Incentive Fees in Hedge Fund Management

by

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Abstract

Portfolio managers are frequently rewarded for increasing the value of assets under management. In this paper, we study the optimal policy of a risk-averse manager who is compensated with an incentive fee subject to exceeding the high-water mark of the fund. We show that the leverage effect of option compensation encourages the manager to seek less risk than if they were trading with their own account, particularly when the value of their portfolio is in the money. We provide a numerical solution to the optimal policy using finite difference.

Keywords: Optimal Investment-Consumption, Dynamic Programming Principle, Hamilton-Jacobi-Bellman Equation, Option Compensation, Risk Taking, Finite Difference Method.

Contents

1 Introduction

The expansion of the hedge fund sector has drawn notice towards a distinctive type of performance compensation scheme. Portfolio managers are typically compensated with a share of the positive profits of the fund, net of a benchmark such as the high-water mark of the fund. These contracts function similarly to a call option, where the portfolio manager receives an incentive fee when the fund surpasses the high-water mark.

In this paper, we explore the effects of option compensation on managerial risk taking. We consider an expected utility maximising manager who controls the allocation of wealth between a riskless and risky asset. The manager has power utility describing their terminal wealth preference. The manager's compensation includes both a management fee and a performance fee based on exceeding the high-water mark of the fund. Under assumptions about the continuous-time financial market and the asset prices, we present a numerical solution of the optimal trading strategy.

Due to the call option structure, the manager's compensation is lower bounded but has unbounded potential gain. Whilst portfolio managers typically aim to be risk-neutral, such managers would take on a portfolio with unbounded volatility in our model. For that reason, we consider a risk-averse manager. They do not strictly prefer to increase the volatility of the portfolio. Instead, they adjust the volatility of the portfolio according to the value of the portfolio. As the value of the portfolio tends to infinity, the portfolio volatility converges to the Merton [13] constant volatility portfolio. When the option is in the money, the manager takes less risk than if they were trading with their own account. This is due to the option providing leverage.

The paper is organised as follows. Chapter 2 discusses the foundational understanding regarding stochastic processes and Brownian motion. Chapter 3 introduces useful results from stochastic analysis such as Itô's Formula. Chapter 4 delves into the theory of stochastic control. We derive a partial differential equation whose solutions coincide with a family of optimal control problems. Chapter 5 illustrates that such optimal control problems can have analytical solutions. We discuss Merton's consumption and terminal wealth problems [13] with a finite evaluation period and an infinite evaluation period. We evaluate how the length of the evaluation period influences the volatility of a participant's portfolio and their consumption. Finally, Chapter 6 formulates the investment problem of a portfolio manager who is compensated with a management fee and a performance fee based on exceeding the high-water mark of the fund. We evaluate how their risk appetite differs to when they trade with their own account. We then present a technique employed in Carpenter [2] to reformulate the problem. Lastly, we illustrate a numerical solution to the reformulated problem using finite difference.

2 Preliminaries

In this section, we present concepts from probability theory that will be essential for subsequent analysis. Let T be the time parameter set: $T = \{0, 1, 2, \dots\}$ in the discrete-time case and $T = [0, \infty)$ in the continuous-time case. Consider a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and a filtration $(\mathscr{F}_t)_{t\in T}$. We call $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in T}, \mathbb{P})$ a filtered probability space.

Definition 2.1. Consider a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in T}, \mathbb{P})$ and a real-valued stochastic process X. X is a *martingale* if

(i) X_t is integrable for every $t \in T$,

(ii) $(X_t)_{t\geq 0}$ is $\mathscr{F}_t$-adapted,

(iii) $\mathbb{E}[X_t \mid \mathscr{F}_s] = X_s$ $\mathbb{P}$-a.s. for every $s \in T$ and $t \in T$ such that $s < t$.

Definition 2.2. Consider a stochastic process X on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$. X is *square integrable* if $\mathbb{E}[|X_t|^2] < \infty$ for every $t \in T$.

Definition 2.3. Suppose $B = (B_t)_{t\geq 0}$ is a continuous, real-valued stochastic process on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$. B is a *One-Dimensional Brownian Motion* if

(i) $B_0 = 0 \quad \mathbb{P}$-a.s,

(ii) For $0 \leq s < t$ the increment $B_t - B_s$ is independent of $\mathscr{F}_s$ and is normally distributed with mean 0 and variance $t - s$.

Remark. Property (ii) is often called *stationary and independent increments*.

Definition 2.4. Let $d > 0$ be a positive integer and μ be a probability measure on $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$. Suppose $B = (B_t)_{t\geq 0}$ is a continuous, adapted, $\mathbb{R}^d$-valued stochastic process on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$. B is a *d-Dimensional Brownian Motion with initial distribution* μ if

(i) $\mathbb{P}(B_0 \in \Gamma) = \mu(\Gamma)$ for every $\Gamma \in \mathscr{B}(\mathbb{R}^d)$,

(ii) For $0 \leq s < t$ the increment $B_t - B_s$ is independent of $\mathscr{F}_s$ and is normally distributed with mean 0 and covariance $(t - s)I_d$ where I_d is a $(d \times d)$ identity matrix.

Moreover, if μ assigns measure one to some singleton $\{x\}$, B is a *d-Dimensional Brownian Motion starting at x*.

3 Stochastic Analysis

In this section, we establish useful results of stochastic analysis. Stochastic differential equations are introduced as a tool to describe some stochastic processes. We also look closer at a result called Itô's Formula. This is often described as a stochastic equivalent of the chain rule. There are many books focusing on the theory presented in this section. We use as main references the following works: Karatzas and Shreve [10] and Ikeda and Watanabe [9].

3.1 Martingales and Itô's Formula

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space and $(\mathscr{F}_t)_{t\geq 0}$ be a right-continuous filtration containing all $\mathbb{P}$-null sets. It is always assumed in this section that $t \mapsto X_t$ is right-continuous $\mathbb{P}$-a.s.

For any $a > 0$, let $\mathscr{S}_a$ be the set of all stopping times σ such that $\sigma \leq a$ $\mathbb{P}$-a.s.

Definition 3.1. Suppose X is a submartingale on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$. X is of *class DL* if, for every $a > 0$, the family $\{X_\sigma : \sigma \in \mathscr{S}_a\}$ is uniformly integrable.

Theorem 3.1. *(Doob-Meyer Decomposition) Suppose X is a submartingale of class DL on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$. Then X admits the decomposition*

$$X_t = M_t + A_t \quad \mathbb{P}\text{-}a.s \quad t \geq 0,$$

where $M = (M_t)_{t\geq 0}$ is a martingale and $A = (A_t)_{t\geq 0}$ is an integrable, increasing stochastic process. Furthermore, A can be chosen to be natural. Under this condition the decomposition is unique.

Definition 3.2. Consider the filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$.

(i) $\mathscr{M}_2 = \{(X_t)_{t\geq 0} : X$ is a square integrable martingale w.r.t $(\mathscr{F}_t)_{t\geq 0}$ and $X_0 = 0$ $\mathbb{P}$-a.s.$\}$,

(ii) $\mathscr{M}_2^c = \{X \in \mathscr{M}_2 : t \to X_t$ is continuous $\mathbb{P}$-a.s.$\}$.

Proposition 3.2. *Suppose X is a non-negative submartingale on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$, then X is of class DL.*

Proof. For any $a > 0$ and $\lambda > 0$ consider some $\sigma \in \mathscr{S}_a$. Then

$$\mathbb{E}\left[X_\sigma \mathbb{1}_{\{X_\sigma > \lambda\}}\right] \leq \mathbb{E}\left[X_a \mathbb{1}_{\{X_\sigma > \lambda\}}\right].$$

Using Markov's inequality

$$\mathbb{P}(X_\sigma > \lambda) \leq \frac{\mathbb{E}[X_\sigma]}{\lambda} \leq \frac{\mathbb{E}[X_a]}{\lambda}.$$

By the integrability of the singleton $\{X_a\}$, given $\varepsilon > 0$ there exists some $\delta > 0$ such that if $\mathbb{P}(B) < \delta$ then $\mathbb{E}[X_a \mathbb{1}_B] < \varepsilon$. Then if $\frac{\mathbb{E}[X_a]}{\lambda} < \delta$ then $\mathbb{P}(X_\sigma > \lambda) < \delta$ and so $\mathbb{E}\left[X_\sigma \mathbb{1}_{\{X_\sigma > \lambda\}}\right] < \varepsilon$. Since a, λ and σ were all arbitrary

$$\lim_{\lambda\to\infty} \sup_{\sigma\in\mathscr{S}_a} \mathbb{E}\left[X_\sigma \mathbb{1}_{\{X_\sigma > \lambda\}}\right] = 0.$$

□

Proposition 3.3. *Suppose $X, Y \in \mathscr{M}_2$. Then*

(i) There exists a integrable, increasing, natural stochastic process $A = (A_t)_{t\geq 0}$ such that $X^2 - A$ is a martingale. Furthermore, A is uniquely determined (up to indistinguishability).

(ii) There exists $A = (A_t)_{t\geq 0}$ that is the difference of two integrable, increasing, natural stochastic processes such that $XY - A$ is a martingale. Furthermore, A is uniquely determined (up to indistinguishability).

Proof. Let $X \in \mathscr{M}_2$. Then, for any $0 \leq s < t$, using conditional Jensen's inequality

$$\mathbb{E}[X_t^2 \,|\, \mathscr{F}_s] \geq (\mathbb{E}[X_t^2 \,|\, \mathscr{F}_s])^2 = X_s^2 \quad \mathbb{P}\text{-a.s.}$$

Then $t \mapsto X_t^2$ is a non-negative submartingale which, by Proposition 3.2, implies that it is of class DL. Hence, by Doob-Meyer's Decomposition theorem, there exists a unique integrable, increasing, natural stochastic process $A = (A_t)_{t\geq 0}$ such that $t \mapsto X_t^2 - A_t$ is an $\mathscr{F}_t$-martingale.

Let $X, Y \in \mathscr{M}_2$. Then $\tilde{X} := \frac{X+Y}{2} \in \mathscr{M}_2$ and $\tilde{Y} := \frac{X-Y}{2} \in \mathscr{M}_2$. Let A^1 and A^2 be the integrable, increasing, natural stochastic processes corresponding to $\tilde{X}$ and $\tilde{Y}$ respectively in the Doob-Meyer Decompositions. Then $A := A^1 - A^2$ satisfies $t \mapsto X_tY_t - A_t$ is an $\mathscr{F}_t$-martingale. The uniqueness of A comes from the uniqueness of A^1 and A^2 in the Doob-Meyer Decompositions. □

Definition 3.3. $A = (A_t)_{t\geq 0}$ in Proposition 3.3 (i) is denoted by $\langle X \rangle = (\langle X \rangle_t)_{t\geq 0}$. $A = (A_t)_{t\geq 0}$ in Proposition 3.3 (ii) is denoted by $\langle X, Y \rangle := (\langle X, Y \rangle_t)_{t\geq 0}$. Notice that $\langle X, X \rangle := \langle X \rangle$. We call $\langle X, Y \rangle$ the *quadratic variational process* corresponding to X and Y.

Definition 3.4. Suppose X is a real-valued stochastic process on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$. X is called a *local martingale* if it is adapted and there exists a sequence of stopping times $(\sigma_n)_{n\geq 0}$ such that $\sigma_n < \infty$, $\sigma_n \uparrow \infty$ and $X_{t\wedge\sigma_n}$ is a martingale for each $n \geq 0$. Moreover, if $X_{t\wedge\sigma_n}$ is a square integrable martingale for each $n \geq 0$, then X is a *locally square integrable martingale*.

Definition 3.5. Consider the filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$. We define the sets

(i) $\mathscr{M}_2^{loc} = \{(X_t)_{t\geq 0} : X$ is a locally square integrable martingale w.r.t $(\mathscr{F}_t)_{t\geq 0}$, $X_0 = 0$ $\mathbb{P}$-a.s.$\}$,

(ii) $\mathscr{M}_2^{c,loc} = \{X \in \mathscr{M}_2^{loc} : t \mapsto X_t$ is continuous $\mathbb{P}$-a.s.$\}$.

Definition 3.6. Suppose X is a real-valued stochastic process on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$. X is a *semimartingale* if X is adapted and has the decomposition

$$X_t = X_0 + M_t + B_t \quad \mathbb{P}\text{-a.s} \quad t \geq 0,$$

where $M = (M_t)_{t\geq 0} \in \mathscr{M}_2^{c,loc}$ and $B = (B_t)_{t\geq 0}$ is the difference of continuous, non-decreasing, adapted stochastic processes $(A_t^{\pm})_{t\geq 0}$ such that

$$B_t := A_t^+ - A_t^- \quad \mathbb{P}\text{-a.s} \quad t \geq 0,$$

and $A_0^{\pm} = 0$. We will assume that this decomposition is the minimal decomposition of B. A^+ is the positive variation of B and A^- is the negative variation of B for $t \geq 0$.

