CS111B - Assignment 1. Linear Systems

February 4, 2020

1 CS111B - Assignment 1. Linear Systems

1.1 Alfonso Santacruz Garcia

Important Notice: Most of the solutions include computations in SageMath. For a reference on these computations specifically please visit any of the links provided in Apprendix B. For instance, the solutions that include proofs and explanations in prose only generally have no algorithm in the #computationaltools link in such appendix.

1.1.1 Glossary Work

• **Linear Functions:** A linear function is any function that depicts a straight line when shown graphsically. For this to mathematically be true, the variables that compose the functions have exponents or power of degree 1. In other words all the terms of the function are linear terms.

• **Linear Systems:** Is a set or collectionf of linear equations or functions that are normally related by their variables.

• **Consistent Systems:** It is a system of two or more linear equations that has at least one solution. If the consistent system has exactly one solution, it is called independent consistent system. On the other hand, if it can have infinitely many solution, it is a dependent consisten system.

• **Augmented Matrix:** It is the matrix obtained by appending the columns of two given matrices. It is usually used to perform elementary row operations and obtain the solution of a linear system of equations.

• **Row Operations:** Are used to solve systems of equation without changing their consistency. There are three, row swapping, scalar multiplication and row addition.

• **Elementary Matrices:** Used in the Gaussian method of elimination to reduce a matrix to its row echelon form (or RREF form). They represent an elementary row operation by multiplying the matrix on the left.

• **Invertible Matrix:** Also called nonsingular. A $n \times n$ matrix is called invertible if there exists another $n \times n$ square matrix such that their multiplication results in an $n \times n$ identity matrix.

• **RREF Matrix:** For a matrix to be RREF, it needs to meet all of the following conditions:

1) If there is a row where every entry is zero, then this row lies below any other row that contains a nonzero entry.
2) The leftmost nonzero entry of a row is equal to 1.

3) The leftmost nonzero entry of a row is the only nonzero entry in its column.

• **Homogeneous Systems:** It is equivalent to a matrix equation of the form $A\vec{x} = \vec{0}$, where A is an $m \times n$ matrix, $\vec{x}$ is a column vector with n entries and $\vec{0}$ is a zero vector with m entries.

• **Determinants:** It is a scalar value that can be computed from the elements of a square matrix and encodes certain properties of the matrix.

• **Singular matrix:** Another way to call an invertible matrix.

1.1.2 Skill-Builder

Sessions 1.1 & 1.2: Solving Linear Systems Using Row Operations

1.1 For each of the following linear systems: i- Define the associated augmented matrix.

ii- Use row operations to reduce the matrix to RREF. Check your work using your favorite computational tool.

iii- Determine whether or not the system is consistent.

iv- If the system is consistent, find all solutions.

(a)

$$x + y + z = 3$$

$$x + 3yz = 1$$

$$3x + 5y + 4z = 10$$

(b)

$$2x4y + 8z = 4$$

$$xy + z = 4$$

$$3x4y + 6z = 7$$

(c)

$$2x + 2y + 6z = 2$$

$$6x + 9y3z = 9$$

$$10x + 14y + 2z = 14$$

1.2 Find numbers a, b, c, and d so that the augmented matrix

$$\begin{bmatrix} 1 & 2 & 3 & a \\ 0 & 4 & 5 & b \\ 0 & 0 & d & c \end{bmatrix}$$

has (a) no solutions, and (b) infinitely-many solutions. Which of the numbers have no effect on the solvability?

1.3 True or False: All homogeneous systems are consistent. Make sure to justify your answer.

Solution to Problem 1.1:

(a) For the following system of equations:

$$x + y + z = 3$$

$$x + 3yz = 1$$

$$3x + 5y + 4z = 10$$

We can define the following augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 3 & -1 & 1 \\ 3 & 5 & 4 & 10 \end{array}\right]$$

Using row operations, we can obtain the matrix RREF. In general, we aim to obtain an identity matrix in the left side of the augmented matrix. We can start by adding R_2 with R_1 and divide the addition by two. Then, we can multiply the R_1 times 4 and then substract R_3 from it.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 3 & -1 & 1 \\ 3 & 5 & 4 & 10 \end{array}\right] \xrightarrow[R_2 = \frac{(R_2+R_1)}{2}]{} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 0 & 2 \\ 3 & 5 & 4 & 10 \end{array}\right] \xrightarrow[R_1 = 4R_1 - R_3]{} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 1 & 2 & 0 & 2 \\ 3 & 5 & 4 & 10 \end{array}\right]$$

As we can see, we have eliminated the z variable from R_1 and R_2 (third column). Plus, the coefficients for the x variable in the first column are the same for R_1 and R_2. Therefore, substracting R_1 from R_2 and then dividing the result by 3, we can obtain the values for the y variable in the second column. With that result we can add R_2 to R_1 to obtain the result for the x variable in the first column.

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 1 & 2 & 0 & 2 \\ 3 & 5 & 4 & 10 \end{array}\right] \xrightarrow[R_2 = \frac{R_2 - R_1}{3}]{} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 3 & 5 & 4 & 10 \end{array}\right] \xrightarrow[R_1 = R_1 + R_2]{} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 3 & 5 & 4 & 10 \end{array}\right]$$

Having the first two variables x and y, we can multiply R_1 times 3 and substract it from R_3 and divide the result by 4 to obtain the variable z in the third column. Notice that we are not using R_2 since we already know from it that $y = 0$. Therefore:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 3 & 5 & 4 & 10 \end{array}\right] \xrightarrow{R_3 = \frac{R_3 - 3R_1}{4}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Notice that since in our resultant matrix there is at least one set of values for the unknowns that satisfies each equation in the system. Or in other words, when substituted into each of the equations makes each equation hold true as an identity. As a result, we can affirm from the augmented matrix that:

$$x = 2; y = 0; z = 1$$

(b) For the following system of equations:

$$2x4y + 8z = 4$$

$$xy + z = 4$$

$$3x4y + 6z = 7$$

We can define the following augmented matrix, which can also be simplified by dividing R_1 by 2:

$$\left[\begin{array}{ccc|c} 2 & -4 & 8 & 4 \\ 1 & -1 & 1 & 4 \\ 3 & -4 & 6 & 7 \end{array}\right] \xrightarrow{R_1 = \frac{R_1}{2}} \left[\begin{array}{ccc|c} 1 & -2 & 4 & 2 \\ 1 & -1 & 1 & 4 \\ 3 & -4 & 6 & 7 \end{array}\right]$$

Next, we can subtract R_1 and $2R_2$ from R_3. Leading to the following result:

$$\left[\begin{array}{ccc|c} 1 & -2 & 4 & 2 \\ 1 & -1 & 1 & 4 \\ 3 & -4 & 6 & 7 \end{array}\right] \xrightarrow{R_3 = R_3 - R_1 - 2R_2} \left[\begin{array}{ccc|c} 1 & -2 & 4 & 2 \\ 1 & -1 & 1 & 4 \\ 0 & 0 & 0 & -3 \end{array}\right]$$

