Design and Simulate the Aerodynamics of Propellers

Udemy Course

Actuator Disk / Momentum Theory

Assumptions :

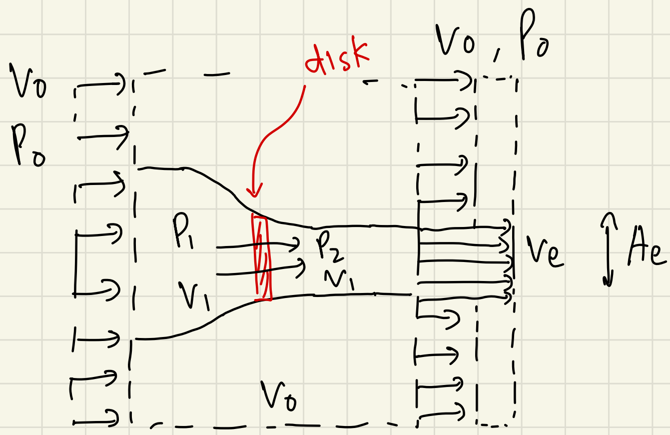
→ neglect any rotational flow imparted to the fluid

→ incompressible flow at low Mach numbers

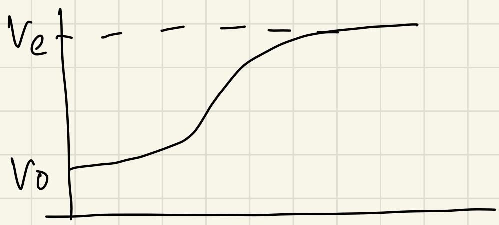
→ flow outside propeller streamtube has constant stagnation pressure

→ steady flow (blades are smeared)

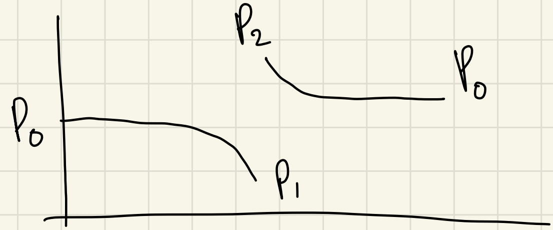
→ velocities across the disk vary in a continuous smooth manner, but the pressure changes discontinuously



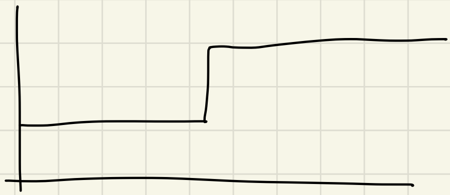
Velocity profile across the disk



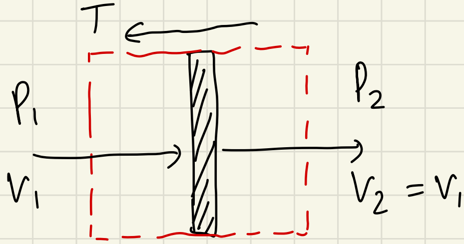
Static pressure across the disk



stagnation pressure / total pressure



thrust, pressure, velocities across the disk



Power due to momentum

power exerted by the disk is equal to the change in kinetic energy as the fluid passes through the propeller

$$P = \frac{1}{2} W \left(V_e^2 - V_0^2 \right)$$

→ P : power (watt)
→ W : mass flow rate (kg/s)
→ V_e : exit velocity (m/s)
→ V_0 : upstream velocity (m/s)

$$W = \rho A V$$

→ ρ : air density (kg/m³)
→ A : disk area (m²)
→ V : velocity at the disk (m/s)

