lecture 11

Augmenting Data Structures
Dynamic Order Statistics
Interval Trees

Dynamic order Statistics

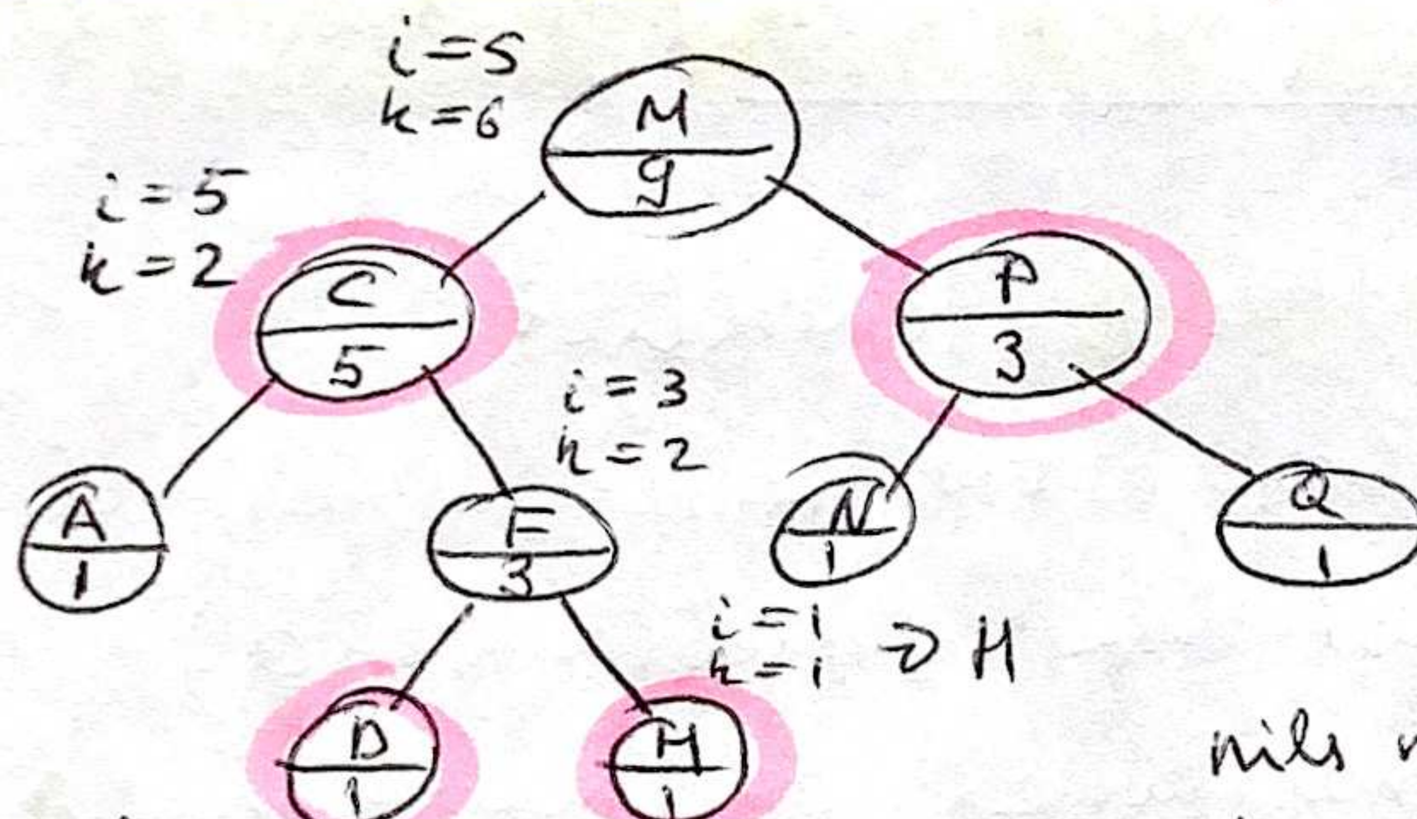
the set is dynamic, not static ⇐ inserts and deletes

OS - Select (i) - returns ith smallest item in dynamic set

OS - Rank (x) - returns rank of x in sorted order

Idea: keep subtree sizes in nodes of r-b tree.

