Integer Division by Constants (32-bit)

0) Power of 2, signed

$q = n \div 2^k \qquad 1 \le k \le 31$ ← although not representable as signed, but works

ex $-7 \div 4$ 2's complement

$11\cdots1001 \div 00\cdots0100$

$n \qquad 2^k \qquad k = 2$

0) shrsi t, n, k-1

t : $11\cdots1100$

1) shri t, t, 32-k

t : $0\cdots011 \; (2^2-1)$

machine with fast shifts

2) add t, n, t

$$\begin{array}{r} 11\cdots1001 \\ 00\cdots0011 \\ \hline 11\cdots1100 \end{array}$$

3) shrsi q, t, k

$11\cdots1111 \qquad -1$

0) bge n, label
1) addi n, n, 2**k - 1
2) label shrsi n, n, k

machine with fast branches and slow shifts

1) Signed Remainder from Division by a known Power of 2

Both quotient and remainder

$n \div 2^k$

$r = n - q \cdot 2^k$

```
0) shli   r, q, k
1) sub    r, n, r
```

← left shift unsigned for $\cdot 2^k$ for both signed and unsigned numbers

two consecutive shifts → replace by end of single shift

2) Signed Division and Remainder by Non-Powers of 2

Division by 3 :

$$M \text{ (magic \#)} = \frac{2^{32}+2}{3}$$

n : numerator

q : quotient

t : temporary register

r : contains the remainder

```
li     M, 0x55555556
mulhs  q, M, n
shri   t, n, 31
add    q, q, t
muli   t, q, 3
sub    r, n, t
```

load $\frac{2^{32}+2}{3}$

$q = \text{floor}\left(\frac{Mn}{2^{32}}\right)$

} add 1 to q if n is negative

} compute remainder from $r = n - q \cdot 3$

Proof (32-bit) division by 3 ②

$n \geq 0$

$$q = \left\lfloor \overbrace{\frac{2^{32}+2}{3}}^{M} \cdot \frac{n}{2^{32}} \right\rfloor = \left\lfloor \frac{n}{3} + \overbrace{\frac{2n}{3 \cdot 2^{32}}}^{\text{error term}} \right\rfloor$$

$$n < 2^{31} \Rightarrow \frac{2n}{3 \cdot 2^{32}} < \frac{1}{3}$$

because $n \geq 0$ $\frac{2n}{3 \cdot 2^{32}} \geq 0$

by theorem (D4)

for n, d integers, $d \neq 0$, and x real

$$\left\lfloor \frac{n}{d} + x \right\rfloor = \left\lfloor \frac{n}{d} \right\rfloor \text{ if } 0 \leq x < \left|\frac{1}{d}\right| \text{ and}$$

$$\left\lceil \frac{n}{d} + x \right\rceil = \left\lceil \frac{n}{d} \right\rceil \text{ if } -\left|\frac{1}{d}\right| < x \leq 0$$

it follows: $q = \left\lfloor \frac{n}{3} \right\rfloor$ ✓

$n < 0$, $-2^{31} \leq n \leq -1$

$$q = \left\lfloor \overbrace{\frac{2^{32}+2}{3}}^{M} \cdot \frac{n}{2^{32}} \right\rfloor + 1 = \left\lfloor \frac{2^{32}n + 2n + 3 \cdot 2^{32}}{3 \cdot 2^{32}} \right\rfloor =$$

$$= \left\lceil \frac{2^{32}n + 2n + 1}{3 \cdot 2^{32}} \right\rceil = \left\lceil \frac{n}{3} + \overbrace{\frac{2n+1}{3 \cdot 2^{32}}}^{\text{error term}} \right\rceil$$

by theorem (D2) for n, d integers, $d > 0$

$$\left\lfloor \frac{n}{d} \right\rfloor = \left\lceil \frac{n-d+1}{d} \right\rceil \text{ and}$$

$$\left\lceil \frac{n}{d} \right\rceil = \left\lfloor \frac{n+d-1}{d} \right\rfloor$$

thus

$$-\frac{1}{3} + \frac{1}{3 \cdot 2^{32}} \le \underbrace{\frac{2n+1}{3 \cdot 2^{32}}} \le -\frac{1}{3 \cdot 2^{32}}$$

error term is
greater than $-\frac{1}{3}$ and
nonpositive

By theorem (D4)
for n, d integers, $d \neq 0$, and x real

$\left\lfloor \frac{n}{d} + x \right\rfloor = \left\lfloor \frac{n}{d} \right\rfloor$ if $0 \le x < \left|\frac{1}{d}\right|$, and

$\left\lceil \frac{n}{d} + x \right\rceil = \left\lceil \frac{n}{d} \right\rceil$ if $-\left|\frac{1}{d}\right| < x \le 0$

$q = \left\lceil \frac{n}{3} \right\rceil$ ✓

remainder follows and cannot overflow

Division by 5

if $M = (2^{32}+4)/5$ the error term is too large

$M = \dfrac{(2^{33}+3)}{5}$ and add a shift right signed

```
li     M, 0x66666667      load magic number, (2^33+3)/5
mulhs  q, M, n            q = floor(M·n / 2^32)
shrsi  q, q, 1
shri   t, n, 31           add 1 to q if n is negative
add    q, q, t

muli   t, q, 5            compute remainder from
sub    r, n, t            r = n - q·5
```

