Advantageous Probability

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The Problem

If two six-sided dice are rolled and the larger number is taken, what is the expected value of the outcome? To approach this, the simple way to think about it is with a table such as the one below.

	1	**2**	**3**	**4**	**5**	**6**
1	1	2	3	4	5	6
2	2	2	3	4	5	6
3	3	3	3	4	5	6
4	4	4	4	4	5	6
5	5	5	5	5	5	6
6	6	6	6	6	6	6

Table 1: Table showing all possible outcomes of six-sided dice thrown.

The table here shows that if taking the larger of the two numbers as the value of the dice, for any two, fair, six-sided dice, the expected value, E(X) is $\frac{161}{36}$. If we compare this to the E(one die) or E(D) which it will now be denoted as, we get the value of 3.5. This value can be found by using this formula: $E(X) = \sum xP(D = x)$ This can be easily visualised using a similar table to that shown above.

x	**1**	**2**	**3**	**4**	**5**	**6**
$P(D = x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
$xP(D = x)$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	1

Table 2: Table showing all probability outcomes of a die thrown.

Summing these fractions gives us 3.5 which can also be shown by finding the mean of the numbers 1 to 6. The mean (μ) can be applied in this scenario as the die is fair and the probability of all of the outcomes is equal. It is also the exact median of the range of numbers and you can find this by doing $\frac{3+4}{2} = 3.5$.

$$\frac{1}{6} + \frac{1}{3} + \frac{1}{2} + \frac{2}{3} + \frac{5}{6} + 1 = \frac{7}{2} = 3.5$$

This is showing that when 2 fair, six-sided dice are thrown, the expected value of two dice or E(2D) is greater than that of the expected value of one die, E(D) as $\frac{161}{36} \approx 4.47$ which is $> 3.5 \Rightarrow E(2D) > E(D)$. However does this change with the number of sides, S of the dice?

Sides > 6

Looking back at Table 1, a pattern is present for the calculation of E(2D). Calculating for 6 sides or 6S is done by finding the sum ($\sum$) of possible outcomes and then dividing by S^2 which in the case of S=6 is,

$$\frac{(1 \times 1) + (2 \times 3) + (3 \times 5) + (4 \times 7) + (5 \times 9) + (6 \times 11)}{6^2} = \frac{161}{36}$$

The total Possible outcomes is S^2 due to the fact that number of sides on both dice are equal to equal other however the numerator is more difficult to see where the numbers come from. By simplifying the problem into a theoretical 'dice' with 3 sides or another object such as a spinner with 3 equally likely options, a smaller table of possible outcomes can be drawn. By visualising the table in

	1	**2**	**3**
1	1	2	3
2	2	2	3
3	3	3	3

Table 3: Table showing all possible outcomes of an object with 3 equally likely outcomes.

this way we can see that for S=3, the total number if all outcomes are added is $(1 \times 1) + (2 \times 3) + (3 \times 5)$ and that with each number of sides, S added to this table the amount it is multiplied by is always an odd number. Starting at S=1, the odd number which denotes the number of appearances of that outcome, is 1. For S=2 it is the next odd number, 3. This can be generalised such that the number of a specific outcome is $2S - 1$ for $S = 3$, the number of outcomes which equal 3 is $2 \times 3 - 1 = 5 \Rightarrow 5$ occurrences of the number 3 which can be seen true from Table 3.

Using this knowledge that each side's odd number can be determined from $2S-1$ (which comes from the sequence of odd numbers $1, 3, 5, 7, 9, ...$) by multiplying this by S, the sum of the outcomes of S can be calculated.

$$S(2S-1) = 2(2 \times 2 - 1) = 6$$

The equation above shows the answer to be 6 and by summing the outcomes which equal 2, an answer of 6 is also reached. As this is true for $S_1 = S_2$ where 1 and 2 denotes which dice S is referring to, by using the equation below we can find the sum of all outcomes from 1 to S.

$$\sum_{s=1}^{S} s(2s-1)$$

The use of s here is such that s can vary from 1 up to S which is the fixed number of sides the dice has. Expanding the bracket gives $2s^2 - s$ which allows calculation of this sum to be easier.

$$\sum_{s=1}^{S}(2s^2 - s) = 2\sum_{s=1}^{S} s^2 - \sum_{s=1}^{S} s$$

Using standards results of series this equation can be simplified by replacing n with the variable S and r with the variable s.

$$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$$

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$$

Starting with the first half of the equation, $2\sum_{s=1}^{S} s^2$, both parts can be solved using these standard results.

