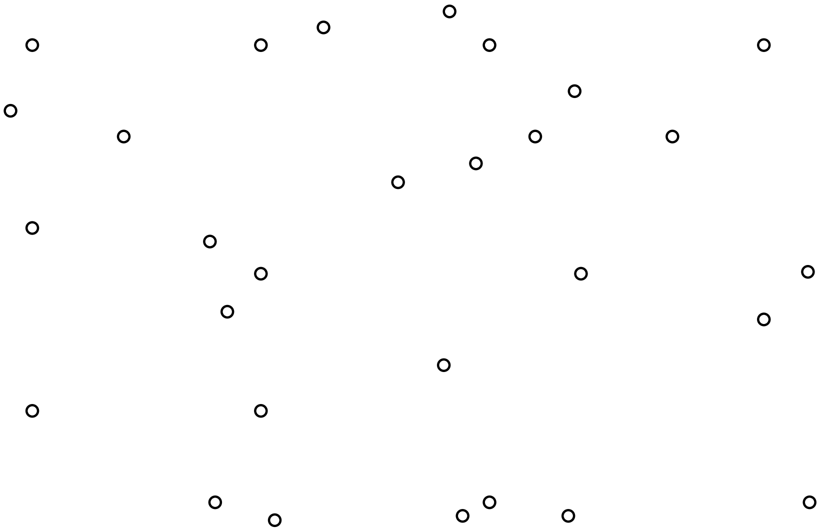
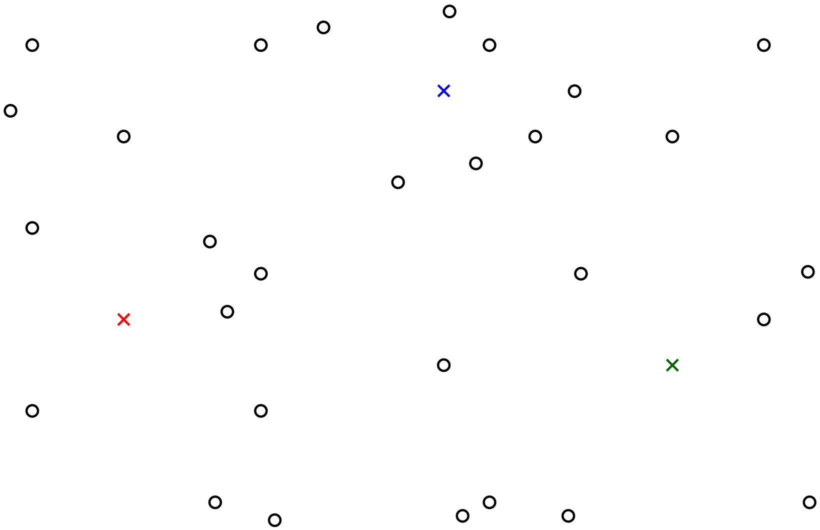
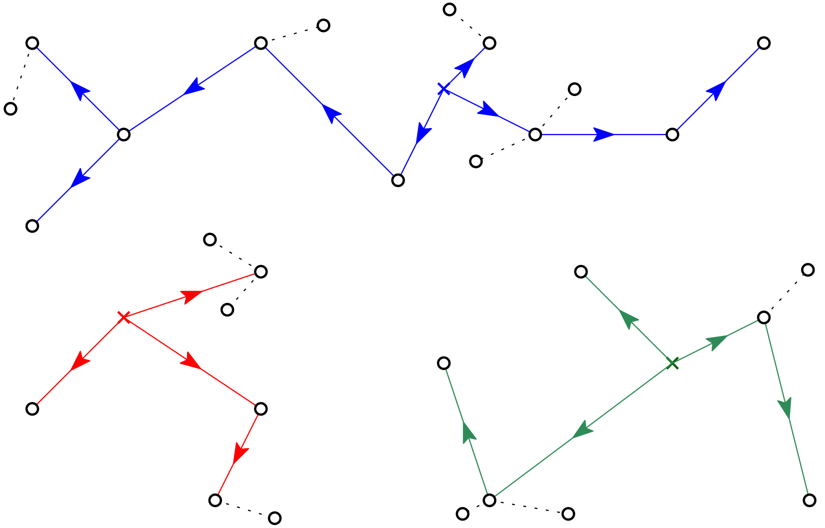
A branch-and-cut algorithm for the κ-connected Forest Star Problem