Power due to thrust

power exerted by the thrust applied to the fluid and the velocity of the fluid

$$P = T\,v_{disk} = \underbrace{(P_2 - P_1)\,A_{disk}}\;V_{disk}$$

thrust = pressure differene

$$P = T\,v_{disk} = \underbrace{W\,(V_e - V_0)}\;V_{disk}$$

thrust = change in momentum

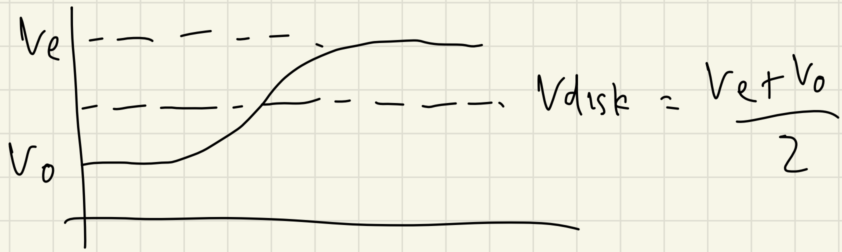
$$P = P$$

$$\frac{1}{2}W\left(V_e^2 - V_0^2\right) = W\,(V_e - V_0)\,V_{disk}$$

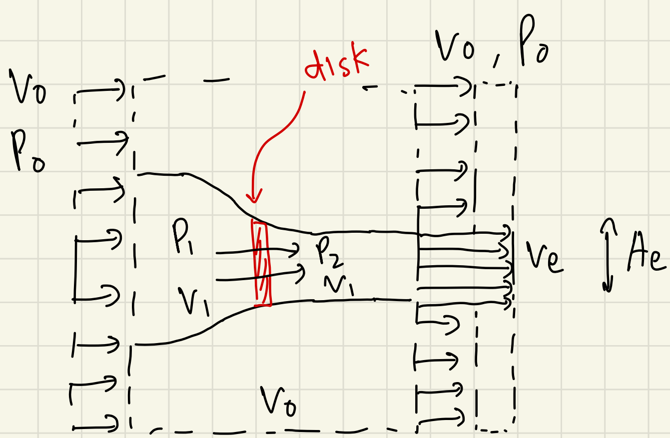
$$\frac{1}{2}\cancel{W}\,(V_e + V_0)\,\cancel{(V_e - V_0)} = \cancel{W}\,\cancel{(V_e - V_0)}\,V_{disk}$$

$$\Rightarrow \quad V_{disk} = \frac{V_e + V_0}{2}$$

which is logical since



Applying Bernoulli eq for total pressure where there is no discontinuity in pressure and velocity



$$P_1 + \frac{1}{2}\rho V_{disk}^2 = P_0 + \frac{1}{2}\rho V_0^2$$

$$\Rightarrow \frac{1}{2}\rho V_{disk}^2 - P_0 = \frac{1}{2}\rho V_0^2 - P_1 \quad \text{---} \quad \text{eq}(1)$$

$$P_2 + \frac{1}{2}\rho V_{disk}^2 = P_0 + \frac{1}{2}\rho V_e^2$$

$$\Rightarrow \frac{1}{2}\rho V_{disk}^2 - P_0 = \frac{1}{2}\rho V_e^2 - P_2 \quad \text{---} \quad \text{eq}(2)$$

from eq (1) & (2)

$$\frac{1}{2}\rho V_e^2 - P_2 = \frac{1}{2}\rho V_0^2 - P_1$$

$$P_2 - P_1 = \frac{1}{2}\rho\left(V_e^2 - V_0^2\right)$$

V_{disk} is not measurable, best to remove it

Calculating thrust from continuity

we know $T = \dot{W}(v_e - V_0)$

and $\dot{W} = \rho A_{disk} V_{disk}$

thus,

$$\dot{W} = \rho A_{disk} \frac{(V_e + V_0)}{2};$$

so : $$T = \rho A_{disk} \frac{(V_e + V_0)}{2} (V_e - V_0)$$

$$T = \rho A_{disk} \frac{(V_e^2 - V_0^2)}{2}$$

ρ, A, V_0 are known, but not for V_e

rewritten :

$$\left(\frac{V_e}{V_0}\right)^2 = \frac{T}{\frac{1}{2} \rho A_{disk} V_0^2} + 1$$

knowing that $V_{disk} = \frac{V_e + V_0}{2}$, we can write

$$V_{disk} = \frac{V_0}{2}\sqrt{\frac{T}{\frac{1}{2}\rho A_{disk} V_0^2} + 1} + \frac{V_0}{2}$$

thus, the minimum power for a specific thrust is:

$$P = TV_{disk} = \frac{TV_0}{2}\sqrt{\frac{T}{\frac{1}{2}\rho A_{disk} V_0^2} + 1} + \frac{TV_0}{2}$$

$$P = T\sqrt{\left(\frac{V_0}{2}\right)^2\left[1 + \frac{T}{\frac{1}{2}\rho A_{disk} V_0^2}\right]} + \frac{TV_0}{2}$$

$$\boxed{P = T\left[\frac{V_0}{2} + \sqrt{\frac{V_0^2}{4} + \frac{T}{2\rho A_{disk}}}\right]}$$

