The Fluctuation-Dissipation Theorem

Friedrich Schüßler

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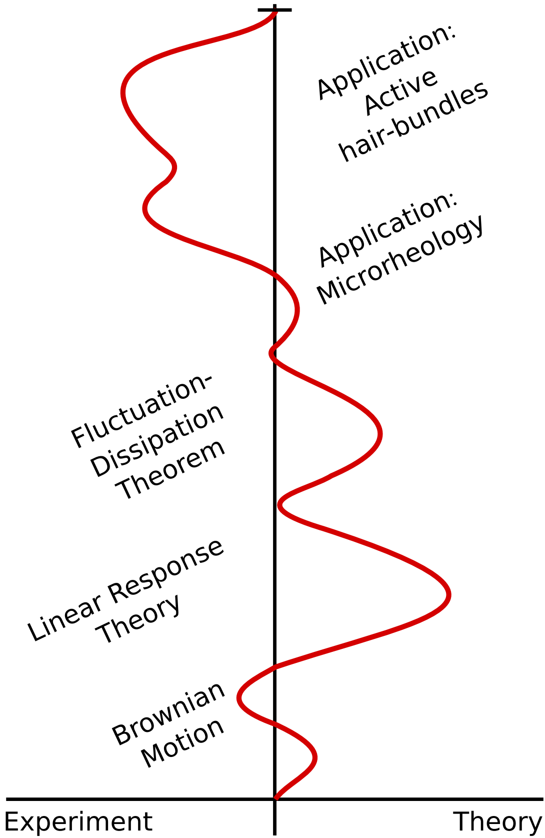
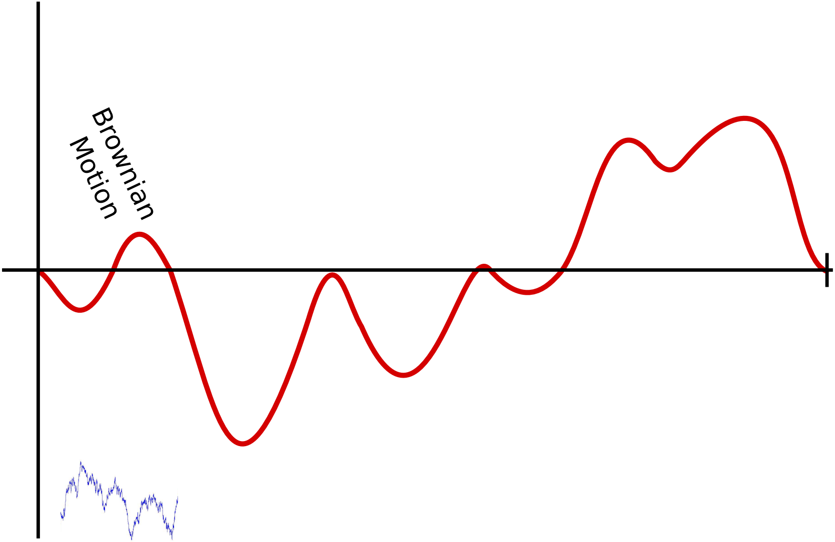
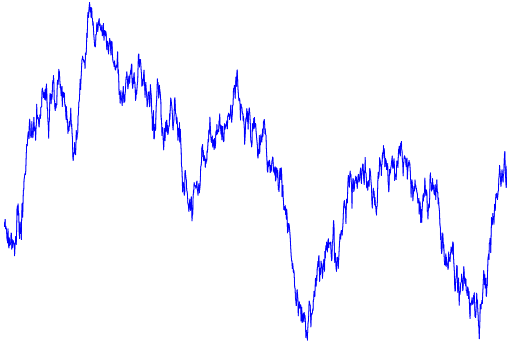
Supervisor: Prof. Oliver Mülken & Dr. Maxim Dolgushev


Application:
Active
hair-bundles
Application:
Microrheology
Fluctuation-
Dissipation
Theorem
Linear Response
Theory
Brownian
Motion
Experiment
Theory


Brownian
Motion

Brownian Motion



Brown 1827

Einstein 1905
Smulochowski 1906
Langevin 1908

Perrin 1908

Brownian Motion

Langevin equation:

$$\partial_t u = -\gamma u + L(t)$$
$$\text{with} \qquad \langle L(t) \rangle = 0; \quad \left\langle L(t) L(t') \right\rangle = \Gamma \delta(t - t')$$

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$$\Longrightarrow \boxed{\Gamma = \frac{2\gamma kT}{m}}$$

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Displacement for $t \gg 1/\gamma$: $$\left\langle x^2(t) \right\rangle = \frac{2kT}{m\gamma} t$$

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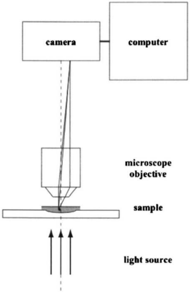

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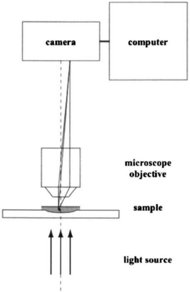