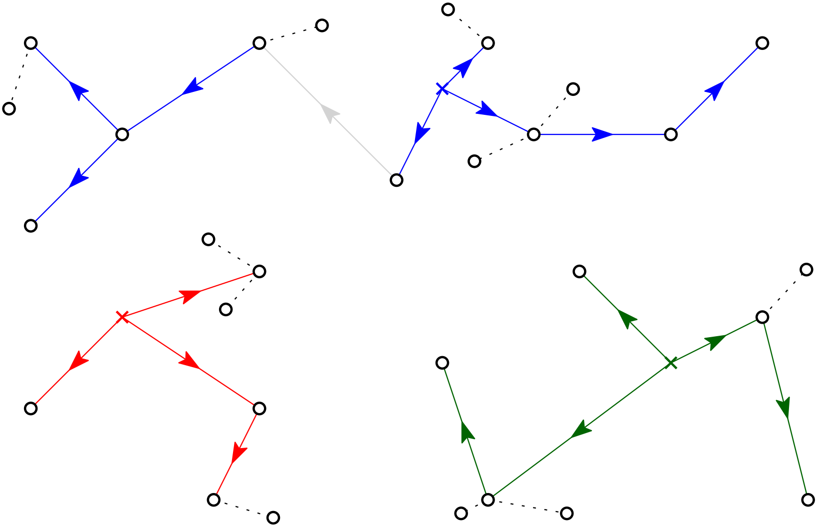
Alfie Plant
University of Edinburgh

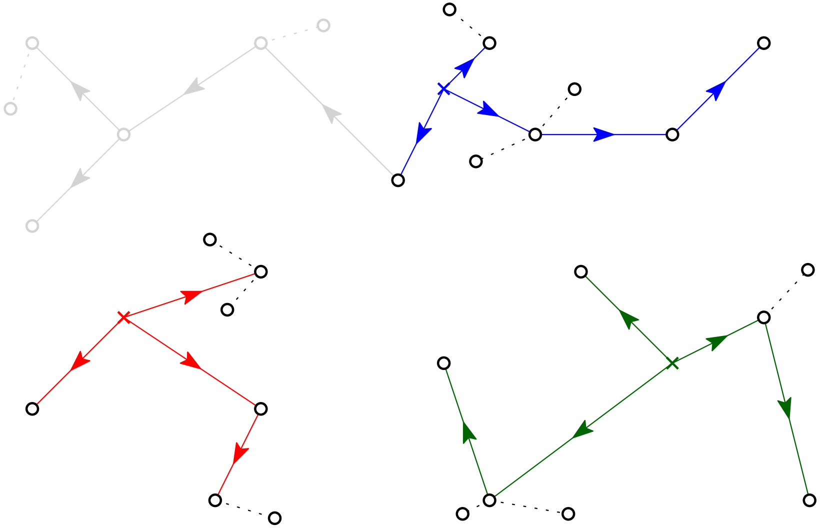
25^{th} March 2025

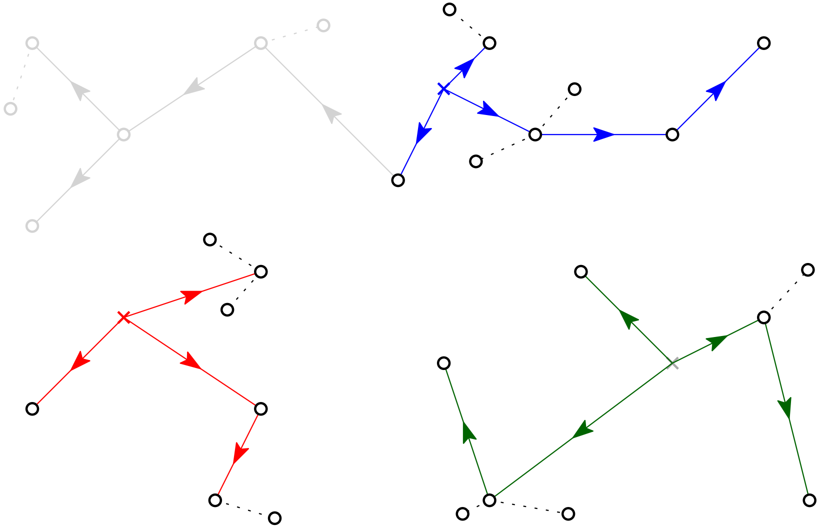


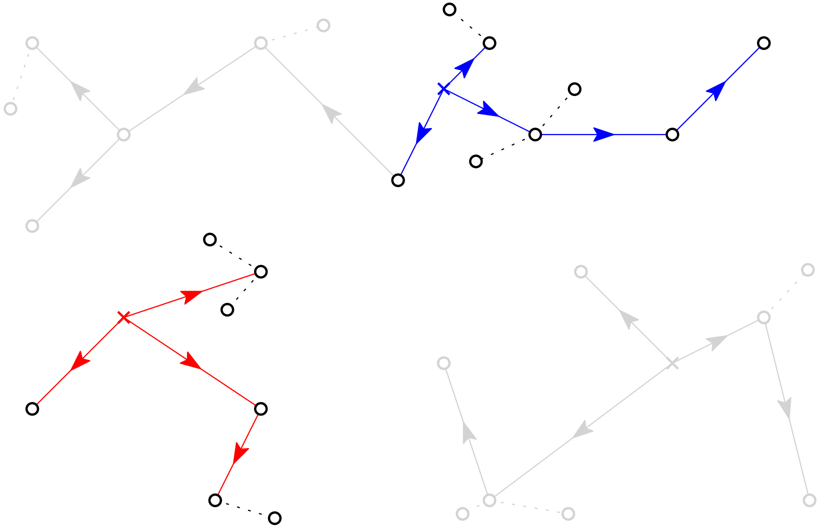


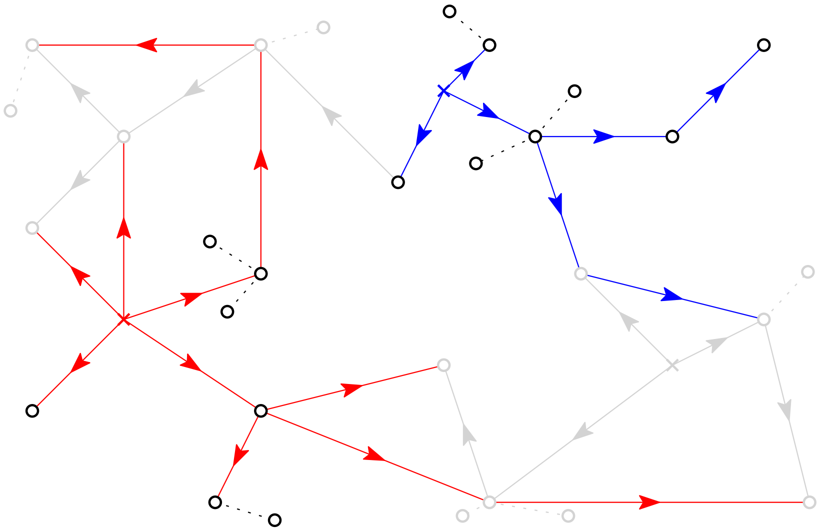


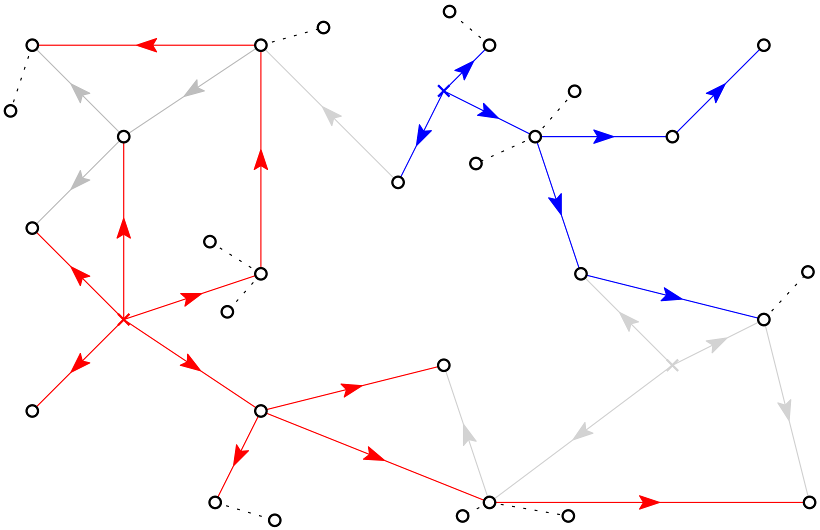












The κ-connected Forest Star Problem

Given a complete directed graph $\mathcal{G} = (\mathcal{V} \cup \mathcal{R}, \mathcal{A})$, find a set of directed trees, each rooted at a different $r \in \mathcal{R}$, such that each vertex $i \in \mathcal{V}$ is either on κ trees, or is assigned a vertex that is, which minimises the total tree and assignment cost.

