Modeling Interval-Scale Item Response Data with the Beta Factor Model

Alfonso J. Martinez

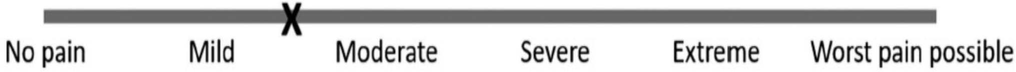
April 28, 2023

Department of Psychological and Quantitative Foundations
Department of Statistics and Actuarial Sciences

IOWA

Overview

- Purpose of research: develop a psychometric model (beta IFA) for analyzing interval-scale data from continuous rating scales (CRS)



- Existing approaches (e.g., normal-theory factor analysis) are not always appropriate for CRS data
- Features of beta IFA model: (1) can directly model skewness in the responses (no data transformations needed) and (2) respects bounds of data

Overview

- Purpose of research (part 2): derive an estimation routine for the beta IFA model (expectation maximization algorithm), provide implementation of algorithm in R, and investigate performance of algorithm in finite sample settings
    - Simulation study I: parameter recovery across multiple R and I, where R is sample size and I is number of indicators
    - Simulation study II: comparative study between beta IFA and NTFA when response distribution was/was not skewed
    - Empirical application of beta IFA model to real dataset

Overview

- Simulation study I: EM algorithm was able to recover parameters in small sample settings (i.e., $R = 100$ and $I = 5$ per factor)
- Simulation study II: beta IFA performed comparable to NTFA in no-skewness conditions (wrt model fit), but outperformed NTFA when data were skewed
- Empirical application: beta IFA model did much better at reproducing observed response distribution than NTFA
- Results provide preliminary evidence that beta IFA model is a suitable alternative for analyzing CRS data

Latent Variable Models

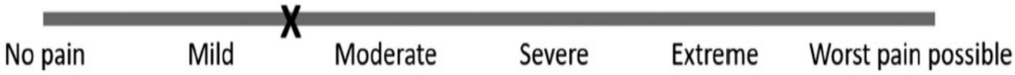
- Latent variable models (LVMs) are among the most widely used statistical methods in social and behavioral sciences for modeling latent constructs, e.g., personality, socioeconomic status, attitudes
- Latent constructs are not directly observable (or are difficult to observe)
- Researchers rely on indirect measurements via observable indicators (items) that serve as proxies of the construct
- Indicators are noisy manifestations (measurements) of latent construct

Latent Variable Models

- LVMs conceptualize latent construct(s) as latent variable(s) θ
- LVMs (specifically beta IFA model) specify a functional relationship between latent variable θ and set of observed indicators y

Continuous Rating Scales

- Measurement of latent constructs with continuous rating scales (CRS) has been found to be more informative than discrete rating scales., e.g., Likert-scales
- Example of CRS format



Analysis of CRS Data

- CRS data is often analyzed with normal-theory factor analysis (NTFA)
- Issue 1: tendency for response distribution to be skewed
- Issue 2: NTFA assumes response range is unbounded; however CRS data typically have well defined endpoints

The Beta Item Factor Analysis Model

- Assume that R respondents are each measured on I observed variables (indicators/items), and the total number of latent variables is $K < I$
- Let $\theta_r = (\theta_{r1}, \ldots, \theta_{rK})^\top$ be respondents r's vector of factor scores (latent variables)
- Assume that $\theta_r \sim N(0, \Sigma)$ where 0 is the $K \times 1$ zero vector and Σ is a $K \times K$ covariance matrix
- Collection of factor scores is $\Theta = (\theta_1^\top, \ldots, \theta_R^\top)^\top$
- The $R \times I$ observed response matrix is denoted $\mathcal{Y} = (y_1^\top, \ldots, y_R^\top)$ where $y_r = (y_{r1}, \ldots, y_{rI})^\top$ is the vector of responses produced by respondent r and y_{ri} is the observed response to the ith item

The Beta Item Factor Analysis Model

- Beta IFA model is constructed by assuming that

$$Y_{ri} \mid \theta_r \sim \text{MP-Beta}(\mu_{ri}, \phi_i) \tag{1}$$

where $0 < \mu_{ri} < 1$ and $\phi_i \in \mathbb{R}^+$ are the mean and precision parameters, respectively, with

$$\mu_{ri} \equiv E(Y_{ri} \mid \theta_r) = F(\beta_i + \lambda_i^\top \theta_r) \tag{2}$$

where (β_i, λ_i) are item parameters and F is a suitable link function

- We use the inverse logit link function is used so that

$$\mu_{ri} \equiv E(Y_{ri} \mid \theta_r) = \frac{\exp(\beta_i + \lambda_i^\top \theta_r)}{1 + \exp(\beta_i + \lambda_i^\top \theta_r)} \tag{3}$$

