Abstract

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Under a continuous-time framework, investors are allowed to trade in the financial market up to finite time T. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space and $(\mathscr{F}_t)_{0\leq t\leq T}$ be a right-continuous filtration containing all $\mathbb{P}$-null sets. In the financial market, there is a risky asset and a riskless asset. We make the following assumptions on the financial market.

2.1 Assumptions

(i) The price of the underlying risky asset follows a geometric Brownian motion

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}W_t,$$

where $\mu \in \mathbb{R}$ is the expected rate of return, $\sigma > 0$ is the volatility and W_t is a Brownian motion,

(ii) The price of the riskless asset follows the differential equation

$$\mathrm{d}B_t = rB_t \mathrm{d}t,$$

where $r > -1$ is the risk-free interest rate,

(iii) The risk-free interest rate r and volatility σ are known functions,

(iv) The asset pays no dividends during the life of the option,

(v) There are no transaction costs,

(vi) There are no arbitrage opportunities within the market,

(vii) The market is complete,

(viii) Short selling is permitted.

Assumption (i) is based on observations made in real life. A pivotal theory in investing is that risky assets such as stocks are mean-reverting. This makes modelling them with SDEs particularly useful. We have a drift term μS_t indicating a general upward or downward trend in the stock price over time and a diffusion term σS_t that captures the random fluctuations around the mean trend. In assumption (ii), we model the riskless asset as an asset that has a constant interest rate with no volatility. This riskless asset is often thought of as a bank account or bond. On the other hand, assumptions (iii)-(viii) are all for the sake of simplicity. It is a huge avenue of research to look at dropping some of these assumptions and can lead to more accurate and detailed models for option pricing.

2.2 Derivation of the Black-Scholes Equation

To derive the governing partial differential equation for the price of a European call option, Black and Scholes [1] use a so called *riskless hedging portfolio* argument. Consider a portfolio that is short one European call option and long Δ_t units of the underlying. The portfolio value $\Pi(S_t,t)$ is then given by

$$\Pi(S_t,t) = -C(S_t,t) + \Delta_t S_t,$$

where $C(S_t,t)$ is the price of the option. Using Itô's Formula

$$\mathrm{d}C(S_t,t) = \frac{\partial C}{\partial t}(S_t,t)\mathrm{d}t + \frac{\partial C}{\partial S_t}(S_t,t)\mathrm{d}S_t + \frac{\sigma^2}{2}S_t^2\frac{\partial^2 C}{\partial S_t^2}(S_t,t)\mathrm{d}t.$$

Thus

$$\begin{aligned}-\mathrm{d}C(S_t,t)+\Delta_t\mathrm{d}S_t &= \left(-\frac{\partial C}{\partial t}(S_t,t) - \frac{\sigma^2}{2}S_t^2\frac{\partial^2 C}{\partial S_t^2}(S_t,t)\right)\mathrm{d}t + \left(\Delta_t - \frac{\partial C}{\partial S_t}(S_t,t)\right)\mathrm{d}S_t. \\ &= \left[-\frac{\partial C}{\partial t}(S_t,t) - \frac{\sigma^2}{2}S_t^2\frac{\partial^2 C}{\partial S_t^2}(S_t,t) + \left(\Delta_t - \frac{\partial C}{\partial S_t}(S_t,t)\right)\mu S_t\right]\mathrm{d}t + \left(\Delta_t - \frac{\partial C}{\partial S_t}(S_t,t)\right)\sigma S_t \mathrm{d}W_t.\end{aligned}$$

