(100)

foo1(x) = maximum(log.(x))

function foo2(x)
    output = -Inf

    for i in eachindex(x)
        output = max(output, log(x[i]))
    end

    return output
end
```

```
julia> @btime foo1($x)
  482.602 ns (1 allocation: 896 bytes)
julia> @btime foo2($x)
  374.961 ns (0 allocations: 0 bytes)
```

```julia
x = rand(100)

foo1(x) = minimum(log.(x))

function foo2(x)
    output = Inf

    for i in eachindex(x)
        output = min(output, log(x[i]))
    end

    return output
end
```

```
julia> @btime foo1($x)
  487.156 ns (1 allocation: 896 bytes)
julia> @btime foo2($x)
  368.502 ns (0 allocations: 0 bytes)
```

REDUCTIONS VIA BUILT-IN FUNCTIONS

In the previous examples, reductions were implemented manually through explicit for-loops. While this approach makes the underlying mechanics transparent, it also introduces considerable verbosity. In real-world scenarios, for-loops tend to compromise code readability. To address this issue, Julia provides several streamlined alternatives for expressing reductions in a more concise and idiomatic way.

In particular, several functions come with an additional method for applying reductions to a transformed version of the input vector. Such methods are present for functions like `sum`, `prod`, `maximum`, and `minimum`. The syntax is given by `foo(<transforming function>, x)`, where `foo` is one of the functions mentioned and `x` is the vector to be transformed. For instance, the following examples consider reductions for a log transformation.

```julia
x      = rand(100)

foo(x) = sum(log, x)        #same output as sum(log.(x))
```

```
julia> @btime foo($x)
  294.889 ns (0 allocations: 0 bytes)
```

```julia
x      = rand(100)

foo(x) = prod(log, x)       #same output as prod(log.(x))
```

```
julia> @btime foo($x)
  294.763 ns (0 allocations: 0 bytes)
```

```
x      = rand(100)

foo(x) = maximum(log, x)    #same output as maximum(log.(x))
```

```
julia> @btime foo($x)
  579.940 ns (0 allocations: 0 bytes)
```

```
x      = rand(100)

foo(x) = minimum(log, x)    #same output as minimum(log.(x))
```

```
julia> @btime foo($x)
  577.516 ns (0 allocations: 0 bytes)
```

To keep matters simple, we've used the built-in function `log` for transforming `x`. However, the approach supports any form of function, including anonymous ones.

```
x      = rand(100)

foo(x) = sum(a -> 2 * a, x)       #same output as sum(2 .* x)
```

```
julia> @btime foo($x)
  6.493 ns (0 allocations: 0 bytes)
```

```
x      = rand(100)

foo(x) = prod(a -> 2 * a, x)      #same output as prod(2 .* x)
```

```
julia> @btime foo($x)
  6.741 ns (0 allocations: 0 bytes)
```

```
x      = rand(100)

foo(x) = maximum(a -> 2 * a, x)   #same output as maximum(2 .* x)
```

```
julia> @btime foo($x)
  172.547 ns (0 allocations: 0 bytes)
```

```
x      = rand(100)

foo(x) = minimum(a -> 2 * a, x)   #same output as minimum(2 .* x)
```

```
julia> @btime foo($x)
  171.490 ns (0 allocations: 0 bytes)
```

All these functions additionally accept transforming functions that require multiple arguments. To incorporate this possibility, it's necessary to call the multiple variables using `zip`, and referring to each variable through indices. We illustrate this below, where the transforming function is `x .* y`.

```
x = rand(100); y = rand(100)

foo(x,y) = sum(a -> a[1] * a[2], zip(x,y))          #same output as sum(x .* y)
```

```
julia> @btime foo($x)
  29.127 ns (0 allocations: 0 bytes)
```

```
x = rand(100); y = rand(100)

foo(x,y) = prod(a -> a[1] * a[2], zip(x,y))         #same output as prod(x .* y)
```

```
julia> @btime foo($x)
  48.031 ns (0 allocations: 0 bytes)
```

```
x = rand(100); y = rand(100)

foo(x,y) = maximum(a -> a[1] * a[2], zip(x,y))      #same output as maximum(x .* y)
```

```
julia> @btime foo($x)
  172.580 ns (0 allocations: 0 bytes)
```

```
x = rand(100); y = rand(100)

foo(x,y) = minimum(a -> a[1] * a[2], zip(x,y))      #same output as minimum(x .* y)
```

```
julia> @btime foo($x)
  166.969 ns (0 allocations: 0 bytes)
```

THE "REDUCE" AND "MAPREDUCE" FUNCTIONS

Beyond the specific functions discussed, reductions can be applied to any operation that meets the requirements of a reduction. In Julia, this generality is provided through the functions `reduce` and `mapreduce`. The function `reduce` applies a binary operation directly to the elements of a collection, combining them into a single result. By contrast, `mapreduce` first transforms each element of the collection and then applies the reduction, thereby unifying the roles of `map` and `reduce` in a single step.

It's worth remarking that reductions involving `sum`, `prod`, `max`, and `min` should still be done via their dedicated functions. These specialized functions have been carefully optimized for their respective tasks and typically outperform the more general alternatives. Consequently, our primary use case of `reduce` and `mapreduce` is for other types of reductions not covered or when packages provide their own implementations for reductions. [2]

FUNCTION "REDUCE"

The function `reduce` uses the syntax `reduce(<function>,x)`, where `<function>` is a two-argument function. The following example demonstrates its use.

```
x      = rand(100)

foo(x) = reduce(+, x)           #same output as sum(x)
```

```
julia> @btime foo($x)
  6.168 ns (0 allocations: 0 bytes)
```

```
x      = rand(100)

foo(x) = reduce(*, x)           #same output as prod(x)
```

```
julia> @btime foo($x)
  6.176 ns (0 allocations: 0 bytes)
```

```
x      = rand(100)

foo(x) = reduce(max, x)         #same output as maximum(x)
```

```
julia> @btime foo($x)
  167.905 ns (0 allocations: 0 bytes)
```

```
x      = rand(100)

foo(x) = reduce(min, x)         #same output as minimum(x)
```

```
julia> @btime foo($x)
  167.440 ns (0 allocations: 0 bytes)
```

Note all the examples presented could've been implemented as we did previously, where we directly applied `sum`, `prod`, `maximum` and `minimum`.

FUNCTION "MAPREDUCE"

The function `mapreduce` combines the functions `map` and `reduce`: before applying the reduction, `mapreduce` transforms vectors via the function `map`. Recall that `map(foo,x)` transforms each element of the collection `x` by applying `foo` element-wise. Thus, `mapreduce(<transformation>, <reduction>,x)` first transforms `x`'s elements through `map`, and then applies a reduction to the resulting output.

To illustrate its use, we make use of a `log` transformation.

```
x      = rand(100)

foo(x) = mapreduce(log, +, x)       #same output as sum(log.(x))
```

```
julia> @btime foo($x)
  294.805 ns (0 allocations: 0 bytes)
```

```
x      = rand(100)

foo(x) = mapreduce(log, *, x)       #same output as prod(log.(x))
```

```
julia> @btime foo($x)
  294.618 ns (0 allocations: 0 bytes)
```

```
x      = rand(100)

foo(x) = mapreduce(log, max, x)     #same output as maximum(log.(x))
```

```
julia> @btime foo($x)
  579.808 ns (0 allocations: 0 bytes)
```

```
x      = rand(100)

foo(x) = mapreduce(log, min, x)     #same output as minimum(log.(x))
```

```
julia> @btime foo($x)
  577.505 ns (0 allocations: 0 bytes)
```

Just like with `reduce`, note that the examples could've been implemented directly as we did previously, through the functions `sum`, `prod`, `maximum`, and `minimum`.

`mapreduce` can also be used with anonymous functions and transformations requiring multiple arguments. Below, we illustrate both possibilities. Note that the implementations are the same as with `sum`, `prod`, `maximum`, and `minimum`.

```
x = rand(100); y = rand(100)

foo(x,y) = mapreduce(a -> a[1] * a[2], +, zip(x,y))       #same output as sum(x .* y)
```

```
julia> @btime foo($x)
  29.165 ns (0 allocations: 0 bytes)
```

```
x = rand(100); y = rand(100)

foo(x,y) = mapreduce(a -> a[1] * a[2], *, zip(x,y))       #same output as prod(x .* y)
```

```
julia> @btime foo($x)
  48.221 ns (0 allocations: 0 bytes)
```

```
x = rand(100); y = rand(100)

foo(x,y) = mapreduce(a -> a[1] * a[2], max, zip(x,y))     #same output as maximum(x .* y)
```

```
julia> @btime foo($x)
  175.634 ns (0 allocations: 0 bytes)
```

```
x = rand(100); y = rand(100)

foo(x,y) = mapreduce(a -> a[1] * a[2], min, zip(x,y))     #same output as minimum(x .* y)
```

```
julia> @btime foo($x)
  166.995 ns (0 allocations: 0 bytes)
```

REDUCE OR MAPREDUCE?

In principle, `reduce` could be considered as a special case of `mapreduce`, where the latter transforms `x` through the identity function `identity(x)`. Likewise, `mapreduce(<transformation>, <operator>,x)` produces the same result as `reduce(<operator>, map(<transformation>,x))`.

Nonetheless, `mapreduce` should be employed in those cases where the vector input must be transformed. The reason is that `mapreduce` avoids the internal memory allocations of the transformed vector. This is demonstrated below, where we compute `sum(2 .* x)` through a reduction.

```
x = rand(100)

foo(x) = mapreduce(a -> 2 * a, +, x)
```

```
julia> @btime foo($x)
  6.372 ns (0 allocations: 0 bytes)
```

```
x = rand(100)

foo(x) = reduce(+, map(a -> 2 * a, x))
```

```
julia> @btime foo($x)
  43.476 ns (1 allocation: 896 bytes)
```

FOOTNOTES

[1.] In the section Lazy Operations, we'll explore an alternative based on broadcasting that doesn't materialize intermediate results either.

[2.] For instance, the package `Folds` provides a parallelized version of both `map` and `mapreduce`, enabling the utilization of all available CPU cores. Its syntax is identical to Julia's built-in functions.

9g. Static Vectors for Small Collections

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INTRODUCTION

The creation of vectors can rapidly become a performance bottleneck, due to the memory-allocation overhead it entails. The issue has far-reaching implications, as vector creation doesn't just occur when we explicitly define a variable holding a new vector. It also happens internally in various scenarios, such as when referencing a slice like `x[1:2]` or computing intermediate results on the fly, like `x .* y` in `sum(x .* y)`.

This section introduces a strategy to address this limitation, while retaining the convenience of vectors for collections. The solution leverages the so-called **static vectors**, provided by the `StaticArrays` package. Unlike built-in vectors, which are allocated on the heap, static vectors are stack-allocated.

Under the hood, static vectors are built on top of tuples. This feature determines that **static vectors are only suitable for collections comprising a few elements**. As a rule of thumb, **you should use static vectors for collections with up to 75 elements**. Exceeding this threshold can lead to increased overhead during creation and access, potentially offsetting any performance benefits or even resulting in a fatal error. [1]

Static vectors offer additional benefits relative to tuples. Firstly, they maintain their performance benefits even at sizes where tuples typically degrade. Secondly, they're manipulated like regular vectors, making them more convenient to work with.

The `StaticArrays` package also supports other array types, including matrices, and provides mutable variants. The latter makes static vectors more flexible than tuples, which are only available in an immutable form. It's worth indicating though that, while the mutable version provides performance benefits relative to regular vectors, the immutable option still offers the best performance.

> **Warning!**
> To avoid repetition, **the entire section assumes all collections are small**. Taking this into account, all the benefits highlighted are contingent upon this assumption. We also suppose that the `StaticArrays` package is already available in the workspace, so the command `using StaticArrays` is omitted from each code snippet.

CREATING STATIC VECTORS

The package `StaticArrays` includes several variants of static arrays. Our focus in particular we'll be on the type `SVector`, whose objects will be referred to as SVectors for simplicity.

There are two approaches to creating an SVector, each serving a distinct purpose: the first one creates an SVector through literal values, while the other option converts a standard vector into an SVector. The implementation of each approach is illustrated below.

```
# all 'sx' define the same static vector '[3,4,5]'


sx = SVector(3,4,5)
sx = SVector{3, Int64}(3,4,5)
sx = SA[3,4,5]
sx = @SVector [i for i in 3:5]
```

```
julia> sx
3-element SVector{3, Int64} with indices SOneTo(3):
 3
 4
 5
```

```
# all 'sx' define a static vector with same elements as 'x'
x = collect(1:10)

sx = SVector(x...)
sx = SVector{length(x), eltype(x)}(x)
sx = SA[x...]
sx = @SVector [a for a in x]
```

```
julia> sx
10-element SVector{10, Int64} with indices SOneTo(10):
  1
  2
  3
  ⋮
  9
 10
```

Of these approaches, we'll primarily rely on the function `SVector`, occasionally employing `SA` for indexing purposes. [2] Note the use of the splatting operator `...` to turn a regular vector into an SVector. This operator is necessary for the function `SVector` to transform a collection into a sequence of arguments. [3]

Regarding slices of SVectors, the approach used for their creation could result in either a regular vector or an SVector. The resulting object depends on the indexing employed: a slice remains an SVector when indices are provided as SVectors, whereas the slice becomes a regular vector when indices are ranges or regular vectors. The sole exception to this rule is when the slice references the whole object (i.e., `sx[:]`), in which case an SVector is returned.

```julia
x = collect(3:10) ; sx = SVector(x...)

# both define static vectors
slice1 = sx[:]
slice2 = sx[SA[1,2]]        # or slice2 = sx[SVector(1,2)]
```

```
julia> slice1
8-element SVector{8, Int64} with indices SOneTo(8):
  3
  4
  ⋮
  9
 10
julia> slice2
2-element SVector{2, Int64} with indices SOneTo(2):
 3
 4
```

```julia
x = collect(3:10) ; sx = SVector(x...)

# both define and ordinary vector
slice2 = sx[1:2]
slice2 = sx[[1,2]]
```

```
julia> slice
2-element Vector{Int64}:
 3
 4
```

SVECTORS DON'T ALLOCATE MEMORY AND ARE FASTER

One of the key advantages of SVectors is that they're internally implemented on top of tuples. Consequently, SVectors don't allocate memory.

```julia
x = rand(10)

function foo(x)
    a = x[1:2]              # 1 allocation (copy of slice)
    b = [3,4]               # 1 allocation (vector creation)

    sum(a) * sum(b)         # 0 allocation (scalars don't allocate)
end
```

```
julia> @btime foo($x)
  29.819 ns (2 allocations: 160 bytes)
```

```julia
x = rand(10)

@views function foo(x)
    a = x[1:2]              # 0 allocation (view of slice)
    b = [3,4]               # 1 allocation (vector creation)

    sum(a) * sum(b)         # 0 allocation (scalars don't allocate)
end
```

```
julia> @btime foo($x)
  15.015 ns (1 allocation: 80 bytes)
```

```julia
x = rand(10);   tup = Tuple(x)

function foo(x)
    a = x[1:2]              # 0 allocation (slice of tuple)
    b = (3,4)               # 0 allocation (tuple creation)

    sum(a) * sum(b)         # 0 allocation (scalars don't allocate)
end
```

```
julia> @btime foo($tup)
  1.400 ns (0 allocations: 0 bytes)
```

```julia
x = rand(10);   sx = SA[x...]

function foo(x)
    a = x[SA[1,2]]          # 0 allocation (slice of static array)
    b = SA[3,4]             # 0 allocation (static array creation)

    sum(a) * sum(b)         # 0 allocation (scalars don't allocate)
end
```

```
julia> @btime foo($sx)
  1.600 ns (0 allocations: 0 bytes)
```

The decrease in memory allocations from SVectors is especially relevant for operations that result in temporary vectors, such as broadcasting.

```julia
x = rand(10)

foo(x) = sum(2 .* x)
```

```
julia> @btime foo($x)
  17.936 ns (1 allocation: 144 bytes)
```

```
x = rand(10);   sx = SVector(x...)

foo(x) = sum(2 .* x)
```

```
julia> @btime foo($sx)
  1.800 ns (0 allocations: 0 bytes)
```

Interestingly, the performance benefits of SVectors extend beyond memory allocation. This means that, even when operations on regular vectors don't allocate memory, SVectors could still deliver significant speedups. Below, we demonstrate this through the function `sum(f, <vector>)`, which sums the elements of `<vector>` after they're transformed via `f`. The example shows that the implementation with SVectors yields faster execution times, despite regular vectors already not incurring memory allocations.

```
x  = rand(10)

foo(x) = sum(a -> 10 + 2a +  3a^2, x)
```

```
julia> @btime foo($x)
  4.400 ns (0 allocations: 0 bytes)
```

```
x  = rand(10);  sx = SVector(x...);

foo(x) = sum(a -> 10 + 2a +  3a^2, x)
```

```
julia> @btime foo($sx)
  2.900 ns (0 allocations: 0 bytes)
```

SVECTOR TYPE AND ITS MUTABLE VARIANT

Similar to tuples, **SVectors are immutable**, meaning that their elements can't be added, removed, or modified. To accommodate mutable collections, the package `StaticArrays` additionally provides a variant given by the type `MVector`. The creation of MVectors and their slices follow the same syntax as SVectors, where the function `SVector` is replaced with `MVector`. This is illustrated below.

```
x  = [1,2,3]
sx = SVector(x...)

sx[1] = 0
```

```
ERROR: setindex!(::SVector{3, Int64}, value, ::Int) is not defined
```

```
x  = [1,2,3]
mx = MVector(x...)

mx[1] = 0
```

```
julia> mx
3-element MVector{3, Int64} with indices SOneTo(3):
 0
 2
 3
```

The mutability of MVectors makes them ideal for initializing a vector that will eventually be filled via a for-loop. In fact, executing `similar(sx)` when `sx` is an SVector automatically returns an MVector.

▼ 'similar' for SVectors

```
sx = SVector(1,2,3)

mx = similar(sx)        # it defines an MVector with undef elements
```

```
3-element MVector{3, Int64} with indices SOneTo(3):
        2073873450800
 -1152921504606846976
                    0
```

TYPE STABILITY: SIZE IS PART OF THE STATIC VECTORS' TYPE

SVectors are formally defined as objects with type `SVector{N,T}`, where `N` specifies the number of elements and `T` denotes the element's type. For instance, `SVector(4,5,6)` has type `SVector{3,Int64}`, indicating that it comprises 3 elements with type `Int64`. Importantly, this implies that **the number of elements is part of the `SVector` type**. This feature, which is shared with MVectors and inherited from tuples, can readily introduce type instabilities if not handled carefully.

To ensure type stability, you can employ approaches similar to those employed for tuples. Essentially, this entails that we should either pass SVectors and MVectors as function arguments, or dispatch by the number of elements through the `Val` technique.

```
x = rand(50)

function foo(x)
    output = SVector{length(x), eltype(x)}(undef)
    output = MVector{length(x), eltype(x)}(undef)

    for i in eachindex(x)
        temp      = x .> x[i]
        output[i] = sum(temp)
    end

    return output
end

@code_warntype foo(x)                       # type unstable
```

```
x = rand(50);   sx = SVector(x...)

function foo(x)

    output = MVector{length(x), eltype(x)}(undef)

    for i in eachindex(x)
        temp      = x .> x[i]
        output[i] = sum(temp)
    end

    return output
end

@code_warntype foo(sx)                      # type stable
```

```
x = rand(50)

function foo(x, ::Val{N}) where N
    sx     = SVector{N, eltype(x)}(x)
    output = MVector{N, eltype(x)}(undef)

    for i in eachindex(x)
        temp      = x .> x[i]
        output[i] = sum(temp)
    end

    return output
end

@code_warntype foo(x, Val(length(x)))      # type stable
```

PERFORMANCE COMPARISON

MVectors offer performance benefits over regular vectors. However, bear in mind that they're never more performant than SVectors. In fact, certain operations on MVectors may still trigger memory allocations. For this reason, the general guideline is to prefer SVectors when the collection doesn't need to be mutated, restricting MVectors when in-place mutation is necessary.

Below, we compare the performance of SVectors and MVectors. The examples demonstrate that they may exhibit similar performance, although SVectors are more performant in certain cases. Likewise, SVectors and MVectors consistently outperform regular vectors for small collections.

```
x  = rand(10)
sx = SVector(x...);  mx = MVector(x...)

foo(x) = sum(a -> 10 + 2a +  3a^2, x)
```

```
julia> @btime foo($x)
  4.400 ns (0 allocations: 0 bytes)
julia> @btime foo($sx)
  2.800 ns (0 allocations: 0 bytes)
julia> @btime foo($mx)
  2.900 ns (0 allocations: 0 bytes)
```

```
x  = rand(10)
sx = SVector(x...);  mx = MVector(x...)

foo(x) = 10 + 2x +  3x^2
```

```
julia> @btime foo.($x)
  19.739 ns (1 allocation: 144 bytes)
julia> @btime foo.($sx)
  1.600 ns (0 allocations: 0 bytes)
julia> @btime foo.($mx)
  6.600 ns (1 allocation: 96 bytes)
```

STATIC VECTORS VS PRE-ALLOCATIONS

Considering the advantages of static vectors over regular vectors, let's compare their performance to other strategies that reduce memory allocations. In particular, we'll examine how they stack up against pre-allocating memory for intermediate outputs. Our examples demonstrate that static vectors can efficiently store intermediate results, making pre-allocation techniques unnecessary. Moreover, the examples reveal that storing the final output in an MVector can lead to performance gains over using a regular vector.

For the illustration, consider a for-loop that requires an intermediate result during each iteration `i`. This involves counting the number of elements in `x` that are greater than `x[i]`, which can be formally implemented as `sum(x .> x[i])`. To make the comparison stark, we'll isolate the computation of the intermediate step `x .> x[i]`. Note that every implementation below requires

pre-allocating the vector `output`, leaving us with only one decision to make: whether to pre-allocate the temporary vector `temp`. This also explains why all the implementations below involve at least one memory allocation.

```
x = rand(50)

function foo(x; output = similar(x))
    for i in eachindex(x)
        temp      = x .> x[i]
        output[i] = sum(temp)
    end

    return output
end
```

```
julia> @btime foo($x)
  3.188 μs (101 allocations: 5.17 KiB)
```

```
x = rand(50)

function foo(x; output = similar(x), temp = similar(x))
    for i in eachindex(x)
        @. temp      = x > x[i]
           output[i] = sum(temp)
    end

    return output
end
```

```
julia> @btime foo($x)
  695.745 ns (2 allocations: 992 bytes)
```

```
x = rand(50);   sx = SVector(x...)

function foo(x; output = Vector{Float64}(undef, length(x)))
    for i in eachindex(x)
        temp      = x .> x[i]
        output[i] = sum(temp)
    end

    return output
end
```

```
julia> @btime foo($sx)
  183.661 ns (1 allocation: 496 bytes)
```

```julia
x = rand(50);   sx = SVector(x...)

function foo(x; output = similar(x))
    for i in eachindex(x)
        temp      = x .> x[i]
        output[i] = sum(temp)
    end

    return output
end
```

```
julia> @btime foo($sx)
  148.817 ns (1 allocation: 448 bytes)
```

```julia
x = rand(50);   sx = SVector(x...)

function foo(x; output = MVector{length(x),eltype(x)}(undef))
    for i in eachindex(x)
        temp      = x .> x[i]
        output[i] = sum(temp)
    end

    return output
end
```

```
julia> @btime foo($sx)
  148.975 ns (1 allocation: 448 bytes)
```

The "No-Preallocation" tab serves as our baseline, providing a reference point for comparison with the other methods. As for the "Pre-allocating" tab, it reuses a regular vector to compute `temp`. In contrast, the "SVector" tab converts `x` to an SVector `sx` without pre-allocating `temp`. The benchmarks reveal that the latter approach is more performant, as it avoids memory allocation and benefits from additional optimizations provided by SVectors.

As for the last two tabs, they continue to define `x` as an SVector, but additionally treat `output` as an MVector. The last tab in particular does this by using `similar(sx)` to initialize `output`, whereas the other tab explicitly specifies an MVector. Comparing these cases, we observe that MVectors deliver additional performance gains.

FOOTNOTES

[1.] The recommended number of elements provided is actually lower than the documentation's suggested (100 elements). The reason for this discrepancy is that, as you approach the upper limit, the performance benefits of static vectors compared to regular vectors decrease significantly. As a result, the time spent benchmarking with collections of 100 elements will likely offset any potential advantage.

[2.] The approach based on the macro `@SVector` requires some caveats. For instance, it doesn't support definitions based on local variables. Because of this, it precludes the use of `eachindex(x)` in an array comprehension, unless `x` is a global variable.

[3.] For instance, `foo(x...)` is equivalent to `foo(x[1], x[2], x[3])` given a vector or tuple `x` with 3 elements.

9h. Lazy Operations

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INTRODUCTION

Computational approaches can be broadly classified into "lazy" and "eager" categories. **Eager operations** are characterized by their immediate execution upon definition, providing instant access to the results. So far, most operations on this website have fallen under this category.

In contrast, **lazy operations** define the code to be executed, deferring computation until the results are actually needed. The approach is particularly valuable for operations involving heavy intermediate computations, as lazy evaluation can **sidestep unnecessary memory allocations**: by fusing operations, it becomes possible to perform intermediate calculations on the fly and feed them directly into the final calculation.

This section provides various implementations for lazy computations. The first approach presented is based on the so-called **generators**, which are the lazy analogue of array comprehensions. After this, we'll introduce several functions from the package **Iterators**, which provides lazy implementations of functions like `map` and `filter`.

GENERATORS

Array comprehensions offer a convenient technique for creating vectors, using a similar syntax to for-loops to define their elements. These elements are computed and stored right away, making array comprehensions an eager operation. For their part, **generators** represent **the lazy counterpart of array comprehensions**, deferring the creation of elements until they're actually needed.

In terms of syntax, generators are identical to array comprehensions, with the sole difference that they're enclosed in parentheses `( )` instead of square brackets `[ ]`. Just like array comprehensions, generators also retain the ability to add conditions and simultaneously iterate over multiple collections.

