Arbitrage in Betting Markets: Implied Probabilities, Stake Allocation, and Arbitrage Criterion

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Abstract

This paper presents a rigorous derivation and proof of the sports betting arbitrage strategy implemented in our Discord bot. We introduce the concept of implied probabilities derived from bookmaker odds, establish the necessary and sufficient condition for arbitrage, and formally derive the stake allocation that guarantees a risk-free profit. All results are proved in a general setting for binary and ternary outcome markets.

1 Introduction

Sports betting arbitrage exploits discrepancies in quoted odds across different bookmakers to secure a guaranteed profit regardless of event outcome. The core insight is that when the sum of implied probabilities falls strictly below one, one may allocate stakes proportionally to ensure a positive net return. In Section 2 we derive implied probabilities; in Section 3 we compute the exact stake allocation and show its optimality; and in Section 4 we prove the arbitrage existence criterion.

2 Odds and Implied Probabilities

Let $o_1, o_2, \ldots, o_n > 1$ denote the decimal odds offered by a bookmaker for each of n mutually exclusive outcomes of a single event. A bettor staking s on outcome i receives $s \cdot o_i$ if i occurs, and otherwise loses the stake s.

Definition 2.1 (Implied Probability)**.** *The implied probability p_i corresponding to odds o_i is defined by*

$$p_i := \frac{1}{o_i} \tag{1}$$

Equation (1) normalizes bookmaker odds to a pseudo-probability mass. Note that bookmakers build in a profit margin (the *overround*), so typically $\sum_i p_i > 1$.

3 Derivation of Stake-Allocation Formulas

In this section, we show analytically the same formulas computed by our code via RREF.

Motivation

Our goal is to allocate a fixed bankroll W among the n outcomes so that no matter which outcome actually occurs, the bettor receives exactly the same total payout R. In particular:

1. **Equal-payout condition.** If we stake s_i on outcome i at decimal odds o_i, then the payout in case i wins is

$$B_i \;=\; s_i\, o_i$$

To eliminate risk, we demand

$$s_1\, o_1 \;=\; s_2\, o_2 \;=\; \cdots \;=\; s_n\, o_n \;=\; R$$

2. **Budget constraint.** We must allocate exactly our total bankroll:

$$\sum_{i=1}^{n} s_i \;=\; W$$

Solving these two sets of equations simultaneously yields closed-form formulas for the s_i. In the next two subsections we carry this out in the special cases $n = 2$ and $n = 3$, and then observe the pattern for general n.

3.1 Two-way Market

Given decimal odds o_1, o_2 and total bankroll W, we solve

$$\begin{cases} s_1\, o_1 \;-\; s_2\, o_2 \;=\; 0 \\ s_1 + s_2 \;=\; W \end{cases}$$

From the first equation $s_1\, o_1 = s_2\, o_2$, hence

$$s_2 = s_1\, \frac{o_1}{o_2}$$

Substitute into $s_1 + s_2 = W$:

$$s_1 + s_1\, \frac{o_1}{o_2} = W \implies s_1\Big(1 + \tfrac{o_1}{o_2}\Big) = W \implies s_1 = \frac{W}{1 + \frac{o_1}{o_2}} = \frac{W\, o_2}{o_1 + o_2}$$

Thus,

$$s_2 = W - s_1 = \frac{W\, o_1}{o_1 + o_2}$$

The common payout R is

$$R = s_1\, o_1 = s_2\, o_2 = \frac{W\, o_1\, o_2}{o_1 + o_2}$$

3.2 Three-way Market

For three decimal odds (o_1, o_2, o_3) and total W, solve

$$\begin{cases} s_1\, o_1 - s_2\, o_2 = 0 \\ -\, s_2\, o_2 + s_3\, o_3 = 0 \\ s_1 + s_2 + s_3 = W \end{cases}$$

From the first two,

$$s_2 = s_1\,\frac{o_1}{o_2} \quad \text{and} \quad s_3 = s_2\,\frac{o_2}{o_3} = s_1\,\frac{o_1}{o_3}$$

Plug into $s_1 + s_2 + s_3 = W$:

$$s_1\Bigl(1 + \tfrac{o_1}{o_2} + \tfrac{o_1}{o_3}\Bigr) = W \implies s_1 = \frac{W}{1 + \frac{o_1}{o_2} + \frac{o_1}{o_3}}$$

Hence

$$s_2 = s_1\,\frac{o_1}{o_2}, \quad s_3 = s_1\,\frac{o_1}{o_3}$$

Equivalently, defining the implied probabilities

$$p_i \;=\; \frac{1/o_i}{\sum_{k=1}^{3}(1/o_k)}$$

one checks

$$s_i \;=\; \frac{W\, p_i}{o_i \sum_{k=1}^{3} p_k} = \frac{W\, p_i}{o_i\,(p_1 + p_2 + p_3)}$$

The common guaranteed return is

$$R = s_i\, o_i = \frac{W}{\sum_{k=1}^{3} p_k}$$

3.3 Profit Rate

In general, for an n-way market the return is

$$R \;=\; \frac{W}{\sum_{k=1}^{n} p_k}$$

so the risk-free profit rate (ROI) is

$$\text{ROI} \;=\; \frac{R - W}{W} \;=\; \frac{1}{\sum_{k=1}^{n} p_k} - 1 \;=\; \frac{1 - \sum_{k=1}^{n} p_k}{\sum_{k=1}^{n} p_k}$$

4 Arbitrage Existence

For an event with n outcomes and m distinct bookmakers, let $o_{i,j}$ be the odds for outcome i at bookmaker j, and let $p_{i,j} = 1/o_{i,j}$ be the corresponding implied probability. Suppose we choose for each outcome i the best available odds

Theorem 4.1 (Arbitrage Criterion)**.** *A risk-free arbitrage opportunity exists if and only if*

$$\sum_{i=1}^{n} p_i \; < \; 1 \tag{2}$$

Proof. ($\Rightarrow$) Suppose an arbitrage exists: there are nonnegative stakes s_i with total stake

$$S \; = \; \sum_{i=1}^{n} s_i$$

and guaranteed profit $\delta > 0$. Then each outcome i pays

$$B_i \; = \; o_i \, s_i \; = \; S + \delta$$

so that

$$s_i \; = \; \frac{S+\delta}{o_i} \;\Longrightarrow\; S \; = \; \sum_{i=1}^{n} s_i \; = \; (S+\delta) \sum_{i=1}^{n} \frac{1}{o_i} \; = \; (S+\delta) \sum_{i=1}^{n} p_i$$

Dividing both sides by $S + \delta$ yields

$$\sum_{i=1}^{n} p_i \; = \; \frac{S}{S+\delta} \; < \; 1$$

($\Leftarrow$) Conversely, if $\sum_i p_i < 1$, choose total capital W and set

$$s_i \; = \; \frac{W \, p_i}{o_i \sum_{k=1}^{n} p_k}$$

whence each payoff $o_i s_i$ equals $R = \dfrac{W}{\sum_k p_k} > W$, so a risk-free profit $R - W > 0$ is secured. $\square$

5 Conclusion

We have shown that an arbitrage exists precisely when the sum of implied probabilities $\sum_{i=1}^{n} p_i < 1$, and derived closed-form stakes $s_i = W \, p_i/(o_i \sum_j p_j)$ that guarantee a uniform payoff $R = W/\sum_j p_j$ and risk-free profit. These results hold for any n-outcome market and directly motivate the implementation in our bot.