The Fluctuation-Dissipation Theorem

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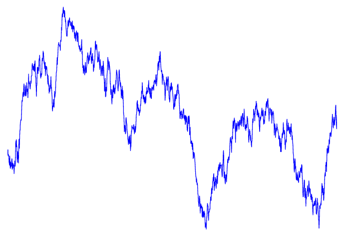
The fluctuation-dissipation theorem (FDT) connects fluctuation properties of a system in equilibrium with the response to a perturbation. The proof is based on linear response theory. Two possible applications are the measurement of the shear modulus of viscoelastic fluids, as well as testing whether a system is actively driven.

Historical introduction: Brownian motion

Einstein-Smoluchowski	$D = \lim_{t\to\infty} \frac{\langle x^2(t)\rangle}{2t} = \frac{kT}{m\gamma}$
Langevin equation	$\partial_t u = -\gamma u + L(t)$
with	$\langle L(t)\rangle = 0$
	$\langle L(t)L(t')\rangle = \Gamma\delta(t-t')$
Equipartition law	$\frac{1}{2}m\langle u^2\rangle = \frac{1}{2}kT$
FDT	$\langle L(t)L(t')\rangle = 2\gamma^2 D\delta(t-t')$

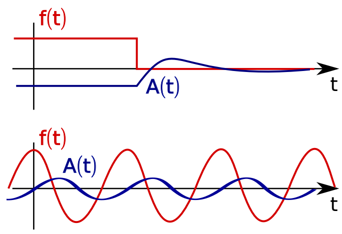


Brownian motion: sample path

Einstein's theory (1905) and Perrin's experiment (1908) confirmed the atomistic paradigm.

Linear Response Theory

Hamiltonian	$H(t) = H_0 - Bh(t)$
Non-equilibrium average*	$\langle A(t)\rangle_{ne} = \frac{1}{Z}\operatorname{Tr}\rho A(t)$
Response	$\delta A(t) := \langle A(t)\rangle_{ne} - \langle A\rangle_{eq}$
Response function**	$\chi_{AB}(t,t') :=$ $\frac{i}{\hbar}\theta(t-t')\langle[A(t),B(t')]\rangle_{eq}$
Linear response	$\delta A(t) = (\chi_{AB} * h)(t)$
or	$\delta A(\omega) = \chi_{AB}(\omega)h(\omega)$



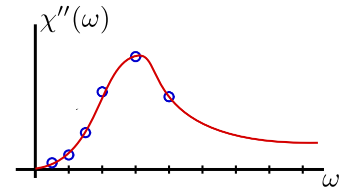
Schematic response of observable $A(t)$ to perturbation $f(t)$.

* Heisenberg picture (const. states ρ); Z = partition function

** Operators evolve according to equilibrium Hamiltonian ($h(t) = 0$).

Fluctuation-dissipation theorem

Observables	$A(t) = e^{iH_0 t/\hbar} A e^{-iH_0 t/\hbar}$
Averages	$\langle A(t)\rangle = \frac{1}{Z}\operatorname{Tr}\rho_0 A(t)$
Correlation function (sym.)	$S_{AB}(t-t') = \langle[A(t),B(t')]_+\rangle$
Response function	$\chi''_{AB}(t-t') = \frac{1}{2\hbar}\langle[A(t),B(t')]\rangle$
FDT (quantum mechanical)	$\chi''_{AB}(\omega) = \frac{\omega}{2E_\beta(\omega)} S_{AB}(\omega)$
FDT (classical)	$\chi''_{AB}(\omega) = \frac{\omega}{2kT} S_{AB}(\omega)$
Kramers-Kronig relations	$\chi_{AB} = \chi'_{AB} + i\chi''_{AB}$
	$\chi'_{AB} = P\int \frac{\chi''_{AB}(\omega')}{\omega'-\omega}\frac{d\omega'}{\pi}$



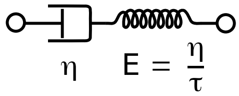
The FDT allows to assess χ'' over a large, continuous range of frequencies, as opposed to mechanical measurements of the response.