$$2\left(\frac{1}{6}S(S+1)(2S+1)\right)$$

$$\frac{1}{3}S(S+1)(2S+1)$$

Subtracting the standard results for s gives:

$$\frac{1}{3}S(S+1)(2S+1) - \frac{1}{2}S(S+1)$$

$$\frac{2}{3}S^3 + S^2 + \frac{1}{3}S - \frac{1}{2}S^2 - \frac{1}{2}S$$

$$\frac{1}{6}S(4S^2+3S-1)$$

$$\frac{1}{6}S(S+1)(4S-1)$$

$$\frac{S(S+1)(4S-1)}{6}$$

To complete the formula for generalising the expected outcome of a set of two dice with side S the total number of possibilities needs to be accounted for so the answer given by the standard results must be divided by S^2 to give the expected value, not just the total of the outcomes.

$$\frac{S(S+1)(4S-1)}{6} \div S^2$$

$$\frac{(S+1)(4S-1)}{6S}$$

This is a general formula for the expected value of two dice with the same number of sides and this can be used to compare to the expected value of one dice with S number of sides.

Expected Value of a die

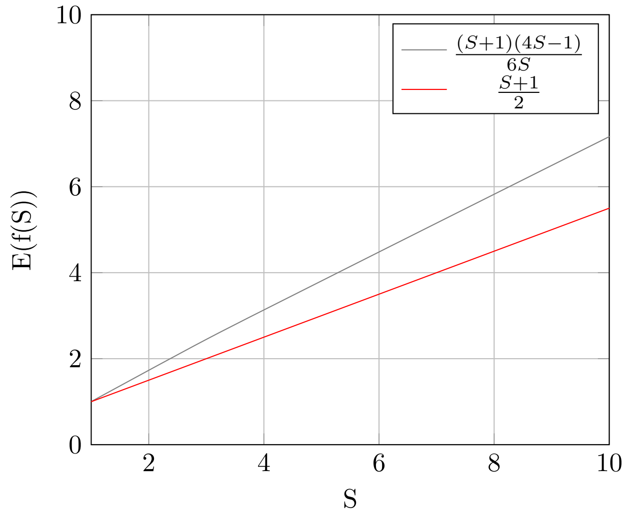
In order to find the expected value of a single die with equal probabilities it is much simpler to figure out than two dice. As stated earlier, μ is equal to E(D) as all of the probabilities are equal for each outcome and to figure out μ it is a lot simpler. To find μ you take the sum of all of the numbers in a set and divide by the total number of the set. This can be expressed as such:

$$\frac{1+2+3+...+n}{n}$$

It is known that the sum from 1 to n of a number is $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$ which was explained before and it can be used here to generalise a formula for μ.

$$\frac{\sum_{r=1}^{n} r}{n} = \frac{n+1}{2}$$

This can be plotted against the expected value of 2 dices to show the difference.



This set of graphs performs as you would expect for $S > 1$ and the expected value is greater with 2 dices compared to one die. With this graphical model, it gets more complicated when $0 < S < 1$ which is to be expected as it is physically impossible to have less than 1 side to a dice so the expected value would be impossible to calculate, this is represented by an asymptote on the graph at x=0. As the sides S are given from a formula, the graph can calculate $S < 0$ values however this too, is not possible in the real world.

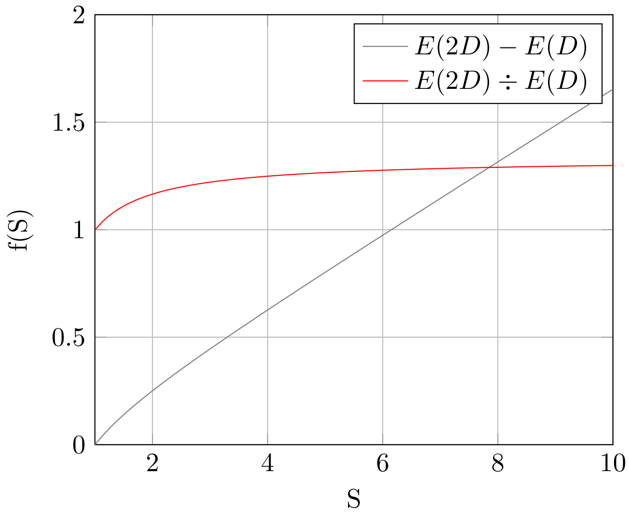
Expanding the Problem

How much bigger is the expected value using advantageous probability than the expected value of one dice?
This can be figured out by either finding the difference between the values or dividing them to see how many times larger the expected value is.