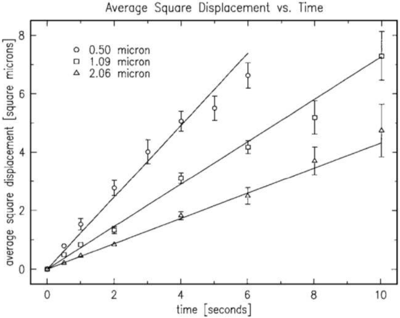

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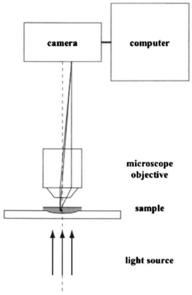

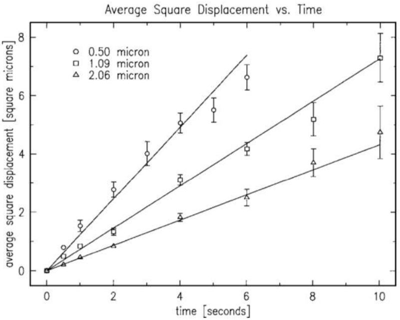

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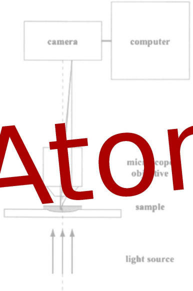
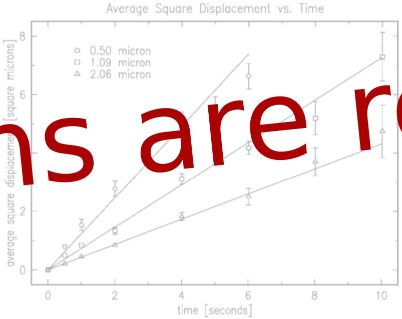
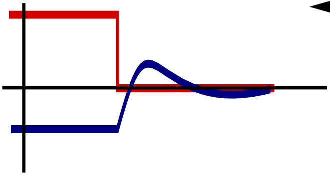




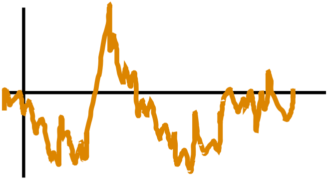
Atoms are real!

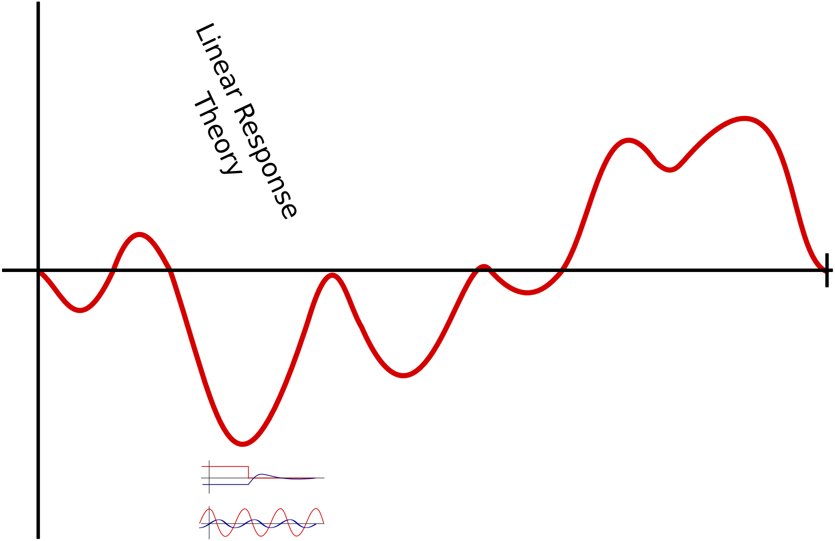
response to perturbation



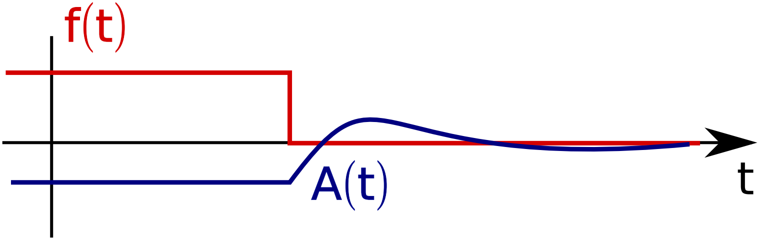
?

equilibrium fluctuations




Linear Response
Theory

Response to a perturbation



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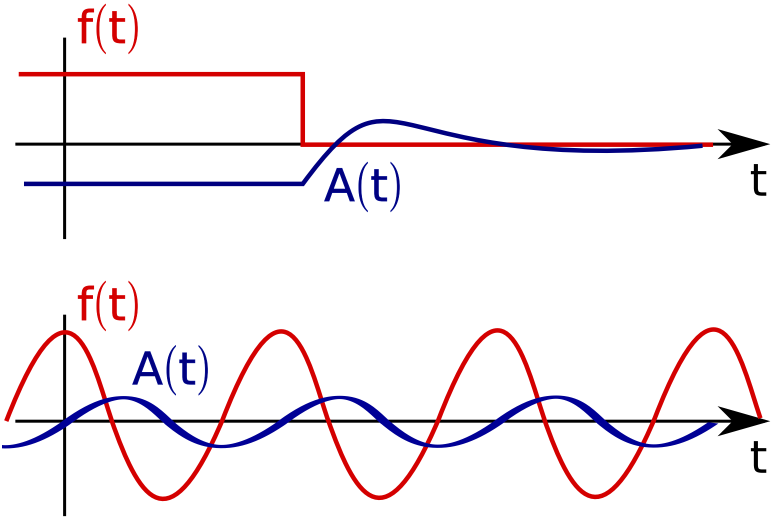

