Lecture 2a: the optimal assignment problem

Alfred Galichon (New York University)

Optimization-Conscious Econometrics Summer School
University of Chicago, June 6, 2023

Companion notebook

https://www.math-econ-code.org/optimal-assignment

- Optimal assignment problem
- Pairwise stability, Walrasian equilibrium
- Computation

- Galichon (2016). *Optimal Transport Methods in Economics*, Ch. 3
- Roth, Sotomayor(1990). *Two-Sided Matching*. Cambridge.
- Koopmans and Beckmann (1957). "Assignment problems and the location of economic activities." *Econometrica*.
- Shapley and Shubik (1972). "The assignment game I: The core." *IJGT*.
- Becker (1993). *A Treatise of the Family*. Harvard.
- Gretsky, Ostroy, and Zame (1992). "The nonatomic assignment model." *Economic Theory*.
- Burkard, Dell'Amico, and Martello (2012). *Assignment Problems*. SIAM.
- Dupuy and Galichon (2014). "Personality traits and the marriage market." *JPE*.

Section 1

Motivation

- Consider the problem of assigning a possibly infinite number of workers and firms.
  - Each worker should work for one firm, and each firm should hire one worker.
  - Workers and firms have heterogenous characteristics; let $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ be the characteristics of workers and firms respectively.
  - Workers and firms are in equal mass, which is normalized to one. The distribution of worker's types is P, and the distribution of the firm's types is Q, where P and Q are probability measures on $\mathcal{X}$ and $\mathcal{Y}$.
- It is assumed that if a worker x matches with a firm y, the total output generated is Φ_{xy}. The questions are then:
  - optimality: what is the optimal assignment in the sense that it maximizes the overal output generated?
  - equilibrium: what are the equilibrium assignment and the equilibrium wages
  - efficiency: do these two notions coincide?
- The same tools have been used by Gary Becker to study the heterosexual marriage market, where x is the man's characteristics, and y is the woman's characteristics, and "wages" are replaced by "transfers".

- In this block, we shall take a first look at marriage data (while a worker-firm example will be seen in next block). Dupuy and Galichon (JPE, 2014) study a marriage dataset where, in addition to usual socio-demographic variables (such as education and age), measures of personality traits are reported.
  - The literature on quantitative psychology argues that one can capture relatively well an individual's personality along five dimensions, the "big 5" – consciousness, extraversion, agreableness, emotional stability, autonomy – assessed though a standardized questionaire.
  - In addition to this, we observed a (self-assessed) measure of health, risk-aversion, education, height and body mass index = weight in kg/ (height in m)^2. In total, the available characteristics x_i of man i and y_j of woman j are both 10-dimensional vectors.
  - It is assumed that the surplus of interaction is given by $\Phi\left(x_i, y_j\right) = x_i^\intercal A y_j$, where A is a *given* 10x10 matrix. (later in this course, we'll see how to estimate A based on matched marital data).
- Today, we solve a central planner's problem (a stylized version of the problem OKCupids would solve): given a population of men and a population of women, how do we mutually assign these in order to 1) maximize matching surplus 2) attain a (hopefully) stable assignment.

Section 2

The discrete Monge-Kantorovich theorem

- Assume that the type spaces $\mathcal{X}$ and $\mathcal{Y}$ are finite, so $\mathcal{X} = \{1, ..., N\}$, and $\mathcal{Y} = \{1, ..., M\}$.
- The total mass of workers and jobs is normalized to one. The mass of workers of type x is p_x; the mass of jobs of type y is q_y, with $\sum_x p_x = \sum_y q_y = 1$.
- Let π_{xy} be the mass of workers of type x assigned to jobs of type y. Every worker is busy and every job is filled, thus

$$\sum_{y \in \mathcal{Y}} \pi_{xy} = p_x \text{ and } \sum_{x \in \mathcal{X}} \pi_{xy} = q_y. \tag{1}$$

