1's in Binary Numbers

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Introduction

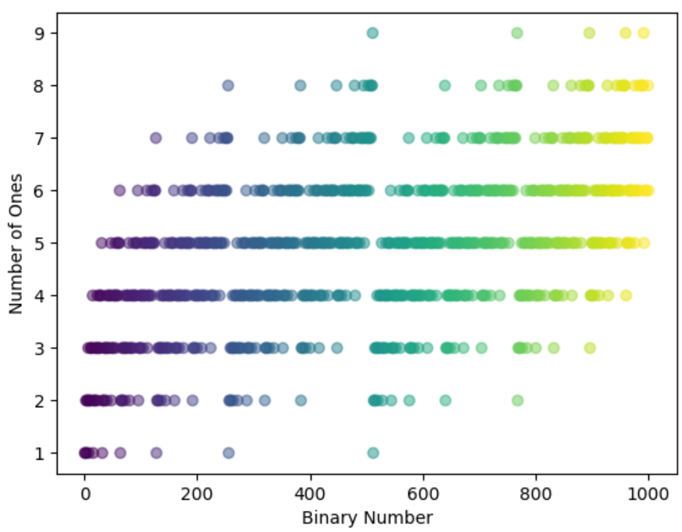
In binary, numbers are classified as a series of 1's and 0's in base 2. The decimal number system is base 10 which means that every digit to the left of another is 10 times the value. An example, in base 2, is 1011 which represents the number 11 as shown below.

2^3	2^2	2^1	2^0
1	0	1	1

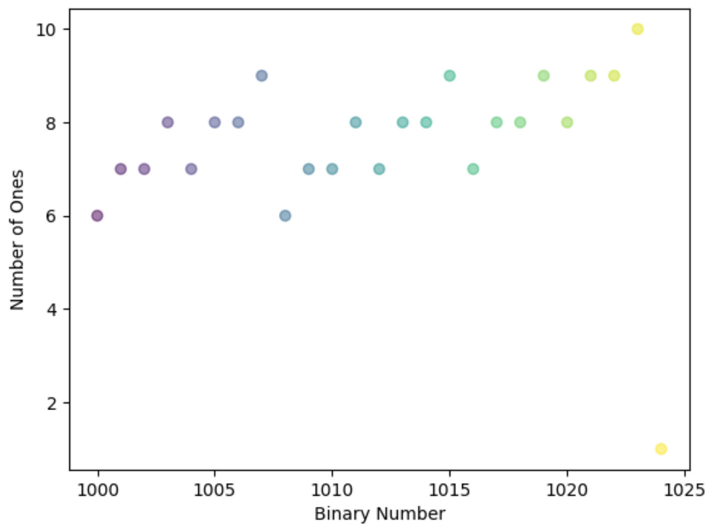
Table 1: To get the number, you add the value of the columns with a 1 in.

1's in Binary

An interesting feature about binary numbers is how the number of 1's can drastically vary from 1 number to the next. To show this extreme variation, I coded a program in Python to plot the number of 1's in the binary numbers up to 1000.



The colour indicates how high the binary number is and it is interesting to note that the maximum number of 1's in the binary numbers is constrained by the powers of 2. The maximum number of 1's in numbers from 1 to 1000 is 9. This is because for there to be 10 1's it would need all of the digits to be a 1. For example if the numbers only go up to 1000, the maximum power of 2 is 512. For there to be 10 1's all the powers of 2 up to and including 2^9 would need to have a place value of 1. This only happens right before the next power of 2 so the maximum number of 1's occurs at $2^9 - 1$ which is 1023. 1023 exceeds the values of the plot so the maximum of 10 1's is not shown. This can be confirmed with an expansion of the plot.



Power Drops

In what I like to call "Power Drops", the number of 1's drops all the way down to 1 at every power of 2. This is seen on the graph above. Although the final point may seem like an outlier, it repeats this pattern of a rapid drop at every power of 2. This is pretty self-explanatory as to why it drops like this. The binary system means that each digit of a number represents whether the number contains that power of 2 in the sum for that number. For a number such as 1024 or 512 or any other power of 2, the binary will look like a singular 1 followed by a series of zeros of length n zeros where n is the power of 2.

