lecture 7 CS124

Kruskal's alg. operations

Makeset (x) - creates set {x}

Find (x) - return "name" of the set containing x

Union (x, y) - union sets containing x and y

Kruskal's alg

Set X = {}

Set E of edges

Sort E — $O(E \log E)$ (assume ordered)

for u ∈ V makeset (u) — $O(V \text{ makeset})$

for (u, v) ∈ E in increasing order, do

  if find (u) ≠ find (v) — $O(E \cdot \text{find})$

    X = X ∪ {(u, v)}

    union (u, v) — $O(V \cdot \text{union})$

edge connects disconnected components ⟺ no cycle → equivalence of data structure and no cycle checking

take cut through disconnected components, take min edge → cut property

Baseline: array implementation

← vertex ix

← set ix

makeset $O(1)$

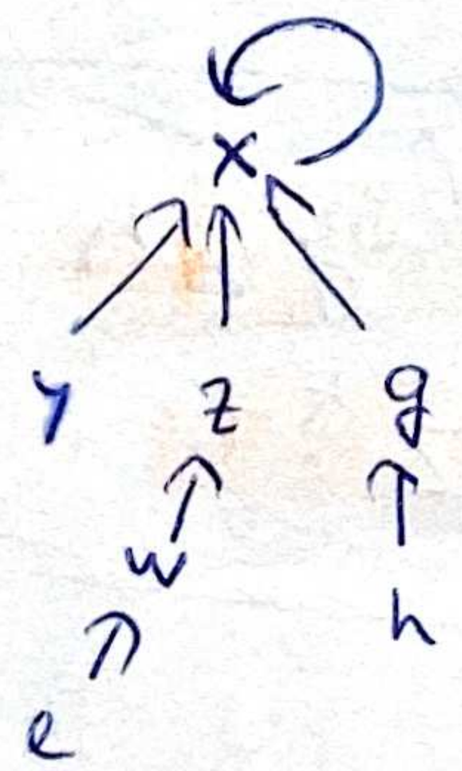
find $O(1)$

union $O(V)$

$O(E + V^2)$

Represent set by a tree

root — name of set



find $O(\text{Depth})$

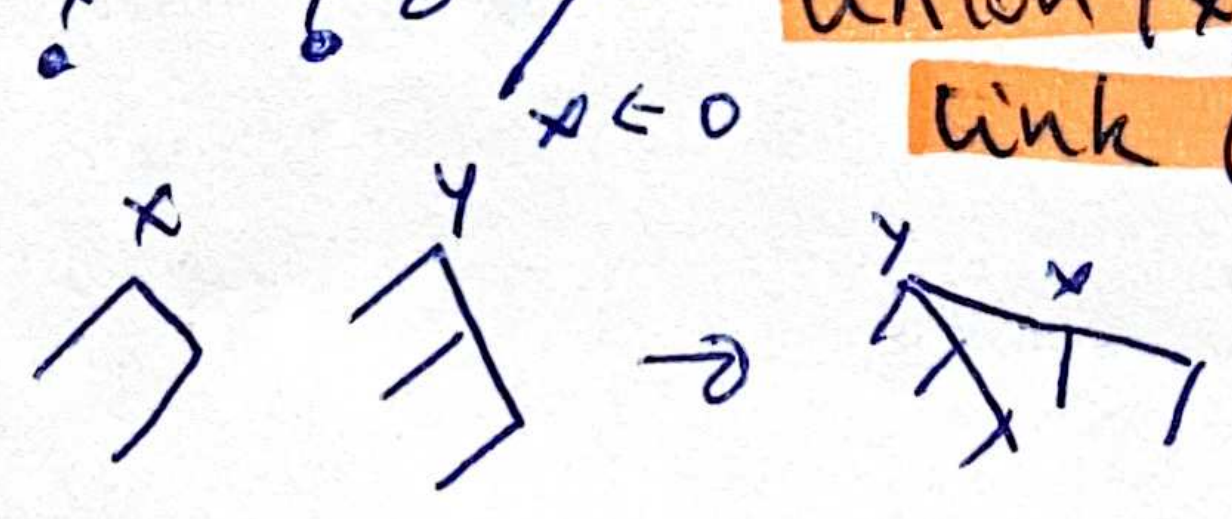
Makeset (x)
  P(x) := x
  Rank (x) := 0

Find (x)
  if x ≠ P(x)
    return (find (P(x)))
  else return (x)

Link (x, y) ← put y on top of x, shallower to deeper
  if rank (x) > rank (y)
    x ⟷ y
  if rank (x) = rank (y)
    rank (y) := rank (y) + 1
  p (x) := y

