Gross Substitutes: Representation and Approximation

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(Google Research)

Joint work with Eric Balkanski, Shahar Dobzinski, Michal Feldman, Uri Feige and Mike Ostrovsky.

What are gross substitutes?

A function $v : 2^{[n]} \to \mathbb{R}$ is gross substitutes iff it satisfies the following discrete inequality: $\partial_i \partial_j v(S) \leq \max\{\partial_i \partial_k v(S), \partial_k \partial_j v(S)\} \leq 0, \forall i, j, k \notin S$

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$$\begin{aligned}\partial_i \partial_j v(S) &= \partial_j v(S \cup \{i\}) - \partial_j v(S) \\ &= v(S \cup \{i, j\}) - v(S \cup \{i\}) - v(S \cup \{j\}) + v(S)\end{aligned}$$

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A function $v : 2^{[n]} \to \mathbb{R}$ is gross substitutes if for any prices $p \in \mathbb{R}^n$, $S \in D(v; p)$ and $p' \geq p$ it exists $S' \in D(v, p')$ s.t. $S' \supseteq S \cap \{i; p_i = p'_i\}$

Demand : $D(v; p) = \underset{S}{\operatorname{argmax}} \left[v(S) - \sum_{i \in S} p_i \right]$

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Necessary and sufficient conditions for Walrasian tatonnement to converge.

[Kelso-Crawford] [Gul-Stachetti]

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Start with $S = \emptyset$ and keep adding $i \in \operatorname{argmax}_i v(S \cup \{i\}) - v(S) - p_i$ while it is positive.

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Other definitions through Discrete Convex Analysis, Tropical Algebraic Geometry, ...

How complex are gross substitutes?

Approach #1: Constructive characterization

Approach #2: Approximation by other classes

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GS function

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simpler functions

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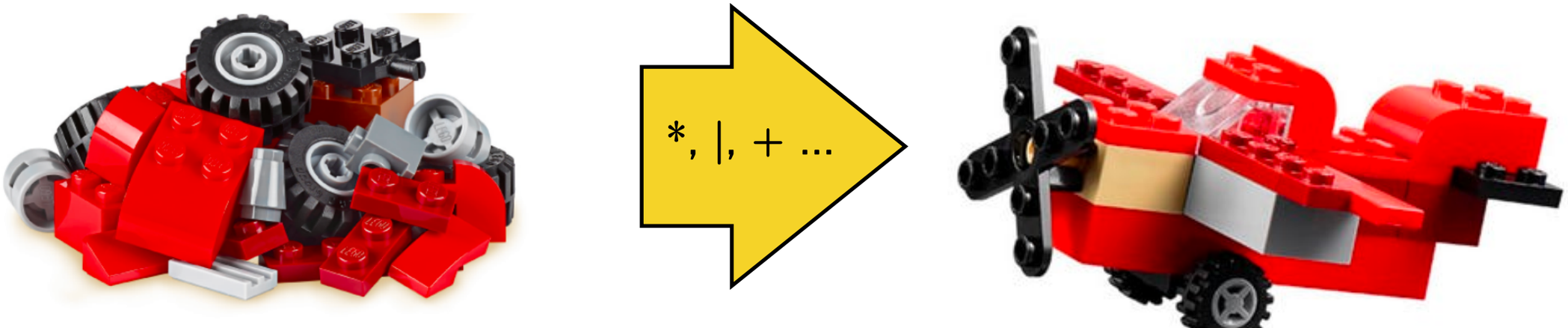
Approach #1: Constructive characterization



simpler functions

GS function

Approach #2: Approximation by other classes

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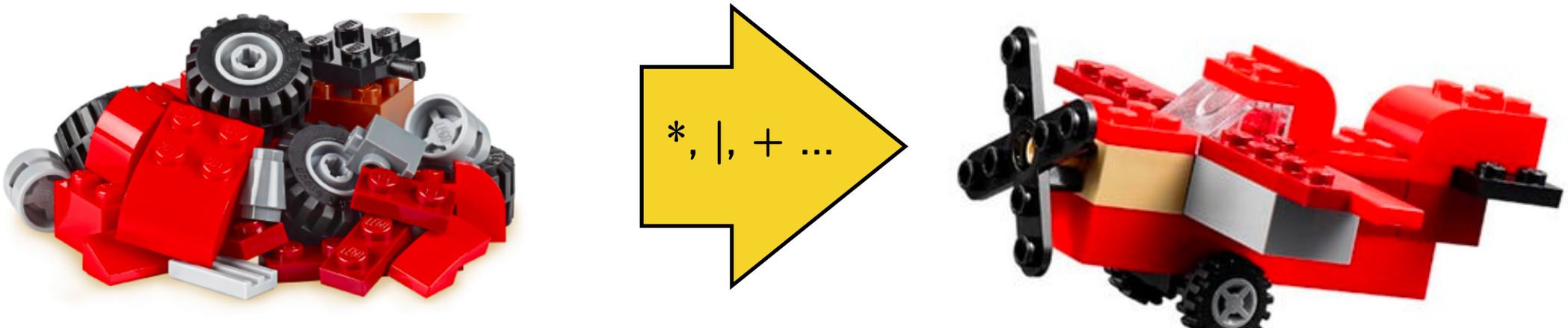