Theorem 3.4. *(Itô's Formula) Let f be a function of class C^2 on $\mathbb{R}^d$ and X be a continuous semimartingale with decomposition as in Definition 3.6 on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$. Then*

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)dM_s + \int_0^t f'(X_s)dB_s + \frac{1}{2}\int_0^t f''(X_s)d\langle M\rangle_s \quad \mathbb{P}\text{-}a.s \quad t \geq 0.$$

Theorem 3.5. *(Multidimensional Itô's Formula) Consider a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$. Suppose $M = (M_t)_{t\geq 0} = \big((M_t^1, M_t^2, \ldots, M_t^d)\big)_{t\geq 0}$ is a vector of local martingales in $\mathscr{M}_2^{c,loc}$ and $B = (B_t)_{t\geq 0} = \big((B_t^1, B_t^2, \ldots, B_t^d)\big)_{t\geq 0}$ is a vector of adapted stochastic processes. Define $X_t := X_0 + M_t + B_t$ for every $t \geq 0$, where X_0 is an $\mathscr{F}_0$-measurable $\mathbb{R}^d$-valued random variable. Let $f(t,x) : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}$ be of class $C^{1,2}$. Then*

$$\begin{aligned} f(t,X_t) = f(0,X_0) &+ \int_0^t \frac{\partial f}{\partial t}(s,X_s)ds + \sum_{i=1}^d \int_0^t \frac{\partial f}{\partial x_i}(s,X_s)dM_s^i + \sum_{i=1}^d \int_0^t \frac{\partial f}{\partial x_i}(s,X_s)dB_s^i \\ &+ \frac{1}{2}\sum_{i=1}^d \sum_{j=1}^d \int_0^t \frac{\partial f}{\partial x_i \partial x_j}(s,X_s)d\langle M^i, M^j\rangle_s \quad \mathbb{P}\text{-}a.s \quad t \geq 0. \end{aligned}$$

3.2 Stochastic Differential Equations

Consider Borel-measurable functions $b_i(t,x), \sigma_{ij}(t,x)$ for $1 \leq i \leq d$ and $1 \leq j \leq r$, that map from $[0,\infty) \times \mathbb{R}^d$ to $\mathbb{R}$. Define the $(d \times 1)$-dimensional *drift* vector and $(d \times r)$-dimensional *dispersion* matrix to be

$$b(t,x) = \{b_i(t,x)\}_{1\leq i\leq d}, \quad \sigma(t,x) = \{\sigma_{ij}(t,x)\}_{1\leq i,j\leq d}.$$

We introduce the concept of a stochastic differential equation. Suppose we have a continuous, adapted, $\mathbb{R}^d$-valued stochastic process $X = (X_t)_{t\geq 0} = \big((X_t^1, X_t^2, \ldots, X_t^d)\big)_{t\geq 0}$ on the filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ that satisfies

$$\mathrm{d}X_t = b(t,X_t)\mathrm{d}t + \sigma(t,X_t)\mathrm{d}W_t, \tag{3.1}$$

expressed componentwise as

$$\mathrm{d}X_t^i = b_i(t,X_t)\mathrm{d}t + \sum_{j=1}^r \sigma_{ij}(t,X_t)\mathrm{d}W_t^j,$$

where $W = (W_t)_{t\geq 0} = \big((W_t^1, W_t^2, \dots, W_t^r)\big)_{t\geq 0}$ is an r-dimensional Brownian motion. Similar to any ordinary/partial differential equation, we call the drift vector and dispersion matrix the *coefficients* of the equation. Also, we define the *diffusion* matrix $a(t,x) := \sigma(t,x)\sigma^T(t,x)$. The aim of this section is to establish the concept of a *strong solution* to the stochastic differential equation.

Definition 3.7. Consider the stochastic differential equation (3.1) on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ with respect to a Brownian motion W and initial condition ζ. We say a continuous stochastic process $X = (X_t)_{t\geq 0}$ is a *strong solution* to the stochastic differential equation if

(i) $(X_t)_{t\geq 0}$ is $\mathscr{F}_t$-adapted,

(ii) $\mathbb{P}(X_0 = \zeta) = 1$,

(iii)
$$\int_0^t |b_i(s, X_s)|\mathrm{d}s + \int_0^t |\sigma_{ij}(s, X_s)|^2 \mathrm{d}s < \infty \quad \mathbb{P}\text{-a.s} \quad t \geq 0 \quad 1 \leq i \leq d \quad 1 \leq j \leq r,$$

(iv)
$$X_t = X_0 + \int_0^t b(s, X_s)\mathrm{d}s + \int_0^t \sigma(s, X_s)\mathrm{d}W_s \quad \mathbb{P}\text{-a.s} \quad t \geq 0,$$

or equivalently componentwise

$$X_t^i = X_0^i + \int_0^t b_i(s, X_s)\mathrm{d}s + \sum_{j=1}^r \int_0^t \sigma_{ij}(s, X_s)\mathrm{d}W_s^j \quad \mathbb{P}\text{-a.s} \quad t \geq 0 \quad 1 \leq i \leq d.$$

Theorem 3.6. *Suppose the coefficients $b(t,x)$ and $\sigma(t,x)$ satisfy global Lipschitz and linear growth conditions*

$$|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq K|x-y|, \tag{3.2}$$

$$|b(t,x)|^2 + |\sigma(t,x)|^2 \leq K^2(1 + |x|^2), \tag{3.3}$$

for every $t \geq 0$, $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^d$. $K > 0$ is a positive constant. Let ζ be an $\mathbb{R}^d$-valued random vector independent of the r-dimensional Brownian motion W on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ and with finite second moment

$$\mathbb{E}\left[|\zeta|^2\right] < \infty.$$

Then there exists a continuous, adapted stochastic process X which is a strong solution to the equation (3.1) relative to W with initial condition ζ. Moreover, this process is square integrable; for every $T > 0$ there exists a constant C, depending on K and T, such that

$$\mathbb{E}\left[|X_t|^2\right] \leq C(1 + \mathbb{E}\left[|\zeta|^2\right])\, e^{Ct} \quad 0 \leq t \leq T.$$

Example 3.1. Consider a strong solution X to the stochastic differential equation (3.1) starting at time t and a function $f : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}$ that is of class $C^{1,2}$. Then by Theorem 3.5, for any $s \geq t$

$$f(s,X_s) = f(t,X_t) + \int_t^s \frac{\partial f}{\partial t}(u,X_u) + \mathscr{L}_u f(u,X_u)\mathrm{d}u + \int_t^s D_x f(u,X_u)^T \sigma(u,X_u)\mathrm{d}W_u,$$

where

$$\mathscr{L}_t f(t,x) = b(t,x)D_x f(t,x) + \frac{1}{2}\mathrm{tr}\left[\sigma(t,x)\sigma^T(t,x)D_{xx}f(t,x)\right],$$

is the infinitesimal operator.

4 Controlled Diffusion Processes

This sub-field of control theory studies dynamic systems subject to random perturbations. Over recent years, the research of control theory has been developed mainly by problems emerging in mathematical finance. We discuss a dynamic programming method for solving controlled diffusion problems formulated within a finite time frame and an infinite time frame. The idea of the approach is to consider a family of optimal control problems and to derive some relations between the associated value functions and initial state values. This yields a second order, non-linear partial differential equation called the Hamilton-Jacobi-Bellman equation. We use as main references for this section the following works: Dana and Jeanblanc [4], Bjork [1] and Pham [18].

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space and $(\mathscr{F}_t)_{t\geq 0}$ be a right-continuous filtration containing all $\mathbb{P}$-null sets. Consider a control model governed by a stochastic differential equation

$$\mathrm{d}X_t = b(t, X_t, \alpha_t)\mathrm{d}t + \sigma(t, X_t, \alpha_t)\mathrm{d}W_t, \tag{4.1}$$

where $W = (W_t)_{t\geq 0} = \left((W_t^1, W_t^2, \ldots, W_t^d)\right)_{t\geq 0}$ is a d-dimensional Brownian motion on $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$, $\alpha = (\alpha_t)_{t\geq 0} = \left((\alpha_t^1, \alpha_t^2, \ldots, \alpha_t^k)\right)_{t\geq 0}$ is an adapted, $\mathbb{R}^k$-valued stochastic process, and b, σ are Borel-measurable functions that map from $[0,\infty)\times\mathbb{R}^d\times\mathbb{R}^k$ to $\mathbb{R}^d$ and $\mathbb{R}^{d\times d}$ respectively. This can be interpreted as having a process X that can be controlled by the choice of a process α. For that reason, we call X a *state process* and α a *control process*. Suppose further that the coefficients $b(t, X_t, \alpha_t)$ and $\sigma(t, X_t, \alpha_t)$ satisfy global Lipschitz and linear growth conditions

$$|b(t,x,a) - b(t,y,a)| + |\sigma(t,x,a) - \sigma(t,y,a)| \leq K|x-y|, \tag{4.2}$$

$$|b(t,x,a)|^2 + |\sigma(t,x,a)|^2 \leq K^2(1+|x|^2), \tag{4.3}$$

for every $t \geq 0$, $x \in \mathbb{R}^d$, $y \in \mathbb{R}^d$ and $a \in \mathbb{R}^k$. $K > 0$ is a positive constant.

4.1 Optimal Control Problem

Consider a finite time frame with horizon $0 < T < \infty$. Define $\mathscr{A}$ to be the set of all control processes α such that

$$\mathbb{E}\left[\int_0^T \left(|b(t,0,\alpha_t)|^2 + |\sigma(t,0,\alpha_t)|^2\right)\mathrm{d}t\right] < \infty. \tag{4.4}$$

Then by Theorem 3.6, properties (4.2), (4.3) and (4.4) ensures the existence and uniqueness of a strong solution to (4.1) for all $\alpha \in \mathscr{A}$ and initial conditions $(t,x) \in [0,T]\times\mathbb{R}^d$. We denote this strong solution as $X_s^{t,x}$ for any $t \leq s \leq T$. Let $f : [0,T]\times\mathbb{R}^d\times\mathbb{R}^k \to \mathbb{R}$ and $g : \mathbb{R}^d \to \mathbb{R}$ be Borel-measurable functions. Let g satisfy a quadratic growth conditions and be lower bounded

$$|g(x)| \leq K(1+|x|^2) \quad g(x) \geq M,$$

for every $x \in \mathbb{R}^d$. $K > 0$ and $M > 0$ are positive constants. Then for some initial condition $(t,x) \in [0,T] \times \mathbb{R}^d$, we define the set of *admissible controls* $\mathscr{A}(t,x)$ as the set of $\alpha \in \mathscr{A}$ such that

$$\mathbb{E}\left[\int_t^T |f(s, X_s^{t,x}, \alpha_s)| \mathrm{d}s\right] < \infty.$$

Assuming that $\mathscr{A}(t,x)$ is non-empty and bounded, we define the *gain function*

$$J(t,x,\alpha) = \mathbb{E}\left[\int_t^T f(s, X_s^{t,x}, \alpha_s)\mathrm{d}s + g(X_T^{t,x})\right],$$

for any $(t,x) \in [0,T] \times \mathbb{R}^d$ and $\alpha \in \mathscr{A}(t,x)$. The objective is to maximise the gain function over all control processes. We define the corresponding *value function*

$$V(t,x) = \sup_{\alpha \in \mathscr{A}(t,x)} J(t,x,\alpha). \tag{4.5}$$

The *optimal control process* is $\alpha^* \in \mathscr{A}(t,x)$ such that $V(t,x) = J(t,x,\alpha^*)$.

Alternatively, consider an infinite time frame with horizon $T = \infty$. Define $\mathscr{A}$ to be the set of all control processes α such that

$$\mathbb{E}\left[\int_0^T \left(|b(t,0,\alpha_t)|^2 + |\sigma(t,0,\alpha_t)|^2\right) \mathrm{d}t\right] < \infty \quad \forall T > 0. \tag{4.6}$$

Then by Theorem 3.6, properties (4.2), (4.3) and (4.6) ensures the existence and uniqueness of a strong solution to (4.1) for all $\alpha \in \mathscr{A}$ and initial condition $x \in \mathbb{R}^d$. We denote this strong solution as X_s^x for any $s \geq 0$. Let $\beta > 0$ be a positive constant and $f : \mathbb{R}^d \times \mathbb{R}^k \to \mathbb{R}$ be a Borel-measurable function. Then for some initial condition $x \in \mathbb{R}^d$, we define the set of *admissible controls* $\mathscr{A}(x)$ as the set of $\alpha \in \mathscr{A}$ such that

$$\mathbb{E}\left[\int_0^\infty e^{-\beta s} |f(s, X_s^x, \alpha_s)| \mathrm{d}s\right] < \infty.$$

Assuming that $\mathscr{A}(x)$ is non-empty and bounded, we define the *gain function*

$$J(x,\alpha) = \mathbb{E}\left[\int_0^\infty e^{-\beta s} f(s, X_s^x, \alpha_s)\mathrm{d}s\right],$$

for any $x \in \mathbb{R}^d$ and $\alpha \in \mathscr{A}(x)$. The objective is to maximise the gain function over all control processes. We define the corresponding *value function*

$$V(x) = \sup_{\alpha \in \mathscr{A}(x)} J(x,\alpha). \tag{4.7}$$

The *optimal control process* is $\alpha^* \in \mathscr{A}(x)$ such that $V(x) = J(x,\alpha^*)$.

4.2 Dynamic Programming

The dynamic programming principle is an essential concept in the theory of analysing stochastic control problems. It refers to simplifying a problem by breaking it down into simpler sub-problems in a recursive manner. In the context of controlled diffusion processes described in the previous subsection, it is formulated as follows.

Theorem 4.1. *(Dynamic Programming Principle) Consider a finite time frame with horizon $0 < T < \infty$ and an initial condition $(t,x) \in [0,T] \times \mathbb{R}^d$. Then*

$$V(t,x) = \sup_{\alpha \in \mathscr{A}(t,x)} \mathbb{E}\left[\int_t^s f(u, X_u^{t,x}, \alpha_u) du + V(s, X_s^{t,x})\right],$$

for any $t \leq s \leq T$. Similarly, for an infinite time frame with horizon $T = \infty$ and an initial condition $x \in \mathbb{R}^d$

$$V(x) = \sup_{\alpha \in \mathscr{A}(x)} \mathbb{E}\left[\int_0^s e^{-\beta u} f(X_u^{t,x}, \alpha_u) du + e^{-\beta s} V(X_s^x)\right],$$

for any $s \geq 0$.