```
x = [a for a in 1:10]

y = [a for a in 1:10 if a > 5]
```

```
julia> x
10-element Vector{Int64}:
  1
  2
  ⋮
  9
 10
julia> y
5-element Vector{Int64}:
  6
  7
  8
  9
 10
```

```
x = (a for a in 1:10)

y = (a for a in 1:10 if a > 5)
```

```
julia> x
Base.Generator{UnitRange{Int64}, typeof(identity)}(identity, 1:10)
```

The examples show that array comprehensions compute all their elements at the moment of defining the vector, giving immediate access to them. In contrast, generators formally define an object with type `Base.Generator`, where operations are described, but no output is materialized.

This characteristic of generators makes them particularly useful for computing reductions with transformed values. By producing values on-demand and fusing them with the reduction function, generators avoid the materialization of temporary vectors, thus reducing memory allocations.

To illustrate the performance benefits this entails, let's compute the sum of all elements in a vector `y`. In particular, `y` is obtained by doubling each element of a vector `x`. One way to compute this operation is by first creating the vector `y` and then summing all its elements. Alternatively, we can describe the transformation through a generator, which bypasses the storage of the intermediate output `y` and instead feeds the transformation directly into the `sum` function. This allows the compiler to perform the addition as a cumulative operation on scalars, thereby reducing memory usage.

```julia
x = rand(100)

function foo(x)
    y = [a * 2 for a in x]                  # 1 allocation (same as y = x .* 2)

    sum(y)
end
```

```
julia> @btime foo($x)
  46.945 ns (1 allocation: 896 bytes)
```

```julia
x = rand(100)

function foo(x)
    y = (a * 2 for a in x)                  # 0 allocations

    sum(y)
end
```

```
julia> @btime foo($x)
  23.996 ns (0 allocations: 0 bytes)
```

```julia
x = rand(100)


foo(x) = sum(a * 2 for a in x)              # 0 allocations
```

```
julia> @btime foo($x)
  23.996 ns (0 allocations: 0 bytes)
```

The last tab shows that generators can be incorporated directly as a function argument, resulting in a compact syntax. Remarkably, this syntax is applicable to *any* function that accepts a collection as its input.

ITERATORS

Iterators are formally defined as lazy objects that create sequential values on demand, rather than storing them all in memory upfront. Throughout the website, we've already encountered numerous scenarios involving iterators. A typical example of an iterator is a range, such as `1:length(x)`, which defines a sequence of numbers to be generated on the fly. Their lazy evaluation explains why the function `collect` is needed when we want to materialize the entire sequence into a vector. Without `collect`, iterators merely describe the numbers to be created, without actually creating and storing them in memory.

Beyond simple ranges, we've also covered other types of iterators that offer more specialized functionality. They included `eachindex` for accessing array indices, `enumerate` for pairing elements with their positions, and `zip` for combining multiple sequences.

The lazy nature of iterators makes them particularly efficient in for-loops: by generating each value as the for-loop progresses, we eliminate unnecessary memory allocations that would arise from materializing the list being iterated over.

```
x = 1:10
```

```
julia> x
1:10
```

```
x = collect(1:10)
```

```
julia> x
10-element Vector{Int64}:
  1
  2
  ⋮
  9
 10
```

The built-in package `Iterators`, which is automatically "imported" in every Julia session, provides multiple functions for generating lazy sequences. Additionally, it offers lazy counterparts of various functions such as `filter` and `map`, which can be accessed as `Iterators.filter` and `Iterators.map`. [1]

The following example demonstrates the use of these functions to avoid memory allocations of intermediate computations.

```
x = collect(1:100)

function foo(x)
    y = filter(a -> a > 50, x)              # 1 allocation

    sum(y)
end
```

```
julia> @btime foo($x)
  53.163 ns (1 allocation: 896 bytes)
```

```
x = collect(1:100)

function foo(x)
    y = Iterators.filter(a -> a > 50, x)    # 0 allocations

    sum(y)
end
```

```
julia> @btime foo($x)
  55.239 ns (0 allocations: 0 bytes)
```

```julia
x = rand(100)

function foo(x)
    y = map(a -> a * 2, x)                  # 1 allocation

    sum(y)
end
```

```
julia> @btime foo($x)
  47.963 ns (1 allocation: 896 bytes)
```

```julia
x = rand(100)

function foo(x)
    y = Iterators.map(a -> a * 2, x)        # 0 allocations

    sum(y)
end
```

```
julia> @btime foo($x)
  23.972 ns (0 allocations: 0 bytes)
```

FOOTNOTES

[1.] The `IterTools` package further extends the functionality of `Iterators`, offering even more tools for working with lazy sequences.

9i. Lazy Broadcasting and Loop Fusion

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INTRODUCTION

This section continues the analysis of lazy and eager operations as a means of reducing memory allocations. The focus now shifts to broadcasting operations, which strike a balance between code readability and performance.

A key feature of broadcasting in Julia is its eager default behavior. This means that broadcasted operations compute their outputs immediately upon execution, inevitably leading to memory allocation when applied to objects such as vectors. The feature becomes especially relevant in scenarios with intermediate broadcasted operations, which can result in allocations that are potentially avoidable.

To address the associated performance cost, we'll present various strategies. One approach highlights the notion of **loop fusion**, which allows multiple broadcasting operations to be combined into a single more efficient operation. After this, we'll explore the `LazyArrays` package, which evaluates broadcasting operations lazily.

HOW DOES BROADCASTING WORK?

Let's first examine the internal mechanics of broadcasting. Under the hood, broadcasting operations are converted into optimized for-loops during compilation, rendering the two approaches computationally equivalent. Essentially, broadcasting serves as syntactic sugar, eliminating the need for explicit for-loops. This allows users to write more concise and expressive code, without compromising performance.

Despite this equivalence, you'll often notice performance differences in practice. These discrepancies are largely driven by compiler optimizations, rather than inherent differences between the two approaches. The reason is that, by supporting a broadcasted form, an operation reveals further information to the compiler about its underlying structure. In this way, the compiler is allowed to apply further optimizations. In contrast, for-loops are conceived as a more general construct, and so these additional assumptions shouldn't be taken for granted.

Note, though, that with careful manual optimization, for-loops can always match or surpass the performance of broadcasting. The following code snippets demonstrate this point.

The first tab describes the operation being performed, while the second tab provides a rough translation of broadcasting's internal implementation. The third tab demonstrates the equivalence by writing a for-loop mirroring the exact code used in broadcasting. This is achieved by adding the `@inbounds` macro, which is automatically applied with broadcasting. The specific role of `@inbounds` will be discussed in a later section. Its inclusion here is to illustrate the equivalence between the two approaches once we account for it.

```
x      = rand(100)

foo(x) = 2 .* x
```

```
julia> @btime foo($x)
  34.506 ns (1 allocation: 896 bytes)
```

```
x      = rand(100)

function foo(x)
    output = similar(x)

    for i in eachindex(x)
        output[i] = 2 * x[i]
    end

    return output
end
```

```
julia> @btime foo($x)
  48.188 ns (1 allocation: 896 bytes)
```

```
x      = rand(100)

function foo(x)
    output = similar(x)

    @inbounds for i in eachindex(x)
        output[i] = 2 * x[i]
    end

    return output
end
```

```
julia> @btime foo($x)
  32.832 ns (1 allocation: 896 bytes)
```

> **Warning!** - About `@inbounds`
>
> In the example provided, `@inbounds` **was added to illustrate the internal implementation of broadcasting, not as a general recommended practice**. In fact, used incorrectly, `@inbounds` **can cause serious issues**.

> To understand what this macro does, recall that Julia by default enforces bounds checks on array indices. For instance, it checks that a vector `x` with 3 elements isn't accessed at index `x[4]`, thus preventing out-of-range access. By olacing `@inbounds` on a for-loop, Julia is instructed to disable these checks. The speed gain from omitting bounds checking, though, comes at the cost of safety: an out-of-range access can silently produce incorrect results or lead to other issues.

A key implication of the example is that Julia's broadcasting is eager by default. In the example, this means that `2 .* x` is immediately computed and then stored in `output`. This also explains the observed memory allocation.

Importantly, **memory allocations under broadcasting arise even if the result isn't explicitly stored**. For example, computing `sum(2 .* x)` involves the computation and internal temporary storage of `2 .* x`.

REMARK (OPTIONAL) Differing Optimizations With Broadcasting and For-Loops

BROADCASTING: LOOP FUSION

While eager broadcasting makes results readily available, their outputs may not be important by themselves. They could simply represent intermediate steps in a larger computation, eventually passed as inputs to subsequent operations.

In the following, we address scenarios like this, where broadcasting is employed for intermediate results. The first approach leverages a technique called **loop fusion**, which combines multiple broadcasting operations into a single loop. By doing so, the compiler can perform all operations in a single pass over the data. This not only eliminates the creation of multiple intermediate vectors, but also provides the compiler with a holistic view of the operations, thus enabling further optimizations.

When all broadcasting operations are nested within a single operation, the compiler automatically implements loop fusion. For complex expressions, however, writing a single lengthy expression can be impractical. To overcome this limitation, we'll show how to break down an operation into partial calculations, while still preserving loop fusion. The method relies on the lazy design of functions definitions, allowing operations to be delayed until their final combination.

```
x        = rand(100)

foo(x)   = x .* 2 .+ x .* 3      # or @. x * 2 + x * 3
```

```
julia> @btime foo($x)
  35.361 ns (1 allocation: 896 bytes)
```

```
x        = rand(100)

function foo(x)
    a      = x .* 2
    b      = x .* 3

    output = a .+ b
end
```

```
julia> @btime foo($x)
  124.420 ns (3 allocations: 2.62 KiB)
```

```
x        = rand(100)

term1(a) = a * 2
term2(a) = a * 3

foo(a)   = term1(a) + term2(a)
```

```
julia> @btime foo.($x)
  34.330 ns (1 allocation: 896 bytes)
```

VECTOR OPERATIONS ALLOCATE AND BREAK LOOP FUSION

A common situation preventing loop fusion is when a single expression mixes broadcasting with vector operations. While combining these operations often result in an error, **some vector operations produce the same results as their broadcasting equivalents**, thus precluding dimensional mismatches. For instance, adding two vectors with `+` yields the same result as summing them element-wise with `.+`.

```
x        = [1, 2, 3]
y        = [4, 5, 6]

foo(x,y) = x .+ y
```

```
julia> foo(x,β)
3-element Vector{Int64}:
 5
 7
 9
```

```
x        = [1, 2, 3]
y        = [4, 5, 6]

foo(x,y) = x + y
```

```
julia> foo(x,β)
3-element Vector{Int64}:
 5
 7
 9
```

The same issue arises with vector-scalar multiplication. When one operand is a scalar, the vector product produces the same element-wise result as its broadcasting form.

```
x        = [1, 2, 3]
β        = 2

foo(x,β) = x .* β
```

```
julia> foo(x,β)
3-element Vector{Int64}:
 2
 4
 6
```

```
x        = [1, 2, 3]
β        = 2

foo(x,β) = x * β
```

```
julia> foo(x,β)
3-element Vector{Int64}:
 2
 4
 6
```

OMITTING DOTS AVOIDS LOOP FUSION

Mixing vector operations and broadcasting is problematic for performance, since it prevents loop fusion and forces memory allocations. Going beyond the examples presented before, you can identify this issue when the following conditions are met:

- The final output requires combining multiple operations
- Broadcasting and vector operations would yield the same result
- Broadcasting and vector operations are effectively mixed, due to the omission of some broadcasting dots

If those conditions hold, Julia partitions the work and computes each part separately, producing multiple temporary vectors and extra allocations.

The following example illustrates this possibility in the extreme case where *all* broadcasting dots `.` are omitted. The example demonstrates that, since vector operations aren't fused, the final output is obtained by calculating separately each intermediate operation.

```
x = rand(100)

foo(x)  = x * 2 + x * 3
```

```
julia> @btime foo($x)
  129.269 ns (3 allocations: 2.62 KiB)
```

```
x = rand(100)

function foo(x)
    term1  = x * 2
    term2  = x * 3

    output = term1 + term2
end
```

```
julia> @btime foo($x)
  130.798 ns (3 allocations: 2.62 KiB)
```

While the previous example exclusively consists of vector operations, the same principle applies when some operations are broadcasted and others not. In such cases, loop fusion is partially achieved, with only a subset of operations being internally computed through a single for-loop.

```
x      = rand(100)

foo(x) = x * 2 .+ x .* 3
```

```
julia> @btime foo($x)
  85.034 ns (2 allocations: 1.75 KiB)
```

```
x      = rand(100)

function foo(x)
    term1  = x * 2

    output = term1 .+ x .*3
end
```

```
julia> @btime foo($x)
  85.763 ns (2 allocations: 1.75 KiB)
```

The key takeaway from these examples is that **guaranteeing loop fusion requires appending a dot to every operator and function to be broadcasted**. Note this can be error-prone, especially in large expressions where a single missing dot can be easily missed. Fortunately, there are two alternatives that mitigate this risk.

One option is to prefix the expression with the macro `@.`, as shown below in the tab "Equivalent 1". This ensures that *all* operators and functions are broadcasted. An alternative solution is to combine all operations into a *scalar* function, which you eventually broadcast. This is presented below in the tab "Equivalent 2".

```
x      = rand(100)

foo(x) = x .* 2 .+ x .* 3
```

```
julia> @btime foo($x)
  36.456 ns (1 allocation: 896 bytes)
```

```
x      = rand(100)

foo(x) = @. x * 2 + x * 3
```

```
julia> @btime foo($x)
  36.573 ns (1 allocation: 896 bytes)
```

```
x      = rand(100)

foo(a) = a * 2 + a * 3
```

```
julia> @btime foo.($x)
  34.536 ns (1 allocation: 896 bytes)
```

When several lengthy operations must be combined, the need to split them becomes inevitable. In such cases, we can still leverage the inherent laziness of function definitions. Loop fusion would then be achieved by defining each operation as a separate scalar function.

```
x        = rand(100)

term1(a) = a * 2
term2(a) = a * 3

foo(a)   = term1(a) + term2(a)
```

```
julia> @btime foo.($x)
  35.346 ns (1 allocation: 896 bytes)
```

LAZY BROADCASTING

To handle intermediate computations efficiently, another possibility is to transform broadcasting into a lazy operation. Similar to a function definition, lazy broadcasting defers computations until their results are explicitly required.

The functionality is provided by the `LazyArrays` package, whose use requires prepending the `@~` macro to the broadcasting operation.

```
x         = rand(100)

function foo(x)
    term1  = x .* 2
    term2  = x .* 3

    output = term1 .+ term2
end
```

```
julia> @btime foo($x,$y)
  109.803 ns (3 allocations: 2.62 KiB)
```

```
x         = rand(100)

function foo(x)
    term1  = @~ x .* 2
    term2  = @~ x .* 3

    output = term1 .+ term2
end
```

```
julia> @btime foo($x,$y)
  37.304 ns (1 allocation: 896 bytes)
```

Lazy broadcasting is useful for some scenarios not considered before. Specifically, note that all cases considered thus far have resulted in a *vector* output. In those cases, memory allocations could at best be reduced to a single unavoidable allocation, necessary for storing the final output.

When the final output is instead given by a scalar, as occurs with reductions, lazy broadcasting enables the complete elimination of memory allocations. This is achieved because nesting lazy broadcasted operations implements loop fusion, as illustrated in the example below.

```
# eager broadcasting (default)
x      = rand(100)

foo(x) = sum(2 .* x)
```

```
julia> @btime foo($x,$y)
  48.012 ns (1 allocation: 896 bytes)
```

```
using LazyArrays
x      = rand(100)

foo(x) = sum(@~ 2 .* x)
```

```
julia> @btime foo($x,$y)
  7.906 ns (0 allocations: 0 bytes)
```

Keep in mind that simply using functions isn't sufficient to completely eliminate allocations. Functions allow for the splitting of broadcasting operations, but don't fuse them with operations like reductions.

```
x = rand(100)

term1(a) = a * 2
term2(a) = a * 3
temp(a)  = term1(a) + term2(a)

foo(x)   = sum(temp.(x))
```

```
julia> @btime foo($x,$y)
  48.307 ns (1 allocation: 896 bytes)
```

```
x = rand(100)

term1(a) = a * 2
term2(a) = a * 3
temp(a)  = term1(a) + term2(a)

foo(x)   = sum(@~ temp.(x))
```

```
julia> @btime foo($x,$y)
  10.474 ns (0 allocations: 0 bytes)
```

```
x = rand(100)

function foo(x)
    term1  = @~ x .* 2
    term2  = @~ x .* 3
    temp   = @~ term1 .+ term2

    output = sum(temp)
end
```

```
julia> @btime foo($x,$y)
  13.766 ns (0 allocations: 0 bytes)
```

Remark

An additional advantage of `@~` is the extra optimizations implemented by the macro when possible. This explains why `@~` tends to be faster than alternatives like a lazy map, despite that neither allocates memory. The performance benefit can be observed in the following comparison.

```
x = rand(100)

term1(a) = a * 2
term2(a) = a * 3
temp(a)  = term1(a) + term2(a)

foo(x)   = sum(@~ temp.(x))
```

```
julia> @btime foo($x,$y)
  10.474 ns (0 allocations: 0 bytes)
```

```
x = rand(100)

term1(a) = a * 2
term2(a) = a * 3
temp(a)  = term1(a) + term2(a)

foo(x)   = sum(Iterators.map(temp, x))
```

```
julia> @btime foo($x,$y)
  28.909 ns (0 allocations: 0 bytes)
```

10a. Overview and Goals

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INTRODUCTION

The previous chapters started our study of techniques for improving performance. The focus was in particular on general strategies, particularly type stability and memory allocations. This chapter transitions to **specialized optimization techniques**.

While the analysis will center on SIMD optimizations, there are two important lessons to carry forward from this chapter.

1. **Inherent Trade-offs**. Once type stability is ensured and memory allocations are reduced, further speed gains almost always require a compromise. These trade-offs typically involve sacrificing *precision* (accepting a less accurate result for a faster calculation), *safety* (bypassing safeguards that prevent errors), or *generality* (writing code that is highly specific to one problem). This is precisely why such techniques aren't applied by default in Julia, which prioritizes correctness and safety above all.
2. **Automated Optimization with Macros**. When the need for speed makes these trade-offs worthwhile, macros provide an elegant way to implement complex strategies. They allow developers to package sophisticated optimization algorithms into simple reusable tools. This makes advanced optimization highly accessible, as users can apply these techniques without the need to understand the underlying intricacies of their implementation.

10b. Macros as a Means for Optimizations

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INTRODUCTION

Customized computational approaches often have an edge over general-purpose built-in solutions, as they can tackle the unique challenges of a given scenario. However, the complexity of specialized techniques often deters their adoption among practitioners, who may lack the necessary expertise to implement them.

Macros offer a practical solution to bridge this gap, making specialized computational approaches more accessible to users. They're particularly well-suited for this purpose, due to their ability to take entire code blocks as inputs and transform them into an optimized execution approach. In this way, practitioners benefit from specialized algorithms, without having to implement them themselves.

In the upcoming sections, the role of macros in boosting performance will be central. By leveraging them, we'll be able to effectively separate the benefits provided by an algorithm from its actual implementation details. This decoupling will let us shift our focus from the nitty-gritty details of how to implement algorithms, to the more practical question of when to apply them. The current section in particular will concentrate on the procedure for applying macros.

USES OF MACROS

Macros, denoted by the `@` symbol before their name, resemble functions in that they take input and produce output. Their primary difference lies in what they operate on and what they can return. Specifically, *functions operate on values*: they take evaluated expressions as arguments and return a final computed value. In contrast, *macros operate on code*: they take unevaluated expressions as input and return a new transformed expression, which is then compiled and executed in place of the original macro call.

This unique feature enables macros to handle tasks that functions can't. Two notable applications are worth mentioning.

First, macros can be used **to simplify code**. By automating repetitive tasks and eliminating boilerplate, macros can make code significantly more readable and maintainable. For instance, suppose a function requires multiple slices of `x` to be converted into views. Without macros, this would involve repeatedly invoking `view(x, <indices>)`, resulting in verbose and error-prone code. Instead, prepending the function definition with `@views` will automatically handle all the slice conversions for us. This is demonstrated below.

```julia
x = rand(1_000)

function foo(x)
    x1 = view(x, x .> 0.7)
    x2 = view(x, x .< 0.5)
    x3 = view(x, 1:500)
    x4 = view(x, 501:1_000)

    x1, x2, x3, x4
end
```

```julia
x = rand(1_000)

@views function foo(x)
    x1 = x[x .> 0.7]
    x2 = x[x .< 0.5]
    x3 = x[1:500]
    x4 = x[501:1_000]

    x1, x2, x3, x4
end
```

Another application of macros is **to modify how operations are computed**, which is the focus of the current section. This lets developers package sophisticated optimization techniques, making advanced solutions accessible. As a result, users unfamiliar with a method's underlying complexity can focus on choosing the most suitable computational approach, rather than grappling with the implementation details.

While macros are powerful tools, they're not without their limitations. Their black-box nature means that **misuse of macros can lead to unexpected results or compromise computational safety**. That's why it's crucial to identify the right scenarios for each macro. Although this requires some upfront work, it's considerably less demanding than implementing the same functionality from scratch.

MACROS APPLIED IN FOR-LOOPS

One distinctive feature of Julia is its ability to execute for-loops with exceptional speed. In fact, carefully optimized for-loops can reach peak performance within the language. This efficiency stems from the versatility of for-loops, which lets users fine-tune them for their specific needs. As a result, it's no surprise that one prominent application of macros is to customize how for-loops are computed.

To illustrate this application, let's consider `@inbounds`. To understand what this macro accomplishes, we first need to understand how for-loops behave in Julia. By default, the language implements **bounds checking**: when an element `x[i]` is accessed during the *i*-th iteration, Julia verifies that *i* falls within the valid range of indices for `x`. This built-in mechanism safeguards against errors and security issues caused by out-of-bounds access.

While bounds checking prevents bugs, it comes at a performance cost: the additional checks not only introduce computational overhead, but also limit the compiler's ability to implement certain optimizations. In situations where iterations are guaranteed to stay within an array's bounds, nonetheless, these safety checks become redundant. Consequently, we can safely boost performance by disabling bounds checking with the `@inbounds` macro.

> **Trade-Offs Entailed by @inbounds**
>
> The `@inbounds` macro perfectly illustrates both the power and risks associated with macro usage. When applied judiciously, it can yield substantial performance gains, especially when multiple slices are involved.
>
> Despite this, disabling bounds checking simultaneously renders code unsafe: it increases the risk of crashes and silent errors, additionally creating security vulnerabilities. In this context, `@inbounds` shifts the responsibility of applying the macro onto the user, who must be absolutely certain that the iteration range is within the arrays' bounds.

@INBOUNDS AS AN EXAMPLE

Using a macro within a for-loop requires its inclusion at the beginning of the for-loop. For instance, to disable bounds checking for every indexed element within a for-loop, we simply need to prepend the for-loop with `@inbounds`.

