6.046

Lecture 3

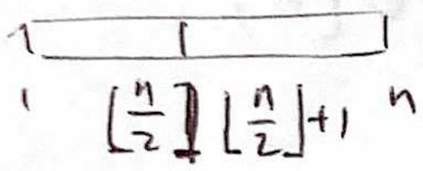
DC, Strassen, Fibonacci, Polynomial Multiplication

Divide and conquer

1) divide the problem (instance) into ≥ 1 subproblems
2) conquer each subproblem recursively
3) combine solution

Merge sort:

1) divide: trivial
2) conquer: recursively sort each subarray
3) combine: linear time merge



Running time: $\lfloor\ \rfloor\ \lceil\ \rceil$ don't matter

$$T(n) = 2T\left(\frac{n}{2}\right) + \Theta(n)$$

size of subproblem — divide & conquer time

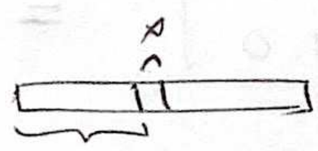
case 2 $(k = 0)$

$$T(n) = \Theta(n \lg n)$$

Binary search:

find x in sorted array

1) divide: compare x with middle
2) conquer: recurse in <u>one</u> subarray
3) combine: trivial



$$T(n) = 1T\left(\frac{n}{2}\right) + \Theta(1)$$

$$n^{\log_2 1} = n^0 = 1$$

case 2 $k=0$

$$\Theta(1) = \Theta(n^0 \lg^0 n) \Rightarrow T(n) = \Theta(\lg n)$$

Powering a number:

given number x

integer $n \geq 0$, compute x^n

Naive alg: $\underbrace{x \cdot x \cdot x \cdots}_{n} = x^n$ $\quad \Theta(n)$

divide and conquer

$$x^n = \begin{cases} x^{n/2} \cdot x^{n/2} & \text{if } n \text{ even} \\ x^{\frac{n-1}{2}} \cdot x^{n-1/2} \cdot x & \text{if } n \text{ is odd} \end{cases}$$

$$T(n) = T\left(\frac{n}{2}\right) + \Theta(1)$$

case 2 $k=0$

$$T(n) = \Theta(\lg n)$$

My comment: can also take powers of two representation of n

$\Rightarrow \leq \lg n$ addition of powers of 2; only the largest power of two needs to be computed and intermediate values stored.

Fibonacci numbers

$$F_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F_{n-1} + F_{n-2} & \text{if } n \geq 2 \end{cases}$$

Naive recursive:

time $\sim (\Phi^n)$ $\quad \Phi = \frac{1+\sqrt{5}}{2} > 1$

Bottom-up algorithm:

compute $F_0, F_1, F_2 \ldots F_n$ time $\Theta(n)$

Naive recursive squaring

$F_n = \Phi^n / \sqrt{5}$ rounded to nearest integer $\to F_n$

floating point — not allowed

Recursive squaring

Thm: [redacted] $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$

implies $\Theta(\lg n)$ time

My comment

$$\begin{vmatrix} (1-\lambda) & 1 \\ 1 & (0-\lambda) \end{vmatrix} = 0$$

$$-\lambda(1-\lambda) - 1 = 0$$

$$\lambda = \frac{1 \pm \sqrt{1+4}}{2}$$

$$\lambda_1 = 1.618\ldots$$

$$\lambda_2 = -0.618\ldots$$

floats, diagonalization for squaring would be on floats not integers

Proof: by induction on n

base: $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^1 = \begin{bmatrix} F_2 & F_1 \\ F_1 & F_0 \end{bmatrix}$ ✓

step: $\begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} = \underbrace{\begin{bmatrix} F_n & F_{n-1} \\ F_{n-1} & F_{n-2} \end{bmatrix}}_{= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-1} \text{ by IH}} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{n-1} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n$$

My comment

not considered the growth of binary representation of F_n

Matrix multiplication

lecture 3

Input: $A = [a_{ij}]$ $B = [b_{ij}]$

Output: $C = [c_{ij}] = A \cdot B$

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Standard alg. $\Theta(n^3)$

for $i \leftarrow 1$ to n
do for $j \leftarrow 1$ to n
do for $k \leftarrow 1$ to n
do $c_{ij} \leftarrow a_{ij} + a_{ik} b_{kj}$

$c_{ij} \leftarrow 0$

Divide and conquer alg:

Idea: $n \times n$ matrix
$= 2 \times 2$ block matrix of $\frac{n}{2} \times \frac{n}{2}$ submatrices

$$\underset{C}{\left[\begin{array}{c|c} r & s \\ \hline t & u \end{array}\right]} = \underset{A}{\left[\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right]} \cdot \underset{B}{\left[\begin{array}{c|c} e & f \\ \hline g & h \end{array}\right]}$$