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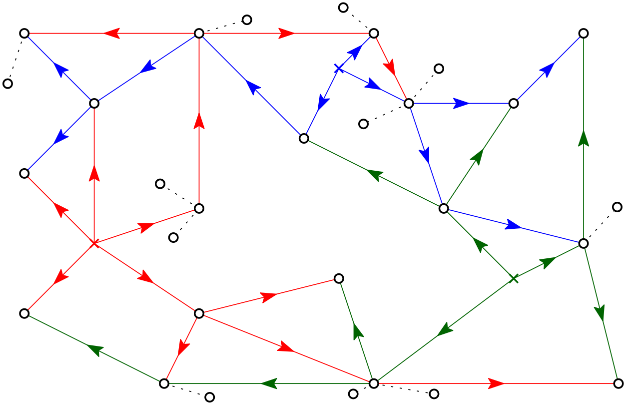
$|\mathcal{V}| = 30$

$|\mathcal{R}| = 3$

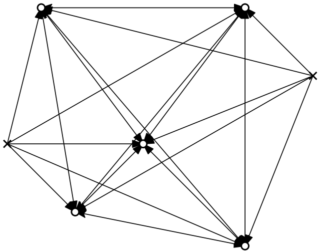
$\kappa = 2$



Agenda

1. Naive IP formulation
2. Reduced IP formulation
3. Branch-and-bound algorithm
4. Outline of proof for the equivalence of formulations
5. Computational results
6. Conclusions

- Consider a directed graph $\mathcal{G} = (\mathcal{V} \cup \mathcal{R}, \mathcal{A})$



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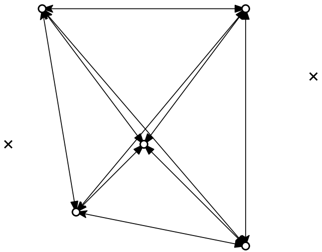

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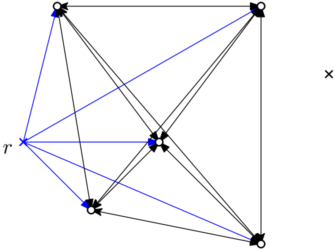

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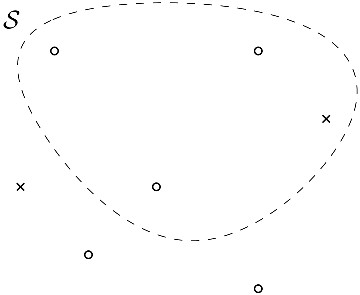