Then, we will subtract R_1 from R_2 followed by adding R_3 and R_2:

$$\left[\begin{array}{ccc|c} 1 & -2 & 4 & 2 \\ 1 & -1 & 1 & 4 \\ 0 & 0 & 0 & -3 \end{array}\right] \xrightarrow{R_2 = R_2 - R_1} \left[\begin{array}{ccc|c} 1 & -2 & 4 & 2 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & -3 \end{array}\right] \xrightarrow{R_2 = R_2 + R_3} \left[\begin{array}{ccc|c} 1 & -2 & 4 & 2 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & -3 \end{array}\right]$$

Lastly, to simplify R_1, we will add R_2 twice to R_1, which will eliminate the coefficient for y in R_1 (the second column). Thus, we can subtract $2R_3$ from R_1 to make the last element of R_1 equal to zero:

$$\left[\begin{array}{ccc|c} 1 & -2 & 4 & 2 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & -3 \end{array}\right] \xrightarrow[R_1 = R_1 + 2R_2]{} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & -3 \end{array}\right]$$

A linear or non linear equation system is called as inconsistent if there is no set of values for the unknowns that satisfies all of the equations. Moreover, in an incosinstent system it is possible to manipulate and combine the equations in such a way as to obtain contradictory information. If we consider R_3 to be consistent, then $0 = 1$. Since this is a contradictory information, we have evidence to affirm tha this system of linear equations is inconsistent after a series of row operations.

(c) For the following system of equations:

$$2x + 2y + 6z = 2$$

$$6x + 9y3z = 9$$

$$10x + 14y + 2z = 14$$

We can define the following augmented matrix, which can also be simplified by dividing R_1 and R_3 by 2 and R_2 by 3:

$$\left[\begin{array}{ccc|c} 2 & 2 & 6 & 2 \\ 6 & 9 & -3 & 9 \\ 10 & 14 & 2 & 14 \end{array}\right] \xrightarrow[R_1 = \frac{R_1}{2}; R_3 = \frac{R_3}{2}; R_2 = \frac{R_2}{3};]{} \left[\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 2 & 3 & -1 & 3 \\ 5 & 7 & 1 & 7 \end{array}\right]$$

We will now subtract $2R_2$ and R_1 from R_3:

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 2 & 3 & -1 & 3 \\ 5 & 7 & 1 & 7 \end{array}\right] \xrightarrow[R_3 = R_3 - 2R_2 - R_1]{} \left[\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 2 & 3 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Last, we will subtract $2R_1$ from R_2 followed by subtracting R_2 from R_1

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 2 & 3 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[R_2 = R_2 - 2R_1]{} \left[\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 0 & 1 & -7 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[R_1 = R_1 - R_2]{} \left[\begin{array}{ccc|c} 1 & 0 & 10 & 0 \\ 0 & 1 & -7 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Let us declare a variable t, where $t \in \mathbb{R}$. Thus, we can say that the following equations hold:

$$x + 10t = 0 \therefore x = -10t$$

$$y - 7t = 1 \therefore y = 1 + 7t$$

$$z = t$$

A linear or nonlinear system of equations is called as consistent if there is **at least one** set of values for the unknowns that satisfies each equation in the system. Since inputting any real number for t will lead to a set of values for x, y and z such that the system of equations may be satisfied, we can say then that this system is consistent. However, there can be infinitely many solutions for this sytem of linear equations, as discussed previously.

Solution to Problem 1.2

(a) A system has no solution if the equations are inconsistent. In other words, they may lead to a contradictory or absurd result. The echelon form of the matrix for an inconsistent system has a row with a nonzero number in the last column and 0's in all other columns. Assuming the matrix given in this problem is an augmented matrix for a linear system of equations, the rref of the matrix leads to an inconsistent result as described when the value of $d = 0$. Moreover, by using Sage, when visualizing matrix in echelon form by setting this value for d, the result is a specific rref regardless of the values of c, a and b. Thus, they have no effect on the solvability.

(b) It is said that a system has infinitely many solutions when it is consistent and the number of variables is more than the number of nonzero rows in the rref of the matrix. Once again, assuming the matrix given in this problem is an augmented matrix and the system consists of three different variables, setting the values of d and c to zero, will lead to a matrix having the last row with all zeroes. Moreover, by using Sage, when visualizing matrix in echelon form by setting these values for c and d, the result is a specific rref regardless of the values of a and b. Thus, both a and b have no effect on the solvability.

Solution to Problem 1.3 True. We define a system as inconsistent if its echelon form contains a row with a nonzero value in the last column and 0's in all other entries of the row. Notice that for this definition we are refering to augmented matrices. Since all linear systems of the form $A\vec{x} = \vec{0}$ are homogeneous, the fact that it equals $\vec{0}$ makes it impossible to have a nonzero value in the last column, which makes the system consistent. Thus, all the homogeneous systems are also consistent.

Sessions 2.1 & 2.2: Elementary and Inverse Matrices

2.1 For each system in Problem 1.1: i- Repeat the process of putting the augmented matrix [A | b] into RREF using multiplication by elementary matrices.

ii- Determine whether the matrix A is invertible. Give a one sentence justification of your conclusion.

iii- If the matrix A is invertible, find A^1. How does A^1 relate to the elementary matrices you found in part (i)?

2.2 Find examples of 2 × 2 matrices that exhibit the following behavior:

(a) a pair of matrices B and C such that $BC = CB$ (excluding the case $BC = 0$).

(b) A non-zero matrix M so that $M^2 = 0$.

(c) Invertible matrices A and B so that $A + B$ is not invertible.

(d) Non-invertible matrices C and D so that $C + D$ is invertible.

Solution to Problem 2.1

(a) (i) Given the system of equations, let us recall its matrix form

$$A = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 3 & -1 & 1 \\ 3 & 5 & 4 & 10 \end{array}\right]$$

We will multiply elementary matrices on the left for each row operations done in the original solution explained in problem 1.1. The first elementary matrix will add R_1 to R_2 while keeping R_1 intact:

$$E_0 A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 3 & -1 & 1 \\ 3 & 5 & 4 & 10 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & 4 & 0 & 4 \\ 3 & 5 & 4 & 10 \end{array}\right]$$

The next elementary matrix will divide the resultant R_2 by 2:

$$E_1 E_0 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & 4 & 0 & 4 \\ 3 & 5 & 4 & 10 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 0 & 2 \\ 3 & 5 & 4 & 10 \end{array}\right]$$

The next elementary matrix will now multiply R_1 by 4 (or in other words, add R_1 by itself four times) and subtract R_3 from R_1 keeping R_3 intact:

$$E_2 E_1 E_0 A = \begin{bmatrix} 4 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 0 & 2 \\ 3 & 5 & 4 & 10 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 1 & 2 & 0 & 2 \\ 3 & 5 & 4 & 10 \end{array}\right]$$