Proof. We first prove the instance with finite horizon $0 < T < \infty$ and the initial condition $(t,x) \in [0,T] \times \mathbb{R}^d$. By the pathwise uniqueness of a solution X to (4.1), we have the Markovian structure

$$X_u^{t,x} = X_u^{s, X_s^{t,x}} \quad u \geq s,$$

for any $t \leq s \leq T$. Let $\alpha \in \mathscr{A}(t,x)$ be an arbitrary admissible control. Then using the Markovian structure of X

$$\begin{aligned} J(t,x,\alpha) &= \mathbb{E}\left[\int_t^s f(u, X_u^{t,x}, \alpha_u) \mathrm{d}u + \int_s^T f(u, X_u^{t,x}, \alpha_u) \mathrm{d}u + g(X_T^{t,x})\right] \\ &= \mathbb{E}\left[\int_t^s f(u, X_u^{t,x}, \alpha_u) \mathrm{d}u\right] + \mathbb{E}\left[J(s, X_s^{t,x}, \alpha)\right] \\ &\leq \inf_{s \in [t,T]} \mathbb{E}\left[\int_t^s f(u, X_u^{t,x}, \alpha_u) \mathrm{d}u + V(s, X_s^{t,x})\right]. \end{aligned}$$

Since α is arbitrarily chosen, we can take the supremum over admissible controls on both sides to obtain

$$V(t,x) \leq \sup_{\alpha \in \mathscr{A}(t,x)} \inf_{s \in [t,T]} \mathbb{E}\left[\int_t^s f(u, X_u^{t,x}, \alpha_u) \mathrm{d}u + V(s, X_s^{t,x})\right]. \tag{4.8}$$

Let $\alpha \in \mathscr{A}(t,x)$ be an arbitrary admissible control and $t \leq s \leq T$. By the definition of the value function, for any sample $\omega \in \Omega$ and $\varepsilon > 0$ there exists a control $\alpha^{\varepsilon,\omega}(\omega) \in \mathscr{A}(s(\omega), X_{s(\omega)}^{t,x}(\omega))$ that is an ε-optimal control for $V(s(\omega), X_{s(\omega)}^{t,x}(\omega))$

$$V(s(\omega), X_{s(\omega)}^{t,x}(\omega)) - \varepsilon \leq J(s(\omega), X_{s(\omega)}^{t,x}(\omega), \alpha^{\varepsilon,\omega}).$$

Define the control process

$$\hat{\alpha}_u = \begin{cases} \alpha_u(\omega) & 0 \le u \le s \\ \alpha_u^{\varepsilon,\omega} & s < u \le T \end{cases}.$$

Using the Markovian structure of X

$$\begin{aligned} V(t,x) \ge J(x,t,\hat{\alpha}) &= \mathbb{E}\left[\int_t^s f(u,X_u^{t,x},\alpha_u)\mathrm{d}u + \int_s^T f(u,X_u^{t,x},\alpha_u^{\varepsilon,\omega})\mathrm{d}u + g(X_T^{t,x})\right] \\ &= \mathbb{E}\left[\int_t^s f(u,X_u^{t,x},\alpha_u)\mathrm{d}u\right] + \mathbb{E}\left[J(s,X_s^{t,x},\alpha^{\varepsilon,\omega})\right] \\ &\ge \mathbb{E}\left[\int_t^s f(u,X_u^{t,x},\alpha_u)\mathrm{d}u + V(s,X_s^{t,x})\right] - \varepsilon. \end{aligned}$$

Since α and s are arbitrarily chosen, we can take the supremum over admissible controls and chosen s on the right-hand side to obtain

$$V(t,x) \ge \sup_{\alpha\in\mathscr{A}(t,x)} \sup_{s\in[t,T]} \mathbb{E}\left[\int_t^s f(u,X_u^{t,x},\alpha_u)\mathrm{d}u + V(s,X_s^{t,x})\right]. \tag{4.9}$$

By combining relations (4.8) and (4.9), we prove the first statement of the theorem. We now prove the instance with an infinite horizon $T=\infty$ and an initial condition $x\in\mathbb{R}^d$. By the pathwise uniqueness of a solution X to (4.1), for any $s\ge 0$ we have the Markovian structure

$$X_u^x = X_u^{X_s^x} \quad u \ge s.$$

Let $\alpha\in\mathscr{A}(x)$ be an arbitrary admissible control. Then using the Markovian structure of X

$$\begin{aligned} J(x,\alpha) &= \mathbb{E}\left[\int_0^s e^{-\beta u} f(u,X_u^x,\alpha_u)\mathrm{d}u + \int_s^\infty e^{-\beta u} f(u,X_u^x,\alpha_u)\mathrm{d}u\right] \\ &= \mathbb{E}\left[\int_0^s e^{-\beta u} f(u,X_u^x,\alpha_u)\mathrm{d}u\right] + \mathbb{E}\left[e^{-\beta s} J(X_s^x,\alpha)\right] \\ &\le \inf_{s\ge 0} \mathbb{E}\left[\int_0^s e^{-\beta u} f(u,X_u^x,\alpha_u)\mathrm{d}u + e^{-\beta s} V(X_s^x)\right]. \end{aligned}$$

Since α is arbitrarily chosen, we can take the supremum over admissible controls on both sides to obtain

$$V(x) \le \sup_{\alpha\in\mathscr{A}(x)} \inf_{s\ge 0} \mathbb{E}\left[\int_0^s e^{-\beta u} f(u,X_u^x,\alpha_u)\mathrm{d}u + e^{-\beta s} V(X_s^x)\right]. \tag{4.10}$$

Let $\alpha\in\mathscr{A}(x)$ be an arbitrary admissible control and $s\ge 0$. By the definition of the value function, for any sample $\omega\in\Omega$ and $\varepsilon>0$ there exists a control $\alpha^{\varepsilon,\omega}(\omega)\in\mathscr{A}(X_{s(\omega)}^x(\omega))$ that is an ε-optimal control for $V(X_{s(\omega)}^x(\omega))$

$$V(X_{s(\omega)}^x(\omega)) - \varepsilon \le J(X_{s(\omega)}^x(\omega),\alpha^{\varepsilon,\omega}).$$

Define the control process

$$\hat{\alpha}_u = \begin{cases} \alpha_u(\omega) & 0 \le u \le s \\ \alpha_u^{\varepsilon,\omega} & s < u \end{cases}.$$

Then using the Markovian structure of X

$$\begin{aligned} V(x) \ge J(x,\tilde{\alpha}) &= \mathbb{E}\left[\int_0^s e^{-\beta u} f(u, X_u^x, \alpha_u)\mathrm{d}u + \int_s^\infty e^{-\beta u} f(u, X_u^x, \alpha_u^{\varepsilon,\omega})\mathrm{d}u\right] \\ &= \mathbb{E}\left[\int_0^s e^{-\beta u} f(u, X_u^x, \alpha_u)\mathrm{d}u\right] + \mathbb{E}\left[e^{-\beta s} J(X_s^x, \alpha^{\varepsilon,\omega})\right] \\ &\ge \mathbb{E}\left[\int_0^s e^{-\beta u} f(u, X_u^x, \alpha_u)\mathrm{d}u + e^{-\beta s} V(X_s^x)\right] - \varepsilon e^{-\beta s}. \end{aligned}$$

Since α and s are arbitrarily chose, we can take the supremum over admissible controls and chosen s on the right-hand side to obtain

$$V(x) \ge \sup_{\alpha \in \mathscr{A}(x)} \sup_{s \ge 0} \mathbb{E}\left[\int_0^s e^{-\beta u} f(u, X_u^x, \alpha_u)\mathrm{d}u + e^{-\beta s} V(X_s^x)\right]. \tag{4.11}$$

By combining relations (4.10) and (4.11), we prove the second statement of the theorem. $\square$

4.3 Hamilton-Jacobi-Bellman Equation

Using the dynamic programming principle, we show that the value function is a solution to the partial differential equation known as the Hamilton-Jacobi-Bellman (HJB) equation. It describes the local behaviour of the dynamic programming principle when s tends to t in Theorem 4.1. We present a formal derivation.

Consider the optimal control problem in a finite time frame with horizon $0 < T < \infty$. Let X be a strong solution to (4.1) with an initial condition $(t,x) \in [0,T] \times \mathbb{R}^d$. Let the value function V be of class $C^{1,2}([0,T] \times \mathbb{R}^d)$. Then for any $s = t + h$ and control process $\alpha \in \mathscr{A}(t,x)$, Theorem 4.1 yields

$$V(t,x) \ge \mathbb{E}\left[\int_t^{t+h} f(u, X_u^{t,x}, \alpha_u)\mathrm{d}u + V(t+h, X_{t+h}^{t,x})\right]. \tag{4.12}$$

By Itô's Formula as in Example 3.1

$$V(t+h, X_{t+h}^{t,x}) = V(t,x) + \int_t^{t+h} \frac{\partial V}{\partial t}(u, X_u^{t,x}) + \mathscr{L}_u^{\alpha} V(u, X_u^{t,x})\mathrm{d}u + \text{local martingale}, \tag{4.13}$$

where $\mathscr{L}^{\alpha}$ is the infinitesimal operator for the control α is defined by

$$\mathscr{L}_t^{\alpha} V(t,x) = b(t,x,\alpha_t) D_x V(t,x) + \frac{1}{2}\mathrm{tr}\left[\sigma(t,x,\alpha_t)\sigma^T(t,x,\alpha_t) D_{xx} V(t,x)\right].$$

By substituting (4.13) into (4.12) we obtain

$$\mathbb{E}\left[\int_t^{t+h}\left(f(u,X_u^{t,x},\alpha_u)+\frac{\partial V}{\partial t}(u,X_u^{t,x})+\mathscr{L}_u^{\alpha}V(u,X_u^{t,x})\right)\mathrm{d}u\right]\leq 0.$$

Dividing by h and considering the case when h tends to 0 yields

$$\frac{\partial V}{\partial t}(t,x)+\mathscr{L}^{\alpha}V(t,x)+f(t,x,\alpha_t)\leq 0.$$

Since α is arbitrarily chosen, we can take the supremum over admissible controls on the left-hand side to obtain,

$$-\frac{\partial V}{\partial t}(t,x)-\sup_{\alpha\in\mathscr{A}(t,x)}[\mathscr{L}^{\alpha}V(t,x)+f(t,x,\alpha_t)]\geq 0.$$

On the other hand, suppose that α^* is an optimal control. Then

$$V(t,x)=J(t,x,\alpha^*)=\mathbb{E}\left[\int_t^{t+h}f(u,X_u^{t,x},\alpha_u^*)\mathrm{d}u+V(t+h,X_{t+h}^{t,x})\right].$$

By repeating the same argument, we find that

$$\frac{\partial V}{\partial t}(t,x)+\mathscr{L}^{\alpha^*}V(t,x)+f(t,x,\alpha_t^*)=0.$$

By the boundedness of the set $\mathscr{A}(t,x)$, we conclude that

$$\frac{\partial V}{\partial t}(t,x)+\sup_{\alpha\in\mathscr{A}(t,x)}[\mathscr{L}^{\alpha}V(t,x)+f(t,x,\alpha_t)]=0\quad\forall(t,x)\in[0,T]\times\mathbb{R}^d.$$

This equation is called the HJB equation and is usually assigned the terminal condition

$$V(T,x)=g(x)\quad\forall x\in\mathbb{R}^d.$$

By using the same method, we derive the HJB equation for the optimal control problem in an infinite time frame with horizon $T=\infty$.

$$\beta V(x)-\sup_{\alpha\in\mathscr{A}(x)}[\mathscr{L}^{\alpha}V(x)+f(0,x,\alpha)]=0\quad\forall x\in\mathbb{R}^d.$$

4.4 Verification Step

In this section, we show that a smooth candidate solution to the HJB equation coincides with the corresponding value function: (4.5) in a finite time frame and (4.7) in an infinite time frame.

