On the Sensitivity of Mean-Variance-Efficient Portfolios to Changes in Asset Means: Some Analytical and Computational Results

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On the Sensitivity of Mean-Variance-Efficient Portfolios to Changes in Asset Means: Some Analytical and Computational Results

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This paper investigates the sensitivity of mean-variance(MV)-efficient portfolios to changes in the means of individual assets. When only a budget constraint is imposed on the investment problem, the analytical results indicate that an MV-efficient portfolio's weights, mean, and variance can be extremely sensitive to changes in asset means. When nonnegativity constraints are also imposed on the problem, the computational results confirm that a positively weighted MV-efficient portfolio's weights are extremely sensitive to changes in asset means, but the portfolio's returns are not. A surprisingly small increase in the mean of just one asset drives half the securities from the portfolio. Yet the portfolio's expected return and standard deviation are virtually unchanged.

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One of the key results derived from the mean-variance (MV) capital asset pricing model (CAPM) is that all investors will hold some fraction of the market portfolio [see Sharpe (1964) and Lintner (1965)]. The normative implication drawn from the theory, as well as from the efficient markets hypothesis, is that investors should hold diversified portfolios. In the simplest case, an investor, who has no special insights that would lead him to generate expectations different from the market consensus, is advised to hold the market portfolio combined with either borrowing or lending. In more realistic settings the investor would hold an index fund as a proxy for the market. Perhaps more important from our point of view, even portfolio managers who are bullish or bearish on a security tend to hold only a little more or a little less than the security's market weight.[1] While this advice has had a major impact on the investments industry,[2] it conflicts with the old adage that a little diversification goes a long way.[3] Furthermore, survey evidence from Friend and Blume (1975a) shows that most individual investors hold fewer than five stocks in their portfolios. This raises two related questions. If an active portfolio manager is bullish or bearish on some security, should he hold a slightly different portfolio or should he hold a portfolio that is radically different? Should he hold all the stocks in his universe or should he hold only a few?

In order to shed light on these questions we investigate how sensitive an investor's MV-efficient portfolio is to a change in asset means. In Section 1, we formulate the MV-portfolio problem and the sensitivity analysis for it as special cases of a parametric quadratic programming (PQP) problem. This permits us to examine the analytical and computational aspects of both problems in a unified framework. In Section 2, we concentrate on the case where only the budget

[1] Based on early empirical studies relating to the efficient markets hypothesis, Black (1971) called for a passive portfolio strategy, whereby an investor would buy a well-diversified portfolio and hold it. Treynor and Black (1973) called for a two-stage process: an active portfolio, for placing bets on mispriced securities, blended with a passive portfolio, to provide diversification. (In this article, the argument for holding a passive portfolio was based on the CAPM.) Active versus passive investment strategies are discussed further in Bodie, Kane, and Marcus (1989, chap. 23), Jacob and Pettit (1988, chap. 21), and Sharpe and Alexander (1990, chap. 22).

[2] Newport (1987) reports that "... since 1981 the value of indexed assets has multiplied 14 times to almost $200 billion, or around 10% of all pension assets" (p. 31). Furthermore, "by late 1986, nearly a third of all equities held by the 200 largest pension funds were indexed" (p. 29).

[3] This idea attracts considerable attention in investment texts. Evans and Archer (1968), among others, showed that randomly selected portfolios of 8 to 20 securities eliminate most "nonmarket" risk. Even with random selection, Statman (1987) argues that the number of securities should be on the order of 30 to 40. While we also reach the conclusion that it may not be desirable for an investor to hold positive amounts of all securities, the MV-optimizing framework we employ to reach this conclusion is very different from the random-selection-market model framework employed by Evans and Archer. Incidentally, we note that transaction-costs arguments serve to buttress the Evans and Archer argument for holding a small number of securities. Ironically, at the other end of the spectrum, they probably also play a role in suggesting that the passive investor should not deviate too far from an index fund.

constraint is active. We begin with a brief statement of the well-known efficient set mathematics with the portfolio problem formulated as a PQP problem. In this framework the subsequent sensitivity analysis follows quite naturally. First, we derive closed-form solutions for the change in the optimal portfolio's weights, mean, and variance as functions of changes in asset means. Then we derive upper bounds on the variability of the variables. Both analyses indicate that all three variables can be extremely sensitive to changes in the asset means. Hence, an optimal portfolio may call for large positive and negative positions in the securities.

In the computational sections, Sections 3–6, we impose nonnegativity conditions on the portfolio problem and systematically document the sensitivity of *positively weighted* MV-efficient portfolios to changes in asset means. In an n-asset universe we report the elasticities of an MV-efficient portfolio's mean, standard deviation, certainty equivalent, and weights with respect to a change in the mean of a particular security, as well as the size of the shifts in the security mean required to eliminate from one up to one half the securities from the portfolio. More specifically, we begin with covariance matrices estimated from historical return data, as Merton (1980) among others argues estimates of variances and covariances are more accurate than estimates of means, and we construct a set of $(\mathbf{\Sigma}, \mathbf{x}^*)$-compatible means that guarantee the positively weighted portfolio $\mathbf{x}^*$ is MV efficient. Then we explore whether an MV decision-maker, who believes that one or more means deviate a little from the $(\mathbf{\Sigma}, \mathbf{x}^*)$-compatible means, will choose an MV-efficient portfolio that contains only a few securities. Clearly, these n-asset examples are analogs of a CAPM setting, where the positively weighted portfolio $\mathbf{x}^*$ plays the role of the market portfolio and the $(\mathbf{\Sigma}, \mathbf{x}^*)$-compatible means correspond to the security market line (SML) means. The results provide a first step in understanding how an investor's MV-efficient portfolio deviates from either the market portfolio or an index fund when he perceives that the means differ from equilibrium means generated from the CAPM or, perhaps, the arbitrage pricing theory.[4] Again, con-

[4] This raises a deep issue regarding the nature of uncertainty, which arises as soon as we deviate from the assumption of homogeneous beliefs. If we view uncertainty as subjective, then there is no sense in which an investor's subjective means can fail to be the "correct" ones. On the other hand, if we assume that the various securities' cash flows have some objective distribution and that investors can simply be wrong, we should really be asking what kind of return distribution the securities will have in equilibrium in such a world. However, if our mistaken investor is small, then so is the issue. While a discussion of models of asset pricing with heterogeneous beliefs is beyond the scope of this study, we note that Lintner (1969) has shown that in such a market equilibrium the variables will be a complex weighted average of all investors' expectations; and in a "rational expectations" framework Grossman and Stiglitz (1976, 1980) have explored how information and expectations become endogenous. Our contribution is to show that the portfolio of an MV decision-maker, who believes that one or more means deviate a little from $(\mathbf{\Sigma}, \mathbf{x}^*)$-compatible or SML means, will be very different from the one envisioned in the traditional homogeneous-beliefs-based CAPM.

sistent with the analytical results, we find that portfolio composition is extremely sensitive to changes in asset means. However, with non-negativity constraints imposed on the problem, portfolio returns are not.

1. The Formulation and Properties of MV-Portfolio Problems and the Sensitivity Analysis for Them

Following Markowitz (1952, 1956, 1959), Sharpe (1970), and Best and Grauer (1990a), we formulate the MV-portfolio problem as a PQP problem. The MV problem, with no riskless asset subject to general linear constraints, is

$$\max\{t\boldsymbol{\mu}'\mathbf{x} - \tfrac{1}{2}\mathbf{x}'\boldsymbol{\Sigma}\mathbf{x} \mid \mathbf{A}\mathbf{x} \leq \mathbf{b}\}, \tag{1}$$

where $\boldsymbol{\mu}$ and $\mathbf{x}$ are n-vectors composed of unities plus expected rates of return and portfolio weights, respectively; $\boldsymbol{\Sigma}$ is an (n, n)-positive definite covariance matrix; $\mathbf{A}$ is an (m, n)-constraint matrix; and $\mathbf{b}$ is an m-vector. (Throughout any vector, $\mathbf{y}$ is assumed to be a column vector unless indicated to the contrary by transposition, e.g., $\mathbf{y}'$.) There are two closely related ways of interpreting (1). First, it may be interpreted as a PQP problem, where the minimum variance frontier is traced out as the PQP parameter t varies from $-\infty$ to ∞. On the other hand, nonnegative values of t are of primary economic interest as they yield MV-efficient portfolios. (Setting $t = 0$ gives the global minimum variance portfolio.) In this case, it is convenient to think of t as an MV investor's risk tolerance parameter, where the larger t is the more tolerant the investor is to risk.[5] When t is fixed at some positive value, say, $t = T$, then the solution to (1) yields the MV-efficient portfolio for the investor with risk tolerance parameter T.[6]

Now assume (1) has been solved for a specific investor with risk tolerance parameter T and suppose the investor wishes to know how the optimal solution for (1) depends on $\boldsymbol{\mu}$. The analysis may be performed conveniently by solving the closely related PQP problem

$$\max\{T(\boldsymbol{\mu} + t\mathbf{q})'\mathbf{x} - \tfrac{1}{2}\mathbf{x}'\boldsymbol{\Sigma}\mathbf{x} \mid \mathbf{A}\mathbf{x} \leq \mathbf{b}\}, \tag{2}$$

[5] Merton (1972) and Roll (1977) developed the efficient set mathematics for the single-constraint case by minimizing the variance for all levels of expected return. It is easily verified from the optimality conditions that the parameter t from (1) is also equivalent to the multiplier associated with the constraint $\mu_p = \boldsymbol{\mu}'\mathbf{x}$ in the Merton–Roll formulation. (While the Merton–Roll formulation is the most widely known, there are distinct advantages to the PQP framework in more complicated cases. See note 7.)