Union (x, y)
  Link (Find (x), Find (y))

Think of rank as depth of the tree from the element (no compression)



$O([E+V] \text{ find})$

$O(E \log V)$

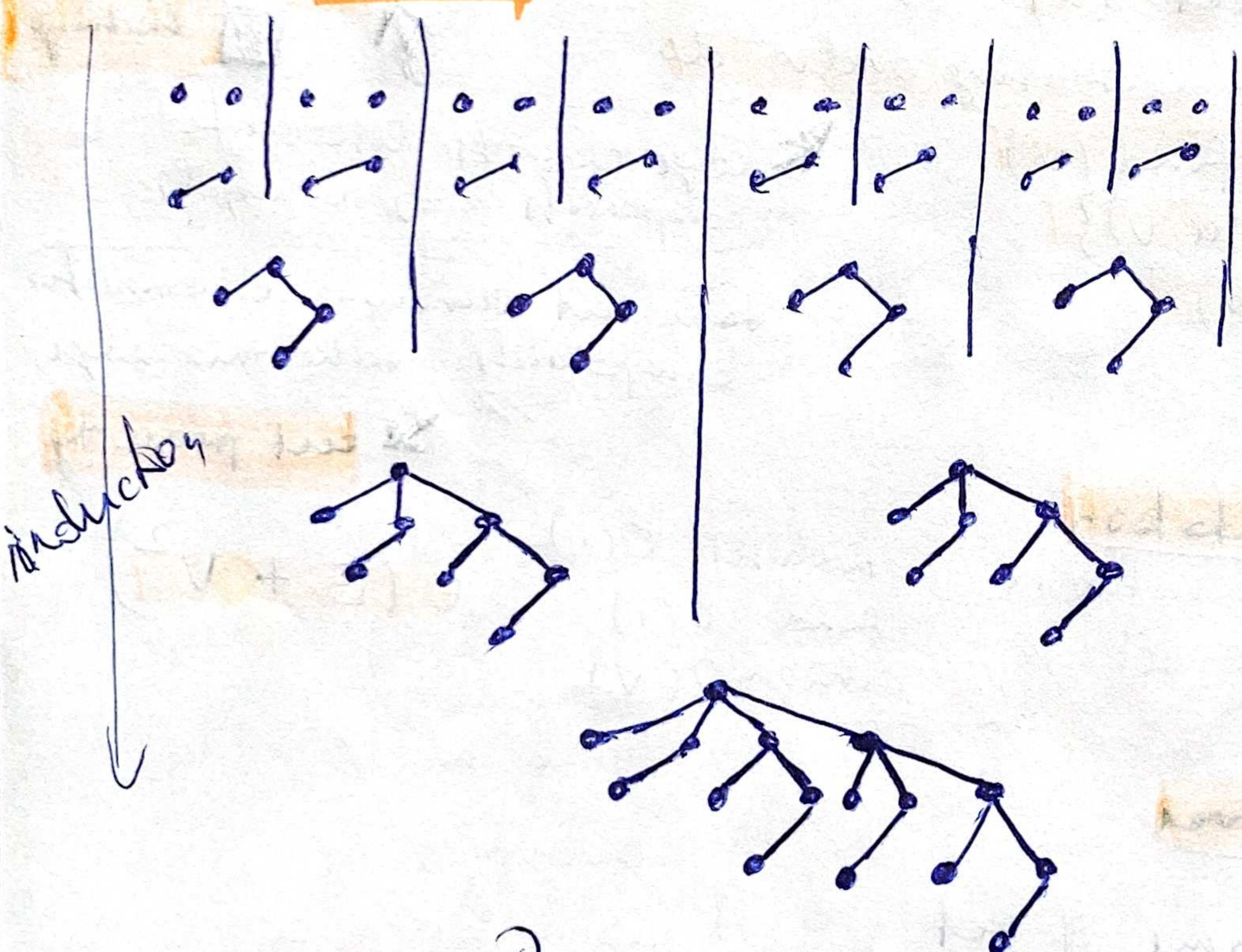
[redacted]

not binary trees, but max depth is $\leq \log_2 n$, where n is # of nodes in a tree

rank increases only if two trees have the same rank, otherwise rank stays.