- θ_r can be thought of as unobserved/missing covariates

The Beta Item Factor Analysis Model with Logit Link

- β_i is item intercept (logit of the expected response when $\theta_r = 0$)
- $\lambda_i = (\lambda_{i1}, \ldots, \lambda_{iK})^\top$ is a $K \times 1$ vector of factor loadings that relate the kth factor to the ith item
- ϕ_i is item precision (degree of separability in the responses)

Maximum Likelihood Estimation via EM Algorithm

- We estimate beta IFA model parameters from observed response matrix $\mathcal{Y}$ via an EM algorithm
- The EM algorithm is an iterative procedure for MLE in the presence of missing, or incomplete, data
- Key in developing EM algorithm for beta IFA model is recognizing that Θ *is* the missing (incomplete) data
- The EM algorithm maximizes incomplete-data likelihood indirectly by maximizing the complete-data likelihood, where the complete data is $(\mathcal{Y}, \Theta)$

Complete Data Log-likelihood

The complete data log-likelihood for the beta IFA model is

$$\log L(\lambda, \beta, \phi \mid \mathcal{Y}, \Theta) = \sum_{r=1}^{R} \sum_{i=1}^{I} \Big\{ \log \Gamma(\phi_i) - \log \Gamma(\mu_{ri}\phi_i) - \log \Gamma(\phi_i(1 - \mu_{ri})) + (\mu_{ri}\phi_i - 1) \log y_{ri} + (\phi_i(1 - \mu_{ri}) - 1) \log(1 - y_{ri}) \Big\}$$

E-step

Let s indicate the iteration number, $s = 1, \ldots, S$. To evolve to the $(s+1)$th iteration, first compute the expected complete-data log-likelihood

$$Q(\lambda, \beta, \phi \mid \lambda^{(s)}, \beta^{(s)}, \phi^{(s)}) = \mathbb{E}\left(\log L(\lambda, \beta, \phi; \mathcal{Y}, \Theta) \mid \lambda^{(s)}, \beta^{(s)}, \phi^{(s)}, \mathcal{Y}\right) \tag{4}$$

where expectation is taken with respect to the posterior of Θ:

$$h(\Theta \mid \lambda^{(s)}, \beta^{(s)}, \phi^{(s)}, \mathcal{Y}) = \frac{L(\lambda^{(s)}, \beta^{(s)}, \phi^{(s)}; \mathcal{Y}, \Theta)}{\int_{\tilde{\Theta}} L(\lambda^{(s)}, \beta^{(s)}, \phi^{(s)}; \mathcal{Y}, \tilde{\Theta}) d\tilde{\Theta}} \tag{5}$$

Posterior expectation is not in closed form and must be evaluated numerically

E-step for Beta IFA Model

The approximate Q function is

$$\hat{Q}\left(\lambda, \beta, \phi \mid \lambda^{(s)}, \beta^{(s)}, \phi^{(s)}\right) = \sum_{r=1}^{R} \sum_{i=1}^{I} \sum_{\vartheta \in \mathcal{G}^K} \hat{h}\left(\vartheta \mid y_r, \lambda^{(s)}, \beta^{(s)}, \phi^{(s)}\right) \Big\{ \log \Gamma(\phi_i) - \log \Gamma(\mu_{ri}\phi_i) - \log \Gamma(\phi_i(1 - \mu_{ri})) + (\mu_{ri}\phi_i - 1) \log y_{ri} + (\phi_i(1 - \mu_{ri}) - 1) \log(1 - y_{ri}) \Big\}, \tag{6}$$

where (...see next slide)

