CS124 — Lecture 3

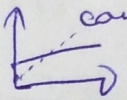
ex n^n is $O(2^n)$ NO

$\lim \frac{n^n}{2^n} \to \infty$

ex
My alg is $O(n^2)$
Your alg is $O(n^2 \log^2 n)$
So mine is faster NO
→ cont
→ not tight bound

$O()$ ≤ asymptotically (up to constant factors)

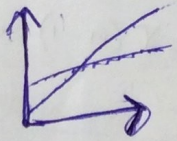
$f(n)$ is $o(g(n))$ << asymptotically
if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$
difference must increase as $n \to \infty$
cannot be a constant difference



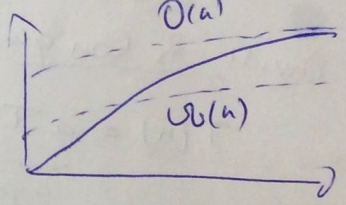
Ω ≥ asymptotically

$f(n)$ is $\Omega(g(n))$
if $\exists c>0, N>0$, s.t
$f(n) \geq c\, g(n) \quad \forall n \geq N$

ω $f(n)$ is $\omega(g(n))$ >> asymptotically
$\lim_{n\to\infty} \frac{g(n)}{f(n)} = 0$



Θ: f is $\Theta(g)$ if
f is $O(g)$ and f is $\Omega(g)$



~~NS~~ Alg runs in $\Theta(n^2)$ time
→ always in $O(n^2)$ time
→ ∃ inputs ∀ large enough n
taking $\Omega(n^2)$ time

ex If $f_1(n)$ is $O(g_1(n))$ and
$f_2(n)$ is $O(g_2(n))$ then
$f_1(n) + f_2(n)$ is $O(g_1(n) + g_2(n))$

Proof:
$\exists\, c_1, c_2, N_1, N_2$ s.t.
$f_1(n) \le c_1 g_1(n) \quad \forall n \ge N_1$
$f_2(n) \le c_2 g_2(n) \quad \forall n \ge N_2$

$c_3 = \max(c_1, c_2)$
$N_3 = \max(N_1, N_2)$

$f_1(n) + f_2(n) \le c_3 [g_1(n) + g_2(n)]$
$\forall n \ge N_3$

※

Recurrence relations

$F(n) = F(n-1) + F(n-2)$

exact $T(n) = T(n-1) + 7n - 3 \Rightarrow n^2$

$T(n) = 2T(n-1) + n - 3 \Rightarrow 2^n$

Divide & Conquer general form

$T(n) = \underbrace{a\,T\left(\frac{n}{b}\right)}_{\text{a pieces of size } \frac{n}{b}} + \underbrace{cn^k}_{\text{glue together}}$

Master Theorem

$T(n) = a\,T\left(\frac{n}{b}\right) + cn^k, \quad a \ge 1, b > 1, c, k \ge 0$

$$T(n) = \begin{cases} \textcircled{1}\ O(n^{\log_b a}) & a > b^k, \text{ lots of subproblems} \\ \textcircled{2}\ O(n^k \log n) & a = b^k, \text{ balance out} \\ & \text{levels of recursion} \\ & \Rightarrow \text{each level contributes to gluing together} \\ \textcircled{3}\ O(n^k) & a < b^k, \text{ not many subproblem,} \\ & \text{most time spent gluing together} \\ & \Rightarrow \text{first term does not matter} \end{cases}$$

$T(n) = a\,T(n/b)$ ← last term does not matter

$a^2 T(\frac{n}{b^2})$ $\quad a^{\log_b n} = n^{\log_b a}$

Mergesort

queue is another form
← stack is one form of the list

List: ordered sequence of elements (array or linked list)

$q : [x_1, x_2 \ldots x_n]$

x_1 = head

x_n = tail

$n = |q|$

$q \circ r$ = concatenation of lists

stack ops:
- head (q) = return (x_1)
- push (q, x) $\quad q := [x] \circ q$
- pop (q) $\quad q := [x_2 \ldots x_n]$, return ($x_1$)

queue ops:
- inject (q, x) $\quad q := q \circ [x]$
- eject (q) $\quad q := [x_1 \ldots x_{n-1}]$, return ($x_n$)

assume these ops can be done in 1 step

← sorted lists

```
merge (s, t)
  if s = [] return t
  else if t = [] return s
  else if s1 ≤ t1 then u := pop(s)
                  else u := pop(t)
  return push (u, merge (s, t))
```

$O(|s| + |t|)$

iterative

```
mergesort (s)
  list q = []
  for x ∈ s
      inject (q, [x])
  while size(q) ≥ 2
      u := pop(q)
      v := pop(q)
      inject (q, merge (u, v))
  if size(q) = 1 return q(1)
```

10 5 9 11 3 8 4 13 1 6 2 12 16 14 7

[10] [5] [9] ... [7]

loop invariant:
input lists are sorted
output list is sorted

[9] [11] ... [7] [5,10]

[3] [8] ... [7] [5,10] [9,11]

[5,10] [9,11] [3,8] ...

[3,8] ... [5,9,10,11]

Phase 1 - list size $1 \to 2$ $O(n)$ touch each elt once

2 $2 \to 4$ $O(n)$

3 $4 \to 8$ $O(n)$

$\log_2 n$ $O(n)$ work per phase

$O(n \log n)$

if $|s|$ is not power of 2, pad with 0s

$\Rightarrow$ each phase will add a constant additional work

$\Rightarrow$ difference between power of 2 and not is a constant factor

recursive

mergesort (s)

if size (s) = 1 return (s)

split (s, s_1, s_2)

s_1 = mergesort (s_1)

s_2 = mergesort (s_2)

merge (s_1, s_2)

correctness proof by induction

$T(n) = 2T(\frac{n}{2}) + O(n)$

$T(1) = 1$

$a = 2, b = 2, k = 1$ (constant don't matter)

$a = b^k \Rightarrow O(n \log n)$ for master theorem

more exact:

let $n = 2^k$, $T(n) = 2T(\frac{n}{2}) + n$

$a n \log_2 n + Bn + c$

when not $n = 2^k$, $T(n) = T(\lceil \frac{n}{2} \rceil) + T(\lfloor \frac{n}{2} \rfloor)$ ...

Graphs + Modeling

$G = (V, E)$ directed, undirected

vertices, edges

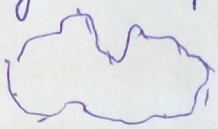
$E \subseteq V \times V$

(u,v) $u \to v$

$(u,v) \in E \Leftrightarrow (v,u) \in E$

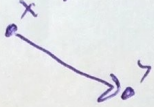
Examples

- TSP (traveling salesperson problem)



$w: E \to \mathbb{R}$
weight function

- Degrees of separation

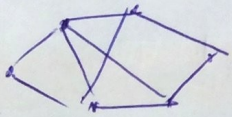


x is friends with y

longest shortest path

- Konigsberg bridge
- Coloring problems (maps, register allocation)

colors



$c: V \to \{0, 1, 2, \ldots k\}$

restriction, no two adjacent vertices can have the same color.