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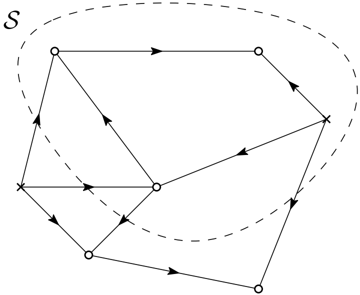

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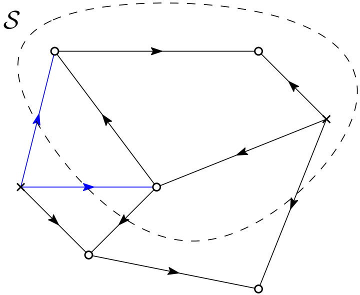

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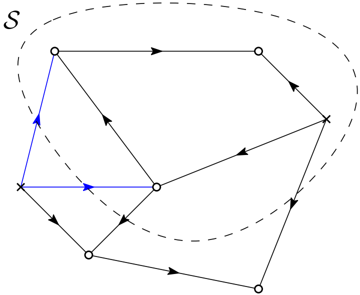
- If $\mathcal{T} = \mathcal{A}$, $\delta^+(\mathcal{S})$ is used for simplicity



Decision variables

$$y_{ij} = \begin{cases} 1 & \text{if vertex } i \in \mathcal{V} \text{ is assigned to vertex } j \in \mathcal{V} \\ 0 & \text{otherwise} \end{cases}$$

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Objective Function

$$\text{minimise} \sum_{(i,j,r) \in \mathcal{A} \times \mathcal{R}} c_{ij} x_{ij}^r + \sum_{(i,j) \in \mathcal{A}_{\mathcal{V}}} a_{ij} y_{ij}$$

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$$\sum_{r \in \mathcal{R}} (x^r_{ij} + x^r_{ji}) \leq 1 \qquad \forall i \in \mathcal{V},\ \forall j \in \mathcal{V}$$

Constraints

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\sum_{(i,j) \in \delta^+(\mathcal{A}_r, i)} x_{ji}^r &= w_{ir} && \forall i \in \mathcal{V},\ \forall r \in \mathcal{R}
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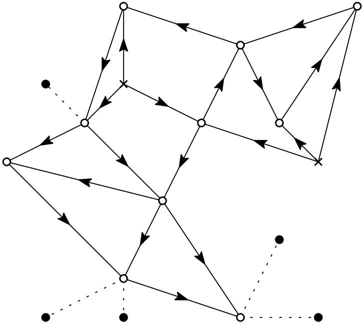
$$\sum_{(i,j) \in \delta^+(\mathcal{A}_r, \mathcal{S})} x_{ij}^r \geq w_{kr} \qquad \forall \mathcal{S} \subseteq \mathcal{V}, \forall k \in \mathcal{S},\ \forall r \in \mathcal{R}$$

Cycle Elimination Constraints

$$\sum_{(i,j)\in\delta^+(\mathcal{A}_r,\mathcal{S})} x^r_{ij} \geq w_{kr} \quad \forall\mathcal{S}\subseteq\mathcal{V},\ \forall k\in\mathcal{S},\ \forall r\in\mathcal{R}$$



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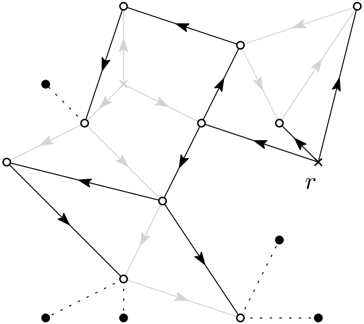

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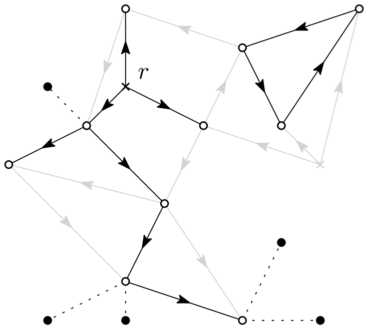

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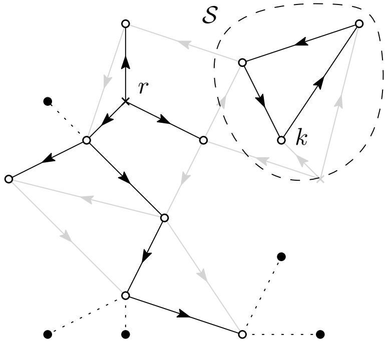

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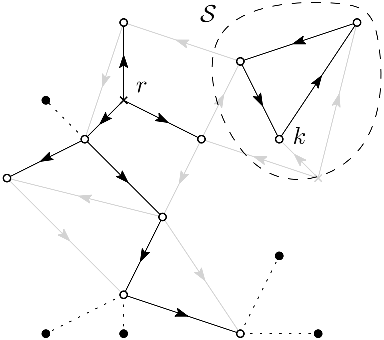