↳ worst case [redacted]

get highest rank from n nodes → increase rank each time two trees are joined [redacted]



n	k (rank)
1 node	0 depth
2 nodes	1 depth
4 nodes	2 depth
8 nods	3 depth
16 nods	4 depth

this case [redacted]

$k = \log_2 n$

$n = 2^k$

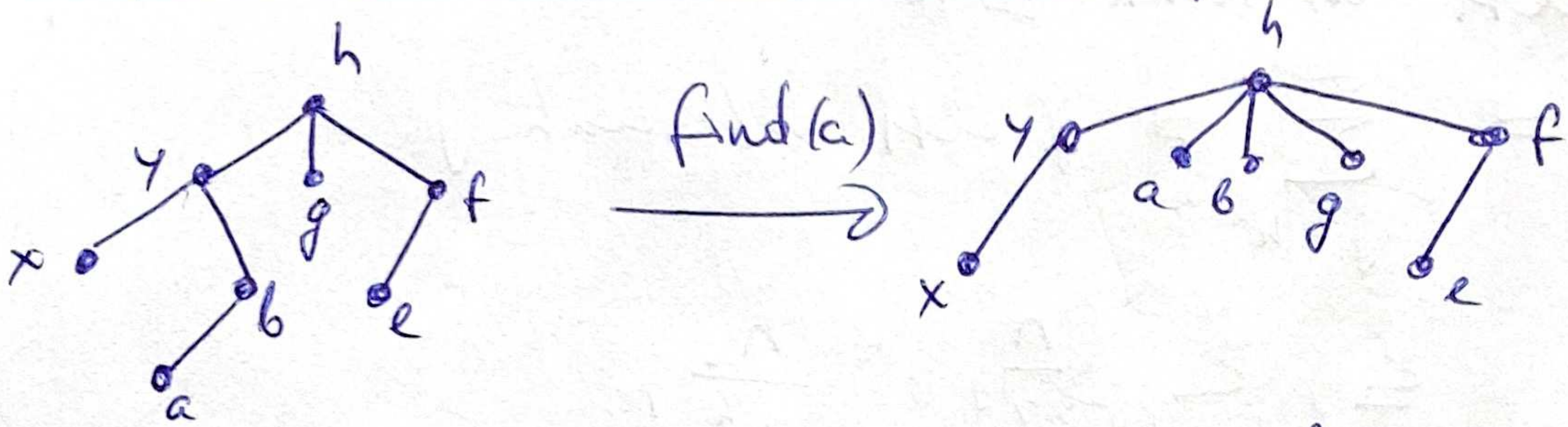
other cases

$k < \log_2 n$

$n > 2^k$

$$\boxed{\begin{array}{c} k \text{ [redacted] } \leq \log_2 n \\ n \geq 2^k \end{array}}$$

improve data structure with path compression



$O([E+V] \log^* ...)$

↗ # of times to take $\log_2$ to get ≤ 1

path compression: $\log n \to 2 \log n$

constant factor cost

Break even point after repeating find(a)

