Contents

1 Mean-Variance Portfolio Optimisation

In this chapter, we seek to answer the question how to optimally invest in a financial market taking into account the mean and the variance of the return of a portfolio.

1.1 Return Of An Asset & Portfolio

Throughout this chapter, we consider a $(1+d)$-dimensional financial market $\bar{S} = (S_t^0, S_t)_{t\in\{0,1\}}$ on some probability space $(\Omega, \mathscr{F}_t, \mathbb{P})$. The asset S^0 is a riskless asset that satisfies

$$S_0^0 = 1, \quad S_0^1 = 1 + r,$$

where $r > -1$ is a risk-free interest rate. S^0 is often thought of as a bank account or bond. The remaining d assets in our financial market are risky assets, often thought of as stocks. We assume that all the assets today are positively priced and have finite second moments

$$S_0^i > 0, \quad \mathbb{E}[(S_t^i)^2] < \infty \quad \forall i \in \{0,1,\dots,d\}, \quad t \in \{0,1\}.$$

Moreover, we assume the financial market $\bar{S}$ satisfies no-arbitrage and risky assets are non-redundant in the sense that

$$\bar{\vartheta} \cdot \bar{S}_1 = 0 \quad \mathbb{P}\text{-a.s.} \implies \bar{\vartheta} = 0.$$

We define the (relative) return of the i-th asset by

$$R^i := \frac{S_1^i - S_0^i}{S_0^i}, \quad \forall i \in \{0,1,\dots,d\}. \tag{1.1}$$

The expected return of the i-th asset is given by

$$\mu^i := \mathbb{E}[R^i], \quad \forall i \in \{0,1,\dots,d\}. \tag{1.2}$$

We define the return vector $\bar{R} = (R^0, R)$ where $R = (R^1, R^2, \dots, R^d)$ and the expected return vector$\bar{\mu} = (\mu^0, \mu)$ where $\mu = (\mu^1, \mu^2, \dots, \mu^d)$. For the risky assets, the return is stochastic. For that reason we define the covariance matrix of the return vector R by $\Sigma = (\Sigma^{ij})_{1\le i,j\le d}$ by

$$\Sigma^{ij} := \mathrm{Cov}[R^i, R^j] = \mathbb{E}[(R^i - \mu^i)(R^j - \mu^j)], \quad \forall i,j \in \{1,2,\dots,d\}. \tag{1.3}$$

The non-redundancy assumption on $\bar{S}$ implies that Σ is positive definite, thus invertible. For a portfolio $\bar{\vartheta} \in \mathbb{R}^{1+d}$ such that $\bar{\vartheta} \cdot \bar{S}_0 \neq 0$, we define the return of the portfolio $\bar{\vartheta}$ by

$$R_{\bar{\vartheta}} := \frac{\bar{\vartheta} \cdot \bar{S}_1 - \bar{\vartheta} \cdot \bar{S}_0}{\bar{\vartheta} \cdot \bar{S}_0}. \tag{1.4}$$

The expected return of the portfolio $\bar{\vartheta}$ is given by

$$\mu_{\bar{\vartheta}} := \mathbb{E}[R_{\bar{\vartheta}}]. \tag{1.5}$$

The variance of the return of the portfolio $\bar{\vartheta}$ is given by

$$\sigma_{\bar{\vartheta}}^2 := \mathrm{Var}[R_{\bar{\vartheta}}] = \mathbb{E}[(R_{\bar{\vartheta}} - \mu_{\bar{\vartheta}})^2]. \tag{1.6}$$

A portfolio $\bar{\vartheta} = (\vartheta^0, \vartheta)$ is risk-only if $\vartheta^0 = 0$. We then identify $\vartheta = (\vartheta_1, \vartheta_2, \dots, \vartheta_d)$ with $(0, \vartheta)$ and call ϑ itself a risk-only portfolio. The return, expected return and variance of returns of a risk-only portfolio is then given by

$$R_{\vartheta} := \frac{\vartheta \cdot S_1 - \vartheta \cdot S_0}{\vartheta \cdot S_0}, \tag{1.7}$$

$$\mu_{\vartheta} := \mathbb{E}[R_{\vartheta}], \tag{1.8}$$

$$\sigma_{\vartheta}^2 := \mathrm{Var}[R_{\vartheta}] = \mathbb{E}[(R_{\vartheta} - \mu_{\vartheta})^2]. \tag{1.9}$$

1.2 Mean-Variance Problems

Consider an investor with initial wealth $x_0 > 0$. They choose a portfolio $\bar{\vartheta} \in \mathbb{R}^{1+d}$ subject to the budget constraint

$$\bar{\vartheta} \cdot \bar{S}_0 = x_0.$$

A portfolio $\bar{\vartheta}$ satisfying this budget constraint is known as an x_0-feasible portfolio. Similarly, a risk-only portfolio ϑ satisfying the budget constraint is a risk-only x_0-feasible portfolio.

There are two versions of the mean-variance problem, each of which has a formulation with risk-only portfolios and general portfolios. For general portfolios

(i) Given an initial wealth $x_0 > 0$ and a minimal desired expected return $\mu_{\min} > 0$, minimise the variance of the return $\sigma^2_{\bar{\vartheta}}$ among all x_0-feasible portfolios $\bar{\vartheta} \in \mathbb{R}^{1+d}$ that satisfy $\mu_{\bar{\vartheta}} \geq \mu_{\min}$.

(ii) Given an initial wealth $x_0 > 0$ and a maximal desired variance of the return $\sigma^2_{\max} \geq 0$, maximise the expected return $\mu_{\bar{\vartheta}}$ among all x_0-feasible portfolios $\bar{\vartheta} \in \mathbb{R}^{1+d}$ that satisfy $\sigma_{\bar{\vartheta}} < \sigma^2_{\max}$.

For risk-only portfolios

(i) Given an initial wealth $x_0 > 0$ and a minimal desired expected return $\mu_{\min} > 0$, minimise the variance of the return σ^2_{ϑ} among all x_0-feasible portfolios $\vartheta \in \mathbb{R}^d$ that satisfy $\mu_\vartheta \geq \mu_{\min}$.

(ii) Given an initial wealth $x_0 > 0$ and a maximal desired variance of the return $\sigma^2_{\max} \geq 0$, maximise the expected return μ_ϑ among all x_0-feasible portfolios $\vartheta \in \mathbb{R}^d$ that satisfy $\sigma_\vartheta < \sigma^2_{\max}$.

