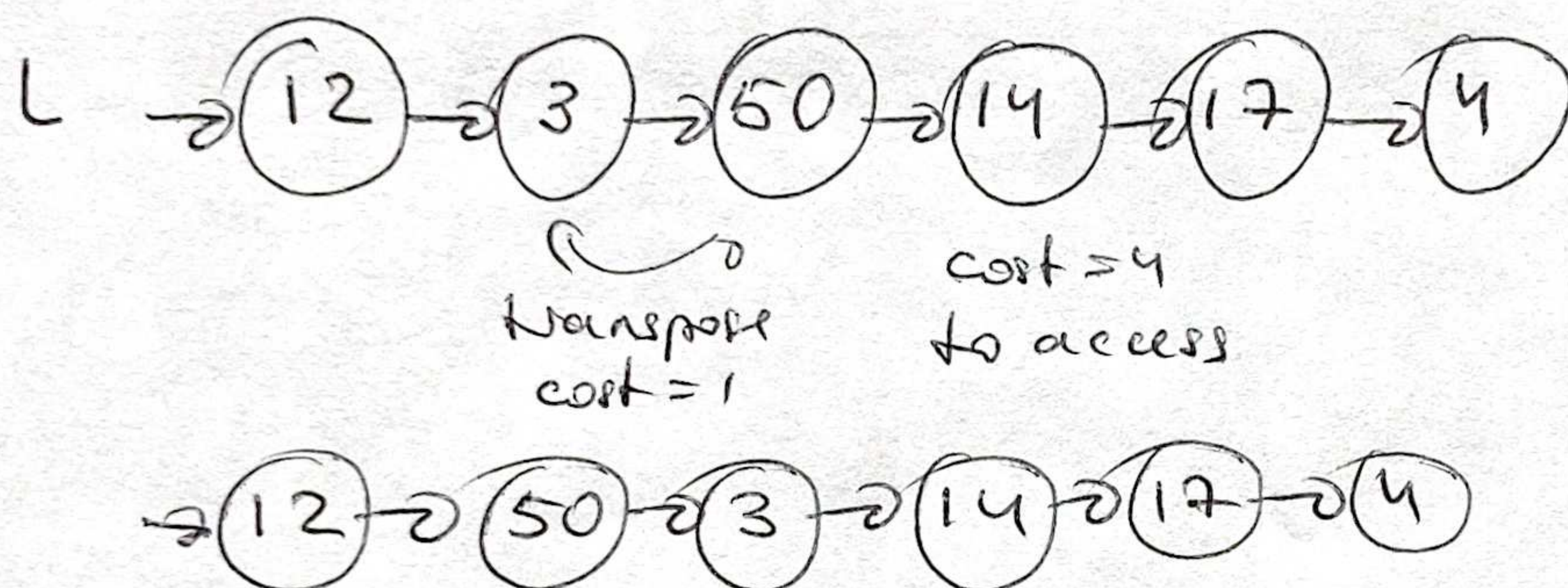


Self-organizing lists

List L of n elements

- Operation $\text{Access}(x)$ costs $\text{rank}(x) = \text{distance of } x \text{ from head of } L$
- L can be reordered by transposing adjacent elements, $\text{cost} = 1$

Ex:



Def: A sequence S of operations is provided one at a time. For each operation, an on-line algorithm must execute the operation immediately (algorithm A).

Off-line algorithm may see all of S in advance.

Tetris: on-line, cannot see in advance.

Goal: minimize total cost $C_A(S)$

Worst-case analysis (online)

Adversary always accesses tail element of L .

$$C_A(S) = \Omega(|S| \cdot n) \text{ worst case}$$

Average-case analysis

Suppose element x is accessed with probability $p(x)$.