```julia
x = rand(1_000)

function foo(x)
    output = 0.

    @inbounds for i in eachindex(x)
                  a       = log(x[i])
                  b       = exp(x[i])
                  output += a / b
    end

    return output
end
```

```
julia> @btime foo($v,$w,$x,$y)
  5.826 μs (0 allocations: 0 bytes)
```

> **Alternative Application of @inbounds**
>
> We can alternatively apply `@inbounds` individually to any specific line within the for-loop. Nonetheless, this possibility is specific to `@inbounds`. It only arises because the macro can actually be employed even outside for-loops.

```
x = rand(1_000)

function foo(x)
    output = 0.

    for i in eachindex(x)
        @inbounds a       = log(x[i])
        @inbounds b       = exp(x[i])
                  output += a / b
    end

    return output
end
```

```
julia> @btime foo($v,$w,$x,$y)
  5.938 μs (0 allocations: 0 bytes)
```

The performance advantages of `@inbounds` come not only from eliminating bounds checking itself, but also from giving the compiler more freedom to implement additional optimizations.

To understand this, note that bounds checking is essentially a conditional statement, where the iteration is executed only if all indices are within range. As we'll see in the next sections, conditional statements limit the compiler's ability to apply the so-called SIMD instructions, which are a form of parallelism within a single core.

MACROS COULD BE APPLIED AUTOMATICALLY OR DISREGARDED BY THE COMPILER

The influence of a macro on code execution isn't always predictable. In many cases, it might have no impact at all. This is because the compiler has the final say on optimization strategy. Thus, it might already be applying the optimization suggested or determine that the macro's recommendation is unhelpful and simply ignore it. In either case, you can infer a macro is ignored when there's no significant change in execution time after applying it. [1]

Both scenarios can arise with `@inbounds` as we show below.

REDUNDANT MACROS

The compiler could prove on its own that a for-loop is safe and therefore disable bounds checking. In those cases, `@inbounds` becomes redundant. This behavior typically occurs in simple cases, such as when iterating over a single collection and using `eachindex`.

```julia
x = rand(1_000)

function foo(x)
    output = 0.

    for i in eachindex(x)
        output += log(x[i])
    end

    return output
end
```

```
julia> @btime foo($v,$w,$x,$y)
  3.384 μs (0 allocations: 0 bytes)
```

```julia
x = rand(1_000)

function foo(x)
    output = 0.

    @inbounds for i in eachindex(x)
        output += log(x[i])
    end

    return output
end
```

```
julia> @btime foo($v,$w,$x,$y)
  3.228 μs (0 allocations: 0 bytes)
```

DISREGARDED MACRO

A macro can also be treated as a mere hint that the compiler is free to disregard. In such cases, a macro signals that the necessary conditions for a particular optimization are satisfied, allowing the compiler to consider more aggressive strategies. The compiler will then carefully analyze the operations and decide if the suggested approach is actually beneficial. This fact highlights how macros can guide the compiler toward better performance, without imposing strict directives.

A prime example along these lines is the `@simd` macro. This suggests the application of SIMD instructions, a subject that will be explored in upcoming sections. When `@simd` is added to a for-loop, the compiler retains full autonomy in deciding whether to implement the suggested optimization. In the example below, the compiler concludes that SIMD would likely degrade performance, thus ignoring `@simd`. This explains why the execution time remains the same with and without the macro. [2]

```julia
x = rand(2_000_000)

function foo(x)
    output = similar(x)

    for i in eachindex(x)
        output[i] = if (200_000 > i > 100_000)
                        x[i] * 1.1
                    else
                        x[i] * 1.2
                    end
    end

    return output
end
```

```
julia> @btime foo($x)
  881.929 μs (3 allocations: 15.259 MiB)
```

```julia
x = rand(2_000_000)

function foo(x)
    output = similar(x)

    @simd for i in eachindex(x)
        output[i] = if (200_000 > i > 100_000)
                        x[i] * 1.1
                    else
                        x[i] * 1.2
                    end
    end

    return output
end
```

```
julia> @btime foo($x)
  863.776 μs (3 allocations: 15.259 MiB)
```

FOOTNOTES

[1.] More formally, one can verify that macros are ignored by examining the generated machine code. Because we haven't covered how to inspect that code, we'll instead use the runtime behavior as an indicator.

[2.] It's possible to confirm that the generated machine code is identical by inspecting the compiler's output.

10c. Introduction to SIMD

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INTRODUCTION

Single Instruction, Multiple Data (SIMD) is an optimization technique widely embraced in modern CPU architectures. At its core, SIMD allows a single CPU instruction to process multiple data points concurrently, rather than sequentially processing them one by one. The parallel approach can yield substantial performance gains, especially for workloads involving simple identical calculations repeated across multiple data elements. [1]

To illustrate the power of SIMD, consider a computation involving four separate addition operations. Without SIMD, the computer would need to execute four distinct instructions, one for each addition. Instead, SIMD makes it possible to bundle the four additions into a single instruction, allowing the CPU to process them all at once. In an ideal scenario, the time required to complete four additions with SIMD would be the same as completing one addition without it.

The efficiency of SIMD lies in its ability to leverage parallelism within a single CPU core. By operating on vectors rather than individual elements, SIMD instructions can execute the same operation on multiple data points simultaneously. This is why the process of applying SIMD is often referred to as **vectorization**.

Throughout the sections, we'll cover two approaches for implementing SIMD instructions.

- Julia's native capabilities.
- The package `LoopVectorization`.

This section will exclusively concentrate on the built-in tools for applying SIMD. In particular, we'll explore the conditions that trigger automatic vectorization and also introduce the `@simd` macro, which lets you manually apply it in for-loops. We'll save our discussion of `LoopVectorization` for later sections. Relative to Julia's built-in tools, this package often implements more aggressive optimizations, but can also introduce bugs if misused.

WHAT IS SIMD?

SIMD is a type of instruction-level parallelism that occurs within a single processor core, applying the same operation to multiple data elements at once. It's particularly effective for basic arithmetic operations, such as addition and multiplication, when the same operation must be applied to multiple data elements. Given the nature of these operations, it's unsurprising that one of SIMD's primary applications is in linear algebra, where operations like matrix multiplication involve applying identical arithmetic steps to multiple elements.

At the heart of SIMD lies the process of vectorization, where data is split into sub-vectors that can be processed as single units. To facilitate this operation, modern processors include specialized SIMD registers designed for this purpose. Today's processors typically feature 256-bit registers for vectorized operations, which are wide enough to hold four values of either `Float64` or `Int64`.

To illustrate the workings of SIMD, consider the task of adding two vectors, `x = [1, 2, 3, 4]` and `y = [10, 20, 30, 40]`. In traditional scalar processing, performing the operation `x + y` would require four separate addition operations, one for each pair of numbers. This means that `1+10`, `2+20`, `3+30`, `4+40` must be executed sequentially, thereby producing the result `[11, 22, 33, 44]` in four steps. In contrast, all four additions can be performed with a single instruction under SIMD in one step. The processor loads all four elements of `x` and `y` into the 256-bit register, and then runs a single "add" instruction to compute all four sums simultaneously.

For larger vectors, the process remains fundamentally the same. The only difference is that the processor first partitions the vectors into sub-vectors that fit within the register's capacity. After this, the processor computes all the operations within each sub-vector simultaneously, repeatedly applying the same the procedure for every sub-vector.

BROADCASTING AND FOR-LOOPS

Based on the previous discussion, we can identify **two types of operations that can potentially benefit from SIMD instructions: for-loops and broadcasting**.

In the case of broadcasting, the compiler implements SIMD automatically, without any user intervention. Instead, the application of SIMD in for-loops isn't guaranteed. This is why the upcoming sections will identify conditions under which SIMD instructions can be applied effectively. If these conditions aren't met, SIMD will become infeasible or substantially reduce its effectiveness. Given this, we'll also provide guidance on how to handle scenarios that aren't well-suited for SIMD.

To pave the way and shift our attention to for-loops, we conclude this section by illustrating the automatic application of SIMD in broadcasting.

SIMD IN BROADCASTING

The decision to apply SIMD instructions is made entirely by the compiler, which relies on a set of heuristics to determine when their use will pay off. One case where Julia strongly favors SIMD is in broadcasting operations.

The following example demonstrates this. It compares the same computation implemented using a for loop and using broadcasting. While broadcasting automatically takes advantage of SIMD, this is not necessarily true for for-loops, and in fact it's not in this case.

```julia
x      = rand(1_000_000)

function foo(x)
    output = similar(x)

    for i in eachindex(x)
        output[i] = 2 / x[i]
    end

    return output
end
```

```
julia> @btime foo($x)
  789.564 µs (2 allocations: 7.629 MiB)
```

```julia
x      = rand(1_000_000)

foo(x) = 2 ./ x
```

```
julia> @btime foo($x)
  414.250 µs (2 allocations: 7.629 MiB)
```

FOOTNOTES

[1.] SIMD isn't exclusive to CPUs, with GPUs also taking advantage of it. GPU's architecture is a natural fit for SIMD, as it was conceived for parallel processing of simple identical operations.

10d. SIMD: Independence of Iterations

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INTRODUCTION

Broadcasting heavily favors the application of SIMD instructions. In contrast, whether and when for-loops apply SIMD is more complex. Furthermore, the heuristics of the compiler, while powerful, aren't without flaws. Indeed, it's entirely possible that SIMD is implemented when it actually degrades performance or not applied when it would've been advantageous. To address this, Julia provides the `@simd` macro to manually implement SIMD, giving developers a more granular control over the optimization process.

An effective application of SIMD requires identifying the conditions under which this optimization can be applied. Failing to meet these criteria can render SIMD infeasible or necessitate internet code adaptations that slow down computation. The ideal conditions for leveraging SIMD instructions are:

- **Independence of Iterations**: Except for reductions, which are specifically handled to ensure their feasibility.
- **Unit Strides**: Elements in collections must be accessed sequentially.
- **No Conditional Statements**: The loop body should consist solely of straight-line code.

In the upcoming sections, we'll elaborate on each of these items, additionally providing guidance on how to address scenarios not conforming to them. This section in particular exclusively focuses on the independence of iterations.

> **Warning!** - Determining Whether SIMD Has Been Implemented
>
> Assessing whether SIMD instructions are implemented requires inspecting the compiled code. Due to the complexity of this approach, we'll instead rely on execution times as a practical indicator.

REMARKS ABOUT @SIMD IN FOR-LOOPS

Recall from the previous section that the impact of macros on computational methods is intricate. The reason is that macros only serve as a hint to the compiler, rather than a strict directive. Consequently, they suggest techniques that the compiler may eventually discard or would have implemented regardless—the compiler has the final say on which optimizations are worth adopting. In this context, the inclusion of `@simd` in a for-loop is far from a guarantee that SIMD will actually be implemented.

Furthermore, it's notoriously difficult to predict whether SIMD instructions are beneficial in particular scenarios. This is due to several factors. Firstly, different CPU architectures provide varying levels of support for SIMD instructions. [1] This diversity in SIMD capabilities means that the benefits of SIMD tend to vary greatly by hardware.

Second, as we already mentioned, it's hard to anticipate when and how SIMD will be applied in our code. The compiler relies on sophisticated heuristics to determine when SIMD may be advantageous, but they aren't infallible. Indeed, it's entirely possible that SIMD is implemented when it actually reduces performance or not applied when it would've been advantageous.

Despite these complexities, structuring operations in certain ways can improve the likelihood of implementing SIMD beneficially. By identifying these conditions, we'll be able to write code that's more amenable to SIMD optimization. It's worth remarking, though, that **the recommendations we'll present should be interpreted as general principles, rather than absolute rules**. Given the complexity of SIMD, benchmarking remains necessary to validate the existence of any performance improvement.

> **Safety of SIMD**
>
> Strictly speaking, SIMD is a form of parallelization. We'll see in subsequent sections that parallelization may render code unsafe and lead to catastrophic errors when used improperly. `@simd` doesn't involve these types of risks, since it's been designed to apply only when it's safe to do so. Specifically, the compiler will disregard SIMD if the conditions for its safe application aren't met.

INDEPENDENCE OF ITERATIONS

To effectively apply SIMD, iterations should be independent. This means that no iteration should depend on the results of previous iterations or affect the results of subsequent ones. When this condition is met, each iteration can be executed in parallel. A typical scenario is when we need to apply some function $f(x_i)$ to each element x_i of a vector `x`.

In the following, we illustrate this case via a polynomial transformation of `x`. The transformation will be done through for-loops with and without SIMD. We'll also compare these approaches with broadcasting, which applies SIMD automatically.

Importantly, as we'll explain in a subsequent section, applying `@simd` in for-loops requires the `@inbounds` macro. We'll see that, essentially, checking index bounds introduces a condition, giving rise to execution branches that hinder or directly prevent the application of SIMD.

```julia
x = rand(1_000_000)

function foo(x)
    output = similar(x)

    for i in eachindex(x)
        output[i] = x[i] / 2 + x[i]^2 / 3
    end

    return output
end
```

```
julia> @btime foo($x)
  806.606 μs (2 allocations: 7.629 MiB)
```

```julia
x = rand(1_000_000)

function foo(x)
    output = similar(x)

    @inbounds @simd for i in eachindex(x)
        output[i] = x[i] / 2 + x[i]^2 / 3
    end

    return output
end
```

```
julia> @btime foo($x)
  464.734 μs (2 allocations: 7.629 MiB)
```

```julia
x = rand(1_000_000)

foo(x) = @. x / 2 + x^2 / 3
```

```
julia> @btime foo($x)
  447.074 μs (4 allocations: 7.629 MiB)
```

A SPECIAL CASE OF DEPENDENCE: REDUCTIONS

SIMD requires that iterations are independent. One exception to this rule is given by reductions, which have been carefully designed for their proper handling.

Julia leverages SIMD automatically for reductions involving integers. Instead, reductions with floating-point numbers require the explicit addition of the `@simd` macro. The following example demonstrates this fact. For the case of integers, we see that there are no differences in execution times with and without `@simd`.

```
x = rand(1:10, 10_000_000)   # random integers between 1 and 10

function foo(x)
    output = 0

    for a in x
        output += a
    end

    return output
end
```

```
julia> @btime foo($x)
  2.606 ms (0 allocations: 0 bytes)
```

```
x = rand(1:10, 10_000_000)   # random integers between 1 and 10

function foo(x)
    output = 0

    @simd for a in x
        output += a
    end

    return output
end
```

```
julia> @btime foo($x)
  2.636 ms (0 allocations: 0 bytes)
```

This behavior contrasts with a sum reduction consisting of floating-point operations, as shown below.

```
x = rand(10_000_000)

function foo(x)
    output = 0.0

    for a in x
        output += a
    end

    return output
end
```

```
julia> @btime foo($x)
  5.033 ms (0 allocations: 0 bytes)
```

```
x = rand(10_000_000)

function foo(x)
    output = 0.0

    @simd for a in x
        output += a
    end

    return output
end
```

```
julia> @btime foo($x)
  2.753 ms (0 allocations: 0 bytes)
```

Why Floating Points Are Treated Differently

Unlike integers, addition of floating-point numbers doesn't obey associativity: due to the inherent imprecision of floating-point arithmetic, `(x+y)+z` may not be exactly equal to `x+(y+z)`. This is one of several reasons why floating-point numbers are distinct from real numbers: they are finite-precision approximations, which don't always follow the same mathematical properties.

The following code shows this feature of floating points.

```
x = 0.1 + (0.2 + 0.3)
```

```
julia> x
0.6
```

```
x = (0.1 + 0.2) + 0.3
```

```
julia> x
0.6000000000000001
```

By instructing the compiler to ignore the non-associativity of floating-point arithmetic, SIMD instructions can optimize performance by reordering terms. However, this approach assumes that the operations do not rely on a specific order of operations.

FOOTNOTES

[1.] For instance, x86 architectures (Intel and AMD processors) offer SSE (Streaming SIMD Extensions) and AVX (Advanced Vector Extensions). In turn, each comprises variants supporting different vector widths and operations (e.g., the variant AVX-512 in Intel Xeon processors).

10e. SIMD: Contiguous Access and Unit Strides

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INTRODUCTION

This section contrasts the performance of copies and views, emphasizing that copies guarantee conditions favorable to SIMD execution.

To understand this, recall that SIMD improves computational performance by simultaneously executing the same operation on multiple data elements. Technically, this is achieved through the use of specialized vector registers, which can hold several values (e.g., 4 floating-point numbers or 4 integers). They allow operations such as multiple additions or multiplications to be completed with a single instruction.

For SIMD to fully exploit this vector-based processing, **data must adhere to specific organizational and access rules in memory**. The two core requirements are contiguous memory layout and unit-stride access.

- **Contiguous Memory Layout**: this means that data elements reside in adjacent memory addresses with no gaps. Freshly allocated arrays meet this requirement, enabling entire segments to load directly into vector registers. In contrast, array views don't guarantee contiguity. By referencing the original data structure, views can result in highly irregular memory access patterns that jump between non-adjacent addresses.

- **Unit Strides**: strides refer to the step size between consecutive memory accesses. Unit strides means in particular that elements are accessed sequentially. For example, iterating through a freshly allocated vector `x` using `eachindex(x)` ensures unit stride: each access moves to the next adjacent address in memory. This contrasts with ranges having a non-unit stride such as `1:2:length(x)` or indices with no predictable pattern (e.g., `[1, 5, 3, 4]` as an index vector).

The fact that SIMD performs best when these two conditions hold creates a key trade-off for developers. On the one hand, using views reduces memory allocations, but makes SIMD less effective as it often violates contiguity or unit stride. Instead, copies create new contiguous arrays that ensure the effectiveness of SIMD, but at the expense of increase memory usage. The current section explores this trade-off.

REVIEW OF INDEXING APPROACHES

The performance trade-offs between copies and views depend on the slicing method employed. For this reason, we begin with a brief overview of the main slicing techniques.

```julia
x         = [10, 20, 30]

indices   = sortperm(x)
elements  = x[indices]    # equivalent to `sort(x)`
```

```
julia> indices
3-element Vector{Int64}:
 1
 2
 3
julia> elements
3-element Vector{Int64}:
 10
 20
 30
```

```julia
x         = [2, 3, 4, 5, 6]

indices_1 = 1:length(x)         # unit strides, equivalent to 1:1:length(x)
indices_2 = 1:2:length(x)       # strides two
```

```
julia> collect(indices_1)
5-element Vector{Int64}:
 1
 2
 3
 4
 5
julia> collect(indices_2)
3-element Vector{Int64}:
 1
 3
 5
```

```julia
x         = [20, 10, 30]

indices   = x .> 15
elements  = x[indices]
```

```
julia> indices
3-element BitVector:
 1
 0
 1
julia> elements
2-element Vector{Int64}:
 20
 30
```

Vector Indexing works by referencing elements directly through their indices. **Ranges** are simply as a special form of vector indexing that lazily references consecutive elements according to a specified stride. The general syntax is `<first index>:<stride>:<last index>`, where omitting `<stride>`

defaults to a step size of 1.

The function `sortperm` is incorporated as it'll starkly illustrate the difference between copies and views. Given some collection `x`, `sortperm(x)` returns the corresponding indices of `sort(x)`. Assuming that `x` hasn't been ordered previously, views that access elements via the indices of `sortperm(x)` will result in an irregular memory access pattern. For instance, given `x = [5, 4, 6]`, we get that `sort(x)` returns `[4, 5, 6]` and `sortperm(x)` provides `[2, 1, 3]`. Therefore, accessing elements of `x` via `sortperm(x)` will involve jumping around within the underlying data.

For its part, **Boolean indexing** returns a Boolean vector, with `1` indicating the element must be kept. This approach is primarily employed for the creation of slices based on broadcasted conditions.

BENEFITS OF SEQUENTIAL ACCESS

In the section on reducing memory allocations, we highlighted the benefits of using views over copies when working with slices: by maintaining references to the original data, views avoid memory allocation. Yet, we briefly remarked that copies could outperform views in some scenarios. We're now in a position to explain in more depth why this occurs.

Creating a copy of some data structure involves allocating the information in a new contiguous block of memory. This layout ensures that elements are stored sequentially, thus offering two key advantages: faster element retrieval and a more effective use of SIMD instructions. Views can't guarantee this property. If the referenced elements are scattered in memory, views will necessarily introduce irregular access patterns.

An analogy may help clarify this. Imagine retrieving books from a library. If every book you need are neatly arranged on a single shelf, collecting them is straightforward—you move once, grab the entire stack, and proceed. This mirrors contiguous memory access. Conversely, if the books are dispersed across different floors and sections, each retrieval demands additional time and effort, akin to non-contiguous access. The analogy extends further: if you also have a cart that allows you to carry multiple books at once, the process becomes even more efficient. SIMD operations play the role of this cart.

ILLUSTRATING EACH BENEFIT IN ISOLATION

The following examples illustrate the potential advantages of copies compared to views. To ensure that the memory allocations of copies aren't distorting the results, slices will be created outside the benchmarked function,

We start with a scenario where views access elements following a pattern, but without this being characterized by unit stride.

```julia
x = rand(1_000_000)
y = @view x[1:2:length(x)]

function foo(y)
    output = 0.0

    for a in y
        output += a
    end

    return output
end
```

```
julia> @btime foo($y)
  884.378 µs (0 allocations: 0 bytes)
```

```julia
x = rand(1_000_000)
y = @view x[1:2:length(x)]

function foo(y)
    output = 0.0

    @simd for a in y
        output += a
    end

    return output
end
```

```
julia> @btime foo($y)
  905.283 µs (0 allocations: 0 bytes)
```

```julia
x = rand(1_000_000)
y = x[1:2:length(x)]

function foo(y)
    output = 0.0

    for a in y
        output += a
    end

    return output
end
```

```
julia> @btime foo($y)
  196.518 µs (0 allocations: 0 bytes)
```

```
x = rand(1_000_000)
y = x[1:2:length(x)]

function foo(y)
    output = 0.0

    @simd for a in y
        output += a
    end

    return output
end
```

```
julia> @btime foo($y)
  41.445 μs (0 allocations: 0 bytes)
```

In some cases, the access of elements doesn't follow any predictable pattern. Such a behavior can be illustrated by accessing elements via `sortperm`.

```
x       = rand(5_000_000)

indices = sortperm(x)
y       = @view x[indices]

function foo(y)
    output = 0.0

    for a in y
        output += a
    end

    return output
end
```

```
julia> @btime foo($y)
  21.584 ms (0 allocations: 0 bytes)
```

```julia
x       = rand(5_000_000)

indices = sortperm(x)
y       = @view x[indices]

function foo(y)
    output = 0.0

    @simd for a in y
        output += a
    end

    return output
end
```

```
julia> @btime foo($y)
  21.333 ms (0 allocations: 0 bytes)
```

```julia
x       = rand(5_000_000)

indices = sortperm(x)
y       = x[indices]

function foo(y)
    output = 0.0

    for a in y
        output += a
    end

    return output
end
```

```
julia> @btime foo($y)
  2.390 ms (0 allocations: 0 bytes)
```

```julia
x       = rand(5_000_000)

indices = sortperm(x)
y       = x[indices]

function foo(y)
    output = 0.0

    @simd for a in y
        output += a
    end

    return output
end
```

```
julia> @btime foo($y)
  1.028 ms (0 allocations: 0 bytes)
```

A similar issue occurs with Boolean indexing.

```
x       = rand(5_000_000)

indices = x .> 0.5
y       = @view x[indices]

function foo(y)
    output = 0.0

    for a in y
        output += a
    end

    return output
end
```

```
julia> @btime foo($y)
  2.196 ms (0 allocations: 0 bytes)
```

```
x       = rand(5_000_000)

indices = x .> 0.5
y       = @view x[indices]

function foo(y)
    output = 0.0

    @simd for a in y
        output += a
    end

    return output
end
```

```
julia> @btime foo($y)
  1.895 ms (0 allocations: 0 bytes)
```

```
x       = rand(5_000_000)

indices = x .> 0.5
y       = x[indices]

function foo(y)
    output = 0.0

    for a in y
        output += a
    end

    return output
end
```

```
julia> @btime foo($y)
  1.017 ms (0 allocations: 0 bytes)
```

```
x       = rand(5_000_000)

indices = x .> 0.5
y       = x[indices]

function foo(y)
    output = 0.0

    @simd for a in y
        output += a
    end

    return output
end
```

```
julia> @btime foo($y)
  279.995 µs (0 allocations: 0 bytes)
```

COPIES VS VIEWS: TOTAL EFFECTS

The previous examples defined slices outside the benchmarked functions, thus avoiding the memory allocations of copies. The goal was to emphasize the benefits of contiguous access and unit strides in isolation. However, the choice between copies and views requires incorporating the overhead of additional memory allocations. Overall, we must weigh these costs against the performance benefits of sequential memory accesses.

In general, benchmarking is the only way to decide whether copies or views deliver better performance. For instance, below we present a case where views are preferred, since operation executed is straightforward enough to benefit from SIMD.