[6] The terms "utility function," "risk tolerance," and "risk aversion" are used in an MV framework. [See Sharpe (1970) or Sharpe and Alexander (1990).] They are not to be confused with von Neumann–Morgenstern utility nor with the Pratt–Arrow measures of risk aversion. On the other hand, the MV-utility function is consistent with the expected utility theorem if the investor has negative exponential utility and makes normal probability assessments. Perhaps more importantly, it serves as an approximation to expected utility (and the isoelastic utility functions in particular) for short holding periods and in continuous time. See the discussion in Section 3.

where in this case T is the investor's fixed risk tolerance parameter from (1) and the new PQP parameter t, through the term $t\mathbf{q}$, captures the change in $\boldsymbol{\mu}$ [i.e., $\boldsymbol{\mu}(t) = \boldsymbol{\mu} + t\mathbf{q} = \boldsymbol{\mu} + \Delta\boldsymbol{\mu}$].[7]

In the computational sections of the paper, we focus primarily on increasing one mean at a time. To increase the mean of security j, the investor would set the jth element of $\mathbf{q}$ equal to 1, the other elements equal to 0, and let t vary between 0 and ∞. Then, the value of t at any point gives the change in μ_j. We also examine what happens when the investor simultaneously increases one half and decreases the other half of the means by the same percent. In this case, the investor would set the relevant halves of $\mathbf{q}$ equal to the corresponding portions (halves) of $(\boldsymbol{\mu} - \boldsymbol{\iota})$ and $-(\boldsymbol{\mu} - \boldsymbol{\iota})$, where $\boldsymbol{\iota}$ is a vector of ones, and let t vary between 0 and ∞. Then, the value of t at any point gives the equal (positive or negative) percentage change in the means. Similarly, if he believes that a single-factor model describes security returns and that the expected return on the market is going to increase, he would set $\mathbf{q} = \boldsymbol{\beta}$, the vector of betas or slope coefficients from the single-factor model, and let t vary between 0 and ∞.

Although a PQP algorithm is required to solve almost any type of realistic MV-investment problem or the sensitivity analysis for it,[8] we require only the most basic properties of PQP problems to develop the analysis. Associated with the solution of *any* PQP is a set of intervals $t_0 \leq t \leq t_1$, $t_1 \leq t \leq t_2, \ldots, t_{\nu-1} \leq t \leq t_\nu$. Each interval corresponds to a different set of active (or binding) constraints. At the end of a typical interval, a previously inactive constraint becomes active, or a previously active constraint becomes inactive.[9] In the ith interval, the optimal solution $\mathbf{x}_i(t)$ is a linear function of the parameter t, that is,

$$\mathbf{x}_i(t) = \mathbf{h}_{0i} + t\mathbf{h}_{1i}, \tag{3}$$

[7] A more general PQP problem subject to general linear constraints is

$$\max\{-(\mathbf{c} + t\mathbf{q})'\mathbf{x} - \tfrac{1}{2}\mathbf{x}'\mathbf{C}\mathbf{x} \mid \mathbf{A}\mathbf{x} \leq \mathbf{b} + t\mathbf{p}\},$$

where $\mathbf{c}$ is an n-vector, $\mathbf{C}$ is an (n, n) matrix, and $\mathbf{p}$ is an m-vector. Equation (1) is a special case of this more general PQP, with $\mathbf{c} = \mathbf{0}$, $\mathbf{q} = -\boldsymbol{\mu}$, $\mathbf{C} = \boldsymbol{\Sigma}$, and $\mathbf{p} = \mathbf{0}$. Similarly, Equation (2) is a special case with $\mathbf{c} = -T\boldsymbol{\mu}$, $\mathbf{q} = -T\mathbf{q}$, $\mathbf{C} = \boldsymbol{\Sigma}$, and $\mathbf{p} = \mathbf{0}$. This framework has two noteworthy properties. First, all the analytics of traditional MV-portfolio problems and the sensitivity analysis for them follow as special cases of this general PQP problem, the optimality conditions for it, and the definitions of the expected return and variance of a portfolio. Given the extreme sensitivity of the portfolio weights to a change in asset means documented in this paper, constraints other than the budget constraint will almost certainly be binding in practice. The analytics of these general linear constraints cases follow quite naturally and are developed in Best and Grauer (1990a, 1990b). Second, this is the framework adopted in computational analyses. Hence, the analytical and computational aspects of both problems may be discussed in a unified framework.

[8] See, for example, Best and Grauer (1990a), Markowitz (1956, 1959), and Sharpe (1970) for a fuller discussion of algorithms. However, the particular algorithm is not of critical importance since Best (1984) showed the equivalence of several QP algorithms.

[9] See Best (1982) for details.

for all t with $t_i \le t \le t_{i+1}$. Each t_i corresponds to a "corner" portfolio. Here, $\mathbf{h}_{0i}$ is the nonvarying (constant) portion of the solution, and $t\mathbf{h}_{1i}$ is the varying (parametric) portion of the solution. In all portfolio problems, traditional MV or sensitivity analysis for them, the budget constraint dictates that the portfolio weights sum to unity (i.e., $\boldsymbol{\iota}'\mathbf{x}_i = 1$). This implies that the nonvarying portion of the portfolio must sum to unity and the varying portion of the portfolio must sum to zero (i.e., $\boldsymbol{\iota}'\mathbf{h}_{0i} = 1$ and $\boldsymbol{\iota}'\mathbf{h}_{1i} = 0$).

For sensitivity analysis on the means, the term $t\mathbf{h}_{1i}$ shows how the portfolio weights change in the ith interval (i.e., in general, the rate of change in the weights depends on the active constraints). The expected return, $\mu_{pi} = (\boldsymbol{\mu} + t\mathbf{q})'\mathbf{x}_i$, and variance, $\sigma^2_{pi} = \mathbf{x}'_i\boldsymbol{\Sigma}\mathbf{x}_i$, of the optimal portfolio are *quadratic* functions of t,

$$\mu_{pi} = \alpha_{0i} + \alpha_{1i}t + \alpha_{2i}t^2, \qquad \sigma^2_{pi} = \gamma_{0i} + \gamma_{1i}t + \gamma_{2i}t^2, \tag{4}$$

where $\alpha_{0i} = \boldsymbol{\mu}'\mathbf{h}_{0i}$ is the mean of the nonvarying portion of the portfolio, $\alpha_{1i}t = \boldsymbol{\mu}'\mathbf{h}_{1i}t + t\mathbf{q}'\mathbf{h}_{0i}$ is the mean of the varying portion of the portfolio plus the change in the mean of the nonvarying portion of the portfolio, and $\alpha_{2i}t^2 = t\mathbf{q}'\mathbf{h}_{1i}t$ is the change in the mean of the varying portion of the portfolio. Turning to the variance terms, $\gamma_{0i} = \mathbf{h}'_{0i}\boldsymbol{\Sigma}\mathbf{h}_{0i}$ is the variance of the nonvarying portion of the portfolio, $\gamma_{1i}t = 2\mathbf{h}'_{0i}\boldsymbol{\Sigma}\mathbf{h}_{1i}t$ is twice the covariance between the nonvarying and varying portions of the portfolio, and $\gamma_{2i}t^2 = t\mathbf{h}'_{1i}\boldsymbol{\Sigma}\mathbf{h}_{1i}t$ is the variance of the varying portion of the portfolio. Clearly, $\alpha_{1i}t + \alpha_{2i}t^2$ and $\gamma_{1i}t + \gamma_{2i}t^2$ show how the portfolio's mean and variance change as a function of t. Similarly, employing the fact that the optimal portfolio is a linear function of t, the MV utility, $U = T(\boldsymbol{\mu} + t\mathbf{q})'\mathbf{x} - 1/2\mathbf{x}'\boldsymbol{\Sigma}\mathbf{x}$, and the certainty equivalent (CE) return can be written as quadratic functions of t:

$$U_i = \delta_{0i} + \delta_{1i}t + \delta_{2i}t^2, \qquad \mathrm{CE}_i = U_i/T. \tag{5}$$

In the computational sections of the paper, we report both rates of change in terms of the expressions in (3) and (4) as well as elasticities. Therefore, before proceeding to the analytics of the single-constraint case and the computational work, we close the section by developing the formulas for the point elasticities of various quantities with respect to a change in the mean of security j. To simplify notation, we drop the i subscript that indicates we are in the ith interval. Then, the jth element of the mean and solution vectors are μ_j and $x_j = h_{0j} + th_{1j}$, respectively. By definition, the point elasticity of any variable y with respect to a change in the mean of security j is

$$E_{y,\mu_j} = \frac{dy}{y}\left(\frac{d\mu_j}{\mu_j - 1}\right)^{-1} = \frac{dy}{dt}\left(\frac{d\mu_j}{dt}\right)^{-1}\frac{\mu_j - 1}{y}. \tag{6}$$

Note that when we increase μ_j only, $q_j = 1$. Hence, $d\mu_j/dt = 1$. Then the point elasticity of the mean of the portfolio with respect to an increase in the mean of security j is

$$E_{\mu_p,\mu_j} = \frac{(\alpha_1 + 2\alpha_2 t)(\mu_j - 1)}{\mu_p - 1}. \tag{7}$$

The elasticity of the standard deviation of the portfolio with respect to an increase in the mean of security j is

$$E_{\sigma_p,\mu_j} = \frac{(\gamma_1 + 2\gamma_2 t)(\mu_j - 1)}{2\sigma_p^2}. \tag{8}$$

The elasticity of the certainty equivalent rate of return on the portfolio with respect to an increase in the mean of security j is

$$E_{\mathrm{CE},\mu_j} = \frac{(\delta_1 + 2\delta_2 t)(\mu_j - 1)}{T(\mathrm{CE} - 1)}, \tag{9}$$

and the elasticity of the portfolio weight invested in security k with respect to an increase in the mean of security j is

$$E_{\mathbf{x}_k,\mu_j} = \frac{h_{1k}(\mu_j - 1)}{x_k}. \tag{10}$$

Finally, in the case of an equally weighted n-asset portfolio, the vector of portfolio-weight elasticities is

$$E_{\mathbf{x},\mu_j} = n\mathbf{h}_1(\mu_j - 1),$$

that is, the vector of elasticities is simply a scalar times $\mathbf{h}_1$.