Theorem 4.2. *(Finite Horizon Verification Theorem) Let Y be a function of class $C^{1,2}([0,T)\times\mathbb{R}^d)\bigcap C^0([0,T]\times\mathbb{R}^d)$ that satisfies a quadratic growth condition*

$$|Y(t,x)|\leq K(1+|x|^2),$$

for every $(t,x) \in [0,T] \times \mathbb{R}^d$*.* $K > 0$ *is a positive constant.*

(i) Suppose that,

$$-\frac{\partial Y}{\partial t}(t,x) - \sup_{\alpha \in \mathscr{A}(t,x)} [\mathscr{L}^{\alpha} Y(t,x) + f(t,x,\alpha_t)] \geq 0 \quad \forall (t,x) \in [0,T) \times \mathbb{R}^d,$$

$$Y(t,x) \geq g(x) \quad \forall x \in \mathbb{R}^d.$$

Then $Y \geq V$ *on* $[0,T] \times \mathbb{R}^d$*.*

(ii) Suppose further that $Y(t,x) = g(x)$ *for every* $x \in \mathbb{R}^d$ *and that there exists a measurable function* $\hat{\alpha}$ *valued in* $\mathbb{R}^k$ *such that*

$$-\frac{\partial Y}{\partial t}(t,x) - \sup_{\alpha \in \mathscr{A}(t,x)} [\mathscr{L}^{\alpha} Y(t,x) + f(t,x,\alpha_t)]$$
$$= -\frac{\partial Y}{\partial t}(t,x) - \mathscr{L}^{\hat{\alpha}} Y(t,x) - f(t,x,\hat{\alpha}_t) = 0,$$

(4.1) with initial condition $(t,x) \in [0,T] \times \mathbb{R}^d$ *admits a unique strong solution* $\hat{X}_s^{t,x}$ *for* $t \leq s \leq T$ *and the process* $(\hat{\alpha}(s, \hat{X}_s^{t,x}))_{s \in [t,T]}$ *is an optimal control that lies in* $\mathscr{A}(t,x)$*. Then* $Y = V$ *on* $[0,T] \times \mathbb{R}^d$*.*

Proof. We first show a proof of (i). Consider a stopping time τ valued in the interval $[t,\infty)$. Then by Itô's Formula as in Example 3.1

$$Y(s \wedge \tau, X_{s\wedge\tau}^{t,x}) = Y(t,x) + \int_t^{s\wedge\tau} \frac{\partial Y}{\partial t}(u, X_u^{t,x}) + \mathscr{L}^{\alpha} Y(u, X_u^{t,x}) \mathrm{d}u$$
$$+ \int_t^{s\wedge\tau} D_x Y(u, X_u^{t,x})^T \sigma(u, X_u^{t,x}, \alpha_u) \mathrm{d}W_u.$$

Consider the sequence of stopping times τ_n

$$\tau_n := \inf\left\{ s \geq t : \int_t^s |D_x Y(u, X_u^{t,x})^T \sigma(u, X_u^{t,x}, \alpha_u)|^2 \mathrm{d}u \geq n \right\}.$$

Then $\tau = \tau_n$ and $\tau_n \uparrow \infty$ as $n \to \infty$. Moreover, the stopped process

$$\left(\int_t^{s\wedge\tau_n} D_x Y(u, X_u^{t,x})^T \sigma(u, X_u^{t,x}, \alpha_u) \mathrm{d}W_u \right)_{s \in [t,T]},$$

is a martingale. Taking expectations on both sides we obtain

$$\mathbb{E}\left[Y(s \wedge \tau_n, X_{s\wedge\tau_n}^{t,x}) \right] = Y(t,x) + \mathbb{E}\left[\int_t^{s\wedge\tau_n} \left(\frac{\partial Y}{\partial t}(u, X_u^{t,x}) + \mathscr{L}^{\alpha} Y(u, X_u^{t,x}) \right) \mathrm{d}u \right].$$

Using our first assumption we obtain

$$\mathbb{E}\left[Y(s\wedge\tau_n, X^{t,x}_{s\wedge\tau_n})\right] \le Y(t,x) - \mathbb{E}\left[\int_t^{s\wedge\tau_n} f(u, X^{t,u}_u, \alpha_u)\mathrm{d}u\right] \quad \forall\alpha\in\mathscr{A}(t,x).$$

Notice that

$$\left|\int_t^{s\wedge\tau_n} f(u, X^{t,u}_u, \alpha_u)\mathrm{d}u\right| \le \int_t^T |f(u, X^{t,x}_u, \alpha_u)|\mathrm{d}u < \infty,$$

which is due to the integrability of controls in $\mathscr{A}(t,x)$. Moreover, Y satisfies a quadratic growth condition

$$\left|Y(s\wedge\tau_n, X^{t,x}_{s\wedge\tau_n})\right| \le K(1+\sup_{s\in[t,T]}|X^{t,x}_s|^2).$$

Then we can apply the dominated convergence theorem and let n tend to infinity

$$\mathbb{E}\left[Y(s, X^{t,x}_s)\right] \le Y(t,x) - \mathbb{E}\left[\int_t^s f(u, X^{t,x}_u, \alpha_u)\mathrm{d}u\right] \quad \forall\alpha\in\mathscr{A}(t,x).$$

Since Y is continuous on $[0,T]\times\mathbb{R}^d$, letting s tend to T

$$\mathbb{E}\left[g(X^{t,x}_T)\right] \le \mathbb{E}\left[Y(T, X^{t,x}_T)\right] \le Y(t,x) - \mathbb{E}\left[\int_t^T f(u, X^{t,x}_u, \alpha_u)\mathrm{d}u\right] \quad \forall\alpha\in\mathscr{A}(t,x).$$

Equivalently, for any $\alpha\in\mathscr{A}(t,x)$

$$Y(t,x) \ge J(t,x,\alpha) = \mathbb{E}\left[\int_t^T f(u, X^{t,x}_u, \alpha_u)\mathrm{d}u + g(X^{t,x}_T)\right],.$$

Thus, $Y \ge V$ on $[0,T]\times\mathbb{R}^d$.

We now prove (ii). Consider $Y(s,\hat{X}^{t,x}_s)$ for some time points $0\le t<T$ and $t\le s<T$ where $\hat{X}$ is defined as in the statement of the theorem. Then by Itô's formula as in Example 3.1

$$\mathbb{E}\left[Y(s,\hat{X}^{t,x}_s)\right] = Y(t,x) + \mathbb{E}\left[\int_t^s \frac{\partial Y}{\partial t}(u,\hat{X}^{t,x}_u) + \mathscr{L}^{\hat{\alpha}}Y(u,\hat{X}^{t,x}_u)\mathrm{d}u\right].$$

By the definition of $\hat{\alpha}$

$$-\frac{\partial Y}{\partial t}(t,x) - \mathscr{L}^{\hat{\alpha}}Y(t,x) - f(t,x,\hat{\alpha}_t) = 0,$$

and by combining these equations we obtain

$$\mathbb{E}\left[Y(s,\hat{X}^{t,x}_s)\right] = Y(t,x) - \mathbb{E}\left[\int_t^s f(u,\hat{X}^{t,x}_u, \hat{\alpha}(u,\hat{X}^{t,x}_u))\mathrm{d}u\right].$$

Since Y is continuous on $[0,T]\times\mathbb{R}^d$, letting s tend to T

$$Y(t,x) = \mathbb{E}\left[\int_t^T f(u,\hat{X}^{t,x}_u, \hat{\alpha}(u,\hat{X}^{t,x}_u))\mathrm{d}u + g(\hat{X}^{t,x}_T)\right] = J(t,x,\hat{\alpha}) \le V(t,x).$$

Thus, $Y = V$ on $[0,T] \times \mathbb{R}^d$ and $\hat{\alpha}$ is an optimal control that lies in $\mathscr{A}(t,x)$. $\square$

We can repeat the same arguments to prove a similar statement when we have an infinite time frame with horizon $T = \infty$.

Theorem 4.3. *(Infinite Horizon Verification Theorem) Let Y be a function of class $C^2(\mathbb{R}^d)$ that satisfies a quadratic growth condition*

$$|Y(x)| \leq K(1+|x|^2),$$

for every $x \in \mathbb{R}^d$. $K > 0$ is a positive constant.

(i) Suppose that

$$-\beta Y(x) - \sup_{\alpha \in \mathscr{A}(x)} [\mathscr{L}^{\alpha} Y(x) + f(0,x,\alpha)] \geq 0 \quad \forall x \in \mathbb{R}^d,$$
$$\limsup_{T \to \infty} e^{-\beta T} \mathbb{E}[Y(X_T^x)] \geq 0 \quad \forall x \in \mathbb{R}^d, \alpha \in \mathscr{A}(x).$$

Then $Y \geq V$ on $\mathbb{R}^d$.

(ii) Suppose further that there exists a measurable function $\hat{\alpha}$ valued in $\mathbb{R}^k$ such that

$$\beta Y(x) - \sup_{\alpha \in \mathscr{A}(x)} [\mathscr{L}^{\alpha} Y(x) + f(0,x,\alpha)] = \beta Y(x) - \mathscr{L}^{\hat{\alpha}} Y(x) - f(0,x,\hat{\alpha}) = 0,$$

(4.1) with initial condition $x \in \mathbb{R}^d$ admits a unique strong solution $\hat{X}_s^x$ for $s \geq 0$ and the process $(\hat{\alpha}(\tilde{X}_s^x))_{s \geq 0}$ is an optimal control that lies in $\mathscr{A}(x)$. Then $Y = V$ on $\mathbb{R}^d$.

Proof. We first show a proof of (i). Consider the sequence of stopping times τ_n

$$\tau_n := \inf\left\{ t \geq 0 : \int_0^t |D_x Y(u, X_u^X)^T \sigma(u, X_u^X, \alpha_u)|^2 \mathrm{d}u \geq n \right\}.$$

Then by Itô's Formula as in Example 3.1

$$e^{-\beta(T \wedge \tau_n)} Y(^{x}_{T \wedge \tau_n}) = Y(x) + \int_0^{T \wedge \tau_n} e^{-\beta u} (\mathscr{L}^{\alpha} Y(X_u^x) - \beta Y(X_u^x)) \, \mathrm{d}u$$
$$+ \int_0^{T \wedge \tau_n} e^{-\beta u} DY(X_u^x)^T \sigma(u, X_u^x, \alpha_u) \mathrm{d}W_u.$$

Then the stopped stochastic integral is a martingale. Taking expectations on both sides we obtain

$$\mathbb{E}\left[e^{-\beta(T \wedge \tau_n)} Y(^{x}_{T \wedge \tau_n}) \right] = Y(x) + \mathbb{E}\left[\int_0^{T \wedge \tau_n} e^{-\beta u} (\mathscr{L}^{\alpha} Y(X_u^x) - \beta Y(X_u^x)) \, \mathrm{d}u \right].$$

Using our first assumption we obtain

$$\mathbb{E}\left[e^{-\beta(T\wedge\tau_n)}Y({}^{x}_{T\wedge\tau_n})\right] \le Y(x) - \mathbb{E}\left[\int_0^{T\wedge\tau_n} e^{-\beta u} f(0, X_u^x, \alpha_u)\mathrm{d}u\right] \quad \forall \alpha \in \mathscr{A}(x).$$

By the same argument as in the proof of Theorem 4.2, we can show that the right-hand side is integrable by the quadratic growth condition and integrability of controls in $\mathscr{A}(x)$. Then we can apply the dominated convergence theorem and let n tend to infinity

$$\mathbb{E}\left[e^{-\beta T}Y(X_T^x)\right] \le Y(x) - \mathbb{E}\left[\int_0^{T} e^{-\beta u} f(0, X_u^x, \alpha_u)\mathrm{d}u\right] \quad \forall \alpha \in \mathscr{A}(x).$$

Letting T tend to infinity

$$Y(x) \ge J(x,\alpha) = \mathbb{E}\left[\int_0^{\infty} e^{-\beta u} f(0, X_u^x, \alpha_u)\mathrm{d}u\right] \quad \forall \alpha \in \mathscr{A}(x),$$

hence $Y \ge V$ on $[0,T] \times \mathbb{R}^d$.

We now prove (ii). We can repeat the method used in the proof of (i) and observe that for $\hat{\alpha}$

$$\mathbb{E}\left[e^{-\beta T}Y(\hat{X}_T^x)\right] \le Y(x) - \mathbb{E}\left[\int_0^{T} e^{-\beta u} f(0, \hat{X}_u^x, \hat{\alpha}(\hat{X}_u^x))\mathrm{d}u\right].$$

Letting T tend to infinity

$$Y(x) \le J(x,\hat{\alpha}) = \mathbb{E}\left[\int_0^{\infty} e^{-\beta u} f(0, \hat{X}_u^x, \hat{\alpha}(\hat{X}_u^x))\mathrm{d}u\right].$$

Hence $Y = V$ on $\mathbb{R}^d$ and $\hat{\alpha}$ is an optimal control that lies in $\mathscr{A}(x)$. $\square$

5 Merton Problems With Consumption

In this section, we present an application of the theory in Chapter 4. In particular, we present examples where the HJB equation has an analytic solution. We discuss and solve the portfolio optimisation problems proposed by Merton in his works [13] and [14]. We use these papers as our main references alongside Rogers [20].

Consider a complete and frictionless 2-dimensional continuous-time financial market $(S_t)_{t\in[0,T]} = ((S_t^0, S_t^1))_{t\in[0,T]}$ composed of a riskless asset S^0 and a risky asset S^1 on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, \mathbb{P})$. We assume that this is a complete probability space and $(\mathscr{F}_t)_{t\in[0,T]}$ is a right-continuous filtration containing all $\mathbb{P}$-null sets. This means that there are no transaction costs; assets can be bought and sold at the same price. Moreover, we shall assume that asset prices are exogenously given and not influenced by the trading activities of other market participants. If an arbitrage opportunity exists in the market, participants need not worry about determining an optimal trading strategy as there is a chance to make risk-free profit. Hence, we consider $(S_t)_{t\in[0,T]}$ to be arbitrage-free.