```
x       = rand(5_000_000)
indices = sortperm(x)

function foo(x, indices)
    y      = @view x[indices]
    output = 0.0

    @simd for a in y
        output += a
    end

    return output
end
```

```
julia> @btime foo($x, $indices)
  23.905 ms (0 allocations: 0 bytes)
```

```
x       = rand(5_000_000)
indices = sortperm(x)

function foo(x, indices)
    y      = x[indices]
    output = 0.0

    @simd for a in y
        output += a
    end

    return output
end
```

```
julia> @btime foo($x, $indices)
  35.922 ms (2 allocations: 38.148 MiB)
```

Instead, the following scenario features a more complex but SIMD-friendly operation, illustrating how copying can actually outperform using views.

```
x       = rand(5_000_000)
indices = sortperm(x)

function foo(x, indices)
    y      = @view x[indices]
    output = 0.0

    @simd for a in y
        output += a^(3/2)
    end

    return output
end
```

```
julia> @btime foo($x, $indices)
  269.828 ms (0 allocations: 0 bytes)
```

```julia
x       = rand(5_000_000)
indices = sortperm(x)

function foo(x, indices)
    y      = x[indices]
    output = 0.0

    @simd for a in y
        output += a^(3/2)
    end

    return output
end
```

```
julia> @btime foo($x, $indices)
  131.374 ms (2 allocations: 38.148 MiB)
```

SPECIAL CASES

Certain patterns allow us to predict the faster approach without exhaustive benchmarking.

One scenario where views always outperform copies is when the view is referencing sequential elements. These scenarios call for the use of view, as they provide the same benefits as copies but additionally avoiding memory allocations.

```julia
x       = rand(1_000_000)

indices = 1:length(x)
y       = @view x[indices]

function foo(y)
    output = 0.0

    @simd for a in y
        output += a
    end

    return output
end
```

```
julia> @btime foo($y)
  83.773 µs (0 allocations: 0 bytes)
```

```
x       = rand(1_000_000)

indices = 1:length(x)
y       = x[indices]

function foo(y)
    output = 0.0

    @simd for a in y
        output += a
    end

    return output
end
```

```
julia> @btime foo($y)
  82.349 μs (0 allocations: 0 bytes)
```

In contrast, a common scenario where copies tend to outpace views is when we need to perform multiple operations on the same slice. In this case, the cost of an additional memory allocation is usually offset by the speed gains from contiguous memory access. This is illustrated below.

```
x       = rand(5_000_000)
indices = sortperm(x)

function foo(x, indices)
    y      = @view x[indices]
    output1, output2, output3 = (0.0 for _ in 1:3)

    @simd for a in y
        output1 += a^(3/2)
        output2 += a / 3
        output3 += a * 2.5
    end

    return output1, output2, output3
end
```

```
julia> @btime foo($y)
  269.302 ms (0 allocations: 0 bytes)
```

```julia
x       = rand(5_000_000)
indices = sortperm(x)

function foo(x, indices)
    y      = x[indices]
    output1, output2, output3 = (0.0 for _ in 1:3)

    @simd for a in y
        output1 += a^(3/2)
        output2 += a / 3
        output3 += a * 2.5
    end

    return output1, output2, output3
end
```

```
julia> @btime foo($y)
  142.894 ms (2 allocations: 38.148 MiB)
```

10f. SIMD: Branchless Code

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BRANCHES

SIMD accelerates computations by executing the same set of instructions in parallel across multiple data elements. Yet, certain programming constructs, particularly conditional statements, can severely degrade SIMD efficiency. The issue arises because conditional statements inherently lead to different instruction paths, thus disrupting the single instruction execution that SIMD relies on. While the compiler attempts to mitigate this issue by transforming code into SIMD-compatible forms, these adaptations often incur a performance penalty.

This section explores strategies for efficiently applying SIMD in the presence of conditional operations. We'll first examine scenarios where the compiler introduces conditional statements as an artifact of its internal computation techniques. By employing alternative coding strategies, we'll show how these conditional statements can be bypassed.

After this, we'll explore conditional statements that are intrinsic to program logic and therefore unavoidable. This includes the standard scenario where conditions are explicitly introduced in the code. In this respect, we'll revisit the usual approaches to expressing conditions, focusing on their internal implementation. We'll outline their relative strengths and limitations, indicating which approaches are more conducive to SIMD optimizations. Finally, we'll show that conditional statements can be equivalently recast as algebraic operations, which effectively removes the branching logic that disrupts SIMD execution.

TYPE STABILITY AND BOUNDS CHECKING AS AVOIDABLE CONDITIONS

Two patterns in Julia introduce hidden branches that hurt SIMD performance: type-unstable functions and bounds checking in array indexing. These conditions arise internally from compiler decisions, rather than explicit code, making them easy to overlook.

When a function is type-unstable, Julia generates multiple execution branches, one for each type. Those extra branches, while invisible to you, still disrupt the uniform instruction flow required by SIMD. The remedies for this case are the same as those for fixing type instabilities. Regardless of any SIMD consideration, recall that you should always strive for type stability. Type instability is a major performance bottleneck, with any attempt to achieve high performance becoming nearly impossible without addressing it.

The other source of hidden conditionals arises in for-loops, which perform bounds checking by default. This operation represents a subtle form of conditional execution, where each iteration is executed only when indices remain within bounds.

In relation to this scenario, the example below demonstrates two key insights. First, merely adding `@inbounds` can be enough to induce the compiler to apply SIMD instructions, rendering `@simd` annotations redundant for performance improvements. This explains why using `@inbounds @simd` in the example has a negligible impact on execution times. [1] Second, the example highlights that **adding `@inbounds` is a necessary precondition for the application of SIMD**. Simply using `@simd` on its own won't trigger the implementation of SIMD instructions, as the compiler may still be hindered by the bounds checks. Overall, if we aim to apply SIMD in a for-loop, we should prepend it with `@inbounds @simd`.

```
x = rand(1_000_000)

function foo(x)
    output = similar(x)

    for i in eachindex(x)
        output[i] = 2/x[i]
    end

    return output
end
```

```
julia> @btime foo($x)
  775.469 µs (2 allocations: 7.629 MiB)
```

```
x = rand(1_000_000)

function foo(x)
    output = similar(x)

    @simd for i in eachindex(x)
        output[i] = 2/x[i]
    end

    return output
end
```

```
julia> @btime foo($x)
  792.687 µs (2 allocations: 7.629 MiB)
```

```
x = rand(1_000_000)

function foo(x)
    output = similar(x)

    @inbounds for i in eachindex(x)
        output[i] = 2/x[i]
    end

    return output
end
```

```
julia> @btime foo($x)
  474.154 μs (2 allocations: 7.629 MiB)
```

```
x = rand(1_000_000)

function foo(x)
    output = similar(x)

    @inbounds @simd for i in eachindex(x)
        output[i] = 2/x[i]
    end

    return output
end
```

```
julia> @btime foo($x)
  452.921 μs (2 allocations: 7.629 MiB)
```

Broadcasting and For-Loops

Broadcasting disables bounds checking and strongly favors SIMD by default, often making it appear more performant than a simple for-loop. Despite this, broadcasting essentially serves as a concise notation for implementing a for-loop. As the example below demonstrates, a for-loop that has been optimized with `@inbounds` and `@simd` will typically exhibit a similar level of performance to a broadcasted operation.

```
x      = rand(1_000_000)
foo(x) = 2 ./ x
```

```
julia> @btime foo($x)
  431.487 μs (2 allocations: 7.629 MiB)
```

```julia
x      = rand(1_000_000)

function foo(x)
    output = similar(x)

    @inbounds @simd for i in eachindex(x)
        output[i] = 2/x[i]
    end

    return output
end
```

```
julia> @btime foo($x)
  435.973 μs (2 allocations: 7.629 MiB)
```

```julia
x      = rand(1_000_000)

function foo(x)
    output = similar(x)

    for i in eachindex(x)
        output[i] = 2/x[i]
    end

    return output
end
```

```
julia> @btime foo($x)
  809.359 μs (2 allocations: 7.629 MiB)
```

APPROACHES TO CONDITIONAL STATEMENTS

When conditions are part of the program's logical flow and therefore unavoidable, we need to inquire on what approach is better for the introduction.

Specifically, conditional statements can be evaluated either eagerly or lazily. To illustrate, let's consider the computation of `1 + 1` but only if certain condition `C` is met. A lazy approach evaluates whether `C` holds true, before proceeding with the computation of `1 + 1`. Thus, the operation is deferred until it's confirmed that `C` holds. In contrast, an eager approach computes `1 + 1`, regardless of whether `C` is satisfied. If `C` turns out to be false, the computation remains unused.

When conditional statements are applied only once, a lazy approach is almost always more performant as it avoids needless computations. However, inside a for-loop, SIMD can compute multiple operations simultaneously. Consequently, it may be beneficial to evaluate all conditions and branches upfront, selecting the relevant branches afterward. The possibility is especially true when branches involve inexpensive computations.

In Julia, whether a conditional statement is evaluated lazily or eagerly depends on how it's written. Next, we explore this nuance in more detail.

IFELSE VS IF

The `ifelse` function in Julia follows an eager evaluation strategy, where both the condition and possible outcomes are computed before deciding which result to return. In contrast, `if` and `&&` favor lazy computations, only evaluating the necessary components based on the truth value of the condition.

The following example demonstrates this computational difference through a reduction operation that's contingent on a condition. [2]

```
x = rand(1_000_000)

function foo(x)
    output = 0.0

    for i in eachindex(x)
        if x[i] > 0.5
            output += x[i]/2
        end
    end

    return output
end
```

```
julia> @btime foo($x)
  415.373 µs (0 allocations: 0 bytes)
```

```
x = rand(1_000_000)

function foo(x)
    output = 0.0

    @inbounds @simd for i in eachindex(x)
        if x[i] > 0.5
            output += x[i]/2
        end
    end

    return output
end
```

```
julia> @btime foo($x)
  414.155 µs (0 allocations: 0 bytes)
```

```
x = rand(1_000_000)

function foo(x)
    output = 0.0

    for i in eachindex(x)
        output += ifelse(x[i] > 0.5, x[i]/2, 0)
    end

    return output
end
```

```
julia> @btime foo($x)
  393.046 μs (0 allocations: 0 bytes)
```

```
x = rand(1_000_000)

function foo(x)
    output = 0.0

    @inbounds @simd for i in eachindex(x)
        output += ifelse(x[i] > 0.5, x[i]/2, 0)
    end

    return output
end
```

```
julia> @btime foo($x)
  87.192 μs (0 allocations: 0 bytes)
```

As the example reveals, an eager computation doesn't automatically imply the application of SIMD. This is precisely why the macro `@simd` is included, which provides a hint to the compiler that vectorizing the operation might be beneficial. In fact, we'll show later that adding `@simd` when conditions comprise multiple statements could prompt the compiler to vectorize conditions, while still relying on a lazy evaluation.

It's also worth remarking that applying SIMD instructions doesn't necessarily increase performance. The example below demonstrates this point, where the compiler adopts a SIMD approach with `ifelse`.

```
x      = rand(5_000_000)
output = similar(x)

function foo!(output,x)
    for i in eachindex(x)
        output[i] = ifelse(x[i] > 0.5, x[i]/2, 0)
    end
end
```

```
julia> foo!($output,$x)
  2.806 ms (0 allocations: 0 bytes)
```

```
x      = rand(5_000_000)
output = similar(x)

function foo!(output,x)
    @inbounds for i in eachindex(x)
        output[i] = ifelse(x[i] > 0.5, x[i]/2, 0)
    end
end
```

```
julia> foo!($output,$x)
  2.888 ms (0 allocations: 0 bytes)
```

```
x      = rand(5_000_000)
output = similar(x)

function foo!(output,x)
    @inbounds @simd for i in eachindex(x)
        output[i] = ifelse(x[i] > 0.5, x[i]/2, 0)
    end
end
```

```
julia> foo!($output,$x)
  16.026 ms (0 allocations: 0 bytes)
```

TERNARY OPERATORS

Ternary operators are an alternative approach for conditional statements, consisting of the form `<condition> ? <action if true> : <action if false>`. Unlike the previous methods, this form relies on heuristics to determine whether an eager or lazy approach should be used. The decision depends on which strategy would more likely be faster in the application considered.

For the illustrations, we'll consider examples where we directly add `@inbounds` and `@simd` in each approach.

DIFFERENT CHOICES

Starting with the same example as above, we show that the ternary operator could choose an eager approach.

```
x = rand(1_000_000)

function foo(x)
    output = 0.0

    @inbounds @simd for i in eachindex(x)
        if x[i] > 0.5
            output += x[i]/2
        end
    end

    return output
end
```

```
julia> foo!($output,$x)
  422.480 μs (0 allocations: 0 bytes)
```

```
x = rand(1_000_000)

function foo(x)
    output = 0.0

    @inbounds @simd for i in eachindex(x)
        output += ifelse(x[i]>0.5, x[i]/2, 0)
    end

    return output
end
```

```
julia> foo!($output,$x)
  85.895 μs (0 allocations: 0 bytes)
```

```
x = rand(1_000_000)

function foo(x)
    output = 0.0

    @inbounds @simd for i in eachindex(x)
        output += x[i]>0.5 ? x[i]/2 : 0
    end

    return output
end
```

```
julia> foo!($output,$x)
  87.881 μs (0 allocations: 0 bytes)
```

Instead, the ternary operator opts for a lazy approach in the following example.

```
x = rand(1_000_000)

function foo(x)
    output = 0.0

    @inbounds @simd for i in eachindex(x)
        if x[i] > 0.99
            output += log(x[i])
        end
    end

    return output
end
```

```
julia> foo!($output,$x)
  405.304 μs (0 allocations: 0 bytes)
```

```
x = rand(1_000_000)

function foo(x)
    output = 0.0

    @inbounds @simd for i in eachindex(x)
        output += ifelse(x[i] > 0.99, log(x[i]), 0)
    end

    return output
end
```

```
julia> foo!($output,$x)
  3.470 ms (0 allocations: 0 bytes)
```

```
x = rand(1_000_000)

function foo(x)
    output = 0.0

    @inbounds @simd for i in eachindex(x)
        output += x[i]>0.99 ? log(x[i]) : 0
    end

    return output
end
```

```
julia> foo!($output,$x)
  421.493 μs (0 allocations: 0 bytes)
```

TERNARY OPERATOR COULD CHOOSE A LESS PERFORMANT APPROACH

It's worth remarking that the method chosen by the ternary operator isn't foolproof. In the following example, the ternary operator actually implements the slowest approach.

```
x      = rand(5_000_000)
output = similar(x)

function foo!(output,x)
    @inbounds @simd for i in eachindex(x)
        if x[i] > 0.5
            output[i] = log(x[i])
        end
    end
end
```

```
julia> foo!($output,$x)
  26.620 ms (0 allocations: 0 bytes)
```

```
x      = rand(5_000_000)
output = similar(x)

function foo!(output,x)
    @inbounds @simd for i in eachindex(x)
        output[i] = ifelse(x[i] > 0.5, log(x[i]), 0)
    end
end
```

```
julia> foo!($output,$x)
  16.864 ms (0 allocations: 0 bytes)
```

```
x      = rand(5_000_000)
output = similar(x)

function foo!(output,x)
    @inbounds @simd for i in eachindex(x)
        output[i] = x[i]>0.5 ? log(x[i]) : 0
    end
end
```

```
julia> foo!($output,$x)
  26.517 ms (0 allocations: 0 bytes)
```

SCENARIOS UNDER WHICH EACH APPROACH IS BETTER

As a rule of thumb, **an eager approach is potentially more performant when branches comprise simple algebraic computations**. On the contrary, conditional statements with **computational-demanding operations will more likely benefit from a lazy implementation**. In fact, this is a heuristic commonly followed by ternary operators.

To demonstrate this, the following example considers a conditional statement where only one branch has a computation, which in turn is straightforward. An eager approach with SIMD is faster, and coincides with the approach chosen when a ternary operator is chosen.

```julia
x              = rand(1_000_000)
condition(a)   = a > 0.5
computation(a) = a * 2

function foo(x)
    output = 0.0

    @inbounds @simd for i in eachindex(x)
        if condition(x[i])
            output += computation(x[i])
        end
    end

    return output
end
```

```
julia> foo!($output,$x)
  416.681 μs (0 allocations: 0 bytes)
```

```julia
x              = rand(1_000_000)
condition(a)   = a > 0.5
computation(a) = a * 2

function foo(x)
    output = 0.0

    @inbounds @simd for i in eachindex(x)
        output += ifelse(condition(x[i]), computation(x[i]), 0)
    end

    return output
end
```

```
julia> foo!($output,$x)
  86.242 μs (0 allocations: 0 bytes)
```

```julia
x              = rand(1_000_000)
condition(a)   = a > 0.5
computation(a) = a * 2

function foo(x)
    output = 0.0

    @inbounds @simd for i in eachindex(x)
        output += condition(x[i]) ? computation(x[i]) : 0
    end

    return output
end
```

```
julia> foo!($output,$x)
  86.370 μs (0 allocations: 0 bytes)
```

Instead, the following scenario considers a branch with more computational-intensive calculations. In this case, a lazy approach is faster, which is the approach implemented by the ternary operator.

```
x               = rand(2_000_000)
condition(a)    = a > 0.5
computation(a) = exp(a)/3 - log(a)/2

function foo(x)
    output = 0.0

    @inbounds @simd for i in eachindex(x)
        if condition(x[i])
            output += computation(x[i])
        end
    end

    return output
end
```

```
julia> foo!($output,$x)
  12.346 ms (0 allocations: 0 bytes)
```

```
x               = rand(2_000_000)
condition(a)    = a > 0.5
computation(a) = exp(a)/3 - log(a)/2

function foo(x)
    output = 0.0

    @inbounds @simd for i in eachindex(x)
        output += ifelse(condition(x[i]), computation(x[i]), 0)
    end

    return output
end
```

```
julia> foo!($output,$x)
  16.552 ms (0 allocations: 0 bytes)
```

```
x              = rand(2_000_000)
condition(a)   = a > 0.5
computation(a) = exp(a)/3 - log(a)/2

function foo(x)
    output = 0.0

    @inbounds @simd for i in eachindex(x)
        output += condition(x[i]) ? computation(x[i]) : 0
    end

    return output
end
```

```
julia> foo!($output,$x)
  12.492 ms (0 allocations: 0 bytes)
```

VECTOR OF CONDITIONS

Next, we consider scenarios where you already have a vector holding conditions. This could occur either because the vector is already part of your dataset, or because the conditions will be reused multiple times over your code, in which case storing the conditions is worthy.

Storing conditions in a vector could be done through an object with type `Vector{Bool}` or `BitVector`. The latter is the default type returned by Julia, as when you define objects like `x .> 0`. Although this type offers certain performance advantages, it can also hinder the application of SIMD. In cases like this, transforming `BitVector` to `Vector{Bool}` could speed up computations.

The following example demonstrates this, where the execution time is faster even considering the vector transformation.

```
x         = rand(1_000_000)
bitvector = x .> 0.5

function foo(x,bitvector)
    output     = similar(x)


    @inbounds @simd for i in eachindex(x)
        output[i] = ifelse(bitvector[i], x[i]/i, x[i]*i)
    end

    return output
end
```

```
julia> foo($x,$bitvector)
  3.393 ms (2 allocations: 7.629 MiB)
```

```
x         = rand(1_000_000)
bitvector = x .> 0.5

function foo(x,bitvector)
    output     = similar(x)
    boolvector = Vector(bitvector)

    @inbounds @simd for i in eachindex(x)
        output[i] = ifelse(boolvector[i], x[i]/i, x[i]*i)
    end

    return output
end
```

```
julia> foo($x,$bitvector)
  862.798 µs (4 allocations: 8.583 MiB)
```

No Vector of Conditions

The conclusions stated here assumes that you already have the vector holding the conditions. If this isn't the case and you want to apply SIMD instructions, you should implement `ifelse` without a vector of conditions. This allows you to avoid memory allocations, while still applying SIMD effectively. The following example illustrates this point. [3]

```
x = rand(1_000_000)

function foo(x)
    output     = similar(x)
    bitvector  = x .> 0.5


    @inbounds @simd for i in eachindex(x)
        output[i] = ifelse(bitvector[i], x[i]/i, x[i]*i)
    end

    return output
end
```

```
julia> foo($x)
  3.628 ms (6 allocations: 7.753 MiB)
```

```julia
x = rand(1_000_000)

function foo(x)
    output     = similar(x)
    boolvector = Vector{Bool}(undef,length(x))
        boolvector .= x .> 0.5

    @inbounds @simd for i in eachindex(x)
        output[i] = ifelse(boolvector[i], x[i]/i, x[i]*i)
    end

    return output
end
```

```
julia> foo($x)
  774.952 μs (4 allocations: 8.583 MiB)
```

```julia
x = rand(1_000_000)

function foo(x)
    output     = similar(x)



    @inbounds @simd for i in eachindex(x)
        output[i] = ifelse(x[i]>0.5, x[i]/i, x[i]*i)
    end

    return output
end
```

```
julia> foo($x)
  501.114 μs (2 allocations: 7.629 MiB)
```

ALGEBRAIC OPERATIONS AS COMPOUND CONDITIONS

We leverage algebraic equivalences to express conditions in ways that allow us to avoid the creation of branches. Mathematically, given a set $\{b_i\}_{i=1}^n$ where $b_i \in \{0,1\}$:

- all conditions are satisfied when

$$\prod_{i=1}^{n} c_i = 1$$

- at least one condition is satisfied when

$$1 - \prod_{i=1}^{n} (1 - c_i) = 1$$

In terms of Julia, given two Boolean scalars `c1` and `c2`, these equivalences become:

- `c1 && c2` is `Bool(c1 * c2)`

- `c1 || c2` is `Bool(1 - !c1 * !c2)`

For instance, with for-loops:

```
x                 = rand(1_000_000)
y                 = rand(1_000_000)


function foo(x,y)
    output = 0.0

    @inbounds @simd for i in eachindex(x)
        if (x[i] > 0.3) && (y[i] < 0.6) && (x[i] > y[i])
            output += x[i]
        end
    end

    return output
end
```

```
julia> foo($x)
  2.116 ms (0 allocations: 0 bytes)
```

```
x                 = rand(1_000_000)
y                 = rand(1_000_000)


function foo(x,y)
    output = 0.0

    @inbounds @simd for i in eachindex(x)
        if (x[i] > 0.3)  * (y[i] < 0.6)  * (x[i] > y[i])
            output += x[i]
        end
    end

    return output
end
```

```
julia> foo($x)
  905.078 μs (0 allocations: 0 bytes)
```

```
x                 = rand(1_000_000)
y                 = rand(1_000_000)


function foo(x,y)
    output = 0.0

    @inbounds @simd for i in eachindex(x)
        if (x[i] > 0.3) || (y[i] < 0.6) || (x[i] > y[i])
            output += x[i]
        end
    end

    return output
end
```

```
julia> foo($x)
  2.724 ms (0 allocations: 0 bytes)
```

```
x                 = rand(1_000_000)
y                 = rand(1_000_000)


function foo(x,y)
    output = 0.0

    @inbounds @simd for i in eachindex(x)
        if Bool(1 - !(x[i] > 0.3)  * !(y[i] < 0.6)  * !(x[i] > y[i]))
            output += x[i]
        end
    end

    return output
end
```

```
julia> foo($x)
  889.879 µs (0 allocations: 0 bytes)
```

While with broadcasting:

```
x              = rand(1_000_000)
y              = rand(1_000_000)


foo(x,y)       = @. ifelse((x>0.3) && (y<0.6) && (x>y), x,y)
```

```
julia> foo($x)
  5.356 ms (2 allocations: 7.629 MiB)
```

```
x               = rand(1_000_000)
y               = rand(1_000_000)


foo(x,y)        = @. ifelse((x>0.3) *  (y<0.6) *  (x>y), x,y)
```

```
julia> foo($x)
  541.621 µs (2 allocations: 7.629 MiB)
```

```
x               = rand(1_000_000)
y               = rand(1_000_000)


foo(x,y)        = @. ifelse((x>0.3) || (y<0.6) || (x>y), x,y)
```

```
julia> foo($x)
  3.354 ms (2 allocations: 7.629 MiB)
```

```
x               = rand(1_000_000)
y               = rand(1_000_000)


foo(x,y)        = @. ifelse(Bool(1 - !(x>0.3) * !(y<0.6) * !(x>y)), x,y)
```

```
julia> foo($x)
  536.276 µs (2 allocations: 7.629 MiB)
```

FOOTNOTES

1. Recall that the compiler may automatically disable bounds checking in some cases, especially in straightforward cases. For instance, this would be the case in our example if only `x` had been indexed and `eachindex(x)` were employed as the iteration range. This is in contrast to scenarios like the one below, where we're indexing both `x` and `output`.

2. Note that `ifelse` requires specifying an operation for when the condition is true and another when it's not. For a sum reduction, this is handled by returning zero when the condition isn't met.

3. Note that the approach for `Vector{Bool}` is somewhat different to the examples we considered above. As we don't have a vector of conditions already defined, it's optimal to create `Vector{Bool}` directly, rather than defining it as a transformation of the `BitVector`. In this way, we avoid unnecessary memory allocations too.

10g. SIMD Packages

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INTRODUCTION

So far, we've relied on the built-in `@simd` macro to apply SIMD instructions. This approach, nonetheless, exhibits several limitations. First, `@simd` acts as a suggestion rather than a strict command: it hints to the compiler that SIMD optimizations might improve performance, but ultimately the implementation decision is up to the compiler's discretion. Second, `@simd` prioritizes code safety over speed, restricting access to advanced SIMD features to avoid unintended bugs. Third, its application is limited to for-loops.

To overcome these shortcomings, we introduce the `@turbo` macro from the `LoopVectorization` package. Unlike `@simd`, the `@turbo` macro enforces SIMD optimizations, guaranteeing that vectorized instructions are actually applied. It also performs more aggressive optimizations than `@simd`, shifting the responsibility for correctness and safety onto the user. Finally, `@turbo` extends beyond for-loops, also supporting broadcasting operations. Finally, `@turbo` integrates with the `SLEEFPirates` package. This provides SIMD-accelerated implementations of common mathematical functions such as logarithms, exponentials, powers, and trigonometric operations.

CAVEATS ABOUT IMPROPER USE OF @TURBO

In contrast to `@simd`, applying `@turbo` requires extra caution, as its misapplication can lead to incorrect results. The risk stems from the additional assumptions that `@turbo` makes to enable more aggressive optimization. In particular:

- **No bound checking**: `@turbo` omits index bound checks, potentially leading to out-of-bounds memory access.
- **Iteration order invariance**: `@turbo` assumes the outcome doesn't depend on the order of iteration, with the sole exception of reduction operations.

The second assumption is particularly relevant for floating-point arithmetic, where non-associativity can cause results to vary with iteration order. Integer operations, by contrast, are unaffected. The following example demonstrates this issue: because each iteration depends on the outcome of the previous one, applying `@turbo` produces incorrect results.

NO MACRO

```
x = [0.1, 0.2, 0.3]

function foo!(x)
    for i in 2:length(x)
        x[i] = x[i-1] + x[i]
    end
end
```

```
julia> foo!(x)

julia> x
3-element Vector{Float64}:
 0.1
 0.3
 0.6
```

@SIMD

```
x = [0.1, 0.2, 0.3]

function foo!(x)
    @inbounds @simd for i in 2:length(x)
        x[i] = x[i-1] + x[i]
    end
end
```

```
julia> foo!(x)

julia> x
3-element Vector{Float64}:
 0.1
 0.3
 0.6
```

@TURBO

```
x = [0.1, 0.2, 0.3]

function foo!(x)
    @turbo for i in 2:length(x)
        x[i] = x[i-1] + x[i]
    end
end
```

```
julia> foo!(x)

julia> x
3-element Vector{Float64}:
 0.1
 0.3
 0.5
```

Considering that `@turbo` isn't suitable for all operations, we next present cases where the macro can be safely applied.

SAFE APPLICATIONS OF @TURBO

There are two safe applications of `@turbo` that cover a wide range of cases. The first applies **when iterations are completely independent**, making execution order irrelevant for the final outcome.

The example below illustrates this case by performing an independent transformation on each element of a vector. Importantly, `@turbo` isn't restricted to for-loops, also allowing for broadcasted operations. We show both applications.

DEFAULT

```
x              = rand(1_000_000)
calculation(a) = a * 0.1 + a^2 * 0.2 - a^3 * 0.3 - a^4 * 0.4

function foo(x)
    output     = similar(x)

    for i in eachindex(x)
        output[i] = calculation(x[i])
    end

    return output
end
```

```
julia> @btime foo($x)
  3.840 ms (3 allocations: 7.629 MiB)
```

@SIMD

```
x              = rand(1_000_000)
calculation(a) = a * 0.1 + a^2 * 0.2 - a^3 * 0.3 - a^4 * 0.4

function foo(x)
    output     = similar(x)

    @inbounds @simd for i in eachindex(x)
        output[i] = calculation(x[i])
    end

    return output
end
```

```
julia> @btime foo($x)
  4.096 ms (3 allocations: 7.629 MiB)
```

@TURBO (FOR-LOOP)

```
x              = rand(1_000_000)
calculation(a) = a * 0.1 + a^2 * 0.2 - a^3 * 0.3 - a^4 * 0.4

function foo(x)
    output     = similar(x)

    @turbo for i in eachindex(x)
        output[i] = calculation(x[i])
    end

    return output
end
```

```
julia> @btime foo($x)
  271.104 μs (3 allocations: 7.629 MiB)
```

@TURBO (BROADCASTING)

```
x              = rand(1_000_000)
calculation(a) = a * 0.1 + a^2 * 0.2 - a^3 * 0.3 - a^4 * 0.4

foo(x)         = @turbo calculation.(x)
```

```
julia> @btime foo($x)
  482.698 μs (3 allocations: 7.629 MiB)
```

The second safe application is **reductions**. While reductions consists of dependent iterations, they're a special case that `@turbo` handles properly.

DEFAULT

```
x              = rand(1_000_000)
calculation(a) = a * 0.1 + a^2 * 0.2 - a^3 * 0.3 - a^4 * 0.4

function foo(x)
    output     = 0.0

    for i in eachindex(x)
        output += calculation(x[i])
    end

    return output
end
```

```
julia> @btime foo($x)
  3.892 ms (0 allocations: 0 bytes)
```

@SIMD

```
x              = rand(1_000_000)
calculation(a) = a * 0.1 + a^2 * 0.2 - a^3 * 0.3 - a^4 * 0.4

function foo(x)
    output     = 0.0

    @inbounds @simd for i in eachindex(x)
        output += calculation(x[i])
    end

    return output
end
```

```
julia> @btime foo($x)
  3.937 ms (0 allocations: 0 bytes)
```

@TURBO

```
x              = rand(1_000_000)
calculation(a) = a * 0.1 + a^2 * 0.2 - a^3 * 0.3 - a^4 * 0.4

function foo(x)
    output     = 0.0

    @turbo for i in eachindex(x)
        output += calculation(x[i])
    end

    return output
end
```

```
julia> @btime foo($x)
  179.364 µs (0 allocations: 0 bytes)
```

SPECIAL FUNCTIONS

Another important application of `LoopVectorization` arises through its integration with the library *SLEEF* (an anocrym for "SIMD Library for Evaluating Elementary Functions"). SLEEF is exposed in `LoopVectorization` via the `SLEEFPirates` package, which accelerates the evaluation of several mathematical functions using SIMD instructions. In particular, it speeds up the computations of the exponential, logarithmic, power, and trigonometric functions.

Below, we illustrate the use of `@turbo` for each type of function. For a complete list of supported functions, see the SLEEFPirates documentation. All the examples rely on an element-wise transformation of `x` via the function `calculation`, which will take a different form depending on the function illustrated.

LOGARITHM

DEFAULT

```
x              = rand(1_000_000)
calculation(a) = log(a)

function foo(x)
    output     = similar(x)

    for i in eachindex(x)
        output[i] = calculation(x[i])
    end

    return output
end
```

```
julia> @btime foo($x)
  3.542 ms (3 allocations: 7.629 MiB)
```

@SIMD

```
x              = rand(1_000_000)
calculation(a) = log(a)

function foo(x)
    output     = similar(x)

    @inbounds @simd for i in eachindex(x)
        output[i] = calculation(x[i])
    end

    return output
end
```

```
julia> @btime foo($x)
  3.546 ms (3 allocations: 7.629 MiB)
```

@TURBO (FOR-LOOP)

```
x              = rand(1_000_000)
calculation(a) = log(a)

function foo(x)
    output     = similar(x)

    @turbo for i in eachindex(x)
        output[i] = calculation(x[i])
    end

    return output
end
```

```
julia> @btime foo($x)
  1.617 ms (3 allocations: 7.629 MiB)
```

@TURBO (BROADCASTING)

```
x               = rand(1_000_000)
calculation(a) = log(a)

foo(x) = @turbo calculation.(x)
```

```
julia> @btime foo($x)
  1.618 ms (3 allocations: 7.629 MiB)
```

EXPONENTIAL FUNCTION

DEFAULT

```
x               = rand(1_000_000)
calculation(a) = exp(a)

function foo(x)
    output     = similar(x)

    for i in eachindex(x)
        output[i] = calculation(x[i])
    end

    return output
end
```

```
julia> @btime foo($x)
  2.608 ms (3 allocations: 7.629 MiB)
```

@SIMD

```
x               = rand(1_000_000)
calculation(a) = exp(a)

function foo(x)
    output     = similar(x)

    @inbounds @simd for i in eachindex(x)
        output[i] = calculation(x[i])
    end

    return output
end
```

```
julia> @btime foo($x)
  2.639 ms (3 allocations: 7.629 MiB)
```

@TURBO (FOR-LOOP)

```
x              = rand(1_000_000)
calculation(a) = exp(a)

function foo(x)
    output     = similar(x)

    @turbo for i in eachindex(x)
        output[i] = calculation(x[i])
    end

    return output
end
```

```
julia> @btime foo($x)
  555.012 μs (3 allocations: 7.629 MiB)
```

@TURBO (BROADCASTING)

```
x              = rand(1_000_000)
calculation(a) = exp(a)

foo(x) = @turbo calculation.(x)
```

```
julia> @btime foo($x)
  544.043 μs (3 allocations: 7.629 MiB)
```

POWER FUNCTIONS

This includes any operation comprising terms x^y.

DEFAULT

```
x              = rand(1_000_000)
calculation(a) = a^4

function foo(x)
    output     = similar(x)

    for i in eachindex(x)
        output[i] = calculation(x[i])
    end

    return output
end
```

```
julia> @btime foo($x)
  3.517 ms (3 allocations: 7.629 MiB)
```

@SIMD

```
x              = rand(1_000_000)
calculation(a) = a^4

function foo(x)
    output     = similar(x)

    @inbounds @simd for i in eachindex(x)
        output[i] = calculation(x[i])
    end

    return output
end
```

```
julia> @btime foo($x)
  3.578 ms (3 allocations: 7.629 MiB)
```

@TURBO (FOR-LOOP)

```
x              = rand(1_000_000)
calculation(a) = a^4

function foo(x)
    output     = similar(x)

    @turbo for i in eachindex(x)
        output[i] = calculation(x[i])
    end

    return output
end
```

```
julia> @btime foo($x)
  371.218 µs (3 allocations: 7.629 MiB)
```

@TURBO (BROADCASTING)

```
x              = rand(1_000_000)
calculation(a) = a^4

foo(x) = @turbo calculation.(x)
```

```
julia> @btime foo($x)
  302.605 µs (3 allocations: 7.629 MiB)
```

The implementation for power functions includes calls to other function, such as the one for square roots.

DEFAULT

```
x              = rand(1_000_000)
calculation(a) = sqrt(a)

function foo(x)
    output     = similar(x)

    for i in eachindex(x)
        output[i] = calculation(x[i])
    end

    return output
end
```

```
julia> @btime foo($x)
  1.159 ms (3 allocations: 7.629 MiB)
```

@SIMD

```
x              = rand(1_000_000)
calculation(a) = sqrt(a)

function foo(x)
    output     = similar(x)

    @inbounds @simd for i in eachindex(x)
        output[i] = calculation(x[i])
    end

    return output
end
```

```
julia> @btime foo($x)
  1.200 ms (3 allocations: 7.629 MiB)
```

@TURBO (FOR-LOOP)

```
x              = rand(1_000_000)
calculation(a) = sqrt(a)

function foo(x)
    output     = similar(x)

    @turbo for i in eachindex(x)
        output[i] = calculation(x[i])
    end

    return output
end
```

```
julia> @btime foo($x)
  590.429 µs (3 allocations: 7.629 MiB)
```

@TURBO (BROADCASTING)

```
x               = rand(1_000_000)
calculation(a) = sqrt(a)

foo(x) = @turbo calculation.(x)
```

```
julia> @btime foo($x)
  578.698 µs (3 allocations: 7.629 MiB)
```

TRIGONOMETRIC FUNCTIONS

Among others, `@turbo` can handle the functions `sin`, `cos`, and `tan`. Below, we demonstrate its use with `sin`.

DEFAULT

```
x               = rand(1_000_000)
calculation(a) = sin(a)

function foo(x)
    output      = similar(x)

    for i in eachindex(x)
        output[i] = calculation(x[i])
    end

    return output
end
```

```
julia> @btime foo($x)
  3.915 ms (3 allocations: 7.629 MiB)
```

@SIMD

```
x               = rand(1_000_000)
calculation(a) = sin(a)

function foo(x)
    output      = similar(x)

    @inbounds @simd for i in eachindex(x)
        output[i] = calculation(x[i])
    end

    return output
end
```

```
julia> @btime foo($x)
  3.895 ms (3 allocations: 7.629 MiB)
```

@TURBO (FOR-LOOP)

```
x              = rand(1_000_000)
calculation(a) = sin(a)

function foo(x)
    output     = similar(x)

    @turbo for i in eachindex(x)
        output[i] = calculation(x[i])
    end

    return output
end
```

```
julia> @btime foo($x)
  1.341 ms (3 allocations: 7.629 MiB)
```

@TURBO (BROADCASTING)

```
x              = rand(1_000_000)
calculation(a) = sin(a)

foo(x) = @turbo calculation.(x)
```

```
julia> @btime foo($x)
  1.315 ms (3 allocations: 7.629 MiB)
```

11a. Overview and Goals

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INTRODUCTION

Programming languages typically execute code sequentially, following a single path of execution that utilizes one core at a time. This linear approach simplifies reasoning about program behavior, as each operation completes before the next begins. However, hardware these days is commonly equipped with multiple processor cores. Consequently, a sequential execution does all the work on one core, while the others sit idle. This leaves substantial computational power untapped.

Multithreading addresses this limitation by running different segments of our program simultaneously across multiple cores. While this capability opens up significant opportunities for performance improvement, it also introduces new challenges that developers need to navigate carefully. In fact, simple operations that work flawlessly in single-threaded programs may yield incorrect results in a multithreaded setting. Furthermore, writing multithreaded code requires a fundamental shift in the user's mindset regarding program execution. All this makes multithreaded code inherently more difficult to write, test, and debug than its single-threaded counterpart.

Despite these challenges, the potential performance benefits of multithreading make it an essential tool in modern programming. This is particularly true for applications that are computationally intensive or demand that the same code be applied to multiple objects.

11b. Introduction to Multithreading

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INTRODUCTION

A correct implementation of multithreading requires a basic understanding of how computers execute instructions. In particular, it's essential to understand this procedure in light of data dependencies: situations where one operation relies on the output of a previous one. If these dependencies aren't properly managed, multithreading could introduce several forms of unsafe code, potentially leading to incorrect results.

This section introduces the preliminary concepts needed to reason about these issues, laying the foundation for the more practical discussions that follow. The emphasis here is on explanation rather than implementation. Accordingly, many of the macros and functions presented will serve only as illustrative examples and won't be reused elsewhere on this website.

NATURE OF COMPUTATIONS

Operations can be broadly classified in terms of their data dependencies. A **dependent operation** is one whose outcome is influenced by the result of another operation. In such cases, the order of execution is critical, because changing the sequence can alter the final outcome. In contrast, an **independent operation** produces the same result, regardless of the order in which it's executed relative to other operations.

The following code gives rise to a dependent or independent operation, based on which values operation B sums.

```
job_A()  = 1 + 1
job_B(A) = 2 + A

function foo()
    A = job_A()
    B = job_B(A)

    return A,B
end
```

```
job_A() = 1 + 1
job_B() = 2 + 2

function foo()
    A = job_A()
    B = job_B()

    return A,B
end
```

Regardless of their dependency status, operations can be computed either sequentially or concurrently. A **sequential** procedure involves executing operations one after the other, ensuring each operation completes before the next one begins. Conversely, **concurrency** allows multiple operations to be processed simultaneously, opening up opportunities for parallel execution.

Like most programming languages, **Julia defaults to a sequential execution**. This is a deliberate design choice that prioritizes result correctness, as **concurrent execution of dependent operations can yield incorrect results if mishandled**. The issue arises because concurrency introduces non-determinism in the execution order, which can lead to timing inconsistencies when reading and writing data. A sequential approach precludes this possibility by guaranteeing a predictable order of execution.

Despite its advantages regarding safety, a sequential approach can be quite inefficient for independent tasks: by restricting computations to one at a time, computational resources may go underutilized. In contrast, **a simultaneous approach allows for operations to be calculated in parallel, thereby fully utilizing all the available computational resources**. This can lead to significant reductions in computation time.

Because most programming languages default to sequential execution, certain nuances of concurrent programming can be difficult to grasp (e.g., concurrency doesn't necessarily imply simultaneity). Such misunderstandings can lead to flawed program design or incorrect handling of concurrent processes. To address this, we next revisit the topic in light of the fundamental concepts of tasks and threads.

TASKS AND THREADS

When computing an operation, Julia internally defines a set of instructions to be processed. This is achieved through the concept of **tasks**. For a task to be computed, it must be assigned to **a computer thread**. Since a single task runs on exactly one thread at a time, **the number of threads available on your computer determines how many tasks can be computed simultaneously**.

Importantly, each session in Julia starts with a predefined pool of threads, defaulting to a single thread regardless of your computer's hardware. We'll begin by examining the single-threaded case, as it provides a clear basis for understanding concurrency.

To build intuition, consider two workers A and B, whom we'll think of as employees working for a company. B's job consists of performing the same operation continuously for a certain period of time. In the code, this is represented by summing `1+1` repeatedly for one second. For its part, A's job consists of waiting for some delivery, which will arrive after a certain period of time. In the code, this job is represented by performing no computations for two seconds, simulated by calling the function `sleep(2)`.

The following code snippet defines functions capturing each worker's job.

```
function job_A(time_working)
    sleep(time_working)        # do nothing (waiting for some delivery in the example)

    println("A completed his task")
end
```

```
function job_B(time_working)
    start_time = time()

    while time() - start_time < time_working
        1 + 1                  # compute `1+1` repeatedly during `time_working` seconds
    end

    println("B completed his task")
end
```

Due to the lazy nature of function definitions, these code snippets merely create a blueprint for a set of operations. This implies that no computation is performed. It's only when we call these functions by lines like `job_A(2)` and `job_B(1)` that the operations are sent for execution.

To lay bare the internal steps Julia follows for computation, let's adopt a lower-level approach where the function calls `job_A(2)` and `job_B(1)` are defined as tasks. As shown below, tasks aren't abstractions to organize our discussion, but actual constructs in Julia's codebase.

```
A = @task job_A(2)      # A's task takes 2 seconds
B = @task job_B(1)      # B's task takes 1 second
```

Once a task is defined, the first step for its computation is **scheduling** it. This means the task is added to the queue of operations awaiting execution by the computer's processor. Then, as soon as a thread becomes available, the machine begins its computation.

Importantly, multiple tasks can be *processed* concurrently, without implying that they'll be *computed* simultaneously. Indeed, this is the case in a single-thread session. The distinction can be understood through an analogy with juggling: a juggler manages multiple balls at the same time, but only holds one ball at any given moment. Similarly, multiple tasks can be processed simultaneously, even when only one is actively executing on the CPU.

The implication of this statement is that true parallelism isn't feasible in single-threaded sessions. Nonetheless, concurrency can still improve efficiency through **task switching**. This is enabled by a **task-yielding** mechanism: when a task becomes idle (e.g., waiting for input or data), it can voluntarily relinquish control of the thread, allowing other tasks to utilize the thread's time. By fostering a cooperative approach, concurrency ensures plenty of computer resource utilization at any given time.

In the following, we describe this mechanism in more detail.

SEQUENTIAL AND CONCURRENT COMPUTATIONS

While code is executed sequentially by default, **tasks are designed to run concurrently**. As a result, to enforce a sequential execution of tasks, we must instruct Julia to run them one at a time. This is achieved by introducing a "wait" instruction immediately after scheduling a task, ensuring that the task completes its calculation before proceeding with the rest of the program.

The code snippet below demonstrates this mechanism by introducing the functions `schedule` and `wait`.

```
A = job_A(2)            # A's task takes 2 seconds
B = job_B(1)            # B's task takes 1 second
```

```
A = @task job_A(2)      # A's task takes 2 seconds
B = @task job_B(1)      # B's task takes 1 second

schedule(A) |> wait
schedule(B) |> wait
```

```
A = @task job_A(2)      # A's task takes 2 seconds
B = @task job_B(1)      # B's task takes 1 second

(schedule(A), schedule(B)) .|> wait
```

Note that `wait` was added even in the concurrent case and after both tasks were scheduled. The purpose is to pause the main program flow until both scheduled tasks have finished, preventing subsequent code from executing prematurely.

The example reveals the benefits of task switching under concurrency: although only one task can run at any moment, task A can yield control of the thread to task B when it becomes idle. In the code, the idle state is simulated by the function `sleep`, during which the computer performs no operations. Once task A becomes idle, its state is saved, allowing it to eventually resume execution from where it left off. In the meantime, task B can use that thread's processing time, explaining why task B finishes first.

By taking turns efficiently and sharing the single available thread, tasks make the most of the CPU's processing power. This contrasts with a sequential approach, where task A must finish before moving to the next task. The difference is reflected in their execution times, resulting in 2 seconds for the concurrent approach and 3 seconds for the sequential one.

Examples of idle states emerge naturally in real-world scenarios. For instance, it's common when a program is waiting for user input, such as a keystroke or mouse click. It can also arise when browsing the internet, where the CPU may idle while waiting for a server to send data. Task switching is so ubiquitous in certain contexts that we often take it for granted. For instance, I bet you never questioned whether you could use the computer while a document prints in the background!

Notice, though, that concurrency with a single thread offers no benefits if both tasks require active computations. This is because the CPU would be fully utilized, leaving no chance for task switching. In such cases, the sequential and concurrent approaches are equivalent. In our example, this would occur if task B consisted of computing `1+1` repeatedly, resulting in an execution time of 3 seconds for both approaches.

```
function job(name_worker, time_working)
    start_time = time()

    while time() - start_time < time_working
        1 + 1                   # compute `1+1` repeatedly during `time_working` seconds
    end

    println("$name_worker completed his task")
end
```

```
function schedule_of_tasks()
    A = @task job("A", 2)       # A's task takes 2 seconds
    B = @task job("B", 1)       # B's task takes 1 second

    schedule(A) |> wait
    schedule(B) |> wait
end
```

```julia
function schedule_of_tasks()
    A = @task job("A", 2)      # A's task takes 2 seconds
    B = @task job("B", 1)      # B's task takes 1 second

    (schedule(A), schedule(B)) .|> wait
end
```

Overall, the key insight from this subsection is the underlying procedure outlined: **when a task is scheduled, the computer attempts to find an available thread to run it**. For concurrency, this implies that **starting a session with multiple threads enables parallel code execution**. This case is simply referred to as **multithreading** and explained next in more detail.

MULTITHREADING

Let's revisit the case where both workers A and B perform meaningful computations. The only change introduced is that Julia's session now starts with more than one thread available. For the concurrent approach, we also specify that the tasks are "non-sticky". This technical detail means a task can run on any available thread, not just the one it started on, allowing for more efficient resource allocation.

Once there's more than one thread available, concurrency implies simultaneity. This means each task runs on a different thread, which is why task B finishes first in the following implementation.