A, C, D, F, H, M, N, P, Q



Size [x] = size [left [x]] + size [right [x]] + 1

Trick: use sentinel (dummy record) for nil s.t. size [nil] = 0

nils not shown [redacted] but are counted

OS - Select (x, i) // ith smallest in subtree rooted at x

$k \leftarrow$ size [left [x]] + 1 // k = rank (x)
if $i = k$ then return x
if $i < k$ then return OS-Select (left [x], i)
else return OS-Select (right [x], $i - k$)

my comment: building a tree = $O(n \lg n)$. given an array of numbers (static case) • Rand-select or BFPRT are $\Theta(n)$, in expectation and worst case respectively.

Ex: OS - Select (root, 5) → pointer to "H"

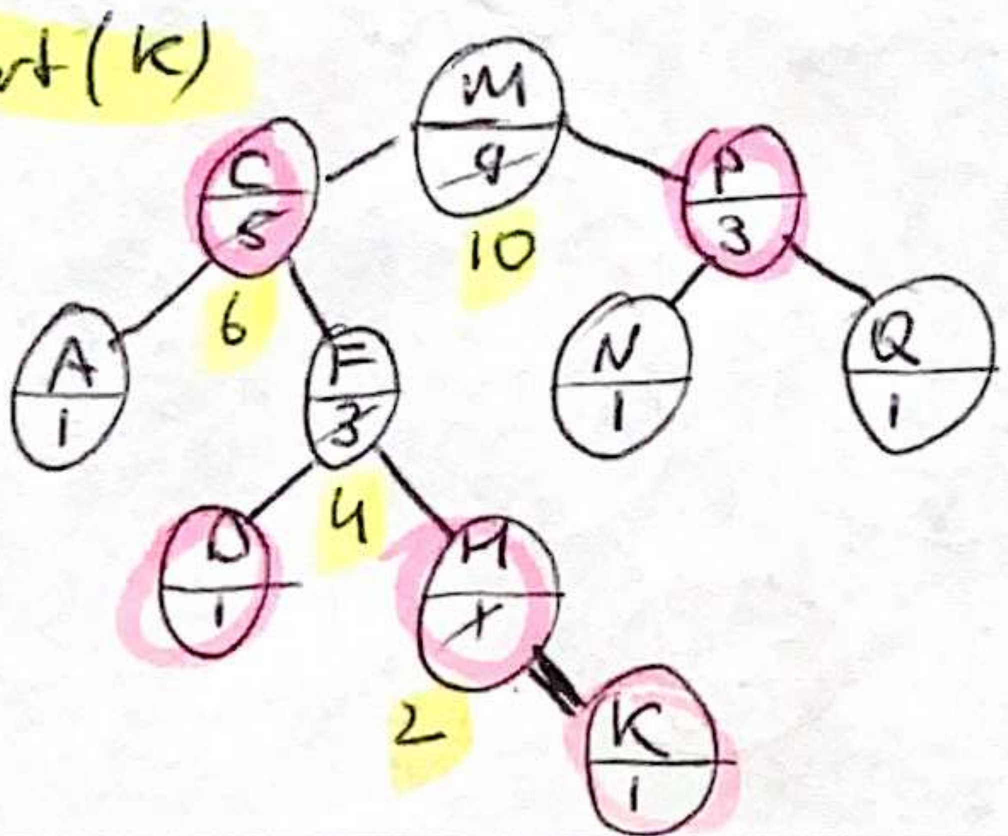
Analysis: $O(\lg n)$
OS-Rank also $O(\lg n)$

Q why not keep ranks in nodes instead of subtree sizes?