The financial gain of the portfolio up to time t is given by

$$G(\Pi(S_t,t)) = \int_0^t -\mathrm{d}C(S_u,u) + \int_0^t \Delta_u \mathrm{d}S_u$$

$$\begin{aligned}= \int_0^t \left[-\frac{\partial C}{\partial u}(S_u,u) - \frac{\sigma^2}{2}S_t^2\frac{\partial^2 C}{\partial S_u^2}(S_u,u) + \left(\Delta_u - \frac{\partial C}{\partial S_u}(S_u,u)\right)\mu S_u\right]\mathrm{d}u \\ + \int_0^t \left(\Delta_u - \frac{\partial C}{\partial S_u}(S_u,u)\right)\sigma S_u \mathrm{d}W_u.\end{aligned}$$

Stochasticity in the portfolio stems from the second integral. Suppose we dynamically hedge our portfolio by choosing $\Delta_u = \frac{\partial C}{\partial S_u}(S_u,u)$ at all times $u < t$, then the financial gain becomes deterministic at all times. Since there is no arbitrage, the financial gain of the portfolio should be the same as the gain investing in the risk-free asset whose value is $-C(S_u,u) + S_u\frac{\partial C}{\partial S_u}(S_u,u)$. The deterministic gain from this position in the risk-free asset is given by

$$B_t = \int_0^t r\left(-C(S_u,u) + S_u\frac{\partial C}{\partial S_u}(S_u,u)\right)\mathrm{d}u.$$

By equating both deterministic gains $G(\Pi(S_t,t))$ and B_t, we obtain

$$-\frac{\partial C}{\partial u}(S_u,u) - \frac{\sigma^2}{2}S_t^2\frac{\partial^2 C}{\partial S_u^2}(S_u,u) = r\left(-C(S_u,u) + S_u\frac{\partial C}{\partial S_u}(S_u,u)\right) \quad 0 < u < t,$$

which is satisfied for any asset price S_t if $C(S_t,t)$ satisfies

$$\frac{\partial C}{\partial t}(S_t,t)+\frac{\sigma^2}{2}S_t^2\frac{\partial^2 C}{\partial S_t^2}(S_t,t)+rS_t\frac{\partial C}{\partial S_t}(S_t,t)-rC(S_t,t)=0.$$

This parabolic partial differential equation is known as the Black-Scholes equation. The terminal payoff of the European call at time T with strike price K is assigned as a terminal condition

$$C(S_T,T)=\max(S_T-K,0).$$

The equation governing the price of a European put option can be derived similarly and the same Black-Scholes equation is obtained. Let $V(S_t,t)$ denote the price of a derivative security, then $V(S_t,t)$ is governed by

$$\frac{\partial V}{\partial t}(S_t,t)+\frac{\sigma^2}{2}S_t^2\frac{\partial^2 V}{\partial S_t^2}(S_t,t)+rS_t\frac{\partial V}{\partial S_t}(S_t,t)-rV(S_t,t)=0. \tag{2.1}$$

The price of a particular derivative can be obtained by setting suitable boundary conditions specific to the derivative.

3 Pricing of European Call & Put Options

3.1 Closed Form Solution Using The Heat Equation

We first present a closed form solution to (2.1) for a European call option using the Heat equation. The payoff function for a European call option becomes our terminal condition $V(S_T,T) = \max(S_T - K, 0)$. Consider the substitution

$$t = T - \frac{2\tau}{\sigma^2}, \quad S_t = Ke^x, \quad V(S_t,t) = Kv(x,\tau).$$

Then we obtain

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + \left(\frac{2r}{\sigma^2} - 1\right)\frac{\partial v}{\partial x} - \frac{2r}{\sigma^2}v,$$

with terminal condition $v(x,0) = \max(e^x - 1, 0)$. We further suppose that v is separable in the form $v(\tau,x) = e^{\alpha x + \beta\tau}u(x,\tau)$, where α and β are constants to be determined. Letting $k = \frac{2r}{\sigma^2}$, we obtain

$$\beta u + \frac{\partial u}{\partial t} = \alpha^2 u + 2\alpha\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + (k-1)\left(\alpha u + \frac{\partial u}{\partial x}\right) - ku.$$