```julia
function schedule_of_tasks()
    A = @task job("A", 2)                        # A's task takes 2 seconds
    B = @task job("B", 1)                        # B's task takes 1 second

    schedule(A) |> wait
    schedule(B) |> wait
end
```

```julia
function schedule_of_tasks()
    A = @task job("A", 2) ; A.sticky = false      # A's task takes 2 seconds
    B = @task job("B", 1) ; B.sticky = false      # B's task takes 1 second

    (schedule(A), schedule(B)) .|> wait
end
```

Previewing some of the approaches we'll introduce later, let's compare Julia's default implementation (sequential) with a multithreaded one. The macro `@spawn`, which will be covered in the next section, offers a simple way to run tasks in a multithreaded environment. Essentially, it's equivalent to creating and scheduling a non-sticky task. The following code snippets demonstrate both the default and multithreaded approaches.

```julia
function schedule_of_tasks()
    A = job("A", 2)             # A's task takes 2 seconds
    B = job("B", 1)             # B's task takes 1 second
end
```

```julia
function schedule_of_tasks()
    A = @spawn job("A", 2)      # A's task takes 2 seconds
    B = @spawn job("B", 1)      # B's task takes 1 second

    (A,B) .|> wait
end
```

THE IMPORTANCE OF WAITING FOR THE RESULTS

Before concluding this section, it's worth stressing a crucial point: you must always instruct your program to wait for operations to complete before proceeding. This holds true even for concurrent computations. **Failing to wait may produce incorrect results, even in a single-threaded environment**.

To illustrate this, consider mutating a vector in a single-threaded session, with a one-second delay for each value update. If we don't wait for the mutation to finish, any subsequent operation will be based on the vector's value at the moment it's accessed. This value doesn't necessarily reflect its final mutated state, but merely its value at the moment of reference.

For instance, suppose we seek to mutate the vector `x = [0,0,0]` into `x = [1,2,3]`. Let's begin considering Julia's default sequential execution. This ensures that the mutation completes before continuing with any other operation.

```
# Description of job
function job!(x)
    for i in 1:3
        sleep(1)      # do nothing for 1 second
        x[i] = 1      # mutate x[i]

        println("`x` at this moment is $x")
    end
end

# Execution of job
function foo()
    x = [0, 0, 0]

    job!(x)           # slowly mutate `x`

    return sum(x)
end

output = foo()
println("the value stored in `output` is $(output)")
```

Let's now consider the same implementation but through tasks. In particular, we define a task to perform a mutation as follows.

```
function job!(x)
    @task begin
        for i in 1:3
            sleep(1)     # do nothing for 1 second
            x[i] = 1     # mutate x[i]

            println("`x` at this moment is $x")
        end
    end
end
```

The following snippets show the consequences of waiting versus not waiting for the task to complete.

```
function foo()
    x = [0, 0, 0]

    job!(x) |> schedule             # define job, start execution, don't wait for job to be
done

    return sum(x)
end

output = foo()
println("the value stored in `output` is $(output)")
```

```
function foo()
    x = [0, 0, 0]

     job!(x) |> schedule |> wait       # define job, start execution, only continue when
finished

    return sum(x)
end

output = foo()
println("the value stored in `output` is $(output)")
```

Without a `wait` call, the main program schedules the mutation task and immediately proceeds to the next line. Since the task contains a `sleep` instruction, the mutation hasn't yet begun when `x` is accessed, resulting in the use of its original value `[0,0,0]`. This demonstrates that properly synchronizing tasks is essential for correctness.

11c. Task-Based Parallelism: @spawn

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INTRODUCTION

The previous section introduced the basics of multithreading, highlighting that operations can be computed either sequentially (Julia's default) or concurrently. The latter enables the processing of multiple operations simultaneously, with each of them running as soon as a thread becomes available. This implies that, when Julia's session is initialized with more than one thread, computations can be executed on different CPU cores in parallel.

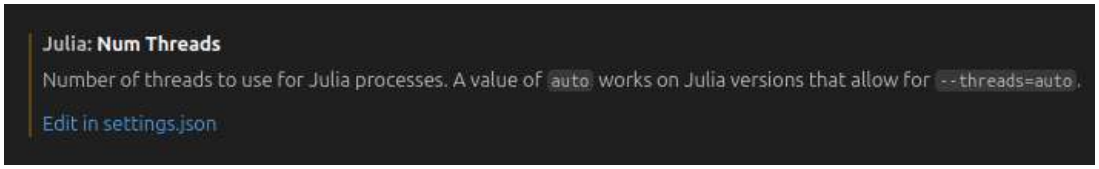
This section will focus on Julia's native multithreading mechanisms, a topic that will span several sections. Our primary goal here is to demonstrate how to write multithreaded code, rather than exploring how and when to apply the technique. We've deliberately structured our explanation in this way to smooth subsequent discussions.

> **Warning!**
> While multithreading can offer significant performance advantages, it's not applicable in all scenarios. In particular, multithreading demands extreme caution in handling dependencies between operations, as mismanagement can lead to silent catastrophic bugs. This is why, after grasping a basic understanding of parallelism techniques, we'll devote an entire section on unsafe-thread operations.

ENABLING MULTITHREADING

Each Julia session is initialized with a pool of threads available for execution. Since each thread can run a set of instructions independently, the total number of threads determines how many instructions can be processed simultaneously.

By default, Julia runs in single-threaded mode. To enable multithreading, you need to configure the environment to launch Julia with more than one thread. In VSCode or VSCodium, this can be done by navigating to *File* > *Preferences* > *Settings*. Then, search for the keyword *threads*, which will display the following option:



After selecting *Edit in settings.json*, add the line `"julia.NumThreads": "auto"` in the JSON file that opens. This setting automatically detects the number of threads supported by your system, typically matching either the logical or physical cores of your CPU.

Notice the effects will take place after starting a new session. Moreover, the changes are permanent, so that every new Julia session will start with the number of threads specified. To check that the effects have taken place, run the command `Threads.nthreads()`, which displays the number of threads available in the session. Any number greater than one will indicate that multithreading is active.

Once a multithreaded session is started, you can perform parallel computations. While several packages exist for this, the current section will focus on the capabilities built directly into Julia. They're provided by the `Threads` package, which is automatically imported in every Julia session.

```julia
# package Threads automatically imported when you start Julia

Threads.nthreads()
```

```
2
```

```julia
using Base.Threads       # or `using .Threads`

nthreads()
```

```
2
```

> **Warning!** - Loaded Package
> All the scripts below assume that you've executed the line `using Base.Threads`. Furthermore, all the examples are based on a session with **two worker threads**.

TASK-BASED PARALLELISM: @SPAWN

The first approach for parallel execution we introduce is the macro `@spawn`. By prefixing an expression with `@spawn`, Julia creates a (non-sticky) task that's immediately scheduled for execution. If a thread is available, the task begins running right away.

Unlike other parallel programming approaches we'll examine later, `@spawn` requires explicit instructions to wait for task completion. The method for doing so depends on the nature of the task output.

The `fetch` function should be employed when tasks produce computational outputs. It serves a dual purpose of waiting for a task to complete and retrieving its return value.Since parallel computation consists of multiple tasks spawned, `fetch` should be broadcasted over a collection containing all the

spawned tasks.

The following example demonstrates `fetch` with two spawned tasks, each returning a vector.

```
x = rand(10); y = rand(10)

function foo(x)
    a = x .* -2
    b = x .*  2

    a,b
end
```

```
x = rand(10); y = rand(10)

function foo(x)
    task_a = @spawn x .* -2
    task_b = @spawn x .*  2

    a,b = fetch.((task_a, task_b))
end
```

Note that `fetch` takes tasks as its input. Consequently, it's essential to distinguish between `task_a` and `a`:

- `task_a` denotes the task creating the vector `a` (i.e., the task itself),
- `a` refers to the vector created (i.e., the task's output).

For tasks that perform actions but don't return any output, we can use either the function `wait` or the macro `@sync`. The function `wait` works analogously to `fetch`, except that it doesn't return any value. Instead, the macro `@sync` requires enclosing all operations to be synchronized using the keywords `begin` and `end`.

To illustrate, consider a mutating function. Such functions are suitable as examples, since they modify the contents of a collection in place, without producing a return value.

```
x = rand(10); y = rand(10)

function foo!(x,y)
    @. x = -x
    @. y = -y
end
```

```
x = rand(10); y = rand(10)

function foo!(x,y)
    task_a = @spawn (@. x = -x)
    task_b = @spawn (@. y = -y)

    wait.((task_a, task_b))
end
```

```julia
x = rand(10); y = rand(10)

function foo!(x,y)
    @sync begin
        @spawn (@. x = -x)
        @spawn (@. y = -y)
    end
end
```

MULTITHREADING OVERHEAD

To see the advantages of `@spawn` in action, let's calculate both the sum and the maximum of a vector `x`. Their computation will be implemented following a sequential and a simultaneous approach. To unveil the benefits of parallelization, we'll also include the execution time of each operation in isolation.

The results establish that the total runtime of the sequential procedure is essentially the sum of the individual runtimes. In contrast, the runtime under multithreading is roughly equivalent to the longer of the two computations, thanks to parallelism.

```julia
x = rand(10_000_000)

function non_threaded(x)
    a           = maximum(x)
    b           = sum(x)

    all_outputs = (a,b)
end
```

```
julia> @btime maximum($x)
  7.705 ms (0 allocations: 0 bytes)
julia> @btime sum($x)
  3.131 ms (0 allocations: 0 bytes)
julia> @btime non_threaded($x)
  10.917 ms (0 allocations: 0 bytes)
```

```julia
x = rand(10_000_000)

function multithreaded(x)
    task_a      = @spawn maximum(x)
    task_b      = @spawn sum(x)

    all_tasks   = (task_a, task_b)
    all_outputs = fetch.(all_tasks)
end
```

```
julia> @btime maximum($x)
  7.705 ms (0 allocations: 0 bytes)
julia> @btime sum($x)
  3.131 ms (0 allocations: 0 bytes)
julia> @btime multithreaded($x)
  7.741 ms (21 allocations: 1.250 KiB)
```

Although the multithreaded runtime is close to the maximum of the individual runtimes, the equivalence isn't exact. This is because **multithreading introduces overhead** due to the creation, scheduling, and synchronization of tasks. As a result, **multithreading isn't advantageous for small workloads**, where the overhead can outweigh any potential gains.

To illustrate this effect, let's compare the execution times of a sequential and multithreaded approach for different sizes of `x`. In the case considered, the single-threaded approach dominates for sizes smaller than 100,000.

```julia
x_small  = rand(    1_000)
x_medium = rand(  100_000)
x_big    = rand(1_000_000)

function foo(x)
    a           = maximum(x)
    b           = sum(x)

    all_outputs = (a,b)
end
```

```
julia> @btime foo($x_small)
  866.758 ns (0 allocations: 0 bytes)
julia> @btime foo($x_medium)
  59.934 µs (0 allocations: 0 bytes)
julia> @btime foo($x_big)
  620.869 µs (0 allocations: 0 bytes)
```

```julia
x_small  = rand(    1_000)
x_medium = rand(  100_000)
x_big    = rand(1_000_000)

function foo(x)
    task_a      = @spawn maximum(x)
    task_b      = @spawn sum(x)

    all_tasks   = (task_a, task_b)
    all_outputs = fetch.(all_tasks)
end
```

```
julia> @btime foo($x_small)
  3.245 μs (14.33 allocations: 1.068 KiB)
julia> @btime foo($x_medium)
  55.853 μs (21 allocations: 1.250 KiB)
julia> @btime foo($x_big)
  549.445 μs (21 allocations: 1.250 KiB)
```

11d. Thread-Safe Operations

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INTRODUCTION

Multithreading allows multiple threads to run simultaneously within a single process, enabling true parallel execution of operations on the same machine. Unlike other forms of parallelization (e.g., multiprocessing), multithreading is characterized by **the sharing of a common memory space among all tasks**.

This shared-memory environment introduces additional complexity, since **parallel execution can produce unintended side effects if not managed carefully**. The core issue arises when multiple threads access and modify shared data concurrently, which can cause unintended consequences in other threads.

These problems motivate the distinction between thread-safe and thread-unsafe operations. An operation is considered **thread-safe** if it can be executed in parallel without causing inconsistencies, crashes, or corrupted results. Conversely, **thread-unsafe** operations require explicit synchronization or restructuring to avoid errors.

This section will focus on identifying key features that render certain operations unsafe. In particular, we'll see that common operations like reductions aren't thread safe, leading to incorrect results if multithreading is applied naively. We'll conclude by exploring the concept of embarrassingly parallel problems, which are a prime example of thread-safe operations. As the name suggests, these problems can be parallelized directly, without requiring significant modifications to program structure.

UNSAFE OPERATIONS

In multithreaded environments, unsafe operations correspond to those that can lead to incorrect behavior, data corruption, or program crashes when executed concurrently. Such issues typically arise when tasks exhibit some degree of dependency, either in terms of operations or shared resources.

WRITING ON A SHARED VARIABLE

One of the simplest examples of a thread-unsafe operation is writing to a shared variable. To illustrate, consider a scenario where a scalar variable `output` is initialized to zero. This value is then updated within a for-loop that iterates twice, with `output` set to `i` in the *i*-th iteration. The corresponding script is shown below.

```julia
function foo()
    output = 0

    for i in 1:2
        sleep(1/i)
        output = i
    end

    return output
end
```

```
julia> foo()
2
```

```julia
function foo()
    output = 0

    @threads for i in 1:2
        sleep(1/i)
        output = i
    end

    return output
end
```

```
julia> foo()
1
```

To illustrate the challenges of concurrent execution, we've deliberately introduced a decreasing delay before updating `output`. This was implemented with `sleep(1/i)`, causing the first iteration to pause for 1 second and the second iteration to pause for half a second. Although this delay is artificially introduced via `sleep`, it represents the potential gap in time caused by intermediate computations, which could preclude an immediate update of `output`.

The delay is inconsequential for a sequential procedure, where `output` takes on the values 0, 1, and 2 as the program progresses. However, when executed concurrently, the first iteration completes after the second iteration has finished. As a result, the sequence of values for `output` is 0, 2, and 1.

While the problem may seem apparent in this simple example, it can manifest in more complex and subtle ways in real-world applications. The core issue is that the order of execution isn't guaranteed in a multithreaded environment. Thus, when multiple threads modify the same shared variable, the final value depends on which thread executes last.

In fact, the issue can be exacerbated when each iteration additionally involves reading a shared variable. Next, we consider a scenario like this.

READING AND WRITING A SHARED VARIABLE

Reading and writing shared data doesn't necessarily cause incorrect results. For instance, a parallel for-loop could safely mutate a vector: even if multiple threads are simultaneously modifying the same shared object, each thread would be operating on distinct elements of the vector. Thus, no two threads would interfere with one another, making the updates remain independent.

Problems arise, however, **when the correctness of reading and writing shared data depends on the order in which threads execute**. This situation is known as a **data race** (also known as **race condition**). he term reflects the fact that the final output may change from one run to the next, depending on which thread finishes and updates the data last.

To illustrate the issue, let's modify our previous example by introducing a variable `temp`, whose value is updated in each iteration. This variable will be shared across threads and used to mutate the *i*-th entry of the vector `output`. By introducing a delay before writing each entry of `output`, a race condition is introduced, where all threads end up using the last value of `temp` (in this case, 2).

```julia
function foo()
    out  = zeros(Int, 2)
    temp = 0

    for i in 1:2
        temp   = i; sleep(i)
        out[i] = temp
    end

    return out
end
```

```
julia> foo()
2-element Vector{Int64}:
 1
 2
```

```julia
function foo()
    out  = zeros(Int, 2)
    temp = 0

    @threads for i in 1:2
        temp   = i; sleep(i)
        out[i] = temp
    end

    return out
end
```

```
julia> foo()
2-element Vector{Int64}:
 1
 1
```

```
function foo()
    out  = zeros(Int, 2)


    @threads for i in 1:2
        temp   = i; sleep(i)
        out[i] = temp
    end

    return out
end
```

```
julia> foo()
2-element Vector{Int64}:
 1
 2
```

In this specific scenario, the issue can be easily circumvented by defining `temp` as a variable local to the for-loop, rather than initializing it outside. This ensures that each thread works with its own private copy of `temp`, thereby eliminating the data race.

Beyond the solution proposed, the example highlights the subtleties of parallelizing operations. Even seemingly simple patterns can introduce hidden dependencies that lead to unsafe behavior when executed concurrently. To make this clearer, we now turn to a more common scenario where data races occur: reductions.

RACE CONDITIONS WITH REDUCTIONS

Reductions are a prime example of thread-unsafe operations. To illustrate, consider summing a collection in parallel. The problem arises because the variable accumulating the sum is shared across all threads. During each iteration, every thread attempts to read its current value, add its contribution, and write the result back. When several threads do this simultaneously, their updates can interleave unpredictably, causing some partial additions to be overwritten rather than combined. As a result, the final sum is nondeterministic and often incorrect, varying from run to run.

```
x = rand(1_000_000)

function foo(x)
    output = 0.

    for i in eachindex(x)
        output += x[i]
    end

    return output
end
```

```
julia> foo(x)
500658.01158503356
```

```julia
x = rand(1_000_000)

function foo(x)
    output = 0.

    @threads for i in eachindex(x)
        output += x[i]
    end

    return output
end
```

```
julia> foo(x)
21534.22602627773
```

```julia
x = rand(1_000_000)

function foo(x)
    output = 0.

    @threads for i in eachindex(x)
        output += x[i]
    end

    return output
end
```

```
julia> foo(x)
21342.557817155746
```

```julia
x = rand(1_000_000)

function foo(x)
    output = 0.

    @threads for i in eachindex(x)
        output += x[i]
    end

    return output
end
```

```
julia> foo(x)
21664.13362271611 2
```

The key insight from this example isn't that reductions are incompatible with multithreading. Rather, that the strategy to apply multithreading needs to be adapted accordingly. While the upcoming sections will present these strategies, we conclude this section by turning to the opposite end of the spectrum: problems that naturally lend themselves to parallel execution

EMBARRASSINGLY PARALLEL PROBLEMS

Programs that can be paralleled seamlessly are referred to as **embarrassingly parallel problems**. They can be broken down into many completely independent tasks, each of which can be executed in parallel without any need for communication, synchronization, or shared state between them. This independence provides complete flexibility in the order of task execution.

In the context of for-loops, one of the simplest ways to parallelize these problems is with the macro `@threads`. This is a form of thread-based parallelism, where the distribution of work is based on the number of threads available. Specifically, `@threads` attempts to balance the workload by dividing the iterations as evenly as possible.

The approach contrasts with `@spawn`, which implements task-based parallelism. With `@spawn`, the programmer explicitly defines tasks and synchronization must be handled manually. By contrast, `@threads` automatically schedules the tasks and waits for their completion before execution proceeds. The following example demonstrates this behavior.

```julia
x_small  = rand(    1_000)
x_medium = rand(  100_000)
x_big    = rand(1_000_000)

function foo(x)
    output = similar(x)

    for i in eachindex(x)
        output[i] = log(x[i])
    end

    return output
end
```

```
julia> @btime foo($x_small)
  3.043 μs (1 allocations: 7.938 KiB)
julia> @btime foo($x_medium)
  315.751 μs (2 allocations: 781.297 KiB)
julia> @btime foo($x_big)
  3.326 ms (2 allocations: 7.629 MiB)
```

```julia
x_small  = rand(    1_000)
x_medium = rand(  100_000)
x_big    = rand(1_000_000)

function foo(x)
    output = similar(x)

    @threads for i in eachindex(x)
        output[i] = log(x[i])
    end

    return output
end
```

```
julia> @btime foo($x_small)
  10.139 μs (122 allocations: 20.547 KiB)
julia> @btime foo($x_medium)
  42.044 μs (123 allocations: 793.906 KiB)
julia> @btime foo($x_big)
  340.589 μs (123 allocations: 7.642 MiB)
```

In the next section, we'll present a detailed comparison of `@threads` and `@spawn`.

11e. Parallel For-Loops: @threads

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INTRODUCTION

For-Loops are fundamental building blocks in scientific computing, often representing the most computationally intensive parts of a program. To accelerate these computations, parallel processing offers a powerful solution by distributing a for-loop's iterations across multiple processor threads. The central challenge, however, lies in how this work is divided, as the optimal strategy depends heavily on the nature of the workload.

This section introduces Julia's `@threads` macro as a straightforward approach to parallelization of for-loops. It operates by dividing the loop iterations evenly among the available threads, before execution begins. For the explanations, we'll contrast its underlying mechanism with the task-based parallelism offered by `@spawn`.

To clearly illustrate the differences between approaches, our examples will use `@spawn` in a simple but inefficient manner, where a separate task is created for every single iteration. While this is enough for the purpose of comparison with `@threads`, you should bear in mind that it's not representative of typical `@spawn` usage. In the next section, we'll revisit the macro by studying how to efficiently define tasks with `@spawn`. This exploration will reveal that the inherent flexibility of `@spawn` enables the construction of sophisticated parallel strategies, capable of replicating even the behavior of `@threads`.

SOME PRELIMINARIES

Parallelism techniques target code that performs multiple operations, making it a natural fit for for-loops. Using the macro `@spawn` introduced in the previous sections, we can parallelize for-loops through a task-based parallelism. In an upcoming section, we'll demonstrate that `@spawn` is flexible enough to split iterations into tasks in various ways. For now, we'll consider a simple (inefficient) case where each iteration defines a separate task. The coding implementing this technique is shown below.

```julia
@sync begin
    for i in 1:4
        @spawn println("Iteration $i is computed on Thread $(threadid())")
    end
end
```

```
Iteration 1 is computed on Thread 1
Iteration 2 is computed on Thread 2
Iteration 4 is computed on Thread 2
Iteration 3 is computed on Thread 2
```

```julia
@sync begin
    @spawn println("Iteration 1 is computed on Thread $(threadid())")
    @spawn println("Iteration 2 is computed on Thread $(threadid())")
    @spawn println("Iteration 3 is computed on Thread $(threadid())")
    @spawn println("Iteration 4 is computed on Thread $(threadid())")
end
```

```
Iteration 1 is computed on Thread 1
Iteration 2 is computed on Thread 2
Iteration 3 is computed on Thread 1
Iteration 4 is computed on Thread 2
```

When there are only a few iterations involved in a for-loop, creating one task per iteration can be a straightforward and effective way to parallelize the code. However, as the number of iterations increases, the approach becomes less efficient due to the overhead of task creation. To mitigate this issue, we need to consider alternative ways of parallelizing for-loops.

One such alternative is to create tasks that encompass multiple iterations, rather than just one iteration per task. The techniques to do this, which will be explored in following sections, offers more granular control, but at the expense of adding substantial complexity to the code.

In light of this complexity, Julia provides the `@threads` macro from the package `Threads`. The goal is to reduce the overhead of task creation, while keeping the parallelization simple. Specifically, `@threads` divides the set of iterations evenly among threads, thereby restricting the creation of tasks to the number of threads available.

The following example demonstrates the implementation of `@threads`, highlighting its difference from the approach with `@spawn`. The scenario considered is based on 4 iterations and 2 worker threads. We also display the thread on which each iteration is executed by using the `threadid()` function, which identifies the thread's ID that's computing the operation.