We also assume that the assets can be modelled by a geometric Brownian motion as in the Black-Scholes model. The riskless asset price is modelled by the differential equation

$$\mathrm{d}S_t^0 = rS_t^0 \mathrm{d}t \quad t \in [0,T].$$

The risky asset price is modelled by the stochastic differential equation

$$\mathrm{d}S_t^1 = \mu S_t^1 \mathrm{d}t + \sigma S_t^1 \mathrm{d}W_t \quad t \in [0,T],$$

where $\mu \in \mathbb{R}$, $\sigma > 0$ and $r \geq 0$. Moreover, $W = (W_t)_{t\in[0,T]}$ is a 1-dimensional Brownian motion on $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,T]}, \mathbb{P})$. At any time t, a market participant invests a proportion of their wealth α_t into the financial market. This is often called a trading strategy. At any time t, they may also consume wealth represented by a non-negative stochastic process c_t. Their wealth process can be expressed as

$$X_t = \alpha_t \cdot S_t - c_t,$$

and evolves according to

$$\begin{aligned}\mathrm{d}X_t = \alpha_t \cdot \mathrm{d}S_t &= \alpha_t^0 \mathrm{d}S_t^0 + \alpha_t^1 \mathrm{d}S_t^1 - c_t \mathrm{d}t = r\alpha_t^0 S_t^0 \mathrm{d}t + \mu \alpha_t^1 S_t^1 \mathrm{d}t + \sigma \alpha_t^1 S_t^1 \mathrm{d}W_t - c_t \mathrm{d}t \\ &= r\alpha_t^0 S_t^0 \mathrm{d}t + r\alpha_t^1 S_t^1 \mathrm{d}t + (\mu - r)\alpha_t^1 S_t^1 \mathrm{d}t + \sigma \alpha_t^1 S_t^1 \mathrm{d}W_t - c_t \mathrm{d}t.\end{aligned}$$

We denote the amount held in the risky asset by $\pi_t := \alpha_t^1 S_t^1$ and the market price of risk of the asset (Sharpe ratio) by $\lambda := \frac{\mu - r}{\sigma}$. The equation governing wealth simplifies to

$$\mathrm{d}X_t = rX_t \mathrm{d}t + \sigma \pi_t (\lambda \mathrm{d}t + \mathrm{d}W_t) - c_t \mathrm{d}t,$$

for all $t \in [0,T]$. We denote by $\mathscr{A}$ the set of adapted stochastic processes (π, c) such that

(i)
$$\int_0^T |\pi_t|^2 \mathrm{d}t < \infty \quad \mathbb{P}\text{-a.s},$$

(ii)
$$\int_0^T c_t \mathrm{d}t < \infty \quad \mathbb{P}\text{-a.s},$$

(iii)
$$X_t \geq 0 \quad \mathbb{P}\text{-a.s} \quad t \in [0,T].$$

Property (iii) states that an admissible control does not allow the participant to fall into debt. We have seen in Chapter 4.1 that assumptions (i) and (ii) assert the existence and uniqueness of a strong solution to the stochastic differential equation governing the wealth process of a market participant. Given a portfolio and consumption $(\pi, c) \in \mathscr{A}$, we denote the corresponding wealth process with initial wealth $x \in \mathbb{R}$ at time $t \in [0,T]$ as $X^{t,x;\pi,c}$. We will often drop the superscript here and write X instead.

5.1 Utility

We want to systematically describe preferences of a market participant who has to compare random outcomes like the return of a portfolio. The axiomatic approach proposed by von Neumann and Morgenstern show that they can be represented through the expectation of some function called utility.

Definition 5.1. (Utility Function) Let $I \subset \mathbb{R}$ be a non-empty interval. A function $U : I \mapsto \mathbb{R}$ is called a utility function if

(i) U is continuous, strictly increasing and strictly concave,

(ii) $U'(x) = 0$ as $x \to \infty$.

Remark. Clearly, (i) asserts that the function U' is strictly decreasing hence admits a continuous inverse U^{-1} defined on $(U'(\infty), U'(0))$. Moreover, (ii) asserts that U^{-1} is defined on $(0, U'(0))$.

Definition 5.2. Suppose $U : I \mapsto \mathbb{R}$ is a utility function. Then the *coefficient of absolute risk aversion* associated to U is
$$A(x) = -\frac{U''(x)}{U'(x)}.$$

The *coefficient of relative risk aversion* associated to U is

$$R(x) = -\frac{xU''(x)}{U'(x)}.$$

Example 5.1. (Standard Utility Functions)

Some standard examples of utility functions are

(i) *Logarithmic Utility*

$$U(x) = \log(x) \quad x > 0.$$

(ii) *Power Utility* with parameter $p < 1$ and $p \neq 0$

$$U(x) = \frac{x^p}{p} \quad x \geq 0.$$

For both the power utility and the logarithmic utility, the coefficient of absolute risk aversion $A(x) \propto \frac{1}{x}$. Hence, they are often called *hyperbolic absolute risk aversion* (HARA) utilities. The coefficient of relative risk aversion associated to power utility is constant, $R(x) = 1 - p$. Hence, power utility is often called a *constant relative risk of aversion* (CRRA) utility.

5.2 Problem Formulation

The goal of the participant in the financial market $(S_t)_{t \in [0,T]}$ is to maximise the expected utility from their consumption and wealth at an evaluation period $t = T$. They start participating in the market at time $t \in [0,T]$ with some wealth $x \in \mathbb{R}$ and continue to participate until the evaluation period. We suppose that the participant may have different utilities describing their consumption and terminal wealth preferences, U_1 and U_2 respectively. The value function of this utility maximisation problem is then defined as

$$V(t,x) = \sup_{(\pi,c) \in \mathscr{A}(t,x)} \mathbb{E}\left[\int_t^T e^{-\delta s} U_1(c_s) \mathrm{d}s + U_2(X_T^{t,x;\pi,c})\right],$$

where $\delta > 0$ is a positive constant representative of a subjective discount rate that measures the desire to spend sooner rather than later (social time preference). Using the derivation of the HJB equation in Chapter 4.3, the corresponding HJB equation is

$$-\frac{\partial V}{\partial t}(t,x) - \sup_{(\pi,c) \in \mathscr{A}(t,x)} \left[e^{-\delta t} U_1(c_t) + \mathscr{L}^{(\pi,c)} V(t,x)\right] = 0,$$

with terminal condition

$$V(T,x) = U_2(x) \quad \forall x \in \mathbb{R},$$

where

$$\mathscr{L}^{(\pi,c)} V(t,x) = (rx + \sigma\lambda\pi_t - c_t)\frac{\partial V}{\partial x}(t,x) + \frac{1}{2}\sigma^2\pi_t^2 \frac{\partial^2 V}{\partial x^2}(t,x).$$

We present an analytical solution to this equation when the utility describing their consumption and terminal wealth is of power type.

Example 5.2. (Consumption problem with a finite evaluation period and power utility)

Suppose $U_1(x) = U_2(x) = \frac{x^p}{p}$ are power utilities where $p < 1$ and $p \neq 0$. Suppose that there is a separable solution $V(t,x) = f(t)U_1(x)$. Then by substituting into the HJB equation

$$\sup_{(\pi,c)\in\mathscr{A}(t,x)} \left[f'(t)\frac{x^p}{p} + e^{-\delta t}\frac{c_t^p}{p} + (rx + \sigma\lambda\pi_t - c_t)f(t)x^{p-1} + \frac{p-1}{2}\sigma^2\pi^2 f(t)x^{p-2} \right] = 0,$$

with terminal condition $f(T) = 1$. Maximizing over π and c respectively yields the optimal feedback control functions

$$\hat{\pi}(t,x) = -\frac{\lambda}{\sigma(p-1)}x, \tag{5.1}$$

and

$$\hat{c}(t,x) = xe^{\frac{\delta t}{p-1}} f^{\frac{1}{p-1}}(t). \tag{5.2}$$

Then by substituting the optimal feedback control functions

$$f'(t)\frac{x^p}{p} + \frac{x^p}{p}e^{\frac{\delta t}{p-1}} f^{\frac{p}{p-1}}(t) + f(t)\left[rx^p - \frac{\lambda^2}{p-1}x^p - x^p e^{\frac{\delta t}{p-1}} f^{\frac{1}{p-1}}(t) \right] + \frac{\lambda^2}{2(p-1)}x^p f(t) = 0,$$

which can be simplified to obtain

$$f'(t) + (1-p)e^{\frac{\delta t}{p-1}} f^{\frac{p}{p-1}}(t) + p\left(r - \frac{\lambda^2}{2(p-1)} \right) f(t) = 0.$$

Using the ansatz $f(t) = g^{1-p}(t)$, we obtain a first order, linear ordinary differential equation which can be solved easily

$$g'(t) + \frac{p}{1-p}\left(r - \frac{\lambda^2}{2(p-1)} \right) g(t) + e^{\frac{\delta t}{p-1}} = 0,$$

with terminal condition $g(T) = 1$. We find that

$$\begin{aligned} g(t) &= e^{A(t-T)} + e^{At}\int_t^T e^{-As}e^{\frac{\delta s}{p-1}}\,\mathrm{d}s \\ &= e^{A(t-T)} + e^{AT}e^{(T-t)\frac{\delta}{p-1}}. \end{aligned}$$

where A is the constant $A = \frac{p}{1-p}\left(r - \frac{\lambda^2}{2(p-1)} \right)$.

5.3 Infinite Evaluation Period

Often, participants do not have finite evaluation periods. Miller [16] considers an individual who invests in a financial market with the hope of raising their family name from obscurity to prominence. Neither they nor their descendants withdraw the investments. This is an example of a portfolio decision making problem which has an infinite evaluation period. We would like to evaluate how the length of a participants evaluation period affects their optimal trading strategy.

Consider the same 2-dimensional continuous-time financial market with an extended time frame $(S_t)_{t\geq 0} = ((S_t^0, S_t^1))_{t\geq 0}$ composed of a riskless asset S^0 and a risky asset S^1 on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$. We assume that this is a complete probability space and $(\mathscr{F}_t)_{t\geq 0}$ is a right-continuous filtration containing all $\mathbb{P}$-null sets. We also assume that the assets can be modelled by a geometric Brownian motion as in the Black-Scholes model. The riskless asset price is modelled by the differential equation

$$\mathrm{d}S_t^0 = rS_t^0\mathrm{d}t \quad t \geq 0.$$

The risky asset price is modelled by the stochastic differential equation

$$\mathrm{d}S_t^1 = \mu S_t^1\mathrm{d}t + \sigma S_t^1\mathrm{d}W_t \quad t \geq 0,$$

where $\mu \in \mathbb{R}$, $\sigma > 0$ and $r \geq 0$. Moreover, $W = (W_t)_{t\geq 0}$ is a 1-dimensional Brownian motion on $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$. At any time t, a market participant invests a proportion of their wealth α_t into the financial market and may consume wealth represented by a non-negative stochastic process c_t. Their wealth process can be expressed as

$$X_t = \alpha_t \cdot S_t - c_t,$$

and evolves according to

$$\mathrm{d}X_t = rX_t\mathrm{d}t + \sigma\pi_t(\lambda\mathrm{d}t + \mathrm{d}W_t) - c_t\mathrm{d}t,$$

for all $t \geq 0$. We denote by $\mathscr{A}$ the set of adapted stochastic processes (π, c) such that

(i)

$$\int_0^T c_t\mathrm{d}t < \infty \quad \mathbb{P}\text{-a.s},$$

(ii)

$$X_t \geq 0 \quad \mathbb{P}\text{-a.s.} \quad t \geq 0.$$

We have seen in Chapter 4.1 that assumption (i) asserts the existence and uniqueness of a strong solution to the stochastic differential equation governing the wealth process of a market participant. The goal of the participant in the financial market $(S_t)_{t\geq 0}$ is to maximise the expected utility from their long-term consumption. They start participating in the market at time $t = 0$ with some wealth $x \in \mathbb{R}$ and continue to participate until the evaluation period $t = T = \infty$. The value function of this utility maximisation problem is then defined as

$$V(x) = \sup_{(\pi,c)\in\mathscr{A}(x)} \mathbb{E}\left[\int_0^\infty e^{-\delta t}U(c_t)\mathrm{d}t\right],$$

where U is the utility of their consumption and $\delta > 0$ is the subjective discount rate. The corresponding HJB equation is

$$\delta V(x) - \sup_{(\pi,c)\in\mathscr{A}(x)} \left[U(c_t) + \mathscr{L}^{(\pi,c)}V(x)\right] = 0.$$

where

$$\mathscr{L}^{(\pi,c)}V(x) = (rx + \sigma\lambda\pi_t - c_t)V'(x) + \frac{1}{2}\sigma^2\pi_t^2V''(x).$$

Since we would like to draw comparisons to when a participant has a finite evaluation period, we present an analytical solution to this equation when the utility describing their consumption is of power type.

Example 5.3. (Consumption problem with an infinite evaluation period and power utility)

Theorem 5.1. *(Scaling Property) Suppose a market participant has power utility describing their consumption $U(x) = \frac{x^p}{p}$ where $p < 1$ and $p \neq 0$. Then for any positive constant $\lambda > 0$*

$$V(\lambda x) = \lambda^p V(x).$$

Furthermore, $V(x) = \kappa U(x)$ where $\kappa \in \mathbb{R}$ is some constant.

