GVPO Explained: Novelties, Advantages, and Mathematical Differences

Executive Summary

GVPO (Group Variance Policy Optimization) addresses the training instability issues of GRPO while providing stronger theoretical guarantees. The key innovation is incorporating the **analytical solution to KL-constrained reward maximization directly into gradient weights** through a clever **zero-sum weight constraint** that eliminates the intractable partition function.

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1. Core Novelties of GVPO

Novel 1: Zero-Sum Weight Constraint Eliminates Partition Function

The Problem: The optimal policy for KL-constrained reward maximization has a closed-form solution:

$$\pi^*(y|x) = \frac{1}{Z(x)}\pi_{\theta'}(y|x)e^{R(x,y)/\beta}$$

where $Z(x) = \sum_y \pi_{\theta'}(y|x)e^{R(x,y)/\beta}$ is computationally intractable (requires summing over all possible responses).

GVPO's Solution: By designing weights such that $\sum_{i=1}^k w_i = 0$, the partition function $\beta \log Z(x)$ becomes **invariant across responses and cancels out** in gradient computations:

$$\nabla_\theta L(\theta) = -\sum_{x,\{y_i\}}\sum_{i=1}^k w_i \nabla_\theta \log \frac{\pi_\theta(y_i|x)}{\pi_{\theta'}(y_i|x)} = -\sum_{x,\{y_i\}}\sum_{i=1}^k w_i \nabla_\theta \frac{R_\theta(x,y_i)}{\beta}$$

Since $\sum w_i = 0$, the $\beta \log Z(x)$ term disappears, making the method computationally tractable.

Novel 2: Gradient Weights Based on Central Distance Differences

GVPO's Weight Design:

$$w_i = (R(x,y_i) - \bar{R}(x)) - \beta\left(\log\frac{\pi_\theta(y_i|x)}{\pi_{\theta'}(y_i|x)} - \overline{\log\frac{\pi_\theta}{\pi_{\theta'}}}\right)$$

where the bar notation denotes group average: $\bar{R}(x) = \frac{1}{k}\sum_{i=1}^k R(x,y_i)$.

Physical Interpretation: The weight is the **difference between actual reward central distance and implicit reward central distance**.

Novel 3: Three Equivalent Loss Interpretations

The paper elegantly shows GVPO's loss has three mathematically equivalent forms:

(a) Negative Log-Likelihood View (Equation 9):

$$\mathcal{L}_{\text{GVPO}}(\theta) = -\beta \sum_{x,\{y_i\}} \sum_{i=1}^{k} \left[(R(x, y_i) - \bar{R}) - \beta \left(\log \frac{\pi_\theta(y_i|x)}{\pi_{\theta'}(y_i|x)} - \overline{\log \frac{\pi_\theta}{\pi_{\theta'}}} \right) \right] \log \pi_\theta(y_i|x)$$

(b) Mean Squared Error View (Middle panel, Figure 1):

$$\nabla_\theta \mathcal{L}_{\text{GVPO}} = \frac{1}{2} \nabla_\theta \sum_{x,\{y_i\}} \sum_{i=1}^{k} \left[(R_\theta(x, y_i) - \bar{R}_\theta) - (R(x, y_i) - \bar{R}) \right]^2$$

Key Insight: Minimizing GVPO loss = minimizing **MSE between implicit and actual reward central distances**.

(c) Reinforcement Learning View (Equation 14, =1):

$$\nabla_\theta \hat{\mathcal{L}}_{\text{GVPO}} = -2\mathbb{E}_{x,y} \left[(R(x, y) - \mathbb{E}_y R) \log \pi_\theta(y|x) + \text{Cov}(\log \pi_\theta, \log \pi_{\theta'}) - 0.5\text{Var}(\log \pi_\theta) \right]$$

Three components: 1. **Group-relative reward term**: Advantage maximization
2. **Covariance term**: Regularization preventing deviation from reference policy
3. **Variance term**: Entropy-like exploration encouragement

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2. Mathematical Comparison: GVPO vs GRPO

GRPO Loss (Equation 2):

$$\mathcal{L}_{\text{GRPO}}(\theta) = - \sum_{x,y_1,\ldots,y_k} \sum_{i=1}^{k} \frac{R(x, y_i) - \text{Mean}(\{R(x, y_i)\})}{\text{Std}(\{R(x, y_i)\})} \log \pi_\theta(y_i|x)$$

Key Differences:

Aspect	GRPO	GVPO
Weight Formula	$w_i = \frac{R(x,y_i)-\bar{R}}{\sigma_R}$ (standardized reward)	$w_i = (R(x, y_i) - \bar{R}) - \beta(\log \frac{\pi_\theta}{\pi_{\theta'}} - \overline{\log \frac{\pi_\theta}{\pi_{\theta'}}})$
Normalization	Divides by standard deviation σ_R	No std normalization (only centering)

Aspect	GRPO	GVPO
Policy Dependency	Weights independent of current policy	Weights depend on $\pi_\theta/\pi_{\theta'}$ ratio
KL Constraint	Applied externally (hyperparameter tuning)	**Built into gradient weights analytically**
Zero-Sum Property	Yes (due to centering)	Yes (by design)

Critical Mathematical Insight:

GRPO's standardization **conflates prompt-level difficulty with reward signals** (cited in paper [17]). For example: - Hard prompt with rewards [8, 9, 10] → all responses get similar standardized scores - Easy prompt with rewards [1, 2, 9] → large standardized score differences

GVPO **removes std normalization** but adds the $\beta(\log \pi_\theta/\pi_{\theta'})$ term to directly encode the optimal policy structure.

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3. Theoretical Advantages of GVPO

Advantage 1: Unique Optimal Solution (Theorem 3.1)

GVPO Guarantee:

$$\text{argmin}_\theta \hat{\mathcal{L}}_{\text{GVPO}}(\theta) = \pi^*(y|x) = \frac{1}{Z(x)}\pi_{\theta'}(y|x)e^{R(x,y)/\beta}$$

Uniqueness is proven by showing: 1. When $\pi_\theta = \pi^*$, the loss equals 0 (minimum achieved) 2. Any other policy yields loss > 0 (contradiction proof in Appendix B.1)

Why This Matters: - DPO fails this: Due to Bradley-Terry model limitations [3, 11], DPO may converge to suboptimal policies - **GRPO lacks this**: No theoretical guarantee of convergence to KL-constrained optimum

Advantage 2: Flexible Sampling Distributions (Corollary 3.2)

GVPO's Condition: Theorem 3.1 holds for **any sampling distribution** π_s satisfying:

$$\forall x, \{y|\pi_{\theta'}(y|x) > 0\} \subseteq \{y|\pi_s(y|x) > 0\}$$

Translation: As long as π_s covers all responses that the reference policy could generate, GVPO maintains theoretical guarantees.

Comparison with GRPO/PPO:

Method	Sampling Requirement	Problem
PPO	On-policy ($\pi_s = \pi_\theta$)	Low sample efficiency, requires fresh trajectories
GRPO	Uses importance sampling $\frac{\pi_\theta}{\pi_{\theta_{\text{old}}}}$	Gradient explosion when policies diverge; requires clipping (introduces bias)
GVPO	Any π_s satisfying mild condition	**No importance sampling, no explosion risk**

Mathematical Detail: Policy gradient methods require:

$$\nabla_\theta[\mathbb{E}_{x,y\sim\pi_\theta}[R(x,y)] - \text{DKL}[\pi_\theta||\pi_{\theta_{\text{old}}}]] = \mathbb{E}_{x,y\sim\pi_\theta}\left[\left(R - \log\frac{\pi_\theta}{\pi_{\theta_{\text{old}}}} - 1\right)\nabla_\theta \log \pi_\theta\right]$$

Off-policy estimation uses importance sampling (Equation 16):

$$\mathbb{E}_{x,y\sim\pi_{\theta_{\text{old}}}}\left[\frac{\pi_\theta(y|x)}{\pi_{\theta_{\text{old}}}(y|x)}\left(R - \log\frac{\pi_\theta}{\pi_{\theta_{\text{old}}}} - 1\right)\nabla_\theta \log \pi_\theta\right]$$

The ratio $\frac{\pi_\theta}{\pi_{\theta_{\text{old}}}}$ can explode $\rightarrow$ gradient clipping needed.

GVPO's Advantage: By using the zero-sum property and central distances, GVPO's gradient becomes:

$$\mathbb{E}_{x,y\sim\pi_s}\left[\left(R - \log\frac{\pi_\theta}{\pi_{\theta'}} - \mathbb{E}_{y\sim\pi_s}\left(R - \log\frac{\pi_\theta}{\pi_{\theta'}}\right)\right)\nabla_\theta \log \pi_\theta\right]$$

No importance sampling ratio appears in the gradient!

