Abstract

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Under a continuous-time framework, investors are allowed to trade in the financial market up to finite time T. Uncertainty is modelled in the filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0\leq t\leq T}, \mathbb{P})$. In the financial market, there is a risky asset and a riskless asset. In their seminal paper, Black and Scholes [1] made the following assumptions on the financial market.

2.1 Assumptions

(i) The price of the underlying asset follows a geometric Brownian motion

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}W_t,$$

where $\mu \in \mathbb{R}$ is the expected rate of return, $\sigma > 0$ is the volatility and W_t is a Brownian motion,

(ii) The price of the riskless asset follows

$$\mathrm{d}B_t = rB_t \mathrm{d}t,$$

where $r > -1$ is the risk-free interest rate,

(iii) The risk-free interest rate r and the volatility σ are known functions,

(iv) The asset pays no dividends during the life of the option,

(v) There are no transaction costs,

(vi) There are no arbitrage opportunities within the market,

(vii) Short selling is permitted.

2.2 Derivation of the Black-Scholes Equation

In addition, Black and Scholes [1] use a so called *riskless hedging portfolio* to derive the governing partial differential equation for the price of a European call option. Consider a portfolio that is short one European call option and long Δ_t units of the underlying. The portfolio value $\Pi(S_t,t)$ is then given by

$$\Pi(S_t,t) = -c(S_t,t) + \Delta_t S_t,$$

where $c(S_t,t)$ is the price of the option. Using Itô's Formula

$$\mathrm{d}c(S_t,t) = \frac{\partial c}{\partial t}(S_t,t)\mathrm{d}t + \frac{\partial c}{\partial S_t}(S_t,t)\mathrm{d}S_t + \frac{\sigma^2}{2}\frac{\partial^2 c}{\partial S_t^2}(S_t,t)\mathrm{d}t.$$

Thus

$$\begin{aligned}
-\mathrm{d}c(S_t,t) + \Delta_t \mathrm{d}S_t &= \left(-\frac{\partial c}{\partial t}(S_t,t) - \frac{\sigma^2}{2}\frac{\partial^2 c}{\partial S_t^2}(S_t,t)\right)\mathrm{d}t + \left(\Delta_t - \frac{\partial c}{\partial S_t}(S_t,t)\right)\mathrm{d}S_t. \\
&= \left[-\frac{\partial c}{\partial t}(S_t,t) - \frac{\sigma^2}{2}\frac{\partial^2 c}{\partial S_t^2}(S_t,t) + \left(\Delta_t - \frac{\partial c}{\partial S_t}(S_t,t)\right)\mu S_t\right]\mathrm{d}t + \left(\Delta_t - \frac{\partial c}{\partial S_t}(S_t,t)\right)\sigma S_t \mathrm{d}W_t.
\end{aligned}$$

The financial gain of the portfolio up to time t is given by

$$G(\Pi(S_t,t)) = \int_0^t -\mathrm{d}c(S_u,u) + \int_0^t \Delta_u \mathrm{d}S_u$$

$$= \int_0^t \left[-\frac{\partial c}{\partial u}(S_u,u) - \frac{\sigma^2}{2}\frac{\partial^2 c}{\partial S_u^2}(S_u,u) + \left(\Delta_u - \frac{\partial c}{\partial S_u}(S_u,u)\right)\mu S_u \right] \mathrm{d}u + \int_0^t \left(\Delta_u - \frac{\partial c}{\partial S_u}(S_u,u)\right)\sigma S_u \mathrm{d}W_u.$$

Stochasticity in the portfolio stems from the second integral. Suppose we dynamically hedge our portfolio by choosing $\Delta_u = \frac{\partial c}{\partial S_u}(S_u,u)$ at all times $u < t$, then the financial gain becomes deterministic at all times. Since there is no arbitrage, the financial gain of the portfolio should be the same as the gain investing in the risk-free asset whose value is $-c + S_u \frac{\partial c}{\partial S_u}(S_u,u)$. The deterministic gain from this position in the risk-free asset is given by

$$B_t = \int_0^t r\left(-c + S_u \frac{\partial c}{\partial S_u}(S_u,u)\right) \mathrm{d}u.$$