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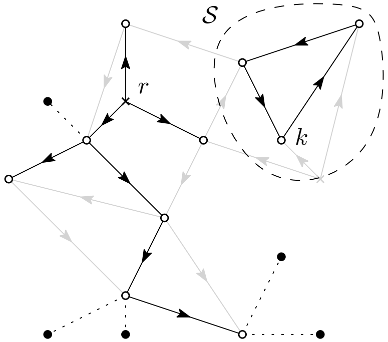

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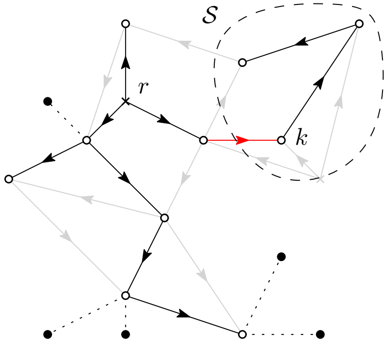

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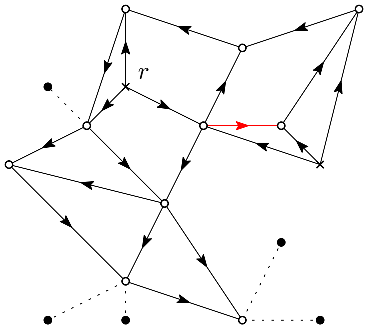

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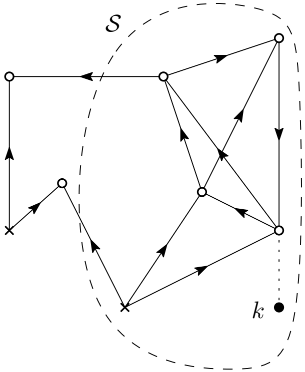

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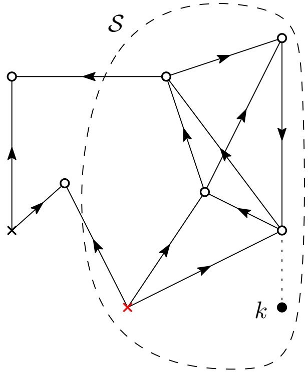
$$\kappa = 2$$
$$|\mathcal{S}\cap\mathcal{R}| = 1$$



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$$\sum_{(o,i)\in\delta^+(\mathcal{S})} x_{oi} \geq \sum_{j\in\mathcal{S}} y_{kj}(\kappa - |\mathcal{S}\cap\mathcal{R}|) \quad \forall \mathcal{S}\subset\mathcal{V}\cup\mathcal{R}, \forall k\in\mathcal{S}$$

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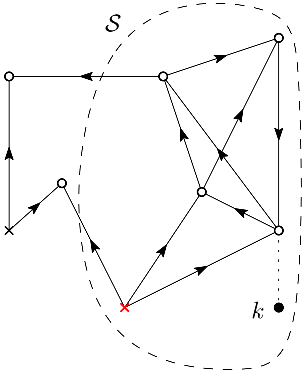

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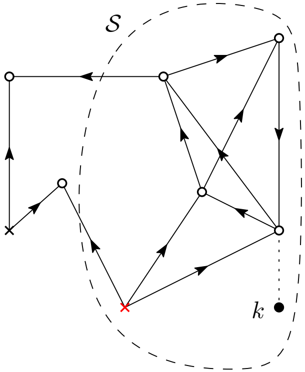

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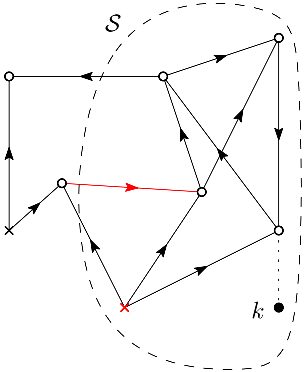

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- Assume $\kappa = |\mathcal{R}|$

Equivalence of Naive and Reduced Formulations

- Assume $\kappa = |\mathcal{R}|$
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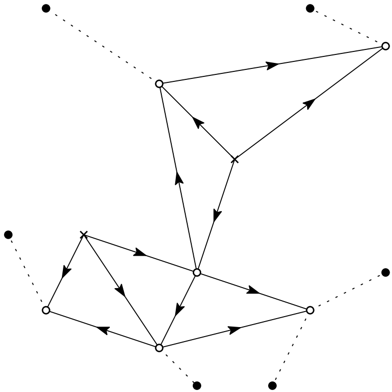
$$\begin{aligned} &\text{Let} \quad \bar{x}_{ij} = \sum_{r \in \mathcal{R}} x_{ij}^{r*} && \forall (i,j) \in \mathcal{A} \\ &\text{and} \quad \bar{y}_{ij} = y_{ij}^* && \forall (i,j) \in \mathcal{A}_{\mathcal{V}}. \end{aligned}$$

Reduced κ-FSP



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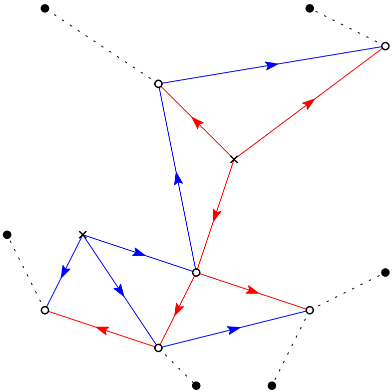