```julia
for i in 1:4
    println("Iteration $i is computed on Thread $(threadid())")
end
```

```
Iteration 1 is computed on Thread 1
Iteration 2 is computed on Thread 1
Iteration 3 is computed on Thread 1
Iteration 4 is computed on Thread 1
```

```julia
@threads for i in 1:4
    println("Iteration $i is computed on Thread $(threadid())")
end
```

```
Iteration 1 is computed on Thread 1
Iteration 2 is computed on Thread 1
Iteration 3 is computed on Thread 2
Iteration 4 is computed on Thread 2
```

```
@sync begin
    for i in 1:4
        @spawn println("Iteration $i is computed on Thread $(threadid())")
    end
end
```

```
Iteration 2 is computed on Thread 2
Iteration 1 is computed on Thread 1
Iteration 4 is computed on Thread 2
Iteration 3 is computed on Thread 2
```

The key distinction between `@threads` and `@spawn` lies in the strategy for thread allocation. Thread assignments with `@threads` are predetermined: before the for-loop begins, the macro pre-allocates threads and distributes iterations evenly. Thus, each thread is assigned a fixed number of iterations upfront, creating a predictable workload distribution. In the example, the feature is reflected in the allocation of two iterations per thread.

In contrast, `@spawn` creates a separate task for each iteration, dynamically scheduling them as soon as a thread becomes available. This method allows for more flexible thread utilization, with task assignments adapting in real-time to the current system load and available thread capacity. For instance, in the previous example, a single thread ended up computing three out of the four iterations.

@SPAWN VS @THREADS

The macros `@threads` and `@spawn` embody two distinct approaches to work distribution, thus catering to different types of scenarios. By comparing the creation of one task per iteration relative to `@threads`, we can highlight the inherent trade-offs involved in parallelizing code.

`@threads` employs a coarse-grained approach, making it well-suited for workloads with similar computational requirements. By reducing the overhead associated with task creation, this approach excels in scenarios where tasks have comparable execution times. However, it's less effective in handling workloads with unbalanced execution times, where some iterations are computationally intensive while others are relatively lightweight.

In contrast, `@spawn` adopts a fine-grained approach, treating each iteration as a separate task that can be scheduled independently. This allows for more flexible work distribution, with tasks dynamically allocated to available threads as soon as they become available. As a result, `@spawn` is particularly well-suited for scenarios with varying computational efforts, where iteration completion times can differ significantly. While this approach has a bigger overhead due to the creation of numerous smaller tasks, it simultaneously enables more efficient resource utilization. This is because no thread remains idle while tasks await computation.

In the following, we demonstrate the efficiency of the approaches under each scenario. With this goal, consider a situation where the *i*-th iteration computes `job(i;time_working)`. This function represents potential calculations performed during `time_working` seconds. It's formally defined as

follows.

```
function job(i; time_working)
    println("Iteration $i is on Thread $(threadid())")

    start_time = time()

    while time() - start_time < time_working
        1 + 1                   # compute `1+1` repeatedly during `time_working` seconds
    end
end
```

Note that `job` additionally identifies the thread on which it's running and displays it on the REPL.

Based on a for-loop with four iterations and a session with two worker threads, we next consider two scenarios. They differ by the computational workload of the iterations.

SCENARIO 1: UNBALANCED WORKLOAD

The first scenario represents a situation with unbalanced work, where some iterations require more computational effort. The feature is captured by assuming that the *i*-th iteration has a duration of `i` seconds.

We start by presenting the coding implementing each approach, and then provide explanations for each.

```
function foo(nr_iterations)
    for i in 1:nr_iterations
      job(i; time_working = i)
    end
end
```

```
Iteration 1 is on Thread 1
Iteration 2 is on Thread 1
Iteration 3 is on Thread 1
Iteration 4 is on Thread 1
  10.000 s (40 allocations: 1.562 KiB)
```

```
function foo(nr_iterations)
    @threads for i in 1:nr_iterations
        job(i; time_working = i)
    end
end
```

```
Iteration 1 is on Thread 1
Iteration 3 is on Thread 2
Iteration 2 is on Thread 1
Iteration 4 is on Thread 2
  7.000 s (51 allocations: 2.625 KiB)
```

```
function foo(nr_iterations)
    @sync begin
        for i in 1:nr_iterations
            @spawn job(i; time_working = i)
        end
    end
end
```

```
Iteration 1 is on Thread 1
Iteration 2 is on Thread 2
Iteration 3 is on Thread 1
Iteration 4 is on Thread 2
  6.000 s (69 allocations: 3.922 KiB)
```

Given the execution times for each iteration, a sequential approach would take 10 seconds. As for parallel implementations, `@threads` ensures that there are as many tasks created as number of threads. In the example, this means that there two tasks are created, with the first task computing iterations 1 and 2, and the second task computing iterations 3 and 4. As a result, the overall execution time is reduced to 7 seconds.

In contrast, `@spawn` creates a separate task for each iteration, which increases the overhead of task creation. Although the overhead is negligible in this example, it can be appreciated in the increased memory allocation. Despite this disadvantage, the approach allows each iteration to be executed as soon as a thread becomes available. Given the varying execution times between iterations, this dynamic allocation becomes advantageous, enabling iterations 3 and 4 to run in parallel.

The example demonstrates this, where iterations 1 and 2 are now executed on different threads. Since the first iteration only requires one second, the thread becomes available to compute the third iteration immediately. The final distribution of tasks on threads is such that iterations 1 and 3 are executed on one thread, while iterations 2 and 4 are executed on the other thread. This results in a total execution time of 6 seconds.

SCENARIO 2: BALANCED WORKLOAD

Consider now a scenario where the execution of `job` requires exactly the same time regardless of the iteration considered. To make the overhead more apparent, we'll use a larger number of iterations. In this context, `@threads` ensures parallelization with a reduced overhead, explaining why it's faster than the approach relying on `@spawn`.

```
function foo(nr_iterations)
    fixed_time = 1 / 1_000_000

    for i in 1:nr_iterations
      job(i; time_working = fixed_time)
    end
end
```

```
julia> @btime foo(1_000_000)
  1.717 s (without a warmup) (0 allocations: 0 bytes)
```

```julia
function foo(nr_iterations)
    fixed_time = 1 / 1_000_000

    @threads for i in 1:nr_iterations
        job(i; time_working = fixed_time)
    end
end
```

```
julia> @btime foo(1_000_000)
  858.399 ms (11 allocations: 1.094 KiB, without a warmup)
```

```julia
function foo(nr_iterations)
    fixed_time = 1 / 1_000_000

    @sync begin
        for i in 1:nr_iterations
            @spawn job(i; time_working = fixed_time)
        end
    end
end
```

```
julia> @btime foo(1_000_000)
  1.270 s (5000021 allocations: 498.063 MiB, 19.16% gc time, without a warmup)
```

11f. Parallelization in Practice

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INTRODUCTION

So far, we've explored two macros for parallelization: `@spawn` and `@threads`. The macro `@spawn` provides granular control over the parallelization process, letting users explicitly define the tasks to be executed concurrently. In contrast, `@threads` offers a simpler approach for parallelizing for-loops, where iterations are automatically partitioned into tasks according to the number of available threads.

Furthermore, we've pointed out that, due to inherent dependencies between computations, not all workloads are equally amenable to parallelization. Focusing on programs that aren't embarrassingly parallel, we've demonstrated that a naive approach to parallelization can lead to severe issues.

Overall, our discussions to this point have largely focused on the *syntax* and *work distribution* of these approaches. Therefore, we have yet to address how to apply multithreading in real scenarios. Furthermore, given the possibility of dependencies between computations, *how* to parallelize is only part of the challenge: knowing *when* to parallelize is equally important.

This section and the next one aim to bridge this gap, providing practical guidance on implementing multithreading. We begin by highlighting the advantages of coarse-grained parallelization over fine-grained parallelization. By dividing the workload into a small number of large tasks, coarse-grained parallelization reduces the scheduling overhead from managing numerous lightweight tasks.

After this, we revisit the parallelization of for-loops, this time using `@spawn`. In particular, leveraging the additional control that `@spawn` provides over task creation, we'll demonstrate how to apply multithreading in the presence of a ubiquitous type of dependency: reductions.

We conclude by showing a performance issue arising with multithreading, known as false sharing. While this doesn't affect the correctness the result, it can slow down computations if not addressed.

BETTER TO PARALLELIZE AT THE TOP

Given the overhead involved in multithreading, there's an inherent trade-off between creating new tasks and fully utilizing machine resources. This is why we must always consider whether parallelizing our code is worthwhile in the first place. For instance, when it comes to operations over collections, multithreading is only justified if the collections are large enough to offset the associated overhead. Otherwise, single-threaded approaches will consistently outperform parallelized ones.

In case multithreading is deemed beneficial, we immediately face another decision: at what level code should be parallelized. Next, we'll demonstrate that **parallelism at the highest possible level is preferable to multithreading individual operations**. In this way, we minimize the overhead of task

creation.

Note that the level of parallelization is always constrained by the degree of dependency between operations. Hence, our qualification of highest **possible** level. For instance, in problems requiring strictly serial computation, the best we can achieve is parallelization within each individual step.

To illustrate this, let's consider a for-loop where each iteration needs to sequentially compute three operations.

JULIA'S DEFAULT

```julia
step1(a) = a ^ 2
step2(a) = sqrt(a)
step3(a) = log(a + 1)

function all_steps(a)
   y      = step1(a)
   z      = step2(y)
   output = step3(z)

   return output
end

function foo(x)
   output = similar(x)

   for i in eachindex(output)
      output[i] = all_steps(x[i])
   end

   return output
end

x_small  = rand(  1_000)
x_large  = rand(100_000)
```

```
julia> @btime foo($x_small)
  5.000 μs (3 allocations: 7.883 KiB)
julia> @btime foo($x_large)
  527.133 μs (3 allocations: 781.320 KiB)
```

PARALLELIZATION AT THE HIGHEST LEVEL POSSIBLE

```julia
step1(a) = a ^ 2
step2(a) = sqrt(a)
step3(a) = log(a + 1)

function all_steps(a)
   y      = step1(a)
   z      = step2(y)
   output = step3(z)

   return output
end

function foo(x)
   output = similar(x)

   @threads for i in eachindex(output)
      output[i] = all_steps(x[i])
   end

   return output
end

x_small  = rand(  1_000)
x_large  = rand(100_000)
```

```
julia> @btime foo($x_small)
  11.289 μs (125 allocations: 20.508 KiB)
julia> @btime foo($x_large)
  54.258 μs (125 allocations: 793.945 KiB)
```

EACH OPERATION PARALLELIZED

```
step1(a) = a ^ 2
step2(a) = sqrt(a)
step3(a) = log(a + 1)

function parallel_step(f, x)
   output = similar(x)

   @threads for i in eachindex(output)
      output[i] = f(x[i])
   end

   return output
end

function foo(x)
   y      = parallel_step(step1, x)
   z      = parallel_step(step2, y)
   output = parallel_step(step3, z)

   return output
end

x_small  = rand(  1_000)
x_large  = rand(100_000)
```

```
julia> @btime foo($x_small)
  33.472 µs (375 allocations: 61.523 KiB)
julia> @btime foo($x_big)
  91.834 µs (375 allocations: 2.326 MiB)
```

The examples illustrate the two-step process outlined. First, it shows that parallelization is advantageous only with large collections. Otherwise, the question of whether to parallelize shouldn't even arise. Second, once multithreading is deemed beneficial, it demonstrates that grouping all operations into a single task is faster than parallelizing each operation individually.

IMPLICATIONS

The strategy of parallelizing code at the highest possible level has significant implications for program design, particularly when the program will eventually be applied to multiple independent objects. It suggests a practical guideline: start by implementing the code for a single object, without introducing parallelism. After thoroughly optimizing the single-case code, parallel execution can then be seamlessly integrated at the top level. The approach not only improves performance, but also simplifies the development by making debugging and testing more straightforward.

A common example of this strategy appears in scientific simulations, where many independent runs of the same model must be executed. In such cases, the most effective method is to maintain a single-threaded codebase for the model itself, and then run multiple instances of that model in parallel. This ensures that each run is as efficient as possible, while still taking full advantage of available computing resources.

THE IMPORTANCE OF WORK DISTRIBUTION

Multithreading performance is heavily influenced by how evenly the computational workload is distributed across iterations. The `@threads` macro is highly effective when each iteration requires roughly equal processing time, since the tasks it spawns will contain an equal amount of iterations. However, scenarios with uneven computational effort can pose significant challenges. In such cases, some threads may remain idle while others are heavily loaded, dramatically reducing the potential performance gains of parallel processing.

To address this issue, we need greater control over how work is distributed among threads. This calls for the use of `@spawn`.

One strategy is to make each iteration a separate task. However, this approach becomes extremely inefficient if there's a large number of iterations: creating far more tasks than there are threads introduces substantial and unnecessary overhead. The following example illustrates this effect, showing that spawning one task per iteration can result in extremely slow execution times.

@THREADS

```julia
x = rand(10_000_000)

function foo(x)
    output = similar(x)

    @threads for i in eachindex(x)
        output[i] = log(x[i])
    end

    return output
end
```

```
julia> @btime foo($x)
  4.907 ms (125 allocations: 76.309 MiB)
```

@SPAWN

```julia
x = rand(10_000_000)

function foo(x)
    output = similar(x)

    @sync for i in eachindex(x)
        @spawn output[i] = log(x[i])
    end

    return output
end
```

```
julia> @btime foo($x)
  11.284 s (60005935 allocations: 5.277 GiB)
```

An alternative strategy, which lets users fine-tune the workload of each task, is to partition iterations into smaller subsets that can be processed in parallel. Before detailing the implementation, we'll first explore how to partition a collection and its indices. This step makes use of the `ChunkSplitters` package.

PARTITIONING COLLECTIONS

The package `ChunkSplitters` makes it possible to partition a collection `x` and its indices. It provides two functions for *lazy* partitioning: `chunks` and `index_chunks`. These functions support `n` and `size` as keyword arguments, depending on the type of partition desired. Specifically, `n` sets the number of subsets to create, with each subset sized to distribute elements evenly. In contrast, `size` specifies the number of elements to be contained in each subset. Since an even distribution across all subsets can't be guaranteed, the package adjusts the number of elements in one of the subsets if necessary.

Below, we apply these functions to a variable `x` containing the 26 letters of the alphabet. Note that outputs require the use of `collect`, since `chunks` and `index_chunks` are lazy.

PARTITION BY NUMBER OF CHUNKS

```julia
x            = string.('a':'z')            # all letters from "a" to "z"

nr_chunks     = 5

chunk_indices = index_chunks(x, n = nr_chunks)
chunk_values  = chunks(x, n = nr_chunks)
```

```julia
julia> collect(chunk_indices)
5-element Vector{UnitRange{Int64}}:
 1:6
 7:11
 12:16
 17:21
 22:26
julia> collect(chunk_values)
5-element Vector{SubArray{String, 1, Vector{String}, Tuple{UnitRange{Int64}}, true}}:
 ["a", "b", "c", "d", "e", "f"]
 ["g", "h", "i", "j", "k"]
 ["l", "m", "n", "o", "p"]
 ["q", "r", "s", "t", "u"]
 ["v", "w", "x", "y", "z"]
```

PARTITION BY SIZE OF CHUNKS

```
x             = string.('a':'z')            # all letters from "a" to "z"

chunk_length  = 10

chunk_indices = index_chunks(x, size = chunk_length)
chunk_values  = chunks(x, size = chunk_length)
```

```
julia> collect(chunk_indices)
3-element Vector{UnitRange{Int64}}:
 1:10
 11:20
 21:26
julia> collect(chunk_values)
3-element Vector{SubArray{String, 1, Vector{String}, Tuple{UnitRange{Int64}}, true}}:
 ["a", "b", "c", "d", "e", "f", "g", "h", "i", "j"]
 ["k", "l", "m", "n", "o", "p", "q", "r", "s", "t"]
 ["u", "v", "w", "x", "y", "z"]
```

A relevant type of partition for multithreading is given by a number of chunks proportional to the number of worker threads. The example below implements this partition. It generates both chunk indices and chunk values.

Since this partition approach will eventually be used with for-loops, we also show how to use `enumerate` to pair each chunk with the values or subindices of its corresponding subset.

PARTITION BY NUMBER OF THREADS

```
x             = string.('a':'z')            # all letters from "a" to "z"

nr_chunks     = nthreads()

chunk_indices = index_chunks(x, n = nr_chunks)
chunk_values  = chunks(x, n = nr_chunks)
```

```
julia> collect(chunk_indices)
24-element Vector{UnitRange{Int64}}:
 1:2
 3:4
 ⋮
 25:25
 26:26
julia> collect(chunk_values)
24-element Vector{SubArray{String, 1, Vector{String}, Tuple{UnitRange{Int64}}, true}}:
 ["a", "b"]
 ["c", "d"]
 ⋮
 ["y"]
 ["z"]
```

PARTITION BY NUMBER OF THREADS - ENUMERATE

```julia
x             = string.('a':'z')            # all letters from "a" to "z"

nr_chunks     = nthreads()

chunk_indices = index_chunks(x, n = nr_chunks)
chunk_values  = chunks(x, n = nr_chunks)

chunk_iter1   = enumerate(chunk_indices)    # pairs (i_chunk, chunk_index)
chunk_iter2   = enumerate(chunk_values)     # pairs (i_chunk, chunk_value)
```

```
julia> collect(chunk_iter1)
24-element Vector{Tuple{Int64, UnitRange{Int64}}}:
 (1, 1:2)
 (2, 3:4)
 ⋮
 (23, 25:25)
 (24, 26:26)
julia> collect(chunk_iter2)
24-element Vector{Tuple{Int64, SubArray{String, 1, Vector{String},
Tuple{UnitRange{Int64}}, true}}}:
 (1, ["a", "b"])
 (2, ["c", "d"])
 ⋮
 (23, ["y"])
 (24, ["z"])
```

WORK DISTRIBUTION: DEFINING TASKS THROUGH CHUNKS

After covering how to partition datasets with the `ChunkSplitters` package, we can now turn to its role in multithreading. By dividing iterations into well-balanced chunks, we can assign each subset as an independent task to be processed concurrently.

Below, we parallelize a for-loop with `@threads`. Then, we present an equivalent implementation with `@spawn`. This requires defining a number of chunks that equals the number of worker threads.

@THREADS

```julia
x = rand(10_000_000)

function foo(x)
    output = similar(x)

    @threads for i in eachindex(x)
        output[i] = log(x[i])
    end

    return output
end
```

```
julia> @btime foo($x)
  4.907 ms (125 allocations: 76.309 MiB)
```

@SPAWN

```julia
x = rand(10_000_000)

function foo(x, nr_chunks)
    chunk_ranges = index_chunks(x, n=nr_chunks)
    output       = similar(x)

    @sync for chunk in chunk_ranges
        @spawn (@views @. output[chunk] = log(x[chunk]))
    end

    return output
end
```

```
julia> @btime foo($x, nthreads())
  5.143 ms (157 allocations: 76.311 MiB)
```

@SPAWN (EQUIVALENT)

```julia
x = rand(10_000_000)

function foo(x, nr_chunks)
    chunk_ranges = index_chunks(x, n=nr_chunks)
    output       = similar(x)
    task_indices = Vector{Task}(undef, nr_chunks)

    for (i, chunk) in enumerate(chunk_ranges)
        task_indices[i] = @spawn (@views @. output[chunk] = log(x[chunk]))
    end

    return wait.(task_indices)
end
```

```
julia> @btime foo($x, nthreads())
  4.816 ms (151 allocations: 76.310 MiB)
```

Unlike `@threads`, the `@spawn` macro offers finer control over how tasks are allocated to threads. This flexibility means we're not constrained to replicate the `@threads`'s behavior. For example, we could adopt different partitioning strategies where the number of chunks is proportional to the number of worker threads, as shown in the following examples.

@SPAWN

```julia
x = rand(10_000_000)

function foo(x, nr_chunks)
    chunk_ranges = index_chunks(x, n=nr_chunks)
    output       = similar(x)

    @sync for chunk in chunk_ranges
        @spawn (@views @. output[chunk] = log(x[chunk]))
    end

    return output
end
```

```
julia> @btime foo($x, 1 * nthreads())
  5.477 ms (157 allocations: 76.311 MiB)
julia> @btime foo($x, 2 * nthreads())
  8.792 ms (302 allocations: 76.325 MiB)
julia> @btime foo($x, 4 * nthreads())
  6.885 ms (590 allocations: 76.352 MiB)
```

@SPAWN

```julia
x = rand(10_000_000)

function compute!(output, x, chunk)
     @turbo for j in chunk
        output[j] = log(x[j])
     end
end

function foo(x, nr_chunks)
    chunk_ranges = index_chunks(x, n=nr_chunks)
    output       = similar(x)

    @sync for chunk in chunk_ranges
        @spawn compute!(output, x, chunk)
    end

    return output
end
```

```
julia> @btime foo($x, 1 * nthreads())
  5.040 ms (133 allocations: 76.310 MiB)
julia> @btime foo($x, 2 * nthreads())
  8.314 ms (254 allocations: 76.323 MiB)
julia> @btime foo($x, 4 * nthreads())
  4.049 ms (494 allocations: 76.347 MiB)
```

HANDLING DEPENDENCIES

So far, our discussion of parallelization has focused on embarrassingly parallel for-loops, where iterations are completely independent. In such cases, each iteration can be executed in isolation, making parallelization straightforward.

In contrast, when operations exhibit dependencies, the situation becomes more complex. Attempting to parallelize without first addressing these dependencies can lead not only to inefficiencies and wasted resources, but most critically to incorrect results.

The issue occurs because there's no one-size-fits-all method for handling dependencies. The appropriate strategy depends on the structure of the specific program. In all cases, though, the approach will require adapting the parallelization technique to work on a reformulated version of the problem. This reformulation must ensure that the tasks to be parallelized are independent—ultimately, parallelization is only possible if independence of operations can be achieved.

Note that, once dependencies are present, some portion of the work will inevitably be non-parallelizable. In fact, in certain cases no subset of independent tasks exists at all, as in computations that are inherently sequential.

HANDLING REDUCTIONS

A prominent example where dependencies between iterations arise is in reductions. To address these dependencies and still benefit from parallelization, the computation must be restructured. The standard approach is to divide the data into chunks, perform partial reductions on each chunk in parallel, and then combine the partial results in a final reduction step. This transformation removes the original dependency between iterations, because each partial reduction operates on a disjoint subset of the data.

To illustrate, let's compute the sum of elements of a vector `x`. The implementation follows a variant of the partitioning techniques discussed earlier, using `ChunkSplitters` to divide the data into independent segments.

JULIA'S DEFAULT (SEQUENTIAL)

```julia
x = rand(10_000_000)

function foo(x)
    output = 0.

    for i in eachindex(x)
        output += x[i]
    end

    output
end
```

```
julia> @btime foo($x)
  5.108 ms (0 allocations: 0 bytes)
```

@THREADS

```julia
x = rand(10_000_000)

function foo(x)
    chunk_ranges    = index_chunks(x, n=nthreads())
    partial_outputs = Vector{Float64}(undef, length(chunk_ranges))

    @threads for (i,chunk) in enumerate(chunk_ranges)
        partial_outputs[i] = sum(@view(x[chunk]))
    end

    return sum(partial_outputs)
end
```

```
julia> @btime foo($x)
  1.194 ms (124 allocations: 13.250 KiB)
```

@SPAWN

```julia
x = rand(10_000_000)

function foo(x)
    chunk_ranges    = index_chunks(x, n=nthreads())
    partial_outputs = Vector{Float64}(undef, length(chunk_ranges))

    @sync for (i, chunk) in enumerate(chunk_ranges)
        @spawn partial_outputs[i] = sum(@view(x[chunk]))
    end

    return sum(partial_outputs)
end
```

```
julia> @btime foo($x)
  1.169 ms (156 allocations: 13.781 KiB)
```

FALSE SHARING

So far, we've focused on constraints on parallelization due to dependencies between operations. But even when those dependencies are eliminated and the parallelized units are logically independent, performance can still be limited by hardware-level effects. One common culprit is cache contention, where multiple processor cores compete for shared cache resources. A particular manifestation of this issue, known as **false sharing**, occurs when multiple cores access data stored in the same cache line. Understanding this issue requires grasping how CPU caches function.

Processors use caches to store copies of frequently accessed data. They represent a smaller and faster memory unit than RAM, and are organized into fixed-size blocks called cache lines (typically 64 bytes). When data is needed, the processor first checks the cache. If the data isn't found, this must be retrieved from RAM and store a copy in the cache, a process that's significantly slower.

When multiple cores access data within the same cache line, the transfer of data follows a cache coherency protocol. This is designed to maintain data consistency across cores. This protocol can lead to situations where one core accesses data that isn't modified by another core, yet shares a cache block with altered data. In such cases, the entire cache line may be invalidated, forcing the cores to reload the entire cache block, despite there being no logical necessity to do so. This phenomenon is known as false sharing, and can cause unnecessary cache invalidations and refetches. The consequence is a significant degradation of the program's performance, particularly if threads frequently modify their variables.

While false sharing can occur in various multithreading scenarios, it's particularly prevalent in reduction operations. This case will be our focus next.

FALSE SHARING IN REDUCTIONS: AN ILLUSTRATION AND SOLUTIONS

Let's consider a simple scenario where the elements of a vector are summed after applying a logarithmic transformation. We'll present two multithreaded implementations to illustrate the impact of false sharing on performance.

The first implementation is a naive approach that closely resembles a typical sequential implementation. Its goal is to illustrate false sharing. The issue arises because multiple threads are repeatedly reading and writing adjacent memory locations in the `partial_outputs` vector. Since CPU cache lines typically span several vector elements, this leads to cache invalidation and forced synchronization between cores.

In contrast, the second implementation avoids false sharing, and we'll analyze why this is so after presenting the code snippets.

SEQUENTIAL