\Rightarrow a priori distribution on elements (input)

$$E[C_A(S)] = \sum_{x \in L} p(x) \cdot \text{rank}_L(x), \text{ which is minimized when } L \text{ is sorted in decreasing order w.r.t. } p.$$

keep count of # times each element is accessed, and maintain list in order of decreasing count.
(assume dist. stays the same; LCN)

Practice

"Move-to-front" heuristic (MTP)

my comment
actually an approximation!
by below analysis

After accessing x , move x to the head of list using transposes. $\text{cost} = 2 \cdot \text{rank}(x)$

↑
access +
transposes

Responds well to locality in s (not static distribution, access x makes x more likely to be accessed again)

Competitive Analysis

Def. An on-line algorithm A is α -competitive

if \exists constant k

s.t. for any sequence s of operations

$$C_A(s) \leq \alpha C_{OPT}(s) + k$$

could be a \uparrow opt. off line algorithm
function or cost

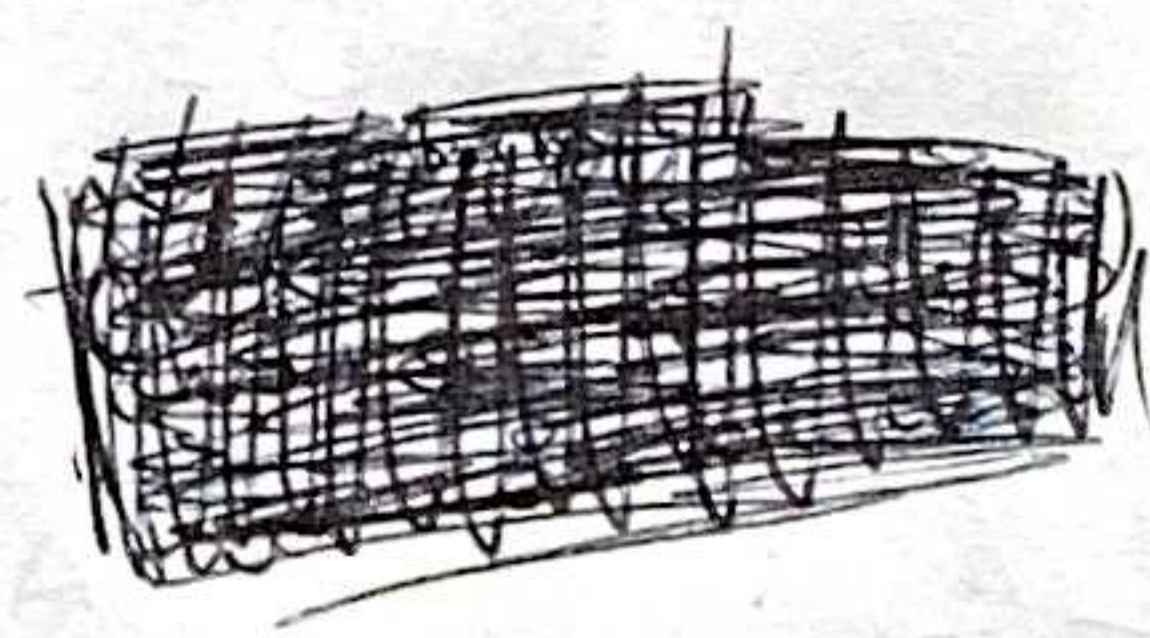
← no assumption of op distribution in s sequence

Theorem: MTP is 4-competitive for self-organizing lists.

Proof: Let L_i be MTP's list after i th access

L_i^* ... OPT's

Let c_i = MTP's cost for i th op
= $2 \cdot \text{rank}_{L_{i-1}}(x)$ if it accesses x



c_i^* = OPT's cost for i th op
= $\text{rank}_{L_{i-1}^*}(x) + t_i$ if OPT performs t_i transposes