insurance: not doing anything that is [redacted] unnecessary

idea [redacted] → add a constant factor to what is necessary anyway

↓

amortized analysis

maybe have to do a $\log n$ operation once but not again

→ amortized
not just worst cost
but consider cost over a sequence of operations

$\underline{\log^*}$

$\log^* 2 = 1$

$\log^* 4 = 2 \quad (\log_2(\log_2 4)) = 1$

$\log^* 16 = 3$

$\log^* 65536 = 4$
$\nearrow 2^{16}$

$\log^* 2^{65536} = 5$
$2^{2^{16}}$

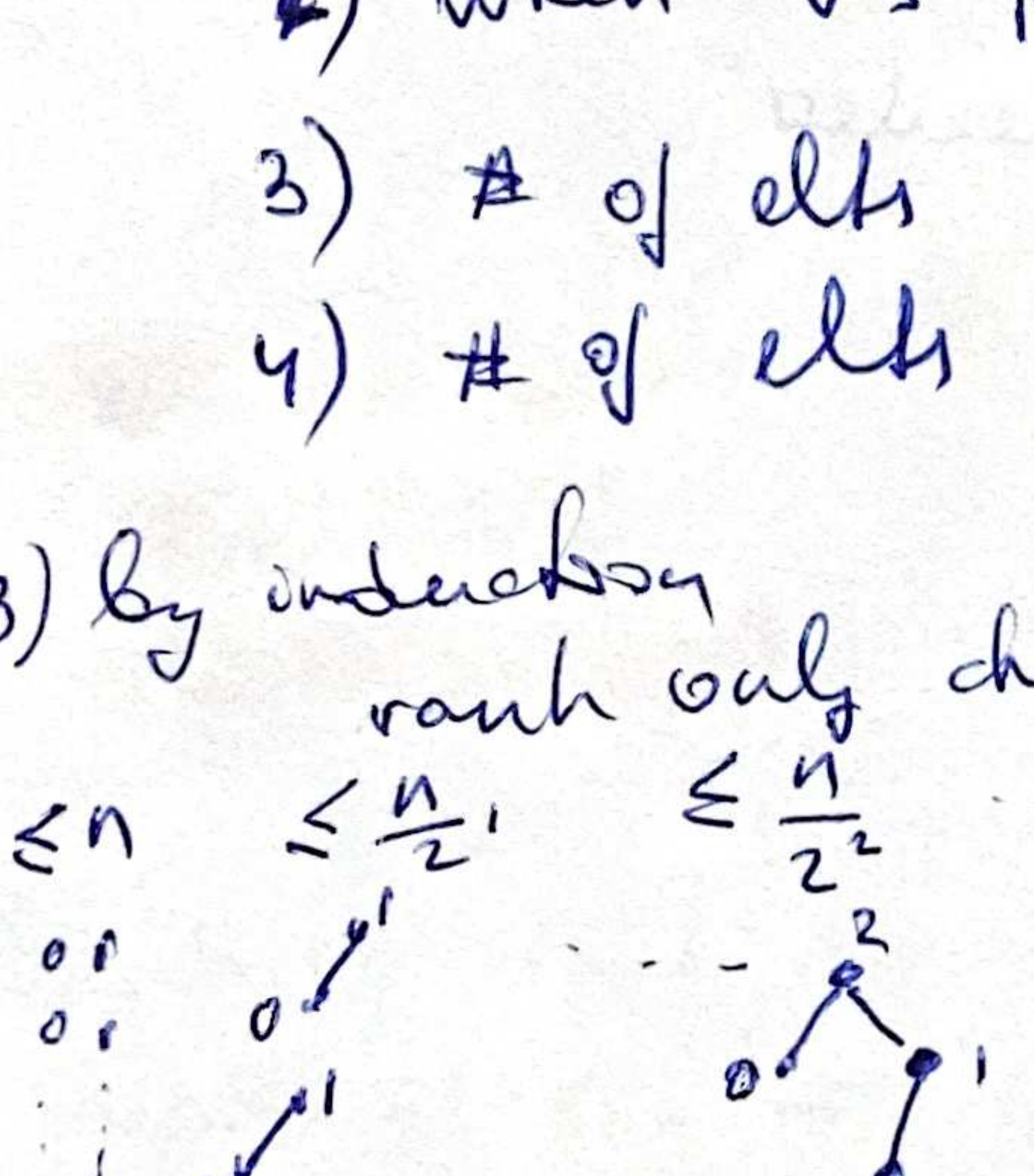
Prove $O([E+V] \log^* n)$

lemmas:

1) if $v \neq p(v)$, then $\text{rank}(p(v)) > \text{rank}(v)$ ← holds with path compression or not
2) when v's parent is updated, $\text{rank}(p(v))$ increases ← linked with higher up the tree
3) # of elts w/ rank $k \leq n/2^k$ ✓ [redacted]
4) # of elts w/ rank $\geq k \leq n/2^{k-1}$

3) by induction
rank only changes, if [redacted] root [redacted]

$\leq n \quad \leq \frac{n}{2^1} \quad \leq \frac{n}{2^2}$ ---



4) # of elts w/ rank $\geq k$

$$= \sum_{j=k}^{\infty} \text{\# of elts w/ rank } j$$

$$\leq \sum_{j=k}^{\infty} \frac{n}{2^j} = \frac{n}{2^{k-1}}$$

$$\sum_{j=0}^{\infty} \frac{n}{2^j} - \sum_{j=0}^{k-1} \frac{n}{2^j} = \overbrace{\frac{n}{1-\frac{1}{2}}}^{2n} - \frac{n\left(1-\frac{1}{2^k}\right)}{\frac{1}{2}}$$

$$= 2n - 2n\left(1-\frac{1}{2^k}\right) =$$

$$= 2n\left(1-1+\frac{1}{2^k}\right) = \frac{n}{2^{k-1}}$$ ✓

Group i = # of non-root elts with rank r satisfying $\log^* r = i$ (group 3 = ranks $(4, 16]$)

$\leq$ group $\log^* n$

① pointer to root (constant)

Type1: follow a pointer from u to v u, v in different groups ← find operation $O(E \log^* n)$

Type2: same group ← group pays for this

take u in group $(k, 2^k]$

assign u 2^k tokens to pay for type 2 links

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group $(k, 2^k]$, has $\leq n/2^k$ elts. (lemma 4)

→ group $(k, 2^k] \leq n$ tokens

⇒ $\leq n \log^* n$ tokens needed

total: $\leq O((E+V) \log^* V)$