$r = ae + bg$
$s = af + bh$
$t = ce + dg$
$u = cf + dh$

8 recursive multiplications of $\frac{n}{2} \times \frac{n}{2}$ matrices
+ 4 additions

$T(n) = 8T(\frac{n}{2}) + \Theta(n^2)$ $\Theta(n^2)$
$= \Theta(n^3)$ case 1, not better

Strassen's algorithm

Idea: reduce # of multiplications $\to 7$

$P_1 = a \cdot (f - h)$
$P_2 = (a+b) \cdot h$
$P_3 = (c+d) \cdot e$
$P_4 = d(g - e)$
$P_5 = (a+d)(e+h)$
$P_6 = [redacted] (b-d)(g+h)$
$P_7 = (a-c)(e+f)$

$r = P_5 + P_4 - P_2 + P_6$
$s = P_1 + P_2$
$t = P_3 + P_4$
$u = P_3 + P_1 - P_3 - P_7$

check u

$u = (\cancel{ae} + \cancel{ah} + \cancel{de} + dh) + (\cancel{af} - \cancel{ah}) - (\cancel{ae} + \cancel{de})$
$- (\cancel{ae} + \cancel{af} - \cancel{ce} - cf) = dh + cf$

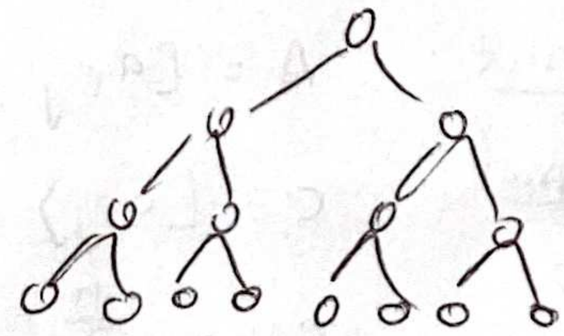
Strassen

1) divide A, B
[redacted] compute terms for products $\Theta(n^2)$
2) conquer recursively computing $P_1 \dots P_7$
3) combine r, s, t, u $\Theta(n^2)$

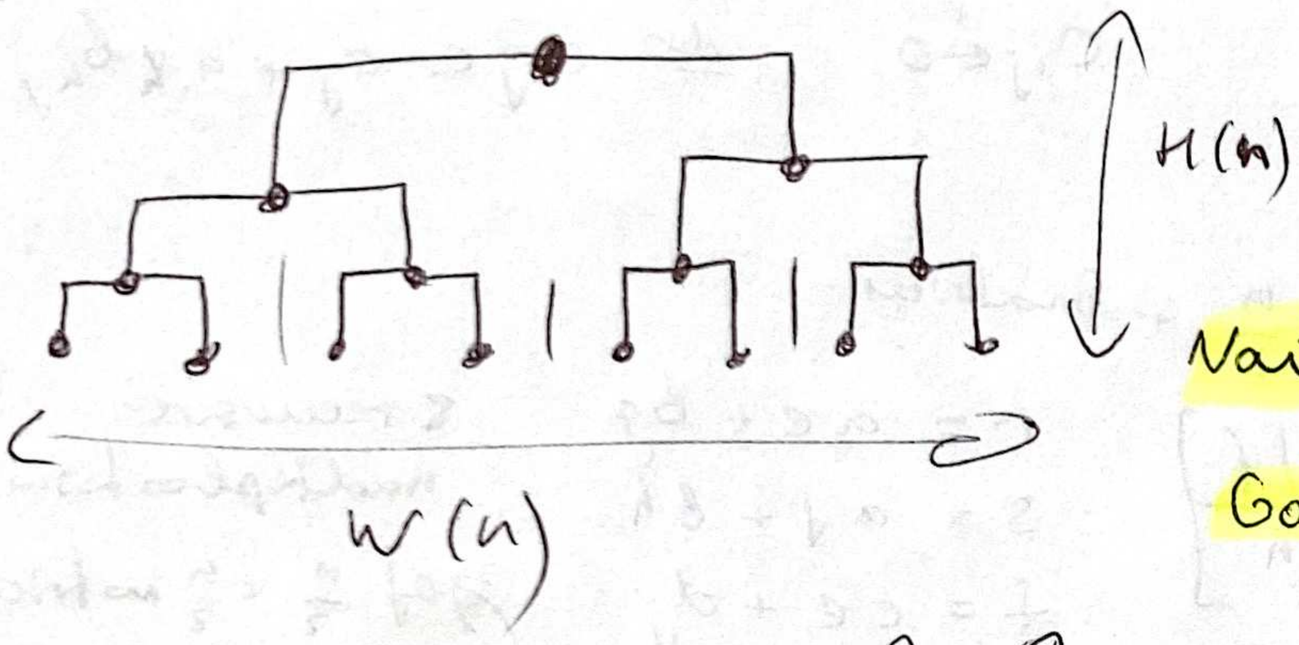
$T(n) = 7T(\frac{n}{2}) + \Theta(n^2) = \Theta(n^{\lg 7}) = \Theta(n^{2.81})$ $n \geq 32$
get improvements

VLSI layout (Very Large Scale Integration)

Problem: Embed a complete binary tree on n leaves in a grid with minimum area (bounding box)
constraints: 1) orthogonal edges,
2) no crossing wires.