Define the potential function

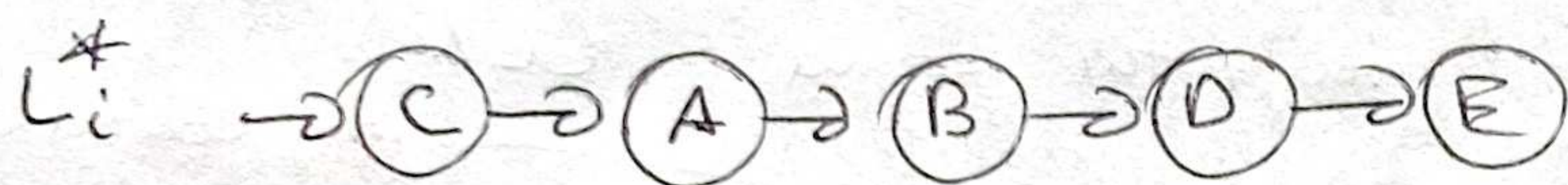
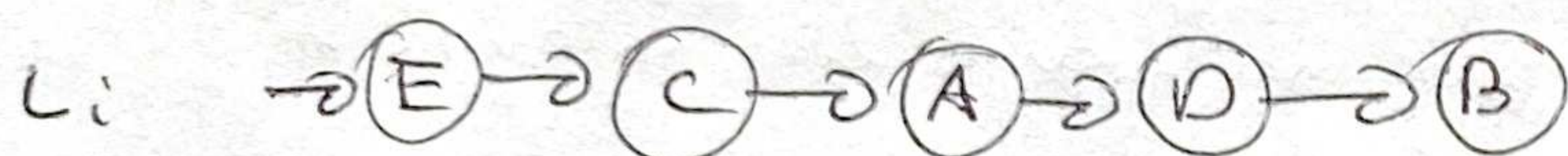
$\Phi: \{L_i\} \rightarrow \mathbb{R}$ by \leftarrow precedes in the L_i list from head

$$\Phi(L_i) = 2 \left| \{ (x, y) : x \prec_{L_i} y \text{ and } y \prec_{L_i^*} x \} \right|$$

← disagreements between L_i and L_i^*

$$= 2 \cdot \# \text{ inversions}$$

Ex:



$$\Phi(L_i) = 2 \cdot |\{(E, C), (E, A), (E, D), (E, B), (B, D)\}| = 10$$

$$= 2 \cdot 5 \text{ inversions}$$

Note: $\Phi(L_i) \geq 0 \quad \forall i$

$\Phi(L_0) = 0$ if MTF and OPT start with same list

the more L_i^* differs from L as i progresses,

the more work is stored for MTF to use

How much does Φ change from 1 transpose?

$$\Delta \Phi = \pm 2 \quad (\text{creates or destroys 1 inversion})$$

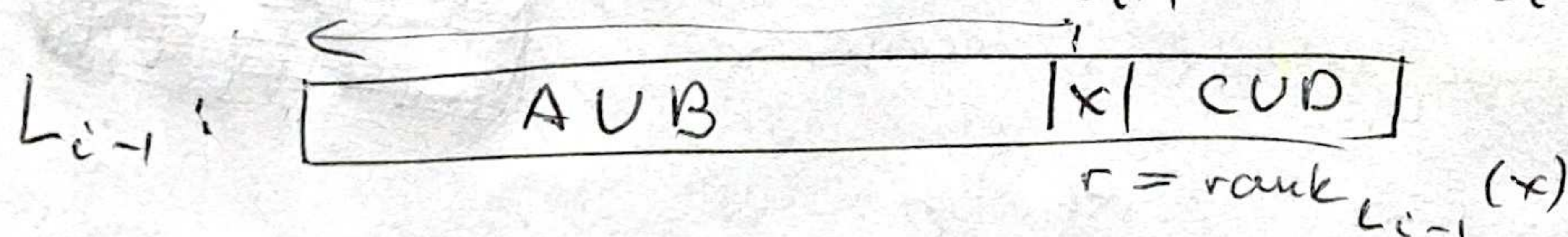
What happens when op i accesses x ?

