Lecture 1: linear programming basics

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Companion notebook

https://www.math-econ-code.org/linear-programming-duality

- Linear programming duality
- Economic interpretation of the dual
- Numerical computation

- Galichon (2016). *Optimal Transport Methods in Economics*. App. B
- Stigler (1945), The cost of subsistence. *Journal of Farm Economics*.
- Dantzig (1990), The diet problem. *Interface*.
- Complements:
  - Gale (1960), *The theory of linear economic models*.
  - Vohra (2011), *Mechanism Design: A Linear Programming Approach*.
- www.gurobi.com
- www.gnu.org/software/glpk/

Section 1

Motivation: the diet problem

Stigler's diet problem

- During World War II, engineers in US Army were wondering how to feed their personnel at minimal cost, leading to what is now called the "optimal diet problem".
  - Nutritionists have identified a number of vital nutrients (calories, protein, calcium, iron, etc.) that matter for a person's health, and have determined the minimum daily intake of each nutrient
  - For each basic food (pasta, butter, bread, etc), nutritionists have characterized the intake in each of the various nutrients
  - Each food has a unit cost, and the problem is to find the optimal diet = combination of foods that meet the minimal intake in each of the nutrients and achieves minimal cost
- The problem was taken on by G. Stigler, who published a paper about it in 1945, giving a first heuristic solution, exhibiting a diet that costs \$39.93 per year in 1939 dollars. Later (in 1947) it was one of the first application of G.B. Dantzig's method (the simplex algorithm), which provided the exact solution (\$39.67). It then took 120 man-day to perform this operation. At the end of this block, the computer will perform it for us in a fraction of second.
- However, don't try this diet at home! Dantzig did so and almost died from it...

- Problem setup:
  - Assume there are nutrients $i \in \{1, ..., m\}$ (calories, protein, calcium, iron, etc.) that matter for a person's health, in such way that the minimum daily intake of nutrient i should be d_i.
  - Nutrients do not come as standalone elements, but are combined into various foods. Each unit of food $j \in \{1, ..., n\}$ yields a quantity N_{ij} of nutrient $i \in \{1, ..., m\}$. The dollar cost of food j is c_j.
- The problem is to find the diet that achieves the minimal intake of each nutrient at a cheapest price. If $q \in \mathbb{R}^n$ is a vector such that $q_j \geq 0$ is the quantity of food i purchased, the quantity of nutrient i ingested is $\sum_{j=1}^n N_{ij} q_j$, and the cost of the diet is $\sum_{j=1}^n q_j c_j$. The optimal diet is therefore given by

$$\min_{q \geq 0} c^\top q \tag{1}$$
$$s.t.\ Nq \geq d.$$

Section 2

A crash course on linear programming

- Let $c \in \mathbb{R}^n$, $d \in \mathbb{R}^m$, A be a $m \times n$ matrix, and consider the following problem

$$V_P = \max_{x \in \mathbb{R}^n_+} c^\top x \qquad (2)$$
$$s.t.\ Ax = d$$

  This problem is a *linear programming problem*, as the objective function, namely $x \to c^\top x$ is linear, and as the constraint, namely $x \in \mathbb{R}^n_+$ and $Ax = d$ are also linear (or more accurately, affine). Problem (**??**) is called *primal program*, for reasons to be explained soon. The set of x's that satisfy the constraint are called *feasible solutions*; the set of solutions of problem (**??**) are called *optimal solutions*.

- Remarks:
  - The previous diet problem can be reformulate into this problem – why?
  - A problem does not necessarily have a feasible solution (e.g. if $A = 0$ and $d \neq 0$), in which case (by convention) $V_P = -\infty$.
  - The whole space may be solution (e.g. if $A = 0$ and $d = 0$), in which case $V_P = +\infty$.

