Graphs with few trivial critical ideals

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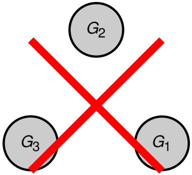
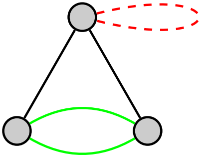
Assumptions on graphs

- are connected,



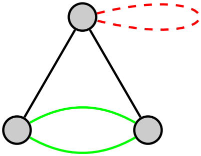
Assumptions on graphs

- multiple edges are allowed, and



Assumptions on graphs

- loops are forbidden.



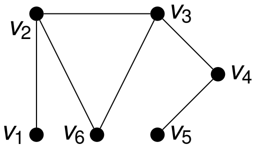
Laplacian Matrix

Definition

Let $G = (V, E)$ be a graph, the Laplacian matrix $L(G)$ of G is the matrix with rows and columns indexed by the vertices of G given by

$$L(G)_{uv} = \begin{cases} \deg_G(u) & \text{if } u = v, \\ -m_{uv} & \text{otherwise,} \end{cases}$$

where $\deg_G(u)$ denote the degree of u, and m_{uv} denote the number of edges from u to v.



$$L(G, X_G) = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & 0 & -1 \\ 0 & -1 & 3 & -1 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 0 & 2 \end{bmatrix}$$

Definition

By considering $L(G)$ as a linear operator on $\mathbb{Z}^n$, the critical group $K(G)$ of G is the torsion part of the cokernel of $L(G)$.

$$coker(L(G)) = \mathbb{Z}^n/\text{Im}L(G) = \mathbb{Z} \oplus K(G).$$

Invarian factors

Theorem

$$K(G) \cong \mathbb{Z}_{f_1} \oplus \mathbb{Z}_{f_2} \oplus \cdots \oplus \mathbb{Z}_{f_{n-1}},$$

where $f_i \geq 0$ *and* $f_i \mid f_j$ *for all* $i \leq j$.

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The $f_1, f_2, ..., f_{n-1}$ are called invariant factors.

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Proposition

If $\Delta_i(G)$ is the g.c.d of the i-minors of $L(G)$, then f_i is equal to $\Delta_i(G)/\Delta_{i-1}(G)$, where $\Delta_0(G) = 1$.

The family $\mathcal{G}_i$

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Let $f_1(G)$ be the number of invariant factors of $L(G)$ equal to 1.

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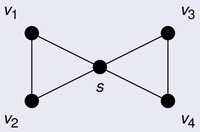
Example

The following graph belongs to $\mathcal{G}_2$.



$$L(G,s) \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

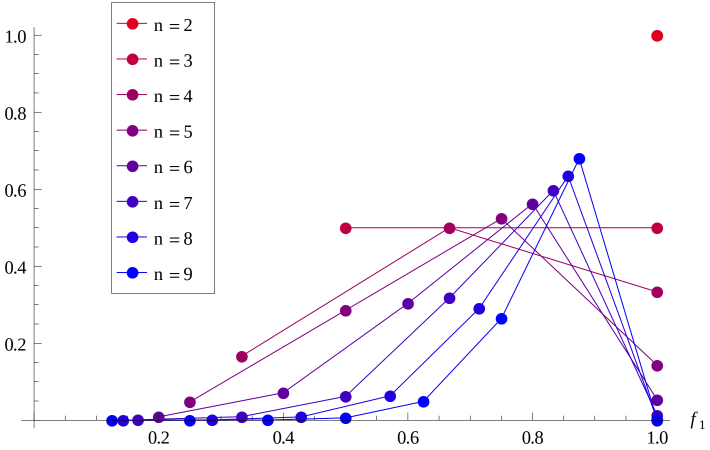


Figura: Normalized number of graphs with f_1 invariant factors equal to 1.

Question

How often the critical group is cyclic? that is, how often $f_1(G)$ is equal to $n-2$ or $n-1$?

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Conjeture (D. Wagner, 2001)

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Theorem (M. Wood, 2014)

The probability that the critical group of a random graph is cyclic is asymptotically at most

$$\zeta(3)^{-1}\zeta(5)^{-1}\zeta(7)^{-1}\zeta(9)^{-1}\zeta(11)^{-1}\cdots \approx 0{,}7935212$$

where ζ is the Riemann zeta function.

Graphs with one invariant factor equal to 1

On the other hand...