We can choose sufficient α and β to remove the $\frac{\partial u}{\partial x}$ and u terms. Setting $\alpha = -\frac{1}{2}(k-1)$ and $\beta = -\frac{1}{4}(k+1)^2$, we recover the Heat equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2},$$

with boundary condition

$$u(x,0) = \max(e^{\frac{1}{2}(k+1)x} - e^{\frac{1}{2}(k-1)x}, 0).$$

Applying the known solution to the initial value problem for the Heat equation, we can express an explicit solution.

$$u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}}\int_{-\infty}^{\infty} \max(e^{\frac{1}{2}(k+1)x} - e^{\frac{1}{2}(k-1)x}, 0)e^{-\frac{(x-s)^2}{4\tau}}\,\mathrm{d}s.$$

Since $u(x,0) = 0$ for $x < 0$, we obtain

$$u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}}\int_{0}^{\infty} (e^{\frac{1}{2}(k+1)x} - e^{\frac{1}{2}(k-1)x})e^{-\frac{(x-s)^2}{4\tau}}\,\mathrm{d}s.$$

Consider the change of variable $z = \frac{s-x}{\sqrt{2\pi}}$. Then

$$u(x,\tau) = -\frac{1}{\sqrt{2\pi}}\int_{-x/\sqrt{2\tau}}^{\infty} e^{\frac{1}{2}(k+1)(z\sqrt{2\tau}+x)-\frac{1}{2}z^2}\mathrm{d}z - \frac{1}{\sqrt{2\pi}}\int_{-x/\sqrt{2\tau}}^{\infty} e^{\frac{1}{2}(k-1)(z\sqrt{2\tau}+x)-\frac{1}{2}z^2}\mathrm{d}z.$$

By completing the square and taking out terms not dependent on z

$$u(x,\tau) = \frac{e^{\frac{1}{2}(k+1)+\frac{1}{4}(k+1)^2\tau}}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{-\frac{1}{2}(z-\frac{1}{2}(k+1)\sqrt{2\tau})^2} \mathrm{d}z \\ - \frac{e^{\frac{1}{2}(k-1)+\frac{1}{4}(k-1)^2\tau}}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{-\frac{1}{2}(z-\frac{1}{2}(k-1)\sqrt{2\tau})^2} \mathrm{d}z.$$

Then using the change of variable $x_1 = z - \frac{1}{2}(k+1)\sqrt{2\tau}$ in the first integral and $x_2 = z - \frac{1}{2}(k-1)\sqrt{2\tau}$ in the second integral, we obtain

$$u(x,\tau) = e^{\frac{1}{2}(k+1)+\frac{1}{4}(k+1)^2\tau} N(d_1) - e^{\frac{1}{2}(k-1)+\frac{1}{4}(k-1)^2\tau} N(d_2),$$

where $N(x)$ is the formula for a cumulative Normal distribution and

$$d_1 = \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k+1)\sqrt{2\tau},$$

$$d_2 = \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k-1)\sqrt{2\tau}.$$

Transforming these variables back to our original choice of variables, we obtain the price of a European call option

$$C(S_t,t) = S_t N(d_1) - K e^{r(T-t)} N(d_2), \tag{3.1}$$

where

$$d_1 = \frac{\log(S_t/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \tag{3.2}$$

$$d_2 = \frac{\log(S_t/K) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}. \tag{3.3}$$

A European put option can be priced in a similar manner. One can also make use of put-call parity. For a pair of European put and call options on the same underlying with the same expiration date and strike price

$$P(S_t,t) = C(S_t,t) - S_t + D + K e^{r(T-t)}. \tag{3.4}$$

Since our underlying is non-dividend paying, we set $D = 0$. Substituting the price of a European call option we find that

$$P(S_t,t) = S_t(N(d_1) - 1) + K e^{r(T-t)}(1 - N(d_2)) = K e^{r(T-t)} N(-d_2) - S_t N(-d_1). \tag{3.5}$$

