

Graphs with few trivial critical ideals

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1 Assumptions

2 Motivation: Critical group

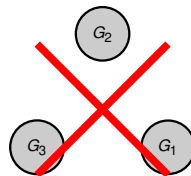
- Invariant factors
- The family \mathcal{G}_i
- Graphs with more than one invariant factor equal to 1

3 Critical ideals

- Definition

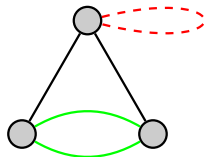
Assumptions on graphs

- are **connected**,



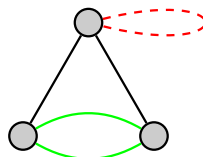
Assumptions on graphs

- **multiple edges** are allowed, and



Assumptions on graphs

- **loops** are forbidden.



Laplacian Matrix

Definition

Let $G = (V, E)$ be a graph, the **Laplacian matrix** $L(G)$ of G is the matrix with rows and columns indexed by the vertices of G given by

$$L(G)_{uv} = \begin{cases} \deg_G(u) & \text{if } u = v, \\ -m_{uv} & \text{otherwise,} \end{cases}$$

where $\deg_G(u)$ denote the degree of u , and m_{uv} denote the number of edges from u to v .

Definition

By considering $L(G)$ as a linear operator on \mathbb{Z}^n , the **critical group** $K(G)$ of G is the torsion part of the cokernel of $L(G)$.

$$\text{coker}(L(G)) = \mathbb{Z}^n / \text{Im} L(G) = \mathbb{Z} \oplus K(G).$$

Invariant factors

Theorem

$$K(G) \cong \mathbb{Z}_{f_1} \oplus \mathbb{Z}_{f_2} \oplus \cdots \oplus \mathbb{Z}_{f_{n-1}},$$

where $f_i \leq 0$ and $f_i \mid f_j$ for all $i \leq j$.

Invarian factors

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The f_1, f_2, \dots, f_{n-1} are called **invariant factors**.

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Proposition

If $\Delta_i(G)$ is the g.c.d of the i -minors of $L(G)$, then f_i is equal to $\Delta_i(G)/\Delta_{i-1}(G)$, where $\Delta_0(G) = 1$.

The family \mathcal{G}_i

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Let \mathcal{G}_i be the family of **simple connected** graphs with $f_1(G) = i$.

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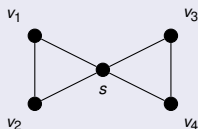
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Let \mathcal{G}_i be the family of **simple connected** graphs with $f_1(G) = i$.

Example

The following graph belongs to \mathcal{G}_2 .



$$L(G, s) \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

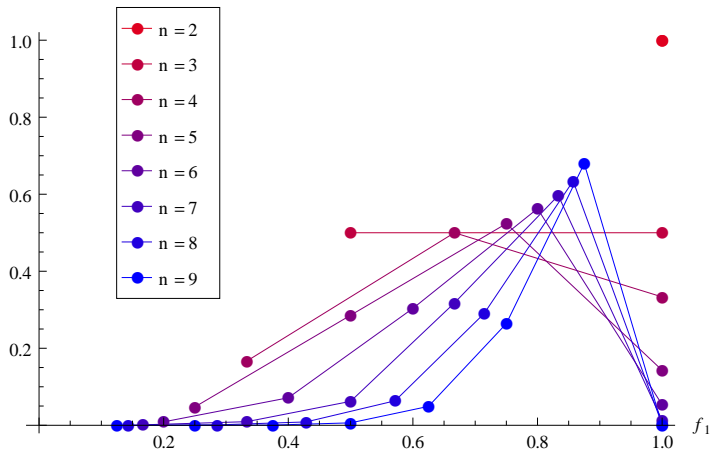


Figure : Normalized number of graphs with f_1 invariant factors equal to 1.

Question

How often the critical group is cyclic? that is, how often $f_1(G)$ is equal to $n - 2$ or $n - 1$?

Conjeture [D. Wagner, 2001]

Almost every connected simple graph has a cyclic critical group.

Theorem [M. Wood, 2014]

The probability that the critical group of a random graph is cyclic is asymptotically at most

$$\zeta(3)^{-1} \zeta(5)^{-1} \zeta(7)^{-1} \zeta(9)^{-1} \zeta(11)^{-1} \dots \approx 0,7935212$$

where ζ is the Riemann zeta function.

Graphs with one invariant factor equal to 1

On the other hand...

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On the other hand...

Question

What we can say about \mathcal{G}_1 ?

Graphs with one invariant factor equal to 1

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What we can say about \mathcal{G}_1 ?