The following elementary matrix will subtract R_1 from R_2 keeping R_1 intact:

$$E_3 E_2 E_1 E_0 A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 1 & 2 & 0 & 2 \\ 3 & 5 & 4 & 10 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 0 & 3 & 0 & 0 \\ 3 & 5 & 4 & 10 \end{array}\right]$$

The next elementary matrix will now divide the resultant R_2 by 3:

$$E_4 E_3 E_2 E_1 E_0 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 0 & 3 & 0 & 0 \\ 3 & 5 & 4 & 10 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 3 & 5 & 4 & 10 \end{array}\right]$$

Our next elementary matrix will add R_2 to R_1 while keeping R_2 intact

$$E_5E_4E_3E_2E_1E_0A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 3 & 5 & 4 & 10 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 3 & 5 & 4 & 10 \end{array}\right]$$

Now the next elementary matrix will subtract R_1 three times from R_3 while keeping R_1 intact:

$$E_6E_5E_4E_3E_2E_1E_0A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 3 & 5 & 4 & 10 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 5 & 4 & 4 \end{array}\right]$$

Our next elementary matrix will subtract R_2 from R_3 five times while leaving R_2 intact:

$$E_7E_6E_5E_4E_3E_2E_1E_0A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 5 & 4 & 4 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 4 \end{array}\right]$$

Last, this elementary matrix will divide the whole R_3 by 4:

$$E_8E_7E_6E_5E_4E_3E_2E_1E_0A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/4 \end{bmatrix} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 4 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

For instance, we can reduce all these operations with one elementary matrix, which is the result of the multiplication of all the individual elementary matrices needed so far. In other terms:

$$E_8E_7E_6E_5E_4E_3E_2E_1E_0 = E_T = \begin{bmatrix} 17/6 & 1/6 & -2/3 \\ -7/6 & 1/6 & 1/3 \\ -2/3 & -1/3 & 1/3 \end{bmatrix}$$

Therefore,

$$E_TA = \begin{bmatrix} 17/6 & 1/6 & -2/3 \\ -7/6 & 1/6 & 1/3 \\ -2/3 & -1/3 & 1/3 \end{bmatrix} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 3 & -1 & 1 \\ 3 & 5 & 4 & 10 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

(ii) A quick way to determine whether a matrix A is invertible is by obtaining its determinant. Nonzero determinants are indicators of invertible matrices. Therefore, using Sage on our matrix A we obtain that $det(A) = 6$. Thus, our matrix A is invertible.

(iii) The process of multiplying for each elementary matrix aimed to return the augmented matrix in its echelon form. In this format, if we take out the last column of the matrix, which is its augmented part (in other words, if we consider only the left side of the system of linear equations) we ideally have an identity matrix. Let us recall that if a matrix M is invertible then the following should be true:

$$M^{-1}M = I$$

Therefore, for our specific case, if the matrix A is invertible, and it is as shown previously, its inverse matrix should be a matrix such that its left multiplication leads to an identity matrix. An example of this is E_T, which is the multiplication of all the elementary matrices in sub problem (i). Furthermore, confirming with Sage $A.inverse()$ function, we conclude that:

$$A^{-1} = \begin{bmatrix} 17/6 & 1/6 & -2/3 \\ -7/6 & 1/6 & 1/3 \\ -2/3 & -1/3 & 1/3 \end{bmatrix} = E_T$$

(b) (i) Given the system of equations, let us recall its matrix form

$$B = \left[\begin{array}{ccc|c} 2 & -4 & 8 & 4 \\ 1 & -1 & 1 & 4 \\ 3 & -4 & 6 & 7 \end{array}\right]$$

We will multiply elementary matrices on the left for each row operations done in the original solution explained in problem 1.1. Our first elementary matrix will divide R_1 by 2 while keeping the rest of the rows intact:

$$E_0B = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left[\begin{array}{ccc|c} 2 & -4 & 8 & 4 \\ 1 & -1 & 1 & 4 \\ 3 & -4 & 6 & 7 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & -2 & 4 & 2 \\ 1 & -1 & 1 & 4 \\ 3 & -4 & 6 & 7 \end{array}\right]$$

Our next elementary matrix will subtract R_1 from R_3. Then, it will subtract R_2 from R_3 two times. In both operations, R_1 and R_2 will remain intact:

$$E_1E_0B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -2 & 1 \end{bmatrix} \left[\begin{array}{ccc|c} 1 & -2 & 4 & 2 \\ 1 & -1 & 1 & 4 \\ 3 & -4 & 6 & 7 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & -2 & 4 & 2 \\ 1 & -1 & 1 & 4 \\ 0 & 0 & 0 & -3 \end{array}\right]$$

The following elementary matrix will subtract R_1 from R_2 and then add R_3 and R_2. Both R_1 and R_3 will remain intact in these operations:

$$E_2E_1E_0B = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \left[\begin{array}{ccc|c} 1 & -2 & 4 & 2 \\ 1 & -1 & 1 & 4 \\ 0 & 0 & 0 & -3 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & -2 & 4 & 2 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & -3 \end{array}\right]$$

Last, this elementary matrix will add R_2 to R_1 keeping R_2 intact:

$$E_3E_2E_1E_0B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left[\begin{array}{ccc|c} 1 & -2 & 4 & 2 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & -3 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & -3 \end{array}\right]$$

For instance, we can reduce all these operations with one elementary matrix, which is the result of the multiplication of all the individual elementary matrices needed so far. In other terms:

$$E_3E_2E_1E_0 = E_T = \begin{bmatrix} -3/2 & -2 & 2 \\ -1 & -1 & 1 \\ -1/2 & -2 & 1 \end{bmatrix}$$

As a result,

$$E_TB = \begin{bmatrix} -3/2 & -2 & 2 \\ -1 & -1 & 1 \\ -1/2 & -2 & 1 \end{bmatrix} \left[\begin{array}{ccc|c} 2 & -4 & 8 & 4 \\ 1 & -1 & 1 & 4 \\ 3 & -4 & 6 & 7 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & -3 \end{array}\right]$$

(ii) A quick way to determine whether a matrix B is invertible is by obtaining its determinant. Nonzero determinants are indicators of invertible matrices. Therefore, using Sage on our matrix B we obtain that $det(B) = 0$. Thus, our matrix B is NOT invertible.