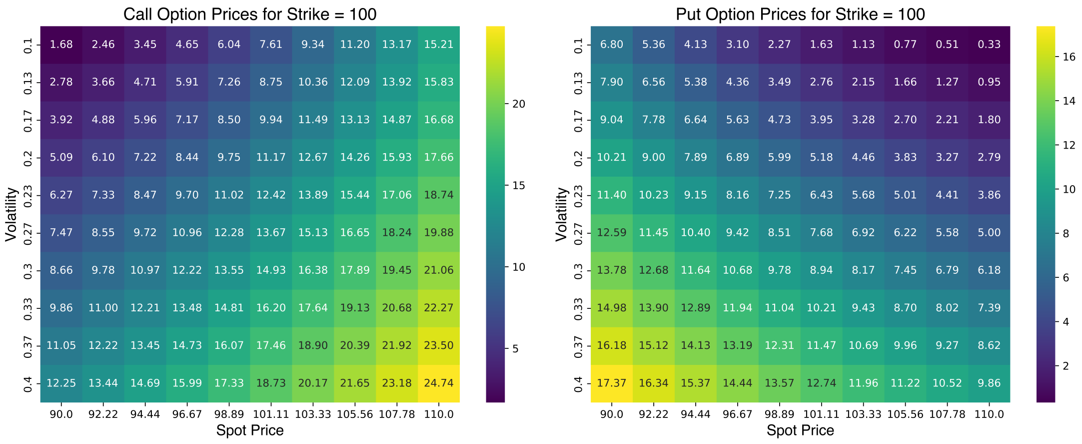

Figure 1: Heat map displaying the arbitrage price of call and put options using the closed form solution to the Black-Scholes equation for varying volatility and spot price. The call/put strike price is 100, the time till expiration is 1 year and the risk-free interest rate is 0.05.

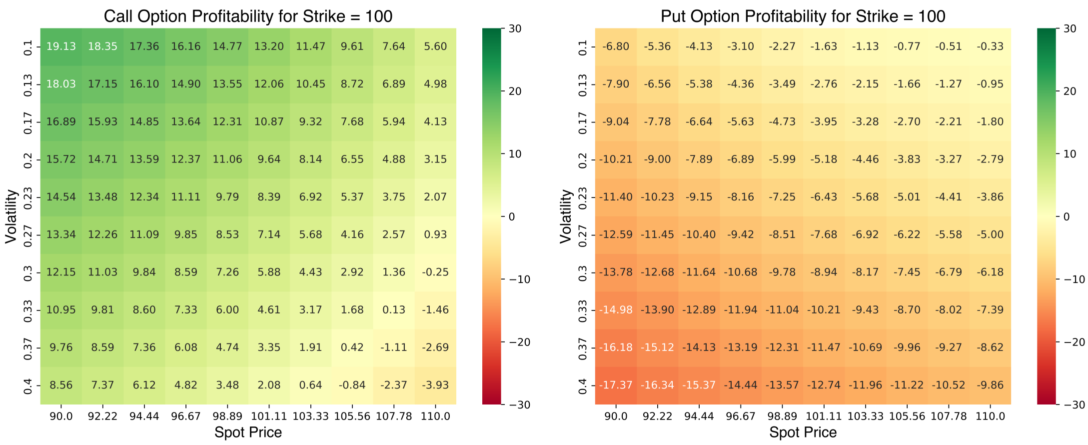

Figure 2: Heat map displaying the profitability of call and put options given an arbitrary price of the underlying at expiration using the closed form solution to the Black-Scholes equation for varying volatility and spot price. The call/put strike price is 100, the time till expiration is 1 year, the risk-free interest rate is 0.05 and the price of the underlying at expiration is 120.32.