1.3 Portfolios In Fractions Of Wealth

In order to study the mean-variance problems, it is convenient to parameterise portfolios in fractions of wealth. For $N \in \mathbb{N}$, define the unit vector $\mathbb{1}_N := (1, 1, \ldots, 1) \in \mathbb{R}^N$. We define the unit hyperplane in $\mathbb{R}^N$ for some $N \in \mathbb{N}$ by

$$H^{N-1} := \{x \in \mathbb{R}^N : x \cdot \mathbb{1}_N = 1\}. \tag{1.10}$$

If $\bar{\vartheta} \in \mathbb{R}^{1+d}$ is a portfolio parameterised in number of shares, we can define the fraction of wealth invested in the i-th asset by

$$\pi^i := \frac{\vartheta^i S^i_0}{\bar{\vartheta} \cdot \bar{S}_0}, \quad \forall i \in \{0, 1, \ldots, d\}. \tag{1.11}$$

We define the portfolio parameterised in fractions of wealth $\bar{\pi} = (\pi^0, \pi)$ where $\pi = (\pi^1, \pi^2, \ldots, \pi^d)$. Notice that $\bar{\pi} \in H^{1+d-1}$. We can also define π, a risk-only portfolio parameterised in fractions of wealth.

Lemma 1.1. *Let $\bar{S} = (S^0_t, S_t)_{t \in \{0,1\}}$ be a non-redundant $1+d$-dimensional market that satisfies no-arbitrage on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Assume that $\bar{S}$ has finite second moments and is positively priced at the initial time step. Denote by μ and Σ the mean vector and covariance matrix of the return vector R of the risky assets. Let $x_0 > 0$ be some fixed initial wealth. Then there is a one-to-one correspondence between x_0-feasible portfolios $\bar{\vartheta} \in \mathbb{R}^{1+d}$ parameterised in numbers of shares and portfolios $\bar{\pi} \in H^{1+d-1}$ parameterised in fractions of wealth. If $\bar{\pi} \in H^{1+d-1}$ denotes the* *fractions of wealth of an x_0-feasible portfolios $\bar{\vartheta} \in \mathbb{R}^{1+d}$, then*

$$R_{\bar{\vartheta}} = R_{\bar{\pi}} := \sum_{i=0}^{d} \pi^i R^i = \bar{\pi} \cdot \bar{R}, \tag{1.12}$$

$$\mu_{\bar{\vartheta}} = \mu_{\bar{\pi}} := \mathbb{E}\left[R_{\bar{\pi}}\right] = \sum_{i=0}^{d} \pi^i \mu^i = \bar{\pi} \cdot \bar{\mu}, \tag{1.13}$$

$$\sigma_{\bar{\vartheta}}^2 = \sigma_{\bar{\pi}}^2 := \operatorname{Var}\left[R_{\bar{\pi}}\right] = \sum_{i=1}^{d} \sum_{j=1}^{d} \pi^i \pi^j \Sigma^{ij} = \pi^\top \Sigma \pi. \tag{1.14}$$

Similarly, there is a one-to-one correspondence between risk-only x_0-feasible portfolios $\vartheta \in \mathbb{R}^d$ parameterised in numbers of shares and risk-only portfolios $\pi \in H^{d-1}$ parameterised in fractions of wealth. If $\pi \in H^{d-1}$ denotes the fractions of wealth of a risk-only x_0-feasible portfolios $\vartheta \in \mathbb{R}^d$, then

$$R_{\vartheta} = R_{\pi} := \sum_{i=1}^{d} \pi^i R^i = \pi \cdot R, \tag{1.15}$$

$$\mu_{\vartheta} = \mu_{\pi} := \mathbb{E}\left[R_{\pi}\right] = \sum_{i=1}^{d} \pi^i \mu^i = \pi \cdot \mu, \tag{1.16}$$

$$\sigma_{\vartheta}^2 = \sigma_{\pi}^2 := \operatorname{Var}\left[R_{\pi}\right] = \sum_{i=1}^{d} \sum_{j=1}^{d} \pi^i \pi^j \Sigma^{ij} = \pi^\top \Sigma \pi. \tag{1.17}$$

Proof. We only prove the case with general portfolios. The risk-only case is analogous. One-to-one correspondence follows from the fact that the map

$$\Phi : \{\bar{\vartheta} \in \mathbb{R}^{1+d} : \bar{\vartheta} \cdot \bar{S}_0 = x_0\} \mapsto H^{1+d-1}, \quad \Phi_i(\bar{\vartheta}) = \frac{\vartheta^i S_0^i}{x_0}, \quad \forall i \in \{0, 1, \dots, d\},$$

is bijective. To prove the following equations, let $\bar{\vartheta} \in \mathbb{R}^{1+d}$ be an x_0-feasible portfolio parameterised in numbers of shares. The corresponding portfolio $\bar{\pi} \in H^{1+d-1}$ parameterised in fractions of wealth is then given by

$$\pi^i := \frac{\vartheta^i S_0^i}{\bar{\vartheta} \cdot \bar{S}_0}, \quad \forall i \in \{0, 1, \dots, d\}.$$

Then

$$R_{\bar{\vartheta}} = \frac{\bar{\vartheta} \cdot \bar{S}_1 - \bar{\vartheta} \cdot \bar{S}_0}{\bar{\vartheta} \cdot \bar{S}_0} = \sum_{i=0}^{d} \frac{\vartheta^i}{\bar{\vartheta} \cdot \bar{S}_0} (S_1^i - S_0^i) = \sum_{i=0}^{d} \frac{\vartheta^i S_0^i}{\bar{\vartheta} \cdot \bar{S}_0} R^i = \sum_{i=0}^{d} \pi^i R^i = \bar{\pi} \cdot \bar{R}.$$

By the linearity of expectation, (1.13) follows. We use that $\operatorname{Cov}[R^i, R^0] = 0$ for any i, because R^0 is deterministic, to obtain

$$\begin{aligned}
\sigma_{\bar{\vartheta}}^2 = \operatorname{Var}[R_{\bar{\vartheta}}] = \operatorname{Var}[R_{\bar{\pi}}] = \operatorname{Var}\left[\sum_{i=0}^{d} \pi^i R^i\right] = \sum_{i=0}^{d} \sum_{j=0}^{d} \pi^i \pi^j \operatorname{Cov}[R^i, R^j] \\
= \sum_{i=1}^{d} \sum_{j=1}^{d} \pi^i \pi^j \operatorname{Cov}[R^i, R^j] = \sum_{i=1}^{d} \sum_{j=1}^{d} \pi^i \pi^j \Sigma^{ij} = \pi^\top \Sigma \pi.
\end{aligned}$$

□

1.4 Optimisation Without A Riskless Asset

We start by considering the risk-only case. First, we characterise the minimum variance portfolio.