```julia
x = rand(10_000_000)

function foo(x)
    output = 0.

    for i in eachindex(x)
        output += log(x[i])
    end

    output
end
```

```
julia> @btime foo($x)
  35.349 ms (0 allocations: 0 bytes)
```

FALSE SHARING

```julia
x = rand(10_000_000)

function foo(x)
    chunk_ranges    = index_chunks(x, n=nthreads())
    partial_outputs = zeros(length(chunk_ranges))

    @threads for (i,chunk) in enumerate(chunk_ranges)
        for j in chunk
            partial_outputs[i] += log(x[j])
        end
    end

    return sum(partial_outputs)
end
```

```
julia> @btime foo($x)
  14.018 ms (124 allocations: 13.250 KiB)
```

LOCAL VARIABLE (@THREADS)

```julia
x = rand(10_000_000)

function foo(x)
    chunk_ranges    = index_chunks(x, n=nthreads())
    partial_outputs = zeros(length(chunk_ranges))

    @threads for (i,chunk) in enumerate(chunk_ranges)
        temp = 0.0
        for j in chunk
            temp += log(x[j])
        end
        partial_outputs[i] = temp
    end

    return sum(partial_outputs)
end
```

```
julia> @btime foo($x)
  3.621 ms (124 allocations: 13.250 KiB)
```

LOCAL VARIABLE (@SPAWN)

```julia
x = rand(10_000_000)

function foo(x)
    chunk_ranges    = index_chunks(x, n=nthreads())
    partial_outputs = zeros(length(chunk_ranges))

    @sync for (i,chunk) in enumerate(chunk_ranges)
        @spawn begin
            temp = 0.0
            for j in chunk
                temp += log(x[j])
            end
            partial_outputs[i] = temp
        end
    end

    return sum(partial_outputs)
end
```

```
julia> @btime foo($x)
  3.407 ms (156 allocations: 13.781 KiB)
```

To address false sharing in parallel reductions, there are several strategies that can be employed. All of them aim to prevent threads from repeatedly accessing the same cache line.

The previous example already presented one solution. It involves introducing a thread-local variable called `temp` to accumulate results. In this way, each thread maintains its own accumulator, writing to the shared array only once at the end.

Two additional solutions are presented below. The first one entails computing the reduction through a separate function. This addresses false sharing by the same logic as before, where the accumulation is done through a variable local to a function. The second solution involves defining `partial_outputs` as a matrix with extra rows (seven in particular), a technique known as vector padding. This approach guarantees that each thread's accumulator is allocated on a different cache line, so that concurrent updates don't interfere with each other at the cache level.

FUNCTION

```
x = rand(10_000_000)

function compute(x, chunk)
    temp = 0.0

    for j in chunk
        temp += log(x[j])
    end

    return temp
end

function foo(x)
    chunk_ranges    = index_chunks(x, n=nthreads())
    partial_outputs = zeros(length(chunk_ranges))

    @threads for (i,chunk) in enumerate(chunk_ranges)
        partial_outputs[i] = compute(x, chunk)
    end

    return sum(partial_outputs)
end
```

```
julia> @btime foo($x)
  3.623 ms (124 allocations: 13.250 KiB)
```

PADDING

```
x = rand(10_000_000)

function foo(x)
    chunk_ranges     = index_chunks(x, n=nthreads())
    partial_outputs  = zeros(7, length(chunk_ranges))

    @threads for (i,chunk) in enumerate(chunk_ranges)
        for j in chunk
            partial_outputs[1,i] += log(x[j])
        end
    end

    return sum(@view(partial_outputs[:,1]))
end
```

```
julia> @btime foo($x)
  3.843 ms (124 allocations: 14.500 KiB)
```

11g. Multithreading Packages

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INTRODUCTION

Parallelizing code may seem straightforward at a first glance. However, once we start delving into its subtleties, it's rapidly revealed that an effective implementation can be a daunting task. As we've discussed, naive implementations can lead to various issues, including performance problems like suboptimal load balancing or false sharing, and more severe concerns such as data races. Furthermore, even if the necessary skills for a correct implementation were mastered, the added complexity can severely impair the code's readability and maintainability.

To assist users in overcoming these obstacles, several packages for parallelization have emerged. These tools aim to simplify the implementation of multithreading, allowing users to leverage its benefits without grappling with low-level intricacies. In this section, we'll present a few of these packages. In particular, the focus will be on those that facilitate the application of multithreading to embarrassingly parallel problems and reductions.

The first package we explore is `OhMyThreads`. This offers a collection of high-level functions and macros that help developers parallelize operations with minimal effort. For instance, it eliminates the need to manually partition tasks and tackles subtle performance issues like false sharing. We'll then examine the `Polyester` package. Thanks to its reduced overhead, this package is capable of streamlining parallelization for small objects. After this, we revisit the `LoopVectorization` package. In particular, we introduce the macro `@tturbo`, which combines the benefits of SIMD instructions with multithreading.

PACKAGE "OHMYTHREADS"

As the package's documentation claims, `OhMyThreads` aims to be a user-friendly package to seamlessly apply multithreading. Consistent with its minimalist approach, the package only introduces a handful of essential functionalities that could easily be part of Julia's `Base`. The goal is to allow users to implement code parallelization, even if they don't possess deep expertise in the subject.

Specifically, the package provides various higher-order functions and macros that internally handle data-race conditions and performance issues like false sharing. Despite its simplicity, `OhMyThreads` still covers a significant range of scenarios, even supporting reduction operations.

In the following, we present the main higher-functions provided by the package. Before introducing them, let's indicate several features that these functions share. Firstly, their names in `OhMyThreads` mirror those in `Base`, but adding a prefix of `t`. For instance, the counterpart to `map` is `tmap`. Secondly, `OhMyThreads` offers the option of customizing the parallelization process. This is achieved

through an integration with `ChunkSplitters`, enabling customized work distributions among tasks via two keyword arguments: `nchunks`(or equivalently `ntasks`) to define the number of subsets in the partition, and `chunksize` to specify the number of elements in each subset.

> **Warning!**
> All the code snippets below assume you've already loaded the package with `using OhMyThreads`.

PARALLEL MAPPING

The `tmap` function serves as the multithreaded counterpart to `map`. Its syntax is `tmap(foo, x)`, where `foo` is the transforming function applied to each element of the collection `x`. Unfortunately, applying `tmap` in this form results in a performance loss. This is due to a technicality, arising from the type instability of the object `Task`.

To circumvent this issue and regain the lost performance, we must then explicitly indicate the output's type. To do this, we need the function method `tmap(foo, T , x)`, where `T` represents the *element type* of the output. Thus, if for instance the output is a `Vector{Float64}`, `T` would be `Float64`. Instead of directly declaring `T`, a more flexible alternative is to use `eltype(x)`, making the output's type mirror that of `x`.

The package also provides an in-place version, `tmap!`. In this case, since `tmap!` requires specifying the output vector, there's no need to do any extra work to avoid performance losses.

Below, we illustrate the application of these functions. To provide a basis for comparison, we include results of `map` and `map!` as single-threaded baselines.

```
x                       = rand(1_000_000)


foo(x)                  =  map(log, x)
foo_parallel1(x)        = tmap(log, x)
foo_parallel2(x)        = tmap(log, eltype(x), x)
```

```
julia> foo($x)
  3.254 ms (2 allocations: 7.629 MiB)
julia> foo_parallel1($x)
  1.494 ms (568 allocations: 16.958 MiB)
julia> foo_parallel2($x)
  337.724 µs (155 allocations: 7.642 MiB)
```

```
x                       = rand(1_000_000)
output                  = similar(x)

foo!(output,x)          =  map!(log, output, x)
foo_parallel!(output,x) = tmap!(log, output, x)
```

```
julia> foo($x)
  3.303 ms (0 allocations: 0 bytes)
julia> foo_parallel!($x)
  334.747 µs (150 allocations: 13.188 KiB)
```

`OhMyThreads` additionally provides the option to control the work distribution among tasks. This is done through the keyword arguments `nchunks` and `chunksize`, which are internally implemented via the package `ChunkSplitters`. Specifically, `nchunks` controls the number of subsets in the partition, while `chunksize` sets the number of elements per task. Note that `nchunks` and `chunksize` are mutually exclusive options, so that only one of them can be used at a time.

To illustrate the use `nchunks`, we'll set its value equal to `nthreads()`. By setting a number of chunks equal to the number of worker threads, we're adopting an even distribution among tasks, similar to how `@threads` operates. To set the same number with `chunksize`, we'll make use of the floor division operator `÷`. This is a binary operator that rounds a division down to the nearest integer towards zero. [1]

```
x      = rand(1_000_000)

foo(x) = tmap(log, eltype(x), x; nchunks = nthreads())
```

```
julia> @btime foo($x)
  339.006 µs (155 allocations: 7.642 MiB)
```

```
x      = rand(1_000_000)

foo(x) = tmap(log, eltype(x), x; chunksize = length(x) ÷ nthreads())
```

```
julia> @btime foo($x)
  355.825 µs (164 allocations: 7.643 MiB)
```

Do-Block Syntax

When `tmap` requires passing more complex functions, we can still use an anonymous function. In this case, the do-block syntax comes in handy. It enables the creation of multi-line functions, making code more readable. Below, we show an example.

```julia
x = rand(1_000_000)

function foo(x)

    output = tmap(a -> 2 * log(a), x)



    return output
end
```

```julia
x = rand(1_000_000)

function foo(x)

    output = tmap(x) do a
                2 * log(a)
             end

    return output
end
```

ARRAY COMPREHENSIONS

`OhMyThreads` also provides an alternative to `tmap` via array comprehensions. Unlike the standard implementation in `Base`, the version from `OhMyThreads` combines a multithreaded variant of `collect` with a generator. Similarly to `tmap`, specifying the output's element type is necessary to prevent performance losses.

```julia
x                = rand(1_000_000)
output           = similar(x)

foo(x)           = [log(a) for a in x]
foo_parallel1(x) = tcollect(log(a) for a in x)
foo_parallel2(x) = tcollect(eltype(x), log(a) for a in x)
```

```
julia> foo($x)
  3.231 ms (2 allocations: 7.629 MiB)
julia> foo_parallel1($x)
  1.489 ms (568 allocations: 16.958 MiB)
julia> foo_parallel2($x)
  336.948 µs (155 allocations: 7.642 MiB)
```

REDUCTIONS AND MAP-REDUCTIONS

`OhMyThreads` also offers multithreaded versions of `reduce` and `mapreduce`. They're respectively referred to as `treduce` and `tmapreduce`. These functions internally handle the race conditions inherent in reductions and address performance issues like false sharing. Notably, unlike `map`, these

functions can achieve optimal performance without requiring a specified output type.

```
x               = rand(1_000_000)

foo(x)          =  reduce(+, x)
foo_parallel(x) = treduce(+, x)
```

```
julia> foo($x)
  86.102 µs (0 allocations: 0 bytes)
julia> foo_parallel($x)
  29.542 µs (513 allocations: 43.047 KiB)
```

```
x               = rand(1_000_000)

foo(x)          =  mapreduce(log, +, x)
foo_parallel(x) = tmapreduce(log, +, x)
```

```
julia> foo($x)
  3.385 ms (0 allocations: 0 bytes)
julia> foo_parallel($x)
  389.624 µs (511 allocations: 43.000 KiB)
```

FOREACH AS A FASTER OPTION FOR MAPPINGS

The package also offers an implementation similar to for-loops through the function `tforeach`. Since we haven't covered the single-threaded version `foreach`, we begin by presenting it. The function follows a syntax identical to `map`, and is usually implemented using a do-block syntax, as shown below.

```
x = rand(1_000_000)

function foo(x)
    output = similar(x)

    for i in eachindex(x)
        output[i] = log(x[i])
    end

    return output
end
```

```
julia> foo($x)
  3.329 ms (2 allocations: 7.629 MiB)
```

```julia
x = rand(1_000_000)

function foo(x)
    output = similar(x)

    foreach(i -> output[i] = log(x[i]), eachindex(x))



    return output
end
```

```
julia> foo($x)
  3.251 ms (2 allocations: 7.629 MiB)
```

```julia
x = rand(1_000_000)

function foo(x)
    output = similar(x)

    foreach(eachindex(x)) do i
        output[i] = log(x[i])
    end

    return output
end
```

```
julia> foo($x)
  3.265 ms (2 allocations: 7.629 MiB)
```

Despite the similarities of `tforeach` and `tmap`, `tforeach` is more performant. Furthermore, it doesn't incur a performance penalty when the output type isn't specified.

```julia
x = rand(1_000_000)

function foo(x)
    output = similar(x)

    for i in eachindex(x)
        output[i] = log(x[i])
    end

    return output
end
```

```
julia> foo($x)
  3.281 ms (2 allocations: 7.629 MiB)
```

```julia
x = rand(1_000_000)

function foo(x)
    output = similar(x)

    tmap(eachindex(x)) do i
        output[i] = log(x[i])
    end

    return output
end
```

```
julia> foo($x)
  1.868 ms (571 allocations: 24.589 MiB)
```

```julia
x = rand(1_000_000)

function foo(x)
    output = similar(x)

    tmap(eltype(x), eachindex(x)) do i
        output[i] = log(x[i])
    end

    return output
end
```

```
julia> foo($x)
  582.144 µs (158 allocations: 15.272 MiB)
```

```julia
x = rand(1_000_000)

function foo(x)
    output = similar(x)

    tmap(eltype(x), eachindex(x)) do i
        output[i] = log(x[i])
    end

    return output
end
```

```
julia> foo($x)
  582.144 µs (158 allocations: 15.272 MiB)
```

Just like `tmap`, `tforeach` offers the keyword arguments `nchunks` and `chunksize` to control the workload distribution among worker threads. For the illustration, we use a distribution analogous to the one used above for `tmap`.

```julia
x = rand(1_000_000)

function foo(x)
    output = similar(x)

    tforeach(eachindex(x); nchunks = nthreads()) do i
        output[i] = log(x[i])
    end

    return output
end
```

```
julia> foo($x)
  340.708 μs (154 allocations: 7.642 MiB)
```

```julia
x = rand(1_000_000)

function foo(x)
    output = similar(x)

    tforeach(eachindex(x); chunksize = length(x) ÷ nthreads()) do i
        output[i] = log(x[i])
    end

    return output
end
```

```
julia> foo($x)
  358.567 μs (161 allocations: 7.643 MiB)
```

POLYESTER: PARALLELIZATION FOR SMALL OBJECTS

> **Warning!**
> All the code snippets below assume you executed `using Polyester` to load the package.

One key limitation of multithreading is its overhead due to the creation and scheduling of tasks. This issue can render parallelization impractical for smaller computational tasks, as the cost of thread management would outweigh any potential performance gain. Considering this, the application of multithreading is commonly reserved for objects sufficiently large to justify the cost.

The `Polyester` package addresses this limitation by implementing techniques that reduce the overhead. In this way, it becomes possible to parallelize objects that, otherwise, would be deemed too small to benefit from multithreading. Importantly, the package requires expressing the code to be parallelized as a for-loop.

To illustrate the benefits of the package, let's compare its performance to traditional methods. The following example considers a for-loop with 500 iterations, a relatively low number for applying multithreading. Indeed, the first tab shows that an approach based on `@threads` is slower than its single-threaded variant. In contrast, `Polyester` achieves comparable performance to the single-threaded variant, despite the low number of iterations. To use `Polyseter`, we simply need to prefix the for-loop with the `@batch` macro.

```
x = rand(500)

function foo(x)
    output = similar(x)

    for i in eachindex(x)
        output[i] = log(x[i])
    end

    return output
end
```

```
julia> foo($x)
  1.552 μs (1 allocations: 4.062 KiB)
```

```
x = rand(500)

function foo(x)
    output = similar(x)

    @threads for i in eachindex(x)
        output[i] = log(x[i])
    end

    return output
end
```

```
julia> foo($x)
  9.362 μs (122 allocations: 16.672 KiB)
```

```
x = rand(500)

function foo(x)
    output = similar(x)

    @batch for i in eachindex(x)
        output[i] = log(x[i])
    end

    return output
end
```

```
julia> foo($x)
  992.000 ns (1 allocations: 4.062 KiB)
```

For larger objects, it's worth noting that `Polyester` may not necessarily outperform (or underperform) alternative methods. In such cases, it's recommend benchmarking your particular application.

REDUCTIONS

`Polyester` also supports reduction operations. These can be implemented by prepending the for-loop with the expression `@batch reduce=(<tuple with operation and variable reduced>)`. Notably, Polyester's implementation has been designed to avoid common pitfalls of reductions, such as data races and false sharing, ensuring both correctness and performance. The following example illustrates its application.

```
x = rand(250)

function foo(x)
    output = 0.0

    for i in eachindex(x)
        output += log(x[i])
    end

    return output
end
```

```
julia> @btime foo($x)
  745.289 ns (0 allocations: 0 bytes)
```

```
x = rand(250)

function foo(x)
    output = 0.0

    @batch reduction=( (+, output) ) for i in eachindex(x)
        output += log(x[i])
    end

    return output
end
```

```
julia> @btime foo($x)
  543.889 ns (0 allocations: 0 bytes)
```

We can also incorporate more than one reduction operation per iteration, as demonstrated below.

```julia
x = rand(250)

function foo(x)
    output1 = 1.0
    output2 = 0.0

    for i in eachindex(x)
        output1 *= log(x[i])
        output2 += exp(x[i])
    end

    return output1, output2
end
```

```
julia> @btime foo($x)
  1.241 μs (0 allocations: 0 bytes)
```

```julia
x = rand(250)

function foo(x)
    output1 = 1.0
    output2 = 0.0

    @batch reduction=( (*, output1), (+, output2) ) for i in eachindex(x)
        output1 *= log(x[i])
        output2 += exp(x[i])
    end

    return output1, output2
end
```

```
julia> @btime foo($x)
  630.302 ns (0 allocations: 0 bytes)
```

```julia
x = rand(250)

function foo(x)
    output1 = 1.0
    output2 = 0.0

    @batch reduction=( (*, output1), (+, output2) ) for i in eachindex(x)
        output1  = output1 * log(x[i])
        output2  = output2 + exp(x[i])
    end

    return output1, output2
end
```

```
julia> @btime foo($x)
  641.075 ns (0 allocations: 0 bytes)
```

LOCAL VARIABLES

`Polyester` also treats variables as local per iteration, unlike `@threads`.

```
function foo()
    out  = zeros(Int, 2)
    temp = 0

    for i in 1:2
        temp   = i; sleep(i)
        out[i] = temp
    end

    return out
end
```

```
julia> foo($x)
2-element Vector{Int64}:
 1
 2
```

```
function foo()
    out  = zeros(Int, 2)


    @threads for i in 1:2
        temp   = i; sleep(i)
        out[i] = temp
    end

    return out
end
```

```
julia> foo($x)
2-element Vector{Int64}:
 1
 2
```

```
function foo()
    out  = zeros(Int, 2)
    temp = 0

    @threads for i in 1:2
        temp   = i; sleep(i)
        out[i] = temp
    end

    return out
end
```

```
julia> foo($x)
2-element Vector{Int64}:
 2
 2
```

```julia
function foo()
    out  = zeros(Int, 2)
    temp = 0

    @batch for i in 1:2
        temp   = i; sleep(i)
        out[i] = temp
    end

    return out
end
```

```
julia> foo($x)
2-element Vector{Int64}:
 1
 2
```

SIMD + MULTITHREADING

> **Warning!**
> All the code snippets below assume you've already loaded the package with `using LoopVectorization`.

We've already covered the package `LoopVectorization` in the section about SIMD instructions. We now revisit this package to demonstrate its ability to combine SIMD with multithreading. The feature is achieved through integration with the `Polyester` package.

The primary approach to implementing the functionality involves the `@tturbo` macro, which provides a parallelized version of `@turbo`. Unlike the @`threads` macro, where the application of SIMD optimizations is left to the compiler's discretion, `@tturbo` automatically applies SIMD.

To illustrate the benefits of `@tturbo`, let's consider an example scenario where SIMD isn't applied automatically by `@threads`, despite that the operation is well-suited for this purpose.

```julia
x = BitVector(rand(Bool, 100_000))
y = rand(100_000)

function foo(x,y)
    output = similar(y)

    for i in eachindex(x)
        output[i] = ifelse(x[i], log(y[i]), y[i] * 2)
    end

    output
end
```

```
julia> foo($x)
  587.694 μs (2 allocations: 781.297 KiB)
```

```julia
x = BitVector(rand(Bool, 100_000))
y = rand(100_000)

function foo(x,y)
    output = similar(y)

    @threads for i in eachindex(x)
        output[i] = ifelse(x[i], log(y[i]), y[i] * 2)
    end

    output
end
```

```
julia> foo($x)
  80.625 μs (123 allocations: 793.906 KiB)
```

```julia
x = BitVector(rand(Bool, 100_000))
y = rand(100_000)

function foo(x,y)
    output = similar(y)

    @tturbo for i in eachindex(x)
        output[i] = ifelse(x[i], log(y[i]), y[i] * 2)
    end

    output
end
```

```
julia> foo($x)
  57.225 μs (2 allocations: 781.297 KiB)
```

The `@tturbo` macro is also available as a broadcasting version. Although the for-loop implementation could be more performant in some scenarios, the broadcasting variant significantly simplifies the syntax. Furthermore, it's particularly useful for parallelizing broadcasting operations,

since no built-in macro currently exists for this purpose. Below, we provide a simple example that demonstrates the improvement in readability achieved by using this variant.

```
x      = rand(1_000_000)

function foo(x)
    output = similar(x)

    @tturbo for i in eachindex(x)
        output[i] = log(x[i]) / x[i]
    end

    return output
end
```

```
julia> foo($x)
  525.304 μs (2 allocations: 7.629 MiB)
```

```
x      = rand(1_000_000)

foo(x) = @tturbo log.(x) ./ x
```

```
julia> foo($x)
  524.273 μs (2 allocations: 7.629 MiB)
```

FLOOPS: PARALLEL FOR-LOOPS (*OPTIONAL*)

> **Warning!**
> All the code snippets below assume you've already loaded the package by executing `using FLoops`.

We conclude this section with a brief overview of the package `FLoops`. The presentation is labeled as optional since its use beyond simple applications could require some workarounds. Moreover, it appears not to be actively maintained.

The primary macro provided by the package is `@floop`, exclusively designed to parallelize for-loops. An example of its usage is provided below.

```julia
x = rand(1_000_000)

function foo(x)
    output = similar(x)

    for i in eachindex(x)
        output[i] = log(x[i])
    end

    return output
end
```

```
julia> foo($x)
  3.353 ms (2 allocations: 7.629 MiB)
```

```julia
x = rand(1_000_000)

function foo(x)
    output = similar(x)

    @floop for i in eachindex(x)
        output[i] = log(x[i])
    end

    return output
end
```

```
julia> foo($x)
  388.563 µs (157 allocations: 7.645 MiB)
```

`@floop` can also be used for reductions by including `@reduce` at the beginning of the line with a reduction operation. The macro addresses the inherent data race of reductions and avoids false sharing issues.

```julia
x = rand(1_000_000)

function foo(x)
    output = 0.0

    for i in eachindex(x)
        output += log(x[i])
    end

    return output
end
```

```
julia> foo($x)
  3.396 ms (0 allocations: 0 bytes)
```

```julia
x = rand(1_000_000)

function foo(x)
    chunk_ranges    = index_chunks(x, n=nthreads())
    partial_outputs = zeros(length(chunk_ranges))

    @threads for (i,chunk) in enumerate(chunk_ranges)
        for j in chunk
            partial_outputs[i] += log(x[j])
        end
    end

    return sum(partial_outputs)
end
```

```
julia> foo($x)
  1.314 ms (122 allocations: 13.234 KiB)
```

```julia
x = rand(1_000_000)

function foo(x)
    output = 0.0

    @floop for i in eachindex(x)
        @reduce output += log(x[i])
    end

    return output
end
```

```
julia> foo($x)
  370.835 µs (252 allocations: 17.516 KiB)
```

FOOTNOTES

[1] For example, `5 ÷ 3` would return `1`.