By equating both deterministic gains $G(\Pi(S_t,t))$ and B_t, we obtain

$$-\frac{\partial c}{\partial u}(S_u,u) - \frac{\sigma^2}{2}\frac{\partial^2 c}{\partial S_u^2}(S_u,u) = r\left(-c + S_u \frac{\partial c}{\partial S_u}(S_u,u)\right) \quad 0 < u < t,$$

which is satisfied for any asset price S_t if $c(S_t,t)$ satisfies

$$\frac{\partial c}{\partial t}(S_t,t) + \frac{\sigma^2}{2}\frac{\partial^2 c}{\partial S_t^2}(S_t,t) + rS_t \frac{\partial c}{\partial S_t}(S_t,t) - rc = 0.$$

This parabolic partial differential equation is known as the Black-Scholes equation. The terminal payoff of the European call at time T with strike price X is assigned as a terminal condition

$$c(S_T,T) = \max(S_T - X, 0).$$

The equation governing the price of a European put option can be derived similarly and the same Black-Scholes equation is obtained. Let $V(S,t)$ denote the price of a derivative security, then V is governed by

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0. \tag{2.1}$$

The price of a particular derivative can be obtained by setting suitable boundary conditions specific to the derivative.

3 Pricing of European Call & Put Options

3.1 Closed Form Solution Using The Heat Equation

We present a solution to (2.1) for a European call option using the Heat equation. As previously mentioned, we consider the terminal condition $V(S,T) = \max(S-X,0)$. Also, note that the derivative security becomes worthless if the value of the underlying is 0, $V(0,t) = 0$ for all $0 \leq t \leq T$. Consider the ansatz $t = T - \frac{\tau}{\frac{1}{2}\sigma^2}$, $S = Xe^x$ and $V = Xv(x,\tau)$. We recover that

$$\frac{\partial V}{\partial t} = -X\frac{\sigma^2}{2}\frac{\partial v}{\partial \tau},$$

$$\frac{\partial V}{\partial S} = \frac{X}{S}\frac{\partial v}{\partial x},$$

$$\frac{\partial^2 V}{\partial S^2} = -\frac{X}{S^2}\frac{\partial v}{\partial x} + \frac{X}{S^2}\frac{\partial^2 v}{\partial x^2}.$$

Substituting this into (2.1), we obtain

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + \left(\frac{2r}{\sigma^2} - 1\right)\frac{\partial v}{\partial x} - \frac{2r}{\sigma^2}v, \tag{3.1}$$

with terminal condition $v(x,0) = \max(e^x - 1, 0)$. To recover the Heat equation, we further suppose that v is separable in the form $v(\tau,x) = e^{\alpha x + \beta \tau}u(x,\tau)$, where α and β are constants to be determined. Letting $k = \frac{2r}{\sigma^2}$, we recover that

$$\frac{\partial v}{\partial t} = \beta e^{\alpha x + \beta \tau}u + e^{\alpha x + \beta \tau}\frac{\partial u}{\partial \tau},$$

$$\frac{\partial v}{\partial x} = \alpha e^{\alpha x + \beta \tau}u + e^{\alpha x + \beta \tau}\frac{\partial u}{\partial x},$$

$$\frac{\partial^2 v}{\partial x^2} = \alpha^2 e^{\alpha x + \beta \tau}u + 2\alpha e^{\alpha x + \beta \tau}\frac{\partial u}{\partial x} + e^{\alpha x + \beta \tau}\frac{\partial^2 u}{\partial x^2}.$$

Substituting this into (3.1), we obtain

$$\beta u + \frac{\partial u}{\partial t} = \alpha^2 u + 2\alpha\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + (k-1)\left(\alpha u + \frac{\partial u}{\partial x}\right) - ku.$$

We can choose sufficient α and β to remove the $\frac{\partial u}{\partial x}$ and u terms. Setting $\alpha = -\frac{1}{2}(k-1)$ and $\beta = -\frac{1}{4}(k+1)^2$, we recover the Heat equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2},$$

with boundary condition

$$u(x,0) = \max(e^{\frac{1}{2}(k+1)x} - e^{\frac{1}{2}(k-1)x}, 0).$$

As such, the Black-Scholes equation can be converted to the Heat equation with substitution $V(S,t) = Xe^{-\frac{1}{2}(k-1)x - \frac{1}{4}(k+1)^2\tau}u(x,\tau)$. Applying the known solution to the initial value problem for the Heat equation

$$u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}}\int_{-\infty}^{\infty}\max(e^{\frac{1}{2}(k+1)x} - e^{\frac{1}{2}(k-1)x}, 0)e^{-\frac{(x-s)^2}{4\tau}}\mathrm{d}s.$$