Lemma 1.2. *Let $\bar{S} = (S_t^0, S_t)_{t\in\{0,1\}}$ be a non-redundant market that satisfies no-arbitrage on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Assume that $\bar{S}$ has finite second moments and is positively priced at the initial time step. Denote by μ and Σ the mean vector and covariance matrix of the return vector R of the risky assets. Then there exists a unique risk-only portfolio $\pi_{\min} \in H^{d-1}$ called the minimum variance portfolio such that*

$$\sigma^2_{\pi_{\min}} \leq \sigma^2_\pi \quad \forall \pi \in H^{d-1}. \tag{1.18}$$

It is given by

$$\pi_{\min} = \frac{\Sigma^{-1}\mathbb{1}}{\mathbb{1}^\top \Sigma^{-1}\mathbb{1}}, \tag{1.19}$$

and satisfies

$$\mu_{\pi_{\min}} = \frac{\mu^\top \Sigma^{-1}\mathbb{1}}{\mathbb{1}^\top \Sigma^{-1}\mathbb{1}}, \quad \sigma^2_{\pi_{\min}} = \frac{1}{\mathbb{1}^\top \Sigma^{-1}\mathbb{1}}. \tag{1.20}$$

Proof. Notice

$$\mathbb{1}^\top \pi_{\min} = \frac{\mathbb{1}^\top \Sigma^{-1}\mathbb{1}}{\mathbb{1}^\top \Sigma^{-1}\mathbb{1}} = 1$$

It follows that $\pi_{\min} \in H^{d-1}$. Let $\pi \in H^{d-1}$ be arbitrary, define y such that

$$y := \pi - \pi_{\min}.$$

Then by the definition of y

$$\mathbb{1}^\top y = \mathbb{1}^\top \pi - \mathbb{1}^\top \pi_{\min} = 1 - 1 = 0. \tag{1.21}$$

Thus y is orthogonal to $\mathbb{1}$. Using that Σ is symmetric, the definition of $\pi_{\min}$ and (1.21)

$$\begin{aligned}
\sigma^2_\pi = \pi^\top \Sigma \pi &= (\pi_{\min} + y)^\top \Sigma (\pi_{\min} + y) = \pi_{\min}^\top \Sigma \pi_{\min} + 2y^\top \Sigma \pi_{\min} + y^\top \Sigma y \\
&= \pi_{\min}^\top \Sigma \frac{\Sigma^{-1}\mathbb{1}}{\mathbb{1}^\top \Sigma^{-1}\mathbb{1}} + 2y^\top \Sigma \frac{\Sigma^{-1}\mathbb{1}}{\mathbb{1}^\top \Sigma^{-1}\mathbb{1}} + y^\top \Sigma y \\
&= \frac{\pi_{\min}^\top \mathbb{1}}{\mathbb{1}^\top \Sigma^{-1}\mathbb{1}} + 2\frac{y^\top \mathbb{1}}{\mathbb{1}^\top \Sigma^{-1}\mathbb{1}} + y^\top \Sigma y \\
&= \frac{1}{\mathbb{1}^\top \Sigma^{-1}\mathbb{1}} + 2 \times 0 + y^\top \Sigma y.
\end{aligned}$$

Since Σ is symmetric and positive definite, $y^\top \Sigma y \geq 0$ with equality if and only if $y = 0$. This shows both that $\pi_{\min}$ is the unique optimiser and yields $\sigma^2_{\pi_{\min}} = \frac{1}{\mathbb{1}^\top \Sigma^{-1}\mathbb{1}}$. The formula for $\mu_{\pi_{\min}}$ follows directly from the lemma. □

We also seek to find the risk-only portfolio which minimises the variance among all risk-only portfolios with a given expected return μ_0. If μ and $\mathbb{1}$ are collinear, then every risk-only portfolio has the same expected return. Indeed, if μ and $\mathbb{1}$ are collinear, then

$$\mu_\pi = \mu \cdot \pi = \mu^1 \mathbb{1}^T \pi = \mu^1.$$

Thus, we assume in the next lemma that μ and $\mathbb{1}$ are not collinear.

Lemma 1.3. *Let $\bar{S} = (S_t^0, S_t)_{t\in\{0,1\}}$ be a non-redundant market that satisfies no-arbitrage on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Assume that $\bar{S}$ has finite second moments and is positively priced at the initial time step. Denote by μ and Σ the mean vector and covariance matrix of the return vector R of the risky assets. Assume that μ and $\mathbb{1}$ are not collinear. Let $\mu_0 \in \mathbb{R}$ be given. Then there exists a unique risk-only portfolio $\pi_{\mu_0} \in H^{d-1}$ such that $\mu_{\pi_{\mu_0}} = \mu_0$ and*

$$\sigma^2_{\pi_{\mu_0}} \leq \sigma^2_\pi \quad \forall \pi \in \{\pi \in H^{d-1} : \mu_\pi = \mu_0\}. \tag{1.22}$$

It is given by

$$\pi_{\mu_0} = \frac{C - B\mu_0}{AC - B^2}\Sigma^{-1}\mathbb{1} + \frac{A\mu_0 - B}{AC - B^2}\Sigma^{-1}\mu, \tag{1.23}$$

where $A = \mathbb{1}^\top\Sigma^{-1}\mathbb{1}, B = \mathbb{1}^\top\Sigma^{-1}\mu$ and $C = \mu^\top\Sigma^{-1}\mu$. Moreover

$$\sigma^2_{\pi_{\mu_0}} = \frac{A\mu_0^2 - 2B\mu_0 + C}{AC - B^2} = \sigma^2_{\pi_{\min}} + \frac{A}{AC - B^2}(\mu_0 - \mu_{\pi_{\min}})^2. \tag{1.24}$$

Proof. Since Σ is positive definite, it is invertible and Σ^{-1} is again positive definite. Thus, it induces a scalar product $\langle\cdot,\cdot\rangle_{\Sigma^{-1}}$ on $\mathbb{R}^d$ given by

$$\langle x, y\rangle_{\Sigma^{-1}} := x^\top\Sigma^{-1}y.$$

By the Cauchy-Schwarz inequality,

$$B^2 = (\langle\mathbb{1}, \mu\rangle_{\Sigma^{-1}})^2 \leq \langle\mathbb{1}, \mathbb{1}\rangle_{\Sigma^{-1}}\langle\mu, \mu\rangle_{\Sigma^{-1}} = AC,$$

where the inequality is an equality if and only if μ and $\mathbb{1}$ are collinear. As they are not, it follows that $AC - B^2 > 0$. Next, we check that π_{μ_0} given by (1.23) is indeed in H^{d-1} and has expected return μ_0. By the definitions of A, B and C

$$\begin{aligned}
\mathbb{1}^\top\pi_{\mu_0} &= \frac{C - B\mu_0}{AC - B^2}\mathbb{1}^\top\Sigma^{-1}\mathbb{1} + \frac{A\mu_0 - B}{AC - B^2}\mathbb{1}^\top\Sigma^{-1}\mu = \frac{C - B\mu_0}{AC - B^2}A + \frac{A\mu_0 - B}{AC - B^2}B \\
&= \frac{AC - AB\mu_0 + AB\mu_0 - B^2}{AC - B^2} = 1. \\
\mu_{\pi_{\mu_0}} &= \mu \cdot \pi_{\mu_0} = \frac{C - B\mu_0}{AC - B^2}\mu^\top\Sigma^{-1}\mathbb{1} + \frac{A\mu_0 - B}{AC - B^2}\mu^\top\Sigma^{-1}\mu = \frac{C - B\mu_0}{AC - B^2}B + \frac{A\mu_0 - B}{AC - B^2}C \\
&= \frac{CB - B^2\mu_0 + AC\mu_0 - BC}{AC - B^2} = \mu_0.
\end{aligned}$$