Perturbed system

Perturbation $f(t)$ coupling to operator A:

$$H(t) = H_0 - A\, f(t)$$

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Non-equilibrium average:

$$\langle A(t) \rangle_{ne} = \frac{1}{Z}\,\mathrm{Tr}\,\rho_0 A(t)$$

where

$$\begin{aligned} Z &= \text{partition function} \\ \rho_0 &= \text{equilibrium density matrix} \end{aligned}$$

Average response

$$\delta A(t) := \langle A(t) \rangle_{ne} - \langle A \rangle_{eq}$$

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$A(t)$ evolves due to H_0:

$$A(t) = e^{itH_0/\hbar} \, A \, e^{-itH_0/\hbar}$$

Linear Response

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$$\boxed{\delta A(\omega) = \chi_{AB}(\omega) f(\omega)}$$

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Kubo

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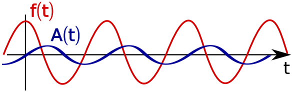
Green

Measure $\chi(\omega)$



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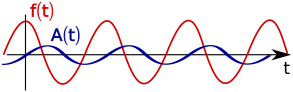

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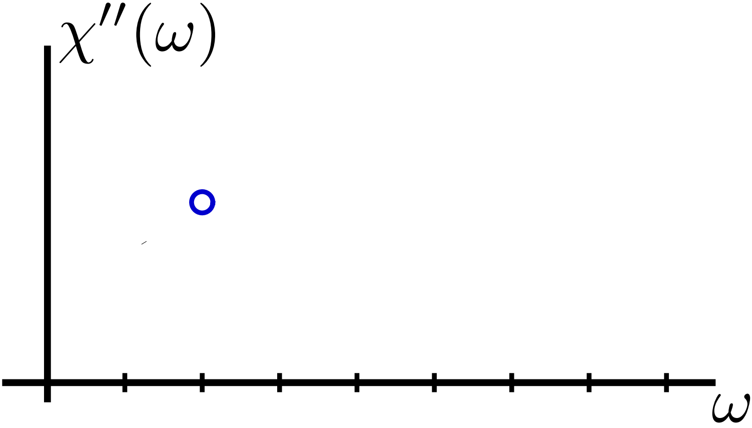

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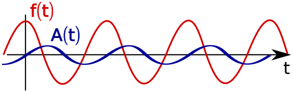

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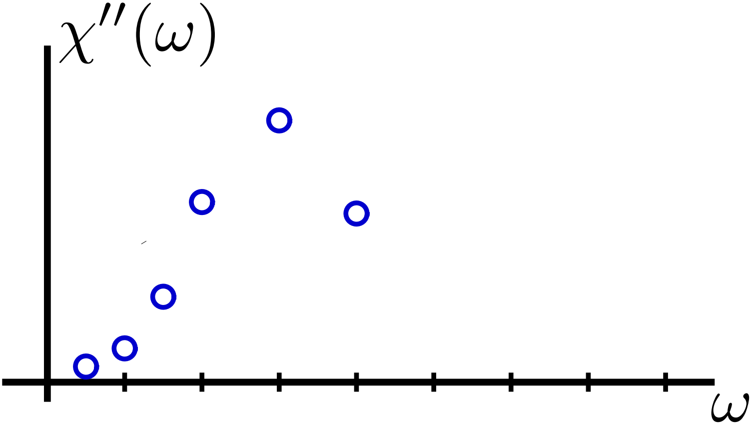

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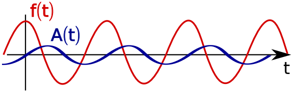

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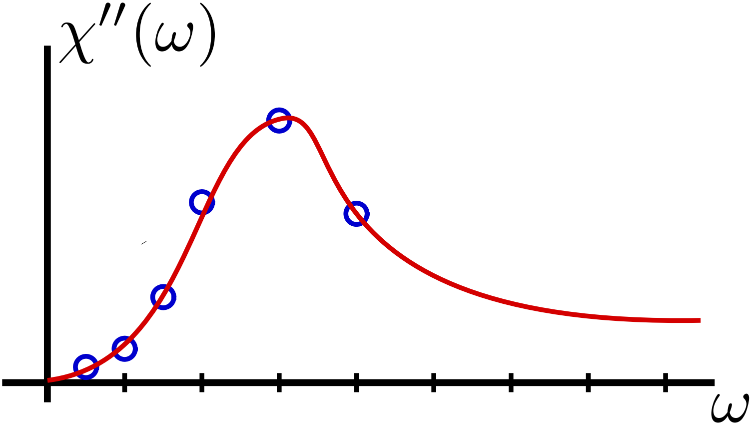