Proof.

$$\begin{aligned}
V(\lambda x) &= \sup_{(\pi,c)\in\mathscr{A}(\lambda x)} \mathbb{E}\left[\int_0^\infty e^{-\delta t} U(c_t)\mathrm{d}t\right] = \sup_{(\lambda\pi,\lambda c)\in\mathscr{A}(x)} \mathbb{E}\left[\int_0^\infty e^{-\delta t} U(c_t)\mathrm{d}t\right] \\
&= \sup_{(\pi,c)\in\mathscr{A}(x)} \mathbb{E}\left[\int_0^\infty e^{-\delta t} \frac{(\lambda c_t)^p}{p}\mathrm{d}t\right] = \lambda^p V(x).
\end{aligned}$$

Notice that $V(1) = V\left(\frac{x}{x}\right) = x^{-p}V(x)$. Then by rearranging

$$V(x) = x^p V(1) = pV(1)U(x) = \kappa U(x).$$

$\square$

Theorem 5.1 motivates there being a solution $V(x) = \kappa U(x)$. We consider this substitution and aim to find κ. Maximizing over π and c respectively yields the optimal feedback control functions

$$\hat{\pi}(t,x) = -\frac{\lambda}{\sigma(p-1)}x, \tag{5.3}$$

and

$$\hat{c}(t,x) = \kappa^{\frac{1}{p-1}}x. \tag{5.4}$$

Then by substituting the optimal feedback control functions

$$-\delta\kappa\frac{x^p}{p} + (1-p)\kappa^{\frac{p}{1-p}}\frac{x^p}{p} + r\kappa x^p - \frac{\lambda^2}{2(p-1)}\kappa x^p = 0,$$

which can be simplified to obtain

$$\kappa^{\frac{1}{p-1}} = \frac{1}{p-1}\left[\delta - p\left(r - \frac{\lambda^2}{2(p-1)}\right)\right].$$

Substituting κ back into the value function, we find that

$$V(x) = \frac{x^p}{p}\left[\frac{1}{p-1}\left[\delta - p\left(r - \frac{\lambda^2}{2(p-1)}\right)\right]\right]^{p-1}.$$

5.4 Influence Of Time Horizon

For the participant with a finite evaluation period, the optimal control processes $\pi^* = (\pi_t^*)_{t\in[0,T]}$ and $c^* = (c_t^*)_{t\in[0,T]}$ are given by

$$\pi_t^* = \hat{\pi}(t, X_t^*) \quad c_t^* = \hat{c}(t, X_t^*),$$

where $X^* \equiv X^{t,x;\pi^*,c^*}$. By (5.1) and (5.2), we find that

$$\pi_t^* = -\frac{\lambda}{\sigma(p-1)}X_t^* \quad c_t^* = X_t^* e^{\frac{\delta t}{p-1}} g(t).$$

If one considers the optimal fraction of wealth to place in the risky asset $\theta_t^* = \frac{\pi_t^*}{X_t^*}$, we notice that this is constant. Hence, the optimal strategy to maximise expected power utility of terminal wealth is to keep a constant proportion of wealth in the risky asset, regardless of the initial starting wealth and value of the portfolio. In other words, they keep the volatility of their portfolio fixed. Note that keeping a fixed volatility requires continuous portfolio re-balancing. Conversely, the optimal consumption is time dependent and non-linear. It's non-linearity is defined explicitly by the function $e^{\frac{\delta t}{p-1}} g(t)$. A change in the parameters μ, σ, r, δ or p will change the relative weights of the portfolio. By construction, they are constant in this specific example.

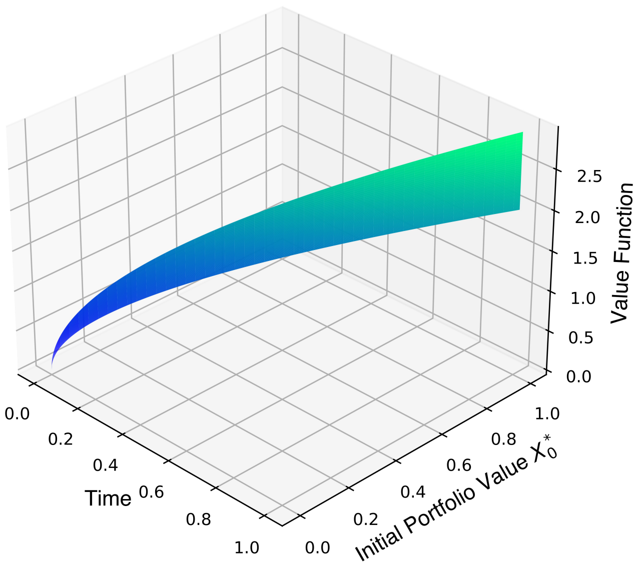

Figure 1: Value function for power utility with termination time $T = 1$ year. The risky asset's expected return is $\mu = 0.08$ and has volatility $\sigma = 0.2$. The riskless asset has no interest rate, $r = 0$. Market participant uses a discount rate $\delta = 1$ and has risk-aversion parameter $p = 0.5$.

For the participant with an infinite evaluation period, the optimal control processes $\pi^* = (\pi_t^*)_{t\geq 0}$ and $c^* = (c_t^*)_{t\geq 0}$ are given by

$$\pi_t^* = \hat{\pi}(t, X_t^*) \quad c_t^* = \hat{c}(t, X_t^*),$$

where $X^* \equiv X^{t,x;\pi^*,c^*}$. By (5.3) and (5.4), we find that

$$\pi_t^* = -\frac{\lambda}{\sigma(p-1)} X_t^* \quad c_t^* = \kappa^{1-p} X_t^*.$$

Like the participant with a finite evaluation period, the optimal fraction of wealth in the risky asset θ_t^* is constant. In addition, it is given by the same constant. However, the participant with an infinite evaluation period has a drastically different optimal consumption process. At any given time point, their optimal consumption is proportional to the value of the portfolio as κ^{1-p} is constant. Hence, the optimal strategy to maximise expected power utility of long-term consumption is to keep the volatility of their portfolio fixed, regardless of the initial starting wealth and value of the portfolio, and consume proportional to the value of the portfolio. The linear relationship between consumption and the value of the portfolio is less surprising due to the scaling property we proved in Theorem 5.1.

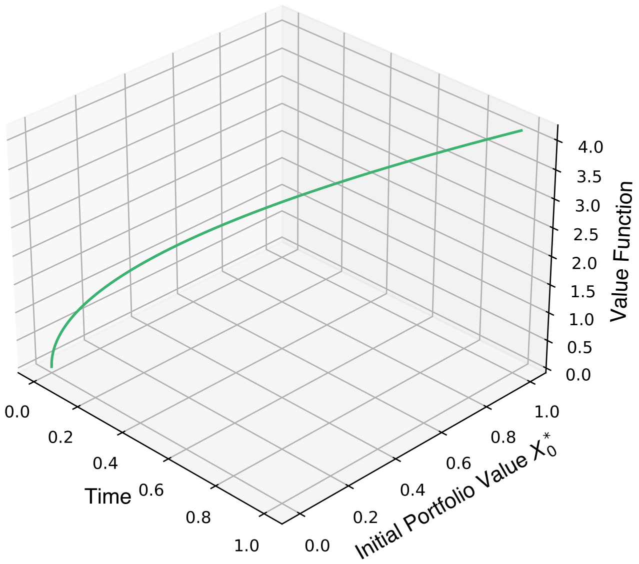

Figure 2: Value function for power utility with infinite horizon $T = \infty$. The risky asset's expected return is $\mu = 0.08$ and it has volatility $\sigma = 0.2$. The riskless asset has no interest rate, $r = 0$. The market participant uses a discount rate $\delta = 1$ and has risk-aversion parameter $p = 0.5$.

We find that participants who are more risk-averse take on a portfolio that is less volatile, regardless of their evaluation period.

$$-\frac{\lambda}{\sigma(p_1 - 1)} < -\frac{\lambda}{\sigma(p_2 - 1)},$$

for any risk-aversion parameter $p_1 < p_2$. In other words, participants with a greater risk appetite use portfolios with greater volatility. Moreover, if participants are risk-neutral then they use portfolios with unbounded volatility

$$\lim_{p\to 1} \theta_t^* = -\lim_{p\to 1} \frac{\lambda}{\sigma(p-1)} = \infty.$$

We assumed in the setup of the problem that a strategy is admissible if the value of the portfolio is positive at all points before the evaluation period. This means that there is no presence of a negative utility outcome. The most undesirable outcome to the participant is to terminate with $X_T = 0$. However, the maximum potential gain is unbounded. Hence, because there is no direct cost associate with an undesirable outcome, there is no disincentive for the risk-neutral participant to invest all their wealth in the risky asset.

5.5 Extension To Multi-Dimensional Market

Up to this point, we have considered a 2-dimensional financial market consisting of a riskless asset and a single risky asset. However, in reality financial markets have multiple risky assets. In this section, we extend our model to include multiple risky assets and evaluate how this affects the participant's optimal trading strategy. We present the extension for a participant with an infinite evaluation period, however the same method can be used to extend the problem with a finite evaluation period.

Consider a complete, frictionless and arbitrage-free $(d+1)$-dimensional continuous-time financial market $(S_t)_{t\geq 0} = ((S_t^0, S_t^1, \ldots, S_t^d))_{t\geq 0}$ composed of a riskless asset S^0 and d many risky assets $S^1, S^2, \ldots, S^d$ on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$. As in Chapter 5.3, we assume that this is a complete probability space and $(\mathscr{F}_t)_{t\geq 0}$ is a right-continuous filtration containing all $\mathbb{P}$-null sets. We also assume that the assets can be modelled by a geometric Brownian motion as in the Black-Scholes model. The riskless asset price is modelled by the differential equation

$$\mathrm{d}S_t^0 = rS_t^0 \mathrm{d}t \quad t \geq 0.$$

The i-th risky asset's price is modelled by the stochastic differential equation

$$\mathrm{d}S_t^i = \mu^i S_t^i \mathrm{d}t + \sum_{j=1}^{d} \sigma^{ij} S_t^i \mathrm{d}W_t^j \quad t \geq 0 \quad 1 \leq i \leq d,$$

where $\mu^i \in \mathbb{R}$, $\sigma^{ij} > 0$ and $r \geq 0$ for all $i, j = 1, 2, \ldots, d$. Moreover, $W = (W_t)_{t\geq 0}$ is an independent 1-dimensional Brownian motion on $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ for all $1 \leq j \leq d$. As in Chapter 5.3, at any time t, a market participant invests a proportion of their wealth α_t into the financial market and may consume wealth represented by a non-negative stochastic process c_t. We denote the amount held in the i-th risky asset by $\pi_t^i = \alpha_t^i S_t^i$. Then the participants wealth evolves according to

$$\mathrm{d}X_t = rX_t \mathrm{d}t + \pi_t^T \left[(\mu - rI_d)\mathrm{d}t + \sigma \mathrm{d}W_t \right] - c_t \mathrm{d}t.$$

Denote by $\mathscr{A}$ the set of adapted stochastic processes (π, c) such that

(i)

$$\int_0^\infty c_t \mathrm{d}t < \infty \quad \mathbb{P}\text{-a.s},$$

(ii)

$$X_t \geq 0 \quad \mathbb{P}\text{-a.s} \quad t \geq 0.$$

As in Chapter 5.3, the goal of the participant in the financial market $(S_t)_{t\geq 0}$ is to maximise the expected utility from their long-term consumption. They start participating in the market at time $t = 0$ with some wealth $x \in \mathbb{R}$ and continue to participate until the evaluation period $t = T = \infty$. The value function of this utility maximisation problem is then defined as

$$V(x) = \sup_{(\pi,c)\in\mathscr{A}(x)} \mathbb{E}\left[\int_0^\infty e^{-\delta t} U(c_t)\mathrm{d}t\right],$$

where U is the utility of their consumption and $\delta > 0$ is the subjective discount rate. The corresponding HJB equation is

$$-\delta V(x) - \sup_{(\pi,c)\in\mathscr{A}(x)} \left[U(c_t) + (rx + \pi_t^T(\mu - rI_d) - c_t)V_x + \frac{1}{2}|\pi_t^T \sigma|^2 V_{xx}\right] = 0.$$

To draw comparisons to Chapter 5.3, we present an analytical solution to this equation when the utility describing their consumption is of power type.

Example 5.4. (Consumption problem in a $(d+1)$-D financial market with an infinite evaluation period and power utility)

Since the scaling property in Theorem 5.1 does not require an assumption on the dimension of the financial market, it motivates there being a solution $V(x) = \kappa U(x)$. We consider this substitution and aim to find κ. Maximizing over π and c respectively yields the optimal feedback control functions

$$\hat{\pi}(t,x) = -\frac{1}{p-1}(\sigma\sigma^T)^{-1}(\mu - rI_d)x,$$

and

$$\hat{c}(t,x) = \kappa^{\frac{1}{p-1}} x.$$

By substituting the optimal feedback control functions, we can find

$$\kappa^{\frac{1}{p-1}} = \frac{1}{p-1}\left[\delta - p\left(r - \frac{|\lambda|^2}{2(p-1)}\right)\right],$$

where $\lambda := \sigma^{-1}(\mu - rI_d)$ is the matrix of asset Sharpe ratios. Substituting κ back into the value function, we find that

$$V(x) = \frac{x^p}{p}\left[\frac{1}{p-1}\left[\delta - p\left(r - \frac{|\lambda|^2}{2(p-1)}\right)\right]\right]^{p-1}.$$

The optimal control processes $\pi^* = (\pi_t^*)_{t\geq 0}$ and $c^* = (c_t^*)_{t\geq 0}$ are given by

$$\pi_t^* = \hat{\pi}(t, X_t^*) \quad c_t^* = \hat{c}(t, X_t^*),$$

where $X^* \equiv X^{t,x;\pi^*,c^*}$. We find that

$$\pi_t^* = -\frac{1}{p-1}(\sigma\sigma^T)(\mu - rI_d)X_t^* \quad c_t^* = \kappa^{1-p}X_t^*.$$

The i-th component of θ_t^* is constant for all i. In other words, the optimal fraction of wealth to place in each risky asset is constant. Hence, the optimal strategy to maximise expected power utility of long-term consumption is to keep the volatility of their portfolio fixed, regardless of the initial starting wealth and value of the portfolio. At any given time point, their optimal consumption is proportional to the value of the portfolio as κ^{1-p} is constant. Notice that the optimal controls are the same as those in Chapter 5.3 when we let $d = 1$. Since our equations are in a vectorised form of the problem with a 1-D financial market, we expects that the same results follow if we had a finite evaluation period.

