Lecture 6   Order Statistics, Median   ①

[redacted] 6.046

given n elements in array
find kth smallest element (elt. of rank k)

Naive algorithm : sort A, return A[k] $\Theta(n \lg n)$

k=1 : minimum
h=n : maximum
} easy in $\Theta(n)$

$h = \lfloor \frac{n+1}{2} \rfloor$ or $\lceil \frac{n+1}{2} \rceil$: median ← harder

Randomized divide & conquer :

```
Rand-Select (A, p, q, i) :    // ith smallest in A[p..q]
    if p = q
        then return A[p]
    r ← Rand-Partition (A, p, q)
    k ← r - p + 1    // k = rank (A[r]) in A[p..q]
                     ↳ # of elts ≤ A[r] including A[r]
    if i = k
        then return A[r]
    if i < k
        then return Rand-Select (A, p, r-1, i)   ← left part recursion, rank remains the same
    else [redacted] return Rand-Select (A, r+1, q, i-k)   ← right part recursion, → rank is off set
```

Rand-Partition
as subroutine of randomized quicksort :
- pick a random elt x
- swap with the first
- partition s.t.

A = $\overset{p}{[}\ \le x \ \overset{r}{\boxed{x}}\ \ge x\ \overset{q}{]}$ A[r] = x

- return index of x

A = [p .... r .... q]

if i<k   if i>k

Rand-Select (A, p, r-1, i)   Rand-Select (A, r+1, q, i-k)

Ex: i = 7

A = | 6 | 10 | 13 | 5 | 8 | 3 | 2 | 11 |

↑ pivot

| 2 | 5 | 3 | 6 | 8 | 13 | 10 | 11 |

↑p   ↑r=4   ↑q

k=4   rank 7-4=3

Intuition for analysis:

(assume elts are distinct)

lucky case : $1/10 : 9/10$ $\quad T(n) \leq T(\frac{9}{10}n) + \Theta(n) = \Theta(n)$ worst case

case 3 $\quad n^{\log_{10/9} 1} = n^0$

unlucky case : $0 : n-1$

$$T(n) = T(n-1) + \Theta(n) = \Theta(n^2) \text{ arithmetic series}$$

↖ much worse than sorting and picking the right element

Analysis of expected time :

- let $T(n)$ be the random variable for running time of Rand-Select on input of size n, assuming random numbers for pivots are independent
- define indicator r.v. X_k for $k = 0, 1, \ldots, n-1$

$$X_k = \begin{cases} 1 & \text{if Partition generates } k : n-k-1 \text{ split} \\ 0 & \text{otherwise} \end{cases}$$

$$T(n) \leq \begin{cases} T(\max\{0, n-1\}) + \Theta(n) & \text{if } 0 : n-1 \text{ split} \\ T(\max\{1, n-2\}) + \Theta(n) & \text{if } 1 : n-2 \text{ split} \\ \vdots \\ T(\max\{n-1, 0\}) + \Theta(n) & \text{if } n-1 : 0 \text{ split} \end{cases}$$

$$= \sum_{k=0}^{n-1} X_k \left[T(\max\{k, n-k-1\}) + \Theta(n) \right]$$

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k \left[T(\max\{k, n-k-1\}) + \Theta(n) \right] \right]$$

$$= [redacted] \sum_{k=0}^{n-1} E\left[\underbrace{X_k}_{\text{a random choice of a split (pivot)}} \underbrace{\left[T(\max\{k, n-k-1\}) + \Theta(n) \right]}_{\text{random choices in the recursive call}} \right]$$

random choices in the recursive call [redacted] are independent as the random generator generates [redacted] successive coinflips

$$= \sum_{k=0}^{n-1} \underbrace{E[X_k]}_{1/n} E\left[T(\max\{k, n-k-1\}) + \Theta(n) \right]$$

$$= \frac{1}{n} \sum_{k=0}^{n-1} E\left[T(\max\{k, n-k-1\}) \right] + \underbrace{\frac{1}{n} \sum_{k=0}^{n-1} \Theta(n)}_{\Theta(n)} \leq \frac{2}{n} \sum_{k=\lfloor \frac{n}{2} \rfloor}^{n-1} E[T(k)] + \Theta(n)$$

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Claim: $E[T(n)] \le cn$ for suff. large const. $c > 0$

Proof: Substitution method

Assume true for $< n$ ⇐ IH

$$E[T(n)] \le \frac{2}{n} \sum_{k=\lfloor n/2 \rfloor}^{n-1} E[T(k)] + \Theta(n)$$

↑ each $k < n$, apply IH

$$\le \frac{2}{n} \sum_{k=\lfloor n/2 \rfloor}^{n-1} ck + \Theta(n) \quad \text{By IH}$$

$$= \frac{2c}{n} \underbrace{\sum_{k=\lfloor n/2 \rfloor}^{n-1} k}_{\le \frac{3}{8} n^2} + \Theta(n)$$

fact for series.. can prove by induction

$$= \underbrace{c \cdot n}_{\text{desired}} - \underbrace{\left(\frac{1}{4} cn - \Theta(n)\right)}_{\text{residual}}$$

positive for c sufficiently large

[redacted] choose $c \ge 4 \cdot$ constant in $\Theta(n)$

my comment: look at [redacted] c necessary for small n [redacted]

choose the [redacted] largest

Rand-Select

$T(n) = \Theta(n)$ in expectation

$\Theta(n^2)$ worst case

worst-case linear-time order statistics

(Blum, Floyd, Pratt, Rivest, Tarjan 1973)