A → hard to maintain after modifications

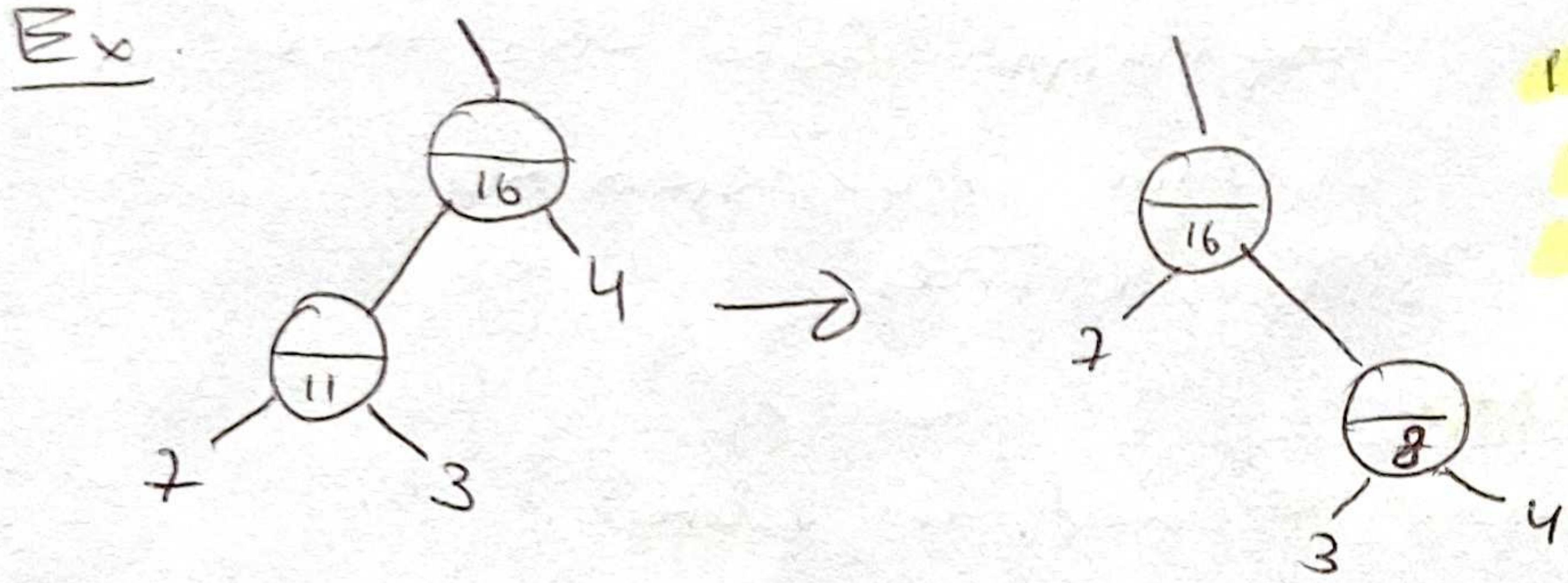
Modifying ops: Insert, Delete
strategy: Update subtree sizes when inserting or deleting

OS_insert (K)



But must handle rebalancing
- r-b color changes: no effect
- rotations, look at children nodes and fix up in $O(1)$ time

Ex.



16 → 16
size of subtree does not change after rotation

other properties (not subtree size) may change after rotation and require upward propagation.

Insert, Delete still $O(\lg n)$ time since $\Theta(1)$ rotations

Data Structure Augmentation (general checklist)

Methodology (Ex. OS trees) – checklist

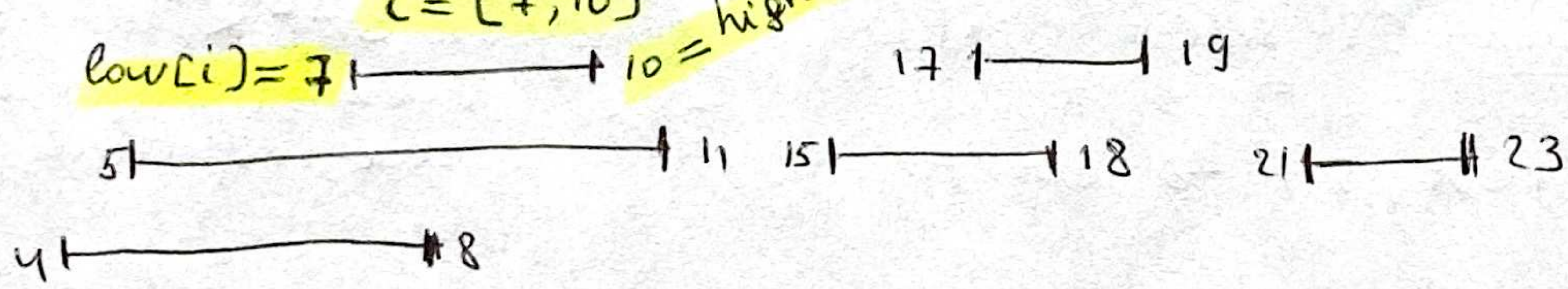
1. Choose underlying data structure (red-black tree)
2. Determine additional info (subtree size)
3. Verify info can be maintained for modifying operations (Insert, Delete, → rotations)
4. Develop new operations that use info (OS-Select, OS-Rank)

