# Graphs with few trivial critical ideals

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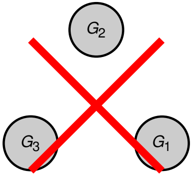
1. **Assumptions**
2. **Motivation: Critical group**
   - Invariant factors
   - The family $\mathcal{G}_i$
   - Graphs with more than one invariant factor equal to 1
3. **Critical ideals**
   - Definition

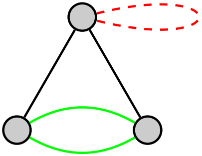
## Assumptions on graphs

- are connected,



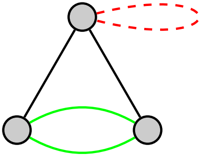
## Assumptions on graphs

- multiple edges are allowed, and



## Assumptions on graphs

- loops are forbidden.



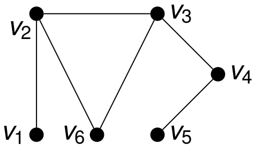
## Laplacian Matrix

### Definition

Let $G = (V, E)$ be a graph, the Laplacian matrix $L(G)$ of $G$ is the matrix with rows and columns indexed by the vertices of $G$ given by

$$L(G)_{uv} = \begin{cases} \deg_G(u) & \text{if } u = v, \\ -m_{uv} & \text{otherwise,} \end{cases}$$

where $\deg_G(u)$ denote the degree of $u$, and $m_{uv}$ denote the number of edges from $u$ to $v$.




$$L(G, X_G) = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & 0 & -1 \\ 0 & -1 & 3 & -1 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 0 & 2 \end{bmatrix}$$

**Definition**

By considering $L(G)$ as a linear operator on $\mathbb{Z}^n$, the critical group $K(G)$ of $G$ is the torsion part of the cokernel of $L(G)$.

$$coker(L(G)) = \mathbb{Z}^n/\mathrm{Im}L(G) = \mathbb{Z} \oplus K(G).$$

## Invarian factors

**Theorem**

$$K(G) \cong \mathbb{Z}_{f_1} \oplus \mathbb{Z}_{f_2} \oplus \cdots \oplus \mathbb{Z}_{f_{n-1}},$$

*where* $f_i \leq 0$ *and* $f_i \mid f_j$ *for all* $i \leq j$.

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## Proposition

If $\Delta_i(G)$ is the g.c.d of the $i$-minors of $L(G)$, then $f_i$ is equal to $\Delta_i(G)/\Delta_{i-1}(G)$, where $\Delta_0(G) = 1$.

## The family $\mathcal{G}_i$

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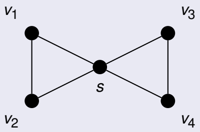
**Example**

The following graph belongs to $\mathcal{G}_2$.




$$L(G, s) \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

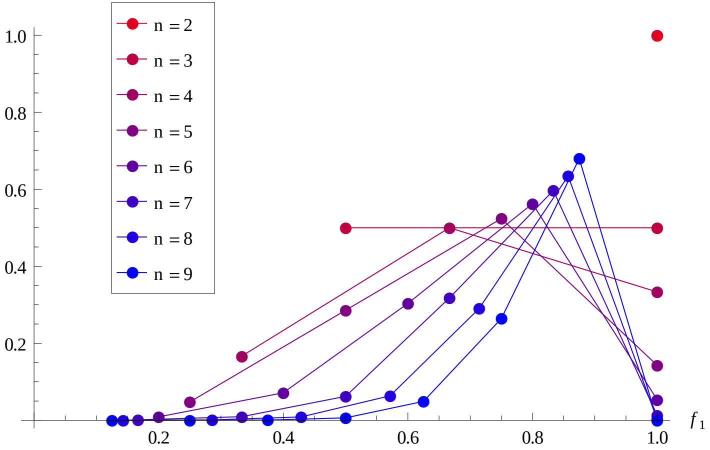



**Figura:** Normalized number of graphs with $f_1$ invariant factors equal to 1.

**Question**

How often the critical group is cyclic? that is, how often $f_1(G)$ is equal to $n-2$ or $n-1$?