2. The Analytics of the Single-Constraint Case

2.1 The traditional efficient set mathematics

We begin with a short statement of the traditional efficient set mathematics, where the portfolio problem is formulated as a PQP problem. In this framework, the subsequent sensitivity analysis follows in a very straightforward manner. The simplest MV problem is

$$\max\{t\boldsymbol{\mu}'\mathbf{x} - \tfrac{1}{2}\mathbf{x}'\boldsymbol{\Sigma}\mathbf{x} \mid \boldsymbol{\iota}'\mathbf{x} = 1\}, \tag{11}$$

where t is a risk tolerance parameter and only the budget constraint is active. The first-order conditions are

$$\boldsymbol{\Sigma}\mathbf{x} + \boldsymbol{\iota}\lambda = t\boldsymbol{\mu}, \qquad \boldsymbol{\iota}'\mathbf{x} = 1. \tag{12}$$

Solving these equations, the optimal portfolio is

$$\mathbf{x}(t) = \boldsymbol{\Sigma}^{-1}\boldsymbol{\iota}/c + t[\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \boldsymbol{\iota}a/c)], \tag{13}$$

where the efficient set constants a, b, and c are defined as

$$a = \iota'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}, \qquad b = \boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}, \qquad c = \iota'\boldsymbol{\Sigma}^{-1}\iota.$$

As there is only one active constraint, there is only one interval. Therefore, we continue to suppress the i subscript that indicates we are in the ith interval. Writing the optimal portfolios as $\mathbf{x}(t) = \mathbf{h}_0 + t\mathbf{h}_1$, we see that $\mathbf{h}_0 = \boldsymbol{\Sigma}^{-1}\iota/c$ is the nonvarying portion of the optimal portfolio, the global minimum variance portfolio in this case, whose components sum to unity; and $t\mathbf{h}_1 = t[\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \iota a/c)]$ is the varying portion of the optimal portfolio, whose components sum to zero.

The expected return, $\mu_p = \boldsymbol{\mu}'\mathbf{x}$, and variance, $\sigma_p^2 = \mathbf{x}'\boldsymbol{\Sigma}\mathbf{x}$, of the optimal portfolio are *linear* and *quadratic* functions of the parameter t:

$$\mu_p = \alpha_0 + \alpha_1 t, \qquad \sigma_p^2 = \gamma_0 + \gamma_1 t + \gamma_2 t^2, \tag{14}$$

where

$$\alpha_0 = \boldsymbol{\mu}'\mathbf{h}_0 = a/c, \qquad \alpha_1 = \boldsymbol{\mu}'\mathbf{h}_1 = b - a^2/c,$$
$$\gamma_0 = \mathbf{h}_0'\boldsymbol{\Sigma}\mathbf{h}_0 = 1/c, \qquad \gamma_1 = 2\mathbf{h}_0'\boldsymbol{\Sigma}\mathbf{h}_1 = 0, \qquad \gamma_2 = \mathbf{h}_1'\boldsymbol{\Sigma}\mathbf{h}_1 = \alpha_1.$$

Eliminating t from (14), we obtain the equation of the minimum variance frontier:

$$(\mu_p - \alpha_0)^2 = \alpha_1(\sigma_p^2 - \gamma_0), \tag{15}$$

which is a parabola (hyperbola) in MV (mean-standard deviation) space.

2.2 Closed-form solutions for changes in the portfolio's mean, variance, and weights induced by changes in the asset means

Now, suppose the investor with risk tolerance parameter T wishes to analyze the sensitivity of the optimal portfolio, its mean, and variance to changes in $\boldsymbol{\mu}$. Investor T's PQP problem is

$$\max\{T(\boldsymbol{\mu} + t\mathbf{q})'\mathbf{x} - \tfrac{1}{2}\mathbf{x}'\boldsymbol{\Sigma}\mathbf{x} \mid \iota'\mathbf{x} = 1\}, \tag{16}$$

where the PQP parameter t, through the term $t\mathbf{q}$, now captures the change in the asset means. The first-order conditions are

$$\boldsymbol{\Sigma}\mathbf{x} + \iota\lambda = T(\boldsymbol{\mu} + t\mathbf{q}), \qquad \iota'\mathbf{x} = 1. \tag{17}$$

Solving these equations, the optimal portfolio is

$$\begin{aligned}\mathbf{x}(t) = {} & \boldsymbol{\Sigma}^{-1}\iota/c + T[\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \iota\, a/c)] \\ & + tT[\boldsymbol{\Sigma}^{-1}\mathbf{q} - \boldsymbol{\Sigma}^{-1}\iota(\mathbf{q}'\Sigma^{-1}\iota)/c].\end{aligned} \tag{18}$$

Again, (18) is a linear function of the parameter t [i.e., $\mathbf{x}(t) = \mathbf{h}_0 + t\mathbf{h}_1$]. From (13),

$$\mathbf{h}_0 = \boldsymbol{\Sigma}^{-1}\iota/c + T[\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \iota\, a/c)]$$

is the solution to investor T's MV-portfolio problem with no change in the means. Then clearly the $\mathbf{h}_1$-vector,

$$T[\mathbf{\Sigma}^{-1}\mathbf{q} - \mathbf{\Sigma}^{-1}\boldsymbol{\iota}(\mathbf{q}'\mathbf{\Sigma}^{-1}\boldsymbol{\iota})/c],$$

gives the rate of change in the investor's portfolio holdings. It is interesting to note that the rate of change is a function of the risk tolerance parameter T, $\mathbf{\Sigma}^{-1}$, $\mathbf{q}$ (which specifies the particular change in $\boldsymbol{\mu}$), $\boldsymbol{\iota}$, and the efficient set constant c *only*. It does not depend on the betas of the assets or, perhaps surprisingly, on the portfolio weights or on the means themselves. Closer inspection shows that the change is composed of a direct and an indirect effect. The direct effect, the first term $T\mathbf{\Sigma}^{-1}\mathbf{q}$, is simply the risk tolerance parameter times a weighted average of the columns of $\mathbf{\Sigma}^{-1}$, where the elements of $\mathbf{q}$ are the weights. The indirect or normalizing effect, the second term, $-T\mathbf{\Sigma}^{-1}\boldsymbol{\iota}(\mathbf{q}'\mathbf{\Sigma}^{-1}\boldsymbol{\iota})/c$, is minus the risk tolerance parameter times the ratio of two scalars $(\mathbf{q}'\mathbf{\Sigma}^{-1}\boldsymbol{\iota})/c$ times a vector $\mathbf{\Sigma}^{-1}\boldsymbol{\iota}$. It is easily verified that the $\mathbf{h}_1$ or rate of change vector sums to zero as it must in all MV-portfolio problems. Finally, the fact that the change in the weights is a function of $\mathbf{\Sigma}^{-1}$, which will almost surely have some large elements, suggests that portfolio composition may be quite sensitive to changes in asset means. (We confirm this intuition in Section 2.3 and in the computational sections.)

Special cases are helpful in understanding the analysis. As noted, we can set $\mathbf{q}$ to anything we want, thereby changing any subset of the means. However, there are degenerate cases. For example, *there are no changes in the portfolio weights* if we change all the means by the same rate $\mathbf{q} = k\boldsymbol{\iota}$, $k \neq 0$, as $\mathbf{h}_1$ will equal $\mathbf{0}$. Now suppose we change the jth mean only (i.e., we set $\mathbf{q}$ equal to a vector containing all zeros with a 1 as the jth element).[10] Then, ignoring the risk tolerance parameter for the moment, the direct effect $\mathbf{\Sigma}^{-1}\mathbf{q}$ contains the jth column of $\mathbf{\Sigma}^{-1}$. Turning to the indirect effect, we see that the kth element of $\mathbf{\Sigma}^{-1}\boldsymbol{\iota}$ contains the sum of the kth column (or, equivalently, row) of $\mathbf{\Sigma}^{-1}$. $\mathbf{\Sigma}^{-1}\boldsymbol{\iota}$ is multiplied by $(\mathbf{q}'\mathbf{\Sigma}^{-1}\boldsymbol{\iota})/c$, where $\mathbf{q}'\mathbf{\Sigma}^{-1}\boldsymbol{\iota}$ is the jth element of $\mathbf{\Sigma}^{-1}\boldsymbol{\iota}$ and c is the sum of all the elements in $\mathbf{\Sigma}^{-1}$. Suppose further that the assets are uncorrelated. Then, $\mathbf{\Sigma}$ is diagonal, with σ_j^2 as the jth diagonal entry, and $\mathbf{\Sigma}^{-1}$ is also diagonal, with σ_j^{-2} as the jth diagonal entry. Furthermore, $\mathbf{\Sigma}^{-1}\mathbf{q}$ is a vector of zeros with σ_j^{-2} as the

[10] An alternative way to find a change in the composition of the optimal portfolio of investor T with respect to a change in the jth mean is to take the gradient of $\mathbf{x}$ with respect to μ_j. Let $\mathbf{G}(T)$ be an (n, n) matrix containing the n gradient vectors $\partial\mathbf{x}(T)/\partial\mu_j$ as columns. From (13),

$$\mathbf{G}(T) = T[\mathbf{\Sigma}^{-1} - \mathbf{\Sigma}^{-1}\boldsymbol{\iota}(\boldsymbol{\iota}'\mathbf{\Sigma}^{-1})/c)].$$

Then we can multiply $\mathbf{G}(T)$ by a vector $\mathbf{q}$ specifying a particular change in $\boldsymbol{\mu}$. For example, if we set $\mathbf{q}$ equal to a vector of 0s with a 1 in the jth row, the product $\mathbf{Gq}$ yields $\partial\mathbf{x}(T)/\partial\mu_j$.

jth entry and the kth entry of $\boldsymbol{\Sigma}^{-1}\boldsymbol{\iota}(\mathbf{q}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\iota})/c$ is $\sigma_k^{-2}(\sigma_j^{-2}/\Sigma_i\sigma_i^2)$. Combining these terms, the jth and kth elements of the gradient $\partial\mathbf{x}/\partial\mu_j$ are

$$T\sigma_j^{-2}\left[1 - \sigma_j^{-2}\left(\sum_i \sigma_i^{-2}\right)^{-1}\right], \qquad -T\sigma_k^{-2}\left[\sigma_j^{-2}\left(\sum_i \sigma_i^{-2}\right)^{-1}\right],$$

respectively. In the case of uncorrelated assets, an increase in μ_j increases $\mathbf{x}_j$ *only*; all other asset holdings decrease. But the analysis shows that this is a very special case. In fact, the computational results indicate that when we increase the mean of the jth asset in portfolios constructed from assets whose returns are correlated to the same degree as the returns on NYSE stocks, about half the asset weights increase and half decrease. And as one might expect, it also shows that the more highly correlated the asset returns, the more sensitive the holdings tend to be to a change in a mean.