Usually, must play with interactions between steps.

Ex: Interval trees.

Maintain a set of intervals, e.g. time intervals.

$i = [7, 10]$

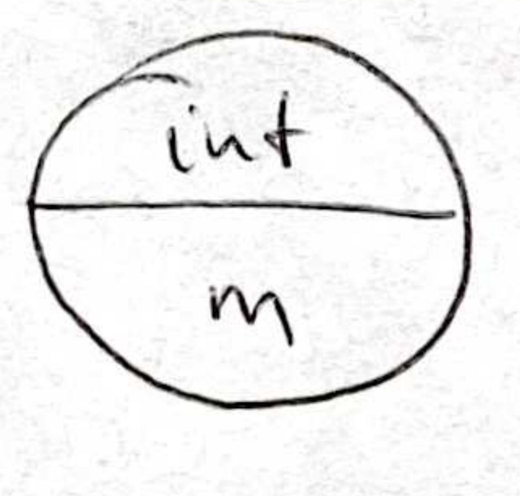


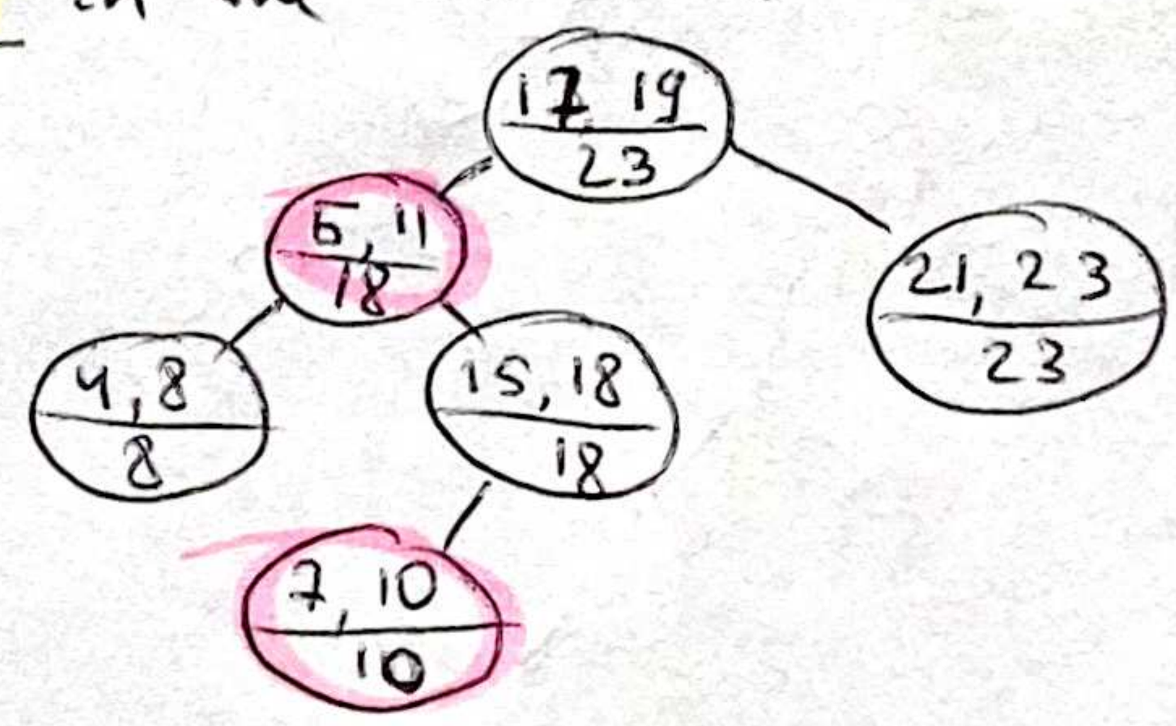
Query: Find an interval in the set that overlaps a given query interval. (have to return one interval)

