Critical ideals of graphs

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Outline

- Critical ideals of graphs
- Applications to Sandpile group
- Applications to Smith group
- Applications to minimum rank and zero-forcing number

Initial motivation

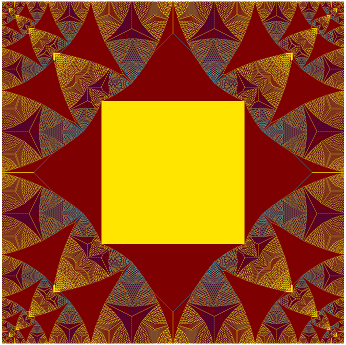
There are many equivalent definitions of the sandpile group $K(G)$ of a graph, we will use the torsion part of the cokernel of the Laplacian matrix.

Many researchers asked for a classification of the graphs whose Laplacian matrix have 2 and 3 invariant factors equal to one.

$$\tau(G) = |K(G)| = T(1,1)$$
$$G \text{ planar} \Rightarrow K(G) \simeq K(G^*)$$

Sandpile groups might detect more properties

than homology groups



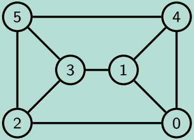
The generalized Laplacian matrix

Let G be a graph with n vertices and $X_G = \{x_u : u \in V(G)\}$ a set of variables.

Definition

The generalized Laplacian matrix $A_X(G)$ of G is the matrix $\operatorname{diag}(X_G) - A(G)$.

Example



G

$$\begin{bmatrix} x_0 & -1 & -1 & 0 & -1 & 0 \\ -1 & x_1 & 0 & -1 & -1 & 0 \\ -1 & 0 & x_2 & -1 & 0 & -1 \\ 0 & -1 & -1 & x_3 & 0 & -1 \\ -1 & 0 & 0 & 0 & x_4 & -1 \\ 0 & 0 & -1 & -1 & -1 & x_5 \end{bmatrix}$$

Critical ideals of graphs

Let $\mathcal{R}[X_G]$ denote the polynomial ring over a commutative ring $\mathcal{R}$ in the variables X_G.

Definition

For $1 \leq k \leq n$, the *k-th critical ideal* $I_k^{\mathcal{R}}(G)$ is the ideal $\langle \mathrm{minors}_k(A_X(G)) \rangle$.

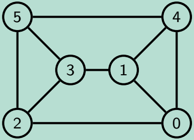
An ideal is said to be *trivial* or unit if it is equal to $\langle 1 \rangle$ $(= \mathcal{R}[X_G])$.

Definition

The *algebraic co-rank* $\gamma_{\mathcal{R}}(G)$ of G is the maximum integer k for which $I_k^{\mathcal{R}}(G)$ is trivial.

Critical ideals of graphs

Example



$$\begin{bmatrix} x_0 & -1 & -1 & 0 & -1 & 0 \\ -1 & x_1 & 0 & -1 & -1 & 0 \\ -1 & 0 & x_2 & -1 & 0 & -1 \\ 0 & -1 & -1 & x_3 & 0 & -1 \\ -1 & 0 & 0 & 0 & x_4 & -1 \\ 0 & 0 & -1 & -1 & -1 & x_5 \end{bmatrix}$$

For our graph, $\gamma_{\mathbb{R}}(G) = \gamma_{\mathbb{Z}}(G) = 3$.
And for the first non trivial $I_4^{\mathbb{R}}(G) = I_4^{\mathbb{Z}}(G)$, we give the Gröbner basis:

$$\langle x_0 + x_5 - 1, x_1 + x_5 - 1, x_2 - x_5, x_3 - x_5, x_4 + x_5 - 1, x_5^2 - x_5 - 1 \rangle.$$

Note $I_n^{\mathcal{R}}(G) = \langle \det(A_X(G)) \rangle$.

Varieties of critical ideals of graphs

Definition

The variety $V(I)$ of an ideal I is the set of common roots between polynomials in I.

Example

The ideal $I_4^{\mathbb{R}}(G)$ for G of our previous example:
$\langle x_0 + x_5 - 1, x_1 + x_5 - 1, x_2 - x_5, x_3 - x_5, x_4 + x_5 - 1, x_5^2 - x_5 - 1\rangle$,
there are only two roots in its variety:
$\left(\frac{1-\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right)$
and
$\left(\frac{1+\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right)$.