The expected return, $\mu_p = (\boldsymbol{\mu} + t\mathbf{q})'\mathbf{x}$, and variance, $\sigma_p^2 = \mathbf{x}'\boldsymbol{\Sigma}\mathbf{x}$, of the optimal portfolio are *quadratic* functions of t:

$$\begin{aligned}
\mu_p &= \alpha_0 + \alpha_1 t + \alpha_2 t^2, \qquad \sigma_p^2 = \gamma_0 + \gamma_1 t + \gamma_2 t^2,\\
\alpha_0 &= \boldsymbol{\mu}'\mathbf{h}_0 = a/c + T(b - a^2/c),\\
\alpha_1 &= \boldsymbol{\mu}'\mathbf{h}_1 + \mathbf{q}'\mathbf{h}_0 = (\mathbf{q}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\iota})/c + 2T[\mathbf{q}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} - (\mathbf{q}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\iota})a/c],\\
\alpha_2 &= \mathbf{q}'\mathbf{h}_1 = T[\mathbf{q}'\boldsymbol{\Sigma}^{-1}\mathbf{q} - (\mathbf{q}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\iota})^2/c],\\
\gamma_0 &= \mathbf{h}_0'\boldsymbol{\Sigma}\mathbf{h}_0 = 1/c + T^2[b - a^2/c],\\
\gamma_1 &= 2\mathbf{h}_0'\boldsymbol{\Sigma}\mathbf{h}_1 = 2T^2[\mathbf{q}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} - (\mathbf{q}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\iota})a/c],\\
\gamma_2 &= \mathbf{h}_1'\boldsymbol{\Sigma}\mathbf{h}_1 = T\alpha_2.
\end{aligned} \tag{19}$$

From (14), α_0 and γ_0 are the mean and variance of investor T's original portfolio. Clearly then, $\alpha_1 t + \alpha_2 t^2$ and $\gamma_1 t + \gamma_2 t^2$ show how the mean and variance of the optimal portfolio change as a function of t. It is interesting to note that, while the change in the portfolio weights is a function of the change in the means only, the changes in the mean and variance of the optimal portfolio are functions of the means themselves as well as the change in the means.

For the investor with risk tolerance T, $\mathbf{x}(t)$, $\mu_p(t)$, and $\sigma_p^2(t)$ give the associated *optimal* MV-efficient portfolios, their expected returns, and variances. The point $[\mu_p(t), \sigma_p^2(t)]$ traces out the optimal μ_p, σ_p^2 path, for the investor with risk tolerance, T, as t (some subset of the means) is varied. We call this path either investor T's mean parameterized efficient frontier or his portfolio expansion path (PEP).

Best and Grauer (1990b) derived the following closed-form expression for the investor's PEP:

$$(\sigma_p^2 - T\mu_p - \delta_1)^2 = \delta_2(T\sigma_p^2 + \mu_p - \delta_3), \tag{20}$$

where the three constants δ_1, δ_2, and δ_3 are

$$\delta_1 = (\gamma_0 - T\alpha_0) + \frac{(T\alpha_1 - \gamma_1)(\alpha_1 + T\gamma_1)}{2\alpha_2(1 + T^2)},$$

$$\delta_2 = \frac{(\gamma_1 - T\alpha_1)^2}{\alpha_2(1 + T^2)} = \frac{T(\mathbf{q}'\mathbf{\Sigma}^{-1}\iota)^2}{(1 + T)^2 c[c(\mathbf{q}'\mathbf{\Sigma}^{-1}\mathbf{q}) - (\mathbf{q}'\mathbf{\Sigma}^{-1}\iota)^2]},$$

$$\delta_3 = \frac{1}{\delta_2}\left[-\delta_1^2 + (\gamma_0 - T\alpha_0)^2 + \frac{\alpha_1\gamma_0 - \gamma_1\alpha_0)(T\alpha_1 - \gamma_1)}{\alpha_2} \right].$$

The PEP is a parabola symmetric about the line

$$\mu_p = -\frac{\delta_1}{T} + \frac{1}{T}\sigma_p^2.$$

For every value of t in (16)–(20), there is a new minimum variance frontier corresponding to the new mean vector $\boldsymbol{\mu}(t) = \boldsymbol{\mu} + t\mathbf{q}$ (and the covariance matrix $\mathbf{\Sigma}$). Figure 1 shows two such minimum variance frontiers and the PEP for an investor with risk tolerance parameter T. The PEP is a parabola symmetric about a line with slope $1/T$. The figure is based on the 10-asset example discussed in Section 4. If the first asset's mean is increased by 0.005, the investor with $T = 0.342$ moves from point A—the equally weighted portfolio—on the lower minimum variance frontier to point B on the upper frontier. Figure 2 shows the same information as Figure 1, as well as the PEPs of two additional investors with risk tolerance parameters $T_1 = 0.1$ and $T_2 = 0.5$. Given the same change in the means that cause investor T to move from point A on the first frontier to point B on the second frontier, investor T_1 (T_2) moves from point E to F (C to D).

Several points follow from Equations (18)–(20) and the two figures. First, Figure 2 illustrates that for each specification of the change in the means, there is a different PEP for each investor. Second, it follows from (18) that the changes in the portfolio weights associated with the movement on the investors' PEPs differ only by the difference in the values of the investors' risk tolerance parameters. [In the limit, an investor with no tolerance for risk would hold the (unchanged) global minimum variance portfolio.] Third, it is clear from the parabolic nature of the PEP that, depending on the level of the means, changes in the means can cause the mean and variance of the optimal portfolio to increase or decrease. Furthermore, large changes in the means can cause dramatic changes in the portfolio's mean and vari-

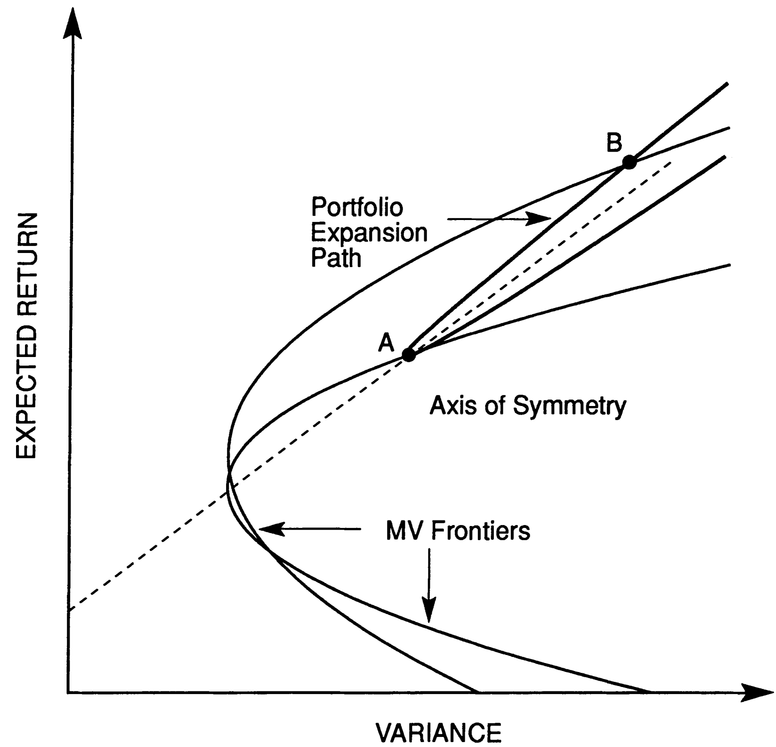

Figure 1
The portfolio expansion path (PEP) for an investor with risk tolerance parameter T
Based on the 10-asset example discussed in Section 4. If the first asset's mean is increased by 0.005, the investor with $T = 0.342$ moves from point A—the equally weighted portfolio—on the lower minimum variance frontier to point B on the upper frontier. The axis of symmetry for the PEP has slope equal to the reciprocal of the risk tolerance parameter T.

ance, or they can leave one or the other unchanged. Finally, for each investor T, there is a different PEP for each possible change in $\boldsymbol{\mu}$ (i.e., for each specification of $\mathbf{q}$). However, each PEP will be a parabola symmetric about a line with slope $1/T$. (In the limit, the PEP for an investor with no tolerance for risk will be a segment of a vertical line whose variance corresponds to that of the global minimum variance portfolio.)

