Sections highlighted in yellow are final answers however for explanatory answers there is no highlighting

1) Consider the function f : R R , f(x) = (x-2)2(x+2)

A. Turn f into a polynomial

$$(x-2)\times(x-2) = x^2-4x+4$$
$$(x+2)\times(x^2-4x+4) = x^3-2x^2+4x+8$$

B. Compute the derivative and second derivative of f

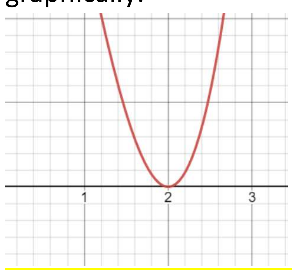
$$f(x) = x^3-2x^2-4x+8$$
$$f'(x) = 3x^2-4x-4$$
$$f''(x) = 6x-4$$

C. Compute the zero crossings, minima and maxima of f (if any).

$$y = x^3-2x^2-4x+8$$
$$0 = x^3-2x^2-4x+8$$
$$x^2(x-2)-4(x-2) = 0$$
$$(x-2)(x^2-4) = 0$$
$$x = 2\ x = -2\ x = 2$$

Zero crossing in x axis = (-2,0)

The occurrence of (2,0) twice means that it does not cross this x axis but it instead touches it and moves away from the x-axis at this point this can be sown graphically:



Zero crossing y axis = (0, 8)

$$y' = 3x^2-4x-4$$
$$y = 0:$$
$$0 = 3x^2-4x-4$$
$$a = 3\ b = -4\ c = -4$$
$$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$\frac{-(-4)\pm\sqrt{(-4)^2-4(3\times -4)}}{2(3)}$$
$$\frac{4\pm\sqrt{16+48}}{6} = \frac{4\pm\sqrt{64}}{6}$$
$$= -\frac{3}{2}, 2$$

$Local\ maxima: -\frac{2}{3}\ \ Local\ minima: 2$

$$\left(-\frac{2}{3},\frac{35}{4}\right)\ and\ (2{,}0) respectively$$

D. Compute the indefinite integral of f

$$\int x^3 - 2x^2 - 4x + 8\,dx$$

$$x^4 - 2x^3 - 4x^2 + 8x$$

$$= \frac{x^4}{4} - \frac{2}{3}x^3 - 2x^2 + 8x + c$$

E. Compute the integral of f between -4 and 3.

$$\int_3^{-4} x^3 - 2x^2 - 4x + 8\,dx$$

$$\tfrac{x^4}{4} - \tfrac{2}{3}x^3 - 2x^2 + 8x + c$$

$$\left[\frac{(-4)^4}{4} - \frac{2}{3}(-4^3) - 2(-4)^2 + 8(-4)\right] - \left[\frac{(3)^4}{4} - \frac{2}{3}(3)^3 - 2(3)^2 + 8(3)\right]$$

$$\frac{256}{4} + \frac{128}{3} - 32 - 32 = \frac{128}{3}$$

$$\frac{81}{4} - 18 - 18 + 24 = \frac{33}{4}$$

$$\frac{128}{3} - \frac{33}{4} = \frac{413}{12}$$

$=\frac{413}{12}$ which is equivalent to 34.416667 (6 D.P.)

F. Determine the limits of f(x) when x approaches +infinity and –infinity.

As x gets infinitely large or small x^3 becomes the most significant value

Therefore:

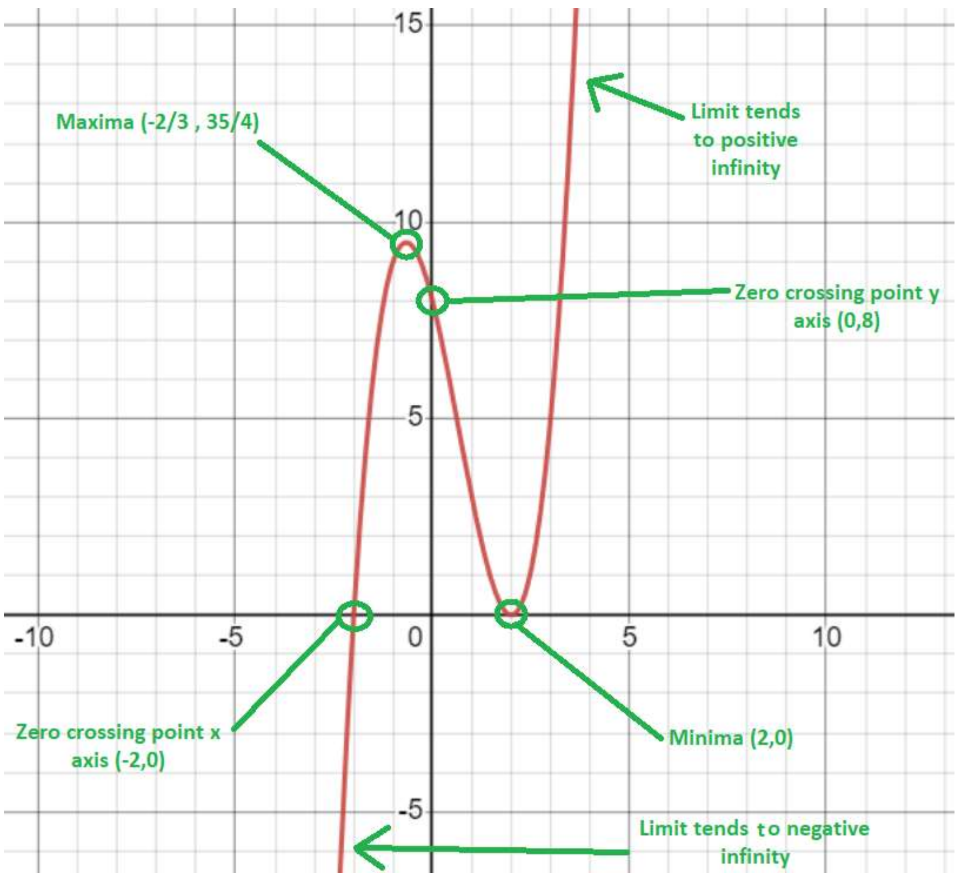
As $x \to +\infty$:

$$[x^3] = +[\infty^3]$$
$$limit = +\infty$$

As $x \to -\infty$:

$$[x^3] = -[\infty^3]$$
$$limit = -\infty$$

G. Draw the graph of f indicating the zero crossings, minima, maxima and limits.


Maxima (-2/3 , 35/4)
Limit tends to positive infinity
Zero crossing point y axis (0,8)
Zero crossing point x axis (-2,0)
Minima (2,0)
Limit tends to negative infinity
15
10
5
0
-5
-10
-5
5
10

2) A. Compute the Union, Intersection, Set difference A/B and B/A, and the Symmetric Difference between A and B.

A = {e1,e2,e5,e8}
B = {e2,e5,e8,e9}

Union – e1, e2, e5, e8, e9
Intersection – e2, e5, e8
Set difference A/B – e1
Set difference B/A – e9
Symmetric difference A Δ B – e1, e9

B. Calculate P(A), P(B) and P(A ∩ B)
For the P(A) I need to add up the probabilities of each of the elements in the set:
P(A) = e1+e2+e5+e8
P(A) = (0.08+0.08+0.1+0.07)
P(A) = 0.33

For the P(B) I need to do the same as above just with the elements in B:
P(B) = e2+e5+e8+e9
P(B) = 0.08 + 0.1 + 0.07 + 0.07
P(B) = 0.32

To calculate P(A ∩ B) I need to add up all the values in the intersection (values that appear in both sets)
P(A ∩ B) = e2+e5+e8
P(A ∩ B) = 0.08+0.1+0.07
P(A ∩ B) = 0.25

C. Using the addition law of probability, calculate P(A U B)
Addition law of probability - P(A U B) = P(A) + P(B) – P(A ∩ B)
P(A) = 0.33
P(B) = 0.32
P(A ∩ B) = 0.25
(0.33 + 0.32) – 0.25 = 0.4

D. List the composition of the event A U B, and calculate P(A U B) by adding the probabilities of the elementary outcomes.