(iii) (i) Given the system of equations, let us recall its matrix form

$$C = \left[\begin{array}{ccc|c} 2 & 2 & 6 & 2 \\ 6 & 9 & -3 & 9 \\ 10 & 14 & 2 & 14 \end{array}\right]$$

We will multiply elementary matrices on the left for each row operations done in the original solution explained in problem 1.1. Our first elementary matrix aims to simplify the equations. It is done by dividing R_1 by 2, R_2 by 3 and R_3 by 2 as follows:

$$E_0C = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \left[\begin{array}{ccc|c} 2 & 2 & 6 & 2 \\ 6 & 9 & -3 & 9 \\ 10 & 14 & 2 & 14 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 2 & 3 & -1 & 3 \\ 5 & 7 & 1 & 7 \end{array}\right]$$

Next, our elementary matrix will subtract R_1 from R_3 and subtract R_2 from R_3 two times keeping both R_2 and R_1 intact:

$$E_1E_0C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -2 & 1 \end{bmatrix} \left[\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 2 & 3 & -1 & 3 \\ 5 & 7 & 1 & 7 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 2 & 3 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Next, our elementary matrix will subtract R_1 from R_2 keeping R_1 intact:

$$E_2E_1E_0C = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left[\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 2 & 3 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 0 & 1 & -7 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Last, our elementary matrix will subtract R_2 from R_1 while keeping R_2 intact:

$$E_3E_2E_1E_0C = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left[\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 0 & 1 & -7 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 10 & 1 \\ 0 & 1 & -7 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

For instance, we can reduce all these operations with one elementary matrix, which is the result of the multiplication of all the individual elementary matrices needed so far. In other terms:

$$E_3E_2E_1E_0 = E_T = \begin{bmatrix} 3/2 & 1/3 & 0 \\ -1 & 1/3 & 0 \\ -1/2 & -2/3 & 1/2 \end{bmatrix}$$

As a result,

$$E_TC = \begin{bmatrix} 3/2 & 1/3 & 0 \\ -1 & 1/3 & 0 \\ -1/2 & -2/3 & 1/2 \end{bmatrix} \left[\begin{array}{ccc|c} 2 & 2 & 6 & 2 \\ 6 & 9 & -3 & 9 \\ 10 & 14 & 2 & 14 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 10 & 1 \\ 0 & 1 & -7 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(ii) A quick way to determine whether a matrix C is invertible is by obtaining its determinant. Nonzero determinants are indicators of invertible matrices. Therefore, using Sage on our matrix C we obtain that $det(B) = 0$. Thus, our matrix C is NOT invertible.

Solution to Problem 2.2

(a) Let us call a two 2×2 identity matrix X and Y such that

$$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{and} Y = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

Therefore,

$$XY = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$

and

$$-YX = \begin{bmatrix} -ae - cf & -be - df \\ -ag - ch & -bg - dh \end{bmatrix}$$

Thus, for the condition of $XY = -YX$ then $XY_{ij} = -YX_{ij}$. As a result, these equations should hold for each i-th and j-th value in the matrices:

$$ae + bg = -ae - cf$$

$$af + bh = -be - df$$

$$ce + dg = -ag - ch$$

$$cf + dh = -bg - dh$$

Reducing the complexity of the next step to solve this system of equations by guessing the value of each equation. An example of one of the many solutions can be:

$$X = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \text{and} Y = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$$

which would lead to:

$$XY = \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix}$$

and

$$-YX = \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix}$$

In general, the format should remain the same. In other words, all the variables should be zero except for b, c, e and h. In addition, $b = c = h = -e$.

(b) Let us call a 2×2 identity matrix P such that

$$P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Thus,

$$P^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix}$$

For P^2 to be a zero-matrix then, we should find values for a, b, c and d such that the following conditions are fulfilled:

$$a^2 + bc = 0$$

$$ab + bd = 0$$

$$ac + cd = 0$$

$$bc + d^2 = 0$$

The number of combinations for n distinct objects taken k at a time can be written as:

$$_nC^k = \frac{n!}{k!(n-k)!}$$

If in this case there are $n = 4$ different values for each element in the 2×2 matrix, and only two can be taken at a time, then there are 6 different combinations for each pair of variables. However, this does not consider reptitions, such as a^2 or b^2. Thus four more possible combinations should be added. Thus, there are 10 different combinations. Translated into a system of equations, this would lead to a matrix with 4 rows and 12 columns, some of which might be empty, for example the columns corresponding to b^2 and c^2. This implies that eventually we could arrive to a consistent homogeneous system with infinitely many solutions. For simplicity, one of the most trivial solutions is:

$$P = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Where,

$$P^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(c) One of the ways we can determine whether a matrix is invertible is by calcualting its determinant. If the determinant of a matrix A equals zero, then the matrix is not invertible. In general, 2×2 matrices which i-th and j-th values are all the same through out the entire matrix. As a result, for A and B to be invertible, they should not fulfill this conditions. All their entries cannot be the same. However a simple way for their addition to be non invertible, or in other words, to have the same values in all the entries, is if B is the swapped-rows version of A, or vice-versa. Therefore, an exmaple of this is:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{and} B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

Formally, for B to be the swapped-rows version of A, then:

$$a = g; b = h; c = e; d = f$$

In addition,

$$ad \neq bc; eh \neq fg$$

which implies:

$$af \neq bh; ec \neq dg$$

An example of these conditions could then be:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \text{and} B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Where,

$$A + B = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}$$

Which, as explained previously, has a determinant equal to zero.

(d) For the addition of the matrices C and D to be invertible, not all the entries in $C + D$ should be the same. As a result, at least one of C and D should not have all its entries equal. Otherwise, the addition would result in a non-invertible matrix with determinant equal zero. A possible solution for this problem is using fractions as follows:

$$C = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \text{and} D = \begin{bmatrix} 1 & 3/2 \\ 2/3 & 1 \end{bmatrix}$$

Where,

$$C + D = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}$$

Notice that according to Sage $det(C)$ and $det(D)$ equal to zero. Plus, $det(C + D) = -\frac{1}{3}$.

Sessions 3.1 & 3.2: Determinants and Introduction to Proofs with Matrix Algebra

3.1 For each of the following statements, prove that the statement is true or give a counterexample demonstrating that it is false.

(a) If $det(A) = 0$ then at least one of the cofactors must be zero.

(b) A matrix whose entries are 0s and 1s has determinant 1, 0, or 1.

(c) If matrices A and B are identical except that $b_{11} = 2a_{11}$, then $det(B) = 2det(A)$.

(d) The determinant of a matrix is the product of its pivots.

3.2 A matrix A is singular if $det(A) = 0$. Prove the following statements are true for 2 × 2 matrices.

(a) If A is invertible and B is singular, then AB is singular.

(b) If A is singular then A^T is singular.

(Optional) Prove that the statements in Problem 3.2 hold for n × n matrices.

Solution to Problem 3.1

(a) False. Using sage, we can get the cofactor matrix of a mtrix M by doing $M.adjugate()$. This shows all the cofactors by "tapping" each of the entries in matrix M. An example of a matrix M could be:

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

The determinant of the matrix M is $det(M) = 0$. Furthermore, the cofactor matrix of M is:

$$M.adjugate() = \begin{bmatrix} -3 & 6 & -3 \\ 6 & -12 & 6 \\ -3 & 6 & -3 \end{bmatrix}$$

Notice that the first element of the cofactor matrix is zero, which contradicts the affirmation presented.

