Network Flows

LP - solvable

Combinatorial alg. augmenting paths, considering residual network

0 + 1

need for residual network

$O(VE^2)$ with BFS

Correctness proof for combinatorial alg.

s-t Max Flow = s-t Min CUT

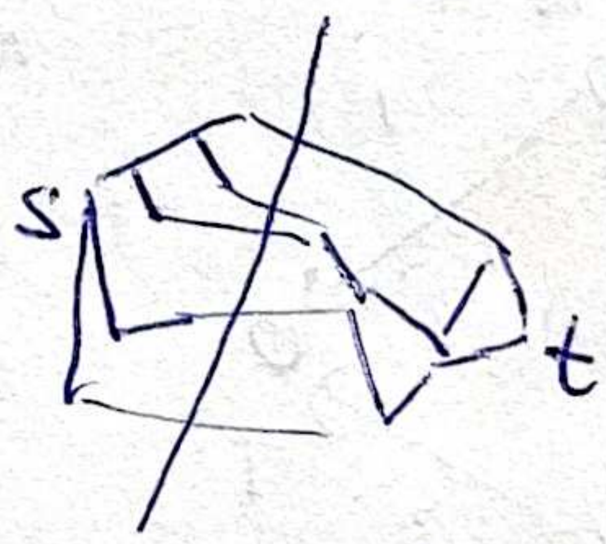
s-t cut is a partition of the vertices

$s \in V_1$, $t \in V_2$

$V_1 \cap V_2 = \emptyset$

$V_1 \cup V_2 = V$

cut



forward direction

max flow ≤ min cut

max flow ≤ any cut, including min cut

amount of flow from s to t goes through a cut, bounds possible flow

capacity of the cut

$$\sum_{\substack{e=(x,y)\\ x\in V_1,\, y \in V_2}} c(e)$$

min cut ≤ max flow

① assume alg. terminates (e.g. BFS, integer weights, flow addition bounded by the sum of all capacities) so must terminate

⇒ cannot get from s to t, otherwise an augmenting path exists [redacted]

⇒ ∃ a cut, saturated with algorithm's flow

⇒ Max flow ≥ algorithm's flow $\overset{\text{definition}}{\geq}$ algorithm's cut ≥ min cut

or =

Max Flow $\overset{=}{\geq}$ Alg's flow $\overset{=}{\geq}$ Alg's cut $\overset{=}{\geq}$ min cut $\overset{=}{\geq}$ max flow

$\Rightarrow$ must be all equalities

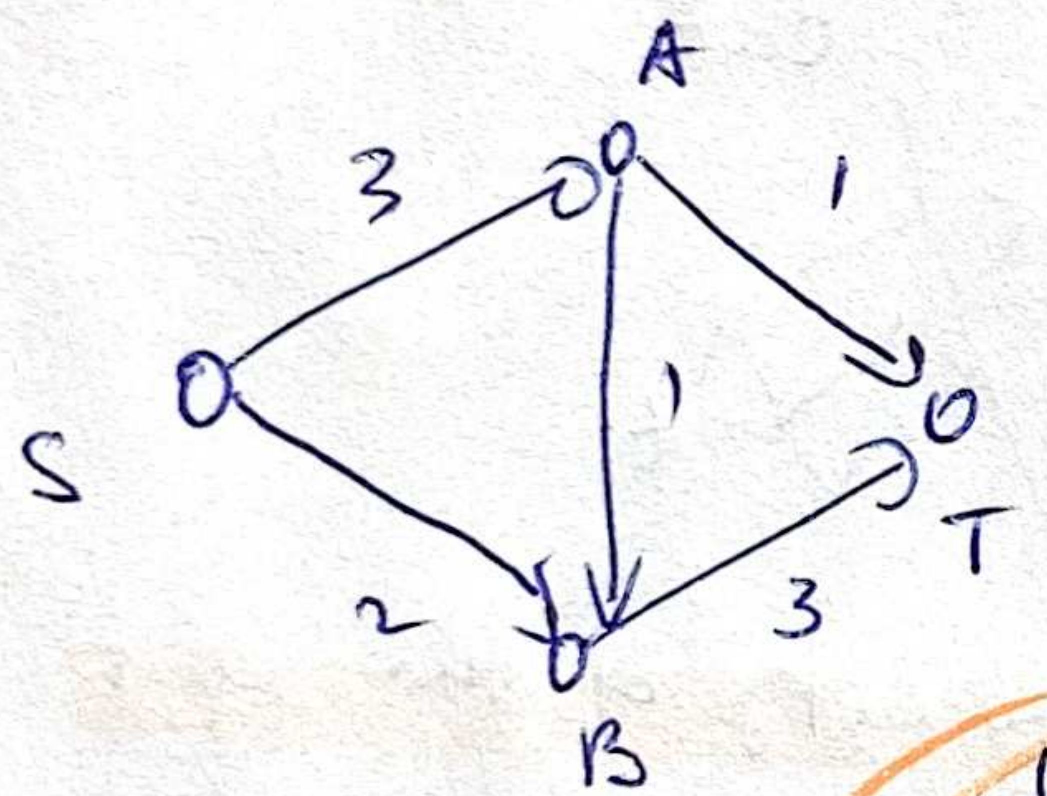
$\Rightarrow$ max flow = min cut
and algorithm returns it, is correct

note on cut construction

start with s, keep including nodes into V_1, where flow can still be pushed from V_1.

