1 Monge-Kantorovich duality in the discrete case

$$\begin{aligned} \max_{\pi_{xy}\geq 0} \quad & \sum_{xy} \pi_{xy}\Phi_{xy} \\ & \sum_{y} \pi_{xy} = p_x \ \ [u_x] \\ & \sum_{x} \pi_{xy} = q_y \ \ [v_y] \end{aligned}$$

$$\max_{\pi_{xy}\geq 0} \sum_{xy} \pi_{xy}\Phi_{xy} + \sum_{x} \min_{u_x} u_x \left(p_x - \sum_{y} \pi_{xy} \right) + \sum_{y} \min_{v_y} v_y \left(q_y - \sum_{x} \pi_{xy} \right)$$

$$\max_{\pi_{xy}\geq 0} \min_{u_x,v_y} \sum_{xy} \pi_{xy}\Phi_{xy} + \sum_{x} u_x p_x + \sum_{y} v_y q_y - \sum_{xy} \pi_{xy} \left(u_x + v_y \right)$$

$$\min_{u_x,v_y} \sum_{x} u_x p_x + \sum_{y} v_y q_y + \max_{\pi_{xy}\geq 0} \sum_{xy} \pi_{xy} \left(\Phi_{xy} - u_x - v_y \right)$$

thus

$$\begin{aligned} \min_{u_x,v_y} \quad & \sum_{x} u_x p_x + \sum_{y} v_y q_y \\ s.t. \quad & u_x + v_y \geq \Phi_{xy} \end{aligned}$$

1.1 Kantorovich duality

Primal

$$\begin{aligned} \max_{\pi_{xy}\geq 0} \quad & \sum_{xy} \pi_{xy}\Phi_{xy} \\ & \sum_{y} \pi_{xy} = p_x \ \ [u_x] \\ & \sum_{x} \pi_{xy} = q_y \ \ [v_y] \end{aligned}$$

Dual

$$\begin{aligned} \min_{u_x,v_y} \quad & \sum_{x} u_x p_x + \sum_{y} v_y q_y \\ s.t. \quad & u_x + v_y \geq \Phi_{xy} \ \ [\pi_{xy} \geq 0] \end{aligned}$$

By complementary slackness, $\pi_{xy} > 0 \implies u_x + v_y = \Phi_{xy}$.

INterpretation: u_x=equilibirum wage of worker x v_y=equilibrium profit of firm y.

$u_x + v_y \geq \Phi_{xy}$.

Claim: If (u, v) is a solution to the dual problem, then

$$v_y = \max_{x \in \mathcal{X}} \{\Phi_{xy} - u_x\} \text{ and } u_x = \max_{y \in \mathcal{Y}} \{\Phi_{xy} - v_y\}.$$

Economic interpretation: u_x and v_y are the worker's and firm's indirect utilities.

2 One-dimensional case

Consider the case $\mathcal{X} = \mathcal{Y} = \mathbb{R}$

and $\Phi(x, y) = xy$

Monge-Kantorovich duality: primal

$$\begin{array}{lcl} \max\limits_{(X,Y)} & & E[XY] \\ s.t.X & \sim & P, Y \sim Q \end{array}$$

and the dual

$$\begin{array}{ll} \min\limits_{u,v} & \int u(x)\, dP(x) + \int v(y)\, dQ(y) \\ s.t. & u(x) + v(y) \geq xy \end{array}$$

If u and v (solutions to the dual) exist then

$$v(y) = \max_{x \in R} \{xy - u(x)\} \text{ and } u(x) = \max_{y \in R} \{xy - v(y)\}.$$

by first order conditions if $x \in \arg\max_{x \in R} \{xy - u(x)\}$, then

$$u'(x) = y.$$

Assume P and Q is absolutely continuous with respect Lebesgue.

Theorem: the solution to the primal problem

$$\begin{array}{lcl} \max\limits_{(X,Y)} & & E[XY] \\ s.t.X & \sim & P, Y \sim Q \end{array}$$

is such that $T(.)$ is increasing and continuous, and such that $T\#P = Q$.

Claim: in this case,

part (1) $T(x) = F_Q^{-1}(F_P(x))$.