Varieties of critical ideals of graphs

Example

For the complete graph K_3 with 3 vertices, $\gamma_{\mathbb{R}}(K_3) = 1$,
$I_2^{\mathbb{R}}(K_3) = \langle x_0 + 1, x_1 + 1, x_2 + 1 \rangle$, and
$I_3^{\mathbb{R}}(K_3) = \langle x_0 x_1 x_2 - x_0 - x_1 - x_2 - 2 \rangle$.
The variety $V(I_2^{\mathbb{R}}(K_3)) = \{(1,1,1)\}$, and the variety $V(I_3^{\mathbb{R}}(K_3))$ is

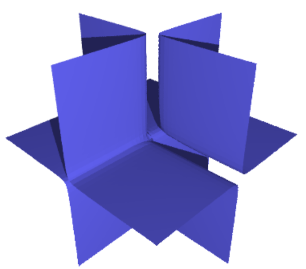


Figure: Partial view of the variety of $I_3^{\mathbb{R}}(K_3)$ in $\mathbb{R}^3$.

Varieties of critical ideals of graphs

We have that

$$\langle 1\rangle \supseteq I_1^{\mathcal{R}}(G) \supseteq \cdots \supseteq I_n^{\mathcal{R}}(G) \supseteq \langle 0\rangle.$$

Then,

$$V(\langle 1\rangle) \subseteq V(I_1^{\mathcal{R}}(G)) \subseteq \cdots \subseteq V(I_n^{\mathcal{R}}(G)) \subseteq V(\langle 0\rangle).$$

Let σ be a permutation on $V(G)$. Thus, σG is a graph on $V(G)$ such that $\{i,j\} \in E(G)$ if and only if $\{\sigma(i), \sigma(j)\} \in E(\sigma G)$.

Corollary

Let G and G' be two graphs with n vertices. Then, G and G' are isomorphic if and only if there exists a permutation σ on $V(G')$ such that the n-th critical ideals of G and $\sigma G'$ are equal.

There is a bijection between the edges of G and the monomials of degree $n-2$ in $\det(A_X(G))$ given by $\{i,j\} \mapsto -\prod_{k \neq i,j} x_k$.

Remember that the term of t^{n-2} in $\det(tI - A(G))$ is the negative of the number of edges of G.

N. Ghareghani, F. Ramezani and B. Tayfeh-Rezaie. Graphs cospectral with starlike trees. *Linear Algebra Appl.* 429 (2008), 2691–2701.

Applications: Sandpile group

Definition

The sandpile group $K(G)$ of a graph G is the torsion part of the cokernel $\mathbb{Z}^n/Im(L(G))$ of the Laplacian matrix $L(G)$.

Proposition

$K(G) \cong \mathbb{Z}/f_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/f_r\mathbb{Z}$, where the integers $f_1, f_2, \ldots, f_r$ are called invariant factors of the Laplacian matrix of G and satisfy $f_i | f_{i+1}$.

Proposition (Elementary divisors' theorem)

$f_i = \Delta_i(L(G))/\Delta_{i-1}(L(G))$, where $\Delta_i(L(G))$ is the g.c.d. of the i-minors of the Laplacian matrix $L(G)$.

Applications: Sandpile group

Proposition (Corrales & Valencia,2013)

By evaluating the i-th critical ideal $I_i^{\mathbb{Z}}(G)$ at the degree vector, we get an ideal that is spanned by $\Delta_i(L(G))$.

Example

$I_1^{\mathbb{Z}}(K_3) = \langle 1 \rangle$,
$I_2^{\mathbb{Z}}(K_3) = \langle x_0 + 1, x_1 + 1, x_2 + 1 \rangle$,
$I_3^{\mathbb{Z}}(K_3) = \langle x_0 x_1 x_2 - x_0 - x_1 - x_2 - 2 \rangle$.

$$I_i^{\mathbb{Z}}(K_3)|_{X_G = \deg(K_3)} = \langle \Delta_i(L(K_3)) \rangle = \begin{cases} \langle 1 \rangle & \text{if } i = 1, \\ \langle 3 \rangle & \text{if } i = 2, \\ \langle 0 \rangle & \text{if } i = 3. \end{cases}$$

Therefore $K(K_3) \cong \mathbb{Z}/3\mathbb{Z}$.