Theorem

If G is a **simple connected** graphs, then the following statements are equivalent:

- I. $G \in \mathcal{G}_1$,
- II. G is P_3 -free, where P_n denote the path with n vertices,
- III. G is a complete graph.

Graphs with more than one invariant factor equal to 1

Question

What we can say about \mathcal{G}_2 and \mathcal{G}_3 ?

Graphs with more than one invariant factor equal to 1

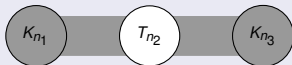
Question

What we can say about \mathcal{G}_2 and \mathcal{G}_3 ?

Theorem

Let G be a *simple connected* graph. Then, $G \in \mathcal{G}_2$ if and only if G is one of the following graphs:

- I. K_{n_1, n_2, n_3} , where n_1, n_2 y n_3 have the same parity.
- II. L_{n_1, n_2, n_3} , where $n_1, n_2, n_3 \geq 3$ have the same parity, and other *eleven special cases*.



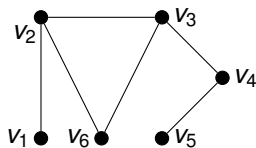
The proof uses critical ideals

The critical ideals of a graph

Definition

Given a graph $G = (V, E)$ and a set of indeterminates $X_G = \{x_u : u \in V\}$, the **generalised Laplacian matrix** $L(G, X_G)$ of G is the matrix given by

$$L(G, X_G)_{uv} = \begin{cases} x_u & \text{if } u = v, \\ -m_{uv} & \text{otherwise.} \end{cases}$$



$$L(G, X_G) = \begin{bmatrix} x_1 & -1 & 0 & 0 & 0 & 0 \\ -1 & x_2 & -1 & 0 & 0 & -1 \\ 0 & -1 & x_3 & -1 & 0 & -1 \\ 0 & 0 & -1 & x_4 & -1 & 0 \\ 0 & 0 & 0 & -1 & x_5 & 0 \\ 0 & -1 & -1 & 0 & 0 & x_6 \end{bmatrix}$$

Definition

For all $1 \leq i \leq |V(G)|$, the i -th critical ideal of G is the determinantal ideal given by

$$I_i(G, X_G) = \langle \{m : m \text{ is an } i \times i \text{ minor of } L(G, X_G)\} \rangle \subseteq \mathbb{Z}[X_G].$$

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Definition

The **algebraic co-rank** $\gamma(G)$ of a graph G is the number of trivial critical ideals of G .

Remark

If H is an induced subgraph of G , then $I_i(H, X_H) \subseteq I_i(G, X_G)$ for all $i \leq |V(H)|$. Thus $\gamma(H) \leq \gamma(G)$.

Definition

$\Gamma_{\leq i} = \{G : G \text{ is a simple connected graph with } \gamma(G) \leq i\}.$

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Remark

$\Gamma_{\leq i}$ is closed under induced subgraphs.

Theorem

If $\deg(G) = (\deg_G(v_1), \dots, \deg_G(v_n))$ is the degree vector of G , and $f_1 \mid \dots \mid f_{n-1}$ are the invariant factors of $K(G)$, then

$$l_i(G, \deg(G)) = \left\langle \prod_{j=1}^i f_j \right\rangle \text{ for all } 1 \leq i \leq n-1.$$

Remark

- If the critical ideal $l_i(G, X_G)$ is trivial, then $f_i = 1$.
- If $f_i \neq 1$, then the critical ideal $l_i(G, X_G)$ is not trivial.
- $\mathcal{G}_i \subseteq \Gamma_{\leq i}$ for all $i \geq 0$.

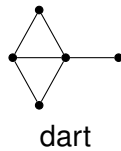
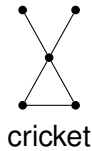
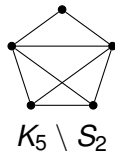
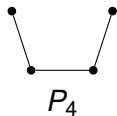


Figure : The family \mathcal{F}_2 of graphs.

Theorem

If G is a simple connected graph. Then the following statements are equivalent:

- $G \in \Gamma_{\leq 2}$,
- G is \mathcal{F}_2 -free,
- G is K_{n_1, n_2, n_3} or L_{n_1, n_2, n_3} .

Our approach to solve this problem

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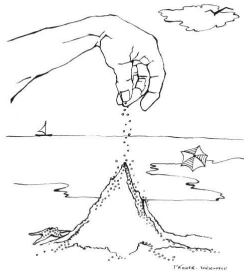
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- 2 Determining the \mathcal{F} -free graphs.

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- 1 Finding the family \mathcal{F} of induced forbidden subgraphs for $\Gamma_{\leq j}$.
- 2 Determining the \mathcal{F} -free graphs.
- 3 Computing the critical group of the \mathcal{F} -free graphs.



¡Gracias!