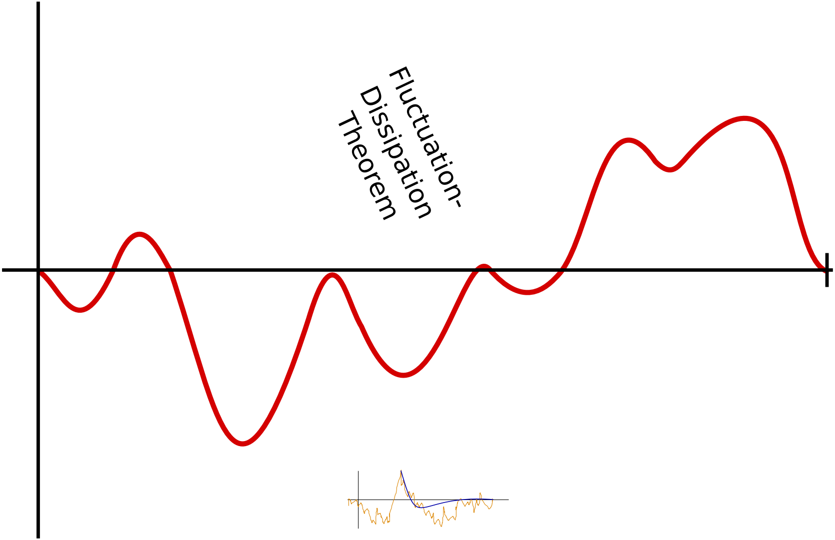
Fluctuation-
Dissipation
Theorem

The fluctuation-dissipation theorem

... is based on thermal equilibrium properties!

- Hamiltonian H_0
- $\rho = \rho_0$ const.,
- $A(t) = e^{iH_0 t/\hbar} A e^{-iH_0 t/\hbar}$.
- Observables: $\langle A(t) \rangle = \frac{1}{Z} \operatorname{Tr} \rho_0 A(t)$

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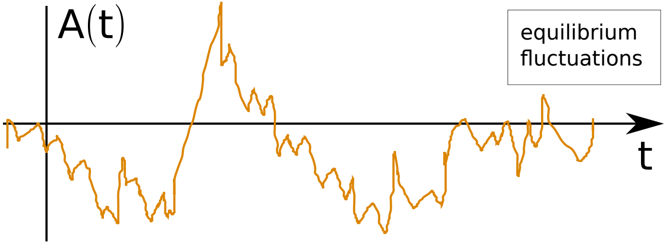

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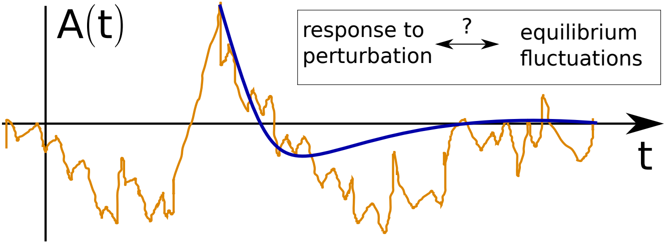

Fluctuation – autocorrelation function:

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In time domain:

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Symmetries of both functions $\Longrightarrow$

$$\boxed{\chi''(\omega) = \frac{1}{\hbar}\tanh\left(\frac{\hbar\omega}{2kT}\right)S(\omega)}$$

The classical limit, $\hbar\omega/kT \to 0$:

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The protagonists next to Green and Kubo:

Onsager

Nyquist

What is experimentally accessible?

δA	or	S
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Kramers–Kronig relations:

$$\chi = \chi' + i\chi''$$

$$\chi' = P \int \frac{\chi''(\omega')}{\omega' - \omega} \frac{\mathrm{d}\omega'}{\pi}$$

$$\chi'' = -P \int \frac{\chi'(\omega')}{\omega' - \omega} \frac{\mathrm{d}\omega'}{\pi}$$

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Dissipation P = work per period T:

$$\begin{aligned} P &= \frac{1}{T} \int_0^T f(t) \dot{\delta x}(t) \, \mathrm{d}t \\ &= \frac{f_0}{2} \omega \chi''(\omega_0) \end{aligned}$$

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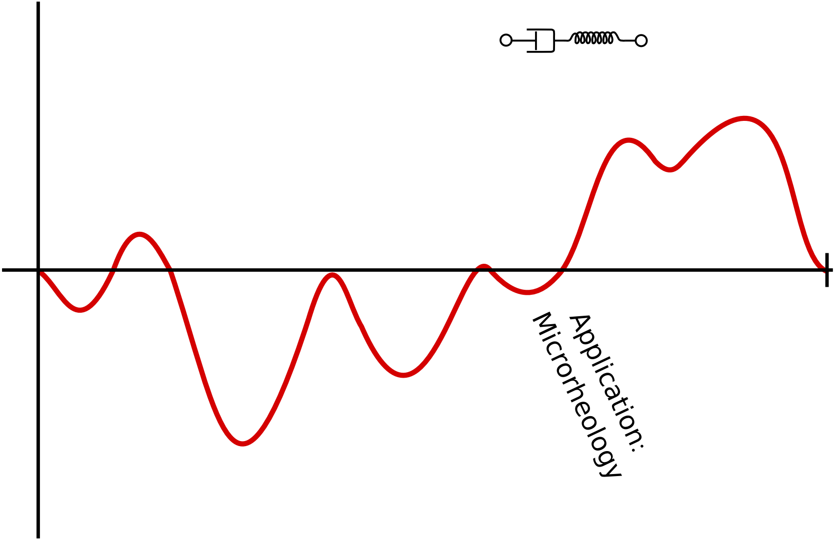
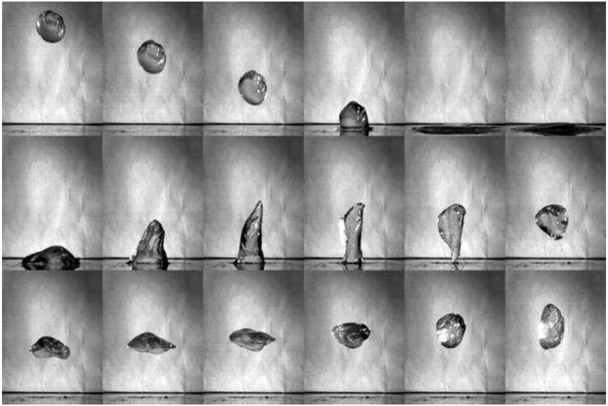
In equilibrium:

$$S(\omega) > 0 \quad \Longrightarrow \quad \chi''(\omega) > 0 \quad \Longrightarrow \quad P > 0$$


Application:
Microrheology

Viscoelastic fluids



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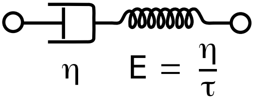
Viscoelastic fluids

Maxwell model:
damper + spring:



$$\sigma = \sigma_D = \sigma_S \quad \text{stress}$$

$$\epsilon = \epsilon_D + \epsilon_S \quad \text{strain}$$

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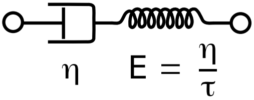

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Stress-strain relation

$$\eta\dot{\epsilon}(t) = \sigma(t) + \tau\dot{\sigma}(t)$$

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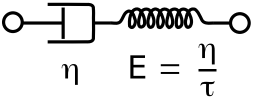

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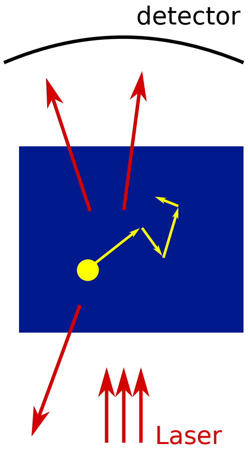
Stress-strain relation

$$\eta \dot{\epsilon}(t) = \sigma(t) + \tau \dot{\sigma}(t)$$

$$\Longrightarrow \quad \tilde{\sigma}(\omega) = -\frac{i\omega\eta}{1 - i\omega\tau}\tilde{\epsilon}(\omega) =: G^*(\omega)\tilde{\epsilon}(\omega)$$

with complex shear modulus $G^*(\omega)$.

Experiment: Bead in viscoelastic fluid



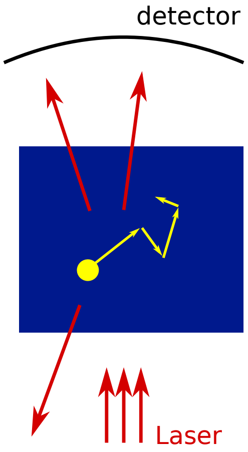
Experiment: Bead in viscoelastic fluid

Generalized Stokes-Einstein equation:

$$\tilde{f}(\omega) = 6\pi a \eta^*(\omega)\tilde{\dot{\epsilon}}(\omega)$$

with complex viscosity

$$\eta^*(\omega) := \frac{\eta}{1 - i\omega\tau}$$



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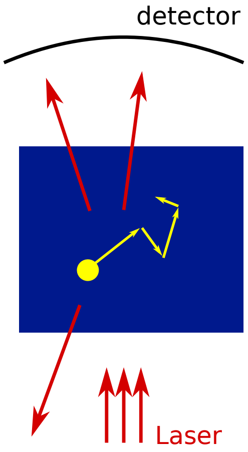

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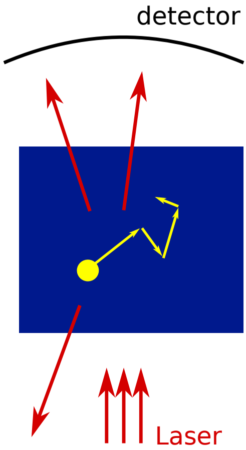
Linear response:

$$\tilde{\epsilon}(\omega) = \chi(\omega)\tilde{f}(\omega) = \frac{\tilde{f}(\omega)}{6\pi a G^*(\omega)}$$



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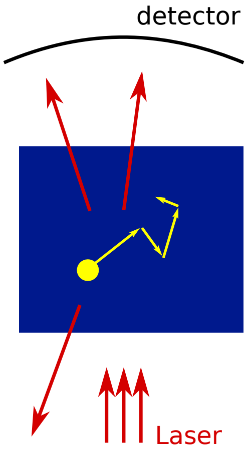
$$\eta^*(\omega) := \frac{\eta}{1 - i\omega\tau} = \frac{G^*(\omega)}{-i\omega}$$

Linear response:

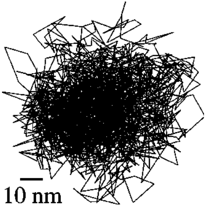
$$\tilde{\epsilon}(\omega) = \chi(\omega)\tilde{f}(\omega) = \frac{\tilde{f}(\omega)}{6\pi a G^*(\omega)}$$

$$\Longrightarrow \qquad G^*(\omega) = \frac{1}{6\pi a \chi(\omega)}$$



Experimental measurement of $G^*(\omega)$

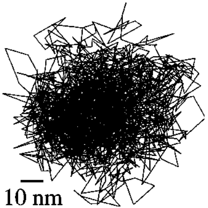
Mason et al., 1997



Trajectory

Experimental measurement of $G^*(\omega)$

Mason et al., 1997

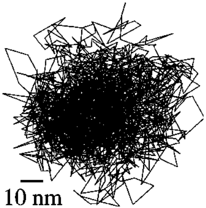


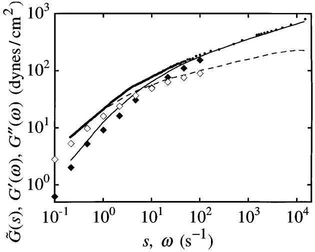
Trajectory

$$r(t) \implies S(\omega) \implies \chi''(\omega) \implies \chi(\omega) \implies G^*(\omega)$$

Experimental measurement of $G^*(\omega)$

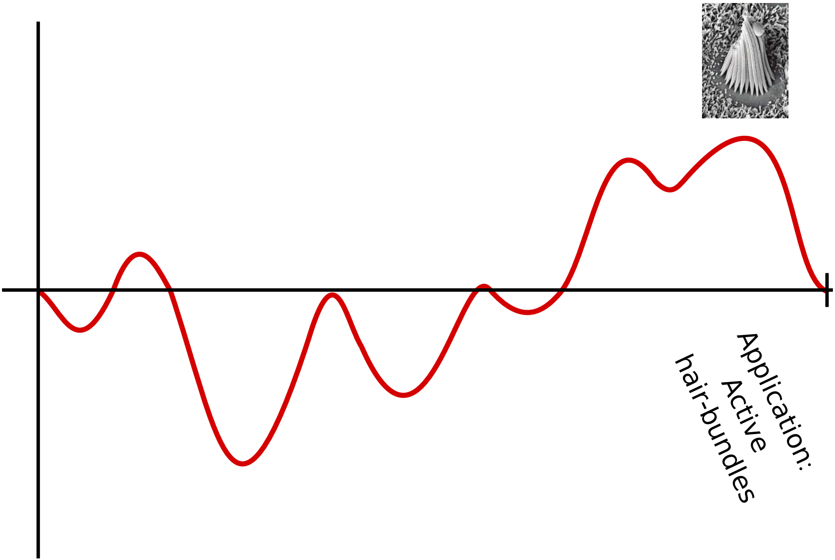
Mason et al., 1997





Trajectory Shear modulus

$$r(t) \implies S(\omega) \implies \chi''(\omega) \implies \chi(\omega) \implies G^*(\omega)$$


Application:
Active
hair-bundles

Is our system in equilibrium?

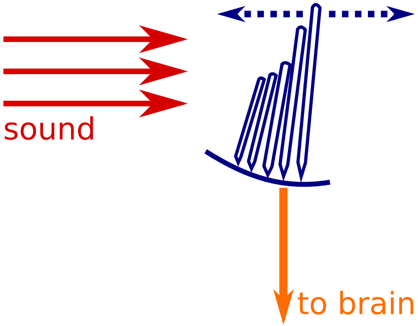
Fluctuation-dissipation theorem:

$$\chi''(\omega) = \frac{\omega}{2kT} S(\omega)$$

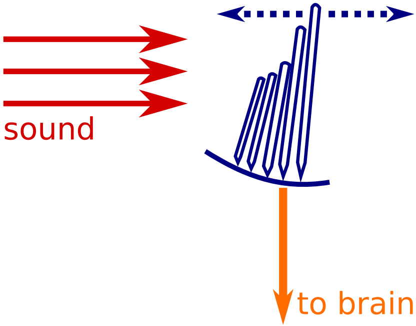
Effective temperature:

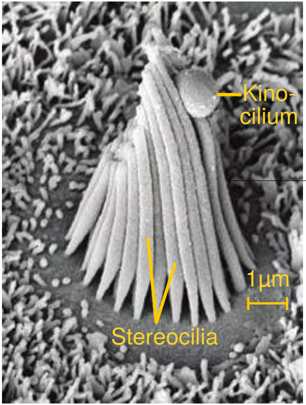
$$T_{\mathrm{eff}}(\omega) := \frac{\omega}{2k} \frac{S(\omega)}{\chi''(\omega)}$$

Active hair-bundle motility



Active hair-bundle motility

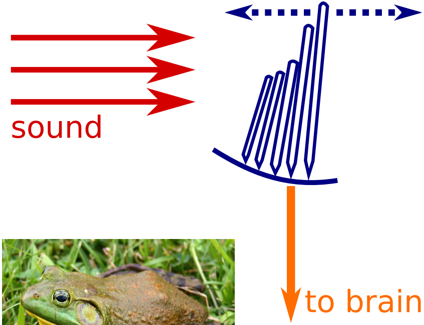


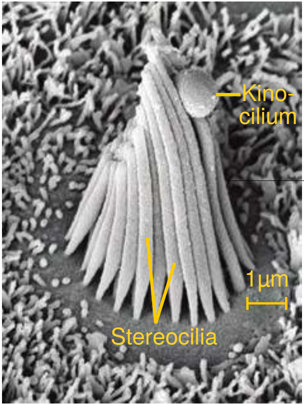


Hair bundle

Hudspeth, 2014

Active hair-bundle motility

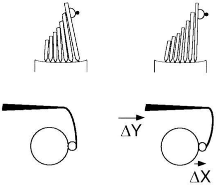




Hair bundle

Hudspeth, 2014

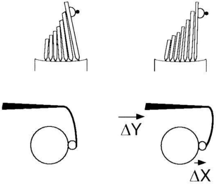
Experimental setup and testing linear response