(b) False. Let M be our example matrix such that:

$$M = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Notice that inputting this matrix in Sage with the function $det(M)$ results in 2. Therefore, it is possible to have a determinant different than 1, 0 and -1 for a matrix made of 1s and 0s only.

(c) False. Using a matrix from the previous sub problem, let us declare:

$$B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Inputting $det(B) == 2det(A)$ in Sage results in a False output. Therefore, the statement is false, as calculated with Sage.

(d) False. If A is a triangular matrix, then $det(A)$ is the product of the diagonal entries.

Take the matrix A in the previous sub-problem as an example:

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

According to Sage, the determinant of this matrix is $det(A) = 3$. Nonetheless, the product of its diagonal is (or its pivots) equals 2. Thus, the statement is false.

Solution to Problem 3.2

(a) Let us declare the following 2×2 matrices for A and B:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

Thus, If A is invertible, then,

$$ad - cb \neq 0$$

On the other hand, if B is singular:

$$eh - gf = 0$$

Let us observe what happens with the indices when both matrices are multiplied:

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$

Thus, for AB to be singular, its determinant has to be singular as well. In other words, the determinant of AB equals zero. Algebraically this means that:

$$det(AB) = -(ce + dg)(af + bh) + (ae + bg)(cf + dh) = 0$$

Therefore,

$$aecf + aedh + bgcf + bgdh - aecf - cebh - dgaf - bgdh = aedh + bgcf - cebh - dgaf = ad(eh - gf) + bc(gf - eh) =$$

Notice this results to zero since we recall the condition such that B is singular, then $eh - gf = 0$ Thus both terms equal to zero, which leads to a resultant determinant of zero for AB. As a result, regardless of the values of the entries in both matrices, ever since both are 2×2 matrices.

(b) Let us use matrix A from the previous sub problem as an example:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Thus, A^T can be written as:

$$A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

Notice that since the determinant of A is $det(A) = ad - bc$ and for a 2×2 matrix the only change in the A^T is a swap between entries A_{12} and A_{21}, bc are still multiplying each other when calculating the determinant of the matrix. In other words, $det(A) = det(A^T)$. Therefore, if A is singular, then A^T is also singular.

1.1.3 Deep Dive

1. A matrix by any other name (#linearsystems, #computationaltools, #theoreticaltools) Two matrices are row equivalent if one matrix can be changed into another matrix by a series of elementary row operations.

(a) Are the following matrices row-equivalent?

$$M1 = \begin{bmatrix} 0 & 2 & 4 & 2 \\ 1 & 0 & 2 & 1 \\ 4 & 1 & 11 & 2 \end{bmatrix}$$

$$M2 = \begin{bmatrix} 2 & 1 & 7 & 0 \\ 0 & 1 & 2 & 1 \\ 5 & 0 & 10 & 5 \end{bmatrix}$$

(b) Removing the third column of each matrix gives

$$N1 = \begin{bmatrix} 0 & 2 & 2 \\ 1 & 0 & 1 \\ 4 & 1 & 2 \end{bmatrix}$$

$$N2 = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ 5 & 0 & 5 \end{bmatrix}$$

Use Sage (or another computational tool) to show whether or not these matrices are row equivalent. Discuss what happens if you remove the fourth column instead: would the resulting matrices be row-equivalent?

(c) Could removing the same column from a pair of equivalent matrices affect row equivalance? If so, give an example. If not, give a short justification.

(d) Show that the system of equations corresponding to the augmented matrix

$$\left[\begin{array}{cc|c} 2 & 4 & 5 \\ 2 & 9 & 1 \\ 4 & 1 & 20 \end{array}\right]$$

has no solutions.

(e) What happens to the solution set of a linear system if you remove a row from the augmented matrix? What does this tell you about the effect of removing a row on row-equivalence?

(f) Determine whether the following statements are true or false. If the statement is true, give a 2 to 3 sentence proof. If the statement is false, give a counter example.

i. Row-equivalent augmented matrices have the same solution set.

ii. Augmented matrices with the same solution set are row-equivalent.

Solution to Deep Dive Problem 1

(a) We say that two $m \times n$ matrices are row equivalent if one can be obtained from the other by a sequence of elementary row operations. In other words, if two matrices have the same echelon form, then one can be tranformed to the other by applying its reverse set of row operations. This implies that our criteria will be two determine whether M_1 and M_2 have the same echelon form. We can start by swapping R_2 and R_1 in the matrix M_1.

$$M1 = \begin{bmatrix} 0 & 2 & 4 & 2 \\ 1 & 0 & 2 & 1 \\ 4 & 1 & 11 & 2 \end{bmatrix} \xrightarrow[R_1, R_2 = R_2, R_1]{} \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 4 & 2 \\ 4 & 1 & 11 & 2 \end{bmatrix}$$

Next, we can add $3R_2$ and R_1 to R_3, which would reduce the z variable in R_3 to one, in our attempt to reduce the matrix to an identity matrix in the left side of the equation. Thus,

$$M1 = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 4 & 2 \\ 4 & 1 & 11 & 2 \end{bmatrix} \xrightarrow[R_3 = R_3 - 4R_1]{} \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 4 & 2 \\ 0 & 1 & 3 & -2 \end{bmatrix} \xrightarrow[R_2 = R_2 + R_3]{} \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 3 & -2 \end{bmatrix}$$

$$M1 = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 3 & -2 \end{bmatrix} \xrightarrow[R_3 = \frac{R_3 + R_2}{2}]{} \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$M_1 = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \xrightarrow[R_2 = R_2 + R_3]{} \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \xrightarrow[R_1 = R_1 - 2R_3]{} \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

For the other matrix M_2, we can start by :

$$M2 = \begin{bmatrix} 2 & 1 & 7 & 0 \\ 0 & 1 & 2 & 1 \\ 5 & 0 & 10 & 5 \end{bmatrix} \xrightarrow[R_3 = R_3 + 2R_1]{} \begin{bmatrix} 2 & 1 & 7 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 2 & -4 & 5 \end{bmatrix} \xrightarrow[R_1 = R_1 - R_3]{} \begin{bmatrix} 1 & 3 & 11 & -5 \\ 0 & 1 & 2 & 1 \\ 1 & 2 & -4 & 5 \end{bmatrix}$$

$$M_2 = \begin{bmatrix} 1 & 3 & 11 & -5 \\ 0 & 1 & 2 & 1 \\ 1 & 2 & -4 & 5 \end{bmatrix} \xrightarrow[R_1 = R_1 + 3R_2]{} \begin{bmatrix} 1 & 0 & 5 & -2 \\ 0 & 1 & 2 & 1 \\ 1 & 2 & -4 & 5 \end{bmatrix}$$

$$M2 = \begin{bmatrix} 1 & 0 & 5 & -2 \\ 0 & 1 & 2 & 1 \\ 1 & 2 & -4 & 5 \end{bmatrix} \xrightarrow[R_3 = R_3 - 2R_2]{} \begin{bmatrix} 1 & 0 & 5 & -2 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 0 & 3 \end{bmatrix}$$

$$M_2 = \begin{bmatrix} 1 & 0 & 5 & -2 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 0 & 3 \end{bmatrix} \xrightarrow[R_3 = \frac{R_3 - R_1}{-5}]{} \begin{bmatrix} 1 & 0 & 5 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \xrightarrow[R_2 = R_2 + 2R_3]{} \begin{bmatrix} 1 & 0 & 5 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$M_2 = \begin{bmatrix} 1 & 0 & 5 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \xrightarrow[R_1 = R_1 - 5R_3]{} \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