Define $A = \{y \in L_{i-1} : y \underset{L_{i-1}}{<} x \text{ and } y \underset{L_{i-1}^*}{<} x\}$

$B = \{y \in L_{i-1} : y \underset{L_{i-1}}{<} x \text{ and } y \underset{L_{i-1}^*}{>} x\}$

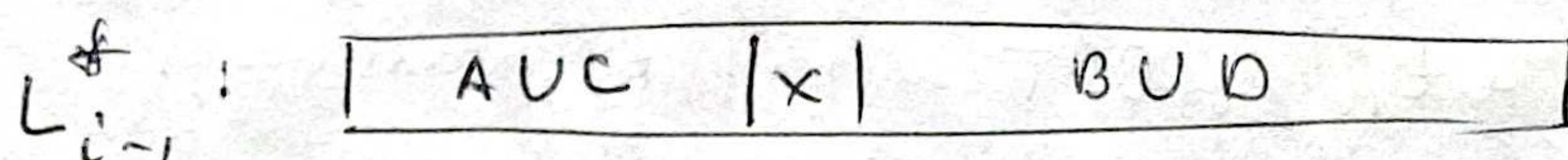
$C = \{y \in L_{i-1} : y \underset{L_{i-1}}{>} x \text{ and } y \underset{L_{i-1}^*}{<} x\}$

$D = \{y \in L_{i-1} : y \underset{L_{i-1}}{>} x \text{ and } y \underset{L_{i-1}^*}{>} x\}$



disjoint
 $\swarrow \searrow$

$$r = |A| + |B| + 1$$



$$r^* = |A| + |C| + 1$$

$$r^* = \text{rank}_{L_{i-1}^*}(x)$$

Access(x)

When MTF moves x to front, it creates

$|A|$ inversions and destroys $|B|$ inversions.

Each transpose by OPT creates ≤ 1 inversion.

Thus

$$\Phi(L_i) - \Phi(L_{i-1})$$

$$\leq 2(|A| - |B| + t_i)$$

OPT can go anywhere
 need to add $t_i \rightarrow$ # transpose by OPT

Amortized cost is

$$\hat{C}_i = C_i + \Phi(L_i) - \Phi(L_{i-1})$$

$$\leq 2r + 2(|A| - |B| + t_i) \leftarrow r = |A| + |B| + 1$$

$$= 2r + 2[|A| - (r-1-|A|) + t_i]$$

$$= \cancel{2r} + 4|A| - \cancel{2r} + 2 + 2t_i$$

$$= 4|A| + 2 + 2t_i$$

$$\leq 4(r^* + t_i) \text{ since } r^* = |A| + |C| + 1$$

$$= 4c_i^*$$

my comment

Amortized cost of $\text{Accum}(x)$ of MTF. The more L^* different from L , the more work we store for MTF to use.

$\Delta \Phi$ accumulates the difference between L^* and L in terms of inversions.

maps to # of transposes

upper bound of transposes of which MTF did too many. Because MTF is not OPT.

Thus

$$C_{\text{MTF}}(S) = \sum_{i=1}^{|S|} C_i = \sum_{i=1}^{|S|} (\hat{C}_i + \Phi(L_{i-1}) - \Phi(L_i))$$

$$\leq \left(\sum_{i=1}^{|S|} 4c_i^* \right) + \underbrace{\Phi(L_0)}_{=0} - \underbrace{\Phi(L_{|S|})}_{\geq 0}$$

$$\leq 4C_{\text{OPT}}(S) \quad \checkmark$$

4-competitive

my comment

runtime comparison to OPT

runtime approximation to OPT (offline) without knowing OPT (algorithm)

If we count transposes that move toward the front of L as "free" (models splicing x in and out of L in const. time), then MTF is 2-competitive.

what if $L_0 \neq L_0^*$?

Then, $\Phi(L_0)$ might be $\Theta(n^2)$

Thus $C_{\text{MTF}}(S) \leq 4C_{\text{OPT}}(S) + \Theta(n^2)$, which is still

4-competitive, since n^2 count as $|S| \rightarrow \infty$

$$O\left(\frac{n-1}{2}\right)$$