Advantage 3: Unbiased and Consistent Estimator (Theorem 3.4)

The empirical loss with finite samples is:

$$\frac{1}{|D|}\sum_{(x,\{y_i\})\in D}\frac{1}{k-1}\sum_{i=1}^{k}\left[(R_\theta(x,y_i) - \bar{R}_\theta) - (R(x,y_i) - \bar{R})\right]^2$$

Note the $\frac{1}{k-1}$ factor (not $\frac{1}{k}$) — this is the **Bessel correction** for unbiased variance estimation.

Why This Matters: - With small k (few samples per prompt), bias becomes significant - Corollary 3.5 extends this to **variable $k(x)$ per prompt**, enabling mixed-source datasets

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4. Algorithm Comparison

Algorithm 1 (GVPO) vs GRPO:

```
GVPO:
1. Sample k responses {yi} ~ s(·|x)
2. Compute weights: wi = (R(x,yi) - R) - (log(/') - log(/'))
3. Update: minimize - wi log (yi|x)

GRPO:
1. Sample k responses {yi} ~ _old(·|x)
2. Compute weights: wi = (R(x,yi) - R) / R
3. Update: minimize - wi log (yi|x)
4. Apply gradient clipping + KL penalty
```

Key Implementation Difference (Listing 1):

GVPO only changes GRPO's loss computation by:

```
# GRPO:
advs = (R - R.mean()) / R.std()  # Standardization
loss = -scores * advs

# GVPO:
advs = (R - R.mean()) - beta * ((scores_new - scores_new.mean())
                              - (scores_old - scores_old.mean()))
loss = -beta * scores * advs / (k-1)  # Note: k-1 for unbiased estimator
```

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5. Empirical Performance (Table 1)

Model	AIME2024	AMC	MATH500	Minerva	OlympiadBench
Base (Qwen2.5-Math-7B)	14.68	38.55	64.00	27.20	30.66
+GRPO	14.79	55.42	**80.00**	41.17	42.07
+Dr.GRPO	16.56	48.19	81.20	44.48	43.40

Model	AIME2024	AMC	MATH500	Minerva	OlympiadBench
+GVPO	**20.72**	**62.65**	**83.80**	**45.95**	**46.96**

Observations: - GVPO achieves **best performance across all 5 benchmarks - 40% relative improvement** on AIME2024 over GRPO (14.79 → 20.72) - Particularly strong on complex reasoning tasks (AIME, OlympiadBench)

Ablation Study Insights:

Figure 2 (sensitivity): - GVPO shows **little performance fluctuation** across [0.01, 0.5] - Suggests **robustness to hyperparameter tuning** (unlike GRPO's high sensitivity)

Figure 3 (Scaling with k): - GVPO **consistently outperforms GRPO** for all k [2, 32] - **Superior scalability**: GVPO on 1.5B model with k=32 matches 7B model performance - **Inference cost reduction**: Can use smaller models with more samples

Figure 4 (Off-policy sampling s): - Tests mixing historical responses with current policy samples - GVPO maintains **robust performance** with ratios from 0:8 to 4:4 (historical:current) - Validates Corollary 3.2's theoretical guarantee

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6. Limitations and Future Work

Acknowledged Limitations:

1. **Computational cost**: Still requires sampling k responses per prompt
2. **Reward model quality**: Performance depends on accurate R(x,y)
3. **Hyperparameter** : Though robust, still requires selection

Unexplored Connections:

- Integration with exploration strategies from classical RL
- Extension to continuous action spaces
- Multi-modal reward signals

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7. Summary: Why GVPO is Better

Criterion	GRPO	GVPO
Training Stability	Documented instability [34, 16]	Implicit regularization via Cov/Var terms

Criterion	GRPO	GVPO
Hyperparameter Sensitivity	High (clip threshold, KL coeff)	Robust to variations
Theoretical Guarantee	No convergence to optimal policy	Unique optimal = KL-constrained optimum
Sampling Flexibility	Uses importance sampling	Any s (no IS needed)
Normalization Bias	Std normalization conflates difficulty	Only centering (no std division)
Gradient Explosion	Requires clipping	No IS ratio → inherently stable
Performance	Baseline	**Best across all benchmarks**

Bottom Line

GVPO's core innovation is **operationalizing the closed-form optimal policy** through a mathematically elegant **zero-sum weight design** that: 1. Eliminates the intractable partition function 2. Embeds KL constraints directly into gradients 3. Enables off-policy training without importance sampling 4. Guarantees convergence to the unique optimal policy

This makes GVPO a **theoretically principled AND empirically superior** alternative to GRPO for LLM post-training.