Chapter Exercises

Combinators

Determine whether each of the following functions are combinators. [Recall the definition of a combinator; a lambda expression that has no free variables. and serves only to combine provided arguments]

1. $\lambda x. xxx$ is a combinator as it does not have free variables
2. $\lambda xy. zx$ is not a combinator. z is a free variable as it is not bounded by the head of the lambda expression.

Normal form or diverge?

Determine whether each of the following expressions can be reduced to a normal form or if they diverge:

1. $\lambda x. xxx$ is already in its beta normal form.
2. $(\lambda z. zz)(\lambda y. yy)$ is beta-reduced as follows:

$$\begin{aligned}(\lambda z. zz)(\lambda y. yy) &= [z := (\lambda y. yy)]\ zz \\ &= (\lambda y. yy)(\lambda y. yy) \\ &= \omega \text{ as defined in Chapter 1.9}\end{aligned}$$

Alternatively, because the expressions containing z and y are alpha-equivalent and variables are only bound within their own lambda expressions, we can change the variable name y to z for a straightforward "transformation" into the the ω function. As such, the lambda term described here is a diverging lambda term.

Beta reduce

Evaluate (beta reduce) each of the following expressions to normal form.

1. $(\lambda abc. cba)zz(\lambda wv. w)$:

$$
\begin{aligned}
(\lambda abc.\, cba)\ z\ z\ (\lambda wv.\, w) &= (\lambda a.\, (\lambda b.\, (\lambda c.\, cba)))\ z\ z\ (\lambda wv.\, w) \\
&= ([a := z](\lambda b.\, (\lambda c.\, cba)))\ z\ (\lambda wv.\, w) \\
&= (\lambda b.\, (\lambda c.\, cbz))\ z\ (\lambda wv.\, w) \\
&= \ldots \\
&= (\lambda wv.\, w)\ z\ z \\
&= \ldots \\
&= (\lambda v.\, z)\ z \\
&= z
\end{aligned}
$$

2. $(\lambda xy.\, xyy)(\lambda a.\, a)b$

$$
\begin{aligned}
(\lambda xy.\, xyy)(\lambda a.\, a)b &= (\lambda y.\, (\lambda a.\, a)yy)b \text{ (applying the outermost lambda)} \\
&= (\lambda a.\, a)bb \\
&= bb
\end{aligned}
$$