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On the other hand...

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Theorem (Lorenzini, 1991)

If G is a simple connected graphs, then the following statements are equivalent:

1. $G \in \mathcal{G}_1$,
2. *G is P_3-free, where P_n denote the path with n vertices,*
3. *G is a complete graph.*

Graphs with more than one invariant factor equal to 1

Question

What we can say about $\mathcal{G}_2$ and $\mathcal{G}_3$?

Graphs with more than one invariant factor equal to 1

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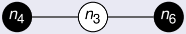
What we can say about $\mathcal{G}_2$ and $\mathcal{G}_3$?

Theorem

Let G be a simple connected graph. Then, $G \in \mathcal{G}_2$ if and only if G is one of the following graphs:

- **I.** K_{n_1,n_2,n_3}, *where n_1, n_2, n_3 have the same parity.*
- **II.** L_{n_1,n_2,n_3}, *where $n_1, n_2, n_3 \geq 3$ have the same parity, and other eleven special cases.*



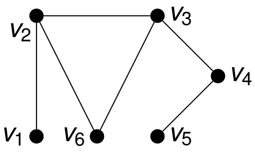
The proof uses critical ideals

The critical ideals of a graph

Definition

Given a graph $G = (V, E)$ and a set of indeterminates $X_G = \{x_u \ : \ u \in V\}$, the generalised Laplacian matrix $L(G, X_G)$ of G is the matrix given by

$$L(G, X_G)_{uv} = \begin{cases} x_u & \text{if } u = v, \\ -m_{uv} & \text{otherwise.} \end{cases}$$



$$L(G, X_G) = \begin{bmatrix} x_1 & -1 & 0 & 0 & 0 & 0 \\ -1 & x_2 & -1 & 0 & 0 & -1 \\ 0 & -1 & x_3 & -1 & 0 & -1 \\ 0 & 0 & -1 & x_4 & -1 & 0 \\ 0 & 0 & 0 & -1 & x_5 & 0 \\ 0 & -1 & -1 & 0 & 0 & x_6 \end{bmatrix}$$

Definition

For all $1 \leq i \leq |V(G)|$, the *i-th critical ideal* of G is the determinantal ideal given by

$$I_i(G, X_G) = \langle \{ m \,:\, m \text{ is an i-minor of } L(G, X_G) \} \rangle \subseteq \mathbb{Z}[X_G].$$

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Remark

If H is an induced subgraph of G, then $I_i(H, X_H) \subseteq I_i(G, X_G)$ for all $i \leq |V(H)|$. Thus $\gamma(H) \leq \gamma(G)$.

Definition

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Remark

$\Gamma_{\leq i}$ is closed under induced subgraphs.

Theorem

If $\deg(G) = (\deg_G(v_1), ..., \deg_G(v_n))$ *is the degree vector of* G*, and* $f_1 \mid \cdots \mid f_{n-1}$ *are the invariant factors of* $K(G)$*, then*

$$I_i(G, \deg(G)) = \left\langle \prod_{j=1}^{i} f_j \right\rangle \text{ for all } 1 \leq i \leq n-1.$$

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Remark

- If the critical ideal $I_i(G, X_G)$ is trivial, then $f_i = 1$.
- If $f_i \neq 1$, then the critical ideal $I_i(G, X_G)$ is not trivial.
- $\mathcal{G}_i \subseteq \Gamma_{\leq i}$ for all $i \geq 0$.

Definition

A graph G is forbidden for $\Gamma_{\leq k}$ when $\gamma(G) \geq k+1$.

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Let **Forb**$(\Gamma_{\leq k})$ be the set of minimal (under induced subgraphs property) forbidden graphs for $\Gamma_{\leq k}$.

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Definition

Let $\mathbf{Forb}(\Gamma_{\leq k})$ be the set of minimal (under induced subgraphs property) forbidden graphs for $\Gamma_{\leq k}$.

Remark

$G \in \Gamma_{\leq k}$ if and only if G is $\mathbf{Forb}(\Gamma_{\leq k})$-free.

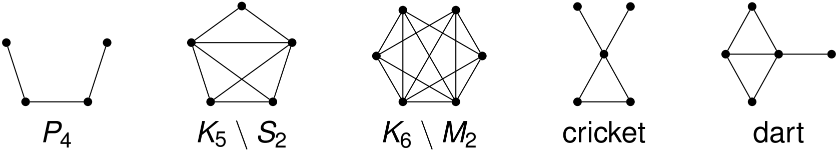

Figura: The family $\mathcal{F}_2$ of graphs.

