Lecture 21 CS124

1

Heuristics, Bubble search

greedy algs → approximations / heuristics

Bubble search: "randomized greedy"

Biggest - 1/2 prob. put it

2nd biggest - 1/4

3rd biggest - 1/8

idea:

① keeps intuition of greedy

but allows to

② restart and have a different order

↳ restart, run, get best soln. → works well in practice

whenever given a greedy alg.

can randomize it like Bubble search

Max-Cut Approximation

$G = (V, E)$

$V_1 \cap V_2 = \emptyset$

$V_1 \cup V_2 = V$

maximize # of crossing edges

(also applies to weighted case)

Randomized approximation:

means: Expected value of soln. is within a factor c of optimal.

flip coin for each vertex

↙ ↘

V_1 V_2

u	v
T	T
T	F
F	T
T	T

given an edge

prob. it is in the cut?

$\frac{1}{2}$

edge ID r.v.s

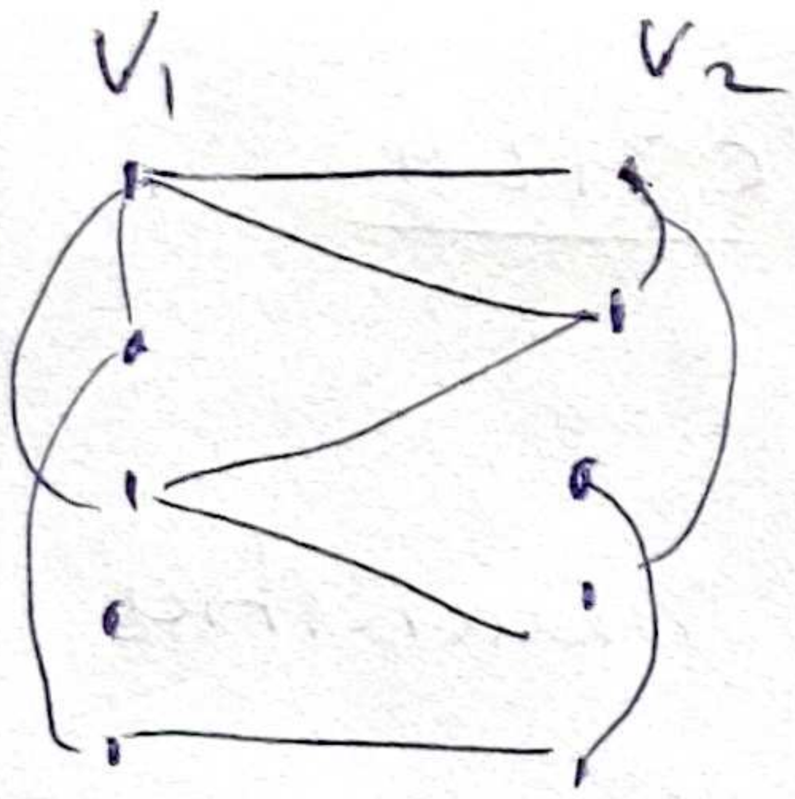
linearity of expectation

$E(\text{Alg. cut}) = \frac{1}{2}|E| \geq \frac{1}{2}$ Max cut

randomized: not guaranteed to get within $\geq \frac{1}{2}$ Max cut, on average

we want a guarantee.

Greedy



Local search

current solution: partition

move : move 1 vertex to other side

if # edges crossing < # edges on the same side

→ derandomization of the randomized algorithm

same $\frac{1}{2}$ approximation

Termination:

- each step, increase # edges on the cut
The cut is finite → terminates in $O(|E|)$

Approximation bound:

$\delta(v) = deg(v)$

sum of all edges crossing ≥ # edges on the same side

each vertex has more edges crossing, than on the same side, otherwise move it to the other side

size of cut

$$C = \left(\sum_{v \in V_1} (v,w)\ w \in V_2 + \sum_{v \in V_2} (v,w)\ w \in V_1\right) \cdot \frac{1}{2}$$

counted twice

$$\geq \frac{1}{2}\left(\sum_{v \in V_1} \frac{1}{2}\delta(v) + \sum_{v \in V_2} \frac{1}{2}\delta(v)\right)$$

count twice

$$= \frac{1}{4}\left(\underbrace{\sum_v \delta(v)}_{2|E|}\right) = \frac{1}{2}|E|$$

$$C \geq \frac{1}{2}|E| \geq \frac{1}{2}\text{ Max cut}$$

guarantee, no expectation

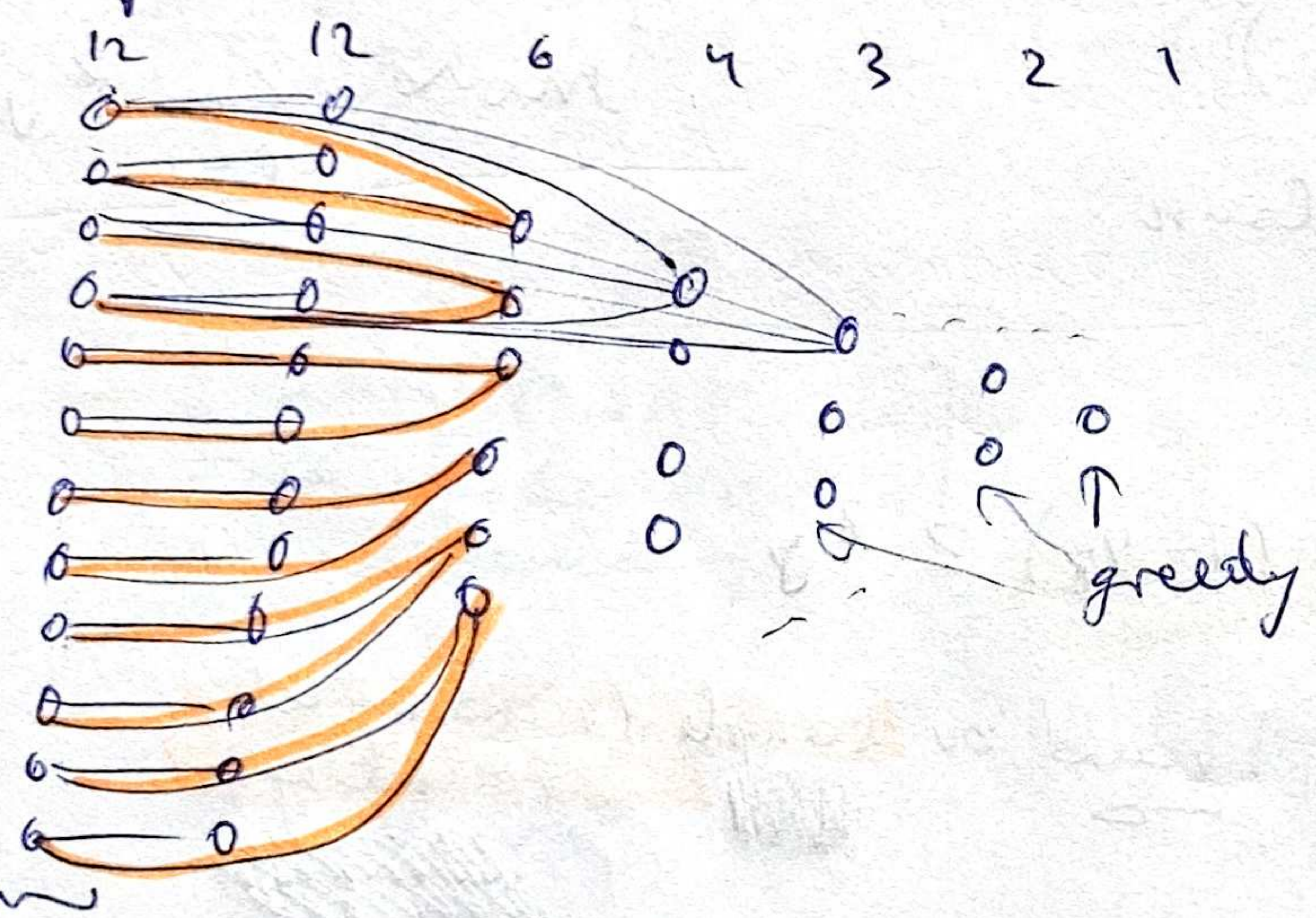
derandomization

Set Cover – log n Approximation
greedy alg.