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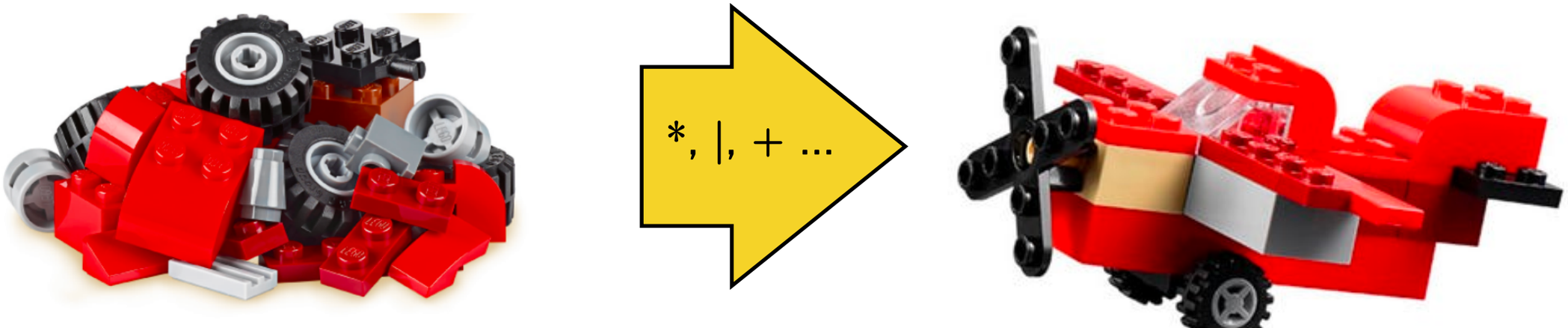

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Approach #2: Approximation by other classes

GS $\subseteq$ Submodular

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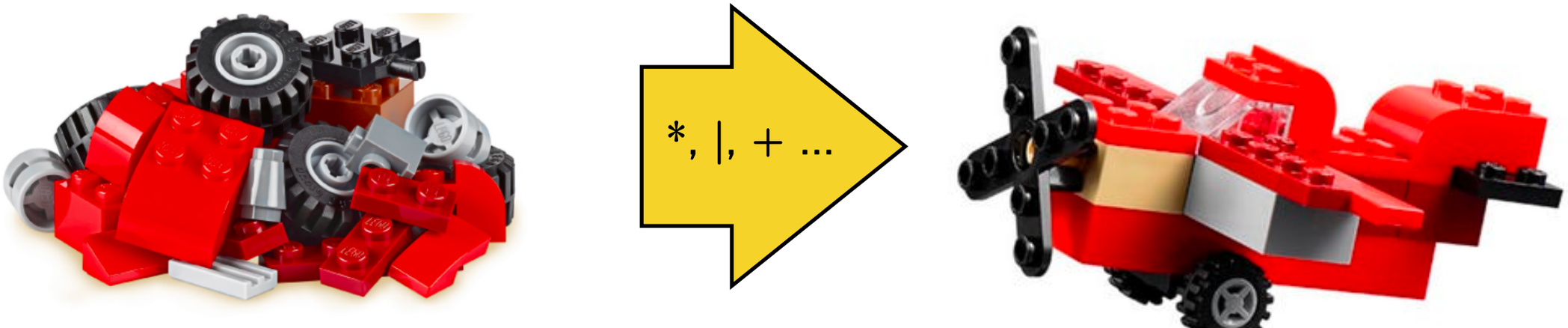

simpler functions

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Approach #2: Approximation by other classes