Let $\pi \in H^{d-1}$ with $\mu_\pi = \mu_0$ and define y such that

$$y := \pi - \pi_{\mu_0}.$$

Then by the definition of y

$$\mathbb{1}^\top y = \mathbb{1}^\top\pi - \mathbb{1}^\top\pi_{\mu_0} = 1 - 1 = 0, \tag{1.25}$$

$$\mu^\top y = \mu^\top\pi - \mu^\top\pi_{\mu_0} = \mu_\pi - \mu_{\pi_{\mu_0}} = \mu_0 - \mu_0 = 0. \tag{1.26}$$

Then using the fact that Σ is symmetric, the definition of π_{μ_0}, the definition of $\pi^T_{\mu_0}\mu = \mu_0$, (1.25) and (1.26)

$$\begin{aligned}
\sigma^2_\pi &= \pi^\top\Sigma\pi = (\pi_{\mu_0} + y)^\top\Sigma(\pi_{\mu_0} + y) = \pi^\top_{\mu_0}\Sigma\pi_{\mu_0} + 2y^\top\Sigma\pi_{\mu_0} + y^\top\Sigma y \\
&= \pi^\top_{\mu_0}\left(\frac{C - B\mu_0}{AC - B^2}\mathbb{1} + \frac{A\mu_0 - B}{AC - B^2}\mu\right) + 2y^\top\left(\frac{C - B\mu_0}{AC - B^2}\mathbb{1} + \frac{A\mu_0 - B}{AC - B^2}\mu\right) + y^\top\Sigma y \\
&= \left(\frac{C - B\mu_0}{AC - B^2} \times 1 + \frac{A\mu_0 - B}{AC - B^2}\mu_0\right) + 2 \times (0 + 0) + y^\top\Sigma y \\
&= \frac{A\mu_0^2 - 2B\mu_0 + C}{AC - B^2} + y^\top\Sigma y.
\end{aligned}$$

As Σ is symmetric and positive definite, $y^\top \Sigma y \geq 0$ with equality if and only if $y = 0$. This shows both that π_{μ_0} is the unique optimiser and yields (1.22). (1.23) follows from the rearrangement $\sigma^2_{\pi_{\min}} = \frac{1}{A}$ and $\mu_{\pi_{\min}} = \frac{B}{A}$. □

For the following result, we introduce the key concept of a risk-only efficient portfolio.

Definition 1.1. A risk-only portfolio $\pi \in H^{d-1}$ is called risk-only efficient if there does not exist another risk-only portfolio $\pi' \in H^{d-1}$ such that $\mu_{\pi'} \geq \mu_\pi$ and $\sigma^2_{\pi'} \leq \sigma^2_\pi$ with at least one inequality being strict.

Theorem 1.4. *Let $\bar{S} = (S^0_t, S_t)_{t\in\{0,1\}}$ be a non-redundant market that satisfies no-arbitrage on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Assume that $\bar{S}$ has finite second moments and is positively priced at the initial time step. Denote by μ and Σ the mean vector and covariance matrix of the return vector R of the risky assets. Assume that μ and $\mathbb{1}$ are not collinear, and set $A = \mathbb{1}^\top \Sigma^{-1}\mathbb{1}$, $B = \mathbb{1}^\top \Sigma^{-1}\mu$ and $C = \mu^\top \Sigma^{-1}\mu$. Define the risk-only efficient frontier by*

$$\mathscr{E} := \left\{ \left(\sigma_0^2, \mu_0\right) \in \mathbb{R}^2 : \mu_0 \geq \frac{B}{A}, \sigma_0^2 = \frac{A\mu_0^2 - 2B\mu_0 + C}{AC - B^2} \right\}. \tag{1.27}$$

For each point $(\sigma_0^2, \mu_0) \in \mathscr{E}$ there exists exactly one risk-only portfolio $\pi \in H^{d-1}$ such that $(\sigma^2_\pi, \mu_\pi) = (\sigma^2_0, \mu_0)$. It is given by

$$\pi = \pi_{\mu_0} = \frac{C - B\mu_0}{AC - B^2}\Sigma^{-1}\mathbb{1} + \frac{A\mu_0 - B}{AC - B^2}\Sigma^{-1}\mu. \tag{1.28}$$

A risk-only portfolio $\pi \in H^{d-1}$ is risk-only efficient if and only if $(\sigma^2_\pi, \mu_\pi) \in \mathscr{E}$.

Proof. To prove (1.28), let $(\sigma_0^2, \mu_0) \in \mathscr{E}$. It follows from Lemma 1.3 that π_{μ_0} satisfies $(\sigma^2_{\pi_{\mu_0}}, \mu_{\pi_{\mu_0}}) = (\sigma_0^2, \mu_0)$. If $\pi' \in H^{d-1}$ is any other portfolio with $\mu_{\pi'} = \mu_0$ but $\pi' \neq \pi_{\mu_0}$, then by Lemma 1.3, $\sigma^2_{\pi'} > \sigma^2_{\pi_{\mu_0}} = \sigma^2_0$. So we have both existence and uniqueness of π.

To prove the final statement, assume that $\pi \in H^{d-1}$ is risk-only efficient. Then $\sigma^2_\pi \geq \sigma^2_{\pi_{\min}}$ by Lemma 1.3, and so $\mu_\pi \geq \mu_{\pi_{\min}} = \frac{B}{A}$ by the definition of efficiency. Set $\mu_0 := \mu_\pi$ and $\sigma_0^2 := \frac{A\mu_0^2 - 2B\mu_0 + C}{AC - B^2}$. Then $(\sigma_0^2, \mu_0) \in \mathscr{E}$ and by the first statement, $(\sigma^2_{\pi_{\mu_0}}, \mu_{\pi_{\mu_0}}) = (\sigma_0^2, \mu_0)$. Efficiency of π together with $\mu_\pi = \mu_0 = \mu_{\pi_{\mu_0}}$ gives $\sigma^2_\pi \leq \sigma^2_{\pi_{\mu_0}} = \sigma_0^2$. On the other hand, Lemma 1.3 gives $\sigma^2_\pi \geq \sigma^2_{\pi_{\mu_0}} = \sigma^2_0$ so $(\sigma^2_\pi, \mu_\pi) \in \mathscr{E}$.