(Note that this formulation allows for mixing, i.e. it allows for $\pi_{xy} > 0$ and $\pi_{xy'} > 0$ to hold simultaneously with $y \neq y'$.) The set of $\pi \geq 0$ satisfying (1) is denoted by

$$\pi \in \mathcal{M}(p, q).$$

- Assume the economic output created when assigning worker x to job y is Φ_{xy}. Hence, under assignment π, the total output is $\sum_{xy} \pi_{xy} \Phi_{xy}$.
- Thus, the optimal assignment is

$$\begin{aligned} & \max_{\pi \geq 0} \sum_{xy} \pi_{xy} \Phi_{xy} \\ s.t. \ & \sum_{y \in \mathcal{Y}} \pi_{xy} = p_x \ \ [u_x] \\ & \sum_{x \in \mathcal{X}} \pi_{xy} = q_y \ \ [v_y] \end{aligned} \tag{2}$$

  and it is now a finite-dimensional linear programming problem.
- Note that it is nothing else than the Monge-Kantorovich problem when P and Q are discrete probability measures on $\mathcal{X} = \{1, ..., N\}$, and $\mathcal{Y} = \{1, ..., M\}$.

Theorem

(i) The value of the primal problem (2) coincides with the value of the dual problem

$$\min_{u,v} \sum_{x\in\mathcal{X}} p_x u_x + \sum_{y\in\mathcal{Y}} q_y v_y. \tag{3}$$

$$s.t. u_x + v_y \geq \Phi_{xy} \ \left[\pi_{xy} \geq 0\right]$$

(ii) Both the primal and the dual problems have optimal solutions. If π is a solution to the primal problem and (u, v) a solution to the dual problem, then by complementary slackness,

$$\pi_{xy} > 0 \textit{ implies } u_x + v_y = \Phi_{xy}. \tag{4}$$

- Note that this result is the min-cost flow duality theorem in the bipartite case, as seen in block 2, after setting transportation cost through $xy \in \mathcal{X} \times \mathcal{Y}$ to $c_{xy} = -\Phi_{xy}$, and $n_t = -p_t 1\left\{t \in \mathcal{X}\right\} + q_t 1\left\{t \in \mathcal{Y}\right\}$. We see various new interpretations of the result.

The proof follows from the min-cost flow duality result, but let us rewrite it anyway. (i) The value of the primal problem (*2*) can be written as $\max_{\pi \geq 0} \min_{u,v} S(\pi, u, v)$, where

$$S(\pi, u, v) := \sum_{xy} \pi_{xy} \Phi_{xy} + \sum_{x \in \mathcal{X}} u_x (p_x - \sum_{y \in \mathcal{Y}} \pi_{xy}) + \sum_{y \in \mathcal{Y}} v_y (q_y - \sum_{x \in \mathcal{X}} \pi_{xy})$$

but by the minmax theorem, this value is equal to $\min_{u,v} \max_{\pi \geq 0} S(\pi, u, v)$, which is the value of the dual problem (3).

(ii) follows by noting that, for a primal solution π and a dual solution (u, v), then $S(\pi, u, v) = \sum_{xy} \pi_{xy} \Phi_{xy}$. ■

- The following statements are equivalent:
  - π is an optimal solution to the primal problem, and (u, v) is an optimal solution to the dual problem, and
  - (i) $\pi \in M(p, q)$
    (ii) $u_x + v_y \geq \Phi_{xy}$
    (iii) $\pi_{xy} > 0$ implies $u_x + v_y \leq \Phi_{xy}$.
- We saw the direct implication. But the converse is easy: take π and (u, v) satisfying (i)–(iii), Then one has

$$dual \leq \sum_x p_x u_x + \sum_y q_y v_y = \sum_{xy} \pi_{xy} (u_x + v_y) \leq \sum_{xy} \pi_{xy} \Phi_{xy} \leq primal$$

but by the MK duality theorem, both ends coincide. Thus π is optimal for the primal and (u, v) for the dual.

Section 3

Some remarks

- A important variant of the problem exists with $\sum_{x\in\mathcal{X}} p_x \neq \sum_{y\in\mathcal{Y}} q_y$ and the primal constraints become inequality constraints. The duality then becomes

$$\begin{array}{rcr} \max_{\pi\geq 0} \sum \pi_{xy}\Phi_{xy} & = & \min_{u,v} \sum_{x\in\mathcal{X}} p_x u_x + \sum_{y\in\mathcal{Y}} q_y v_y \\ s.t.\ \sum_{y\in\mathcal{Y}} \pi_{xy} \leq p_x & & u \geq 0,\ \ v \geq 0 \\ \sum_{x\in\mathcal{X}} \pi_{xy} \leq q_y & & u_x + v_y \geq \Phi_{xy} \end{array}$$