S	**E(D)**	**E(2D)**	**E(2D)-E(D)**	**E(2D)/E(D)**
1	1	1	0	1
2	1.5	1.75	0.25	1.167
3	2	2.444	0.444	1.222
4	2.5	3.125	0.625	1.25
5	3	3.8	0.8	1.267
6	3.5	4.472	0.972	1.278
7	4	5.143	1.143	1.286
8	4.5	5.8125	1.3125	1.292
9	5	6.481	1.481	1.296
10	5.5	7.15	1.65	1.3

Table 4: Table showing comparison between expected values.

These expected value differences can be graphed to clearly show the relationship between them. The difference between the expected values shows a linear graph with the difference increasing exponentially as $S \to \infty$ whereas the number of times greater E(2D) is than E(D) diverges as it approaches infinity to be $\frac{1}{3}$. Using this model, it can be seen that there is one point of intersection which is when $S \approx 7.873$ which is the only intersection of the two values on this graph.



The formulas for each of these graphs can be found algebraically using the formulas derived for E(2D) and E(D), $\frac{(S+1)(4S-1)}{6S}$ and $\frac{S+1}{2}$ respectively.

$$E(2D) - E(D)$$

$$\frac{(S+1)(4S-1)}{6S} - \frac{(S+1)}{2}$$

$$\frac{(S+1)(4S-1)}{6S} - \frac{3S(S+1)}{6S}$$

$$\frac{(S+1)(4S-1) - 3S(S+1)}{6S}$$

$$\frac{(S+1)[(4S-1) - 3S]}{6S}$$

$$\frac{(S+1)(S-1)}{6S}$$

$$E(2D) \div E(D)$$

$$\frac{(S+1)(4S-1)}{6S} \div \frac{(S+1)}{2}$$

$$\frac{(S+1)(4S-1)}{6S} \times \frac{2}{(S+1)}$$

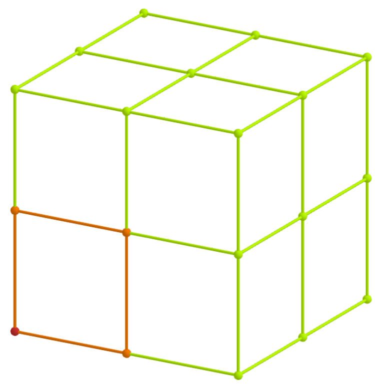
$$\frac{2(S+1)(4S-1)}{6S(S+1)}$$

$$\frac{(4S-1)}{3S}$$

This shows how the two graph functions are derived however, the question then becomes if there are more dice what happens to the expected value?

More than two dice

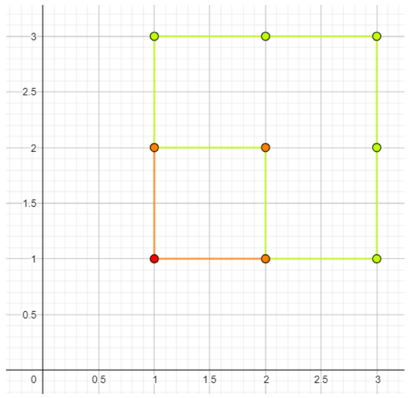
Expanding this idea to three dice becomes a lot more complicated. A table cannot be created for the outcomes of three dice so it is necessary to start thinking in 3D. Graphically, this is interpreted by the different combinations of dice outcomes represented as points on a 3D plane and the number of these points is the number of outcomes of a particular number.



It can be seen that this number is the total, S cubed take away the cube of S-1. For S=2 this is not as clear as dice rolls cannot be 0 and on a graph this results in the points not going through any axis. 1^3 however is 1 so the assumption that the pattern is $S^3 - (S-1)^3$ holds true. The graph however only goes up to S=3 but at that point it is clear how the pattern continues $\rightarrow \infty$ as more are more points almost 'cover' the previous set of points which is why it is not the complete S^3 but the cube before it subtracted from it.

This can be generalised for any number of dice assuming it follows the same pattern that 2 dice and 3 dice do. 4 Dice graphically would be difficult to visualise the world can only be seen in 3 dimensions. The general formula for the number of outcomes of a specific number is $S^n - (S-1)^n$ where n is the number of dice used.

Using the same graphical technique of using the possible combinations as coordinates, the graph for two dice can be drawn which shows how this formula can be generalised.