Conversely, let $\pi \in H^{d-1}$ be such that $(\sigma^2_\pi, \mu_\pi) \in \mathscr{E}$. Set $\mu_0 := \mu_\pi$ and $\sigma_0^2 := \sigma^2_\pi$. Then $\pi = \pi_{\mu_0}$ by (1.28). Seeking a contradiction, suppose there is $\pi' \in H^{d-1}$ such that $\mu_{\pi'} \geq \mu_0$ and $\sigma^2_{\pi'} \leq \sigma^2_0$ with at least one of the inequalities being strict. If $\mu_{\pi'} = \mu_0$, then $\sigma^2_{\pi'} < \sigma^2_0 = \sigma^2_{\pi_{\mu_0}}$, and by Lemma 1.3, we arrive at a contradiction. Otherwise, if $\mu_{\pi'} > \mu_0$, set $\mu_1 := \mu_{\pi'}$ and $\sigma_1^2 = \frac{A\mu_1^2 - 2B\mu_1 + C}{AC - B^2}$. Then $\sigma_1^2 > \sigma_0^2$ because the function $x \mapsto \frac{Ax^2 - 2Bx + C}{AC - B^2}$ is strictly increasing for $x \geq \frac{B}{A}$. This means, that $\sigma^2_{\pi'} < \sigma_1^2$. But $\sigma_1^2 = \sigma^2_{\pi_{\mu_1}}$ by the first statement. Thus, $\mu_{\pi'} = \mu_1 = \mu_{\pi_{\mu_1}}$ and $\sigma^2_{\pi'} < \sigma^2_1 = \sigma^2_{\pi_{\mu_1}}$, and again by Lemma 1.3, we arrive at a contradiction. □

We can now fully solve the risk-only versions of the mean-variance problems.

Theorem 1.5. *Let $\bar{S} = (S^0_t, S_t)_{t\in\{0,1\}}$ be a non-redundant market that satisfies no-arbitrage on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Assume that $\bar{S}$ has finite second moments and is positively priced at the initial time step. Denote by μ and Σ the mean vector and covariance matrix of the return vector R of the risky assets. Assume that μ and $\mathbb{1}$ are not collinear, and set $A = \mathbb{1}^\top \Sigma^{-1}\mathbb{1}$, $B = \mathbb{1}^\top \Sigma^{-1}\mu$ and $C = \mu^\top \Sigma^{-1}\mu$.*

(i) Let $\mu_{\min} \geq \frac{B}{A}$ be given. Then the risk-only mean-variance problem

$$\underset{\pi \in H^{d-1}}{\operatorname{argmin}} \sigma_\pi^2 \quad \textit{subject to } \mu_\pi \geq \mu_{\min},$$

has a unique solution π_* given by

$$\pi_* = \pi_{\mu_{\min}} = \frac{C - B\mu_{\min}}{AC - B^2}\Sigma^{-1}\mathbb{1} + \frac{A\mu_{\min} - B}{AC - B^2}\Sigma^{-1}\mu. \tag{1.29}$$

It is risk-only efficient and satisfies

$$\mu_{\pi_*} = \mu_{\min}, \quad \sigma_{\pi_*}^2 = \frac{A\mu_{\min}^2 - 2B\mu_{\min} + C}{AC - B^2}. \tag{1.30}$$

(ii) Let $\sigma_{\max}^2 \geq \frac{1}{A}$ be given. Then the risk-only mean-variance problem

$$\underset{\pi \in H^{d-1}}{\operatorname{argmax}} \mu_\pi \quad \textit{subject to } \sigma_\pi^2 \leq \sigma_{\max}^2,$$

has a unique solution π_* given by

$$\pi_* = \pi_{\mu_{\sigma_{\max}^2}} = \frac{C - B\mu_{\sigma_{\max}^2}}{AC - B^2}\Sigma^{-1}\mathbb{1} + \frac{A\mu_{\sigma_{\max}^2} - B}{AC - B^2}\Sigma^{-1}\mu, \tag{1.31}$$

where

$$\mu_{\sigma_{\max}^2} = \frac{B}{A} + \frac{\sqrt{(AC - B^2)\,(A\sigma_{\max}^2 - 1)}}{A}.$$

It is risk-only efficient and satisfies

$$\mu_{\pi_*} = \mu_{\sigma_{\max}^2} = \frac{B}{A} + \frac{\sqrt{(AC - B^2)\,(A\sigma_{\max}^2 - 1)}}{A}, \quad \sigma_{\pi_*}^2 = \sigma_{\max}^2. \tag{1.32}$$

Proof. We only establish the proof of (ii). The proof of (i) is very similar. First, it follows from Theorem 1.4, that $\pi_* = \pi_{\mu_{\sigma_{\mathrm{m}}^2}}$ is risk-only efficient and $\sigma_{\pi_*}^2 = \sigma_{\max}^2$. So π_* satisfies the constraint in (1.31) with equality. Let $\pi' \in H^{d-1}$ be any portfolio satisfying the constraint in (1.31). Then by efficiency of π_* it follows that $\mu_{\pi'} \leq \mu_{\pi_*}$, which gives that π_* is optimal for (1.31). To establish uniqueness, note that if $\mu_{\pi'} = \mu_{\pi_*}$, then $\sigma_{\pi'}^2 \geq \sigma_{\pi_*}^2$ by efficiency of π_* and hence by the constraint in (1.31), $\sigma_{\pi'}^2 = \sigma_{\pi_*}^2$. But this then implies that $(\sigma_{\pi'}^2, \mu_{\pi'}) = (\sigma_{\pi_*}^2, \mu_{\pi_*}) \in \mathscr{E}$, and by Theorem 1.4, it follows that $\pi' = \pi_*$. □

1.5 Optimisation With A Riskless Asset

We proceed to study the case with a riskless asset. The analogue of Lemma 1.2 is trivial. With a riskless asset, we can achieve zero risk by just investing in the riskless asset. This is choosing the portfolio $\bar{\pi}_{\min,r} := (1, 0, \ldots, 0) \in H^{1+d-1}$. So we directly consider the analogue of Lemma 1.3. Since we can now also invest in the riskless asset, we do not need to assume that μ and $\mathbb{1}_d$ are not collinear but only that $\bar{\mu}$ and $\mathbb{1}_{1+d}$ are not collinear. By the fact that $\mu^0 = r$, this is equivalent to assuming that $\mu \neq r\mathbb{1}_d$.