GS $\subseteq$ Submodular $\subseteq$ XOS

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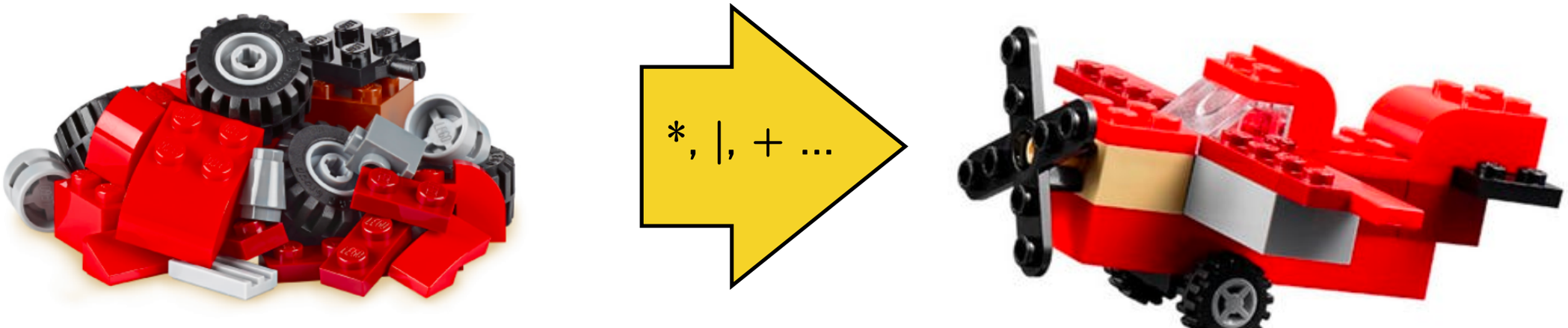

simpler functions

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Approach #2: Approximation by other classes

GS $\subseteq$ Submodular $\subseteq$ XOS $\subseteq$ Subadditive

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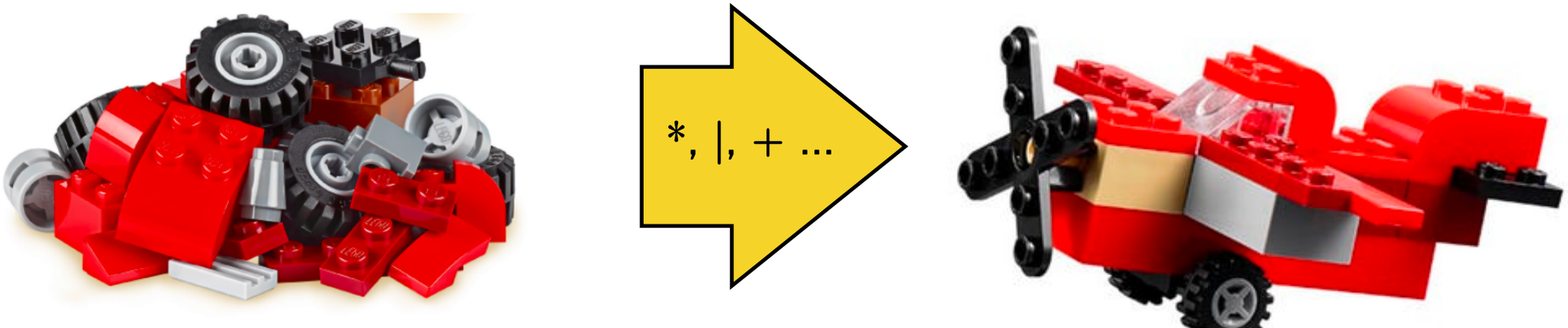

simpler functions

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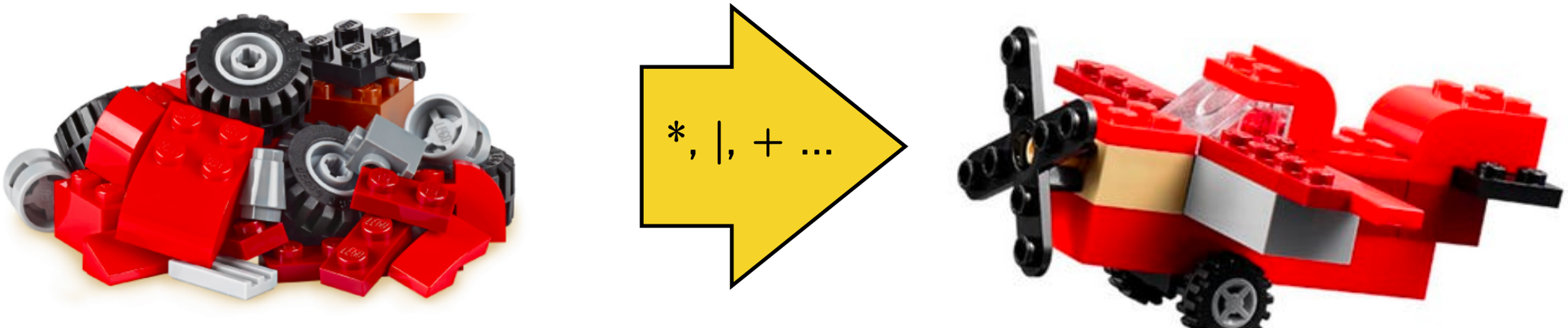
Approach #2: Approximation by other classes

Rado
Valuations $\subsetneq$ **GS** $\subsetneq$ Submodular $\subsetneq$ XOS $\subsetneq$ Subadditive

How complex are gross substitutes?