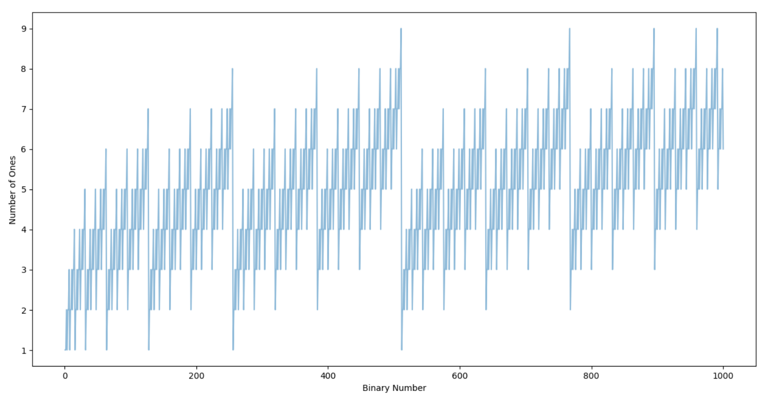
Here are some examples:

Decimal	Binary
1	1
2	10
4	100
...	...
256	100000000
512	1000000000

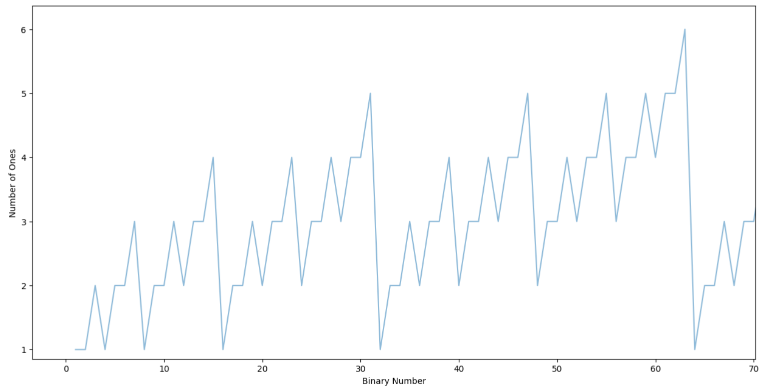
Table 2: Powers of 2 Binary Conversion

Plotting

After having noticed this relationship, I plotted these steep drops again using Python but this time using lines between the data points. As shown below, the plot seems to have a repeating pattern getting slighter higher each period.



An enlargement, such as the one below, can be analysed more precisely. As seen below, the pattern between powers of 2 is of a very similar shape. The only major difference is the magnitude of the drop increasing with each power of 2.



Pascal's Triangle

After further research into these patterns, I found an article [1] which relates the number of 1's in binary to Pascal's Triangle. The relationship found in the article is as follows,

$$\text{Number of odd integers} = 2^{\text{Number of 1's}} \tag{1}$$

A key thing to note about this relationship is that the binary numbers start from 0. The first row of Pascal's Triangle has 1 odd number (which is equal to 2^0) which shows the binary numbers in this equation starts at 0 not 1.

Number	0	1	2	3	...
$2^{\text{Number of 1's}}$	1	2	2	4	...
Number of Odds	1	2	2	4	...

Table 3: Comparison of the Two Sides of the Equation.

The triangle has a direct link to powers of two because the expansion of $(1+1)^x$ is equivalent to saying $\binom{n}{r} \times 1 \times 1$ and the LHS is equivalent to 2^x so Pascal's Triangle is formed of rows of powers of 2.

2^0					1				
2^1				1		1			
2^1			1		2		1		
2^2		1		3		3		1	
2^1	1		4		6		4		1

Table 4: Powers of 2 in relation to Pascal's Triangle

One relationship noted in the article is that, the odd numbers in Pascal's Triangle resembles the Sierpiński triangle and it is interesting that every 2^nth row (starting at n=1) is only made up of odd numbers. This fits with the binary plots where every number, 1 less than a power of 2, has a maximum number of 1's.