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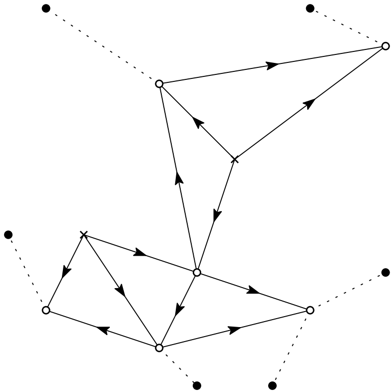

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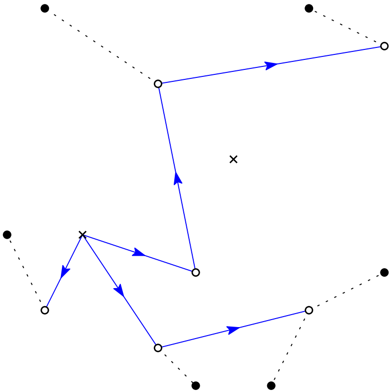

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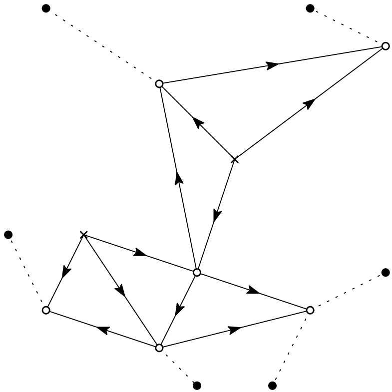

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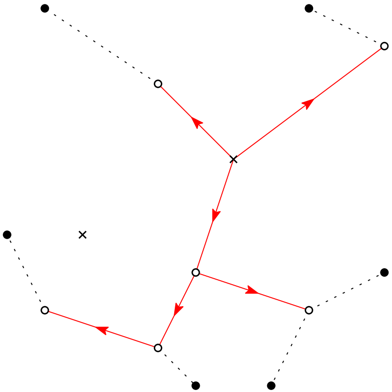

Theorem (Max-flow Min-cut)

Given $\mathcal{G} = (\mathcal{V} \cup \mathcal{R}, \mathcal{A})$ such that $\omega_{ij} \geq 0$ for $(i, j) \in \mathcal{A}$, together with some $s \in \mathcal{V} \cup \mathcal{R}$ and $t \in \mathcal{V} \cup \mathcal{R}$, the maximum flow in $\mathcal{G}$ from s to t is equal to the minimum s-t cut of $\mathcal{G}$.

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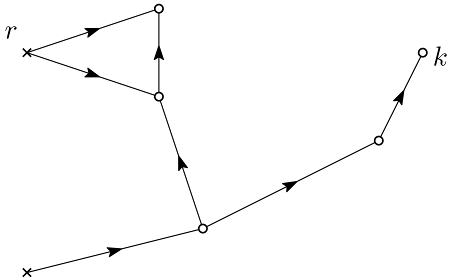

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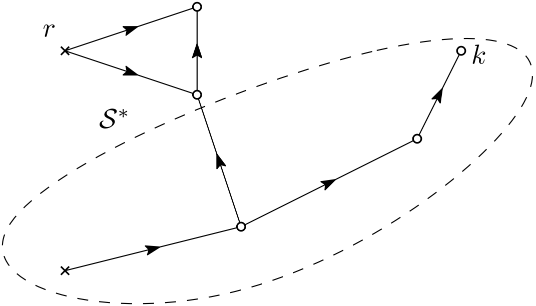

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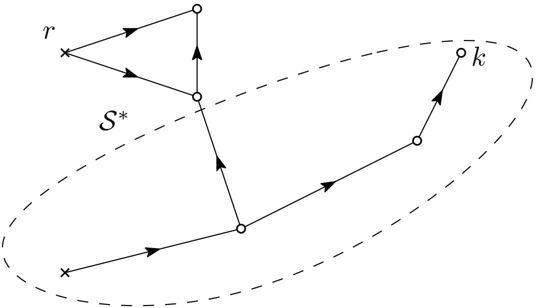

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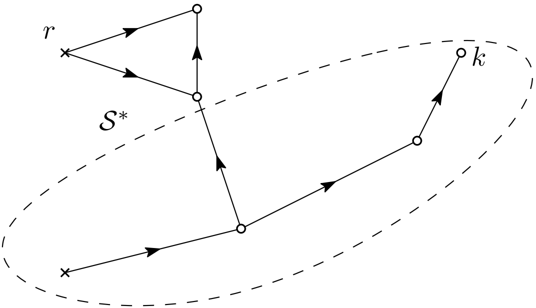

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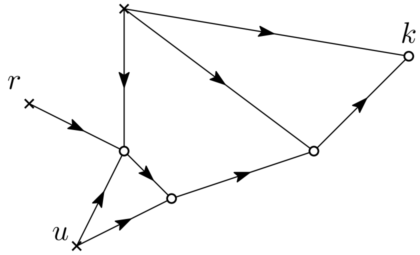

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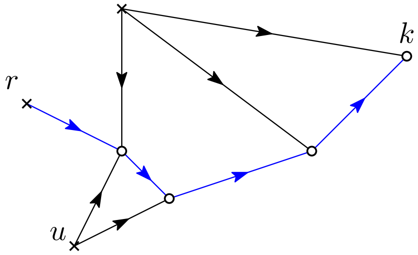