2.3 Bounds on changes in the portfolio's mean, variance, and weights induced by changes in the asset means

Further insight into the sensitivity of the optimal portfolio's weights, mean, and variance may be obtained in terms of upper bounds on

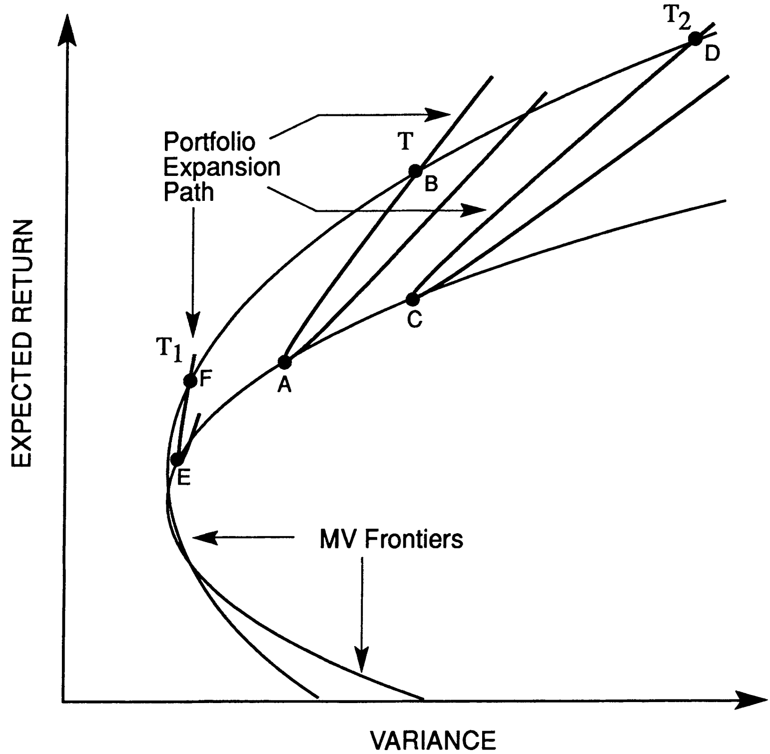

Figure 2
Portfolio expansion paths for three investors
On the lower (upper) minimum variance frontier, the investors with risk tolerance parameters T = 0.342, T_2 = 0.5, and T_1 = 0.1 hold the portfolios at points A, C, and E (B, D, and F), respectively.

the variation in $\mathbf{x}$, μ_p, and σ_p^2, corresponding to the variation in the mean vector $\boldsymbol{\mu} + t\mathbf{q}$. We employ standard analytic techniques to obtain the bounds [e.g., see Stewart (1973)]. They include the use of the Cauchy–Schwartz inequality, the triangle inequality, and the matrix norm induced by the Euclidean vector norm. Let $\|\mathbf{y}\|$ denote the Euclidean norm of the vector $\mathbf{y}$ and $\lambda_{\min}$ and $\lambda_{\max}$ be the minimum and maximum eigenvalues of $\boldsymbol{\Sigma}$, respectively. Then, the bounds are derived from the definitions of $\mathbf{h}_0$ and $\mathbf{h}_1$ in (18). The bound for $\mathbf{h}_0$ is

$$\|\mathbf{h}_0\| \leq \frac{\lambda_{\max}}{n^{1/2}\lambda_{\min}} + \frac{T\|\boldsymbol{\mu}\|}{\lambda_{\min}}\left(1 + \frac{\lambda_{\max}}{\lambda_{\min}}\right).$$

The bounds for the change in investor T's optimal portfolio's weights, mean, and variance are

$$\|\mathbf{x} - \mathbf{x}^*\| \leq t\|\mathbf{h}_1\| \leq t\left[\frac{T\|\mathbf{q}\|}{\lambda_{\min}}\left(1 + \frac{\lambda_{\max}}{\lambda_{\min}}\right)\right], \tag{21}$$

$$|\mu_p - \mu_p^*| \leq \frac{T\|\mathbf{q}\|}{\lambda_{\min}}\left[T\left(1 + \frac{\lambda_{\max}}{\lambda_{\min}}\right)(2\|\boldsymbol{\mu}\| + t\|\mathbf{q}\|) + \frac{\lambda_{\max}}{n^{1/2}}\right], \tag{22}$$

$$\begin{aligned} |\sigma_p^2 - \sigma_p^{2*}| \leq\ & tT\|\mathbf{q}\| \frac{\lambda_{\max}}{\lambda_{\min}^2}\left(1 + \frac{\lambda_{\max}}{\lambda_{\min}}\right) \\ & \times \left[T\left(1 + \frac{\lambda_{\max}}{\lambda_{\min}}\right)\left\{(2\|\boldsymbol{\mu}\| + t\|\mathbf{q}\|)\right.\right. \\ & \left.\left.\times \left(1 + \frac{\lambda_{\max}}{\lambda_{\min}}\right)\right\} + \frac{\lambda_{\max}}{n^{1/2}}\right], \end{aligned} \tag{23}$$

where $\mathbf{x}^*$, μ_p^*, μ_p^{2*} denote the original, or initial, values of the portfolio's weights, mean, and variance.

Inspection of (21)–(23) again reveals that the changes in the optimal portfolio's weights, mean, and variance could be extremely large as they are functions of the ratio of the largest to the smallest eigenvalue as well as the reciprocal of the smallest eigenvalue. Furthermore, there is reason to believe that the more assets there are in the universe, the more sensitive MV-efficient portfolios will be to changes in the means. For example, the minimum and maximum eigenvalues in the 10-asset universe considered below are 0.677×10^{-3} and 0.323×10^{-1}, respectively. In the 100-asset universe, the eigenvalues are 0.321×10^{-4} and 0.282. Assume one asset's mean is increased, so that $\|\mathbf{q}\| = 1$. Then, from (21), in the 10-asset universe, $\|\mathbf{x} - \mathbf{x}^*\| \leq tT\,72{,}000$. In the 100-asset universe, $\|\mathbf{x} - \mathbf{x}^*\| \leq tT\,(2.7 \times 10^8)$. In other words, the sensitivity of the change in the portfolio weights in the 100-asset universe could be on the order of 4000 times greater than in the 10-asset universe.

3. The Computational Methodology

In the remainder of the paper we adopt a computational approach and systematically document the sensitivity of *positively weighted* MV-efficient portfolios to changes in the means when the portfolio problem is subject to nonnegativity constraints. We operationalize the idea in two ways. First, for a positively weighted MV-efficient portfolio, we calculate the elasticities of the portfolio's mean, standard deviation, certainty equivalent, and weights with respect to a change in the mean of security *j*.[11] Second, with nonnegativity constraints imposed on the *n*-asset MV-portfolio problem, we calculate the size of the shifts in the asset means required to drive from one up to one half the securities from the optimal portfolio.

Four steps are involved. First, we select random samples of *n* stocks from the Center for Research in Security Prices (CRSP) tape. Each of these *n*-asset samples serves as an investment universe for a sensitivity experiment. Second, given an *n*-asset universe, we estimate the covariance matrix from the historical return data. Third, we calculate a set of $(\mathbf{\Sigma}, \mathbf{x}^*)$-compatible means so that the positively weighted (and, more specifically, equally weighted) portfolio $\mathbf{x}^*$ will be MV efficient. Fourth, we increase one of the means and perform the sensitivity or comparative statics analysis.

Turning to the details, first, we identify the 958 firms on the CRSP tape for which there are no missing rate-of-return data over the 120-month period from January 1976 to December 1985. From these 958 securities, we select random samples (without replacement) of size *n*, where $n = 10$, 20, 50, or 100 securities. (The smaller random samples are proper subsets of the larger random samples.) Second, for a given *n*-asset universe, we calculate the covariance matrix from historical data. Third, following Best and Grauer (1985), we calculate a set of $(\mathbf{\Sigma}, \mathbf{x}^*)$-compatible means from the equation

$$\boldsymbol{\mu} = \theta_1 \boldsymbol{\iota} + \theta_2 \mathbf{\Sigma} \mathbf{x}^*, \tag{24}$$

where in general $\mathbf{x}^*$ is a prespecified portfolio, and in this paper an equally weighted portfolio. The $(\mathbf{\Sigma}, \mathbf{x}^*)$-compatible means, together with the given covariance matrix, guarantee that $\mathbf{x}^*$ is MV efficient.[12] Or, to put it in a CAPM setting, in the *n*-asset universe the equally weighted portfolio $\mathbf{x}^*$ plays the role of the market portfolio, and the $(\mathbf{\Sigma}, \mathbf{x}^*)$-compatible means correspond to the SML means.

While there are an infinite number of $(\mathbf{\Sigma}, \mathbf{x}^*)$-compatible means from the two-parameter family in (24), we must recognize that our choice of θ_1 and θ_2 values is constrained by the following economics. The intercept of (24) must equal unity plus the zero-beta rate, r_z, and the slope must equal the excess return of the optimal portfolio divided by its variance as well as the reciprocal of investor risk tolerance. That is,

[11] These changes are for unconstrained (other than the budget constraint) problems. With other binding constraints, a change in the means may have no impact on the optimal portfolio.

[12] It is interesting to note that, while the sensitivity of the optimal portfolio's weights depends on $\mathbf{\Sigma}$, $\mathbf{\Sigma}$ itself does not tell us whether there are positively weighted minimum variance portfolios. It is clear from (24) that there are positively weighted minimum variance portfolios for any positive definite $\mathbf{\Sigma}$, as long as the means are $(\mathbf{\Sigma}, \mathbf{x}^*)$-compatible.

$$\theta_1 = r_z, \qquad \theta_2 = \frac{\mu_p - r_z}{\sigma_p^2} = \frac{1}{T}. \tag{25}$$

We focus on one set of realistic θ_1 and θ_2 values that have been chosen to yield reasonable expected returns, excess returns, and risk tolerance parameters. Here, θ_1 was chosen so that the zero-beta rate is 0.75 percent per month (9 percent per annum), and θ_2 was chosen so that the excess return on the optimal MV portfolio is also 0.75 percent per month. This latter value corresponds to the excess return of the Standard & Poor's 500 Index over Treasury bills during the last half century. Having chosen θ_2 in this way, it is clear from (25) that there will be an implied risk tolerance parameter for each data set. Under certain conditions, it can be shown that[13] if (1) is used to approximate expected utility for an isoelastic function of the form

$$u(w) = (1/\gamma)\, w^{\gamma}, \qquad \gamma < 1, \tag{26}$$

then the risk tolerance parameter T is the reciprocal of the Pratt–Arrow measure of relative risk aversion (RRA), where RRA $= -wu''(w)/u'(w)$. That is,

$$T = 1/(1 - \gamma). \tag{27}$$

The data yielded values of RRA ranging between 2.92 and 3.73, which is consistent with Friend and Blume's (1975b) estimate that RRA is equal to 2.

Finally, given the equally weighted MV-efficient portfolio, *we systematically increase the means of all n risky securities, one at a time.* For each asset j, we record the point elasticities of the portfolio's mean, standard deviation, certainty equivalent, and weights with respect to the increase in μ_j. (These elasticities are evaluated at the point where the portfolio is equally weighted.) But, given that there are n portfolio weights, we only report the minimum, maximum, and standard deviation of the weight elasticities and the average of their absolute values.[14]

As μ_j increases, assets are systematically driven from the MV portfolio.[15] We record the percentage changes in $(\mu_j - 1)$ needed to drive from one up to one half the assets from the portfolio. In addition, we record the corresponding percentage changes in the mean, standard deviation, and certainty equivalent of the portfolio and the two largest components of the weights vector.