There is a powerful tool called duality which provides much insight into the analysis of problem (**??**). The idea is to rewrite the problem as

$$V_P = \max_{x \in \mathbb{R}_+^n} \left\{ c^\top x + L_P \left(d - Ax \right) \right\}$$

where $L_P(z)$ is a penalty function whose value is zero if the constraint is met, that is if $z = 0$, and $-\infty$ if it is not, namely if $z \neq 0$. The simplest choice of such penalty function is given by $L_P(z) = \min_{y \in \mathbb{R}^m} \left\{ z^\top y \right\}$. One has

$$V_P = \max_{x \in \mathbb{R}_+^n} \min_{y \in \mathbb{R}^m} \left\{ c^\top x + \left(d - Ax \right)^\top y \right\}.$$

However, the minimax inequality $\max_x \min_y \leq \min_y \max_x$ always holds, thus

$$V_P \leq \min_{y \in \mathbb{R}^m} \max_{x \in \mathbb{R}^n_+} \left\{ c^\top x + (d - Ax)^\top y \right\} = \min_{y \in \mathbb{R}^m} \max_{x \in \mathbb{R}^n_+} \left\{ x^\top \left(c - A^\top y \right) + d^\top y \right\}$$

$$\leq \min_{y \in \mathbb{R}^m} \left\{ d^\top y + L_D \left(c - A^\top y \right) \right\} =: V_D$$

where $L_D(z) = \max_{x \in \mathbb{R}^n_+} \left\{ x^\top z \right\}$ is equal to 0 if $z \in \mathbb{R}^n_-$, and to $+\infty$ if not. Therefore, the value V_D is expressed by the *dual program*

$$V_D = \min_{y \in \mathbb{R}^m} d^\top y, \tag{3}$$
$$s.t. \; A^\top y \geq c$$

and the weak duality inequality $V_P \leq V_D$ holds. It turns out that as soon as either the primal or dual program has an optimal solution, then both programs have an optimal solution and the values of the two programs coincide, so the weak duality becomes an equality $V_P = V_D$ called strong duality. Further, if $x^* \in \mathbb{R}^n_+$ is an optimal primal solution, and $y^* \in \mathbb{R}^m$ is an optimal dual solution, then complementary slackness holds, that is $x^*_i > 0$ implies $\left(A^\top y^* \right)_i = c_i$.

We summarize these results into the following statement.
Theorem. In the setting described above:
(i) The weak duality inequality holds:

$$V_P \leq V_D.$$

(ii) As soon as the primal or the dual program have an optimal solution, then both programs have an optimal solution, and strong duality holds:

$$V_P = V_D.$$

(iii) If $x^* \in \mathbb{R}^n_+$ is an optimal primal solution, and $y^* \in \mathbb{R}^m$ is an optimal dual solution, then complementary slackness holds:

$$x_i^* > 0 \text{ implies } \left(A^\top y^*\right)_i = c_i.$$

- Recall the optimal diet problem

$$\min_{q\geq 0} c^\top q$$
$$s.t.\ Nq \geq d.$$

which has minimax formulation $\min_{q\geq 0}\max_{\pi\geq 0} c^\top q + d^\top \pi - q^\top N^\top \pi$, so the dual is

$$\max_{\pi\geq 0} d^\top \pi$$
$$s.t.\ N^\top \pi \leq c$$

- Interpretation: imagine that there is a new firm called Nutrient Shoppe, who sells raw nutrients. Let π_i be the price of nutrient i. The cost of the diet is $d^\top \pi$. Consumer purchase raw nutrients and can generate "synthetic" foods. The cost of the synthetic version of food j is $\sum_{i=1}^m N_{ij}\pi_i = (N^\intercal \pi)_j$. The constraint thus means that each "synthetic" food is more affordable than its natural counterpart.

- The duality means that it is possible to price the nutrients so that the synthetic foods are cheaper than the natural ones, in such a way that the price of the synthetic diet equals the price of the natural diet.
- Complementary slackness yields:
  - $q_j > 0$ implies $\left(N^{\intercal}\pi\right)_j = c_j$; that is, if natural food j is actually purchased, then the prices of its synthetic and natural versions coincide
  - $\pi_i > 0$ implies $\left(Nq\right)_i = d_i$; that is, if nutrient i has a positive price, then the natural diet has the "just right" amount.