Applying EM Algorithm to the Beta IFA Model

$$\hat{h}\left(\vartheta \mid y_r, \lambda^{(s)}, \beta^{(s)}, \phi^{(s)}\right) = \frac{\varphi(\vartheta) \prod_{i=1}^{I} B\left(\phi_i^{(s)}, \mu_{\bullet i}^{(s)}\right) y_{ri}^{\mu_{\bullet i}^{(s)} \phi_i^{(s)} - 1} (1 - y_{ri})^{(1-\mu_{\bullet i}^{(s)})\phi_i^{(s)} - 1}}{\sum_{\vartheta' \in \mathcal{G}^K} \varphi(\vartheta') \prod_{i=1}^{I} B\left(\phi_i^{(s)}, \mu_{\bullet i}^{(s)}\right) y_{ri}^{\mu_{\bullet i}^{(s)} \phi_i^{(s)} - 1} (1 - y_{ri})^{(1-\mu_{\bullet i}^{(s)})\phi_i^{(s)} - 1}}$$

is the approximate posterior density and

$$\mu_{\bullet i}^{(s)} = \frac{\exp(\beta_i^{(s)} + (\lambda_i^{(s)})^\top \vartheta)}{1 + \exp(\beta_i^{(s)} + (\lambda_i^{(s)})^\top \vartheta)}$$

is the expected response evaluated at the quadrature point ϑ.

M-step

Given provisional item estimates $(\lambda^{(s)}, \beta^{(s)}, \phi^{(s)})$, the parameter estimates are updated at the $(s+1)$th iteration by solving the optimization problem:

$$\left(\lambda^{(s+1)}, \beta^{(s+1)}, \phi^{(s+1)}\right) = \underset{\lambda,\beta,\phi}{\arg\max}\ \hat{Q}\left(\lambda, \beta, \phi \mid \lambda^{(s)}, \beta^{(s)}, \phi^{(s)}\right). \tag{7}$$

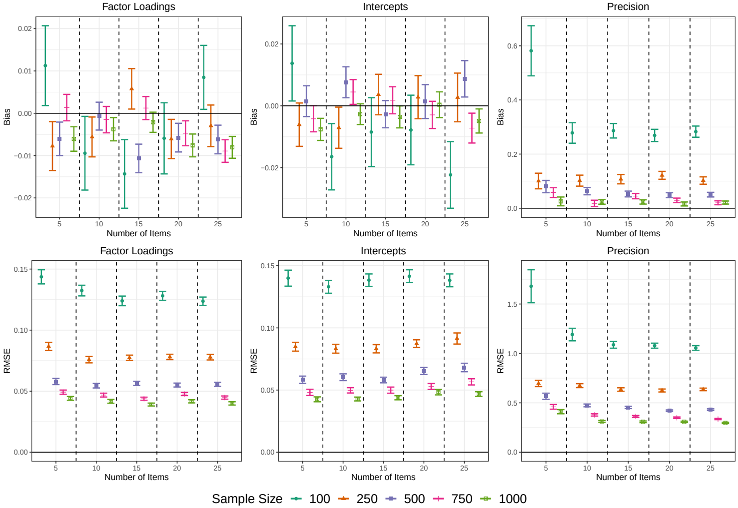
Repeat E- and M- step until convergence (i.e., parameters stabilize). BFGS algorithm was used in M-step.

Simulation Studies

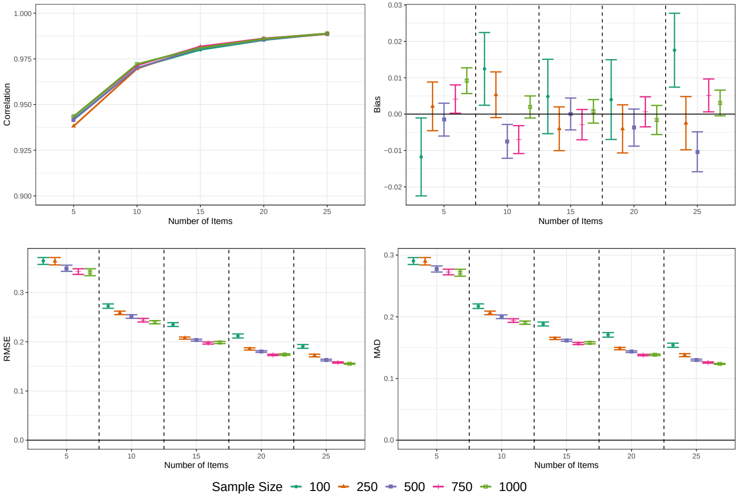
Simulation Study I: Parameter Recovery

- Purpose: evaluate performance of EM algorithm in recovering item parameters in small and large sample size settings
- 25 Conditions: $(R, I) \in \{100, 250, 500, 750, 1000\} \times \{5, 10, 15, 20, 25\}$
- Specifications: $K = 1$, $\lambda \sim U(0.3, 1.8)$, $\beta \sim U(-1.5, 1.5)$, $\phi \sim U(2, 10)$
- $J = 100$ independent datasets per condition
- Measures of estimation quality
    - Bias
    - RMSE
    - AE
    - Correlation