[13] See Pulley (1981), Grauer (1986), and Merton (1980), as well as note 6.

[14] The average of the weight elasticities themselves is zero. This follows from the fact that $\iota' \mathbf{h}_{1j} = 0$ and the definition of the weight elasticity in (10).

[15] If the shift in μ_j is large enough, everything will be invested in asset j.

Table 1
Input data for a 10-asset mean-variance portfolio problem

Mean vector

Asset	1	2	3	4	5
	1.010720	1.017618	1.018270	1.010761	1.019845
Asset	6	7	8	9	10
	1.014452	1.009910	1.016353	1.013755	1.018315

Covariance matrix

Asset	1	2	3	4	5
1	0.251561E−02				
2	0.765454E−03	0.137432E−01			
3	0.110378E−02	0.284738E−02	0.139958E−01		
4	0.131391E−02	0.930502E−03	0.102651E−02	0.192771E−02	
5	0.157145E−03	0.561023E−02	0.424560E−02	0.450576E−03	0.159810E−01
6	0.554516E−03	0.345666E−02	0.276910E−02	0.897736E−03	0.349022E−02
7	0.936570E−03	0.253434E−03	0.758764E−03	0.100989E−02	0.713579E−03
8	0.164603E−02	0.175684E−02	0.319972E−02	0.164109E−02	0.421274E−02
9	0.509158E−03	0.180949E−02	0.327502E−02	0.993300E−03	0.297823E−02
10	0.151493E−02	0.344478E−02	0.362698E−02	0.965801E−03	0.439917E−02

Asset	6	7	8	9	10
6	0.487226E−02				
7	0.643392E−03	0.166439E−02			
8	0.266937E−02	0.101965E−02	0.901327E−02		
9	0.178289E−02	0.635091E−03	0.153437E−02	0.573117E−02	
10	0.265065E−02	0.611154E−03	0.359597E−02	0.215377E−02	0.140409E−01

The means are recorded as 1 plus expected rates of return (per month). Given the covariance matrix, the means were constructed to make an equally weighted portfolio of the 10 assets MV efficient for an investor with a risk tolerance parameter equal to 0.342. The data have been rounded from 14 decimal places of accuracy.

We repeat this process for each asset and report summary measures of the variables. More specifically, we report the variables for the most and least sensitive assets as well as the average and standard deviation of each of the variables calculated over the n assets.[16]

4. Computational Results for a 10-Asset Universe

Before considering the computational results in detail, it is helpful to consider a 10-asset example in order to get a feel for the magnitude of the variables involved. The input data for the portfolio problem is presented in Table 1. Given the covariance matrix, the mean vector was constructed to make an equally weighted portfolio of the 10 assets MV efficient.

Now, suppose that we increase the mean of asset 1. As we have

[16] There are many ways to define the most sensitive asset. In the computational sections, we define the most (least) sensitive asset as the one that requires the smallest (largest) increase in its mean to drive 1, 5, 10, 25, or 50 assets from the portfolio. This means the most sensitive asset may change, depending on the number of assets driven from the portfolio.

Table 2
Results of the sensitivity analysis when the mean of asset 1 is increased

Interval 0: $0 \le t \le 0.7645883\text{E}-03$

		Optimal portfolio: Constant		Optimal portfolio: Parametric		Optimal portfolio: At lower bound	Optimal portfolio: At upper bound
x(1)	=	0.1000000	+ t	228.2552	x(1)	0.1000000	0.2745213
x(2)		0.1000000		−7.853754	x(2)	0.1000000	0.0939951
x(3)		0.1000000		−6.609941	x(3)	0.1000000	0.0949461
x(4)		0.1000000		−130.7893	x(4)	0.1000000	0
x(5)		0.1000000		9.848154	x(5)	0.1000000	0.1075298
x(6)		0.1000000		11.74796	x(6)	0.1000000	0.1089824
x(7)		0.1000000		−80.99048	x(7)	0.1000000	0.0380756
x(8)		0.1000000		−15.21627	x(8)	0.1000000	0.0883658
x(9)		0.1000000		7.307490	x(9)	0.1000000	0.1055872
x(10)		0.1000000		−15.69901	x(10)	0.1000000	0.0879967
$\mu_{p0} = 1.015000 + 0.007116t + 228.2552t^2$					μ_{p0}	1.015000	1.015139
$\sigma^2_{p0} = 0.002566 - 0.063559t + 78.09603t^2$					σ^2_{p0}	0.002566	0.002563

Interval 4: $0.4868319\text{E}-02 \le t \le 0.5289392\text{E}-02$

		Optimal portfolio: Constant		Optimal portfolio: Parametric		Optimal portfolio: At lower bound	Optimal portfolio: At upper bound
x(1)	=	0.3316803	+ t	66.44265	x(1)	0.6551443	0.6831215
x(2)		0.0898674		−6.095846	x(2)	0.0601909	0.0576241
x(3)		0.1254902		−9.983543	x(3)	0.0768872	0.0726833
x(4)		0		0	x(4)	0	0
x(5)		0.1358275		1.758001	x(5)	0.1443860	0.4151262
x(6)		0.2085763		−39.43294	x(6)	0.0166042	0
x(7)		0		0	x(7)	0	0
x(8)		0		0	x(8)	0	0
x(9)		0		0	x(9)	0	0
x(10)		0.1085584		−12.68833	x(10)	0.0467875	0.0414448
$\mu_{p4} = 1.015130 - 0.013248t + 66.44265t^2$					μ_{p4}	1.016640	1.016919
$\sigma^2_{p4} = 0.002852 - 0.236030t + 22.73292t^2$					σ^2_{p4}	0.002242	0.002239

The input data for this problem is recorded in Table 1. In the ith interval where $t_i \le t \le t_{i+1}$, $\mathbf{x}_i(t) = \mathbf{h}_{0i} + t\mathbf{h}_{1i}$, $\mu_{pi} = \alpha_{0i} + \alpha_{1i}t + \alpha_{2i}t^2$, and $\sigma^2_{pi} = \gamma_{0i} + \gamma_{1i}t + \gamma_{2i}t^2$. The table shows the results for intervals 0 and 4. In each interval the parametric results are shown on the left, and $\mathbf{x}_i(t)$, $\mu_{pi}(t)$, and $\sigma^2_{pi}(t)$, evaluated at t_i and t_{i+1}, are shown on the right. In the first (fourth) interval, asset 1's mean increased from roughly 1.01072 to 1.011485 (1.015588 to 1.016009).

noted, in the ith interval, where $t_i \le t \le t_{i+1}$, $\mathbf{x}_i(t) = \mathbf{h}_{0i} + t\mathbf{h}_{1i}$, $\mu_{pi}(t) = \alpha_{0i} + \alpha_{1i}t + \alpha_{2i}t^2$, and $\sigma^2_{pi}(t) = \gamma_{0i} + \gamma_{1i}t + \gamma_{2i}t^2$. Table 2 shows the values of these parameters in intervals 0 and 4. In each interval, the parametric results are shown on the left; and $\mathbf{x}_i(t)$, $\mu_{pi}(t)$, and $\sigma^2_{pi}(t)$, evaluated at t_i (the lower) and t_{i+1} (the upper bound on the interval), are shown on the right. The parametric portion of the optimal portfolio in interval 0 shows that small changes in the mean of an asset generate large changes in the portfolio weights. For example, asset 1's (4's) weight increases (decreases) at a rate of 228 (131) times the rate of increase in the mean of asset 1. [From (10), we can convert the rates of change in the weights of an equally weighted

Table 3
A comparison of selected variables for the 10-asset example with and without nonnegativity constraints

	Shift to drive 5 assets out		Shift to drive 9 assets out[1]	
	With nonnegativity constraints	Without nonnegativity constraints	With nonnegativity constraints	Without nonnegativity constraints
% change μ_1	49.34	49.34	149.43	149.43
% change μ_p	12.79	42.83	78.25	391.21
% change σ_p^2	−12.73	72.05	−1.97	741.23
$\lVert \mathbf{x} - \mathbf{x}^* \rVert$	0.6309	1.4643	0.9487	4.4345
$\lVert \boldsymbol{\mu} - \boldsymbol{\mu}^* \rVert$	0.0053	0.0053	0.0160	0.0160
$\lVert \mathbf{x} - \mathbf{x}^* \rVert / \lVert \boldsymbol{\mu} - \boldsymbol{\mu}^* \rVert$	59.2	276.8	119.3	276.8
Largest weight	0.66	1.31	1	3.76
Second largest weight	0.15	0.16	0	0.29
Smallest weight	0	−0.59	0	−2.00

The input data for these problems is recorded in Table 1.

[1] Specifically, the size of the shift in the mean of asset 1 required to drive nine assets from the 10-asset portfolio when the investor imposes nonnegativity constraints on the problem.

portfolio to elasticities by multiplying the elements of $\mathbf{h}_1$ by $n(\mu_j - 1)$. Thus, in 10-asset cases the elasticities will be roughly one tenth the size of the rates of change contained in $\mathbf{h}_1$, and in 100-asset cases the elasticities will be roughly the same size as the rates of change.] When asset 4 is driven from the portfolio and interval 0 ends, the mean of the portfolio has increased by less than 1 percent and the variance of the portfolio has decreased by less than 1 percent. In interval 4, we see that the rates of change in the portfolio weights are smaller than in interval 0; four rates of change are zero, reflecting the fact that four nonnegativity constraints are active; and asset 6's rate of change has changed from positive in interval 0 to negative. In sum, a 50 percent increase in the mean of asset 1 has driven half the assets from the portfolio, while the mean and variance of the portfolio have changed by less than 13 percent in absolute value.