Proposition 1.6. *Let $\bar{S} = (S_t^0, S_t)_{t \in \{0,1\}}$ be a non-redundant market that satisfies no-arbitrage on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Assume that $\bar{S}$ has finite second moments and is positively priced at the initial time step. Denote by μ and Σ the mean vector and covariance matrix of the return vector R of the risky assets, and let r be the risk-free interest rate. Let $\bar{\pi} = (\pi^0, \pi) \in H^{1+d-1}$. Then*

$$\mu_{\bar{\pi}} - r = (\mu - r\mathbb{1}_d) \cdot \pi$$

Proof. By the fact that $\pi^0 = 1 - \pi \cdot \mathbb{1}_d$ and $\mu^0 = r$

$$\begin{aligned}(\mu - r\mathbb{1}_d) \cdot \pi &= \mu \cdot \pi + r(-\pi \cdot \mathbb{1}_d) = \mu \cdot \pi + r(1 - \pi \cdot \mathbf{1}_d) - r = \mu \cdot \pi + \mu^0 \pi^0 - r \\ &= \bar{\mu} \cdot \bar{\pi} - r = \mu_{\bar{\pi}} - r.\end{aligned} \tag{1.33}$$

□

We now seek to find the portfolio in H^{1+d-1} which minimises the variance among all portfolios in H^{1+d-1} with a given expected return μ_0. Note that below $\mathbb{1}$ will always be $\mathbb{1}_d$.

Lemma 1.7. *Let $\bar{S} = (S_t^0, S_t)_{t \in \{0,1\}}$ be a non-redundant market that satisfies no-arbitrage on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Assume that $\bar{S}$ has finite second moments and is positively priced at the initial time step. Denote by μ and Σ the mean vector and covariance matrix of the return vector R of the risky assets, and let r be the risk-free interest rate. Assume that $\mu \neq r\mathbb{1}$. Let $\mu_0 \in \mathbb{R}$ be given. Then there exists a unique portfolio $\bar{\pi}_{\mu_0,r} \in H^{1+d-1}$ such that $\mu_{\pi_{\mu_0,r}} = \mu_0$ and*

$$\sigma^2_{\bar{\pi}_{\mu_0,r}} \leq \sigma^2_{\bar{\pi}} \quad \forall \bar{\pi} \in \{\bar{\pi} \in H^{1+d-1} : \mu_{\bar{\pi}} = \mu_0\}.$$

It is given by $\bar{\pi}_{\mu_0,r} = (1 - \pi_{\mu_0,r} \cdot \mathbb{1}, \pi_{\mu_0,r})$ where

$$\pi_{\mu_0,r} = \frac{\mu_0 - r}{(\mu - r\mathbb{1})^\top \Sigma^{-1} (\mu - r\mathbb{1})} \Sigma^{-1} (\mu - r\mathbb{1}). \tag{1.34}$$

It satisfies

$$\sigma^2_{\bar{\pi}_{\mu_0,r}} = \frac{(\mu_0 - r)^2}{(\mu - r\mathbb{1})^\top \Sigma^{-1} (\mu - r\mathbb{1})} = \frac{(\mu_0 - r)^2}{Ar^2 - 2Br + C}, \tag{1.35}$$

where $A = \mathbb{1}^\top \Sigma^{-1} \mathbb{1}$, $B = \mathbb{1}^\top \Sigma^{-1} \mu$ and $C = \mu^\top \Sigma^{-1} \mu$.

Proof. First, it follows from the condition $\mu \neq r\mathbb{1}$ and the fact that Σ^{-1} is symmetric and positive definite that $(\mu - r\mathbb{1})^\top \Sigma^{-1} (\mu - r\mathbb{1}) > 0$. We proceed to check that $\bar{\pi}_{\mu_0,r} \in H^{1+d-1}$ and $\mu_{\pi_{\mu_0,r}} = \mu_0$. By the definition of $\bar{\pi}_{\mu_0,r}$ and Proposition 1.6

$$\begin{aligned}\bar{\pi}_{\mu_0,r} \cdot \mathbb{1}_{1+d} &= \pi^0_{\mu_0,r} + \pi_{\mu_0,r} \cdot \mathbb{1}_d = (1 - \pi_{\mu_0,r} \cdot \mathbb{1}_d) + (\pi_{\mu_0,r} \cdot \mathbb{1}_d) = 1, \\ \mu_{\bar{\pi}_{\mu_0,r}} &= (\mu_{\pi_{\mu_0,r}} - r) + r = (\mu - r\mathbb{1}) \cdot \pi_{\mu_0,r} + r \\ &= (\mu_0 - r) \frac{(\mu - r\mathbb{1})^\top \Sigma^{-1} (\mu - r\mathbb{1})}{(\mu - r\mathbb{1})^\top \Sigma^{-1} (\mu - r\mathbb{1})} + r = (\mu_0 - r) + r = \mu_0.\end{aligned}$$

Let $\bar{\pi} \in H^{1+d-1}$ with $\mu_{\bar{\pi}} = \mu_0$ and define y such that

$$\bar{y} := \bar{\pi} - \bar{\pi}_{\mu_0,r}.$$

Proposition 1.6 obtains

$$\begin{aligned}(\mu - r\mathbb{1}) \cdot y &= (\mu - r\mathbb{1}) \cdot \pi - (\mu - r\mathbb{1}) \cdot \pi_{\mu_0,r} = (\mu_{\bar{\pi}} - r) - (\mu_{\pi_{\mu_0,r}} - r) \\ &= (\mu_0 - r) - (\mu_0 - r) = 0\end{aligned}$$

Then by Theorem 1.5, the fact that Σ is symmetric, the definition of $\pi_{\mu_0,r}$, Proposition 2.8 and the fact that y is orthogonal to $\mu - r\mathbb{1}$

$$
\begin{aligned}
\sigma_\pi^2 = \pi^\top \Sigma \pi &= (\pi_{\mu_0,r} + y)^\top \Sigma (\pi_{\mu_0,r} + y) = \pi_{\mu_0,r}^\top \Sigma \pi_{\mu_0,r} + 2y^\top \Sigma \pi_{\mu_0,r} + y^\top \Sigma y \\
&= (\mu_0 - r) \frac{\pi_{\mu_0,r}^\top (\mu - r\mathbb{1})}{(\mu - r\mathbb{1})^\top \Sigma^{-1} (\mu - r\mathbb{1})} + 2(\mu_0 - r) \frac{y^\top (\mu - r\mathbb{1})}{(\mu - r\mathbb{1})^\top \Sigma^{-1} (\mu - r\mathbb{1})} + y^\top \Sigma y \\
&= (\mu_0 - r) \frac{(\mu_0 - r)}{(\mu - r\mathbb{1})^\top \Sigma^{-1} (\mu - r\mathbb{1})} + 2(\mu_0 - r) \times 0 + y^\top \Sigma y \\
&= \frac{(\mu_0 - r)^2}{(\mu - r\mathbb{1})^\top \Sigma^{-1} (\mu - r\mathbb{1})} + y^\top \Sigma y.
\end{aligned}
$$