Methodology:

1. red-black tree keyed on low endpoint
2. store in node the largest value m in the subtree rooted at that node.

$$m[x] = \max \begin{cases} high[int[x]] \\ m[left[x]] \\ m[right[x]] \end{cases}$$





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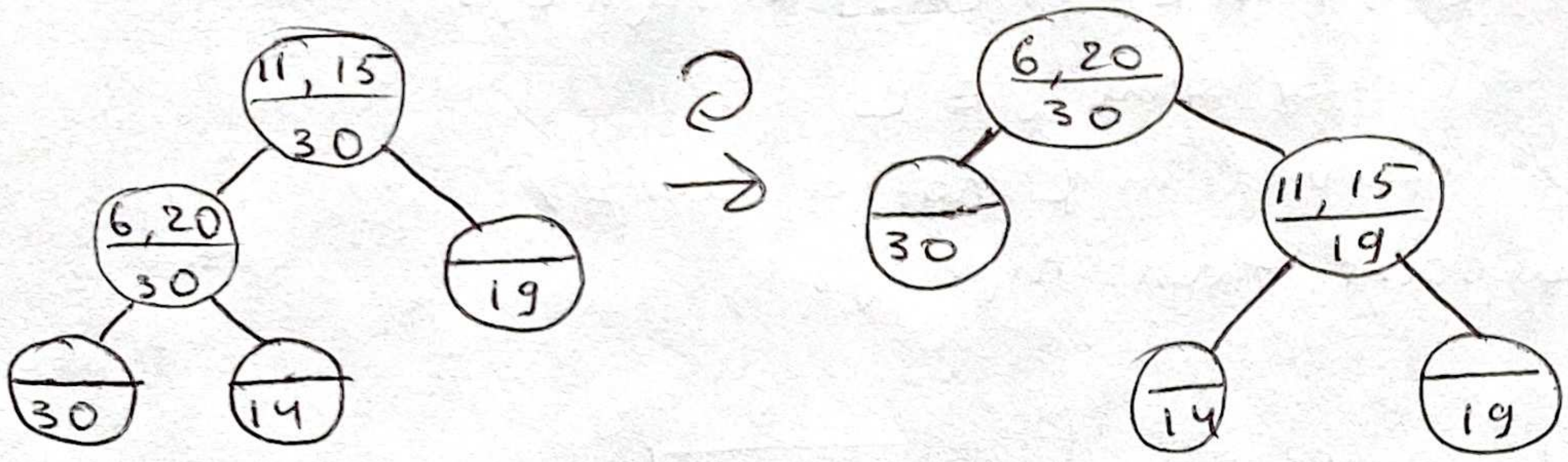
3. Modifying ops:

Insert: BST insert, fix m's on way down
But, also need to handle rotations.



Fixing up m's during rotations takes $O(1)$ time.

Insert time $= O(\lg n)$

Delete: similar

4. Interval-Search (i) // Finds an interval that overlaps i

$x \leftarrow$ root

while $x \neq$ nil and (low[i] > high[int[x]]
or low[int[x]] > high[i])

do // i and int[x] don't overlap, otherwise just return

if left[x] $\neq$ nil and low[i] $\leq$ m[left[x]]