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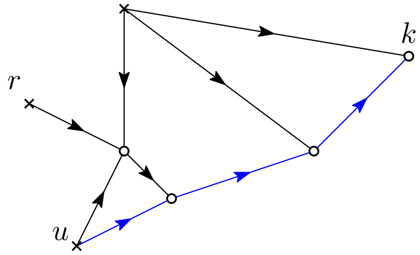

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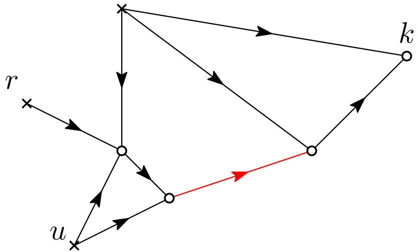

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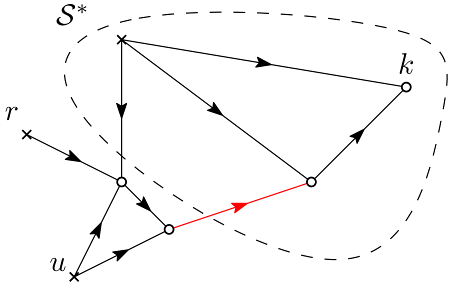

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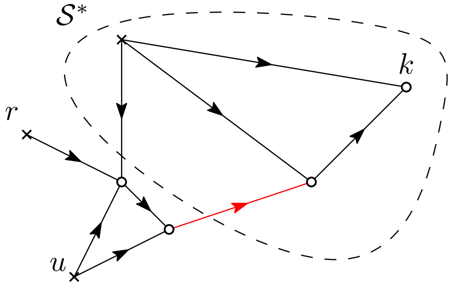

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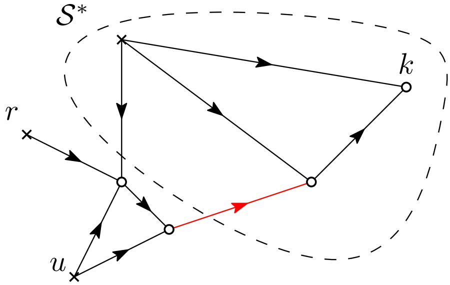

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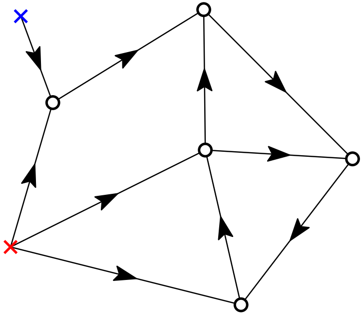
$$\sum_{r \in \mathcal{R}} (x_{ij}^r + x_{ji}^r) \leq 1 \quad \forall i \in \mathcal{V},\ \forall j \in \mathcal{V}$$

- A chokepoint is an arc that, when removed from the graph, there is no directed path from some $r \in \mathcal{R}$ to some $k \in \mathcal{V}$.



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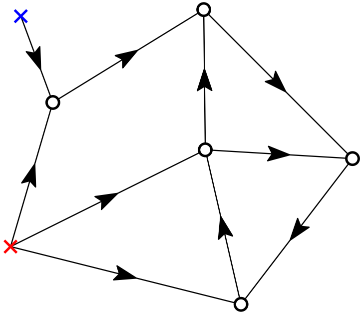

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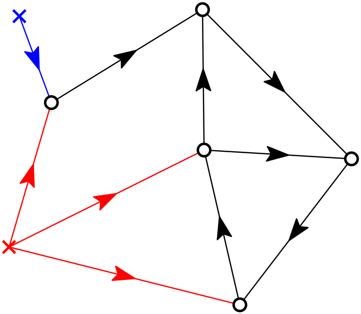

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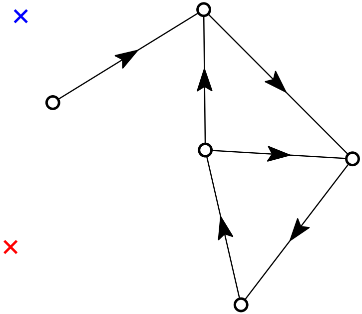

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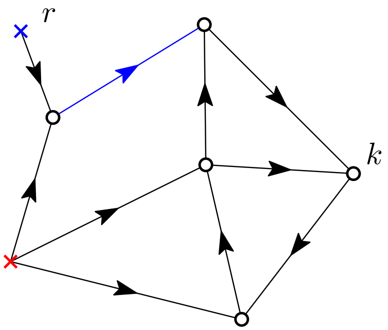

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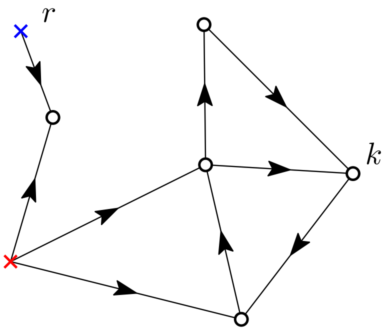

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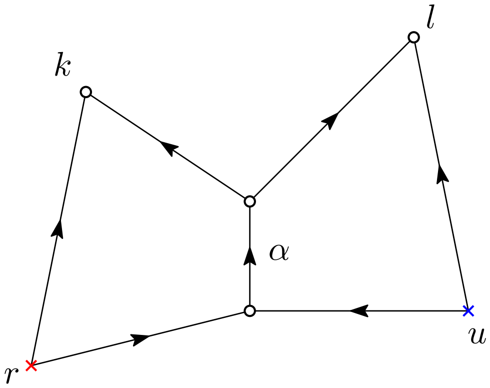
For any solution $(\bar{x}, \bar{y})$ of the reduced formulation there does not exist a chokepoint arc associated to $k \in \mathcal{V}$ and $r \in \mathcal{R}$ and some other $l \in \mathcal{V}$ and $u \in \mathcal{R}$.