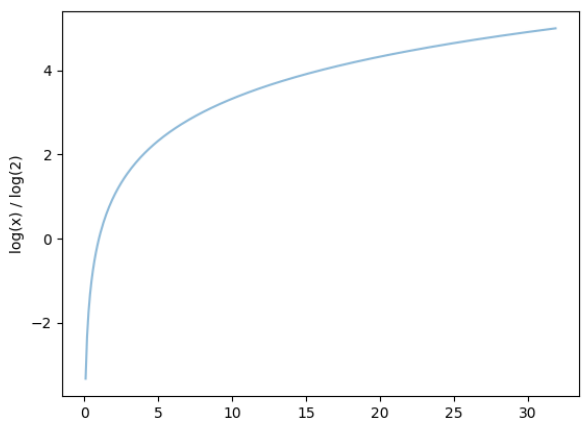
Another interesting thing to note is that the only time there is an odd number of odd numbers in the row is the first row. After this, there will only be even amounts of odd numbers. This can be seen in the triangle from the y-axis symmetry and, the fact that, the only time there is an odd number in the middle is on the first line.

Logarithms

Rearranging (1) gives:

$$\text{Number of 1's} = \frac{\log(\text{Number of odd integers})}{\log(2)} \tag{2}$$

The log equation can be graphed with the log being to any base and it gives a plot like the one below.



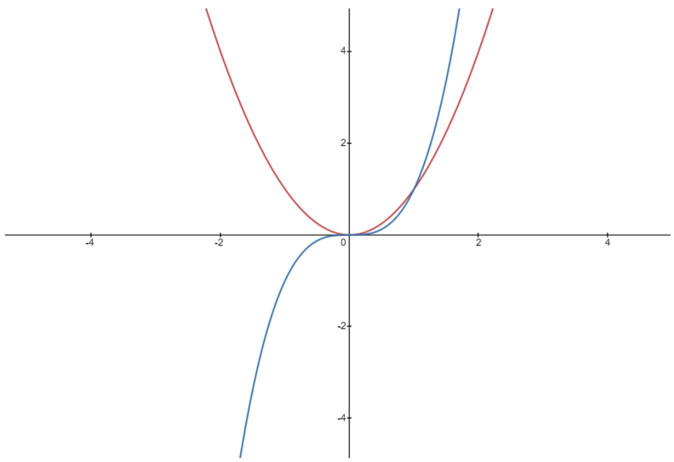
This graph is interesting because the only integer solutions are powers of 2. This is because the amount of odd numbers will always be a power of 2. There is also an asymptote at $x = 0$ and this aligns with Pascal's Triangle as there will always be at least 1 odd number. This graph also shows that the number of ones increases rapidly to begin with but very quickly has a decreasing gradient. It is still > 0 but the increase to the next number on the y-axis gets slower and slower by a power of 2 each time.

A Formula

To piece everything that I have learnt about binary numbers together, I have come up with a formula which counts the number of 1's in any binary number. To start with it needs a way to count the number of odd numbers in a row of Pascal's Triangle. This can be done using a summation such as the one below:

$$\sum_{r=0}^{n} \binom{n}{r} \tag{3}$$

However, this cannot distinguish between odd and even numbers. My first thought was the graphs of polynomials which change based on whether a power is odd or even.



Because of the way powers work, even powers will take any negative input and make it positive whereas if a negative number is raised to an odd power it remains negative. I used this logic for the next part of the formula.

$$\sum_{r=0}^{n} (-1)^{\binom{n}{r}} \tag{4}$$

This outputs -1 if the binomial coefficient is odd and 1 if it is even. If this is all added together, they will cancel out. To make it so that the -1's are considered separately, I square rooted the sum so that all of the -1's become i and the 1's become the real part of a complex number.

$$\sum_{r=0}^{n} \sqrt{(-1)^{\binom{n}{r}}} \tag{5}$$

This equation is nearly finished and if only the imaginary part of the complex number is taken into account then it gives the number of odds in the n'th row of Pascal's Triangle. This is almost an equation for the number of 1's, it just needs to have the same equation but $\log_2$ as shown from equation (1). This gives a final equation of.

$$\text{Number of 1's in Binary} = \log_2 \left(Im \left(\sum_{r=0}^{n} \sqrt{(-1)^{\binom{n}{r}}} \right) \right) \tag{6}$$

References

[1] Daniel Mathews. Adventures with pascal's triangle and binary numbers. 2004.