Naive embedding



one subtree have to add 1

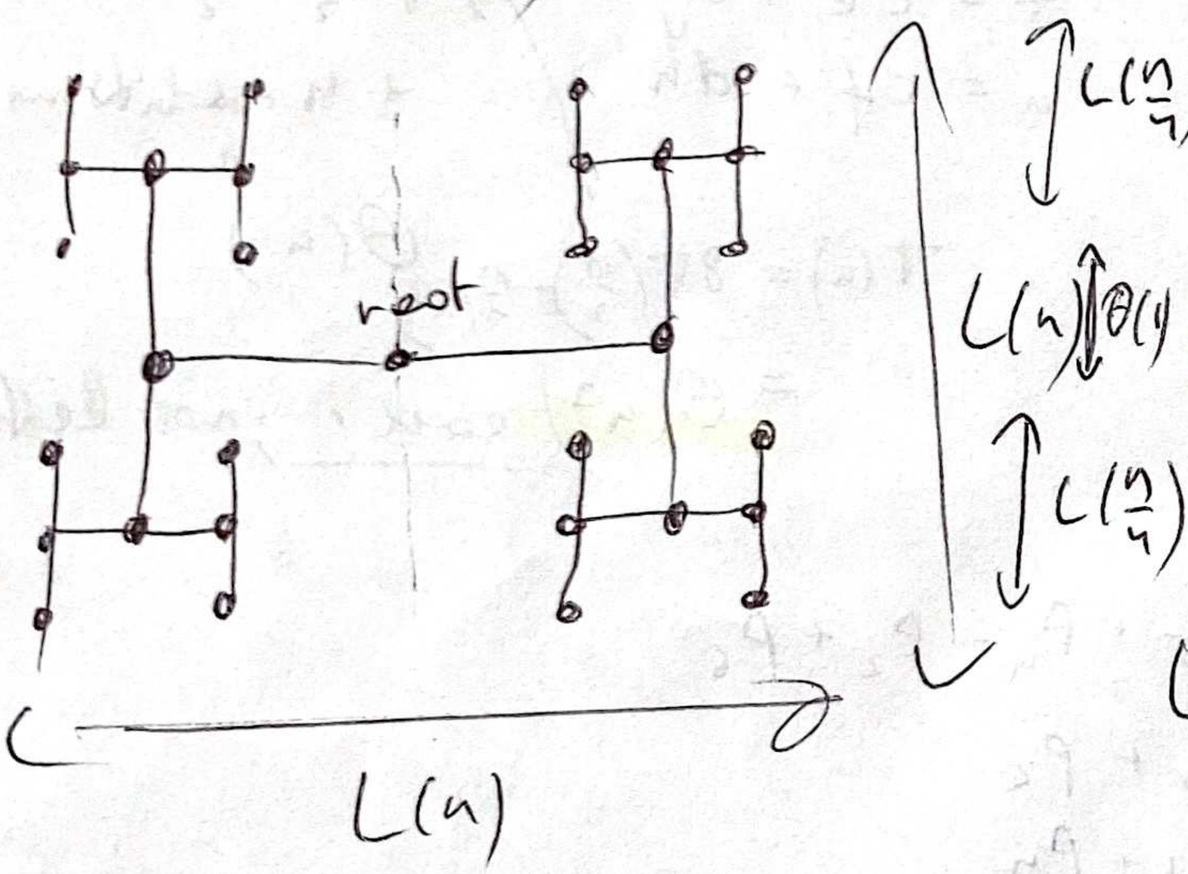
$$H(n) = H\left(\frac{n}{2}\right) + \Theta(1) = \Theta(\lg n)$$

$$W(n) = 2W\left(\frac{n}{2}\right) + \Theta(1) = \Theta(n)$$

do not have to add 1

Naive Area $= \Theta(n \lg n)$

Goal: $W(n) = \Theta(\sqrt{n})$
$H(n) = \Theta(\sqrt{n})$
$\Rightarrow$ Area $= \Theta(n)$



e.g. $\log_4 2 = \frac{1}{2}$, then $n^{\log_4 2} = \sqrt{n}$, other a, b possible

$$T(n) = 2T\left(\frac{n}{4}\right) + O(n^{1/2 - \varepsilon})$$

need care!

recurrence provides a formulation of design constraints.

$$L(n) = 2L\left(\frac{n}{4}\right) + \Theta(1) = \Theta(\sqrt{n})$$

care!

from a recurrence [redacted] to a design [redacted] that is scalable