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Computer Networks and Telematics
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for exercise on July 13th, 2023

Exercise No. 9
Peer-To-Peer Networks
Summer 2023

Exercise 1 *Invertibility of a Random Matrix:* Consider an $m \times n$ matrix M over a given Galois field $GF[b]$ of size b. The rank of a matrix is the maximum number of linearly independent row vectors. It is defined to be 0 if all entries are 0. All entries of a random matrix are chosen independently at random from the finite field of size b.

1. What is the maximum rank of an $m \times n$ matrix M?
2. What is the probability that a random $1 \times n$ matrix has rank 0?
3. Given m linearly independent vectors $v_1, \ldots, v_m \in GF[b]^n$, what is the probability that a random vector $r \in GF[b]^n$ is linearly dependent on $v_1, \ldots, v_m$? (Hint: Consider the number of possible points in F^n that can be described by a linear combination $\sum_{i=1}^{m} \alpha_i v_i$.)
4. For $m < n$, compute the probability that an $(m+1) \times n$ matrix has maximal rank under the condition that the $m \times n$ matrix for the first m rows already has maximal rank.
5. Combine all these observations to compute the probability that an $n \times n$ matrix has rank n and is thus invertible.
6. Consider a $(n+x) \times n$ random matrix M. How do you have to choose x such that M has rank n with high probability?