6.046
Lecture 1

Insertion Sort,
Mergesort

Analysis of Algorithms:

theoretical study of computer program performance and resource usage.

What's more important than performance?

- correctness, simplicity, maintainability, modularity, security, ... user friendliness

why study algs [redacted] performance

- feasible vs. infeasible
- currency to pay for other important [redacted] properties
- bottom of the heap

Sorting Problem

Input: sequence $\langle a_1, a_2 \ldots a_n \rangle$ of numbers

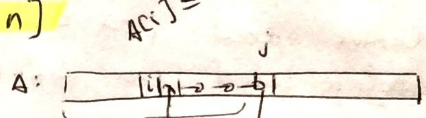
Output: permutation $\langle a_1', a_2' \ldots a_n' \rangle$

s.t. ($) $a_1' \leq a_2' \leq \ldots \leq a_n'$

Insertion Sort (A, n) // sorts A[1 ... n]

```
for j ← 2 to n
    do key ← A[j]
       i ← j-1
       while i > 0 and A[i] > key
           do A[i+1] ← A[i]
              i ← i-1
       A[i+1] ← key
```

$A[i] \leq$ key



Ex:

8 2 4 9 3 6

2 8 4 9 3 6

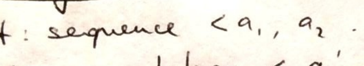
2 4 8 9 3 6

2 4 8 9 3 6

2 3 4 8 9 6

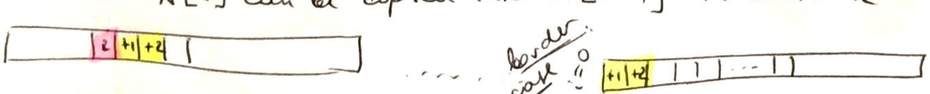
2 3 4 6 8 9

after completing a [redacted] while loop iteration?



A[i+1] = A[i+2]

- so key can be inserted into A[i+1] if $A[i] \leq k$
- or A[i] can be copied into A[i+1] to continue



border case ?!

Running time

- depends on input (already sorted, or reverse-sorted)
- depends on input size (6 elem. vs. 6×10^9)
  - parameterize in input size
- want upper bounds
  - represents a guarantee to user

Kinds of analysis

Worst-case (usually):

$T(n)$ = max time on any input of size n. max turns relation to function

Average case (sometimes):

$T(n)$ = expected time over all inputs of size n

(Need an assumption of statistical distribution of inputs)

Best case: (bogus)

What is insertion sort's worst-case time?

Depends on computer
- relative speed (on same machine)
- absolute speed (on diff. machines)

Big Idea: Asymptotic analysis

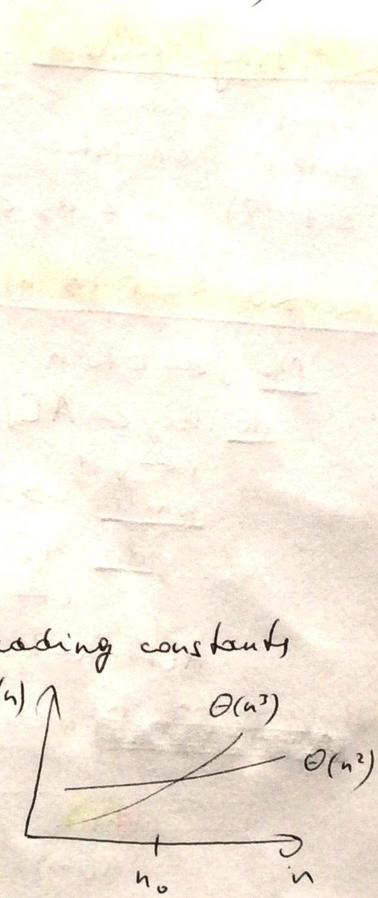
- ignore machine-dependent constants
- look at <u>growth</u> of $T(n)$ as $n \to \infty$

Asymptotic notation

Θ-notation: Drop low-order terms, ignore leading constants

Ex: $3n^3 + 90n^2 - 5n + 6046 = \Theta(n^3)$

as $n \to \infty$, $\Theta(n^2)$ alg always beats a $\Theta(n^3)$ alg.



lecture 1

Insertion sort analysis

worst case : input reverse-sorted

$$T(n) = \sum_{j=2}^{n} \Theta(j) = \Theta(n^2)$$

arithmetic series

$$\sum_{k=1}^{m} k = \frac{m(m+1)}{2}$$

50 times

0,1 - - - 99,100

Is insertion sort fast ?
- moderately so
- not for large n

Merge sort

$T(n)$

abuse of notation → $\Theta(1)$

sloppy → $2T(n/2)$

$\Theta(n)$

Merge sort A [1 - - - n]

1. If $n = 1$ done
2. recursively sort $A[1 - - - \lceil n/2 \rceil]$ and $A[\lceil n/2 \rceil + 1 - - - n]$
3. merge 2 sorted lists

Recurrence:

usually omit const. base case

$$T(n) = \begin{cases} \Theta(1), & \text{if } n = 1 \\ 2T(\frac{n}{2}) + \Theta(n), & n > 1 \end{cases}$$

key subroutine : Merge

20	12
13	11
7	9
2	1

1 2 7 - - - -

Time $= \Theta(n)$ on n total elements

Recursion tree:

$T(n) = 2T(\frac{n}{2}) + cn, \quad c > 0$

assume n is power of 2

$$T(n) = \begin{array}{c} cn \\ / \quad \backslash \\ T(n/2) \quad T(n/2) \end{array} = \begin{array}{c} cn \\ \frac{cn}{2} \qquad \frac{cn}{2} \\ T(\frac{n}{4}) \; T(\frac{n}{4}) \; T(\frac{n}{4}) \; T(\frac{n}{4}) \end{array} = \cdots = \begin{array}{c} cn \\ \frac{cn}{2} \qquad \frac{cn}{2} \\ \frac{cn}{4} \; \frac{cn}{4} \; \frac{cn}{4} \; \frac{cn}{4} \\ \vdots \\ \Theta(1) \end{array}$$

cn

cn

$\log n$

$\Theta(n)$

leaves = n

Total: $cn \log n + \Theta(n) = \Theta(n \log n)$

optimally combine insertion sort at lower levels

on a large n merge sort is faster than insertion sort $\sim n > 30$