Graphs

Adjacency matrix representation

$V = \{1 \ldots n\}$

$$\begin{array}{c|c} & 1 \quad \cdots \quad n \\ 1 & \begin{bmatrix} 0 & 1 & 1 & \cdots & 0 & 1 \\ & & & & & \\ & & & & & \end{bmatrix} \\ n & \end{array}$$

$a_{ij} = 1$ if there is a directed edge $i \to j$

Is $(i,j) \in E$? $O(1)$

what are i's neighbors ? $O(n)$

$O(n^2)$ space ← bad for sparse graphs

Adjacency list representation

$(i,j) \in E$? $O(\deg(i))$

potentially binary search tree $O(\log \deg(i))$

what are i's neighbors ? $O(1)$ pointer

$\Theta(\deg(i))$ list copy

$\Theta(E+V)$ space

graph/data representation choice:
→ graph sparsity
→ which operations expected

Depth First Search (DFS)

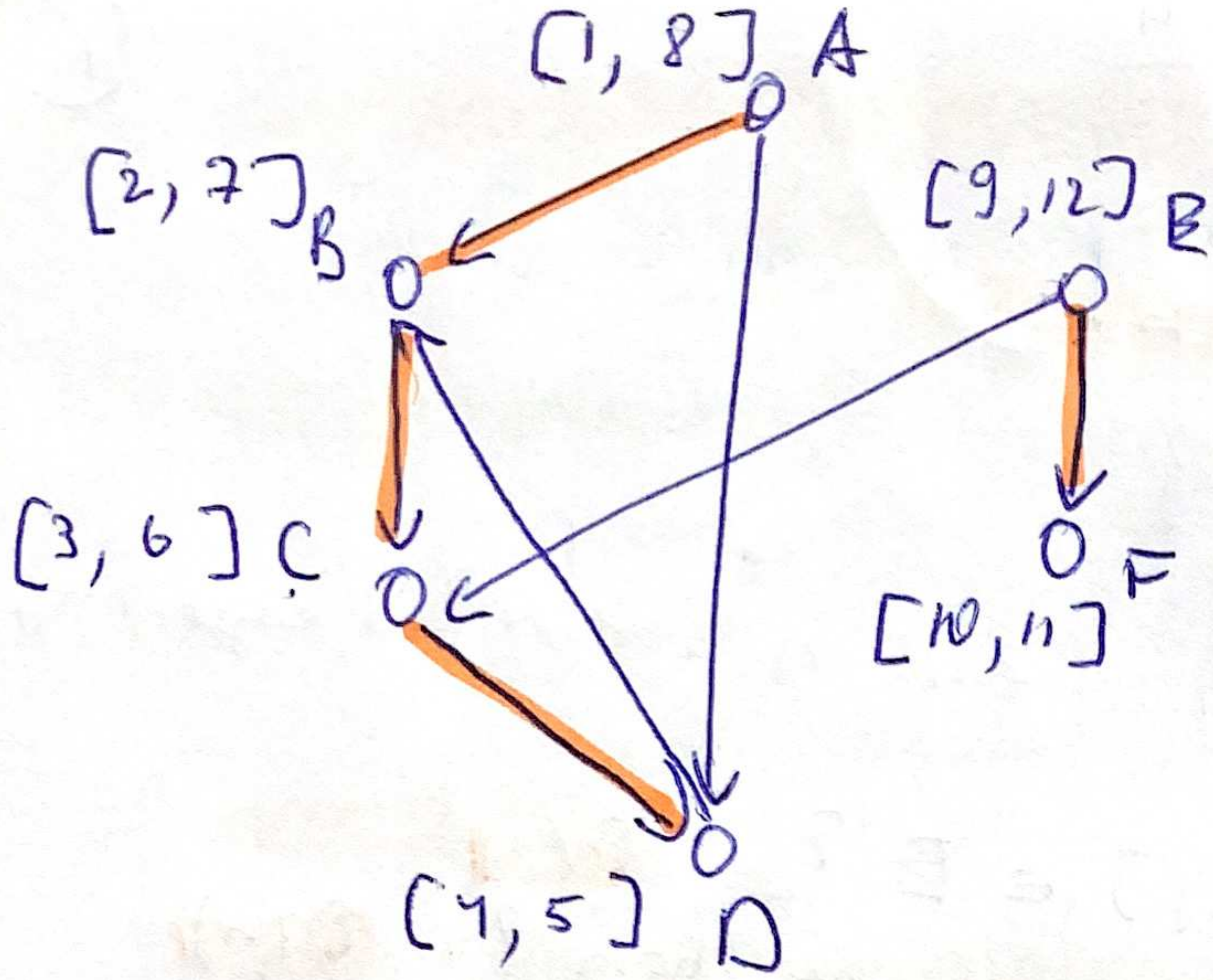
→ adjacency list representation
→ a local search algorithm
↳ exploit local information to traverse, can keep track record of the past

```
Proc Search (v)
  explored (v) := 1
  previsit (v)
  for (v, w) ∈ E          ← O(E)
    if explored (w) = 0
      search (w)
  postvisit (v)
```

every edge crossed only once

```
DFS (G)
  for each v ∈ V               O(V)
    explored (v) := 0
  for each v ∈ V               O(V)
    if explored (v) = 0
      search (v)
```

$O(E+V)$ ← can be graph with only vertices, or with a lot of edges.