Table 3 shows the analysis with and without nonnegativity constraints. We note two points. First, from Tables 2 and 3, we see by any metric—Euclidean norms, rates of change, or elasticities—that the portfolio weights are extremely sensitive to changes in the means. Second, without nonnegativity constraints, a large change in the mean of an asset generates large changes in the portfolio's weights, expected return, and variance. But there is so much action in the weights that, with nonnegativity constraints imposed on the problem, assets are driven out of the portfolio before large changes in the portfolio mean and variance can occur.

5. More Extensive Computational Results

Table 4 shows the average elasticities with respect to a change in the mean of an asset for universes ranging from 10 to 100 assets. Two

Table 4
Average elasticities with respect to changes in the mean of an asset for equally weighted portfolios

Port. size	Elast. of mean of port.	Elast. of SD of port.	Elast. of CE of port.	Smallest elast. of weights	Greatest elast. of weights	Ave. abs. elast. of weights
10	0.127	0.053	0.133	−7.20	15.04	5.69
20	0.066	0.032	0.067	−12.01	30.58	7.27
50	0.028	0.017	0.027	−41.75	107.02	20.00
100	0.015	0.010	0.013	−523.61	826.00	218.00

Elast., elasticity; mean, mean rate of return; SD, standard deviation of rate of return; CE, certainty equivalent rate of return; ave. abs., average of absolute values; port., portfolio. For an n-asset portfolio, all n-asset means are increased one at a time. The reported elasticities are averages for the increases in the n-asset means. The ave. abs. of the elast. of weights is a "double" average. First, the absolute values of the n-portfolio weights are averaged for an increase in one mean. Then, the averages are averaged for the increases in all the means.

points stand out. First, the average elasticities of portfolio returns are small (and decrease as the portfolio size increases). Second, the average elasticities of the portfolio weights are large (and increase as the portfolio size increases). The largest average elasticity of a portfolio return is less than 0.15; and, for a 100-asset portfolio, the average elasticities are on the order of 0.015. On the other hand, the average elasticities of the weights range from −7 to 15 for a 10-asset portfolio, and from −524 to 826 for a 100-asset portfolio. The average of the average of the absolute values of the elasticities of the portfolio weights ranges from under 6 for a 10-asset portfolio to 218 for a 100-asset portfolio. Furthermore, the average of the average of the absolute values of the elasticities of the weights is over 40 times as large as the average elasticities of any of the return variables for a 10-asset portfolio, and over 14,000 times as large for a 100-asset portfolio.

Table 5 shows the average increases in the mean of an asset (plus the accompanying changes in the mean, standard deviation, and certainty equivalent of the portfolio) required to drive from one up to one half the assets from equally weighted MV-efficient portfolios. The table also shows the two largest components of the portfolio-weights vector at each iteration. On average, it takes an increase of over 77 percent in the mean of an asset to drive half the assets from a 10-asset portfolio. The increase is accompanied by relatively large changes in the mean, standard deviation, and certainty equivalent of the portfolio. Furthermore, 83 percent of the portfolio's weight is concentrated in just two assets. On the other hand, on average, it only takes an increase of 11.6 percent in the mean of an asset to drive half the assets from a 100-asset portfolio. The accompanying changes in the portfolio returns are about 2 percent, and 26 percent of the optimal MV-efficient portfolio's weight is concentrated in two assets.

Table 5
Average increase in the mean of an asset required to drive from one to one half the assets from equally weighted portfolios

Port. size	No. of assets out	% change in mean of asset j	% change in mean of port.	% change in SD of port.	% change in CE of port.	Weight of asset j	Second highest weight
10	1	22.70	10.14	8.71	7.26	0.32	0.14
	5	77.68	66.55	39.80	50.10	0.71	0.12
20	1	12.80	3.25	3.04	2.24	0.19	0.08
	5	30.24	11.66	9.54	8.61	0.35	0.10
	10	58.00	33.68	23.70	25.64	0.55	0.11
50	1	2.97	0.25	0.23	0.18	0.07	0.04
	5	5.83	0.69	0.59	0.53	0.12	0.05
	10	9.44	1.48	1.22	1.15	0.16	0.06
	25	29.64	9.72	7.41	7.70	0.34	0.08
100	1	0.23	0.01	0.00	0.01	0.02	0.02
	5	0.52	0.02	0.01	0.02	0.04	0.03
	10	0.90	0.04	0.02	0.04	0.05	0.03
	25	2.92	0.23	0.13	0.22	0.09	0.04
	50	11.60	2.14	1.52	1.82	0.20	0.06

See Table 4 for definitions of symbols and an explanation of the averaging process.

To put the size of these changes in perspective, note that the average mean return for the assets is 1.5 percent per month, or approximately 18 percent per annum. Thus, if the "average" asset's mean increases by 11.6 percent (i.e., from 18 to 20.1 percent per annum), one half of the assets are driven from the optimal MV-efficient portfolio, but there is virtually no change in its expected return or standard deviation. This is illustrated in Figure 3. The figure plots the efficient frontier; the individual securities, as dots; and the equally weighted MV-efficient portfolio, as the dark circle. On average, if the mean of an asset increases by 11.6 percent, then, in spite of the dramatic change in the composition of the optimal MV-efficient portfolio, it still plots within the dark circle.

Tables 4 and 5 show averages, but averages tell only part of the story. Therefore, we report more detailed statistics on the cross-sectional variation in the sensitivities of portfolio returns and composition for a 100-asset portfolio in Tables 6 and 7. Note, for example, consistent with the analytical results, that when an asset's mean increases, the elasticities of the portfolio mean and standard deviation can be negative.[17]

[17] The extreme sensitivities of the portfolio weights reported in this paper may be related to the estimation risk problem [see Bawa, Brown, and Klein (1979)]. When the mean vector and covariance matrix are estimated from historical return data, estimation risk is taken into account by multiplying Σ by a constant $k = (\tau + 1)(\tau - 1)/\tau(\tau - n - 2)$, where τ is the number of time-series observations and n the number of securities [see Bawa, Brown, and Klein (1979, pp. 88–89)]. This causes the efficient frontier to shift to the right, and an investor with given risk tolerance T to hold a less risky portfolio, where in this case risk is synonymous with variance.

In general, we view the sensitivity, or comparative statics, analysis as complementary to estimation

Table 6
Elasticities with respect to changes in the mean of an asset for an equally weighted portfolio of 100 assets

	Elast. of mean of port.	Elast. of SD of port.	Elast. of CE of port.	Smallest elast. of weights	Greatest elast. of weights	Ave. abs. elast. of weights	SD of elast. of weights
Most sensitive asset	0.053	0.093	0.009	−1254.96	2586.17	343.53	371.37
Least sensitive asset	−0.048	−0.118	0.015	−192.77	195.54	85.02	69.98
Average	0.015	0.010	0.013	−523.61	826.00	218.00	191.84
SD	0.076	0.150	0.003	224.71	521.01	84.96	79.41

See Table 4 for definition of symbols. By definition, the most (least) sensitive asset is the one that requires the smallest (largest) increase in its mean to drive 1, 5, 10, 25, or 50 assets from the portfolio. The averages are explained in Table 4. The SDs in the last row are taken across securities.

6. The Robustness of the Computational Results

We examined the robustness of the results in three ways. First, we examined whether the results were affected by expanding the opportunity set to allow borrowing or lending at the riskless rate. With 50 risky or more assets, there was virtually no difference in the sensitivity of the optimal portfolio to increases in the means of risky assets whether or not the riskless opportunity was included.[18] Furthermore, on average, the increase in the riskless rate required to drive assets

risk analysis. There appears to be no reason the two forms of analysis could not be combined. However, we have chosen not to follow that route in the text for the following reason. The computational analysis focuses on what happens to a nonnegatively weighted MV-efficient portfolio as some subset of the means changes. If we wish to begin the analysis with an MV investor holding a nonnegatively weighted portfolio, then we must employ $(\boldsymbol{\Sigma}, \mathbf{x}^*)$-compatible means. We have taken considerable care to choose $(\boldsymbol{\Sigma}, \mathbf{x}^*)$-compatible means that embody realistic expected returns, excess returns, and risk tolerance parameters. If we now choose to treat the means, variances, and covariances as if they were measured with error (i.e., if we now combine our analysis with the estimation risk analysis), it will imply that either (a) the initial MV-efficient portfolio is no longer nonnegatively weighted; or (b) the $(\boldsymbol{\Sigma}, \mathbf{x}^*)$-compatible means must be unrealistically high; or (c) the investor must be unrealistically tolerant of risk.

To see this, we note that if $\boldsymbol{\Sigma}$ is replaced by $k\boldsymbol{\Sigma}$, then many of the other scalars, vectors, and matrices in the efficient set mathematics and sensitivity analysis are multiplied by either k or k^{-1}. Therefore, we consider four cases. Case (i), the case examined in the text, has $\boldsymbol{\mu}$, $\boldsymbol{\Sigma}$, and risk tolerance parameter T. [Throughout this note $\boldsymbol{\mu}$ is assumed to be $(\boldsymbol{\Sigma}, \mathbf{x}^*)$ compatible.] Case (ii), which corresponds to the classic estimation risk case, has $\boldsymbol{\mu}$, $k\boldsymbol{\Sigma}$, and T. Case (iii) has $\boldsymbol{\mu}$, $k\boldsymbol{\Sigma}$, and kT. Case (iv) has $\boldsymbol{\mu}(k) = r_z\boldsymbol{\iota} + k(\mu_m - r_z)\boldsymbol{\beta}$, $k\boldsymbol{\Sigma}$, and T. In Cases (i), (iii), and (iv), the equally weighted portfolio is MV efficient given T or kT. With 100 assets and the elements of $\boldsymbol{\Sigma}$ estimated from 120 time-series observations, $k = 20/3$. If we were to embody $k\boldsymbol{\Sigma}$ in the analysis as in Case (ii), then the investor with risk tolerance parameter T would hold a portfolio with less risk (variance), but it would no longer be a nonnegatively weighted portfolio. Case (iii) yields identical results to Case (i) in terms of portfolio weights, expected returns, and their elasticities. However, given the parameters involved, the investor would have to have a risk aversion parameter γ, from (26), on the order of 0.8, which would make him much less risk-averse than almost anyone in the literature has suggested as being realistic. Case (iv), on the other hand, has a realistic risk tolerance value, but the excess return on the optimal portfolio would have to be on the order of 60 rather than 9 percent per annum. Hence, we confine our analysis to Case (i) and its variants in the text.