Theorem

If G is a simple connected graph. Then the following statements are equivalent:

- $G \in \Gamma_{\leq 2}$,
- G *is* $\mathcal{F}_2$*-free,*
- G *is* K_{n_1,n_2,n_3} *o* L_{n_1,n_2,n_3}.

Our approach to solve this problem

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2. Determining the structure of the $\mathcal{F}$-free graphs.
3. Computing the critical group of the $\mathcal{F}$-free graphs.

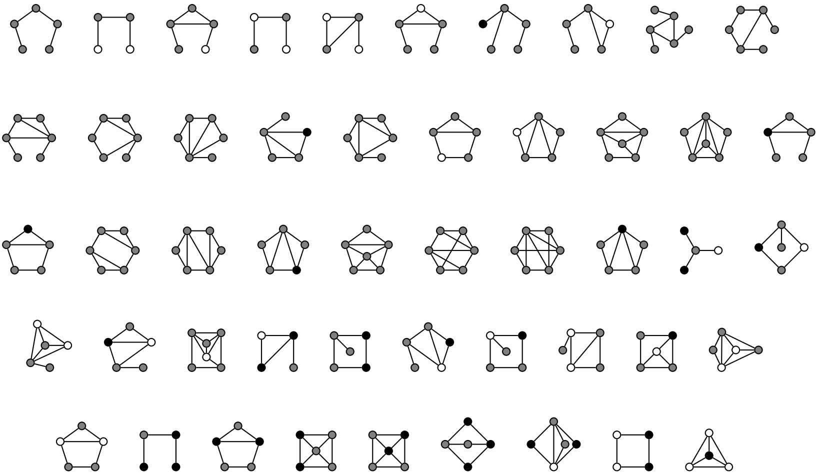

Figura: The family of graphs $\mathfrak{F}$.

Proposition

Each graph in $\mathfrak{F}$ belongs to $\mathbf{Forb}(\Gamma_{\leq 3})$.

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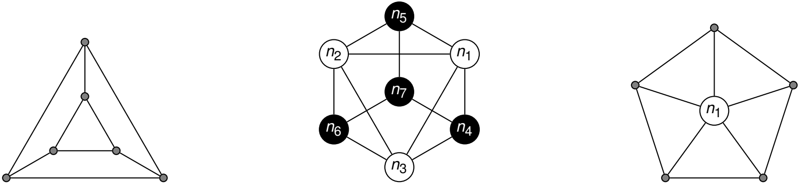
Conjecture

$\mathfrak{F} = \textbf{Forb}(\Gamma_{\leq 3})$.

Theorem

If a graph $G \in \Gamma_{\leq 3}$ has clique number at most 3, then G is an induced subgraph of a graph in the family of graphs $\mathfrak{C}$.



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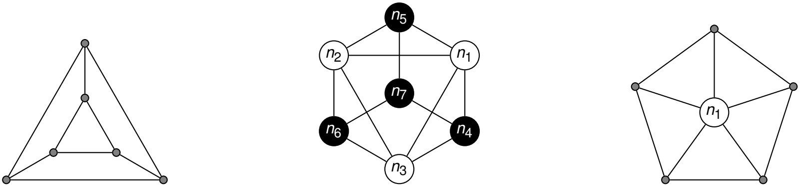

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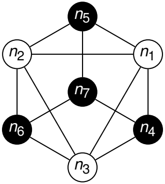
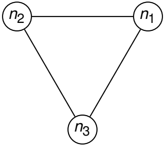
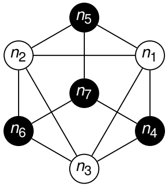
Theorem

Each induced subgraph of a graph in $\mathfrak{C}$ belongs to $\Gamma_{\leq 3}$.


n_5
n_2
n_1
n_7
n_6
n_4
n_3


n_2
n_1
n_3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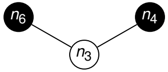
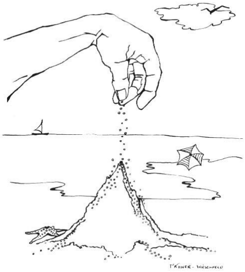

n_6
n_4
n_3



¡Gracias!