Graph Signal Processing - Graph Laplacian

Prof. Luis Gustavo Nonato

August 23, 2017

Adjacency Matrix

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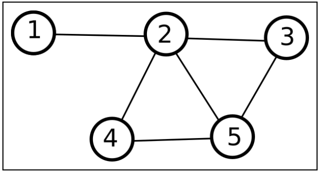

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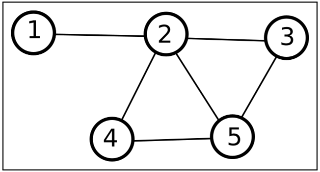

$$\mathbf{A} = \begin{array}{c c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

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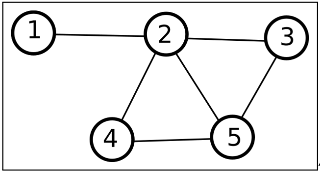

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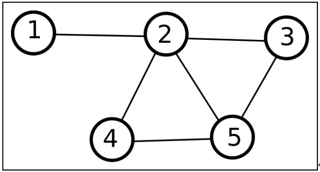

;

$$\underbrace{\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 4 & -1 & -1 & -1 \\ 0 & -1 & 2 & 0 & -1 \\ 0 & -1 & 0 & 2 & -1 \\ 0 & -1 & -1 & -1 & 3 \end{bmatrix}}_{\mathbf{L}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}}_{\mathbf{D}} - \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}}_{\mathbf{A}}$$

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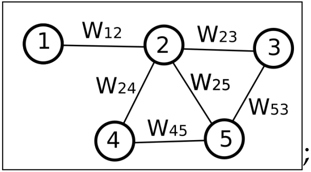

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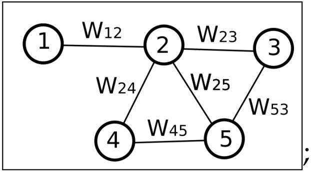

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$$\begin{bmatrix} w_{12} & -w_{12} & 0 & 0 & 0 \\ -w_{21} & \sum_{j\neq 2} w_{2j} & -w_{23} & -w_{24} & -w_{25} \\ 0 & -w_{32} & \sum_{j\neq 3} w_{3j} & 0 & -w_{35} \\ 0 & -w_{42} & 0 & \sum_{j\neq 4} w_{4j} & -w_{45} \\ 0 & -w_{52} & -w_{53} & -w_{54} & \sum_{j\neq 5} w_{5j} \end{bmatrix}$$

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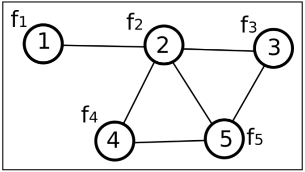

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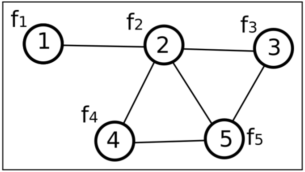

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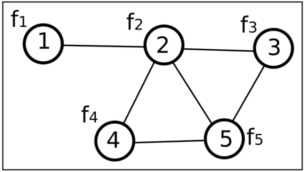

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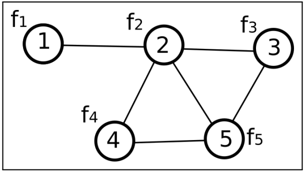

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or equivalently

$$f_i = \frac{1}{l_{ii}} \sum_{j \neq i} f_j$$

(the value in each node is the average of values in neighbor nodes)

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  Eigenvalues are non-negative
- The eigenvectors are "nice" functions defined on the graph.

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Property 1

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$$\mathbf{L}\mathbf{u}_0 = \begin{bmatrix} & & & & \vdots & & & & \\ -w_{i1} & 0 & -w_{i3} & \cdots & \sum_j w_{ij} & \cdots & 0 & -w_{3(n-1)} & 0 \\ & & & & \vdots & & & & \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = 0\mathbf{u}_0$$

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Imposing the additional constraint $\|\mathbf{f}\|^2 = 1$, the Courant-Fiecher theorem ensures that the minimum is reached when $\mathbf{f}$ is the eigenvector associated to the second smallest eigenvalue.

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where $\mathbf{u}_i$ are the eigenvectors of $\mathbf{L}$.

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- A positive (or negative) *strong nodal domain* of a function f defined on V is a maximal connected subgraph of G where $f(v) > 0$ (or $f(v) < 0$). The number of strong nodal domains of f is $\mathcal{S}(f)$.

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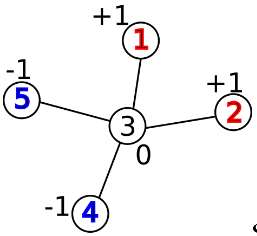

$$\mathcal{S}(f) = \{\{1\}, \{2\}, \{4\}, \{5\}\}$$
$$\mathcal{W}(f) = \{\{1,2,3\}, \{3,4,5\}\}$$

Property 4: Discrete Courant's Nodal Theorem

Let G be a connected graph with n vertices. Any Graph Laplacian eigenvector $\mathbf{u}_k$ with corresponding eigenvalue λ_k with multiplicity r has at most $k+1$ weak nodal domains and $k+r$ strong nodal domains, i.e.,

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This theorem was proved by Davies, Gladwell, Leydold, Stadler in 2001 and it is the discrete version of the Courant's Nodal Theorem for the Laplace operator on manifolds.

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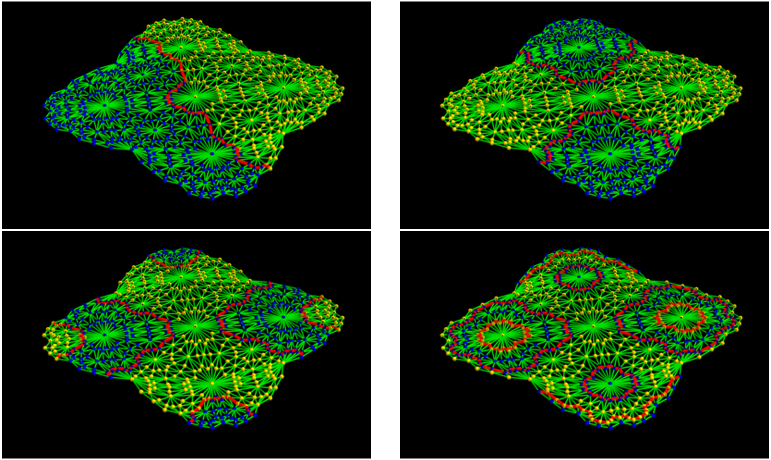

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- In the case of normalized graph Laplacian, there are a multitude of results giving upper and lower bounds for the eigenvalues, specially related to the diameter of G.
- Normalized Graph Laplacian is also closely related with random walks on graphs.

  $$\mathbf{P} = \mathbf{D}^{-1/2}(\mathbf{I} - \mathcal{L})\mathbf{D}^{1/2}$$