Duality

(min into max problems, and vice versa)



max flow formulation

$$\max f_{SA} + f_{SB}$$

$$\begin{aligned}
f_{SA} &\leq 3 \\
f_{SB} &\leq 2 \\
f_{AB} &\leq 1 \\
f_{AT} &\leq 1 \\
f_{BT} &\leq 3 \\
f_{SA} - f_{AB} - f_{AT} &= 0 \\
f_{SB} + f_{AB} - f_{BT} &= 0
\end{aligned}$$

$f \geq 0$

min cut

$y_e = 1$ if crosses cut
$\quad\;\; 0$ otherwise

$u_A = 1$ if A is on the set with S for the cut
$\quad\;\; 0$ otherwise

$$\min 3y_{SA} + 2y_{SB} + y_{AB} + y_{AT} + 3y_{BT}$$

$$\begin{aligned}
y_{SA} + u_A &\geq 1 \\
y_{SB} + u_B &\geq 1 \\
y_{AB} - u_A + u_B &\geq 0 \\
y_{AT} - u_A &\geq 0 \\
y_{BT} - u_B &\geq 0
\end{aligned}$$

$y \geq 0$

$$
\begin{array}{l|ccccc|c|c}
\text{max} & 1 & 1 & 0 & 0 & 0 & & \\
\hline
 & 1 & 0 & 0 & 0 & 0 & \le & 3 \\
 & 0 & 1 & 0 & 0 & 0 & \le & 2 \\
 & 0 & 0 & 1 & 0 & 0 & \le & 1 \\
 & 0 & 0 & 0 & 1 & 0 & \le & 1 \\
 & 0 & 0 & 0 & 0 & 1 & \le & 3 \\
 & 1 & 0 & -1 & -1 & 0 & = & 0 \\
 & 0 & 1 & 1 & 0 & -1 & = & 0 \\
\hline
 & \ge & \ge & \ge & \ge & \ge & &
\end{array}
$$

getting a just dual is a mechanical process

→ get a differently conceptual problem formulation

→ an LP has a dual

transposes (other [redacted] rearranged)

$$
\begin{array}{l|ccccccc|c|c}
\text{min} & 3 & 2 & 1 & 1 & 3 & 0 & 0 & & \\
\hline
 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & \ge & 1 \\
 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & \ge & 1 \\
 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & \ge & 0 \\
 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & \ge & 0 \\
 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & \ge & 0 \\
\hline
 & \ge & \ge & \ge & \ge & \ge & \text{unr} & \text{unr} & &
\end{array}
$$

Dual formulation

- transpose coeff. matrix
- max ⟺ min
- interchange obj. function (outer rearrange)
- non-neg. var for inequality, unrestr. variable for equality

2-Player Matrix Games (zero sum)

$$
\begin{array}{cc}
 & \text{column} \\
 & \begin{array}{ccc} R & P & S \end{array} \\
\text{row} \begin{array}{c} R \\ P \\ S \end{array} & \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}
\end{array}
$$

other games

$$
\begin{bmatrix} 3 & -2 & 4 \\ 4 & -1 & 2 \end{bmatrix}
$$

a strategy is a prob. distribution over choices

eg.
$$
\begin{array}{ccc} R & P & S \\ \left[\tfrac{1}{3} \right. & \tfrac{1}{3} & \left. \tfrac{1}{3} \right] \end{array}
$$

opt. strategy: no matter what strategy the opponent has, achieve optimal outcome

best opponent assumption
{ the opponent has, achieve optimal outcome
→ assume the opponent knows strategy and chooses optimal counter-strategy

Game

G_{ij} = $m \times n$ matrix

· strategy for row player
$(x_1 \text{ ---- } x_m)$
strategy column player
$(y_1 \text{ ---- } y_n)$

Expected payoff

$$\sum_{i,j} G_{ij} x_i y_j$$

ex. column player

$$\begin{array}{cc} y_1 & y_2 \\ \end{array}$$
$$\begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$$

ex row player

$$\begin{array}{c|c} x_1 & \\ x_2 & \end{array} \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$$

similarly
$-x_1 + x_2$

if column player picks
its exp. loss to row player is
$3x_1$ [redacted] $-2x_2$

transposes, duals

column player [redacted] [redacted] picks column minimizing loss

column player

min w
$w \geq 3y_1 - y_2$
$w \geq -2y_1 + y_2$
$y_1 + y_2 = 1$
$y_1, y_2 \geq 0$

minimize exp. loss

optimal player column strategy [redacted]

optimal row player strategy

row player

max z
$z \leq 3x_1 - 2x_2$
$z \leq -x_1 + x_2$

column player guarantees that expected win is $\leq$ than the smaller of two [redacted] it can have a strategy to always pick one or the other column whichever minimizes row player exp. winning

