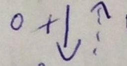
Network Flows

LP - solvable

Combinatorial alg. augmenting paths, considering residual network



need for residual network

$O(VE^2)$ with BFS

Correctness proof for combinatorial alg.

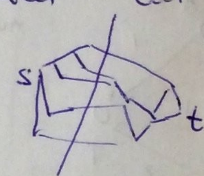
$\overset{s-t}{\text{Max Flow}} = \overset{s-t}{\text{Min CUT}}$

s-t cut is a partition of the vertices

$s \in V_1, \ t \in V_2$

$V_1 \cap V_2 = \emptyset$

$V_1 \cup V_2 = V$



capacity of the cut

$$\sum_{\substack{e=(x,y) \\ x \in V_1, y \in V_2}} c(e)$$

forward direction

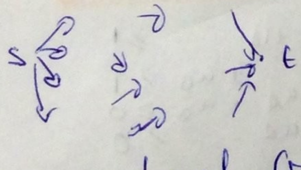
max flow ≤ min cut

max flow ≤ any cut, including min cut

amount of flow from s to t goes through a cut, bounds possible flow

min cut ≤ max flow

① assume alg. terminates (e.g. BFS, integer weights, flow addition bounded by the sum of all capacities) ↳ must terminate



⇒ cannot get from s to t, otherwise an augmenting path exists [redacted]

⇒ ∃ a cut, saturated with algorithm's flow

⇒ Max flow $\overset{\text{definition}}{\geq}$ algorithm's flow $\underset{\text{or } =}{\geq}$ algorithm's cut $\geq$ min cut

Max Flow $\geq$ Alg's flow $\geq$ Alg's cut $\geq$ min cut $\geq$ max flow

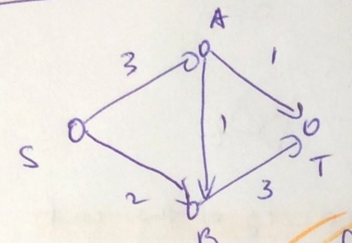
$\Rightarrow$ must be all equalities

$\Rightarrow$ max flow = min cut
and algorithm returns it, is correct

note on cut construction

start with S, keep including nodes into V_1, where flow can still be pushed from V_1.

Duality (min into max problems, and vice versa)



max flow formulation

$$\max f_{SA} + f_{SB}$$

$$\begin{array}{lllllll}
f_{SA} & & & & & \leq & 3 \\
& f_{SB} & & & & \leq & 2 \\
& & f_{AB} & & & \leq & 1 \\
& & & f_{AT} & & \leq & 1 \\
& & & & f_{BT} & \leq & 3 \\
f_{SA} & & - f_{AB} & - f_{AT} & & = & 0 \\
& f_{SB} & + f_{AB} & & - f_{BT} & = & 0
\end{array}$$

$f \geq 0$

min cut

$y_e = 1$ if crosses cut
0 otherwise

$u_A = 1$ if A is on the set with S for the cut
0 otherwise

$$\min 3y_{SA} + 2y_{SB} + y_{AB} + y_{AT} + 3y_{BT}$$

$$\begin{array}{llllllll}
y_{SA} & & & & & + u_A & & \geq 1 \\
& y_{SB} & & & & & + u_B & \geq 1 \\
& & y_{AB} & & & - u_A & + u_B & \geq 0 \\
& & & y_{AT} & & - u_A & & \geq 0 \\
& & & & y_{BT} & & - u_B & \geq 0
\end{array}$$

$y \geq 0$

max

	1	1	0	0	0	
	1	0	0	0	0	≤ 3
	0	1	0	0	0	≤ 2
	0	0	1	0	0	≤ 1
	0	0	0	1	0	≤ 1
	0	0	0	0	1	≤ 3
	1	0	−1	−1	0	= 0
	0	1	1	0	−1	= 0
	≥	≥	≥	≥	≥	

getting a just dual is a mechanical process

→ get a differently conceptual problem formulation
→ an LP has a dual

transposes
(other [redacted] rearranged)

min

3	2	1	1	3	0	0	
1	0	0	0	0	1	0	≥ 1
0	1	0	0	0	0	1	≥ 1
0	0	1	0	0	−1	1	≥ 0
0	0	0	1	0	−1	0	≥ 0
0	0	0	0	1	0	−1	≥ 0
≥	≥	≥	≥	≥	unr	unr	