As discussed previously, since both M_1 and M_2 have the same echelon form, we can affirm both are row-equivalent according to our definition. Thus,

$$M_1 \; M_2$$

(b) Declaring the matrices N_1 and N_2 in Sage and computing $N_1.rref() == N_2.rref()$ returns "True".

Similarly, delcaring another couple of matrices as P and Q which are similar to M_1 and M_2 just without the column as:

$$P = \begin{bmatrix} 0 & 2 & 4 \\ 1 & 0 & 2 \\ 4 & 1 & 11 \end{bmatrix}$$

$$Q = \begin{bmatrix} 2 & 1 & 7 \\ 0 & 1 & 2 \\ 5 & 0 & 10 \end{bmatrix}$$

Once again, computing $P.rref() == Q.rref()$ returns "True", which means these matrices are row equivalent despite of the modifications.

(c) It should not affect the row equivalence since the operations that lead to it are row operations. In other words, row operations only affect entries with their j-ths other entries, it is, with the entries that are in the same column. The entries of one column will not produce any changes on the elements of other columns while performing row operations. Thus, none of the operations used to obtain row equivalence in the original matrices would need to be ommitted after a column deletion.

(d) A way to find the solutions of linear system of equations in an augmented matrix is by transofrming the matrix to its echelon form. Using Sage, the echelon form of the matrix leads to a 3×3 identity matrix as shown:

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

However, since this is an augmented matrix, the last row indicates a contradiction. In which two zeroed elements result in one. This inconsisntency is interpreted as the linear system having no solutions.

(e) Let us remove the last row in our augmented matrix solved in the last sub problem. Tranforming this new matrix into an echelon form results in a consistent system of linear equations with a unique solution as follows:

$$\left[\begin{array}{cc|c} 2 & 4 & 5 \\ 2 & 9 & 1 \end{array}\right] \xrightarrow[RREF]{} \left[\begin{array}{cc|c} 1 & 0 & 49/10 \\ 0 & 1 & -6/5 \end{array}\right]$$

Given this example, we can infer that removing a row can change the rref form of a matrix. This changes the whole set of possible row operations to be performed on the matrix. Since this may lead to different rref forms for two different matrices, our condition for row equivalence is not met. Thus, removing a row may lead to not row-equivalence. Take as an example our original matrices M_1 and M_2 but now the last row was removed in both.

$$K_1 = \begin{bmatrix} 0 & 2 & 4 & 2 \\ 1 & 0 & 2 & 1 \end{bmatrix}$$

$$K_2 = \begin{bmatrix} 2 & 1 & 7 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

Computing the command $K_1.rref() == K_2.rref()$ returns "False". Therefore, their rref forms are different, which breaks our condition for row equivalence.

(f) (i) True. If both matrices are augmented matrices, then they represent a linear system of equations. If two matrices are row equivalent, then they have the same rref form. If both have the same rref form, then if one of them has no solutiones, the other one has not too. If one of them has inifinite many solutions, the set of inifnitaley many solutions is also the same. If the set of solutions is unique (they have an identity matrix on the left side of the linear system), then their variables have the same values. Thus, they have the same solution set.

(ii) False. A counter example could be augmented matrices with no solution. In theory, two augmented matrices with no solution hold the same set of solutions, which is an empty set. However, the other rows in the system might not be the same on their echelon form. As a result, the condition is not fulfiled and they are not row-equivalent even thouth they are both augmentec matrices with the same solution set.

2. Opening the model (#linearsystems, #computationaltools, #theoreticaltools) In Session 1.2 you explored a simple Closed Leontief Exchange Model with five interdependent industries. In the closed exchange model no goods or services enter or leave a national economy. However, most nations import and export goods. For this problem you will develop an open exchange model for the newly formed nation of Minervalia, then generalize the model to an economy with n sectors.

(a) Suppose Minervalia has 5 sectors: agriculture (A), energy (N), real estate (R), education (E), and entertainment (T). The following table represents the number of units of goods from other sectors used to produce one unit with a given sector. The table also lists the demand for each good from Minervalia's trading partners.

[1]:
```python
from IPython.display import Image
Image(filename = 'Minervalia Table.png')
```

[1]:

Internal and External Unit Demand Levels						
	Internal Consumption Levels					Outside Demand
	A	N	R	E	T	
Agriculture (A)	0.2	0.1	0.05	0.3	0.1	15
Energy(N)	0.1	0.2	0.1	0.05	0.2	10
Real Estate (R)	0.4	0.3	0.1	0.1	0.25	0
Education (E)	0.3	0.1	0.2	0.4	0.2	25
Entertainment (T)	0.1	0.1	0.1	0.05	0.25	5

Let x_A, x_N , x_R, x_E, and x_T denote the number of units produced by the agriculture (A), energy (N), real estate (R), education (E), and entertainment (T) sectors respectively. Write a system of equations that defines the number of units each sector should produce to satisfy both internal and external demand.

(b) Rewrite your system of equations in the form $M\vec{x} = \vec{d}$ where M is a matrix of coefficients. What do $\vec{x}$ and $\vec{d}$ represent in this model?

(c) Does the system have a solution? In other words, is there a set of production levels that would satisfy both internal and external demand? Is this solution unique?

(d) Adjust the parameters in the Demand Table to find levels that lead to (i) infinitely many solutions, and (ii) no solution. Interpret the resulting models in this context.

(e) Now suppose Minervalia has n sectors. Let x_i represent the number of units produced by the i-th sector. Assume that Sector j uses a_{ij} units from Sector i to produce one unit. Let d_i be the external demand for goods from Sector i.

i. Write a system of equations that models internal and external demand in this n sector system. Rewrite your system of equations in the form $M\vec{x} = \vec{d}$ where M is a matrix of coefficients.

ii. Let $A = [a_{ij}]$ be the matrix whose entries are the internal production requirements. Express M in terms of A.

iii. Find a condition you can use to quickly determine whether the model has a unique solution.

iv. Derive a formula for the vector of $\vec{x}_i$ in terms of A and the demand vector $\vec{d}$ that holds when the system has a unique solution.