Indeed, $T(X) \sim Q$ when $X \sim P$ implies

$$P(T(X) \leq y) = F_Q(y)$$

thus

$$\begin{aligned}\Pr\left(X \leq T^{-1}(y)\right) &= F_Q(y)\\ F_P\left(T^{-1}(y)\right) &= F_Q(y)\\ T^{-1}(y) &= F_P^{-1} \circ F_Q\\ T &= F_Q^{-1} \circ F_P.\end{aligned}$$

part (2) a solution (u, v) to the dual problem is given using

$$u(x) = \int_0^x F_Q^{-1}(F_P(z))\, dz$$

and $v(y) = \max_x \{xy - u(x)\}$.

Particular case: when $P = \mathcal{U}([0,1])$, $F_P(x) = x$ on $[0,1]$, and

$$T(x) = F_Q^{-1}(x)$$

is the quantile map of Q and also

$$u'(x) = F_Q^{-1}(x).$$

This could have given us a definition of quantile!

3 Multivariate case and a notion of multivariate quantiles

Now assume that $\mathcal{X} = \mathcal{Y} = \mathbb{R}^d$ and $\Phi(x, y) = x^\top y$.

Assume that we are in the conditions that give existence of dual solutions (u, v) (e.g. compactness of support of P and Q).

Then we have

$$v(y) = \max_{x \in \mathbb{R}^d} \left\{x^\top y - u(x)\right\} \text{ and } u(x) = \max_{y \in \mathbb{R}^d} \left\{x^\top y - v(y)\right\}.$$

Note that $u(x)$ and $v(y)$ are convex functions.

If x is matched to y, we get by FOC

$$\frac{\partial u(x)}{\partial x_i} = y_i$$

hence

$$\nabla u(x) = y$$

Hence, we are looking for a convex function $u : R^d \to R$ such that if $X \sim P$, then

$$\nabla u(X) \sim Q.$$

Because $\nabla u(x)$ is the gradient of a convex function

$$\left(\nabla u(x) - \nabla u(x')\right)^{\top}(x - x') \geq 0$$

Let's see what this problem is in dimension 1. In dimension 1, $\nabla u(x) = u'(x)$. The derivative of a convex function is a nondecreasing function, call it $T(x)$. Thus, in dimension 1, this problem consists of looking for a nondecreasing function T such that

$$T(X) \sim Q$$

which is $T(x) = F_Q^{-1} \circ F_P(x)$.

4 Day 2

4.1 Computation of discrete OT problems as LP problems

$$\begin{aligned}
\max_{\pi_{xy} \geq 0} \quad & \sum_{xy} \pi_{xy} \Phi_{xy} \\
& \sum_{y} \pi_{xy} = p_x \;\; [u_x] \\
& \sum_{x} \pi_{xy} = q_y \;\; [v_y]
\end{aligned}$$

$vec(\Pi)$ using the Fortran convention $\Pi_{11}, \Pi_{21}, \Pi_{31}$ -stack the columns

$$\begin{aligned}
\max_{\pi_{xy} \geq 0} \quad & \sum_{xy} \pi_{xy} \Phi_{xy} \\
& \Pi 1_{\mathcal{Y}} = p \\
& \Pi^{\top} 1_{\mathcal{X}} = q
\end{aligned}$$

One has

$$vec\left(BXA^{\top}\right) = (A \otimes B)\, vec(X)$$

thus

$$\begin{aligned}
vec\left(I_{\mathcal{X}} \Pi 1_{\mathcal{Y}}\right) &= p \\
\left(1_{\mathcal{Y}}^{\top} \otimes I_{\mathcal{X}}\right) vec(\Pi) &= p
\end{aligned}$$

similarly

$$\begin{aligned}
vec\left(1_{\mathcal{X}}^{\top} \Pi I_{\mathcal{Y}}\right) &= vec\left(q^{\top}\right) = q \\
\left(I_{\mathcal{Y}} \otimes 1_{\mathcal{X}}^{\top}\right) vec(\Pi) &= q
\end{aligned}$$

therefore, calling $\pi = vec\,(\Pi)$

$$\begin{aligned} & \max_{\pi \geq 0} \pi^\top vec\,(\Phi) \\ s.t. \quad & \left(1_{\mathcal{Y}}^\top \otimes I_{\mathcal{X}}\right)\pi = p \\ & \left(I_{\mathcal{Y}} \otimes 1_{\mathcal{X}}^\top\right)\pi = q \end{aligned}$$

thus the constraint matrix

$$A = \begin{pmatrix} 1_{\mathcal{Y}}^\top \otimes I_{\mathcal{X}} \\ I_{\mathcal{Y}} \otimes 1_{\mathcal{X}}^\top \end{pmatrix}$$