Calculating efficiency

- propulsive efficiency is the measure of how much of the power given to the actuator is converted to thrust

$$\eta_{propulsive} = \frac{2}{1 + \frac{V_e}{V_0}} = \frac{2}{1 + \left(\frac{T}{\frac{1}{2}\rho A_{disk} V_0^2} + 1\right)^{1/2}}$$

Dimensionless Numbers

1. Advance ratio (distance advanced per revolution)

$$J = \frac{V_0}{nD}$$; n = revolutions per second

D = diameter of propeller

2. Thrust coefficient

$$C_T = \frac{T}{\rho n^2 D^4}$$

T = thrust in N

ρ = air density in kg/m^3

3. Torque coefficent

$$C_Q = \frac{Q}{\rho n^2 D^5}$$

4. Propulsive efficiency : ratio of useful power out to mechanical power supplied to the shaft :

$$\eta_{prop} = \frac{P_{out}}{P_{in}} = \frac{T v_0}{2\pi n Q}$$

$$= \frac{C_T \rho n^2 D^4 v_0}{C_Q \rho n^2 D^5 2\pi n} = \frac{C_T}{C_Q}\frac{J}{2\pi}$$

$$\boxed{\eta_{prop} = \frac{C_T}{C_Q}\frac{J}{2\pi}}$$

Blade Element Theory (BET)

- BET allows a more detailed analysis compared to momentum theory by considering the shape of the blade
- the blade is divided into a number of small sections along its length that act independently of surrounding elements
- considers the flow as two dimensional
- works well for lightly loaded two- or three-bladed propellers, except neart the hub

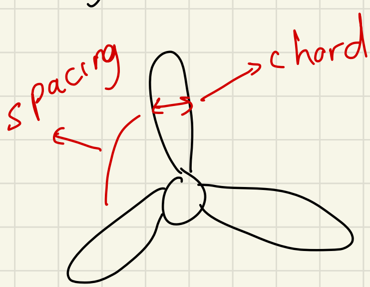
Solidity & validity

individual propeller blades can be assumed to operate in isolation without interference from other blades when the spacing-to-chord ratio is sufficiently high

$$\frac{s}{c} \gg 1$$

s : spacing / circumferential distance between blades

c : chord length of the blade



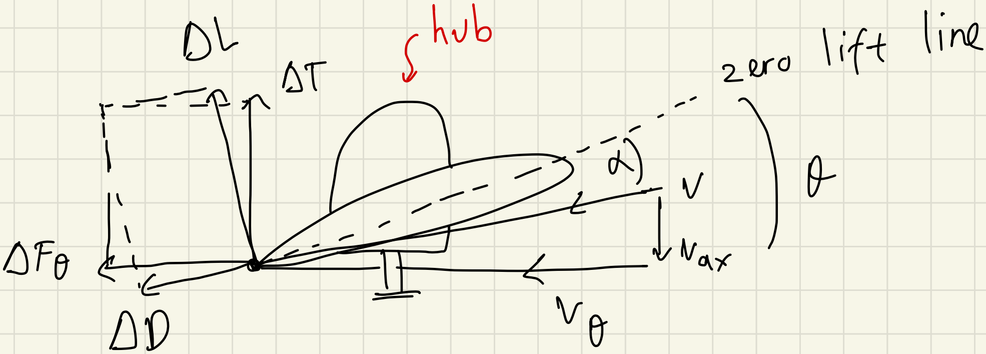
Solidity

$$\sigma = \frac{Bc}{\pi r}$$; B : number of blades

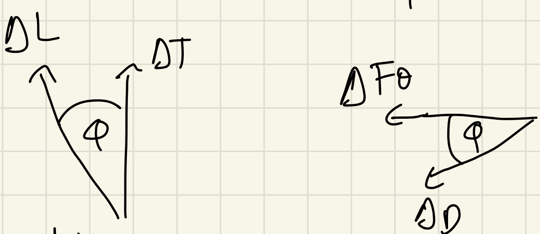
BET is valid if $\sigma \ll 1$

Blade Elements

- blades are divided into smaller elements and sections
- a force balance is applied to each section to produce Lift & Drag and therefore the propeller's thrust & torque
- the section local flow velocity is the vector summation of the axial flow velocity V_{ax} and the angular flow velocity V_θ
- as the propeller's blades are set at a given geometric pitch angle θ, the local flow velocity creates a flow angle of attack α