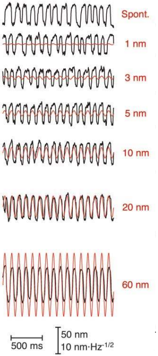


Experimental Setup

Benser et al., 1996

Experimental setup and testing linear response





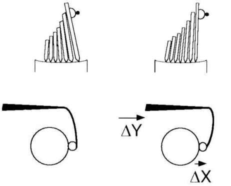
Experimental Setup

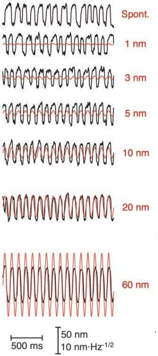
Benser et al., 1996

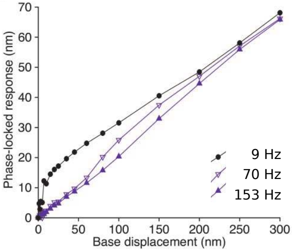
Applying force

Martin & Hudspeth, 2001

Experimental setup and testing linear response







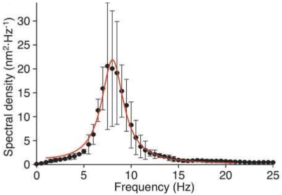
Experimental Setup
Benser et al., 1996

Applying force

Linear Response
Martin & Hudspeth, 2001

Experimental results

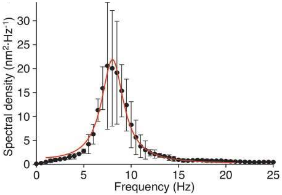
Martin et al., 2001



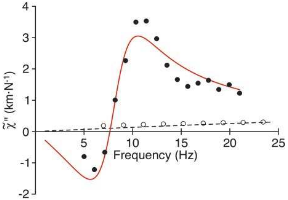
Power Spectrum

Experimental results

Martin et al., 2001



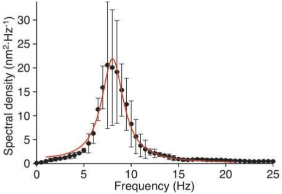
Power Spectrum



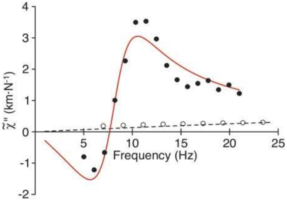
Response function

Experimental results

Martin et al., 2001



Power Spectrum



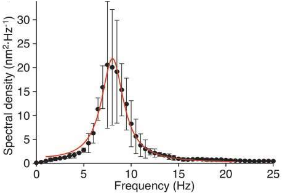
Response function

$$T_{\mathrm{eff}}(\omega) = \frac{\omega}{2k}\frac{S(\omega)}{\chi''(\omega)}$$

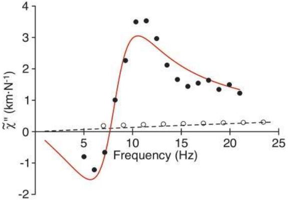
Experimental results

Martin et al., 2001

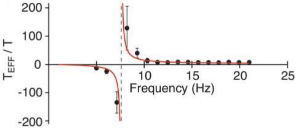
$$T_{\text{eff}}(\omega) = \frac{\omega}{2k} \frac{S(\omega)}{\chi''(\omega)}$$

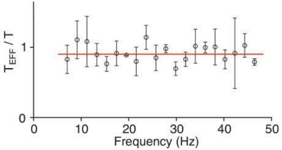


Power Spectrum

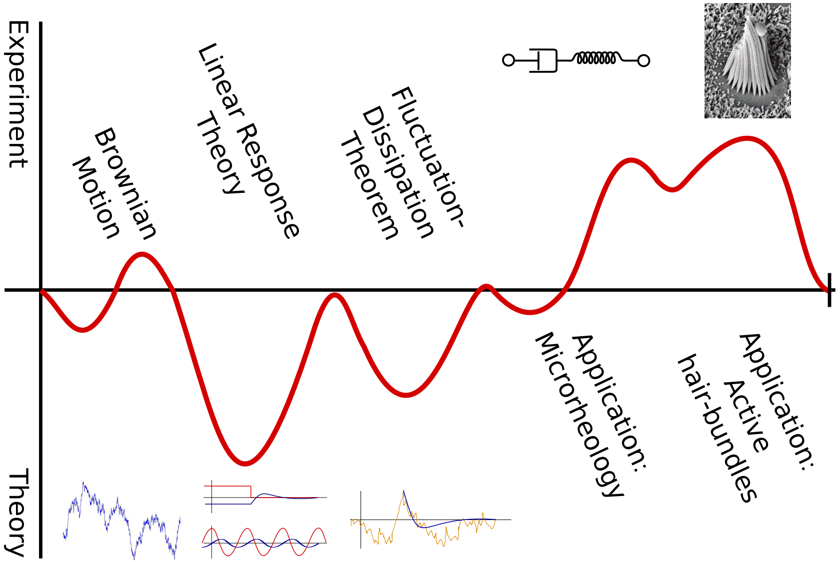


Response function





Effective temperature


Experiment
Theory
Brownian Motion
Linear Response Theory
Fluctuation-Dissipation Theorem
Application: Microrheology
Application: Active hair-bundles