3.2 Risk Neutral Valuation

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space and $(\mathscr{F}_t)_{0 \leq t \leq T}$ be a right-continuous filtration containing all $\mathbb{P}$-null sets. Throughout this section, we consider a trading strategy $\theta_t = (\theta_t^0, \theta_t^1)$ to be an $\mathscr{F}_t$-adapted stochastic process. θ_t^0 represents the number of units of the riskless asset held in the portfolio, and θ_t^1 represents the number of units of the risky asset held in the portfolio. Contingent claims are modelled as $\mathscr{F}_t$-measurable random variables. A European call option is an example of a contingent claim whose payoff function is $h(S_T) = \max(S_T - K, 0)$.

Definition 3.1. (Self-Financing Portfolio) The value process associated with a trading strategy $\theta_t = (\theta_t^0, \theta_t^1)$ is

$$V_t(\theta) = \theta_t^0 B_t + \theta_t^1 S_t \quad 0 \le t \le T.$$

Moreover, the gain process is

$$G_t(\theta) = \int_0^t \theta_s^0 \mathrm{d}B_s + \int_0^t \theta_s^1 \mathrm{d}S_s \quad 0 \le t \le T.$$

The trading strategy θ_t is self-financing if and only if

$$V_t(\theta) = V_0(\theta) + G_t(\theta),$$

or equivalently,

$$\mathrm{d}V_t = \theta_t^0 \mathrm{d}B_t + \theta_t^1 \mathrm{d}S_t.$$

Taking the riskless asset B_t as the numeraire, we define the discounted price process $X_t = \frac{S_t}{B_t}$. The discounted value process $V_t^*(\theta)$ is obtained by dividing $V_t(\theta)$ by B_t. Moreover, the discounted gain process $G_t^*(\theta)$ is given by

$$G_t^*(\theta) = V_t^*(\theta) - V_0^*(\theta).$$

Definition 3.2. (Admissible Trading Strategy) A trading strategy $\theta_t = (\theta_t^0, \theta_t^1)$ is $\mathbb{Q}$-admissible if it is self-financing and the discounted gain process $G_t^*(\theta)$ is a $\mathbb{Q}$-martingale.

Definition 3.3. (Attainable Contingent Claim) A contingent claim, defined by the payoff function h, is attainable if there is an admissible trading strategy worth h at time T. A financial market is complete if every contingent claim is attainable.

In Chapter 2, we assumed that the financial market is complete. Thus, all contingent claims within our market can be replicated by an admissible trading strategy. Consider the discounted prices process X_t defined in 3.1. Then X_t satisfies

$$\mathrm{d}X_t = (\mu - r)X_t \mathrm{d}t + \sigma X_t \mathrm{d}W_t. \tag{3.6}$$

We would like to find an equivalent measure $\mathbb{Q}$ under which X_t is a $\mathbb{Q}$-martingale. We can do this using a change of measure theorem called Girsanov's Theorem.

Theorem 3.1. *(Girsanov's Theorem) Let W_t be a $\mathbb{P}$-Brownian motion and let $\mathscr{F}_t$ be the natural filtration generated by W_t. Consider an $\mathscr{F}_t$-adapted stochastic process γ_t that satisfies the Novikov condition*

$$\mathbb{E}\left[e^{\int_0^T \gamma_t^2 dt}\right] < \infty.$$

Then there exists a measure $\mathbb{Q}$ such that *(i)* $\mathbb{Q}$ *is equivalent to* $\mathbb{P}$*,*

(ii) $\frac{d\mathbb{Q}}{d\mathbb{P}} = e^{-\int_0^T \gamma_t dW_t - \frac{1}{2}\int_0^T \gamma_t^2 dt}$*,*

(iii) $W_t + \int_0^t \gamma_s ds = \tilde{W}_t$ *is a* $\mathbb{Q}$*-Brownian motion.*