- the difference between the lift and thrust vectors is $\varphi = \theta - \alpha$



- elemental thrust

$$\Delta T = \Delta L \cos \varphi - \Delta D \sin \varphi$$

$$\Delta F_\theta = \Delta D \cos \varphi + \Delta L \sin \varphi$$

- the torque required to turn that element

$$\Delta Q = r \, \Delta F_\theta$$

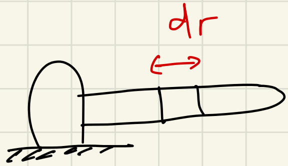
r : distance between element and the axis of rotation of the propeller

∘) Elemental lift & drag equations

$$\Delta L = \frac{\rho V^2}{2} \times C_l \times c \times dr$$

$$\Delta D = \frac{\rho V^2}{2} \times C_d \times c \times dr$$

dr is element width



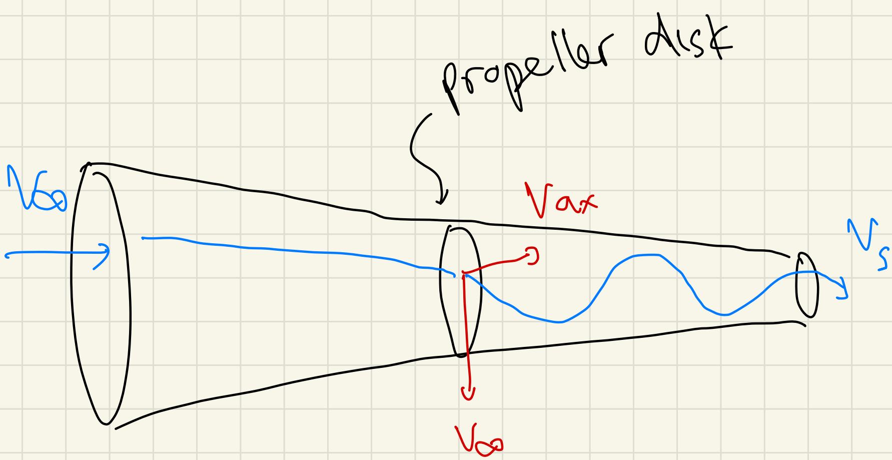
Total thrust & torque

$$\Delta T = \frac{\rho V^2}{2} \left(C_l \cos\varphi - C_d \sin\varphi \right) B\, c\, dr$$

$$\Delta Q = \frac{\rho V^2}{2} \left(C_l \sin\varphi + C_d \cos\varphi \right) B\, c\, r\, dr$$

Real Axial and Radial Velocities

- V_{ax} is roughly equal to the forward velocity of the aircraft but is increased by the propeller's own induced axial flow
- V_θ is roughly equal to the blade section's angular speed (Ωr) but is reduced slightly due to the swirling nature of the flow induced by the propeller



•) In order to calculate V_{ax} & V_θ we apply both axial and angular momentum equations

•) the induced components can be defined as factors increasing or decreasing the major flow components