[Start time, Finish time] intervals either disjoint or within each other.

(Forest)
Tree edges : edge that is passed in DFS

Back edges : go from a descendent to ancestor in the forest (tree) $\overrightarrow{DB}$

Forward edges : from ancestor to a descendent in the forest $\overrightarrow{AD}$

cross edges : between cousins, no ancestor / descendent relationship $\overrightarrow{EC}$

DFS
⇕
Nodes on a stack : intervals either disjoint or within each other.

[redacted] $(u,v) \in E$, [redacted]

prove $postorder(u) < postorder(v) \Leftrightarrow (u,v)$ is a back edge.

[redacted]

⇒ u finishes before v
v put on stack before u

[redacted]

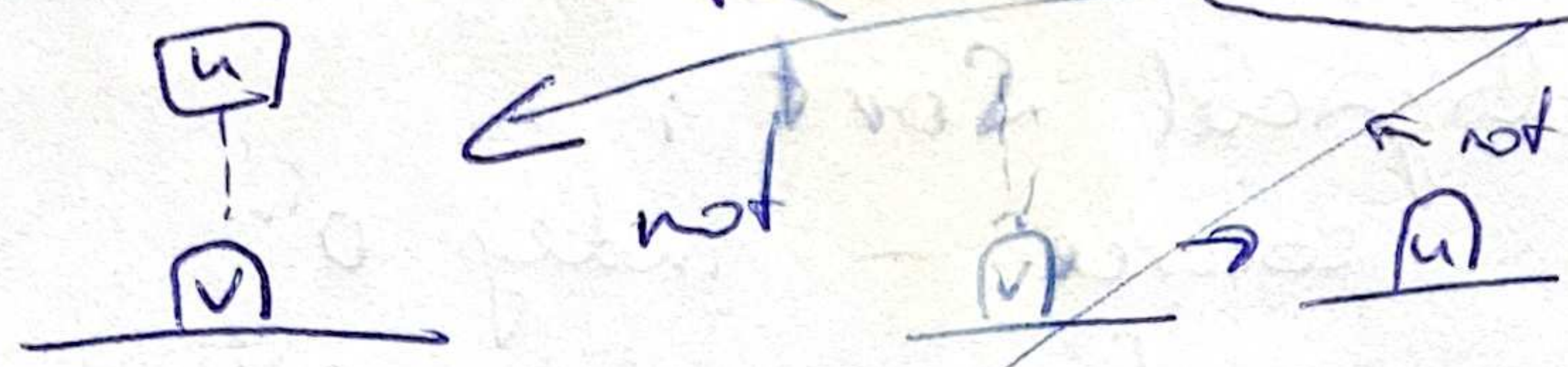
⇒ ∃ path from v to u that is used

⇒ (u, v) must be a back edge

v on stack before u   u on stack before v

$\Leftarrow$ [redacted]

by definition
u is descendant of v
in DFS forest



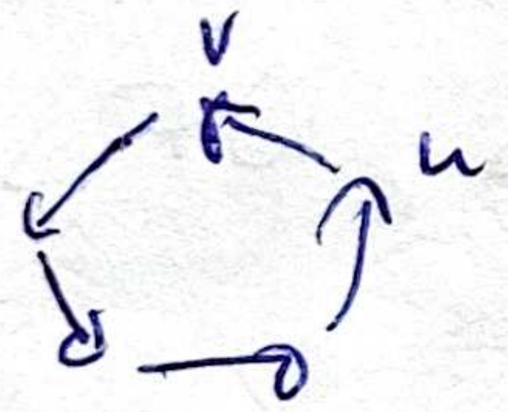
$\Rightarrow$ postorder(u) $\not<$ postorder(v)

Tree, back, cross, forward edges are relative to a specific search. complete and disjoint classification