Dual formulation

- transpose coeff. matrix
- max ⟺ min
- interchange obj. function (outer rearrange)
- non-neg. var for inequality, unrestr. variable for equality

2-Player Matrix Games (zero sum)

column

$$\begin{array}{cc} & \begin{array}{ccc} R & P & S \end{array} \\ \text{row} \begin{array}{c} R \\ P \\ S \end{array} & \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \end{array}$$

other games

$$\begin{bmatrix} 3 & -2 & 7 \\ 4 & -1 & 2 \end{bmatrix}$$

a strategy is a prob. distribution over choices

eg.
$$\begin{array}{ccc} R & P & S \\ \end{array}$$
$$\begin{bmatrix} 1/3 & 1/3 & 1/3 \end{bmatrix}$$

opt. strategy: no matter what strategy the opponent has, achieve optimal outcome

best opponent assumption { the opponent knows strategy → assume the opponent knows strategy and chooses optimal counter-strategy

Game

G_{ij} = $m \times n$ matrix

· strategy for row player

$(x_1, \ldots, x_m)$

strategy column player

$(y_1, \ldots, y_n)$

Expected payoff

$$\sum_{i,j} G_{ij} x_i y_j$$

ex row player

$$\begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$$

similarly $-x_1 + x_2$

if column player picks its ~~exp~~ loss to row player is

$3x_1$ [redacted] $-2x_2$

column player [redacted] [redacted] picks column minimizing loss

ex. column player

$$\begin{array}{cc} y_1 & y_2 \end{array}$$
$$\begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$$

transposes, duals

column player

min w

$w \geq 3y_1 - y_2$

$w \geq -2y_1 + y_2$

$y_1 + y_2 = 1$

$y_1, y_2 \geq 0$

$\boxed{\text{minimize exp. loss}}$

optimal player column strategy [redacted]

optimal row player strategy

row player

max z

$z \leq 3x_1 - 2x_2$

$z \leq -x_1 + x_2$

column player guarantees that expected win is $\leq$ ~~than~~ the smaller of two [redacted] it can have a strategy to always pick one or the other column whichever minimizes row player exp. winning

$x_1, x_2 \geq 0$

$x_1 + x_2 = 1$

$\boxed{\text{maximize exp. win}}$

duals => same solution

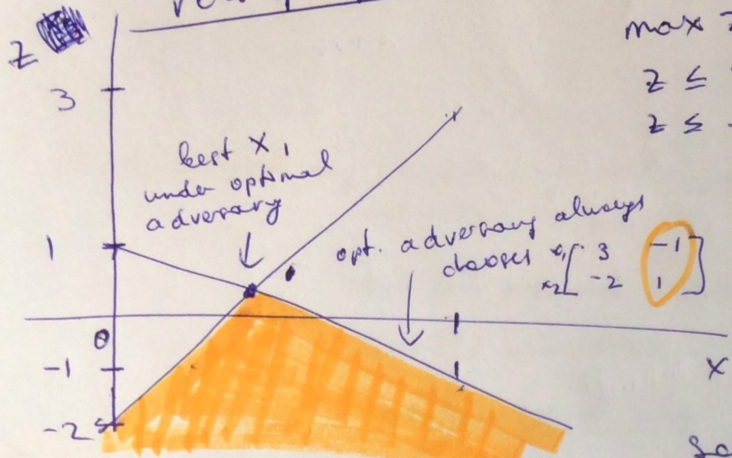
value of the game!

$$\max_x \min_y \sum x_i y_j G_{ij} = \min_y \max_x \sum x_i y_j G_{ij}$$

[redacted]

order does not matter

row player



max z

$z \le 3x_1 - 2x_2$

$z \le -x_1 + x_2$

$x_1 + x_2 = 1$

$x_1, x_2 \ge 0$

solve game

$3x_1 - 2x_2 = -x_1 + x_2$

$4x_1 = 3x_2$

$x_1 = \frac{3}{7} \quad x_2 = \frac{4}{7}$

value: $-\frac{3}{7} + \frac{4}{7} = \frac{1}{7}$

column player

$3y_1 - y_2 = -2y_1 + y_2$

$5y_1 = 2y_2$

$y_1 = \frac{2}{7} \quad y_2 = \frac{5}{7}$

value: $\frac{1}{7}$ ✓

duals:
player perspectives