Inflow factors

$$V_{ax} = (1+a) V_\infty \quad ; \quad V_\theta = (1-a_\Omega)\Omega r$$

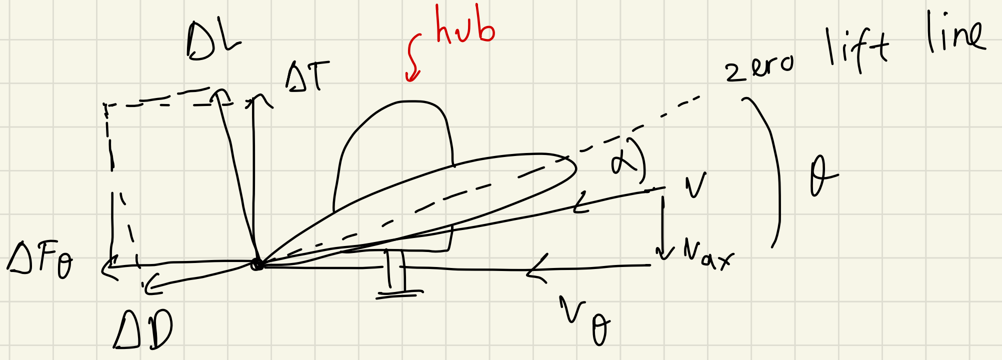
a : axial inflow factor
a_Ω : angular inflow factor (swirl factor)

local flow velocity V :

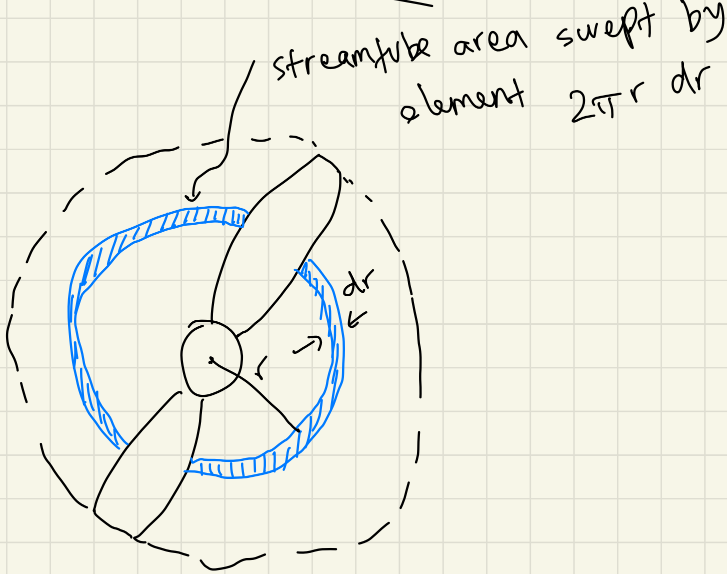
$$V = \sqrt{V_{ax}^2 + V_\theta^2}$$

angle of attack of the blade :

$$\alpha = \theta - \tan^{-1}\left(\frac{V_{ax}}{V_\theta}\right)$$



Conservation of axial momentum



$$\Delta T = \rho V_{ax} (V_s - V_\infty)\, 2\pi r\, dr$$

by applying bernoulli : $V_s = V_\infty (1 + 2a)$

hence,

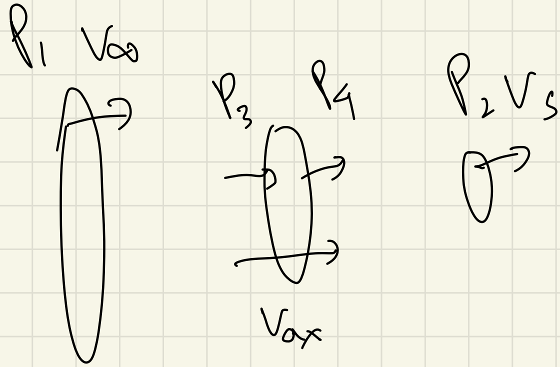
$$\Delta T = \rho V_\infty (1+a) \left(V_\infty (1+2a) - V_\infty \right) 2\pi r\, dr$$

thus,

$$\Delta T = \rho V_\infty^2 a (1+a)\, 4\pi r\, dr$$

Proof for $V_s = V_\infty(1+2a)$

Bernoulli



$$P_1 + \frac{1}{2}\rho V_\infty^2 = P_3 + \frac{1}{2}\rho V_{ax}^2$$

$$P_2 + \frac{1}{2}\rho V_s^2 = P_4 + \frac{1}{2}\rho V_{ax}^2$$

$$\Rightarrow \quad P_1 = P_2$$

$$\Rightarrow \quad P_4 - P_3 = \frac{1}{2}\rho\left(V_s^2 - V_\infty^2\right)$$

we also know that thrust is equal to change in axial momentum flux

$$T = \frac{d}{dt}\left(m\left(V_S - V_\infty\right)\right)$$

$$= \frac{dm}{dt}\left(V_S - V_\infty\right)$$

$$= \rho V_{ax} A_{disk} \left(V_S - V_\infty\right)$$

and $T = \left(P_4 - P_3\right) A_{disk}$

thus

$$\rho V_{ax}\left(V_S - V_\infty\right) = \frac{1}{2}\rho\left(V_S^2 - V_\infty^2\right)$$