Energy of harmonic oscillator:
$E_\beta(\omega) = \frac{\omega\hbar}{2}\coth\left(\frac{\beta\hbar\omega}{2}\right)$

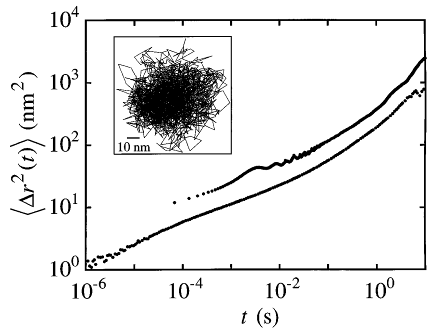
Effective temperature:
$T_{\text{eff}}(\omega) := \frac{\omega}{2k}\frac{S_{AB}(\omega)}{\chi''_{AB}(\omega)}$

Application: Mircorheology

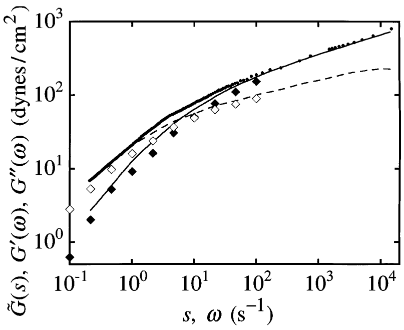
Maxwell model	$\eta\dot{\epsilon}(t) = \sigma(t) + \tau\dot{\sigma}(t)$
Stress-strain relation	$\tilde{\sigma}(\omega) = G^*(\omega)\tilde{\epsilon}(\omega)$
Complex shear modulus	$G^*(\omega) = -\frac{i\omega\eta}{1-i\omega\tau}\tilde{\epsilon}(\omega)$
Stokes-Einstein equation	$\tilde{f}(\omega) = 6\pi a G^*(\omega)\tilde{\epsilon}(\omega)$
Linear response	$G^*(\omega) = \frac{1}{6\pi a\chi(\omega)}$



Maxwell model for viscoelastic fluid: damper (viscosity η) and spring (elastic modulus E) in series;
σ = stress,
ϵ = strain.



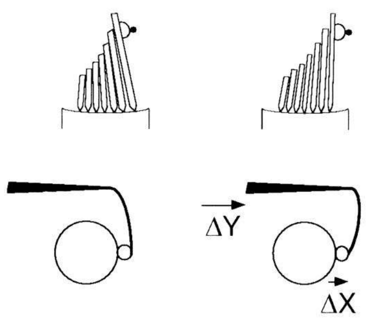
Mean square displacement of bead in viscoelastic fluid. Inset: Recorded 2d trajectory.



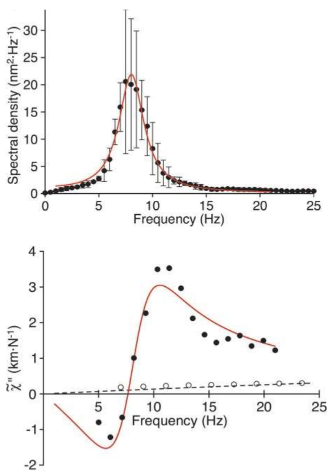
Complex shear modulus $G^*(\omega)$ of DNA using bead tracking. Diamonds: mechanical measurements.

$$G^*(\omega) = G'(\omega) + iG''(\omega)$$

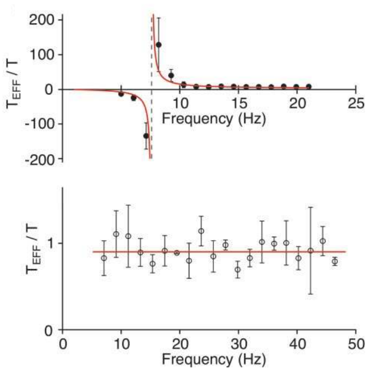
Application: Active hair-bundles – Is hearing an active process?



Experimental setup: flexible glass fiber connected to tip of hair-bundle.



Power spectrum $S(\omega)$ and response function $\chi''(\omega)$.



Effective temperature of hair-bundle with (upper) and without (lower) spontaneous oscillations.

References

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