Lecture 21 CS124

Heuristics, Bubble search

greedy algs → approximations / heuristics

Bubble search: "randomized greedy"

Biggest – 1/2 prob. put it

2nd biggest – 1/4

3rd biggest – 1/8

idea:

1. **keeps intuition of greedy**
2. but **allows to restart and have a different order**

↳ **restart, run, get best soln. → works well in practice**

whenever given a greedy alg.
can randomize it like Bubble search

Max-Cut Approximation

$G = (V, E)$ maximize # of crossing edges
(also applies to weighted case)

$V_1 \cap V_2 = \emptyset$

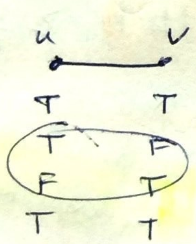
$V_1 \cup V_2 = V$

Randomized approximation:

means: Expected value of soln. is within a factor C of optimal.

flip coin for each vertex

V_1 V_2



given an edge
prob. it is in the cut?

$\frac{1}{2}$

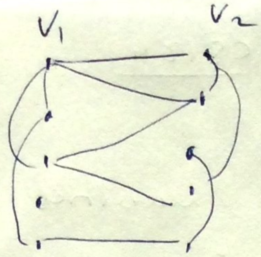
edge ID r.v.s linearity of expectation

$$E(\text{Alg. cut}) = \frac{1}{2}|E| \geq \frac{1}{2} \text{ Max cut}$$

randomized: **not guaranteed to get within $\geq \frac{1}{2}$ Max cut**, on average

we want a guarantee.

Greedy



Local search

current solution : partition

move : move 1 vertex to other side if # edges crossing < # edges on the same side

→ derandomization of the randomized algorithm

same $\frac{1}{2}$ approximation

Termination

- each step, increase # edges on the cut

The cut is finite → terminates in $O(|E|)$

Approximation bound

$\delta(v) = deg(v)$

sum of all edges crossing ≥ # edges on the same side

each vertex has more edges crossing, than on the same side, otherwise move it to the other side

size of cut →

$$C = \left(\sum_{v \in V_1} (v,w)\ w \in V_2 + \sum_{v \in V_2} (v,w)\ w \in V_1 \right) \cdot \frac{1}{2}$$

counted twice

$$\geq \frac{1}{2} \left(\sum_{v \in V_1} \frac{1}{2} \delta(v) + \sum_{v \in V_2} \frac{1}{2} \delta(v) \right)$$

count twice →

$$= \frac{1}{4} \Big(\underbrace{\sum_{v} \delta(v)}_{2|E|} \Big) = \frac{1}{2} |E| \checkmark$$

$$\boxed{C \geq \frac{1}{2} |E|} \geq \frac{1}{2} \text{ Max cut}$$

guarantee, no expectation

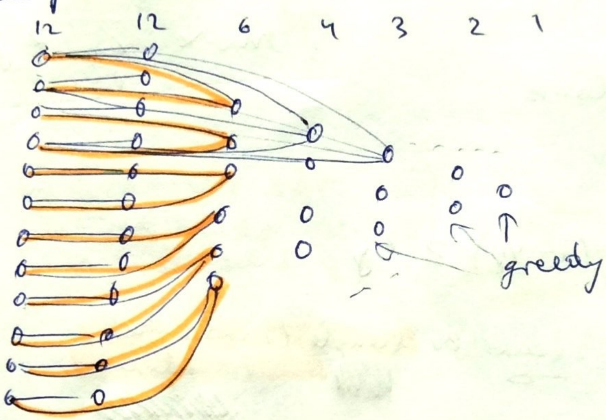
derandomization

Set Cover – $\log n$ Approximation
greedy alg.

vertex Cover

$G = (V, E)$
Find $U \subseteq V$, s.t.
every edge is adjacent to a vertex in U
↳ [redacted] **reduction** [redacted] to **set cover**
for each vertex create set corresponding
to the edges that it covers
[redacted]

Greedy on vertex cover? <u>NOT great</u>



$|VC| = 12$ $\quad m \quad \frac{m}{2} \quad \frac{m}{3} \quad \frac{m}{4} \quad \cdots \quad \frac{m}{m}$

log m factor off the OPT

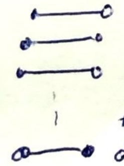
↳ not a const. factor

2-Approximation for vertex cover

- take any edge
- put both vertices in cover
- remove edge
- remove vertices and all adjacent edges

each edge taken is <u>disjoint</u>, since two vertices and adjacent edges thrown out

at least one vertex in <u>each</u> pair has to be in VC to cover the connecting edge



$2 \cdot \#$ disjoint edges $\geq$ OPT VC $\geq$ # of disjoint edges

Alg $\leq$ 2 OPT VC

Max-SAT (at least NP hard)

satisfy as many clauses as possible

Randomized alg. (baseline)

[redacted] flip coin for each variable

count satisfied clauses

k literals in a clause

$(x_1 \lor x_2 \lor \dots x_k)$

$\underbrace{1-2^{-k}}$

prob. for [redacted] satisfied clause

my comment:

$E[\#\text{sat clauses}] = \underset{\#\text{all clauses}}{n} \cdot (1-2^{-k})$

$\max \sum_{i=1}^{n} z_j$

Linear program (integer)

Pretend it is a linear program

$(x_2 \lor \bar{x}_4 \lor x_6 \lor \bar{x}_8)$

z_j = var for jth clause

$0 \le z_j \le 1$

$0 \le y_i \le 1$

$y_2 + (1-y_4) + y_6 + (1-y_8) \ge z_j$

solution: $y_1 = 0.7$ $\xrightarrow{\text{round or}}$ sample (randomized [redacted] rounding) [redacted]

$y_4 = 0.2$

$y_6 = 0.1$

$\vdots$

rounding vs. randomized rounding

$x_1 \lor x_2 \lor x_3$

$x_1 = 0.4$

$x_2 = 0.4$

$x_3 = 0.4$

rounding sets all to 0

but to LP, this clause has value 1

but with randomized rounding

$0.6^3 = 0.216$

$\Rightarrow$ expected value of 0.784 for the clause

Suppose:

← positive without loss of generality

$C = (x_1 \vee x_2 \ldots \vee x_k)$ $\quad z_i = \beta$ [redacted]

$y_1 \ldots y_k$ [redacted]

$\max \prod_i (1 - y_i) \quad \sum y_i \geq \beta$ $\quad$ worst case for randomized rounding in a clause

solve $y_i = \frac{\beta}{k}$ ↖ maximizes prob. that the clause is unsatisfied

clause $\Rightarrow$ prob rounds to 0 $\leq \left(1 - \frac{\beta}{k}\right)^k$

[redacted] fact

prob rounds to 1 $\geq 1 - \left(1 - \frac{\beta}{k}\right)^k \geq \left(1 - \frac{1}{e}\right)\beta$

$\geq$ chance that a clause rounds to SAT [redacted] in randomized rounding

formula

E[clauses SAT in randomized rounding]

By linearity of expectation

$$\geq \sum_j \left(1 - \frac{1}{e}\right)\beta_j = \left(1 - \frac{1}{e}\right)\sum \beta_j =$$

$$= \left(1 - \frac{1}{e}\right)\sum z_j$$

by linear program (if taking integers, than worse)

$$\geq \underbrace{\left(1 - \frac{1}{e}\right)}_{\text{within a constant factor of optimum in its expectation}} OPT$$