$$V_{ax} = \frac{\left(V_S + V_\infty\right)}{2}$$

because $V_{ax} = V_\infty(1+a)$

$$2V_\infty(1+a) = V_s + V_\infty$$

$$\boxed{V_s = V_\infty(1+2a)}$$

QED!

Remember for conservation of axial momentum:

$$\Delta T = \rho V_\infty^2 a(1+a) 4\pi r\, dr$$

Conservation of angular momentum

$$\Delta Q = \Delta F_\theta \, r$$

$$\Delta Q = \rho V_{ax} \left(V_{\theta,s} - V_{\theta,\infty} \right) r \cdot 2\pi r \, dr$$

since

$$\Delta F_\theta = \rho V_{ax} \left(V_{\theta,s} - V_{\theta,\infty} \right) 2\pi r \, dr$$

force = delta momentum

with $V_{\theta,s}$ is angular velocity in the slipstream and $V_{\theta,\infty}$ is the angular flow velocity in the upstream

we know

$$V_{\theta,\infty} = 0$$

$$V_{\theta,s} = 2 a_\Omega \Omega r$$

Proof for $V_{\theta,s} = 2a_\Omega \Omega r$

- blade tangential speed : Ωr
- air tangential speed at disk : $V_{\theta,d}$
- relative tangential speed

$$V_\theta = \Omega r - V_{\theta,d}$$

- by definition

$$V_\theta = \Omega r (1 - a_\Omega)$$

hence

$$\Omega r - V_{\theta,d} = \Omega r - \Omega r a_\Omega$$

So

$$V_{\theta,d} = \Omega r a_\Omega$$

→ actuator disk theory says that for an inviscid thin disk velocity at disk takes average of its upstream & downstream

thus $$V_{\theta,d} = \frac{(0 + V_{\theta,s})}{2} = \frac{V_{\theta,s}}{2}$$

since $V_{\theta,\infty} = 0$

so, $$a_{\Omega}\,\Omega r = \frac{V_{\theta,s}}{2}$$

$$\boxed{V_{\theta,s} = 2\,a_{\Omega}\,\Omega r}$$

QED!

hence for conservation of angular momentum (remember):

$$\Delta Q = \rho V_\infty (1+a)(2 a_\Omega \Omega r) 2\pi r^2 dr$$

$$\Delta Q = \rho V_\infty (1+a) a_\Omega \Omega 4\pi r^3 dr$$

In a nutshell

T : thrust
Q : torque

$$\Delta T = \rho V_\infty^2 a(1+a) 4\pi r dr$$

$$\Delta Q = \rho V_\infty (1+a) a_\Omega \Omega 4\pi r^3 dr$$

Iterative Solution Process

1. Make initial guess values for inflow factors a and a_Ω
2. Use these guess values to find the flow angle on the blade element
3. Use blade section properties to produce first estimate of the element thrust ΔT & torque ΔQ
4. With these approximate ΔT & ΔQ we will produce next estimate values for inflow factors a & a_Ω

5. Steps are repeated until a & a_Ω converge

6. Once they converge, we will have final values for ΔT & ΔQ which will be integrated along the blade to get T & Q

Reynolds Number

$$Re = \frac{V_\infty c}{\nu}$$

V_∞ : airflow velocity (m/s)

c : chord length

ν : kinematic viscosity

airfoil's C_L & C_D are calculated using XFOIL (MIT Aero)

Calculating Reynold's number for the following case

Propeller

- In this course we will look at a Jabiru light aircraft propeller.
- The following dimensions and characteristics were measured in lab taking 12 sections of 0.05m (5cm):