Since $u(x,0)=0$ for $x<0$, we obtain

$$u(x,\tau)=\frac{1}{2\sqrt{\pi\tau}}\int_0^\infty (e^{\frac{1}{2}(k+1)x}-e^{\frac{1}{2}(k-1)x})e^{-\frac{(x-s)^2}{4\tau}}\mathrm{d}s.$$

Consider the change of variable $z=\frac{s-x}{\sqrt{2\pi}}$. Then

$$u(x,\tau)=-\frac{1}{\sqrt{2\pi}}\int_{-x/\sqrt{2\tau}}^\infty e^{\frac{1}{2}(k+1)(z\sqrt{2\tau}+x)-\frac{1}{2}z^2}\mathrm{d}z-\frac{1}{\sqrt{2\pi}}\int_{-x/\sqrt{2\tau}}^\infty e^{\frac{1}{2}(k-1)(z\sqrt{2\tau}+x)-\frac{1}{2}z^2}\mathrm{d}z.$$

By completing the square and taking out terms not dependent on z

$$u(x,\tau)=\frac{e^{\frac{1}{2}(k+1)+\frac{1}{4}(k+1)^2\tau}}{\sqrt{2\pi}}\int_{-x/\sqrt{2\tau}}^\infty e^{-\frac{1}{2}(z-\frac{1}{2}(k+1)\sqrt{2\tau})^2}\mathrm{d}z$$
$$-\frac{e^{\frac{1}{2}(k-1)+\frac{1}{4}(k-1)^2\tau}}{\sqrt{2\pi}}\int_{-x/\sqrt{2\tau}}^\infty e^{-\frac{1}{2}(z-\frac{1}{2}(k-1)\sqrt{2\tau})^2}\mathrm{d}z.$$

Then using the change of variable $x_1=z-\frac{1}{2}(k+1)\sqrt{2\tau}$ in the first integral and $x_2=z-\frac{1}{2}(k-1)\sqrt{2\tau}$ in the second integral, we obtain that

$$u(x,\tau)=e^{\frac{1}{2}(k+1)+\frac{1}{4}(k+1)^2\tau}N(d_1)-e^{\frac{1}{2}(k-1)+\frac{1}{4}(k-1)^2\tau}N(d_2),$$

where $N(x)$ is the formula for a cumulative Normal distribution and

$$d_1=\frac{x}{\sqrt{2\tau}}+\frac{1}{2}(k+1)\sqrt{2\tau},$$
$$d_2=\frac{x}{\sqrt{2\tau}}+\frac{1}{2}(k-1)\sqrt{2\tau}.$$

Transforming these variables back to our original choice of variables, we obtain the price of a European call option

$$C(S,t)=V(S,t)=SN(d_1)-Xe^{r(T-t)}N(d_2), \tag{3.2}$$

where

$$d_1=\frac{\log(S/X)+(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \tag{3.3}$$
$$d_2=\frac{\log(S/X)+(r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}. \tag{3.4}$$

A European put option can be priced in a similar manner. One can also make use of put-call parity. For a pair of European put and call options on the same underlying with the same expiration date and strike price

$$p(S,t)=c(S,t)-S+D+Xe^{r(T-t)}. \tag{3.5}$$

Since our underlying is nondividend paying, we set $D=0$. Substituting the price of a European call option we find that

$$p(S,t)=S(N(d_1)-1)+Xe^{r(T-t)}(1-N(d_2))=Xe^{r(T-t)}N(-d_2)-SN(-d_1). \tag{3.6}$$

3.2 Risk Neutral Valuation

Taking the riskless asset B_t as the numeraire, we define the discounted price process $X_t=\frac{S_t}{B_t}$. We assume the existence of an equivalent martingale measure $\mathbb{Q}$ under which the discounted price process is a $\mathbb{Q}$-martingale. By Itô's formula

$$\mathrm{d}X_t=(\mu-r)X_t\mathrm{d}t+\sigma X_t\mathrm{d}W_t.$$

4 Conclusion

References

[1] Fischer Black and Myron Scholes. The pricing of options and corporate liabilities. *Journal of political economy*, 81(3):637–654, 1973.

[2] Yue-Kuen Kwok. *Mathematical models of financial derivatives*. Springer, 2008.