G has a cycle $\Leftrightarrow$ DFS has a back edge

$\Leftarrow$ obvious, by definition of cycle

$\Rightarrow$



let u be the vertex in the cycle with the smallest postorder

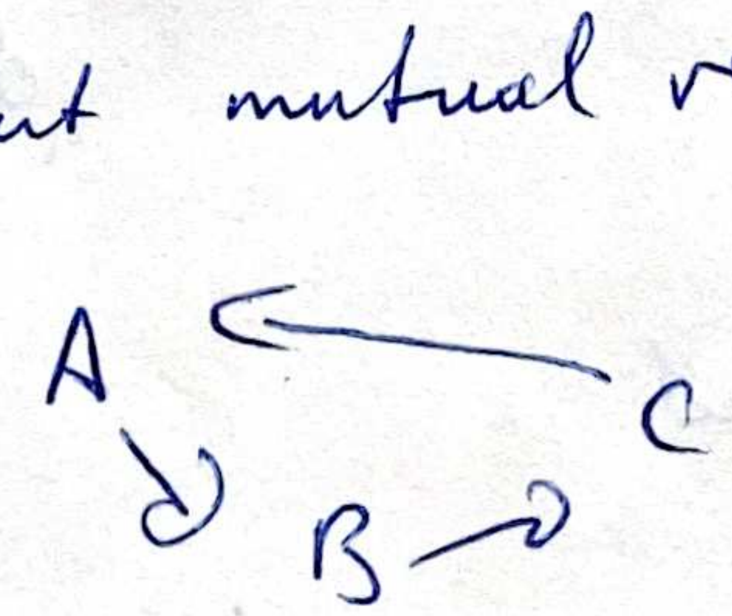
since u is in the cycle $\exists (u, v) \in E$
$\Rightarrow$ where v is part of the cycle.

$\Rightarrow$ postorder(u) < postorder(v)

$\Rightarrow$ (u, v) is a back edge

<u>Debugging</u> Proc A calls Proc B, debug B before A,
[redacted] A depends on B

but mutual recursion problem



assume no mutual recursion for now
↳ no cycles
↳ directed acyclic graph (DAG)

Given a DAG G, sort it

if $\exists$ $(a,b) \in E$, then a comes before b in the list

Topological Sort:

source – indeg 0

sink – out deg 0

> Do a DFS
> Sort decreasing postorder #

$O(E+V)$

Proof

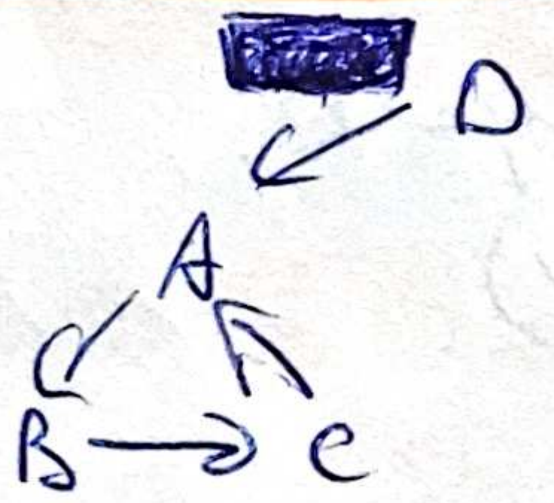
acyclic [redacted] $\overset{(\Rightarrow)}{}$ no back edges

[redacted] $(u,v) \in E$ [redacted] $\Rightarrow$ postorder $(u) >$ postorder (v)

Reduction: Topological search to DFS

Now assume mutual recursion

[redacted]