SECTION (m)	Thickness to chord ratio	PITCH (rad)	CHORD LENGTH (m)	THICKNESS (m)
0,150	0,539	0,570	0,106	0,057
0,200	0,514	0,540	0,111	0,057
0,250	0,485	0,506	0,115	0,056
0,300	0,457	0,475	0,118	0,054
0,350	0,427	0,441	0,119	0,051
0,400	0,401	0,413	0,115	0,046
0,450	0,381	0,391	0,105	0,040
0,500	0,349	0,356	0,100	0,035
0,550	0,349	0,356	0,092	0,032
0,600	0,342	0,349	0,082	0,028
0,650	0,345	0,352	0,072	0,025
0,700	0,335	0,342	0,063	0,021

- Number of Blades: 2
- Diameter: 1.52m

Aircraft

When necessary, we will consider that the Jabiru propeller is mounted on the popular Cirrus SR-22 with the following characteristics:

Cirrus SR22 characteristics and performance	
Wing aspect ratio	9.9
Gross wing area	13.9 m^2
Max Takeoff Weight	1542 kg
Takeoff run	314 m
Rate of climb	427 m/min
Cruising speed	185 kt = 95.17 $m.s^{-1}$
Stalling speed (flaps down)	60 kt = 30.87 $m.s^{-1}$
Takeoff speed	72 kt = 37.04 $m.s^{-1}$
Climb speed	88 kt = 45.27 $m.s^{-1}$
Engine Type and performance	
Company	Continental
Model	IO-550-N
Power	310 HP= 231kW
Max RPM	2700

Flight phase	Takeoff	Climb	Cruise
Altitude	Sea level	6000 ft	12000 ft
Density	1.225 $kg.m^{-3}$	1.024 $kg.m^{-3}$	0.849 $kg.m^{-3}$
Velocity	37.04 $m.s^{-1}$	45.27 $m.s^{-1}$	95.17 $m.s^{-1}$
RPM	2700	2650 (2600 initially)	2680 (2500 initially)
Advance ratio	0.540	0.639	1.49

Re calculation at cruise

assumption

$$V_{ax} = V_{cruise} = 95.17 \text{ m/s}$$

weighted average around this

$$V_\theta = RPM \frac{2\pi}{60} r_w = 2500 \cdot \frac{2\pi}{60} \cdot \frac{1.52}{2} \cdot 0.7$$

weighted average

$$= 139.3 \text{ m/s}$$

thus $V = \sqrt{V_{ax}^2 + V_\theta^2} = 168.7 \text{ m/s}$

→ average chord value of 0.095 m (approximate)

→ kinematic viscosity at 12000 ft is 1.9687×10^{-5}

$$Re = \frac{168.7 \cdot 0.095}{1.9687 \times 10^{-5}} = 814{,}080$$

Xfoil Results

Using the NACA 4412 airfoil and the Reynolds Number calculated in the previous slide of $Re = 814080$ we obtain the following results:

Alfa (deg)	Cl	Cd
-13	-0,5609	0,09348
-12	-0,5198	0,08719
-11	-0,4634	0,08593
-8	-0,4024	0,01286
-7	-0,2971	0,0113
-6	-0,1886	0,0101
-5	-0,0784	0,00924
-4	0,0323	0,00847
-3	0,1434	0,00797
-2	0,2541	0,00761
-1	0,3642	0,00739
0	0,4726	0,00694
3	0,8031	0,00711
4	0,9104	0,00765
5	1,0169	0,00822
6	1,1208	0,00893
7	1,2194	0,00996
8	1,3035	0,01192
9	1,3692	0,0148
10	1,4235	0,01754
11	1,476	0,02056
12	1,5216	0,02434
13	1,556	0,02952
14	1,5802	0,03634
15	1,5856	0,04603

Eliott Wertheimer

Core Equations

⇒ Inflow factors

$$V_{ax} = (1 + a)\, V_\infty$$

$$V = \sqrt{V_{ax}^2 + V_\theta^2} \quad ; \quad V_\theta = (1 - a_\Omega)\, \Omega r$$

⇒ Inflow angles

$$\varphi = \tan^{-1}\left[\frac{V_{ax}}{V_\theta}\right] \quad ; \quad AoA = \overset{\text{local + global}}{\text{pitch}} - \varphi$$