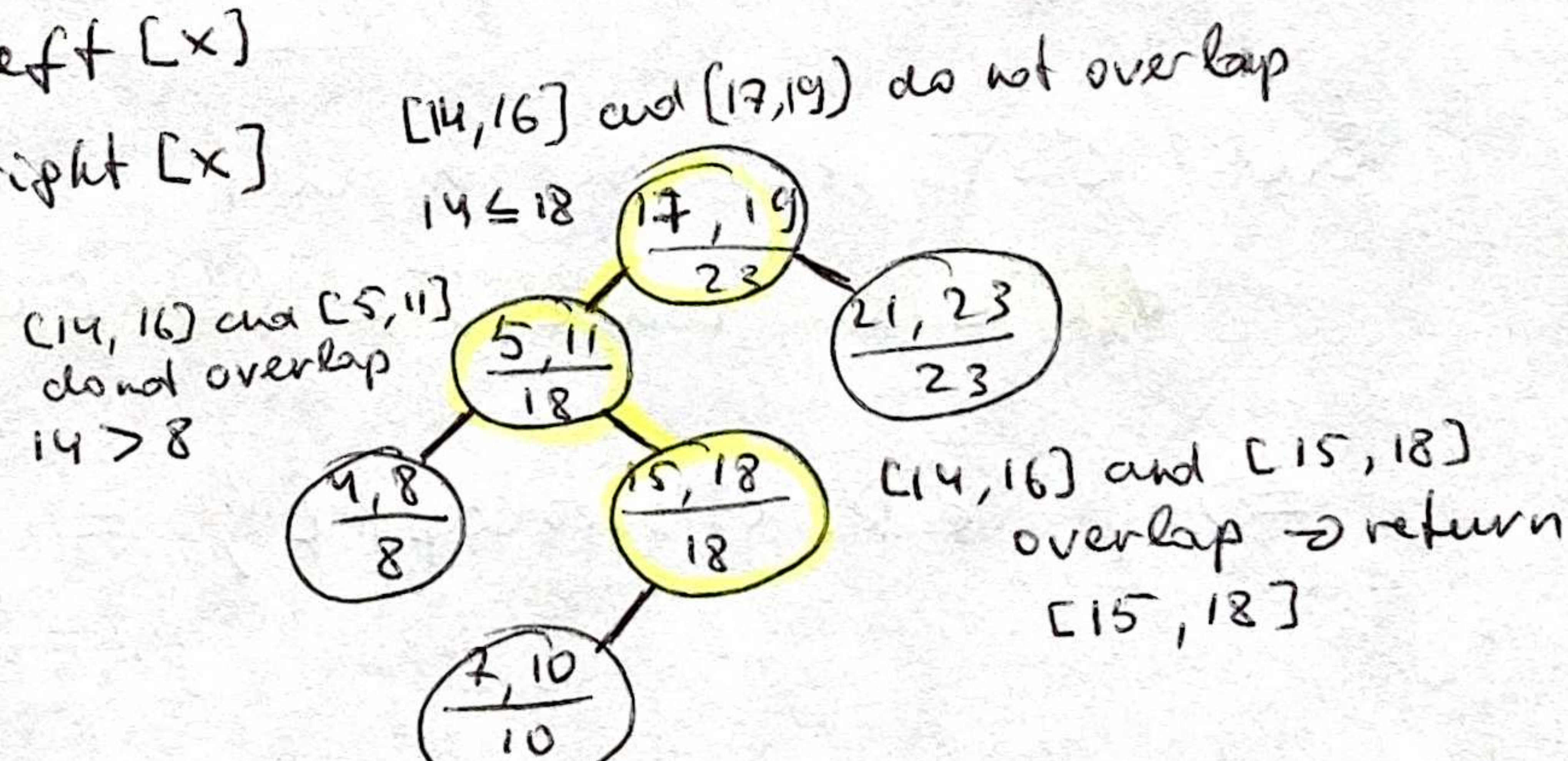
then $x \leftarrow$ left[x]

else $x \leftarrow$ right[x]

return x

Ex [14, 16] → [15, 18]

$\overset{i}{[12, 14]}$ → nil



Time $O(\lg n)$

List all overlaps: $O(k \lg n)$ after finding, delete, put on [redacted] temp. storage then insert

" output sensitive "

Best to date $= O(k + \lg n)$

Correctness

Theorem let $L = \{ i' \in \text{left}[x] \}$
$R = \{ i' \in \text{right}[x] \}$

(1) - If search goes right, then $\{ i' \in L : i' \text{ overlaps } i \} = \emptyset$