Simulation Study I Selected Results: Item Parameter Recovery



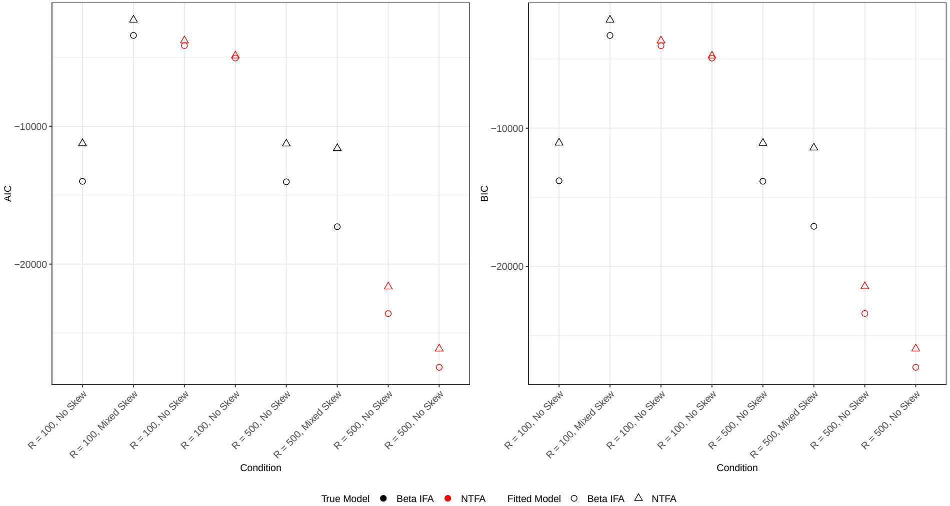
Simulation Study I Selected Results: Factor Recovery



Simulation Study II: Comparison with Normal-Theory Factor Analysis

- Purpose: Investigate performance of beta IFA relative to normal-theory factor analysis with true data-generating model (TDGM): (1) beta IFA or (2) normal-theory model
- NTFM: $Y_{ri} \mid \theta_r \sim N(\tau_i + \lambda_i \theta_r, \psi_i)$
- Specifications: $K = 1$, $I = 15$, $\lambda_i = 1$ for all $i = 1, \ldots, 15$ Conditions:
  - TDGM: beta IFA with $R = 100$ (500) and $\beta_i = 0$ for all $i = 1, \ldots, 15$ (**No Skewness**)
  - TDGM: beta IFA with $R = 100$ (500) and $\beta_i = -1.5$ for all $i = 1, \ldots, 5$, $\beta_i = 0$ for all $i = 6, \ldots, 10$, and $\beta_i = 1.5$ for all $i = 11, \ldots, 5$ (**Mixed Skewness**)
  - TDGM: NTFM with $R = 100$ (500) and $\tau_i = 0$ for all $i = 1, \ldots, 15$ (**No Skewness**)
  - TDGM: NTFM with $R = 100$ (500) and $\tau_i = -1.5$ for all $i = 1, \ldots, 5$, $\tau_i = 0$ for all $i = 6, \ldots, 10$, and $\tau_i = 1.5$ for all $i = 11, \ldots, 5$ (**No Skewness**)

Simulation Study II Results: Model Fit



Empirical Application

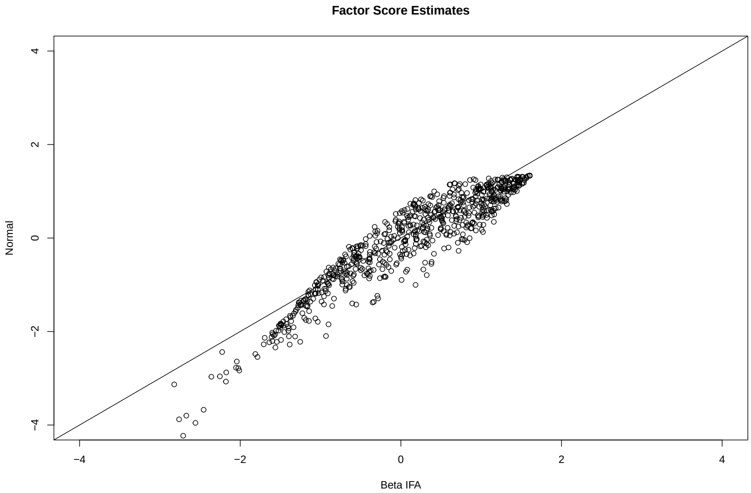
Empirical Application: Self-determination