Solution to Deep Dive Problem 2 NOTE: Questions (a) and (b) will be answered as one since I beleive one follows the other ver closely

(a) (b) In this problem we are dealing with a consumption matrix A. The j-th column lists the inputs (A, N, R, E, T) consumed by each industry to produce one unit of output. For example, the second column (E) lists the separate inputs from from all the other industries. Given that the consumption matrix A is a square matrix, we can say that the n-th industry consumed from industry j-th to produce one unit of its own product. Thus, this inputs are costs for the j-th industry. As a result, the total sum of entries in the j- column represents the total input from all the industries consumed by the j-th industry in order for this industry to produce one unit of its product. In other words, this is the cost of the industry j-th industry to produce one unit. For instance, if the sum of the j-th column is less than one, then we can say that industry if profitable, since its production costs are less than the 100% of its total input.

In an equilibrium state, the total production $\vec{x}$ is set such that both the internal demand of the industries to produce $\vec{x}$ and the outside demand are satisfied with zero waste. Therefore, we want to find an $\vec{x}$ such that:

$$\vec{x} = A\vec{x} + \vec{d}$$

Where A is the internal consumption matrix. Therefore, $A\vec{x}$ represents the internal demand from the industries to produce $\vec{x}$ and $\vec{d}$ represents the outside demand. Thus, the following equation on its matrix equivalence can be written as:

$$(I - A)\vec{x} = M\vec{x} = \vec{d}$$

As a result, given the variables and from the table (internal consumption matrix) we can conclude that the following system of equations should hold:

$$x_A = 0.2x_A + 0.1x_N + 0.05x_R + 0.3x_E + 0.1x_T + 15$$

$$x_N = 0.1x_A + 0.2x_N + 0.1x_R + 0.05x_E + 0.2x_T + 10$$

$$x_R = 0.4x_A + 0.3x_N + 0.1x_R + 0.1x_E + 0.25x_T + 0$$

$$x_E = 0.3x_A + 0.1x_N + 0.2x_R + 0.4x_E + 0.2x_T + 25$$

$$x_T = 0.1x_A + 0.1x_N + 0.1x_R + 0.05x_E + 0.25x_T + 5$$

Notice that this system of equations represents the $(IA_{\vec{x}=\vec{d}}$ form showed before. Transorming this system of equations to the $(IA)\vec{x} \equiv M\vec{x} = \vec{d}$ form would be as follows:

$$0.8x_A - 0.1x_N - 0.05x_R - 0.3x_E - 0.1x_T = 15$$

$$-0.1x_A + 0.8x_N - 0.1x_R - 0.05x_E - 0.2x_T = 10$$

$$-0.4x_A - 0.3x_N + 0.9x_R - 0.1x_E - 0.25x_T = 0$$

$$-0.3x_A - 0.1x_N - 0.2x_R + 0.6x_E - 0.2x_T = 25$$

$$-0.1x_A - 0.1x_N - 0.1x_R - 0.05x_E + 0.75x_T = 5$$

Which is in the matrix $M\vec{x} = \vec{d}$ form can be written as:

$$M\vec{x} = \begin{bmatrix} 0.8 & -0.1 & -0.05 & -0.3 & -0.1 \\ -0.1 & 0.8 & -0.1 & -0.05 & -0.2 \\ -0.4 & -0.3 & 0.9 & -0.1 & -0.25 \\ -0.3 & -0.1 & -0.2 & 0.6 & -0.2 \\ -0.1 & -0.1 & -0.1 & -0.05 & 0.75 \end{bmatrix} = \begin{bmatrix} 15 \\ 10 \\ 0 \\ 25 \\ 5 \end{bmatrix}$$

$$\equiv \left[\begin{array}{ccccc|c} 0.8 & -0.1 & -0.05 & -0.3 & -0.1 & 15 \\ -0.1 & 0.8 & -0.1 & -0.05 & -0.2 & 10 \\ -0.4 & -0.3 & 0.9 & -0.1 & -0.25 & 0 \\ -0.3 & -0.1 & -0.2 & 0.6 & -0.2 & 25 \\ -0.1 & -0.1 & -0.1 & -0.05 & 0.75 & 5 \end{array}\right]$$

(c) After augmenting the vector $\vec{d}$ with the matrix M, we can set the augmented matrix M_d in its rref form as follows:

$$M_d = \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 89.35 \\ 0 & 1 & 0 & 0 & 0 & 54.93 \\ 0 & 0 & 1 & 0 & 0 & 86.57 \\ 0 & 0 & 0 & 1 & 0 & 139.95 \\ 0 & 0 & 0 & 0 & 1 & 46.78 \end{array}\right]$$

Notice that since we we have an identity matrix in the left side of the linear system (in the not agumented part of the matrix), we can say that the system is consistent the number of variables is equal to the number of nonzero rows. As a result, it has a unique solution.

(d) (i) Recall that a system has infinitely many solutions when it is consistent and the number of variables is more than the number of nonzero rows in the rref of the matrix. Thus, our tasks reduces to set the matrix A such that the rref form of the augmented matrix M_d has a row full of zeroes. Thus, the following consumption matrix A_{inf} would lead to to infinitely many solutions:

$$A_{\text{inf}} = \begin{bmatrix} 0.2 & 0.1 & 0.05 & 0.3 & 0.1 \\ 0.1 & 0.2 & 0.1 & 0.05 & 0.2 \\ 0 & 0 & 1 & 0 & 0 \\ 0.3 & 0.1 & 0.2 & 0.4 & 0.2 \\ 0.1 & 0.1 & 0.1 & 0.05 & 0.25 \end{bmatrix}$$

Notice that this is not a realistic scenario. This implies that none of the other inustries consume from Real Estate and there is also no outside demand from this industry. Thus, everything that is produced by this industry is consumed by itself only. However, this industry consumes from the other industries as indicated on the other columns. The discepancy allows to other solutions to satisfy the overflow.

(ii) Recall that a system has no solution if the equations are inconsistent, they are contradictory. Thus, the rref of the matrix for an inconsistent system has a row with a nonzero number in the last column and 0's in all other columns. Thus, our tasks reduces to set the matrix A such that the rref form of the augmented matrix M_d has a row full of zeroes, with a non zero value in the augmented section of the matrix. Thus, the following consumption matrix A_0 would lead to to no solutions:

$$A_0 = \begin{bmatrix} 0.2 & 0.1 & 0.05 & 0 & 0.1 \\ 0.1 & 0.2 & 0.1 & 0 & 0.2 \\ 0.4 & 0.3 & 0.1 & 0 & 0.25 \\ 0.3 & 0.1 & 0.2 & 1 & 0.2 \\ 0.1 & 0.1 & 0.1 & 0 & 0.25 \end{bmatrix}$$

In this case, the Education industry does not consume from other industries but other sectors do consume from Education. In fact, Education only consumes from itself, but produces for the other industries. This industry is then not profitable. The system has no solutions.