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**Conjeture (D. Wagner, 2001)**

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**Theorem (M. Wood, 2014)**

*The probability that the critical group of a random graph is cyclic is asymptotically at most*

$$\zeta(3)^{-1}\zeta(5)^{-1}\zeta(7)^{-1}\zeta(9)^{-1}\zeta(11)^{-1}\cdots \approx 0{,}7935212$$

*where $\zeta$ is the Riemann zeta function.*

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**Theorem (Lorenzini, 1991)**

*If G is a simple connected graphs, then the following statements are equivalent:*

I. $G \in \mathcal{G}_1$,
II. $G$ *is* $P_3$*-free, where* $P_n$ *denote the path with n vertices,*
III. $G$ *is a complete graph.*

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### Question

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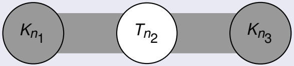
**Question**

What we can say about $\mathcal{G}_2$ and $\mathcal{G}_3$?

**Theorem**

*Let $G$ be a simple connected graph. Then, $G \in \mathcal{G}_2$ if and only if $G$ is one of the following graphs:*

- I. $K_{n_1,n_2,n_3}$, *where $n_1$, $n_2$ y $n_3$ have the same parity.*
- II. $L_{n_1,n_2,n_3}$, *where $n_1, n_2, n_3 \geq 3$ have the same parity, and other eleven special cases.*



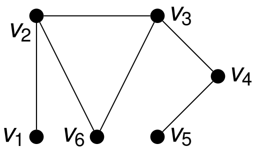

The proof uses critical ideals

# The critical ideals of a graph

## Definition

Given a graph $G = (V, E)$ and a set of indeterminates $X_G = \{x_u : u \in V\}$, the generalised Laplacian matrix $L(G, X_G)$ of $G$ is the matrix given by

$$L(G, X_G)_{uv} = \begin{cases} x_u & \text{if } u = v, \\ -m_{uv} & \text{otherwise.} \end{cases}$$




$$L(G, X_G) = \begin{bmatrix} x_1 & -1 & 0 & 0 & 0 & 0 \\ -1 & x_2 & -1 & 0 & 0 & -1 \\ 0 & -1 & x_3 & -1 & 0 & -1 \\ 0 & 0 & -1 & x_4 & -1 & 0 \\ 0 & 0 & 0 & -1 & x_5 & 0 \\ 0 & -1 & -1 & 0 & 0 & x_6 \end{bmatrix}$$

## Definition

For all $1 \leq i \leq |V(G)|$, the *$i$-th critical ideal* of $G$ is the determinantal ideal given by

$$I_i(G, X_G) = \langle \{ m \; : \; m \text{ is an } i \times i \text{ minor of } L(G, X_G) \} \rangle \subseteq \mathbb{Z}[X_G].$$

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The *algebraic co-rank* $\gamma(G)$ of a graph $G$ is the number of trivial critical ideals of $G$.

**Remark**

If $H$ is an induced subgraph of $G$, then $I_i(H, X_H) \subseteq I_i(G, X_G)$ for all $i \leq |V(H)|$. Thus $\gamma(H) \leq \gamma(G)$.

**Definition**

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**Remark**

$\Gamma_{\leq i}$ is closed under induced subgraphs.

**Theorem**

*If* $\deg(G) = (\deg_G(v_1), ..., \deg_G(v_n))$ *is the degree vector of* $G$*, and* $f_1 \mid \cdots \mid f_{n-1}$ *are the invariant factors of* $K(G)$*, then*

$$I_i(G, \deg(G)) = \left\langle \prod_{j=1}^{i} f_j \right\rangle \text{ for all } 1 \leq i \leq n-1.$$

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### Remark

- If the critical ideal $I_i(G, X_G)$ is trivial, then $f_i = 1$.
- If $f_i \neq 1$, then the critical ideal $I_i(G, X_G)$ is not trivial.
- $\mathcal{G}_i \subseteq \Gamma_{\leq i}$ for all $i \geq 0$.

**Definition**

A graph $G$ is forbidden for $\Gamma_{\leq k}$ when $\gamma(G) \geq k+1$.

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**Remark**

$G \in \Gamma_{\leq k}$ if and only if $G$ is $\mathbf{Forb}(\Gamma_{\leq k})$-free.

