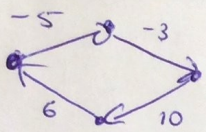
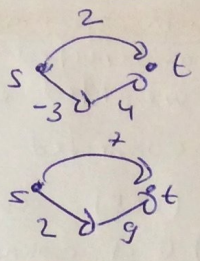
CS 124 — Lecture 6



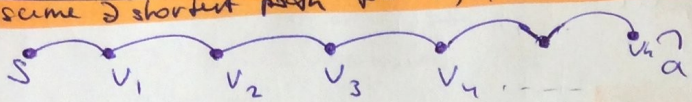
how to handle negative edges

idea: add smallest weight to all
- → wrong
- priority queue update won't work



Bellman - Ford

assume ∃ shortest path from $s, v_1, v_2, \dots v_k \dots a$



property: if $s \to a$ is shortest path, then each subpath is shortest

↳ inductively find shortest path

1st time loop — 2nd time loop

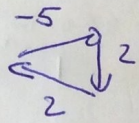
SSD2

```
for v ∈ V do
    dist[v] := ∞
    prev[v] := nil
dist[s] := 0
for i = 1 ... n-1                    O(V)
    for (v,w) ∈ E update (v,w)       O(E)
```

$O(E \cdot V)$

↓ After k rounds 1st k edges of the shortest path found

↑ one with negative weights, but does not allow for negative cycles



→ there is no finite # of steps because there is no shortest path

how to deal with negative cycles?

run through E one extra time

[redacted] no neg. cycles ⟺ nothing changes in nth loop

also once nothing changes, it cannot change in the next round

↳ [redacted] stop early

Graph Algs

Minimum spanning trees

* undirected graph
  weighted edges

Tree: connected and acyclic

Lemma (LTR): any 2 of these ⇒ 3rd

1) G is connected
2) G is acyclic
3) $|E| = |V| - 1$

Spanning tree ; given G:

$T \subseteq E$ on all of V (vertices of G)

Min spanning tree:

spanning tree minimizing $\sum_{e \in T} w(e) = w(T)$

Baseline alg:

for each spanning tree calculate its weight, keep the minimum

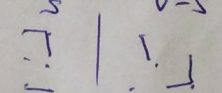
n	# spanning trees
2	1
3	3
4	16
5	125
n	n^{n-2}

↖ super exponential growth of the problem size

MST - an edge at a time

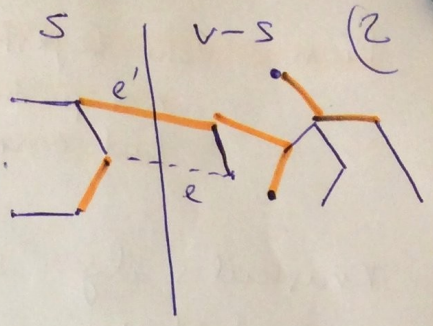
Cut property:

Let $X \subseteq T$, where T is an MST of G
Let $S \subset V$, such that no edge in X crosses from S to V−S
Let e be a min weight edge from S to V−S
Then $X \cup \{e\} \subseteq T'$ for some MST T'



→ can construct MST in a greedy fashion

Proof by contradiction



assume $e \notin T$ [redacted]

adding e creates a cycle

from $S \to V-S \to S$, since

$\exists e' \in T$, going from $S \to V-S$

let $T' = T \cup \{e\} - \{e'\}$

T' is a spanning tree, since the single cycle eliminated

$w(e) \leq w(e')$

$w(T') \leq w(T)$

if $w(e) < w(e')$ contradiction $\Rightarrow e \in T$

if $w(e) = w(e')$, then T' is another MST

= not yet added

= added

$|E| = |V|-1$ and acyclic $\Rightarrow$ connected $\Rightarrow$ spanning tree

$\Rightarrow$ e is in an MST

General algorithm form

$X = \emptyset$

Repeat until $|X| = n-1$

cut $\Rightarrow$ pick $S \subseteq V$ w/ no edge in X crossing $S, V-S$

exploit cut property $\Rightarrow$ get lightest edge e [redacted] between S and $V-S$

$X = X \cup \{e\}$

One way: force X at each step to be a connected subtree

Prim's

$H := \{S : 0\}$

for $v \in V$

  dist[v] := ∞, prev[v] := nil

dist[s] := 0

while $H \neq \emptyset$

  v := deletemin(H)

  S := S $\cup$ {v}

  for $(v,w) \in E$ and $v \in V-S$ do

    if dist[w] > length(v,w)

      dist[w] := ~~dist[v] + length(v,w)~~ length(v,w)

      prev[w] := v

      insert(w, dist[w], H)

go trough E once

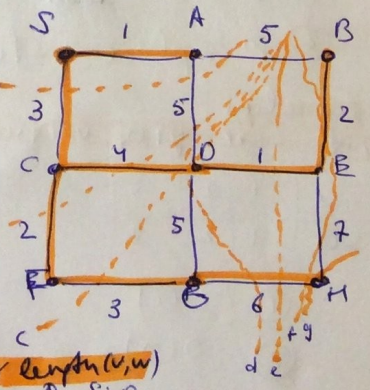
like Dij before but without adding to previous sum

$\Rightarrow$ Prim's alg. use priority queue (e.g. heap)

O(E insert + V delete)

heap O(E log V)

list $O(V^2)$



S: 0

A: 1, C: 3

~~A: 1~~, B: 5, D: 5

~~C: 3~~, B: 5, D: 5

~~D: 2~~, D: 4, B: 5

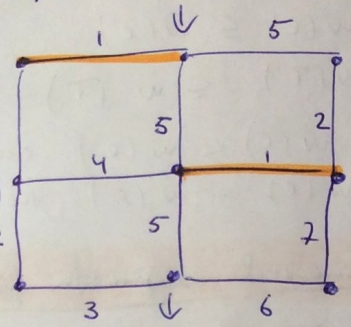
~~B: 3~~, D: 4, B: 5

~~D: 4~~, B: 5, H: 6

Prim: like Dijkstra, distance notion of different
not path length (sum of previous edges)
but connection length to the subtree

Kruskal's alg: second way

sort edges
in increasing order of edges
if the edge does not add a cycle
add it
} implicit cut



Proof

■ since it does not create a cycle

no cycle ⇒ not connected and |V|-1 > |E| G

if crosses a cut

connect disconnected parts, otherwise cycle

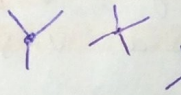
and

of all ■ a smallest edge

to cross this cut

⇒ cut property satisfied

checking if no cycle:

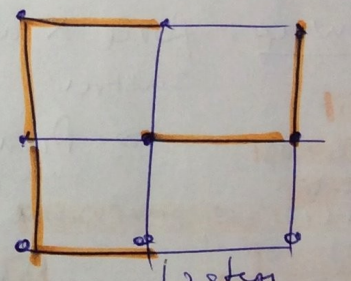


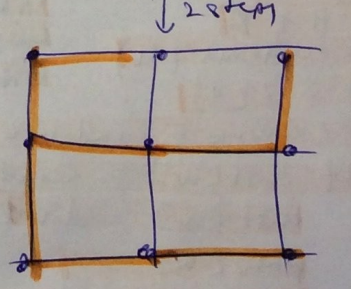
disjoint set data structures

sets: components of vertices

① ⇒ are 2 ■ elts in the same set

② ⇒ replace 2 sets by their union





baseline
with array of vertices:

v	1	2	3	...	n
set	1	2	2		3

① O(1)

② O(n) ← bad for Kruskal