Conclusion

perturbed system $\Longleftrightarrow$ response function $\Longleftrightarrow$ autocorrelation

$$\delta A \underset{1.}{\Longleftrightarrow} \chi \underset{2.}{\Longleftrightarrow} \chi'' \underset{3.}{\Longleftrightarrow} S$$

Conclusion

$$\text{perturbed system} \iff \text{response function} \iff \text{autocorrelation}$$

$$\delta A \underset{1.}{\iff} \chi \underset{2.}{\iff} \chi'' \underset{3.}{\iff} S$$

1. Linear response theory
2. Kramers-Kronig relations
3. Fluctuation-dissipation theorem

Conclusion

$$\begin{array}{ccccc} \text{perturbed system} & \Longleftrightarrow & \text{response function} & \Longleftrightarrow & \text{autocorrelation} \\ \delta A & \underset{1.}{\Longleftrightarrow} & \chi \ \underset{2.}{\Longleftrightarrow} \ \chi'' & \underset{3.}{\Longleftrightarrow} & S \end{array}$$

1. Linear response theory
2. Kramers-Kronig relations
3. Fluctuation-dissipation theorem

- Microrheology:

$$r(t) \implies S(\omega) \implies \chi''(\omega) \implies \chi(\omega) \implies G^*(\omega)$$

Conclusion

$$\begin{array}{ccccc} \text{perturbed system} & \Longleftrightarrow & \text{response function} & \Longleftrightarrow & \text{autocorrelation} \\ \delta A & \underset{1.}{\Longleftrightarrow} & \chi \underset{2.}{\Longleftrightarrow} \chi'' & \underset{3.}{\Longleftrightarrow} & S \end{array}$$

1. Linear response theory
2. Kramers-Kronig relations
3. Fluctuation-dissipation theorem

- Microrheology:

$$r(t) \implies S(\omega) \implies \chi''(\omega) \implies \chi(\omega) \implies G^*(\omega)$$

- Active hair-bundles:

$$S(\omega), \chi(\omega) \implies T_{\text{eff}} \neq T \implies \text{actively driven}$$

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Thank you for your attention!

Appendix

Heisenberg picture

All time dependency carried by operators, while $\rho = \rho_{eq}$ constant.
Operators:

$$A(t) = U^{-1}(t, t_0) A U(t, t_0)\,,$$

following the equations of motion

$$i\hbar \partial_t A(t) = [H(t), A(t)]\,.$$

Average measurements:

$$\langle A(t) \rangle_{ne} = \mathrm{Tr}\, \rho_{eq} A(t)\,.$$

Deduction FDT

Start with correlation function

$$\bar{S}_{AB}(t-t') = \left\langle A(t)B(t')\right\rangle \,.$$

Using $A(t) = e^{itH/\hbar}A(t)e^{-itH/\hbar}$, one shows that

$$\begin{aligned}
\bar{S}_{AB}(t-t') &= \frac{1}{Z}\,\mathrm{Tr}\, e^{-\beta H}A(t)B(t') \\
&= \frac{1}{Z}\,\mathrm{Tr}\, e^{+i(t+i\beta\hbar)H/\hbar}Ae^{-i(t+i\beta\hbar)H/\hbar}e^{-\beta H}B(t') \\
&= \frac{1}{Z}\,\mathrm{Tr}\, A(t+i\beta\hbar)e^{-\beta H}B(t') \\
&= \frac{1}{Z}\,\mathrm{Tr}\, e^{-\beta H}B(t')A(t+i\beta\hbar) \\
&= \bar{S}_{BA}(t'-t-i\beta\hbar)\,.
\end{aligned}$$

Then, with $\tau := t - t'$,

$$\begin{aligned}
\bar{S}_{AB}(\omega) &= \int e^{i\omega\tau}\,\bar{S}_{AB}(-\tau - i\beta\hbar)\,\mathrm{d}\tau \\
&= \int e^{i\omega\tau}\left(\int \bar{S}_{AB}(\omega')e^{-i\omega'(-\tau - i\beta\hbar)}\frac{\mathrm{d}\omega'}{2\pi}\right)\mathrm{d}\tau \\
&= \int e^{-\beta\hbar\omega'}\,\bar{S}_{AB}(\omega')2\pi\delta(\omega+\omega')\frac{\mathrm{d}\omega'}{2\pi} \\
&= \bar{S}_{AB}(-\omega)e^{+\beta\hbar\omega}\,.
\end{aligned}$$

And inserting yields

$$\begin{aligned}
\chi''_{AB}(t-t') &= \frac{1}{2\hbar}\left\langle [A(t), B(t')] \right\rangle \\
&= \frac{1}{2\hbar}\left\langle \bar{S}_{AB}(\tau) - \bar{S}_{BA}(-\tau) \right\rangle\ , \\
\Rightarrow \quad \chi''_{AB}(\omega) &= \frac{1}{2\hbar}\left(\bar{S}_{AB}(\omega) - \bar{S}_{BA}(-\omega)\right) = \frac{1 - e^{-\beta\hbar\omega}}{2\hbar}\bar{S}_{AB}(\omega)\,.
\end{aligned}$$