A U B = e1, e2, e5, e8, e9
P(A U B) = 0.08+0.08+0.1+0.07+0.07
P(A U B) = 0.4

E. Calculate P(B') from P(B), also calculate P(B') directly from the elementary outcomes of B'

Calculating based of the value of P(B):
P(B) = 0.32 therefore P(B') = 1-P(B)
P(B') = 1-0.32
P(B') = 0.68

Calculating directly from the elementary outcomes of B':

B' = All elements – set b
B' = e1,e2,e3,e4,e5,e6,e7,e8,e9 – e2,e5,e8,e9
B' = e1,e3,e4,e6,e7
B' = 0.08+0.1+0.1+0.2+0.2
B' = 0.68

F. Calculate the probability of the symmetric difference between A and B.

Symmetric difference A Δ B – e1, e9
A Δ B = e1+e9
A Δ B = 0.08 + 0.07
A Δ B = 0.15

3) A. Prove that divisibility over the natural numbers induces a partial order

For it to induce a partial order it must be: reflexive, antisymmetric and transitive

$$\frac{a}{b} \leftrightarrow a \times n = b$$

Reflexivity - To prove that the divisibility is reflexive it must be shown that when a/a it is always true for every natural number

$$if\,\frac{a}{a}\; is\; true\;\; \leftrightarrow a \times n = a$$
$$\rightarrow n = 1$$

For every natural number 'a' n=1 which means that every natural number is divisible by itself which means that the divisibility is reflexive.

Antisymmetry – To prove that the divisibility is antisymmetric it must be shown that if:

$$\frac{a}{b}\; and\, \frac{b}{a}\; \rightarrow a = b$$

Therefore based on the assumption that a/b and b/a:

$$\frac{a}{b} \, and \, \frac{b}{a} \leftrightarrow a \times n = b \; and \; b \times m = a$$
$$\rightarrow a \times n \times m = a$$

In the instance that a≠0:

$$n \times m = 1 \rightarrow n = 1 \; and \; m = 1$$
$$\rightarrow a = b$$

In the instance that a=0:

$$a \times 0 = b \rightarrow a = b = 0$$

Therefore in the case that a = 0 a=b which proves antisymmetry

Transitivity – Finally it is necessary to show that the divisibility is transitive which means it needs to be proved that if $\frac{a}{b} \, and \, \frac{b}{c} \rightarrow \frac{a}{c}$

First it is necessary to assume that $\frac{a}{b} \, and \, \frac{b}{c} \leftrightarrow a \times n = b \; and \; b \times m = c$

$$a \times n \times m = c$$

Combine n x m into another natural number called j

$$a \times j = c \leftrightarrow \frac{a}{c}$$

As it has been proven that a/c transitivity has been proven

Therefore as the Reflexivity divisibility, Antisymmetry divisibility and Transitivity divisibility has been proven for natural numbers it is the case that divisibility over the natural numbers induces a partial order.

B. Prove that the lexicographic ordering on alphanumeric strings is an order or disprove this if it is not true.

In order to prove that lexicographic ordering on alphanumeric strings is an order it must satisfy three properties:

1) Reflexivity – Any string must be equal to itself
2) Antisymmetry – If string A is ≤ to string B and B ≤ A then A = B
3) Transitivity – If A ≤ B and B ≤ C then A ≤ C

Reflexivity is satisfied because a string is always equal to itself.

Antisymmetry is satisfied because two equal strings will have characters in the same order which will be in the same order under lexicographical ordering.

Transitivity is satisfied because a character to character comparison of two strings allows us to demine which sting is larger or smaller. If A < B and B < C then A < C.

As all three requirements are satisfied lexicographic ordering on alphanumeric strings is an order.

4) A. compute Iv, IM, Mv, v'v, v'M, MM, vv', and det(M)

M = $\begin{matrix} 2 & 2 \\ 1 & 3 \end{matrix}$ v = (1,-2)' I = -3

Iv:

$$Iv = I \times v$$
$$= -3 \times (1, -2)' = (-3, 6)'$$
$$Iv = (-3, 6)'$$

IM:

$$IM = I \times M$$
$$= -3 \times \begin{matrix} 2 & 2 \\ 1 & 3 \end{matrix}$$
$$IM = \begin{matrix} -6 & -6 \\ -3 & -9 \end{matrix}$$

Mv:

$$Mv = M \times v$$
$$\begin{matrix} 2 & 2 \\ 1 & 3 \end{matrix} \times (1, -2)'$$
$$((2 \times 1) + (2 \times -2)) = -2$$
$$((1 \times 1) + (3 \times -2)) = -5$$
$$Mv = (-2, -5)$$

v'v:

$$v'v = v \times v$$
$$[a, b] \times [c, d] = ac + bd$$
$$a = 1 \; b = -2 \; c = 1 \; d = -2$$
$$(1 \times 1) + ((-2) \times (-2))$$
$$= 5$$

v'M:

$$(1, -2) \times \begin{matrix} 2 & 2 \\ 1 & 3 \end{matrix}$$
$$((1 \times 2) + (-2 \times 1))$$
$$((1 \times 2) + (-2 \times 3))$$
$$= (0, -4)$$

