6.046 Universal Hashing, Perfect Hashing

Lecture 8

Weakness of hashing

For any choice of hash function
∃ a bad set of keys that all hash to same slot. ↪ potential vulnerability against an adversary

Idea: choose hash function at random, independently from keys

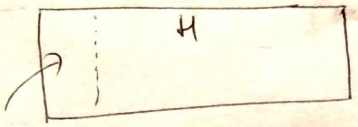
Universal hashing

Def. Let U be a universe of keys, and let H be a finite collection of hash functions mapping U to $\{0, 1, \ldots, m-1\}$

H is universal if $\forall x, y \in U$, where $x \neq y$,

$$|\{h \in H : h(x) = h(y)\}| = \frac{|H|}{m}$$

i.e. if h is chosen randomly from H, the prob. of collision between x and y is $1/m$



$\{h : h(x) = h(y)\}$ ← this subset of H would be different for each x, y distinct pair

$\frac{1}{m}|H|$

Thm: Choose h randomly from H. Suppose hashing n keys into m slots in table T. Then, for a given key x,

$$E[\#\text{ collisions with } x] < \frac{n}{m}$$

Pf. Let C_x be r.v. denoting total # collisions of keys in T with x,

let $c_{xy} = \begin{cases} 1 & \text{if } h(x) = h(y) \\ 0 & \text{otherwise} \end{cases}$

$E[c_{xy}] = 1/m$

$C_x = \sum_{y \in T - \{x\}} c_{xy}$

$$E[C_x] = E\left[\sum_{y \in T-\{x\}} c_{xy}\right]$$

$$= \sum_{y \in T-\{x\}} E[c_{xy}] = \sum_{y \in T-\{x\}} \frac{1}{m}$$

$$= \frac{n-1}{m} < \frac{n}{m}$$

↙ exp. guarantee against adversary

Constructing a universal hash function

Let m be prime. Decompose key k into $r+1$ digits:

$k = \langle k_0, k_1, \ldots, k_r \rangle$, where $0 \leq k_i \leq m-1$

↖ Representation of k base m

base $m, m^2, \ldots, m^{r+1}$ then use mod

- pick $a = \langle a_0, a_1, \ldots, a_r \rangle$, each a_i is chosen randomly from $\{0, 1, \ldots, m-1\}$
- Define $h_a(k) = \left(\sum_{i=0}^{r} a_i k_i\right) \bmod m$

↖ dot product a and k, then take mod m

How big is H?

$|H| = m^{r+1}$ ← # of all a

Thm: H is universal

↙ base m representations

Pf. Let $x = \langle x_0, x_1, \ldots, x_r \rangle$

$y = \langle y_0, y_1, \ldots, y_r \rangle$ be distinct keys

$\Rightarrow$ they differ in at least one digit,

wlog position 0.

For how many $h_a \in H$ do x and y collide?

Must have $h_a(x) = h_a(y)$

$$\Rightarrow \sum_{i=0}^{r} a_i x_i \equiv \sum_{i=0}^{r} a_i y_i \pmod m$$

↑ congruent

$$\Rightarrow \sum_{i=0}^{r} a_i (x_i - y_i) \equiv 0 \pmod m$$

$$\Rightarrow a_0(x_0 - y_0) + \sum_{i=1}^{r} a_i (x_i - y_i) = 0 \bmod (m)$$

$$\Rightarrow a_0(x_0 - y_0) \equiv -\sum_{i=1}^{r} a_i (x_i - y_i) \bmod (m)$$

Number theory fact:

Let m be prime. For any $z \in \mathbb{Z}_m$ (integers mod m)

s.t. $z \neq 0$, $\exists$ unique $z^{-1} \in \mathbb{Z}_m$ s.t. $z \cdot z^{-1} \equiv 1 \pmod m$.

$z \equiv -5 \bmod 7$

Ex $m = 7$

z	1	2	3	4	5	6
z^{-1}	1	4	5	2	3	6

↖ not true if m is not prime since any $z \in \mathbb{Z}_m$ is relatively prime to m

$a \pmod b$

↖ if not relatively prime a does not have an inverse mod b.

6.046
Lecture 8

Since $x_0 \neq y_0$, $\exists\ (x_0 - y_0)^{-1}$

$$\Rightarrow a_0 \equiv \left(-\sum_{i=0}^{r} a_i (x_i - y_i)\right) \cdot (x_0 - y_0)^{-1} \pmod m$$

- if x, y hash to the same place (assumed) then a_0 has a particular value as a function of other a_i
- Thus, for any choice of $a_1, a_2 \ldots a_r$ exactly 1 of m choices of a_0 that causes x and y to collide, and no collision for other $m-1$ choices for a_0.