As Σ is symmetric and positive definite, $y^\top \Sigma y \geq 0$ with equality if and only if $y = 0$. Moreover, since $\bar{\pi}, \bar{\pi}_{\mu_0,r} \in H^{1+d-1}$ it follows that

$$
\sum_{k=0}^{d} y_k = \bar{y} \cdot \mathbb{1}_{1+d} = \bar{\pi} \cdot \mathbb{1}_{1+d} - \bar{\pi}_{\mu_0,r} \cdot \mathbb{1}_{1+d} = 1 - 1 = 0.
$$

This implies in particular that $\bar{y} \neq 0$ if and only if $y \neq 0$. Hence, $\bar{\pi}_{\mu_0,r}$ is the unique optimiser and yields the first equality in (1.34). The second equality in (1.35) follows by expanding the denominator and using the definitions of A, B and C. $\square$

We proceed to formulate the analogue of Theorem 1.5. To this end, we also need to consider the notion of efficiency for general portfolios.

Definition 1.2. A portfolio $\bar{\pi} \in H^{1+d-1}$ is called efficient (in the mean-variance sense) if there does not exist another portfolio $\bar{\pi}' \in H^{1+d-1}$ such that $\mu_{\bar{\pi}'} \geq \mu_{\bar{\pi}}$ and $\sigma_{\bar{\pi}'}^2 \leq \sigma_{\bar{\pi}}^2$ with one inequality being strict.

Theorem 1.8. *Let $\bar{S} = (S_t^0, S_t)_{t \in \{0,1\}}$ be a non-redundant market that satisfies no-arbitrage on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Assume that $\bar{S}$ has finite second moments and is positively priced at the initial time step. Denote by μ and Σ the mean vector and covariance matrix of the return vector R of the risky assets, and let r be the risk-free interest rate. Assume that $\mu - r\mathbb{1} \neq 0$, and set $A = \mathbb{1}^\top \Sigma^{-1} \mathbb{1}$, $B = \mathbb{1}^\top \Sigma^{-1} \mu$ and $C = \mu^\top \Sigma^{-1} \mu$. Define the efficient frontier by*

$$
\overline{\mathscr{E}} := \left\{ (\sigma_0^2, \mu_0) \in \mathbb{R}^2 : \mu_0 \geq r, \sigma_0^2 = \frac{(\mu_0 - r)^2}{Ar^2 - 2Br + C} \right\}. \tag{1.36}
$$

For each point $(\sigma_0^2, \mu_0) \in \overline{\mathscr{E}}$, there exists exactly one portfolio $\bar{\pi} \in H^{1+d-1}$ such that $(\sigma_{\bar{\pi}}^2, \mu_{\bar{\pi}}) = (\sigma_0^2, \mu_0)$. It is given by $\bar{\pi} = (1 - \pi_{\mu_0,r} \cdot 1, \pi_{\mu_0,r})$, where

$$
\pi_{\mu_0,r} = \frac{\mu_0 - r}{Ar^2 - 2Br + C} \Sigma^{-1} (\mu - r\mathbb{1}). \tag{1.37}
$$

A portfolio $\bar{\pi} \in H^{1+d-1}$ is efficient if and only if $(\sigma_{\bar{\pi}}^2, \mu_{\bar{\pi}}) \in \overline{\mathscr{E}}$.

Proof. The argument for (1.36) and (1.37) is almost identical to the proof of Theorem 1.5. The only difference is that we replace the minimum variance portfolio by the riskless portfolio and use Lemma 1.7 instead of Lemma 1.3 . $\square$

Remark. $\bar{\pi} \in H^{1+d-1}$ is efficient if and only if $\pi = \gamma \Sigma^{-1} (\mu - r\mathbf{1})$ for some $\gamma \in (0, \infty)$ and $\bar{\pi} = (1 - \pi \cdot 1, \pi)$.

We proceed to formulate the analogue of Theorem 1.5, the solution to the mean-variance problems with a riskless asset.

Theorem 1.9. *Let $\bar{S} = (S_t^0, S_t)_{t\in\{0,1\}}$ be a non-redundant market that satisfies no-arbitrage on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Assume that $\bar{S}$ has finite second moments and is positively priced at the initial time step. Denote by μ and Σ the mean vector and covariance matrix of the return vector R of the risky assets, and let r be the risk-free interest rate. Assume that $\mu \neq r1$. Set $A = \mathbb{1}^\top \Sigma^{-1}\mathbb{1}$, $B = \mathbb{1}^\top \Sigma^{-1}\mu$ and $C = \mu^\top \Sigma^{-1}\mu$.*

(i) Let $\mu_{\min} \geq r$ be given. Then the mean-variance problem with a riskless asset

$$\operatorname*{argmin}_{\bar{\pi}\in H^{1+d-1}} \sigma_{\bar{\pi}}^2 \quad \textit{subject to } \mu_{\bar{\pi}} \geq \mu_{\min},$$

has a unique solution $\bar{\pi}_*$ given by $\bar{\pi}_* = (1 - \pi_{\mu_{\min},r} \cdot 1, \pi_{\mu_{\min},r})$, where

$$\pi_{\mu_{\min},r} = \frac{\mu_{\min} - r}{Ar^2 - 2Br + C}\Sigma^{-1}(\mu - r\mathbb{1}). \tag{1.38}$$

It is efficient and satisfies

$$\mu_{\bar{\pi}_*} = \mu_{\min} \quad \textit{and} \quad \sigma_{\bar{\pi}_*}^2 = \frac{(\mu_{\min} - r)^2}{Ar^2 - 2Br + C} \tag{1.39}$$

(ii) Let $\sigma_{\max}^2 \geq 0$ be given. Then the mean-variance problem with riskless asset

$$\operatorname*{argmax}_{\bar{\pi}\in H^{1+d-1}} \mu_{\bar{\pi}} \quad \textit{subject to } \sigma_{\bar{\pi}}^2 \leq \sigma_{\max}^2,$$

has a unique solution $\bar{\pi}_*$ given by $\bar{\pi}_* = (1 - \pi_{\sigma_{\max}^2,r} \cdot \mathbb{1}, \pi_{\sigma_{\max}^2,r})$, where

$$\pi_{\mu_{\sigma_{\max}^2},r} = \frac{\mu_{\sigma_{\max}^2} - r}{Ar^2 - 2Br + C}\Sigma^{-1}(\mu - r\mathbb{1}), \tag{1.40}$$

and

$$\mu_{\sigma_{\max}^2} = r + \sigma_{\max}\sqrt{Ar^2 - 2Br + C}. \tag{1.41}$$

It is efficient and satisfies

$$\mu_{\bar{\pi}_*} = \mu_{\sigma_{\max}^2} = r + \sigma_{\max}\sqrt{Ar^2 - 2Br + C}, \quad \sigma_{\bar{\pi}_*}^2 = \sigma_{\max}^2. \tag{1.42}$$