Applications: Smith group

Definition

The Smith group $S(G)$ of a graph G is the cokernel $\mathbb{Z}^n/Im(A(G))$ of the adjacency matrix $A(G)$.

Proposition

$S(G) \cong \mathbb{Z}/g_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/g_r\mathbb{Z} \oplus \mathbb{Z}^{n-r}$, where the integers $g_1, g_2, \ldots, g_r$ are called invariant factors of the adjacency matrix and satisfy $g_i | g_{i+1}$.

Proposition

$g_i = \Delta_i(G)/\Delta_{i-1}(G)$, where $\Delta_i(G)$ is the g.c.d. of the i-minors of $A(G)$.

Applications: Smith group

Proposition

By evaluating the i-th critical ideal $I_i^{\mathbb{Z}}(G)$ at the zero vector, we get an ideal that is spanned by $\Delta_i(A(G))$.

Example

$$I_i^{\mathbb{Z}}(K_3)|_{\deg(K_3)} = \langle \Delta_i(K_3) \rangle = \begin{cases} \langle 1 \rangle & \text{if } i = 1, \\ \langle 1 \rangle & \text{if } i = 2, \\ \langle 2 \rangle & \text{if } i = 3. \end{cases}$$

Therefore $S(A(K_3)) \cong \mathbb{Z}/2\mathbb{Z}$.

Applications: Sandpile group

Definition

Let $f(G)$ denote the number of invariant factors of the Laplacian matrix of G equal to 1.

Problem

How often the critical group is cyclic? that is, how often $f(G)$ is equal to $n-2$ or $n-1$?

Conjeture (D. Wagner, 2001 & D. Lorenzini, 2008)

Almost every connected simple graph has a cyclic critical group.

Applications: Sandpile group

Theorem (M. Wood, 2017)

The probability that the critical group of a random graph is cyclic is asymptotically at most

$$\zeta(3)^{-1}\zeta(5)^{-1}\zeta(7)^{-1}\zeta(9)^{-1}\zeta(11)^{-1}\cdots \approx 0.7935212$$

where ζ is the Riemann zeta function.

Applications: Sandpile group

On the other hand...

Problem

What are the graphs whose critical group has i invariant factors equal to 1?

Dino Lorenzini noticed that the graphs with $f(G) = 1$ are only the complete graphs.

Applications: Sandpile group

Observation

It is not always true that if H is an induced subgraph of G, then $K(H) \triangleleft K(G)$.

Proposition (Corrales & Valencia, 2013)

If H is an induced subgraph of G, then $I_i^{\mathcal{R}}(H) \subseteq I_i^{\mathcal{R}}(G)$.

Observation

- $\gamma_{\mathbb{Z}}(G) \leq f(G)$.
- $\{$*graphs with* $f(G) \leq i\}$ *is contained in* $\{$*graphs with* $\gamma_{\mathbb{Z}}(G) \leq i\}$.

Applications: Sandpile group

Theorem (Alfaro & Valencia, 2014)

Let G be a connected graph. Then, the following are equivalent:

1. $\gamma_{\mathbb{Z}}(G) \leq 2$,
2. $\{P_4, \ltimes, \text{dart}, K_{2,2,1,1}, K_5 \setminus P_3\}$*-free graphs,*
3. *G is isomorphic to K_{n_1,n_2,n_3} or to $\overline{K_{n_1}} \vee (K_{n_2} + K_{n_3})$.*

Theorem (Alfaro & Valencia, 2014)

Let G be a connected graph. Then, $f(G) = 2$ if and only if G is one of the following graphs:

1. *K_{n_1,n_2,n_3}, where n_1, n_2 and n_3 have the same parity.*
2. *$\overline{K_{n_1}} \vee (K_{n_2} + K_{n_3})$, where $n_1, n_2, n_3 \geq 3$ and have the same parity*
3. *few other cases.*

Applications: Sandpile group

Theorem (Alfaro, Barrus, Sinkovic & Villagrán, 2021)

Let G be a connected simple regular graph. Then $G \in \mathcal{K}_{\leq 3}$ if and only if G is one of the following:

(a) C_5,

(b) $K_3 \square K_2$,

(c) *a complete graph* K_r,

(d) *a regular complete bipartite graph* $K_{r,r}$,

(e) *a regular complete tripartite graph* $K_{r,r,r}$,

(f) *a regular complete 4-partite graph* $K_{r,r,r,r}$,

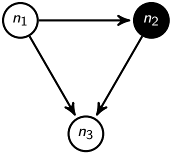
(g) $C_4^{(-r,-r,-r,-r)}$, *for any* $r \in \mathbb{N}$.

where $C_4^{(-r,-r,-r,-r)}$ *is the graph obtained by replacing the vertices in* C_4 *for cliques of size* r *and preserving adjacency.*

Applications: Sandpile group

Theorem (Alfaro, Valencia & Vazquez, 2017)

Let D be a connected digraph. Then, $\gamma_{\mathbb{Z}}(D) \leq 1$ if and only if D is isomorphic to Λ_{n_1,n_2,n_3}.



Theorem (Alfaro, Valencia & Vázquez,2017)

Let D be a connected digraph. Then, $f(D) = 1$ if and only if D is one of the following graphs:

1. $n_1, n_2, n_3 \geq 1$,
2. *other few cases.*

Minimum rank

Definition

The minimum rank $\mathrm{mr}_{\mathcal{R}}(G)$ of G is the smallest possible rank among all the $n \times n$ symmetric matrices with entries in the field $\mathcal{R}$, whose u, v-entry $(u \neq v)$ is nonzero whenever u is adjacent to v and zero otherwise.

Observation

We might focus when $\mathcal{R}$ is either $\mathbb{R}$ or $\mathbb{C}$.

Zero-forcing number

Definition

The zero forcing game is a color-change game where vertices can be blue or white. At the beginning, a set of vertices B are colored blue while others remain white. The goal is to color all vertices blue through repeated applications of the color change rule: If u is a blue vertex and v is the only white neighbor of u, then v is forced to become blue. An initial set of blue vertices B is called a zero forcing set if starting with B one can make all vertices blue.

Definition

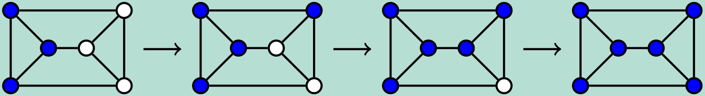
The zero forcing number $Z(G)$ is the minimum cardinality of a zero forcing set.

Minimum rank & zero-forcing

Definition

$\mathrm{mz}(G) = |V(G)| - Z(G)$.

Example



Since there is no zero forcing set of size 2, then $\mathrm{mz}(G) = Z(G) = 3$.

Theorem (AIM Minimum Rank Work Group, 2008)

For every graph G, $\mathrm{mz}(G) \leq \mathrm{mr}_{\mathcal{R}}(G)$ for any field $\mathcal{R}$.

Minimum rank & critical ideals

Lemma (The Weak Nullstellensatz)

Let $\mathcal{R}$ be an algebraically closed field and let $I \subseteq \mathcal{R}[X]$ be an ideal satisfying $V(I) = \emptyset$. Then I is trivial.

Theorem (Alfaro & Lin, 2019)

If $\mathcal{R}$ is an algebraically closed field, then $\text{mr}_{\mathcal{R}}(G) \leq \gamma_{\mathcal{R}}(G)$.

Theorem (Alfaro & Lin, 2019)

For every graph G, $\text{mz}(G) \leq \gamma_{\mathcal{R}}(G)$ for any commutative ring $\mathcal{R}$.

Minimum rank & critical ideals

Conjecture

$\mathrm{mr}_{\mathbb{R}}(G) \leq \gamma_{\mathbb{R}}(G)$.

Theorem (Alfaro & Lin, 2019)