Theorem 3.1 describes a way to change a regular Brownian motion into one with a drift of $-\gamma_t$. We utilise this theorem in the following way. Suppose we choose the Radon-Nikodym derivative in Theorem 3.1 to be $\gamma_t = \frac{\mu - r}{\sigma}$ (often known as the market price of risk). Then we recover a $\mathbb{Q}$-Brownian motion $\tilde{W}_t$ such that

$$\mathrm{d}W_t = -\frac{\mu - r}{\sigma}\mathrm{d}t + \mathrm{d}\tilde{W}_t.$$

Substituting into (3.6), we recover

$$\mathrm{d}X_t = \sigma X_t \mathrm{d}\tilde{W}_t.$$

We can check that X_t is indeed a $\mathbb{Q}$-martingale. We can express X_t in Itô integral form and take the expectation under $\mathbb{Q}$.

$$\mathbb{E}^{\mathbb{Q}}[X_t|\mathscr{F}_s] = \mathbb{E}^{\mathbb{Q}}\left[X_0 + \int_0^t \sigma X_s \mathrm{d}\tilde{W}_s \,\middle|\, \mathscr{F}_s\right] = X_0 + \int_0^s \sigma X_u \mathrm{d}\tilde{W}_u + \mathbb{E}^{\mathbb{Q}}\left[\int_s^t \sigma X_u \mathrm{d}\tilde{W}_u \,\middle|\, \mathscr{F}_s\right] = X_s.$$

We now have an equivalent martingale measure, however is it unique? Due to the assumptions in the Black-Scholes model of no-arbitrage and market completeness, we can in fact guarantee the uniqueness of equivalent martingale measures. This theorem is known as the Fundamental Theorem of Asset Pricing (FTAP) and will be stated without proof. We call this unique equivalent martingale measure the *risk neutral measure*.

Theorem 3.2. *(Fundamental Theorem of Asset Pricing) In a complete market that satisfies no-arbitrage, there exists a unique equivalent martingale measure* $\mathbb{Q}$*.*

We consider a contingent claim Y with payoff function h. Consider an admissible trading strategy $\theta_t = (\theta_t^0, \theta_t^1)$ that replicates the contingent claim Y. Then by the definition of an admissible trading strategy, the discounted gain process $G_t^*(\theta)$ is a martingale under the risk neutral measure

$$\mathbb{E}^{\mathbb{Q}}[G_t^*(\theta)|\mathscr{F}_s] = G_s^*(\theta).$$

Using this, we can also show that the discounted value process $V_t^*(\theta)$ is a martingale under the risk neutral measure. By definition

$$\mathbb{E}^{\mathbb{Q}}[V_t^*(\theta)|\mathscr{F}_s] = \mathbb{E}^{\mathbb{Q}}[G_t^*(\theta) + V_0^*(\theta)|\mathscr{F}_s] = G_s^*(\theta) + V_0^*(\theta) = V_s^*(\theta).$$

Since θ_t replicates the contingent claim Y, the value process at time T must equal the contingent claim $V_T(\theta) = h(S_T)$. Thus the risk neutral valuation of the contingent claim Y is given by

$$V_t(\theta) = B_t V_t^*(\theta) = B_t \mathbb{E}^{\mathbb{Q}}[V_T^*(\theta)|\mathscr{F}_t] = B_t \mathbb{E}^{\mathbb{Q}}\left[\frac{Y}{B_T} \,\middle|\, \mathscr{F}_t \right].$$

Using the example of a European call option whose payoff function is $h(S_T) = \max(S_T - K, 0)$ we obtain

$$C(S,t) = e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}}[\max(S_T - K, 0)]. \tag{3.7}$$

Similarly for a put option whose payoff function is $h(S_T) = \max(K - S_T, 0)$ we obtain

$$P(S,t) = e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}}[\max(K - S_T, 0)]. \tag{3.8}$$

4 Conclusion

References

[1] Fischer Black and Myron Scholes. The pricing of options and corporate liabilities. *Journal of political economy*, 81(3):637–654, 1973.

[2] Yue-Kuen Kwok. *Mathematical models of financial derivatives*. Springer, 2008.