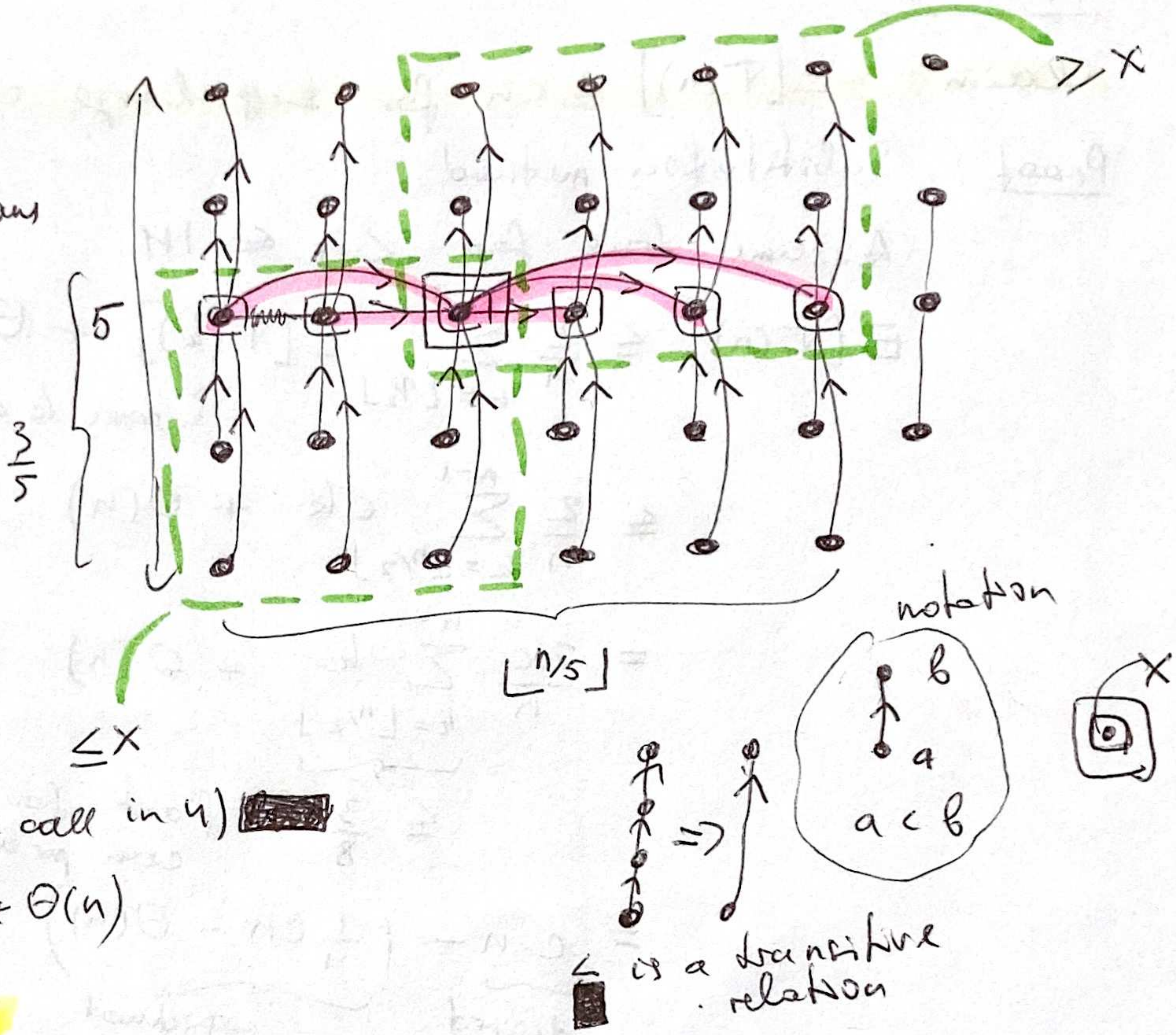
- idea: generate good pivot recursively

```
Select(i, n):
  1) Divide the n elements into ⌊n/5⌋ groups of 5 elts each
     [redacted] Find the median of each group → $\Theta(n)$
  2) Recursively Select the median x of the ⌊n/5⌋ group medians → $T(\frac{n}{5})$
  3) Partition with x as pivot                     } $\Theta(n)$
     Let $k = rank(x)$   ↖ median of medians        }
  4) if $i = k$ then return x
     if $i < k$ then recursively Select ith smallest elts in the lower part of the array
     else ———— $(i-k)$th smallest elt ---- upper part of the array      $T(\frac{3}{4}n)$
```

as Rand-Select

$T(n) \le T(n/5) + T(\frac{3}{4}n) + \Theta(n)$

my comment: add up to less than 1, need it for $\Theta(n)$ [redacted] [redacted] (see case 3) not directly applicable, because of different sizes



$\geq 3\lfloor\lfloor n/5\rfloor/2\rfloor$ elts $\leq x$

$\geq x$

$\geq \underbrace{\lfloor\lfloor n/5\rfloor/2\rfloor}_{\lfloor n/10\rfloor}$ group medians $\leq x$

$\Rightarrow \geq 3\lfloor n/10\rfloor$ elts $\leq x$

Simplification

for $n \geq 50$

$3\lfloor n/10\rfloor \geq n/4$

my comment

guarantee in the recursive call in 4) [redacted]

$T(n) \leq T(\frac{n}{5}) + T(\frac{3}{4}n) + \Theta(n)$

Claim $T(n) \leq cn$

Proof: Substitution

$T(n) \leq \frac{c}{5}n + \frac{3}{4}cn + \Theta(n)$ by IH

$= \frac{19}{20}cn + \Theta(n)$

$= cn - (\frac{1}{20}cn - \Theta(n))$

set $c \geq 20$ const in $\Theta(n)$, and check base case if need a larger c

$\leq cn$ for c sufficiently large