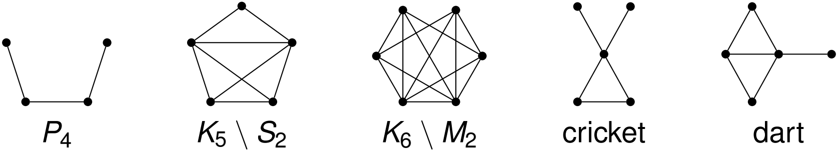


**Figura:** The family $\mathcal{F}_2$ of graphs.

**Theorem**

*If $G$ is a simple connected graph. Then the following statements are equivalent:*

- $G \in \Gamma_{\leq 2}$,
- $G$ is $\mathcal{F}_2$-free,
- $G$ is $K_{n_1,n_2,n_3}$ o $L_{n_1,n_2,n_3}$.

**Our approach to solve this problem**

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2. Determining the structure of the $\mathcal{F}$-free graphs.
3. Computing the critical group of the $\mathcal{F}$-free graphs.

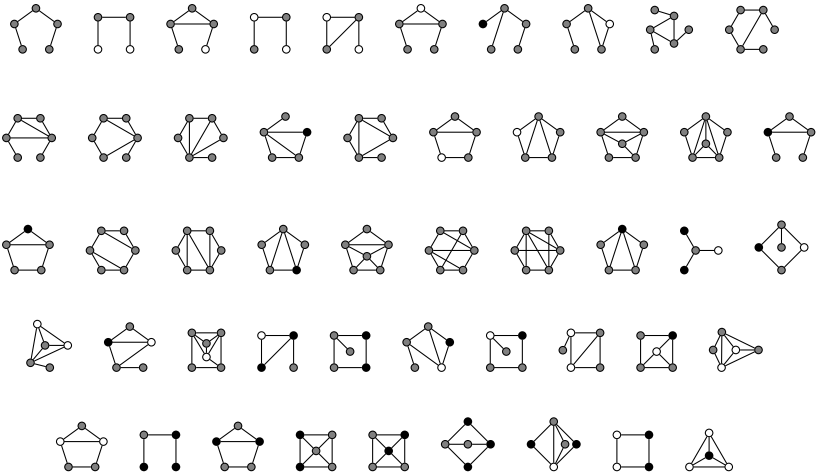

**Figura:** The family of graphs $\mathfrak{F}$.

**Proposition**

Each graph in $\mathfrak{F}$ belongs to $\mathbf{Forb}(\Gamma_{\leq 3})$.

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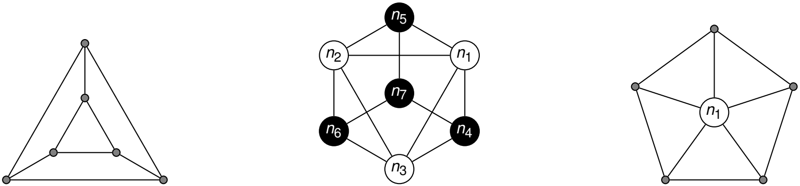
**Conjecture**

$\mathfrak{F} = \textbf{Forb}(\Gamma_{\leq 3})$.

### Theorem

*If a graph $G \in \Gamma_{\leq 3}$ has clique number at most 3, then $G$ is an induced subgraph of a graph in the family of graphs $\mathfrak{C}$.*

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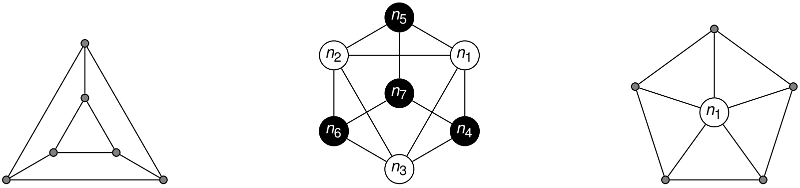


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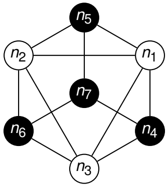
$n_5$
$n_2$
$n_1$
$n_7$
$n_6$
$n_4$
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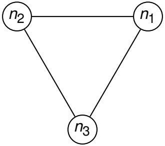
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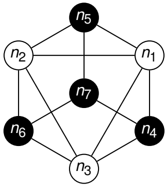
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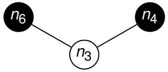
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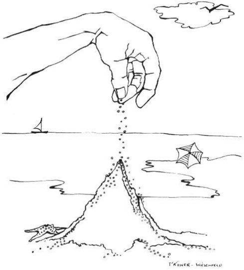

¡Gracias!