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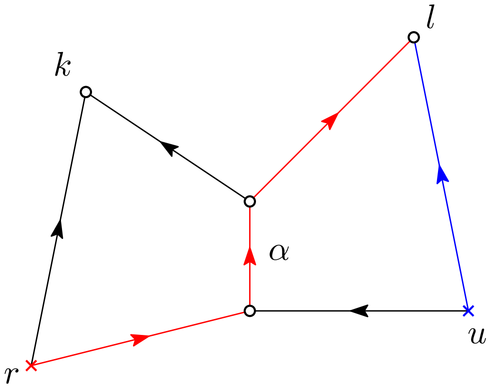

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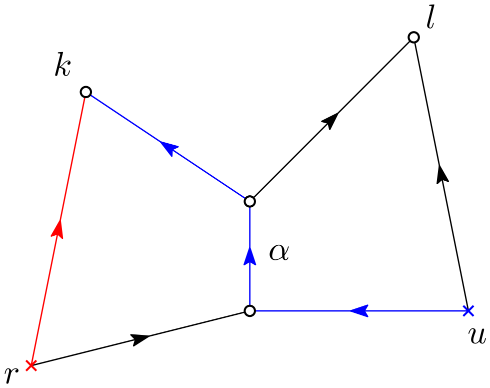

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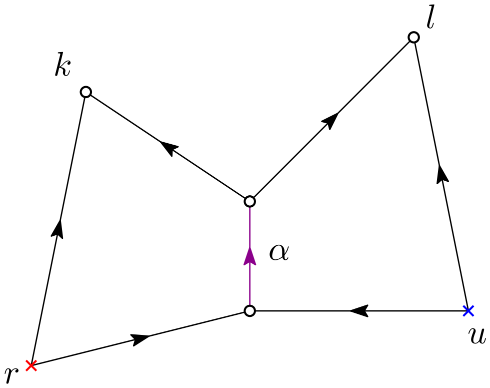

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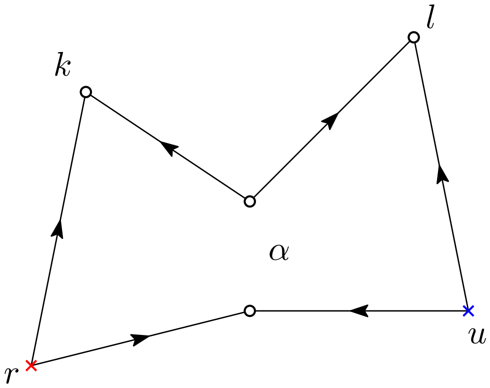

Proposition 2

For any optimal solution $(\bar{x}, \bar{y})$ of the reduced κ-FSP with $\kappa = |\mathcal{R}|$, there exists a mapping to a feasible solution of the naive κ-FSP.

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- **Step 3b**: Assign Remaining Paths

- Performance of naive and reduced formulations tested on five different seeds across 15 instances of data
- $|\mathcal{V}| = 20, 40, 60, 80, 100$
- $\kappa = 2, 4, 6$

Computational Results

instance		naive					reduced				
$\lvert\mathcal{V}\rvert$	κ	build	cuts	cut time	opt.	solve	build	cuts	cut time	opt.	solve
20	2	0.07	187	00.19	5	00.60	0.06	40	00.05	5	00.12
	4	0.10	718	02.96	5	06.89	0.07	28	00.16	5	00.75
	6	0.16	656	03.19	5	08.28	0.06	11	00.11	5	00.76
40	2	0.39	912	01.87	5	04.27	0.33	140	00.35	5	00.80
	4	1.35	–	–	0	–	0.35	434	26.15	5	49.41
	6	1.23	–	–	0	–	0.37	165	06.72	5	19.43
60	2	1.47	2755	17.79	5	31.03	1.15	400	02.04	5	04.34
	4	2.85	10848	01:27.39	2	12:11.83	1.26	3930	06:06.37	4	16:17.40
	6	4.22	–	–	0	–	1.48	2307	02:14.70	5	09:14.83
80	2	6.83	8371	05:14.50	5	07:09.84	3.07	1289	11.32	5	18.97
	4	13.67	14111	02:36.59	3	18:00.77	2.96	–	–	0	–
	6	16.69	–	–	0	–	3.90	3190	03:50.63	1	25:50.09
100	2	11.60	9274	11:54.36	3	14:34.47	4.08	1306	11.80	5	19.52
	4	25.09	–	–	0	–	4.24	–	–	0	–
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- Future research could explore the development of a primal heuristic for the κ-FSP and prove the result of equivalence for $\kappa < |\mathcal{R}|$