Enumeration of cospectral and coinvariant graphs

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- $D(G)$, **distance** matrix
- $D^L(G) = Tr(G) - D(G)$, **distance Laplacian** matrix
- $D^Q(G) = Tr(G) + D(G)$, **signless distance Laplacian** matrix

where $Tr(G)$ denotes the diagonal matrix of the vertex transmissions in G.
The transmission $Tr(v)$ of a vertex v is defined to be the sum of the distances from v to all other vertices in G.

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Some enumeration results

- Godsil and McKay (1976) $n \leq 9$ for A
- Haemers and Spence (2004) $n = 10, 11$ for A
- Brouwer and Spence (2009) $n = 12$ for A
- Aouchiche and Hansen (2018) $n \leq 10$ for D, D^L, D^Q

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We will provide numerical evidence that this algebraic graph representation may do a better job in distinguishing graphs.

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The **Smith normal form** of a integer matrix M, denoted by $\mathrm{SNF}(M)$, is the unique diagonal matrix $\mathrm{diag}(f_1, \ldots, f_r, 0, \ldots, 0)$ equivalent to M such that $r = rank(M)$ and $f_i | f_j$ for $i < j$.

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We say that the graphs G and H are M-**coinvariant** if the SNFs of $M(G)$ and $M(H)$, computed over $\mathbb{Z}$, are the same.

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Coinvariant graphs were introduced by Andrew Vince in “Elementary Divisors of Graphs and Matroids” (1991).

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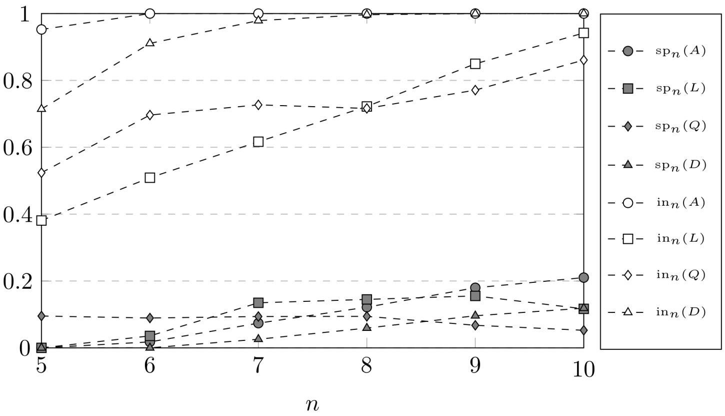

Figure 1: The fraction of graphs on n vertices that have at least one cospectral mate with respect to a certain associated matrix is denoted as *sp*. The fraction of graphs on n vertices with respect to a certain associated matrix that have at least one coinvariant mate is denoted as *in*.

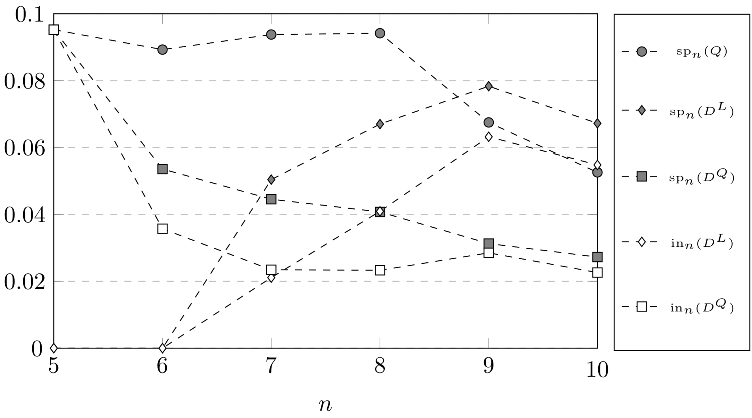

Figure 2: The fraction of graphs on n vertices that have at least one cospectral mate with respect to a certain associated matrix is denoted as *sp*. The fraction of graphs on n vertices with respect to a certain associated matrix that have at least one coinvariant mate is denoted as *in*.

Cospectral trees

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Conjecture (Aouchiche and Hansen, 2013)

Every tree is determined by its distance Laplacian spectrum, and by its distance signless Laplacian spectrum.

Coinvariant trees

Hou and Woo extended the Graham and Pollak celebrated formula, $\det(D(T_{n+1})) = (-1)^n n 2^{n-1}$ for any tree T_{n+1} with $n+1$ vertices, to the SNF of the distance matrix.

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Theorem

Let T_{n+1} be a tree with $n+1$ vertices, then
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Theorem

Let T_{n+1} be a tree with $n+1$ vertices, then
$\mathrm{SNF}(D(T_{n+1})) = \mathrm{I}_2 \oplus 2\mathrm{I}_{\mathrm{n}-2} \oplus (2\mathrm{n})$.

Corollary

All trees with n vertices are D-coinvariant mates.

Coinvariant trees

After enumerating all trees with at most 20 vertices with respect to D^L and D^Q, we found no D^L-coinvariant mates and no D^Q-coinvariant mates among all trees with up to 20 vertices.

Conjecture

Trees are determined by the SNF of its D^L, and analogously, by the SNF of its D^Q.

Theorem (Abiad, Alfaro, Heysse & Vargas, 2019)

Complete graphs are determined by the SNF of the D^L matrix.

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Theorem (Abiad & Alfaro, 2020)

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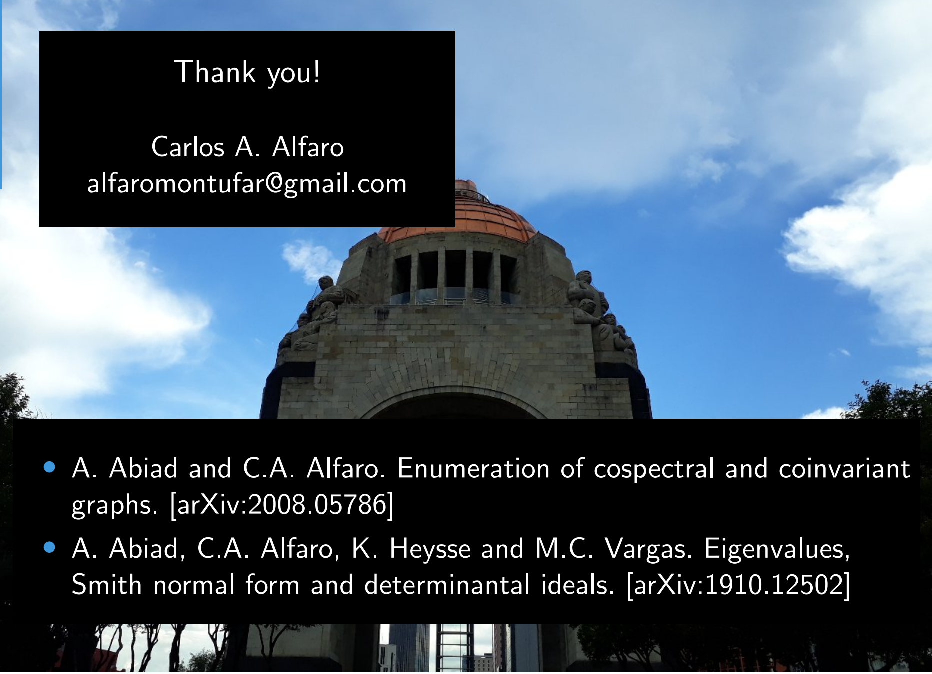

Thank you!

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- A. Abiad, C.A. Alfaro, K. Heysse and M.C. Vargas. Eigenvalues, Smith normal form and determinantal ideals. [arXiv:1910.12502]