- Self-Determination Inventory (SDI) is a 21-item measure of self-determination developed to document change in the self-determination of adolescents with and without disabilities
- Ratings are made on a slider scale with anchors of “Disagree” and “Agree” with numeric ratings represented on a scale ranging from 0 to 99
- Analysis with data from $R = 739$ students who participated in a randomized control trial in which the SDI was used as an outcome measure

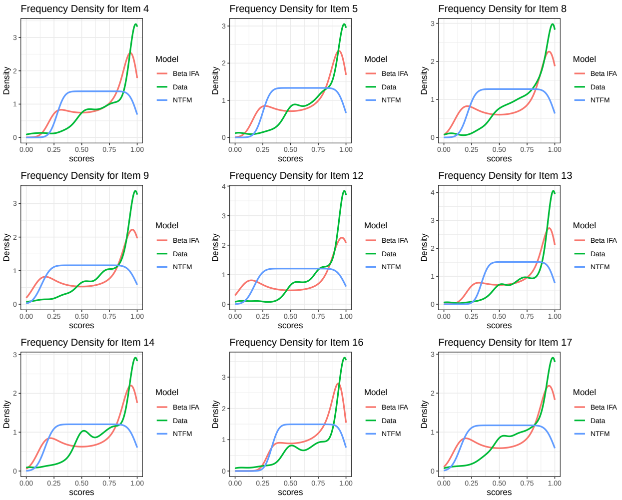
Empirical Application: Self-determination

- SDI was estimated with the beta IFA model assuming a one-factor structure ($K = 1$) using the EM algorithm
- For comparative purposes, SDI was also estimated under a normal-theory factor model (lavaan was used for estimation)
- Models compared via AIC and BIC (not shown here)
- Compute correlation of factor score estimates (not shown here)
- Superimposed plots of marginal density functions produced by both models

Empirical Application: Self-determination



Empirical Application: Self-determination



Wrapping Up

- Propose and develop the beta item factor analytic model (beta IFA) for modeling interval item response data
- Derive a maximum likelihood-based estimation algorithm

Simulation study I: Algorithm was capable of accurately recovering model parameters in small sample size settings ($R = 100$, $I = 5$)

Simulation study II: Beta IFA model performed similar to the NTFA in terms of model fit when NTFA was the true data-generating model but outperformed NTFA when data were skewed

Empirical applications provide preliminary evidence that model may be useful for interval scale data

Wrapping Up

Future directions:

- Improve efficiency of EM algorithm
  - Currently based on fixed quadrature points
  - Possible extension: adaptive quadrature
  - Possible extension: stochastic EM
- Development of diagnostic procedures for evaluating model fit
- Possible extension: allow observed covariates x to enter measurement model
- Possible extension: model precision as function of latent factor θ and observed covariates x
- Investigating if and when beta IFA model is more suitable than NTFA

Thank You!

Selected References

Bartholomew, D. J., Knott, M., & Moustaki, I. (2011). Latent variable models and factor analysis: A unified approach. John Wiley Sons.

Bock, R. D., & Gibbons, R. D. (2021). Item response theory. John Wiley Sons.

Dempster, A. P., Laird, N. M., & Rubin, D. B. (1977). Maximum likelihood from incomplete data via the em algorithm. Journal of the Royal Statistical Society: Series B (Methodological), 39(1), 1–22.

Ellerby, Z., Wagner, C., & Broomell, S. B. (2021). Capturing richer information: On establishing the validity of an interval-valued survey response mode. Behavior Research Methods, 1–23.

Selected References

Ferrari, S., & Cribari-Neto, F. (2004). Beta regression for modelling rates and proportions. Journal of applied statistics, 31(7), 799–815.

McLachlan, G. J., & Krishnan, T. (2007). The EM algorithm and extensions. John Wiley & Sons.

Rosseel, Y. (2012). Lavaan: An R package for structural equation modeling. Journal of statistical software, 48, 1–36.

Shogren, K. A., Little, T. D., Grandfield, E., Raley, S., Wehmeyer, M. L., Lang, K. M., & Shaw, L. A. (2020). The self-determination inventory–student report: Confirming the factor structure of a new measure. Assessment for Effective Intervention, 45(2), 110–120.