⇒ From Blade Element

$$\Delta T = \frac{\rho V^2}{2}\left(C_l \cos\varphi - C_d \sin\varphi\right) B\, c\, dr$$

$$\Delta Q = \frac{\rho V^2}{2}\left(C_l \sin\varphi + C_d \cos\varphi\right) B\, c\, r\, dr$$

⇒ From Momentum

$$\Delta T = \rho V^2 a (1 + a)\, 4\pi r\, dr$$

$$\Delta Q = \rho V (1 + a)\, a_\Omega\, \Omega\, 4\pi r^3\, dr$$

Steps :

1. Guess initial values for a and a_Ω
2. Calculate inflow velocities V_{ax} & V_θ and then $V = \sqrt{V_{ax}^2 + V_\theta^2}$
3. Calculate φ and AoA from local pitch angle (+ global pitch)
4. Calculate C_L & C_D from AoA using interpolated C_L & C_D functions
5. Calculate ΔT & ΔQ using equations from Blade Element
6. Calculate new inflow factors from momentum equations

$$a_{new} = \Delta T / \left(\rho V^2 (1+a) 4\pi r \, dr \right)$$

$$a_{\theta,new} = \Delta Q / \left(\rho V (1+a) \Omega 4\pi r^3 dr \right)$$

7. Calculate middle values for inflow factors

$$a_{middle} = (a + a_{new})/2$$

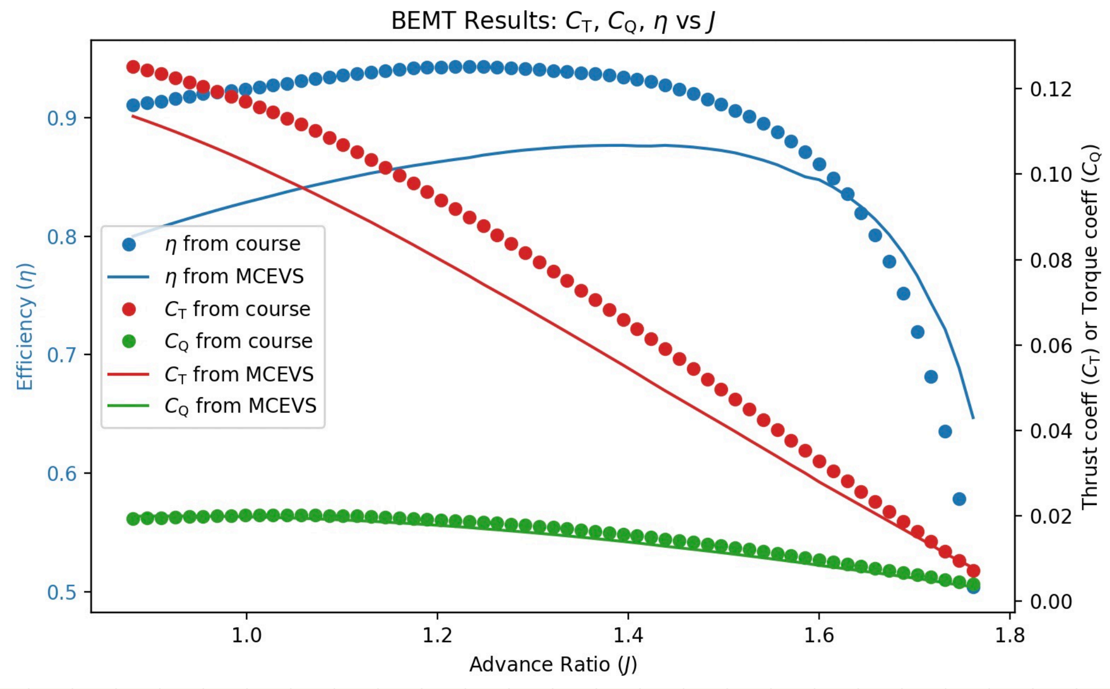
$$a_{0,middle} = (a_0 + a_{0,new})/2$$

8. Check for convergence

If $abs(a_{middle} - a) < 1 \times 10^{-5}$

and $abs(a_{0,middle} - a_0) < 1 \times 10^{-5}$

then STOP!



The reason for the discrepancy between the course & MCEVS:

→ MCEVS implements Prandtl's tip/root loss factors

→ which is why MCEVS underpredicts the efficiency