debug D first

then treat A B C as one

→ debugging order

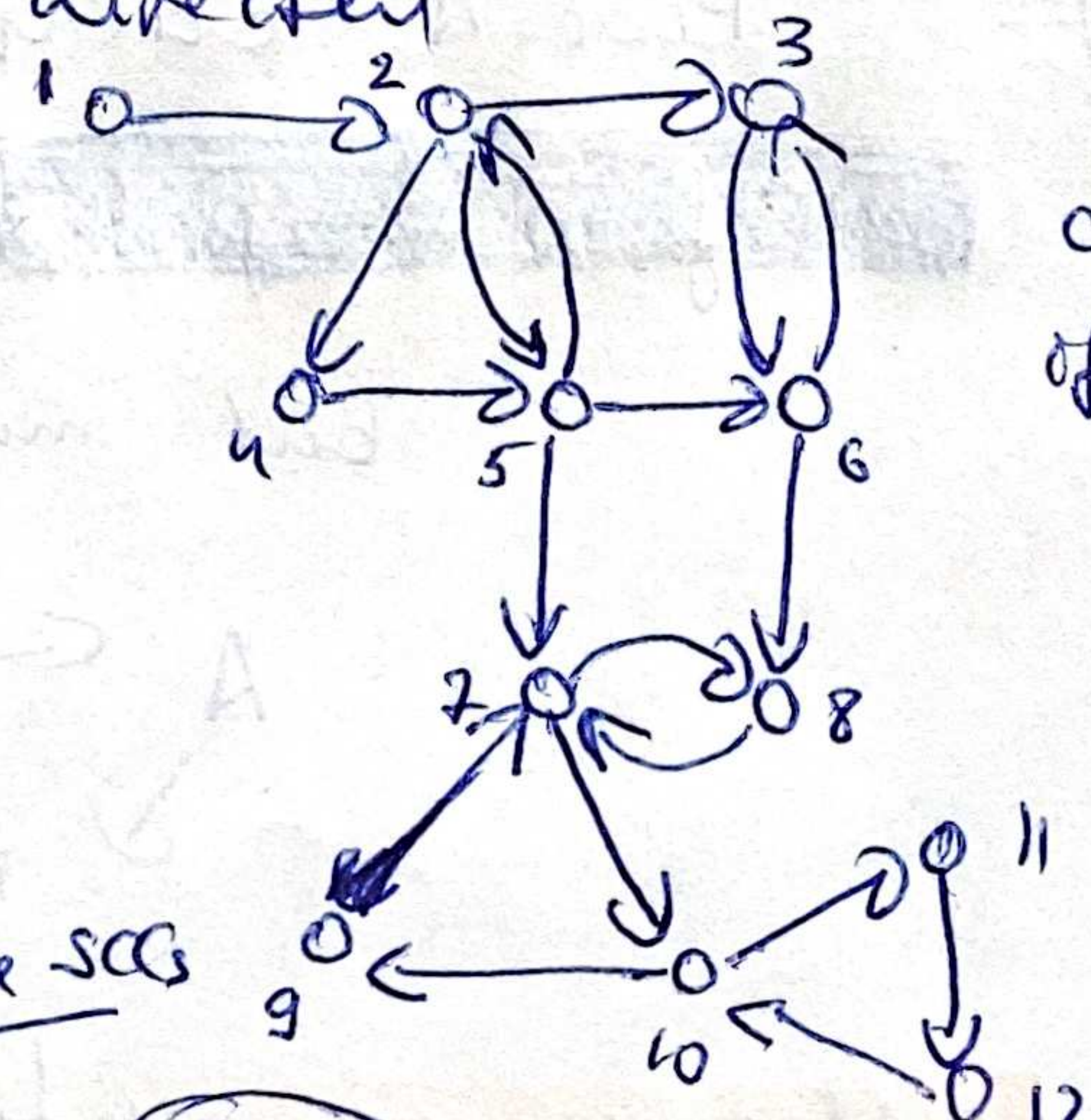
Strongly Connected Components (SCC)

undirected

connected

$\Leftrightarrow$ strongly connected

directed

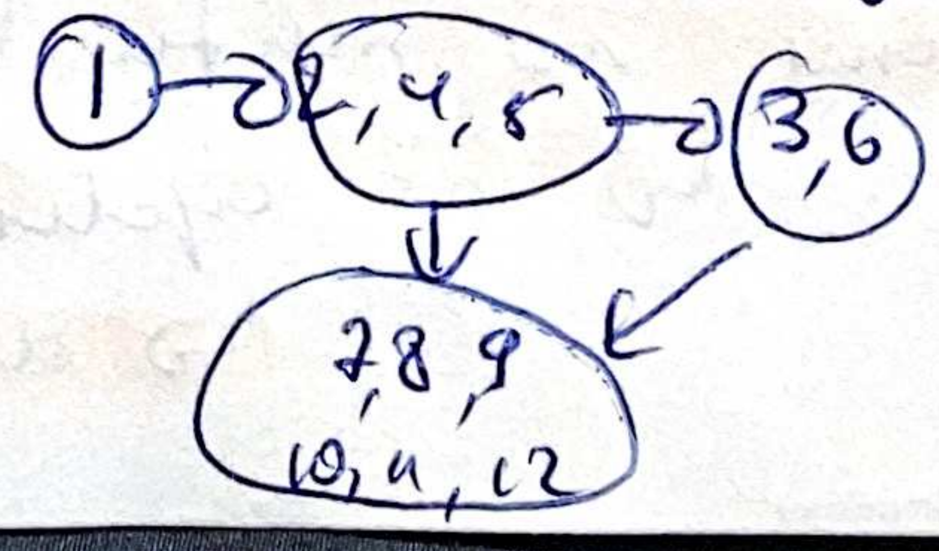


a is strongly connected to b if $\exists$ path $a \to b$ and $b \to a$

- reflexive
- symmetric
- transitive

definition

collapse SCCs



get DAG → do topological sort