MM:

$$\begin{matrix} a & b \\ c & d \end{matrix} \times \begin{matrix} e & f \\ g & h \end{matrix} = \begin{matrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{matrix}$$

$$a = 2\ b = 2\ c = 1\ d = 3\ e = 2\ f = 2\ g = 1\ h = 3$$

$$(2 \times 2) + (2 \times 1) = 6$$

$$(2 \times 2) + (2 \times 3) = 10$$

$$(1 \times 2) + (3 \times 1) = 5$$

$$(1 \times 2) + (3 \times 3) = 11$$

$$= \begin{matrix} 6 & 10 \\ 5 & 11 \end{matrix}$$

vv':

$$(1, -2) \times (1, -2)'$$

$$\begin{matrix} (1 \times -2) & (1 \times -2) \\ (-2 \times 1) & (-2 \times -2) \end{matrix}$$

$$= \begin{matrix} 1 & -2 \\ -2 & 4 \end{matrix}$$

det(M):

det(A) = ad – bc

$$a = 2\ b = 2\ c = 1\ d = 3$$

$$(2 \times 3) - (2 \times 1)$$

$$= 4$$

B. Compute the eigenvalues and eigenvectors of M and normalise the eigenvectors. Check orthogonality using scalar products. What are the angles between pairs of eigenvectors?

M = $\begin{matrix} 2 & 2 \\ 1 & 3 \end{matrix}$

$$\begin{matrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{matrix}$$

$$(2-\lambda)(3-\lambda) - 2 = \lambda^2 - 5\lambda + 4$$

$$= (\lambda - 1)(\lambda - 4)$$

Eigenvalues: λ_1 =1 and $\lambda_2 = 4$

Eigenvectors can be computed from the definition: Au = λu

(M - λI)v = 0

For λ_1 =1:

M-λ_1I = $\begin{matrix} 1 & 2 \\ 1 & 2 \end{matrix}$

(A- λI).v=0

$$x_1 + 2 \times x_2 = 0$$

$$x_1 = -2x_2$$

$$x_2 = x_2$$

$$x_2 = 1:$$

$$v_1 = \begin{matrix} -2 \\ 1 \end{matrix}$$

Normalise the vector with the formula: x/|x|

-2/2 = 1

For $\lambda_2 = 4$:

M-λ_2I = $\begin{matrix} -2 & 2 \\ 1 & -1 \end{matrix}$

(A- λI).v=0

$$x_1 - x_2 = 0$$

$$x_2 \times \begin{matrix} 1 \\ 1 \end{matrix}$$

$$x_2 = 1:$$

$$v_2 = \begin{matrix} 1 \\ 1 \end{matrix}$$

Normalise the vector with the formula: x/|x|

1/1 = 1

As the vectors do not create a right angle they are not orthogonal because $\cos\theta \neq 0$

$\theta = \arccos(x.y/|x||y|)$

$\arccos((-2\times1)/(2\times1)) = 180$

$\theta = 180$

$\cos 180 = -1$

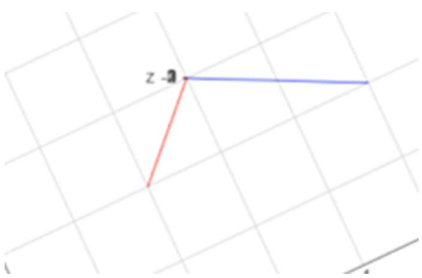
$\theta = \arccos(x.y/|x||y|)$

$\arccos((1\times1)/(1\times1))$

$\theta = 0$

$\cos 0 = 1$

This can also be seen in the diagram below:



Angles between pairs of eigenvectors:

Formula: $\theta = \arccos(x.y/|x||y|)$

$|a|=\sqrt{-2^2+1^2} = \sqrt{5}$

$|a|=\sqrt{1^2+1^2} = \sqrt{2}$

$$\frac{-1}{\sqrt{5}\times\sqrt{2}}$$

$$= -\frac{\sqrt{10}}{10}$$

$\arccos(-\frac{\sqrt{10}}{10})$ = 108.4349° (4 d.p.)