This is a visual proof of $2S-1$ from earlier by using $S^2-(S-1)^2$ and expanding and simplifying. $S^2-(S^2-2S+1)=2S-1$. This formula to find out the possible number of outcomes of the dice can be used to find the expected value like before. Calculations similar to two dice can be used here however the summations get more complicated the higher the power of n.

To eventually find the expected value, first the sum of all of the outcomes is needed. The expression for this would be:

$$\sum_{s=1}^{S} s[s^3-(s-1)^3]$$

It is necessary to times the formula by s as it gives the value of that outcome. For example if you substitute 3 into $s^3-(s-1)^3$ an answer of 19 is reached which is the number of combinations which are equal to 3 so to get the value of all the outcomes which equal 3, it is necessary to do $3\times$ number of occurrences. This applies for all s and the expression above is calculated all of the possible outcomes and adding them together to get the total of all the outcomes which can later be divided by the number of outcomes to get an expected value, E(3D).

In order to solve this summation it is required to expand the brackets and simplify so that the standard results can be later used.

$$s[s^3-(s-1)^3]$$
$$s[s^3-(s^3-3s^2+3s-1)]$$
$$s(3s^2-3s+1)$$
$$3s^3-3s^2+s$$

Now that the expression has been expanded the standard results of series used as before however this time it involves 3 terms and the use of standard results for cubes.

$$\sum_{r=1}^{n} r^3=\frac{1}{4}n^2(n+1)^2$$

This can be used for $3s^3 - 3s^2 + s$:

$$\sum_{s=1}^{S} 3s^3 - 3s^2 + s = 3\sum_{s=1}^{S} s^3 - 3\sum_{s=1}^{S} s^2 + \sum_{s=1}^{S} s$$

$$\frac{3S^2(S+1)^2}{4} - \frac{S(S+1)(2S+1)}{2} + \frac{S(S+1)}{2}$$

$$\frac{3S^2(S+1)^2 - 2S(S+1)(2S+1) + 2S(S+1)}{4}$$

$$\frac{S(S+1)[3S(S+1) - 2(2S+1) + 2]}{4}$$

$$\frac{S(S+1)(3S^2 - S)}{4}$$

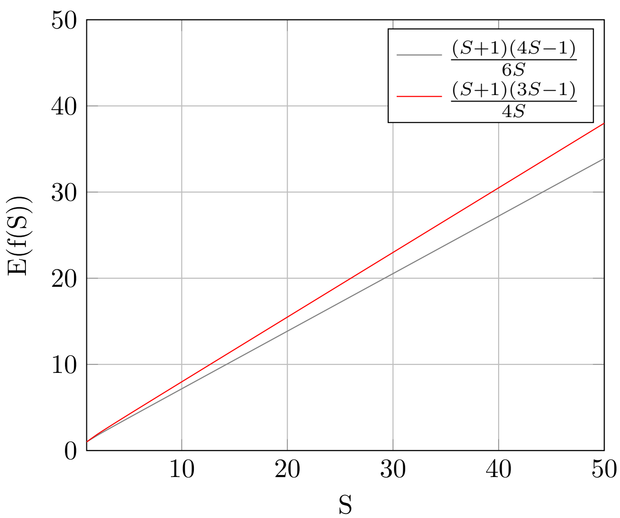
$$\frac{S^2(S+1)(3S-1)}{4}$$

This value denotes the total of the outcomes now if it is divided by S^3 it gives E(3D) which can be written as the following:

$$\frac{S^2(S+1)(3S-1)}{4} \div S^3$$

$$\frac{(S+1)(3S-1)}{4S}$$

This is E(3D) which is a lot more complicated than $\frac{(S+1)(4S-1)}{6S}$. The difference between the two can be shown side-by-side.



The E(3D) is always greater than E(2D) but when S=1, the expected value is the same which is to be expected as there is only one outcome for S=1 which is 1 for any power. 3 dice with one outcome will always give the same outcome no matter what as it is 100% certain for any number of dice.

Conclusion

To begin, this started with the simple problem of rolling two dice and now this simple idea has expanded to solve the expected value of any number of dice with any number of sides. The only difficulty with these sorts of calculations is knowing the standard results of $S, S^2, S^3...S^4$ but this can be calculated using Faulhaber's formula which is where the standard results for series come from. To summarise my findings in one expression it would be the following:

$$\frac{\sum_{s=1}^{S} s[s^n - (s-1)^n]}{S^n}$$

Which is the general formula for finding the expected value of any number of dice with any number of sides, simply let S be the total number of sides and n be the number of dice and the Expected value can be found.