vertex cover

$G = (V, E)$
Find $U \subseteq V$, s.t.
every edge is adjacent to a vertex in U
↳ [redacted]. **reduction** [redacted] to **set cover**
for each vertex create set corresponding
to the edges that it covers
[redacted]

Greedy on vertex cover? NOT great



log m factor off the OPT

↳ not a const. factor

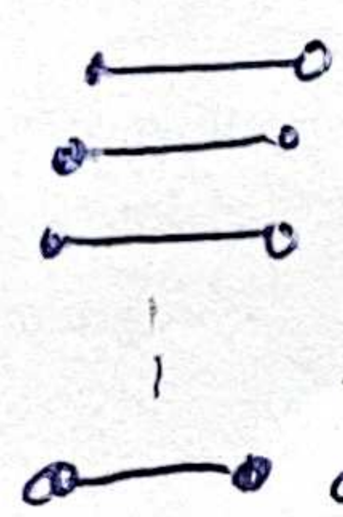
$|VC| = 12$ m $\frac{m}{2}$ $\frac{m}{3}$ $\frac{m}{4}$ ... $\frac{m}{m}$

2-Approximation for vertex cover

- **take any edge**
- **put both vertices in cover**
- **remove edge**
- **remove vertices and all adjacent edges**

each edge taken is disjoint, since two vertices and adjacent edges thrown out

at least one vertex in each pair has to be in VC to cover the connecting edge



2· # disjoint edges $\geq$ OPT VC $\geq$ # of disjoint edges

Alg $\leq$ 2 OPT VC

Max-SAT (at least NP hard)

satisfy as many clauses as possible

Randomized alg. (baseline)

flip coin for each variable
count satisfied clauses

k literals in a clause $(x_1 \vee x_2 \vee \ldots x_k)$

$$\underbrace{1-2^{-k}}_{\text{prob. for satisfied clause}}$$

Linear program (integer)

Pretend it is a linear program

my comment: $E[\#\text{ sat clauses}] = \overset{\#\text{ all clauses}}{n} \cdot (1-2^{-k})$

$(x_2 \vee \bar{x}_4 \vee x_6 \vee \bar{x}_8)$

$\max \sum_{i=1}^{n} z_j$

z_j = var for jth clause

$0 \le z_j \le 1$

$0 \le y_i \le 1$

$y_2 + (1-y_4) + y_6 + (1-y_8) \ge z_j$

solution:

$y_1 = 0.7$ round or sample (randomized rounding)

$y_4 = 0.2$

$y_6 = 0.1$

$\vdots$

rounding vs. randomized rounding

$x_1 \vee x_2 \vee x_3$

$x_1 = 0.4$
$x_2 = 0.4$
$x_3 = 0.4$

rounding sets all to 0
but to LP, this clause has value 1

but with randomized rounding
$0.6^3 = 0.216$
$\Rightarrow$ expected value is 0.784 for the clause

Suppose:

$C = (x_1 \vee x_2 \ldots \vee x_k)$ $\quad z_i = \beta$ [redacted]

positive without loss of generality

$y_1 \ldots y_k$ [redacted]

max $\prod_i (1-y_i)$ $\quad \sum y_i \geq \beta$

worst case for randomized rounding in a clause

solve $y_i = \frac{\beta}{k}$ maximizes prob. that the clause is unsatisfied

clause $\Rightarrow$ prob rounds to 0 $\leq \left(1-\frac{\beta}{k}\right)^k$

[redacted]. fact

prob rounds to 1 $\geq 1-\left(1-\frac{\beta}{k}\right)^k \geq \left(1-\frac{1}{e}\right)\beta$

$\geq$ chance that a clause rounds to SAT [redacted] in randomized rounding

formula

E [clauses SAT in randomized rounding]

By linearity of expectation

$$\geq \sum_j \left(1-\frac{1}{e}\right)\beta_j = \left(1-\frac{1}{e}\right)\sum \beta_j =$$

$$= \left(1-\frac{1}{e}\right)\sum z_j$$

$$\geq \left(1-\frac{1}{e}\right) OPT$$

by linear program (if taking integers, than worse)

within a constant factor of optimum in its expectation