$\Rightarrow$ # h_a's that cause x, y to collide $= \underset{a_1}{m} \cdot \underset{a_2}{m} \cdots \underset{a_r}{m} \cdot \underset{a_0}{1}$

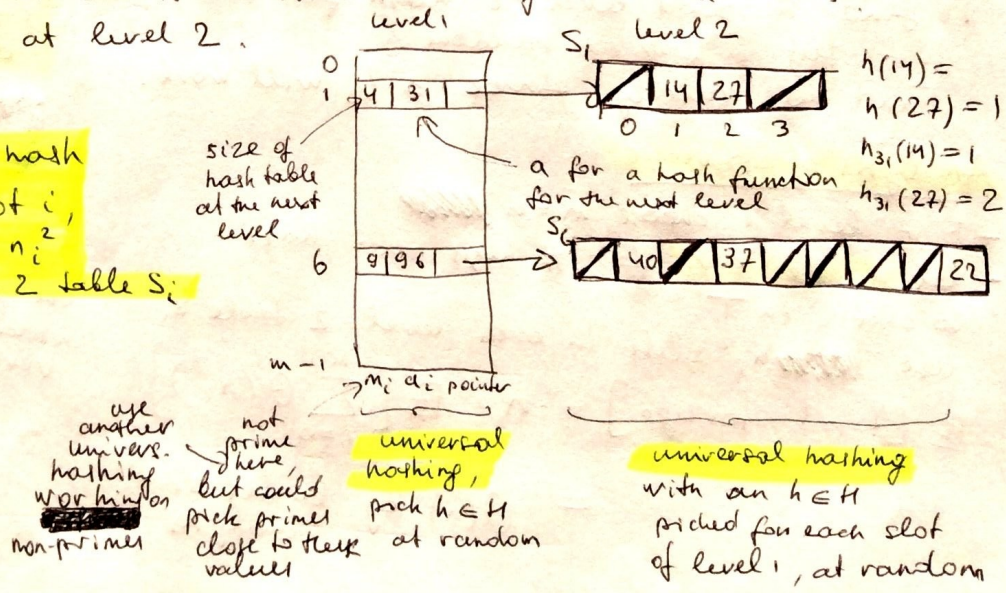
$$= m^r = \frac{|H|}{m} \checkmark$$

Perfect hashing (given a fixed set of keys, built a static table with good worst case search time) $\rightarrow$ not in expectation

Given n keys, construct a static hash table of size $m = O(n)$, s.t. search takes $O(1)$ time in the worst case.

Two-level scheme with universal hashing at both levels. No collisions at level 2.

If n_i items hash to level 1 slot i, then use $m_i = n_i^2$ slots in level 2 table S_i



use another univers. hashing worst-case non-primes

not prime here but could pick primes close to these values

universal hashing, pick $h \in H$ at random

universal hashing with an $h \in H$ picked for each slot of level 1, at random

Level 2 analysis:

Thm: Hash n keys into $m = n^2$ slots, using random h in universal $H \Rightarrow E[\#\text{collisions}] < \frac{1}{2}$

Pf: Prob. 2 given keys collide under h is $\frac{1}{m} = \frac{1}{n^2}$

$\binom{n}{2}$ pairs of keys

$$E[\#\text{collisions}] = \binom{n}{2}\frac{1}{n^2} = \frac{n(n-1)}{2}\frac{1}{n^2} = \frac{n^2}{2n^2} - \frac{n}{2n^2} =$$

$$= \frac{1}{2} - \frac{1}{2n} < \frac{1}{2} \checkmark$$

Markov ineq:

For r.v. $X \geq 0$, $Pr\{X \geq t\} \leq \frac{E(X)}{t}$

Pf. $E[X] = \sum_{x=0}^{\infty} x \, Pr\{X = x\} \geq \sum_{x=t}^{\infty} x \, Pr\{X = x\}$

↖ throw away lower terms

$$\geq \sum_{x=t}^{\infty} t \, Pr\{X = x\} = t \, Pr\{X \geq t\} \checkmark$$

Corollary

$Pr\{\text{no collisions}\} \geq \frac{1}{2}$

Pf: $Pr\{\geq 1 \text{ collision}\} \leq \frac{E[\#\text{collisions}]}{1}$

$< \frac{1}{2} \checkmark$

To find a good level-2 hash function, just try a few at random. Find one quickly, since $\geq 1/2$ will work.

} randomized construction

[redacted] check a fixed # of hash functions for each slot, s.t. the prob. to find one without collisions is very high

Analysis of storage

- For level 1, choose $m = n$ [redacted]
- Let n_i be r.v. for # keys that hash to slot i in T.
- Use [redacted] n_i^2 slots in each level-2 table S_i

$$E[\text{total storage}] = n + E\left[\sum_{i=0}^{m-1} \Theta(n_i^2)\right]$$

$= \Theta(n)$ by bucket sort analysis