$x_1, x_2 \geq 0$
$x_1 + x_2 = 1$

maximize exp. win

duals => same solution

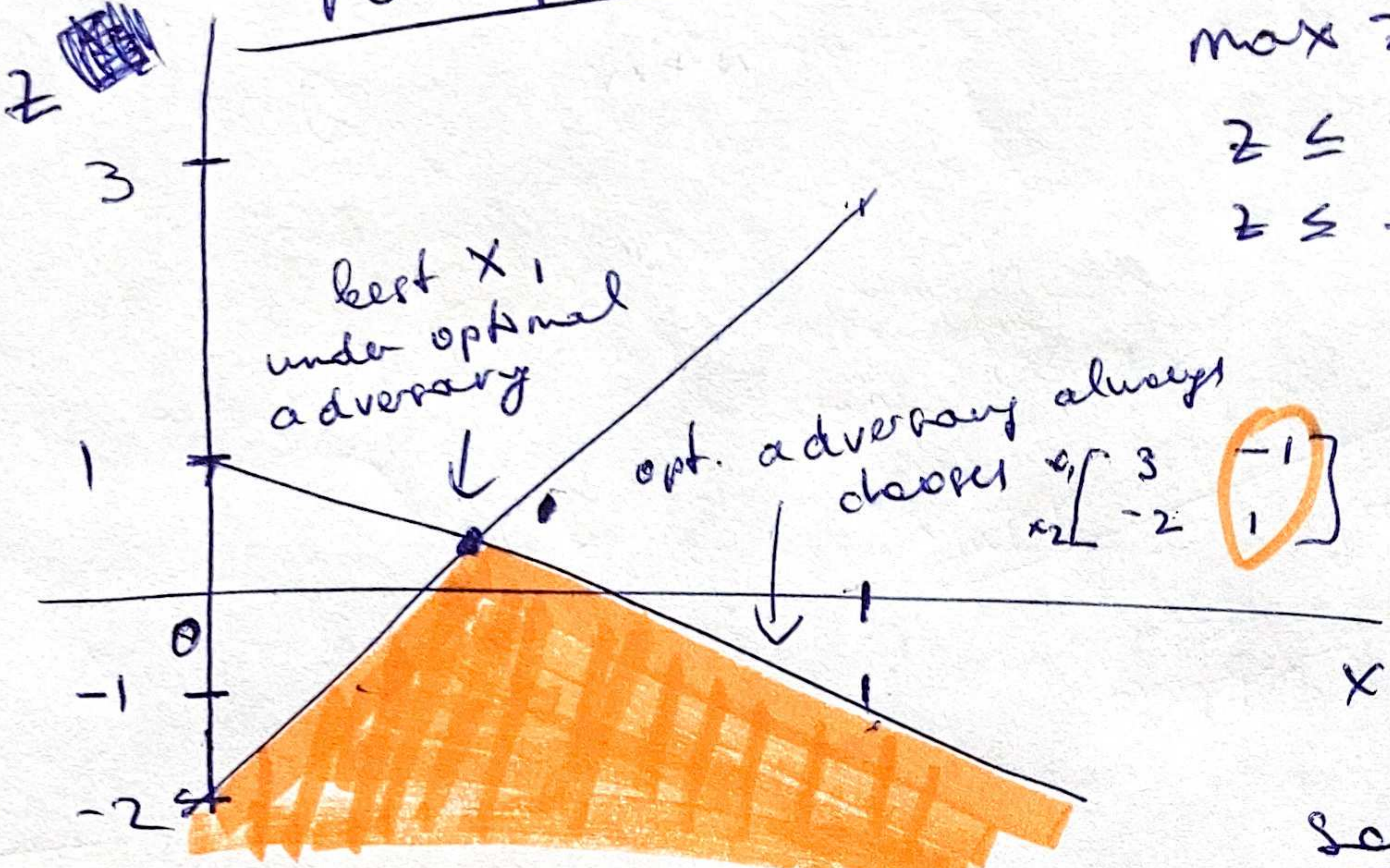
value of the game!

$$\max_x \min_y \sum x_i y_j G_{ij} = \min_y \max_x \sum x_i y_j G_{ij}$$

[redacted]

order does not matter

row player



$$\begin{array}{c c} x_1 & \\ x_2 & \end{array}\begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$$

max z

$z \leq 3x_1 - 2x_2$

$z \leq -x_1 + x_2$

$x_1 + x_2 = 1$

$x_1, x_2 \geq 0$

solve game

$3x_1 - 2x_2 = -x_1 + x_2$

$4x_1 = 3x_2$

$x_1 = \frac{3}{7} \quad x_2 = \frac{4}{7}$

value: $-\frac{3}{7} + \frac{4}{7} = \frac{1}{7}$

column player

$3y_1 - y_2 = -2y_1 + y_2$

$5y_1 = 2y_2$

$y_1 = \frac{2}{7} \quad y_2 = \frac{5}{7}$

value: $\frac{1}{7}$ ✓

duals:

player perspectives