(e) (i) In a system where there are n different industries one linked to the other, a generalization of the system may be represented as follows:

$$x_1 = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1j}x_j + \cdots + a_{1n}x_n + d_1$$

$$x_2 = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2j}x_j + \cdots + a_{2n}x_n + d_2$$

$$\vdots$$

$$x_i = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{ij}x_j + \cdots + a_{in}x_n + d_i$$

$$\vdots$$

$$x_n = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nj}x_j + \cdots + a_{nn}x_n + d_n$$

In order to write it in the $M\vec{x} = \vec{d}$ form, the previous system of equations should have in general the following form:

$$d_1 = (1 - a_{11})x_1 - a_{12}x_2 - \cdots - a_{1j}x_j - \cdots - a_{1n}x_n$$

$$d_2 = -a_{21}x_1 + (1 - a_{22})x_2 - \cdots - a_{2j}x_j - \cdots - a_{2n}x_n$$

$$\vdots$$

$$d_i = -a_{i1}x_1 - a_{i2}x_2 - \cdots + (1 - a_{ij})x_j - \cdots - a_{in}x_n$$

$$\vdots$$

$$d_n = -a_{n1}x_1 - a_{n2}x_2 - \cdots - a_{nj}x_j - \cdots + (1 - a_{nn})x_n$$

The system of equations above can be translated to the form $M\vec{x} = \vec{d}$ as follows:

$$\begin{bmatrix} (1-a_{11})x_1 & -a_{12}x_2 & \cdots & -a_{1j}x_j & \cdots & -a_{1n}x_n \\ -a_{21}x_1 & (1-a_{22})x_2 & \cdots & -a_{2j}x_j & \cdots & -a_{2n}x_n \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ -a_{i1}x_1 & -a_{i2}x_2 & \cdots & (1-a_{ij})x_j & \cdots & -a_{in}x_n \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ -a_{n1}x_1 & -a_{n2}x_2 & \cdots & -a_{nj}x_j & \cdots & (1-a_{nn})x_n \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_i \\ \vdots \\ d_n \end{bmatrix}$$

Note that in general the matrix form should not include the production units, but only the coefficients. This representation aims to provide a complete panorama of the matrix form. However, notice that this should only include the coefficient for computing the rref form of the matrix, after augmenting the matrix. For instance, could look as follows:

$$\begin{bmatrix} (1-a_{11}) & -a_{12} & \cdots & -a_{1j} & \cdots & -a_{1n} \\ -a_{21} & (1-a_{22}) & \cdots & -a_{2j} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ -a_{i1} & -a_{i2} & \cdots & (1-a_{ij}) & \cdots & -a_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & -a_{nj} & \cdots & (1-a_{nn}) \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_i \\ \vdots \\ d_n \end{bmatrix}$$

(ii) As mentioned in previous sub problems, given an internal demand or consumption matrix $A = [a_{ij}]$, the matrix M can be expressed as:

$$M = I - A$$

Where I represents an identity matrix.

(iii) Let us assume that K is a square $(n \times n)$ invertible matrix. Therefore, given:

$$K\vec{s} = \vec{p}$$

The following should also be true by associativity of matrix multiplication and by the definition of an inverse matrix:

$$K^{-1}(K\vec{s}) = (K^{-1}K)\vec{s} = I\vec{s}$$

Notice that $I\vec{s} = \vec{s}$ by the definition of an identity matrix. Therefore, a solution for $K\vec{s} = \vec{p}$ is $\vec{s} = K^{-1}\vec{p}$. Let us assume that $\vec{r}$ is another solution to the linear system. Similarly, then it is also true that $K\vec{r} = \vec{p}$. By multiplying both sides by K^{-1} and as demonstrated before results in $\vec{r} = K^{-1}\vec{p} = \vec{s}$. As a result, if K is a square invertible matrix, then the system of linear equations given by $K\vec{s} = \vec{p}$ has a unique solution given by $\vec{s} = K^{-1}\vec{p}$.

For our specific sub problem, assuming that for a system like this we are always dealing with square matrices (n $\times n), we then only need to know whether the matrix of coefficients M$ is invertible. A fast way to know this is by using computational tools to calculate its determinant. Nonzero determinants are indicators of invertible matrices. Thus, a condition we can use to quickly determine whether the model has a unique solution is whether the determinant of its coefficient matrix M is not zero.

(iv) As explained in the previous sub problem, a solution to the system of equations is one such that:

$$\vec{s} = K^{-1}\vec{p}$$

Where K^{-1} is the iverse of the coefficient matrix of the system. Recall that for a generalization of system like this, the coefficient matrix can be represented as $M = I - A$. Thereore, a unique solution to the system can be represented as:

$$\vec{X} = (I - A)^{-1}\vec{d}$$

1.1.4 References

MIT. (2017, September 7). Determinants. Retrieved from http://web.mit.edu/18.06/www/Spring17/Determinants.pdf

projects, C. to W. (2020, January 15). LaTeX/Mathematics. Retrieved from https://en.wikibooks.org/wiki/LaTeX/Mathematics#Mathematics_environments

The three types of solution sets. (n.d.). Retrieved from https://www.math.tamu.edu/~jlewis/Threetypesofsolsets.htm

1.1.5 Appendix A. HC's

#deduction: Several proofs were developed and discussed. Based on research and course knowledge as premises, deductive reasoning with clear explanations of the steps was provided. The identification of the premises and conclusions was also justified. In addition, deductive techniques for the realm of linear algebra were effectively and accurately applied to demonstrate mathematical expressions, justify examples and support sophisticated derivations.

#plausability: Applied to answer the questions requiring True or False. The premises and assumptions around the answer were identified and explained in order to provide a proper justification. The plausibility of our hypothesis and final answers was justified by mathematical demonstrations and the computation of examples which are included in the links on Appendix B.

#variables: For the Deep Dive Problem 2, a thorough explanation of the behavior of the system, including the interaction between the variables in the consumption matrix was provided. When mathematical expressions were presented, the description of the parameter of the system or model were descibed. In addition, the interdependance of these variables or even principles, including the determinants, was presented.

1.1.6 Appendix B. #computationaltools

You can find the code used to compute and demonstrate all the conclusions and findings in this paper by accessing the following download options for CoCalc and Github gist:

CoCalc Jupyter Notebook Download Link:

https://cocalc.com/c54ed430-726a-40be-9959-fbd443c3cd72/raw/CS111B%20-%20Assignment%201%20OFF.sa

Github Gist:

https://gist.github.com/alfonsosantacruz/bb16a1443611330d6ff82eac5a7cc55f

```
[ ]:
```