[18] However, two points are worth noting. First, with an increase in the mean of a risky asset, the opportunity set is more favorable. Hence, the investor borrows to help finance the investment in the risky assets. Second, with more risky assets in the portfolio, borrowing plays a smaller role. Hence, the cases with and without the riskless asset converge.

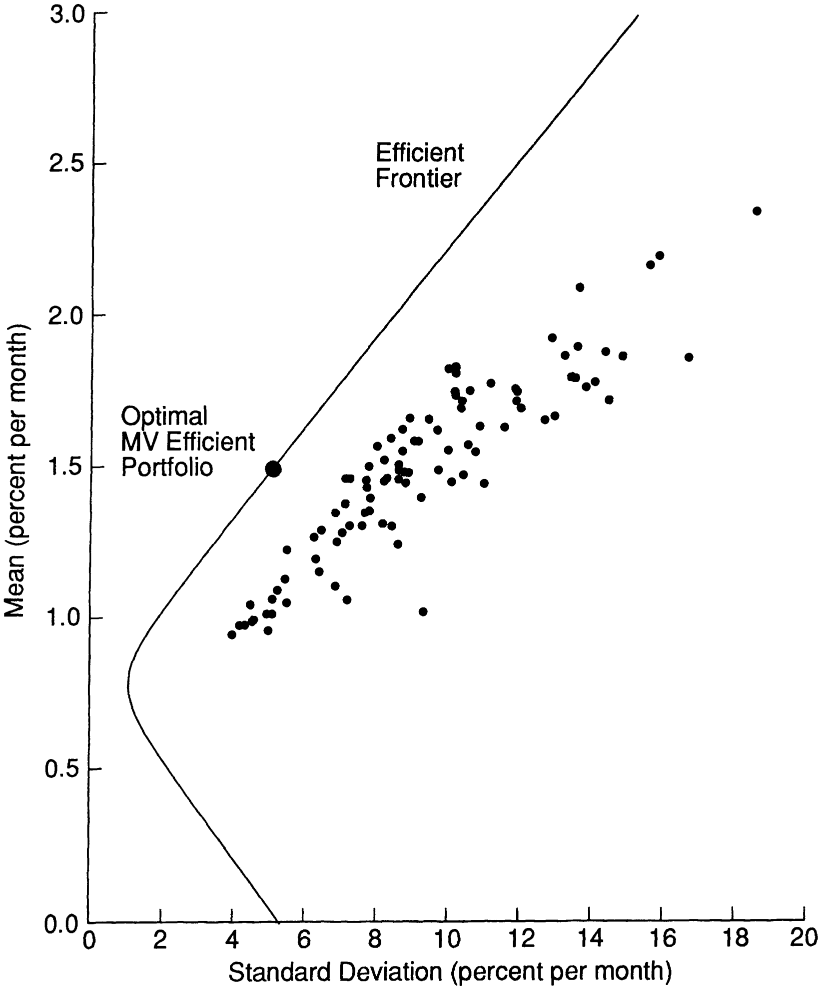

Figure 3
Results of increasing the "average" asset's mean
On average, if an asset's mean increases by 11.6 percent, half the assets are driven from an equally weighted MV-efficient portfolio. Yet the optimal portfolio still plots within the dark circle.

from the portfolio was about half the size of the increase in the mean of a risky asset required to drive the same number of assets from the portfolio.

Second, we examined the robustness of the results to three different covariance structures. The benchmark was the random sample of 20 securities drawn from the CRSP tapes, which were correlated on the order of 0.25. From an empirical point of view, it is clearly the relevant case. The other two cases are extreme. A zero-correlation case used

Table 7
Increase in the mean of an asset required to drive from one to one half the assets from an equally weighted portfolio of 100 assets

No. of assets out	% change in mean of asset *j*	% change in mean of port.	% change in SD of port.	% change in CE of port.	Weight of asset *j*	Second highest weight
Most sensitive asset						
1	0.08	0.01	0.01	0.00	0.03	0.02
5	0.18	−0.02	−0.05	0.00	0.05	0.02
10	0.30	0.03	0.04	0.01	0.07	0.02
25	1.28	−0.06	−0.26	0.09	0.09	0.03
50	6.52	0.88	−0.19	1.05	0.18	0.05
Least sensitive asset						
1	0.52	−0.02	−0.06	0.01	0.02	0.02
5	1.41	−0.03	−0.13	0.05	0.03	0.03
10	2.11	0.21	0.29	0.09	0.05	0.04
25	5.69	0.74	0.86	0.41	0.05	0.05
50	21.44	2.90	2.26	2.34	0.10	0.06
Averages						
1	0.23	0.01	0.01	0.01	0.02	0.02
5	0.52	0.02	0.01	0.02	0.04	0.03
10	0.90	0.04	0.02	0.04	0.05	0.03
25	2.92	0.23	0.13	0.22	0.09	0.04
50	11.60	2.14	1.52	1.82	0.20	0.06
SD						
1	0.10	0.02	0.03	0.00	0.01	0.00
5	0.23	0.04	0.07	0.01	0.01	0.00
10	0.37	0.06	0.12	0.01	0.01	0.01
25	0.98	0.19	0.35	0.06	0.03	0.01
50	3.46	0.95	1.47	0.44	0.05	0.01

See Tables 4 and 6 for definition of symbols.

the variances of the 20 securities in the random sample as the diagonal elements in the covariance matrix, and a more highly correlated case (on the order of 0.85) employed 20 beta-ranked portfolios that include most of the stocks on the New York Stock Exchange. [See Cheng and Grauer (1980) for a complete description of the grouping procedure.] The results showed that portfolios constructed from uncorrelated data were relatively insensitive to an increase in the mean of an asset, while portfolios constructed from highly correlated data were extremely sensitive to an increase in the mean of an asset. The return elasticities were similar. But the weight elasticities were very different. While the directions of the results were what one might expect (i.e., the more highly correlated the data, the more sensitive a portfolio is to increases in the means), the magnitudes of the differences were striking. The average of the average absolute values of the weight elasticities were 0.44, 7.27, and 105.22 as we moved from the uncorrelated to the highly correlated data. On average, it took an increase of 673, 58, and 2.6 percent in the mean of an asset to drive half the assets from the three portfolios.

Finally, instead of increasing one asset's mean, we examined the effect of simultaneously increasing or decreasing all of the means by the same percent—that is, one half of the means were increased and one half decreased. For a 10-, 20-, 50-, and 100-asset universe, it took a 1.94, 0.99, 0.20, and 0.01 percent change in *all of the means* to drive *one asset* from the portfolio, respectively. (The corresponding average increases in just one mean to drive one asset from the portfolio were 22.70, 12.80, 2.97, and 0.23 percent, respectively.) The simultaneous changes in all the means were 9 and 4 percent the size of the average increase in one mean for the 10- and 100-asset portfolios, respectively. For a 10-, 20-, 50-, and 100-asset universe, it took a 19.72, 15.77, 4.85, and 2.11 percent change in *all of the means* to drive *half the assets* from the portfolio, respectively. (The corresponding average increases in one mean to drive out half the assets were 77.68, 58.00, 29.64, and 11.60 percent, respectively.) In this case, the simultaneous changes in all the means were 25 and 18 percent the size of the average increase in one mean for the 10- and 100-asset portfolios. Furthermore, the absolute size of the shifts was minuscule. Using an 18 percent per annum return as a benchmark, a 2.11 percent shift in the mean would yield means of between 17.62 and 18.38 percent per annum.[19]

7. Summary and Concluding Comments

The Sharpe–Lintner version of the CAPM asserts that all investors will hold some portion of the market portfolio. The normative implication drawn from the theory, as well as from the efficient markets hypothesis, is that investors should hold diversified portfolios. Even active portfolio managers tend to hold the market portfolio and to shade into or out of those securities that they feel are under- or overpriced. However, both our analytical and computational comparative statics results indicate that MV investors plunge, rather than shade, into or out of under- or overpriced securities. For example, our computational analysis showed that with a 100-asset portfolio, on average, the elasticities of the portfolio weights are on the order of 14,000 times the magnitude of the average elasticity of any of the portfolio return variables. Furthermore, it takes surprisingly small changes in the means of assets to drive half the securities from an equally weighted MV-

[19] However, while the shifts are small, they may not appear small relative to the average shift in the asset's mean. The mitigating influence is the degree of substitution between asset holdings. For example, when all the means were shifted at once, some of the holdings of assets whose means increased (decreased) actually decreased (increased).

efficient portfolio. On average, an increase of 11.6 percent in the mean of any one asset, say, from 18 to 20.1 percent per annum, drives half the assets from the equally weighted portfolio. Yet, despite this dramatic change in its composition, the portfolio's expected return and standard deviation only change by about 2 percent.

This extreme sensitivity of the composition of MV-efficient portfolios to changes in the asset means has at least three implications that extend beyond the diversification issue explored here. Two are concerned with portfolio management; the third with tests of the MV efficiency of a portfolio. First, with a perfectly functioning short sales mechanism, Best and Gauer (1990c) showed almost any deviation from $(\mathbf{\Sigma}, \mathbf{x}^*)$-compatible means implies that there are no positively weighted MV-efficient portfolios. Second, the evidence presented here and in that paper almost certainly guarantees that binding constraints will be the rule for real-world portfolio problems. If constraints are almost certain to be binding, any viable sensitivity analysis must be able to deal explicitly with multiple constraints. As a practical matter, the parametric quadratic programming analysis described in this paper is ideally suited to deal with problems of this sort. Third, turning to the question of testing, Roll (1977) has argued that there is a single testable hypothesis associated with the CAPM: the market portfolio is MV efficient. Yet Grauer (1990) showed that multivariate tests of MV efficiency have little power to reject the hypothesis that a positively weighted portfolio is MV efficient when samples are drawn from populations where no positively weighted portfolio is MV efficient.

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