4.2 Quantile regression with shape constraints

$$V\,(\tau) = 1\left\{Y \geq X^\top \beta\,(\tau)\right\}$$

Natural shape restriction

$$\tau \in [0,1] \to x^\top \beta\,(\tau) \text{ is increasing}$$

this implies that

$$\tau \to V\,(\tau) = 1\left\{Y \geq X^\top \beta\,(\tau)\right\} \text{ is weakly decreasing a.s.}$$

We have transformed the problem into

$$\begin{aligned} \max_{\pi \geq 0} \quad & U^\top \pi Y \\ s.t. \quad & \pi X = \mu\,\bar{x} \\ & \pi^\top 1_T = p \end{aligned}$$

The objective function

$$\sum_{tj} \pi_{tj} U_t Y_j = E_\pi\,[UY]$$

the constraints

$$\sum_j \pi_{tj} X_{jk} = \mu_t \bar{x}_k$$

Let's assume 1 is included in the regressors, i.e. $X_{j1} = 1$. In this case

$$\sum_j \pi_{tj} = \mu_t$$

Finally

$$\sum_j \pi_{tj} X_{jk} = \mu_t \bar{x}_k,\; k \geq 2$$

rewrites as

$$\frac{\sum_j \pi_{tj} X_{jk}}{\sum_j \pi_{tj}} = \bar{x}_k,\; k \geq 2$$

that is

$$E_\pi \left[X_k | U\right] = \bar{x}_k$$

therefore, we are led

$$\begin{aligned} \max_{\pi(u,(x,y))\geq 0} \quad & E_\pi \left[UY\right] \\ s.t. \quad & U \sim \mu \\ & (X,Y) \sim P \\ & E\left[X^k | U\right] = E\left[X^k\right], k \geq 2 \end{aligned}$$

The last constraint $E\left[X^k | U\right] = E\left[X^k\right]$ means that X should be mean-independent from U.

Let's see a version of the problem with full independence

$$\begin{aligned} \max_{\pi(u,(x,y))\geq 0} \quad & E_\pi \left[UY\right] \\ s.t. \quad & U \sim \mu \\ & (X,Y) \sim P \\ & X \text{ and } U \text{ independent} \end{aligned}$$

write

$$E_\pi \left[UY\right] = E\left[E\left[UY|X\right]\right]$$

this is maximized by taking for each x the optimal coupling between $U \sim \mathcal{U}\left([0,1]\right)$ and $P_{Y|X=x}$. This is given to us by $Y = Q_{Y|X}\left(U|x\right)$. Thus the soluton to the problem with full independence is given to us by

$$Y = Q_{Y|X}\left(U|X\right)$$

and hence

$$U = F_{Y|X}\left(Y|X\right).$$

Back to mean-independence, write the dual of the above primal problem

$$\begin{aligned} \max_{\pi(u,(x,y))\geq 0} \quad & E_\pi \left[UY\right] \\ s.t. \quad & U \sim \mu \\ & (X,Y) \sim P \\ & E\left[X^k | U\right] = E\left[X^k\right], k \geq 2 \end{aligned}$$

the dual to this problem is

$$\begin{aligned} \min_{\varphi(u),\psi(x,y),b^k(u)} \quad & E\left[\varphi\left(U\right)\right] + E\left[\psi\left(X,Y\right)\right] + E\left[\bar{x}^\top b\left(U\right)\right] \\ s.t. \quad & \varphi\left(u\right) + \psi\left(x,y\right) + x^\top b\left(u\right) \geq x^\top y \end{aligned}$$