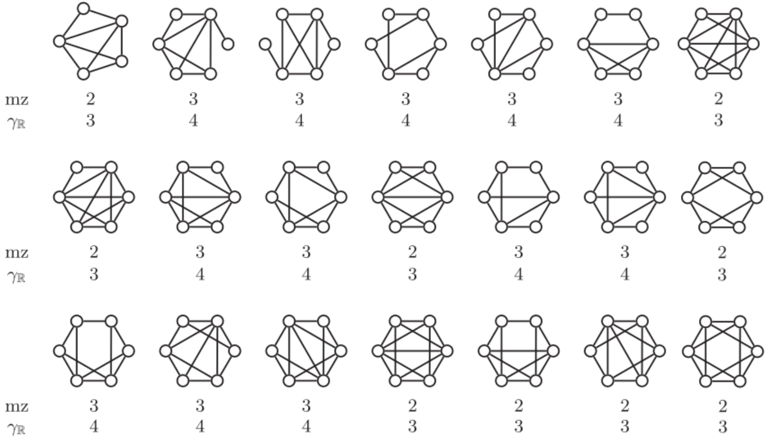
If G is a connected graph with $\mathrm{mr}_{\mathbb{R}}(G) \leq 2$, then $\mathrm{mr}_{\mathbb{R}}(G) \leq \gamma_{\mathbb{R}}(G)$.

Theorem (Alfaro, 2018)

If G is a connected graph with $\mathrm{mr}_{\mathbb{R}}(G) \leq 3$, then $\mathrm{mr}_{\mathbb{R}}(G) \leq \gamma_{\mathbb{R}}(G)$.

Computational results

From the 143 connected graphs with at most 6 vertices, only 21 graphs have $\mathrm{mz}(G) < \gamma_{\mathbb{R}}(G)$. For the other graphs, $\mathrm{mz}(G) = \mathrm{mr}(G) = \gamma_{\mathbb{R}}(G)$.



Graphs with equal mz, mr and γ

Theorem (Alfaro & Lin, 2019)

For any tree T, $\mathrm{mz}(T) = \mathrm{mr}(T) = \gamma_{\mathbb{R}}(T)$.

Theorem (Alfaro & Lin, 2019)

For any cycle C_n with $n \geq 3$, $\mathrm{mz}(C_n) = \mathrm{mr}(C_n) = \gamma_{\mathbb{R}}(C_n)$.

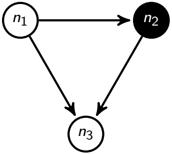
Theorem (Alfaro & Lin, 2019)

Let G be the line graph of a tree. Then $\mathrm{mz}(G) = \mathrm{mr}(G) = \gamma_{\mathbb{R}}(G)$.

Graphs with equal mz, mr and γ

Theorem (Alfaro, Valencia & Vazquez, 2018)

Let D be a connected digraph. Then, $\gamma_{\mathbb{Z}}(D) \leq 1$ if and only if D is isomorphic to Λ_{n_1,n_2,n_3}.



Graphs with equal mz, mr and γ

Theorem (Alfaro & Lin, 2019)

Let $\mathcal{R}$ be a commutative ring with unity. The following are equivalent:

1. *D is isomorphic to Λ_{n_1,n_2,n_3},*
2. $\mathrm{mr}_{\mathcal{R}}(D) \leq 1$,
3. $\mathrm{mz}(D) \leq 1$,
4. $\gamma_{\mathcal{R}}(D) \leq 1$.

Sandpile group of outerplanar graphs

Given a plane graph G with s interior faces $F_1, \ldots, F_s$, let $c(F_i)$ denote the length of the cycle which bounds interior face F_i.
The *cycle-intersection matrix*, $C(G) = (c_{ij})$ is the $s \times s$ matrix, where $c_{ii} = c(F_i)$, and c_{ij} is the negative of the number of common edges in the cycles bounding the interior faces F_i and F_j, for $i \neq j$.

Lemma

Let G be a connected plane graph. Then $K(G) \cong \operatorname{coker}(C(G))$ and $\tau(G) = \det(C(G))$.

Sandpile group of outerplanar graphs

The following result turns out by evaluating critical ideals of trees at $X = (c_{11}, \dots, c_{ss})$.

Theorem (Alfaro & Villagrán, 2021)

Let G be a biconnected outerplane graph with $F_1, \dots, F_n$ interior faces and whose weak dual is the tree T with n vertices. Let

$$\Delta_k = \gcd\left(\left\{d(\mathcal{M}) : \mathcal{M} \in 2M_k^*\left(T^l\right)\right\}\right),$$

for $k \in [n]$. Then $K(G) \cong \mathbb{Z}_{\Delta_1} \oplus \mathbb{Z}_{\frac{\Delta_2}{\Delta_1}} \oplus \cdots \oplus \mathbb{Z}_{\frac{\Delta_n}{\Delta_{n-1}}}$ and $\tau(G) = \Delta_n$.

Main references

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Thank you

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