1.6 Markowitz Tangency Portfolio & Capital Market Line

Theorem 1.10. *Let $\bar{S} = (S_t^0, S_t)_{t\in\{0,1\}}$ be a non-redundant market that satisfies no-arbitrage on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Assume that $\bar{S}$ has finite second moments and is positively priced at the initial time step. Denote by μ and Σ the mean vector and covariance matrix of the return vector R of the risky assets, and let r be the risk-free interest rate. Assume that $r < \frac{B}{A}$ where $A = \mathbb{1}^\top \Sigma^{-1}\mathbb{1}$ and $B = \mathbb{1}^\top \Sigma^{-1}\mu$.*

(i) There exists a unique efficient portfolio $\bar{\pi}_{tan}$ called the Markowitz tangency portfolio that is risk only $bar\pi_{tan} = (0, \pi_{tan})$. It satisfies

$$\pi_{tan} = \frac{1}{B - rA}\Sigma^{-1}(\mu - r\mathbb{1}). \tag{1.43}$$

(ii) A portfolio $\bar{\pi} \in H^{1+d-1}$ is efficient if and only if it can be written as

$$\bar{\pi} = \lambda \bar{\pi}_{tan} + (1-\lambda)\bar{\pi}_{tan,r} = (1-\lambda, \lambda \pi_{tan}), \tag{1.44}$$

where $\bar{\pi}_{tan,r} = (1,0)$ denotes the riskless portfolio, and $\lambda \geq 0$.

Proof. Let $\bar{\pi} \in H^{1+d-1}$ be an efficient portfolio. It follows from Theorem 1.8 that there is $\mu_0 \geq r$ such that $\bar{\pi} = \bar{\pi}_{\mu_0,r}$. Moreover, $\bar{\pi}_{\mu_0,r}$ is risk-only if and only if $\bar{\pi}_{\mu_0,r} \cdot \mathbb{1} = 1$. Using the definitions of $\bar{\pi}_{\mu_0,r}$, A and B

$$\bar{\pi}_{\mu_0,r} \cdot \mathbb{1} = \frac{\mu_0 - r}{Ar^2 - 2Br + C} \mathbb{1}^\top \Sigma^{-1}(\mu - r\mathbb{1}) = \frac{\mu_0 - r}{Ar^2 - 2Br + C}(B - rA).$$

Solving $\bar{\pi}_{\mu_0,r} \cdot \mathbb{1} = 1$, we obtain that $\bar{\pi}_{\mu_0,r} \in H^{1+d-1}$ if and only if

$$\mu_0 = r + \frac{Ar^2 - 2Br + C}{B - rA} := \mu_{\tan}.$$

Notice that $\mu_{\tan} > 0$ since $Ar^2 - 2Br + C > 0$ and $B - rA > 0$. Theorem 1.8 implies that $\bar{\pi}_{\mu_{\tan,r}}$ is indeed efficient. We can conclude that an efficient portfolio is risk-only if and only if $\bar{\pi} = \bar{\pi}_{\mu_{\tan,r}}$. Moreover,

$$\pi_{\mu_{\tan,r}} = \frac{1}{B - rA}(\mu - r\mathbb{1}) = \pi - \tan.$$

This recovers (1.43). Let $\bar{\pi} \in H^{1+d-1}$ be of the form

$$\bar{\pi} = \lambda \bar{\pi}_{\tan} + (1-\lambda)\bar{\pi}_{\min,r},$$

for some $\lambda \geq 0$. Using that $\bar{\pi}_{\min,r} = (1,0)$ and $\bar{\pi}_{\tan} = \bar{\pi}_{\mu_{\tan,r}}$ where $\mu_{\tan} = r + \frac{Ar^2-2Br+C}{B-rA}$, we obtain

$$\begin{aligned}
\pi &= \lambda \pi_{\tan} + (1-\lambda)\pi_{\min,r} \\
&= \lambda \frac{\mu_{\tan} - r}{Ar^2 - 2Br + C} \Sigma^{-1}(\mu - r\mathbb{1}) + (1-\lambda) \times 0 \\
&= \frac{\mu_\lambda - r}{Ar^2 - 2Br + C} \Sigma^{-1}(\mu - r\mathbb{1}),
\end{aligned}$$

where $\mu_\lambda := \lambda_{\mu_{\tan}} + (1-\lambda)r$. Thus, $\pi = \pi_{\mu_\lambda,r}$ and $\bar{\pi} = \bar{\pi}_{\mu_\lambda,r}$. Since $\mu_\lambda \geq r$, it follows from the remark after Theorem 1.8 that $\bar{\pi}$ is efficient. Conversely, suppose that $\bar{\pi} \in H^{1+d-1}$ is efficient. Then by Theorem 1.8, there is $\mu_0 \geq r$ such that $\bar{\pi} = \bar{\pi}_{\mu_0,r}$. Set

$$\lambda := \frac{\mu_0 - r}{\mu_{\tan} - r},$$

where $\mu_{\tan}$ is defined as before. Then $\lambda \geq 0$ and $\mu_0 = \lambda \mu_{\tan} + (1-\lambda)r$. The same calculation as above shows that $\pi = \lambda \pi_{\tan} + (1-\lambda)\pi_{\min,r}$ and then also $\lambda \bar{\pi} + (1-\lambda)\bar{\pi}_{\min,r}$. □

Remark. Theorem 1.10 is usually referred to as the mutual fund theorem because it states every efficient portfolio is a combination of the mutual funds $\bar{\pi}_{\tan}$ and $\bar{\pi}_{\min,r}$ (the first containing only risky assets and the second containing only the riskless asset). In the seeting of the mutal fund theorem, define the capital market line (CML) as

$$\mathrm{CML} = \{(\lambda \mu_{\bar{\pi}_{\tan}} + (1-\lambda)r, \lambda \sigma_{\bar{\pi}_{\tan}}, \lambda \geq 0)\}. \tag{1.45}$$

Then it follows from the mutual fund theorem that a portfolio $\bar{\pi}$ is efficient if and only if it lies on the capital market line in the sense that $(\mu_{\bar{\pi}}, \sigma_{\bar{\pi}}) \in \mathrm{CML}$. Notice the CML is just a reparameteristion of the efficient frontier $\overline{\mathscr{E}}$.