Graphs with few trivial critical ideals

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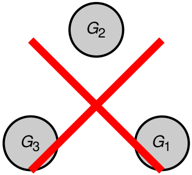
1. **Assumptions**

2. **Motivation: Critical group**
   - Invariant factors
   - The family $\mathcal{G}_i$

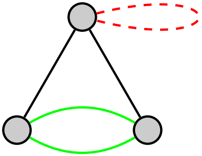
Assumptions on graphs

- are connected,



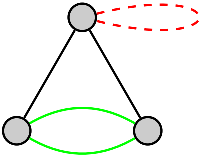
Assumptions on graphs

- multiple edges are allowed, and



Assumptions on graphs

- loops are forbidden.



Laplacian Matrix

Definition

Let $G = (V, E)$ be a graph, the Laplacian matrix $L(G)$ of G is the matrix with rows and columns indexed by the vertices of G given by

$$L(G)_{uv} = \begin{cases} \deg_G(u) & \text{if } u = v, \\ -m_{uv} & \text{otherwise,} \end{cases}$$

where $\deg_G(u)$ denote the degree of u, and m_{uv} denote the number of edges from u to v.

Definition

By considering $L(G)$ as a linear operator on $\mathbb{Z}^n$, the critical group $K(G)$ of G is the torsion part of the cokernel of $L(G)$.

$$coker(L(G)) = \mathbb{Z}^n / \mathrm{Im} L(G) = \mathbb{Z} \oplus K(G).$$

Invarian factors

Theorem

$$K(G) \cong \mathbb{Z}_{f_1} \oplus \mathbb{Z}_{f_2} \oplus \cdots \oplus \mathbb{Z}_{f_{n-1}},$$

where $f_i \leq 0$ and $f_i \mid f_j$ for all $i \leq j$.

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The $f_1, f_2, \ldots, f_{n-1}$ are called invariant factors.

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Proposition

If $\Delta_i(G)$ is the g.c.d of the i-minors of $L(G)$, then f_i is equal to $\Delta_i(G)/\Delta_{i-1}(G)$, where $\Delta_0(G) = 1$.

The family $\mathcal{G}_i$

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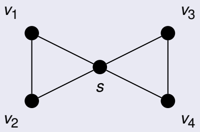
Example

The following graph belongs to $\mathcal{G}_2$.



$$L(G,s) \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

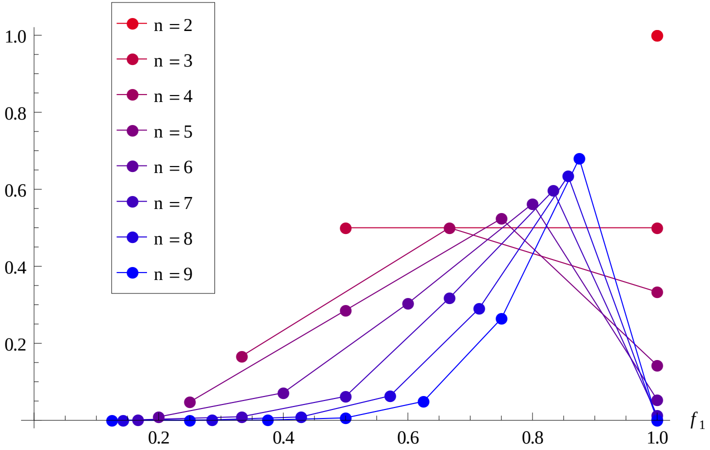


Figura: Normalized number of graphs with f_1 invariant factors equal to 1.

Pregunta

¿Qué tan frecuente es cíclico el grupo crítico? es decir, ¿Qué tan frecuente $f_1(G)$ es igual a $n-1$ o $n-2$?

Conjetura [D. Wagner, 2001]

Casi todas las gráficas simples y conexas tienen grupo crítico ciclico.

Teorema [M. Wood, 2014]

La probabilidad de que el grupo crítico de una gráfica aleatoria sea cíclica es asintóticamente a lo más

$$\zeta(3)^{-1}\zeta(5)^{-1}\zeta(7)^{-1}\zeta(9)^{-1}\zeta(11)^{-1}\cdots \approx 0{,}7935212$$

donde ζ es la función zeta de Riemann.

Gráficas con un factor invariante igual a 1

Por otro lado...

Pregunta

¿Qué podemos decir sobre $\mathcal{G}_1$?

Teorema [Lorenzini, 1989]

Si G es una gráfica simple conexa , entonces los siguientes enunciados son equivalentes:

I. $G \in \mathcal{G}_1$,

II. G es P_2-libre,

III. G es la gráfica completa.

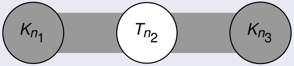
Pregunta

¿Qué podemos decir sobre $\mathcal{G}_2$ y $\mathcal{G}_3$?

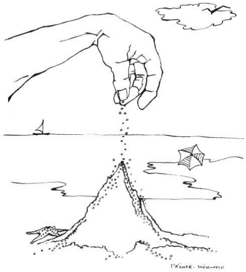
Teorema

Sea G una gráfica simple conexa . Entonces, $G \in \mathcal{G}_2$ si y solamente si G es una de las siguientes gráficas:

I. K_{n_1,n_2,n_3}, donde n_1, n_2 y n_3 tienen la misma paridad.
II. L_{n_1,n_2,n_3}, si $n_1, n_2, n_3 \geq 3$ tienen la misma paridad, u otros once casos.



La demostración usa los ideales críticos.



¡Gracias!