1 Monge-Kantorovich duality in the discrete case

$$\begin{aligned} \max_{\pi_{xy}\geq 0} \quad & \sum_{xy} \pi_{xy}\Phi_{xy} \\ & \sum_{y} \pi_{xy} = p_x \ \ [u_x] \\ & \sum_{x} \pi_{xy} = q_y \ \ [v_y] \end{aligned}$$

$$\max_{\pi_{xy}\geq 0} \sum_{xy} \pi_{xy}\Phi_{xy} + \sum_{x} \min_{u_x} u_x \left(p_x - \sum_{y} \pi_{xy} \right) + \sum_{y} \min_{v_y} v_y \left(q_y - \sum_{x} \pi_{xy} \right)$$

$$\max_{\pi_{xy}\geq 0} \min_{u_x, v_y} \sum_{xy} \pi_{xy}\Phi_{xy} + \sum_{x} u_x p_x + \sum_{y} v_y q_y - \sum_{xy} \pi_{xy} \left(u_x + v_y \right)$$

$$\min_{u_x, v_y} \sum_{x} u_x p_x + \sum_{y} v_y q_y + \max_{\pi_{xy}\geq 0} \sum_{xy} \pi_{xy} \left(\Phi_{xy} - u_x - v_y \right)$$

thus

$$\begin{aligned} \min_{u_x, v_y} \quad & \sum_{x} u_x p_x + \sum_{y} v_y q_y \\ s.t. \quad & u_x + v_y \geq \Phi_{xy} \end{aligned}$$

1.1 Kantorovich duality

Primal

$$\begin{aligned} \max_{\pi_{xy}\geq 0} \quad & \sum_{xy} \pi_{xy}\Phi_{xy} \\ & \sum_{y} \pi_{xy} = p_x \ \ [u_x] \\ & \sum_{x} \pi_{xy} = q_y \ \ [v_y] \end{aligned}$$

Dual

$$\begin{aligned} \min_{u_x, v_y} \quad & \sum_{x} u_x p_x + \sum_{y} v_y q_y \\ s.t. \quad & u_x + v_y \geq \Phi_{xy} \ \ [\pi_{xy} \geq 0] \end{aligned}$$

By complementary slackness, $\pi_{xy} > 0 \implies u_x + v_y = \Phi_{xy}$.

INterpretation: u_x=equilibirum wage of worker x v_y=equilibrium profit of firm y.

$u_x + v_y \geq \Phi_{xy}$.

Claim: If (u, v) is a solution to the dual problem, then

$$v_y = \max_{x \in \mathcal{X}} \{\Phi_{xy} - u_x\} \text{ and } u_x = \max_{y \in \mathcal{Y}} \{\Phi_{xy} - v_y\}.$$

Economic interpretation: u_x and v_y are the worker's and firm's indirect utilities.

2 One-dimensional case

Consider the case $\mathcal{X} = \mathcal{Y} = \mathbb{R}$

and $\Phi(x, y) = xy$

Monge-Kantorovich duality: primal

$$\begin{aligned} \max_{(X,Y)} & \quad E[XY] \\ s.t. X & \sim P, Y \sim Q \end{aligned}$$

and the dual

$$\begin{aligned} \min_{u,v} & \quad \int u(x)\, dP(x) + \int v(y)\, dQ(y) \\ s.t. & \quad u(x) + v(y) \geq xy \end{aligned}$$

If u and v (solutions to the dual) exist then

$$v(y) = \max_{x \in R} \{xy - u(x)\} \text{ and } u(x) = \max_{y \in R} \{xy - v(y)\}.$$

by first order conditions if $x \in \arg\max_{x \in R} \{xy - u(x)\}$, then

$$u'(x) = y.$$

Assume P and Q is absolutely continuous with respect Lebesgue.

Theorem: the solution to the primal problem

$$\begin{aligned} \max_{(X,Y)} & \quad E[XY] \\ s.t. X & \sim P, Y \sim Q \end{aligned}$$

is such that $T(.)$ is increasing and continuous, and such that $T \# P = Q$.

Claim: in this case,

part (1) $T(x) = F_Q^{-1}(F_P(x))$.

Indeed, $T(X) \sim Q$ when $X \sim P$ implies

$$P(T(X) \leq y) = F_Q(y)$$

thus

$$\begin{aligned}
\Pr\left(X \leq T^{-1}(y)\right) &= F_Q(y) \\
F_P\left(T^{-1}(y)\right) &= F_Q(y) \\
T^{-1}(y) &= F_P^{-1} \circ F_Q \\
T &= F_Q^{-1} \circ F_P.
\end{aligned}$$

part (2) a solution (u, v) to the dual problem is given using

$$u(x) = \int_0^x F_Q^{-1}(F_P(z))\, dz$$

and $v(y) = \max_x \{xy - u(x)\}$.

Particular case: when $P = \mathcal{U}([0,1])$, $F_P(x) = x$ on $[0,1]$, and

$$T(x) = F_Q^{-1}(x)$$

is the quantile map of Q and also

$$u'(x) = F_Q^{-1}(x).$$

This could have given us a definition of quantile!

3 Multivariate case and a notion of multivariate quantiles

Now assume that $\mathcal{X} = \mathcal{Y} = \mathbb{R}^d$ and $\Phi(x, y) = x^\top y$.

Assume that we are in the conditions that give existence of dual solutions (u, v) (e.g. compactness of support of P and Q).

Then we have

$$v(y) = \max_{x \in \mathbb{R}^d} \left\{x^\top y - u(x)\right\} \text{ and } u(x) = \max_{y \in \mathbb{R}^d} \left\{x^\top y - v(y)\right\}.$$

Note that $u(x)$ and $v(y)$ are convex functions.

If x is matched to y, we get by FOC

$$\frac{\partial u(x)}{\partial x_i} = y_i$$

hence

$$\nabla u(x) = y$$

Hence, we are looking for a convex function $u : R^d \to R$ such that if $X \sim P$, then

$$\nabla u(X) \sim Q.$$

Because $\nabla u(x)$ is the gradient of a convex function

$$(\nabla u(x) - \nabla u(x'))^\top (x - x') \geq 0$$

Let's see what this problem is in dimension 1. In dimension 1, $\nabla u(x) = u'(x)$. The derivative of a convex function is a nondecreasing function, call it $T(x)$. Thus, in dimension 1, this problem consists of looking for a nondecreasing function T such that

$$T(X) \sim Q$$

which is $T(x) = F_Q^{-1} \circ F_P(x)$.