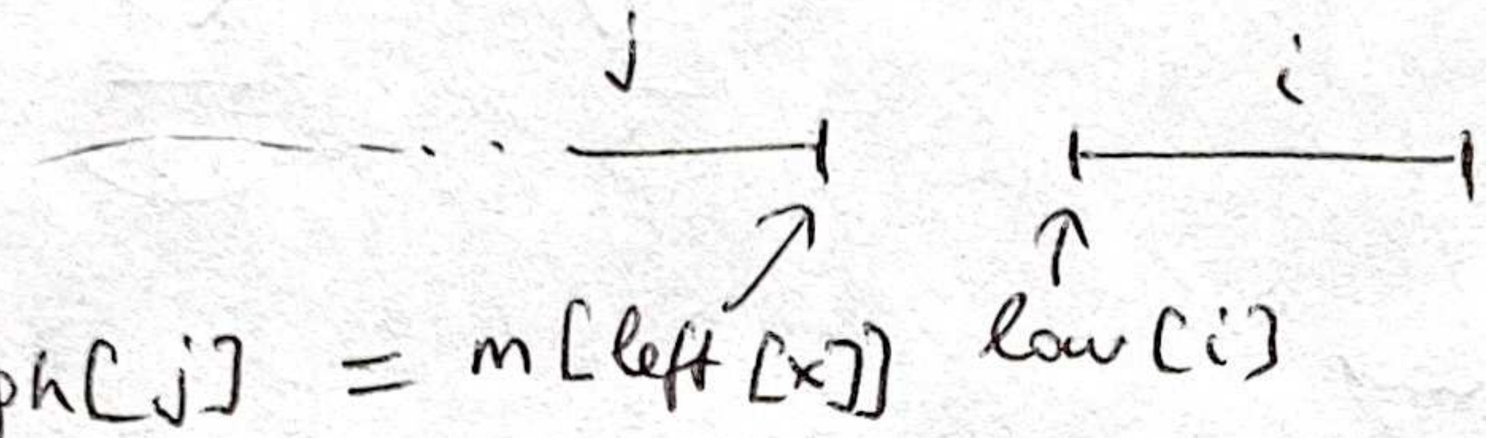
(2) - If search goes left, then $\{ i' \in L : i' \text{ overlaps } i \} = \emptyset$
$\Rightarrow \{ i' \in R : i' \text{ overlaps } i \} = \emptyset$

Pf: Suppose search goes right.

(1) - If $left[x] = nil$, done since $L = \emptyset$

- Otherwise $low[i] > m[left[x]] = high[j]$, for some $j \in L$.

No other interval in L has a larger high endpoint than $high[j]$.

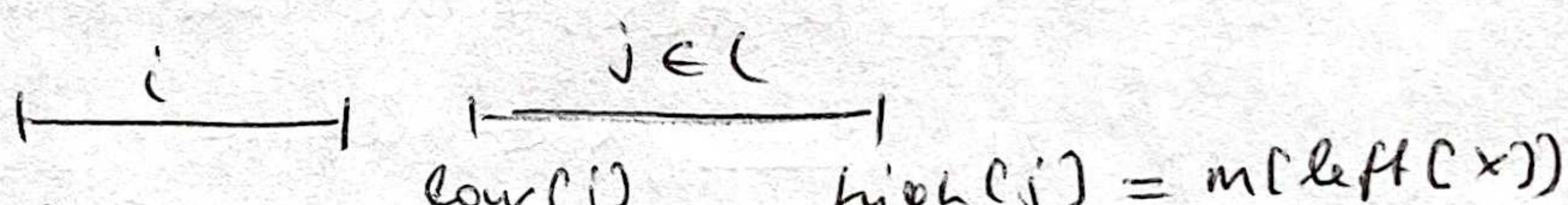


$\Rightarrow \{ i' \in L : i' \text{ overlaps } i \} = \emptyset$

(2) Suppose search goes left and $\{ i' \in L : i' \text{ overlaps } i \} = \emptyset$

Then, $low[i] \le m[left[x]] = high[j]$ for some $j \in L$

Since $j \in L$, j does not overlap i $\Rightarrow high[i] < low[j]$



But, BST property implies

$\forall i' \in R$, $low[j] \le low[i']$ ← red-black tree keyed on low endpoint

$\Rightarrow \{ i' \in R : i' \text{ overlaps } i \} = \emptyset$