- In a marriage context, an important concept is stability:
  - An outcome is a vector (π, u, v), where u_x and v_y are x's and y's payoffs, and π is a matching that is
$$\pi \in \mathcal{M}(p, q). \tag{5}$$
  - A pair xy is blocking if x and y can find a way of sharing their joint surplus Φ_{xy} in such a way that x gets more than u_x and y gets more than v_y. Hence there is no blocking pair if and only if for every x and y, one has
$$u_x + v_y \geq \Phi_{xy}. \tag{6}$$
  - If x and y are actually matched, their utilities u_x and v_y need to be feasible, i.e. the above inequality should be saturated. Hence
$$\pi_{xy} > 0 \text{ implies } u_x + v_y = \Phi_{xy} \tag{7}$$
- **Definition**: A matching that satisfies (5), (6), and (7) is called a stable matching.
- As it turns out, these conditions are precisely the conditions that express complementarity slackness in the Monge-Kantorovich problem. Therefore, outcome(π, u, v) is stable if and only if π is a solution to the primal problem, and (u, v) is a solution to the dual problem.

- Back to the workers / firms interpretation and assume for now that workers are indifferent between any two firms that offer the same salary. We argue that $u(x)$ can be interpreted as the equilibrium wage of worker x, while $v(y)$ can be interpreted as the equilibrium profit of firm y. Indeed:

Proposition

If (u, v) is a solution to the dual of the Kantorovich problem, then

$$u_x = \sup_{y \in \mathcal{Y}} \left(\Phi_{xy} - v_y \right) \tag{8}$$

$$v_y = \sup_{x \in \mathcal{X}} \left(\Phi_{xy} - u_x \right). \tag{9}$$

- Therefore, u_x can be interpreted as equilibrium wage of worker x, and v_y as equilibrium profit of firm y. In this interpretation, all workers get the same wage at equilibrium.

Equilibrium wages when workers are not indifferent between firms

- Assume now that if a worker of type x works for a firm of type y for wage w_{xy}, then gets $\alpha_{xy} + w_{xy}$, where α_{xy} is the nonmonetary payoff associated with working with a firm of type y. The firm's profit is $\gamma_{xy} - w_{xy}$, where γ_{xy} is the economic output.
- If an employee of type x matches with a firm of type y, they generate joint surplus Φ_{xy}, given by

$$\Phi_{xy} = \underbrace{\alpha_{xy} + w_{xy}}_{\text{employee's payoff}} + \underbrace{\gamma_{xy} - w_{xy}}_{\text{firm's payoff}} = \alpha_{xy} + \gamma_{xy}$$

which is independent from w.
- Workers choose firms which maximize their utility, i.e. solve

$$u_x = \max_y \{\alpha_{xy} + w_{xy}\} \tag{10}$$

and $u_x = \alpha_{xy} + w_{xy}$ if x and y are matched. Similarly, the indirect payoff vector of firms is

$$v_y = \max_x \{\gamma_{xy} - w_{xy}\} \tag{11}$$

and, again, $v_y = \gamma_{xy} - w_{xy}$ if x and y are matched.

- As a result,

$$u_x + v_y \geq \alpha_{xy} + \gamma_{xy} = \Phi_{xy}$$

  and equality holds if x and y are matched. Thus, if w_{xy} is an equilibrium wage, then the triple (π, u, v) where π is the corresponding matching, and u_x and v_y are defined by (10) and (11) defines a stable outcome.
- Conversely, let (π, u, v) be a stable outcome. Then let $\bar{w}_{xx}$ and $\underline{w}_{xy}$ be defined by

$$\bar{w}_{xy} = u_x - \alpha_{xy} \text{ and } \underline{w}_{xy} = \gamma_{xy} - v_y.$$

- One has $\bar{w}_{xy} \geq \underline{w}_{xy}$. Any w_{xy} such that $\bar{w}_{xy} \geq w_{xy} \geq \underline{w}_{xy}$ is an equilibrium wage. Indeed, $\pi_{xy} > 0$ implies $\bar{w}_{xy} = \underline{w}_{xy}$, thus (10) and (11) hold. Given u and v, w_{xy} is uniquely defined on the equilibrium path (ie. when x and y are such that $\pi_{xy} > 0$), but there are multiple choices of w outside the equilibrium path.
- Note that all workers of the same type get the same indirect utility, but not necessarily the same wage.