6 Incentive Fees and High-Water Marks

In this section, we find the optimal trading strategy for a risk-averse portfolio manager compensated with a call option on the assets they control. Fund managers typically manage investments without consumption. Thus, we first discuss the terminal wealth problem proposed by Merton in his work [13] and [14] without consumption. This is similar to the problem in Chapter 5 but we do not have a consumption process, $c_t = 0$. We extend this problem to the case where a portfolio manager receives a management fee and an incentive fee based on exceeding the high-water mark of the fund.

6.1 Merton's Terminal Wealth Problem

We consider the same financial market $(S_t)_{t \in [0,T]}$ as in Chapter 5.2. At any time t, a market participant invests a proportion of their wealth α_t into the financial market. This time they have no consumption: $c_t = 0$. Their wealth process can be expressed as

$$X_t = \alpha_t \cdot S_t,$$

and evolves according to

$$\mathrm{d}X_t = rX_t \mathrm{d}t + \sigma \pi_t (\lambda \mathrm{d}t + \mathrm{d}W_t)\mathrm{d}t.$$

We denote by $\mathscr{A}$ the set of adapted stochastic processes π such that

(i)

$$\int_0^T |\pi_t|^2 \mathrm{d}t < \infty \quad \mathbb{P}\text{-a.s},$$

(ii)

$$X_t \geq 0 \quad \mathbb{P}\text{-a.s} \quad t \in [0,T].$$

Given a portfolio $\pi \in \mathscr{A}$, we denote the corresponding wealth process with initial wealth $x \in \mathbb{R}$ at time $t \in [0,T]$ as $X^{t,x;\pi}$. The goal of the participant in the financial market $(S_t)_{t \in [0,T]}$ is to maximise the expected utility from their wealth at an evaluation period $t = T$. They start participating in the market at time $t \in [0,T]$ with some wealth $x \in \mathbb{R}$ and continue to participate until the evaluation period. We suppose that the participant has utility describing their terminal wealth preference, U. The value function of this utility maximisation problem is then defined as

$$V(t,x) = \sup_{\pi \in \mathscr{A}(t,x)} \mathbb{E}\left[U(X_T^{t,x;\pi})\right].$$

The corresponding HJB equation is

$$-\frac{\partial V}{\partial t}(t,x) - \sup_{\pi \in \mathscr{A}(t,x)} \left[(rx + \sigma \lambda \pi_t)\frac{\partial V}{\partial x}(t,x) + \frac{1}{2}\sigma^2 \pi_t^2 \frac{\partial^2 V}{\partial x^2}(t,x)\right] = 0,$$

with terminal condition

$$V(T,x) = U(x) \quad \forall x \in \mathbb{R}.$$

Maximising over π yields the optimal feedback control function

$$\hat{\pi}(t,x) = -\frac{\lambda}{\sigma}\frac{V_x(t,x)}{V_{xx}(t,x)}. \tag{6.1}$$

Then by substituting the optimal feedback control functions

$$V_t(t,x) + rxV_x(t,x) - \frac{\lambda^2}{2}\frac{(V_x(t,x))^2}{V_{xx}(t,x)} = 0.$$

One can find an explicit solution to this equation by considering logarithmic or power utility.

Example 6.1. (Terminal wealth problem with a finite evaluation period and logarithmic utility)

Consider the logarithmic utility function $U(x) = \log(x)$. Suppose that there is a separable solution $V(t,x) = f(t) + U(x)$. Then by substituting into the HJB equation, we obtain a first order, linear ordinary differential equation that can be solved easily

$$f'(t) + r + \frac{1}{2}\lambda^2 = 0,$$

with terminal condition $f(T) = 0$. We find that

$$f(t) = (T-t)\left(r + \frac{1}{2}\lambda^2\right).$$

Substituting $f(t)$ back into the value function, we find that

$$V(t,x) = \log(x) + (T-t)\left(r + \frac{1}{2}\lambda^2\right).$$

The optimal control process $\pi^* = (\pi_t^*)_{t\in[0,T]}$ is given by

$$\pi_t^* = \hat{\pi}(t, X_t^*)$$

where $X^* \equiv X^{t,x;\pi^*}$. By (6.1), we find that

$$\pi_t^* = -\frac{\lambda}{\sigma(p-1)}X_t^*.$$

We can repeat the argument presented in Example 6.1 to find an explicit solution when the market participant has power utility describing their terminal wealth preference. Considering the separable solution $V(t,x) = f(t)U(x)$, we obtain

$$V(t,x) = \frac{x^p}{p}\exp\left[p(t-T)\left(r + \frac{\lambda^2}{2(p-1)}\right)\right].$$

6.2 Call Option Compensation Scheme

We now consider a portfolio manager whose terminal wealth is

$$\alpha(X_T - B_T)^+ + K,$$

where X_T is the value of the assets at time T, $\alpha > 0$ is the number of options, $K > 0$ is the management fee and B_t is the option strike price or high-water mark. The portfolio manager earns an incentive fee when the terminal value of assets exceeds the high-water mark of the fund. The goal of the portfolio manager in the financial market $(S_t)_{t\in[0,T]}$ is to maximise the expected utility from their terminal wealth at an evaluation period $t = T$. They start participating in the market at time $t \in [0,T]$ with some wealth $x \in \mathbb{R}$ and continue to participate until the evaluation period. We suppose that the manager has power utility describing their terminal wealth preference, $U(x) = \frac{x^p}{p}$ where $p < 1$ and $p \neq 0$. The value function of this utility maximisation problem is then defined as

$$V(t,x) = \sup_{\pi\in\mathscr{A}(t,x)} \mathbb{E}\left[U(\alpha(X_T^{t,x;\pi} - B_T)^+ + K)\right].$$

We define the portfolio manager's *objective function* $\Phi : \mathbb{R}\times(0,\infty) \mapsto \mathbb{R}$ as

$$\Phi(x,b) = \begin{cases} -\infty & x < 0 \\ U(\alpha(x-b)^+ + K) & x \geq 0 \end{cases}.$$

With the option compensation scheme, the manager's objective function is clearly not strictly concave nor strictly increasing below the high-water mark ($x < b$). However, Carpenter [2] proposes a concavfication technique that can be used to solve this problem. This technique allows us to work on the concave envelope and apply the methodology used in Chapter 6.1.

Definition 6.1. The *concave envelope* $f^c : I \mapsto \mathbb{R}$ of a function $f : I \mapsto \mathbb{R}$ is the smallest concave function that is greater than or equal to f

$$f^c := \inf\{g : I \mapsto \mathbb{R} : g \text{ is concave and } g(x) \geq f(x) \text{ for all } x \in I\}.$$

Proposition 6.1. *For every $b > 0$, the concave envelope of $\Phi(x,b)$ has the following properties*

(i) $\Phi(x,b) \leq \Phi^c(x,b) \quad \forall x \in \mathbb{R}$,

(ii) $\Phi^c(x,b)$ *is continuous, strictly increasing and strictly concave.*

We prove this proposition by constructing the concave envelope $\Phi^c(x,b)$. Let $\rho(b)$ be the tangent point of a straight line starting connecting the coordinate $(0,\Phi(0,b))$ to the curve $\Phi(x,b)$ for all $x \geq b$.

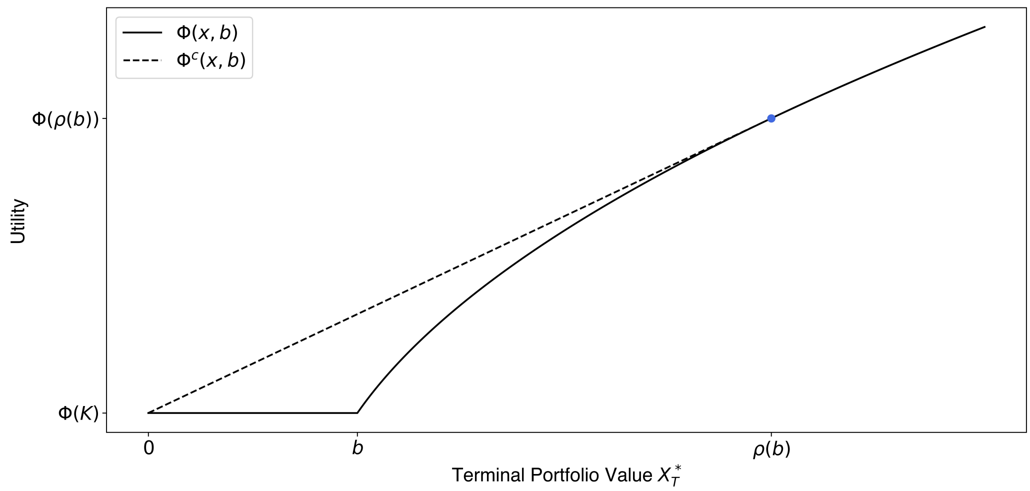

Figure 3: Concave envelope of Φ, Φ^c. The manager receives $\alpha = 0.5$ options, has risk-aversion parameter $p = 0.5$ and receives a management fee $K = 0.03$. The high-water mark is at $B_T = 0.25$.

Proposition 6.2. *For each $b > 0$, there exists a unique point $\rho(b) > b$ such that,*

$$\frac{\Phi(\rho(b), b) - \Phi(0, b)}{\rho(b)} = \frac{\partial \Phi}{\partial x}(\rho(b), b)$$

Proof. Fix a $b > 0$. Consider the tangent of $\Phi(x, b)$ that intersects the point $(0, \Phi(0, b))$. Suppose that such a tangential point is $(\rho(b), \Phi(\rho(b), b))$. Then it is clear that $\rho(b) > b$. Hence, the derivative $\frac{\partial \Phi}{\partial x}(\rho(b), b)$ exists. Using the equation for a line

$$y' - \Phi(\rho(b), b) = (x' - \rho(b)) \frac{\partial Y}{\partial x}(\rho(b), b).$$

Substituting the point $(x', y') = (0, \Phi(0, b))$ and rearranging, we obtain the required statement. $\square$

It follows that $\Phi^c : \mathbb{R} \times (0, \infty) \mapsto \mathbb{R}$ defined by

$$\Phi^c(x, b) = \begin{cases} -\infty & x < 0 \\ \Phi(0, b) + x \frac{\partial \Phi}{\partial x}(\rho(b), b) & 0 \leq x < \rho(b) \\ \Phi(x, b) & \rho(b) \leq x \end{cases},$$

is continuous, strictly concave and strictly increasing in x. Moreover, $\Phi^c(x, b) \geq \Phi(x, b)$ for all $(x, b) \in \mathbb{R} \times (0, \infty)$ which proves Proposition 6.1. Reichlin [19] investigates the relationship between Φ and Φ^c further. They find the following result.

Theorem 6.3. *(Reichlin [19] Theorem 5.1.) Assume that*

$$V(t,x) = \sup_{\pi\in\mathscr{A}(t,x)} \mathbb{E}\left[\Phi(X_T^{t,x;\pi}, B_T)\right] \quad V^c(t,x) = \sup_{\pi\in\mathscr{A}(t,x)} \mathbb{E}\left[\Phi^c(X_T^{t,x;\pi}, B_T)\right].$$

Then $V(t,x) = V^c(t,x)$ for all $(t,x) \in [0,T]\times\mathbb{R}^d$.

Theorem 6.3 states that the optimal terminal wealth for the concavified value function is also optimal for the original value function as it never takes on values where the two objective functions, Φ and Φ^c, disagree. In other words, the optimal payoff has an all-or-nothing property. Either the option is as far out of the money as possible, or it is in the money by at least $\rho(B_T) - B_T$. It does not pay for our portfolio manager to be just marginally in the money, because he must expend substantial resources to bring asset value into the money at all.