Proof:

adding shrsi by 1 position is equivalent to dividing by 2^{33} instead of 2^{32} and taking $\lfloor\ \rfloor$

thus: for $n \geq 0$, $n < 2^{31}$

$$q = \left\lfloor \frac{2^{33}+3}{5} \cdot \frac{n}{2^{33}} \right\rfloor = \left\lfloor \frac{n}{5} + \frac{3n}{5 \cdot 2^{33}} \right\rfloor$$

$$0 \leq \frac{3n}{5 \cdot 2^{33}} < \frac{1}{5} \overset{D4}{\Rightarrow} \left\lfloor \frac{n}{5} \right\rfloor = q \checkmark$$

for $n < 0$, $n \geq -2^{31}$

$$q = \left\lfloor \frac{2^{33}+3}{5} \cdot \frac{n}{2^{33}} \right\rfloor + 1 = \left\lfloor \frac{2^{33} n + 3n + 5 \cdot 2^{33}}{5 \cdot 2^{33}} \right\rfloor$$

$$\overset{D2}{=} \left\lceil \frac{2^{33} n + 3n + \cancel{5 \cdot 2^{33}} - \cancel{5 \cdot 2^{33}} + 1}{5 \cdot 2^{33}} \right\rceil = \left\lceil \frac{2^{33} n + 3n + 1}{5 \cdot 2^{33}} \right\rceil$$

$$= \left\lceil \frac{n}{5} + \frac{3n+1}{5 \cdot 2^{33}} \right\rceil$$

$$-\left|\frac{1}{5}\right| < \frac{3n+1}{5 \cdot 2^{33}} \leq 0 \overset{D4}{\Longrightarrow} q = \left\lceil \frac{n}{5} \right\rceil \checkmark$$

Division by 7

$\frac{2^{32}+3}{7}$ and $\frac{2^{33}+6}{7}$ give too large errors

$\frac{2^{34}+5}{7}$ works ~~by~~ but is too large to represent in a 32-bit signed word

→ split by multiplying by $\frac{2^{34}+5}{7} - 2^{32}$ (negative #) and the add

```
li    M, 0x92492493
mulhs q, M, n
add   q, q, n
shrsi q, q, 2
shri  t, n, 31
add   q, q, t

muli  t, q, 7
sub   r, n, t
```

$\frac{(2^{34}+5)}{7} - 2^{32}$ (negative)

$q = \left\lfloor \frac{M \cdot n}{2^{32}} \right\rfloor + n$ cannot overflow

opposite signs

$q = \left\lfloor \frac{q}{4} \right\rfloor$

add 1 to q if n is negative

compute remainder from
$r = n - 7q$

Proof:

$n \geq 0$ $\quad q = \left\lfloor \frac{\left\lfloor \left(\frac{2^{34}+5}{7} - 2^{32} \right) \frac{n}{2^{32}} \right\rfloor + n}{4} \right\rfloor = \left\lfloor \frac{\left\lfloor \frac{2^{34} n + 5n \cancel{- 7 \cdot 2^{32} n} \cancel{+ 7 \cdot 2^{32} n}}{7 \cdot 2^{32}} \right\rfloor}{4} \right\rfloor$

$\lfloor \lfloor \rfloor \rfloor \to \lfloor \rfloor$ valid consider cases $\quad = \left\lfloor \frac{n}{7} + \frac{5n}{7 \cdot 2^{34}} \right\rfloor$

D3: Corollary for a, b real, $b \neq 0$, d an integer > 0

$$\left\lfloor \left\lfloor \frac{a}{b} \right\rfloor / d \right\rfloor = \left\lfloor \frac{a}{bd} \right\rfloor \text{ and } \left\lceil \left\lceil \frac{a}{b} \right\rceil / d \right\rceil = \left\lceil \frac{a}{bd} \right\rceil$$

D3: for x real, d an integer > 0: $\lfloor \lfloor x \rfloor / d \rfloor = \left\lfloor \frac{x}{d} \right\rfloor$, $\lceil \lceil x \rceil / d \rceil = \left\lceil \frac{x}{d} \right\rceil$

$$0 \le \frac{5n}{7 \cdot 2^{34}} < \frac{1}{7} \quad \text{for} \quad 0 \le n < 2^{31}$$

D4

$$\Rightarrow \quad q = \left\lfloor \frac{n}{7} \right\rfloor \checkmark$$

n < 0

$$q = \left\lfloor \frac{\left\lfloor \left(\frac{2^{34}+5}{7} - 2^{32} \right) \frac{n}{2^{32}} \right\rfloor + n}{4} \right\rfloor + 1 =$$

$$= \left\lfloor \frac{2^{34}n + 5n + 7 \cdot 2^{34}}{7 \cdot 2^{34}} \right\rfloor \overset{D2}{=} \left\lceil \frac{2^{34}n + 5n + \cancel{7 \cdot 2^{34}} - \cancel{7 \cdot 2^{34}} + 1}{7 \cdot 2^{34}} \right\rceil$$

$$= \left\lceil \frac{2^{34}n + 5n + 1}{7 \cdot 2^{34}} \right\rceil = \left\lceil \frac{n}{7} + \frac{5n+1}{7 \cdot 2^{34}} \right\rceil$$

$$-\frac{1}{7} < \frac{5n+1}{7 \cdot 2^{34}} \le 0 \quad \text{for} \quad -2^{31} \le n < 0$$

$$q = \left\lceil \frac{n}{7} \right\rceil \checkmark$$