Using Theorem 6.3, we can apply the theory of Chapter 6.1. The HJB equation for the value function $V^c(t,x)$ is

$$-\frac{\partial V^c}{\partial t}(t,x) - \sup_{\pi\in\mathscr{A}(t,x)} \left[(rx+\sigma\lambda\pi_t)\frac{\partial V^c}{\partial x}(t,x) + \frac{1}{2}\sigma^2\pi_t^2\frac{\partial^2 V^c}{\partial x^2}(t,x)\right] = 0,$$

with terminal condition

$$V^c(T,x) = \Phi^c(x,B_T) \quad \forall x\in\mathbb{R}.$$

Maximising over π yields the optimal feedback control function

$$\hat{\pi}(t,x) = -\frac{\lambda}{\sigma}\frac{V_x^c(t,x)}{V_{xx}^c(t,x)}, \tag{6.2}$$

and the partial differential equation

$$V_t^c(t,x) + rxV_x^c(t,x) - \frac{\lambda^2}{2}\frac{(V_x^c(t,x))^2}{V_{xx}^c(t,x)} = 0. \tag{6.3}$$

6.3 Numerical Approximation

This section illustrates the optimal trading strategy and value function for the portfolio manager's problem. To do this, we calculate a numerical solution to the HJB equation (6.3). Finite difference is the main method to approximate the solution of a partial differential equation. This method consists of making approximate derivatives through finite differences. Consider a function $f:\mathbb{R}\mapsto\mathbb{R}$ that belongs in $C^\infty(\mathbb{R})$. Then by Taylor's theorem

$$f(x_0+h) - f(x_0) = \sum_{k=1}^{n}\frac{1}{k!}f^{(k)}(x_0)h^k + R_n(x),$$

where $R_n(x)$ is an error term. To approximate the first derivative of f, consider $k=1$

$$f(x_0+h) - f(x_0) = f'(x_0)h + R_1(x).$$

Assuming that the error term $\frac{R_1(x)}{h}$ tends to 0 as h tends to 0, we conclude that

$$f'(x_0) \approx \frac{f(x_0+h)-f(x_0)}{h}.$$

We will use this method naively to solve (6.3). Consider the change in time variable $\tilde{t} = T - t$. Then, omitting tilda, we obtain

$$-V_t^c(t,x) + rxV_x^c(t,x) - \frac{\lambda^2}{2}\frac{(V_x^c(t,x))^2}{V_{xx}^c(t,x)} = 0, \tag{6.4}$$

with initial condition

$$V^c(0,x) = \Phi^c(x,B_T) \quad \forall x \in \mathbb{R}. \tag{6.5}$$

Consider the finite time interval $[0,T]$ and define the time step $h := \frac{T}{N}$. Then we have $N+1$ time steps. Suppose the existence of some maximum wealth the manager may start with $x_{\max}$. Define the interval $[0,x_{\max}]$ and spacial step $k := \frac{x_{\max}}{M}$. Then we have $M+1$ spacial steps. Denote the time at node j by $t_j := jh$, and the space at node i by $x_i := ik$. Then the domain $[0,T]\times[0,x_{\max}]$ is discretized into the uniform grid

$$\Gamma = \left\{(t_j,x_i) : i,j \in \mathbb{Z} \text{ and } 0 \le j \le N, 0 \le i \le M\right\}.$$

Let $V^c(t_j,x_i) = V_i^j$. Then the semi-discretized version of (6.4) can be written as

$$-(V_t)_i^j + rik(V_x)_i^j - \frac{\lambda^2}{2}\frac{((V_x)_i^j)^2}{(V_{xx})_i^j} = 0,$$

with initial condition

$$V_i^0 = \Phi^c(ik,B_T) \quad 0 \le i \le M.$$

By rearranging, we obtain a forward-time central-space scheme given by

$$V_i^{j+1} = V_i^j + \frac{rh}{2}\left(V_{i+1}^j - V_{i-1}^j\right) - \frac{\lambda^2 h}{8}\frac{\left(V_{i+1}^j - V_{i-1}^j\right)^2}{V_{i+1}^j - 2V_i^j + V_{i-1}^j} \quad 0 \le j \le N \quad 1 \le i \le M,$$

with boundary conditions

$$V_i^0 = \Phi^c(ik,B_T) \quad 0 \le i \le M,$$

$$V_0^j = U(K) \quad 0 \le j \le N.$$

This is an explicit scheme because V at time t_j can be explicitly solved in terms of known quantities at time t_{j-1}.

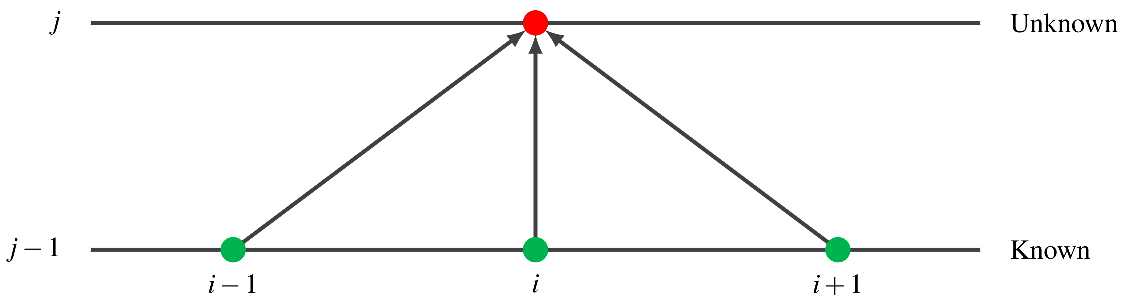

Figure 4: Explicit forward-time central-space scheme for V.

The left boundary ($x = 0$) is representative of the portfolio manager participating with no initial capital. Since the portfolio manager only manages the distribution of wealth between assets, they do not consume or add capital, $X_T = 0$ and $\Phi^c(0, B_T) = U(K)$. The right boundary $(x = x_{\max})$ has no obvious answer. Instead, we calculate points on the right boundary using a *cone of influence*.

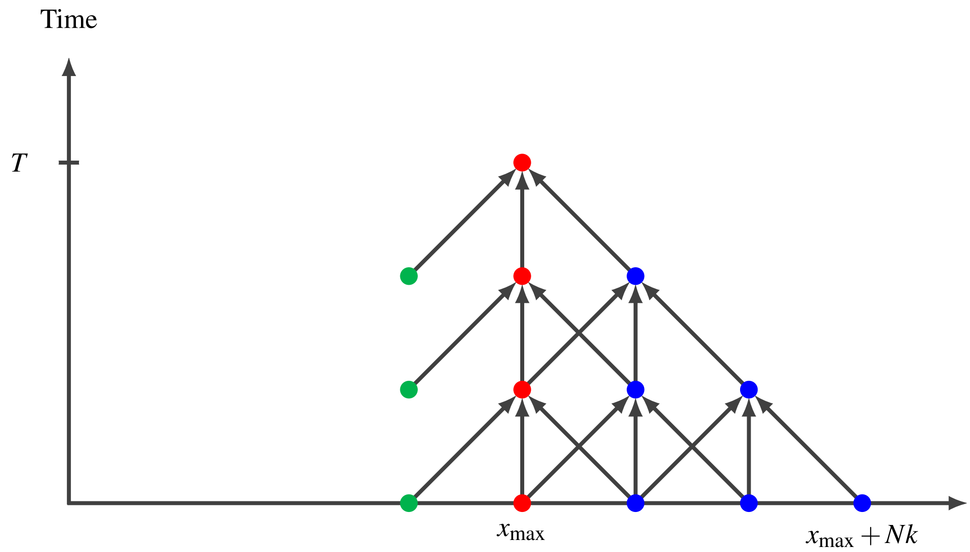

Figure 5: Cone of influence to find the right boundary. Red indicates the points on the boundary, green indicates the known points within the original spacial interval and blue indicates the points from the extended spacial interval $[0, x_{\max} + Nk]$.

We extend the spacial interval to $[0, x_{\max} + Nk]$ and use the numerical scheme to calculate points on the right boundary. The extension is dependent on the number of time steps, $N + 1$. We construct the approximation of the value function

$$V^c_{\text{approx}} = \left\{ V_i^j : i, j \in \mathbb{Z} \text{ and } 0 \leq j \leq N, 0 \leq i \leq M \right\}.$$

In addition to approximating the value function, we would like to approximate the optimal control function (6.2). Let $\hat{\pi}(t_j, x_i) = \hat{\pi}_i^j$. Then using V^c_{approx}, the semi-discretized version of (6.2) can be written as

$$\hat{\pi}_i^j = -\frac{\lambda k}{2\sigma}\frac{V_{i+1}^j - V_{i-1}^j}{V_{i+1}^j - 2V_i^j + V_{i-1}^j} \quad 0 \le j \le N \quad 1 \le i \le M-1.$$

Using this scheme, we construct the approximation of the optimal control function

$$\hat{\pi}_{\text{approx}} = \left\{ \hat{\pi}_i^j : i, j \in \mathbb{Z} \text{ and } 0 \le j \le N, 1 \le i \le M-1 \right\}.$$

6.4 Economic Interpretation

The optimal control process $\pi^* = (\pi_t^*)_{t\in[0,T]}$ is given by

$$\pi_t^* = \hat{\pi}(t, X_t^*),$$

where $X^* \equiv X^{t,x;\pi^*}$. We notice that the optimal fraction of wealth to place in the risky asset, θ_t^* is dependent on the value of the assets X_t^*.

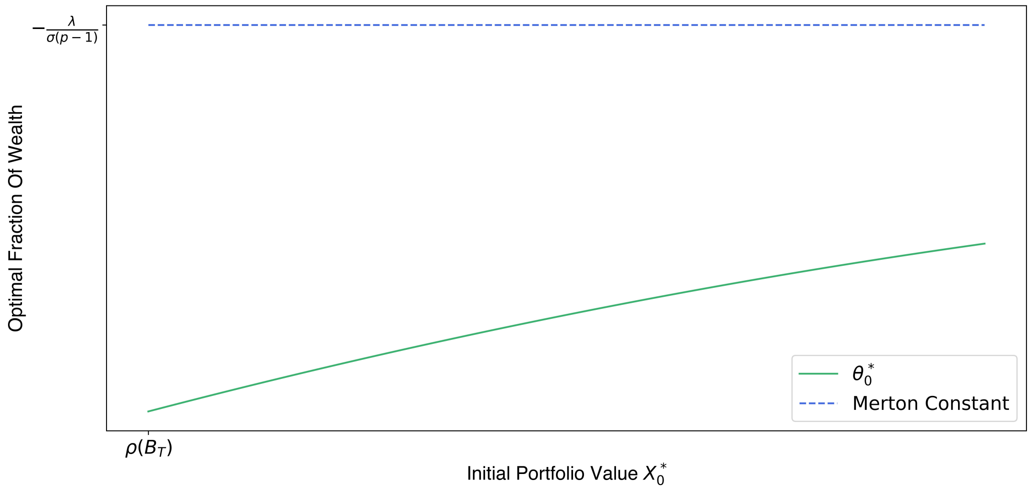

Figure 6: Optimal fraction of wealth for power utility with termination time $T = 1$ year. The risky asset's expected return is $\mu = 0.08$ and has volatility $\sigma = 0.2$. The riskless asset has an interest rate $r = 0.1$. The manager receives $\alpha = 0.15$ options, has risk-aversion parameter $p = 0.5$ and receives a management fee $K = 0.03$. The high-water mark is at $B_T = 0.5$.

We observe the following features

(i)

$$X_t^* \to \infty \quad \Rightarrow \quad \theta_t^* = \frac{\pi_t^*}{X_t^*} \to -\frac{\lambda}{\sigma(p-1)},$$

(ii)

$$X_t^* \to \rho(B_T) \quad \Rightarrow \quad |\theta_t^*| = \left|\frac{\pi_t^*}{X_t^*}\right| \to \infty.$$

The first observation states that as the value of the portfolio tends to infinity, the portfolio converges to the constant volatility portfolio that they would follow if they were trading with their own account (as in Chapter 6.1). The second observation indicates that when the manager's portfolio is performing poorly (close to the high-water mark), the portfolio manager takes on a portfolio with unbounded volatility.

When the option is in the money, the portfolio manager takes less risk than if they were trading with their own account. This is due to the option providing leverage for the portfolio manager, who in turn reduces the volatility of the trading strategy to offset this. Rather than strictly increasing the volatility of the portfolio to exceed the high-water mark, we see that the portfolio manager adjusts the volatility to the value of the portfolio. In some cases, they even take less risk than if they were trading with their own account. We can extend the financial market using the same method as in Chapter 5.5 and expect to find the same result.

7 Conclusion

This paper describes the optimal trading strategy for a portfolio manager compensated with a call option contract. The option structure affects the manager's risk appetite drastically. An expected utility maximising manager with power utility describing their terminal wealth preference does not strictly prefer to increase the volatility of their portfolio. Instead, they dynamically adjust the volatility of their portfolio according to the value of the portfolio. As the value of the portfolio increases, the manager moderates volatility. At times, the volatility of the portfolio is lower than if they were trading with their own account. However, extreme risk taking is incentivised when the value of the manager's portfolio is below the high-water mark.

We use finite difference to calculate a numerical solution to the HJB equation (6.3). However, we find that the forward-time central-space scheme is very unstable. For difficult equations like the HJB equation, we need more complicated numerical integration schemes.

In our setup, we assume the manager participates in a frictionless, complete and arbitrage-free market. In reality, markets are likely to be incomplete and there are numerous constraints such as transaction costs, borrowing and insider trading. This restricts the strategies at a portfolio managers disposal to achieve optimal payoff. Considering this, the manager's optimal payoff with option compensation might appear somewhat unrealistic. Extensive research has been dedicated to studying the Merton problem within an incomplete market framework, taking transaction costs into account. Magill and Constantinides [12] and Choi et al. [3] reduce the problem to the solution of a free-boundary problem for a first-order ordinary differential equation.

We also assume that the risky assets follow the Black-Scholes model. This means that the risky assets have a constant underlying volatility that is unaffected by the changes in the price of the asset. While the Black-Scholes model has demonstrated high tractability, it assumes that the log-returns of the risky asset prices follow a normal distribution with constant variance and independent increments. However, this assumption does not align with the distributions implied by real-world financial markets. To address this discrepancy, stochastic volatility models are used. Kraft [11] explicitly calculates the optimal trading strategy to the Merton problem with Heston model [7] describing the risky asset's price. They find that the optimal trading strategy is unbounded. This raises the question as to how managerial risk taking is affected in stochastic volatility models.

In addition, our model illustrates a manager facing a single evaluation period at $t = T$. In reality, portfolio managers may face random, indefinite or even multiple evaluation periods. Furthermore, performance based compensation schemes often contain multiple incentives such as restricted stock and annual bonus plans. Panageas [17] and Hodder [8] explore how the incentives of the manager are affected if the evaluation period of the compensation contract is infinite or indefinite. They find that a manager facing an infinite evaluation period has a reduced risk appetite.

While it is clear that the risk-taking behavior of the manager depends on the specific problem, the model we have presented offers a useful example of the strategy a manager adopts when presented with option compensation. It would be fascinating to investigate whether this behavior accurately reflects the actions of portfolio managers in practical settings. Often, portfolio managers face consequences for poor performance; investors may lose confidence in the fund and withdraw their assets. In terms of future research, it would be interesting to impose a constraint reflecting investors liquidating their assets due to poor asset management. This could lead to different behavioral patterns from both the risk-averse and risk-neutral managers, providing a more accurate illustration of what we can expect in practice.

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