Approach #1: Constructive characterization



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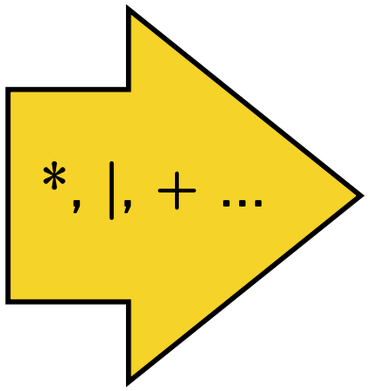
Approach #2: Approximation by other classes

Weighted Matroids
Assignment Valuations $\subseteq$ Rado Valuations $\subseteq$ **GS** $\subseteq$ Submodular $\subseteq$ XOS $\subseteq$ Subadditive

Part I: Toward a Constructive View

simpler functions



GS function

Operations Preserving GS

- Convolution: given $u, v : 2^{[n]} \to \mathbb{R}$, then we define $u * v : 2^{[n]} \to \mathbb{R}$ as: $u * v(S) = \max_{T \subseteq S} u(T) + v(S \setminus T)$

- Endowment: given $v : 2^{[n]} \to \mathbb{R}$ and a set S we define $u : [n] \setminus S \to \mathbb{R}$ as: $u(T) = v(T|S)$

- Affine Transformations: given $v : 2^{[n]} \to \mathbb{R}$, $p \in \mathbb{R}^n$ and $v_0 \in \mathbb{R}$ we can define $u(S) = v_0 + p(S) + v(S)$

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but in general $u + v$ may not be in GS

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- Strong Quotient Sum: given v in GS and a matroid rank function r s.t. $v(S|T) \leq r(S|T), \forall S, T$ then $\alpha v + \beta w$ in GS for $\alpha, \beta \geq 0$

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$$v(S) = \max_{\text{matching} M \subseteq E, \partial_V(M) \subseteq S} [u(\partial_U(M)) + \sum_{e \in M} w_e]$$

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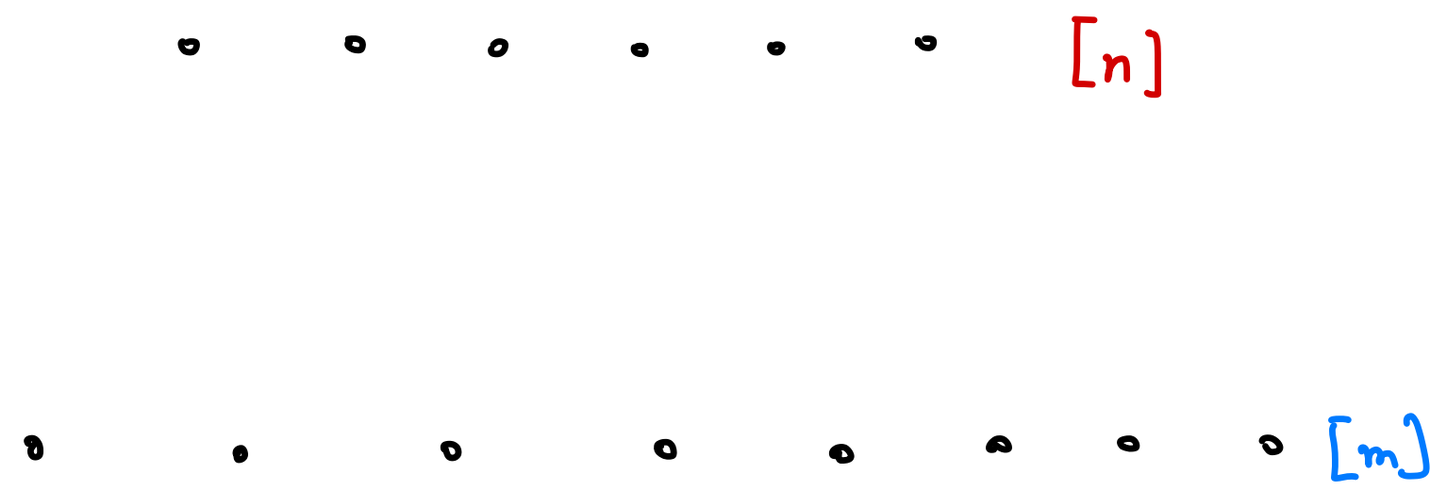

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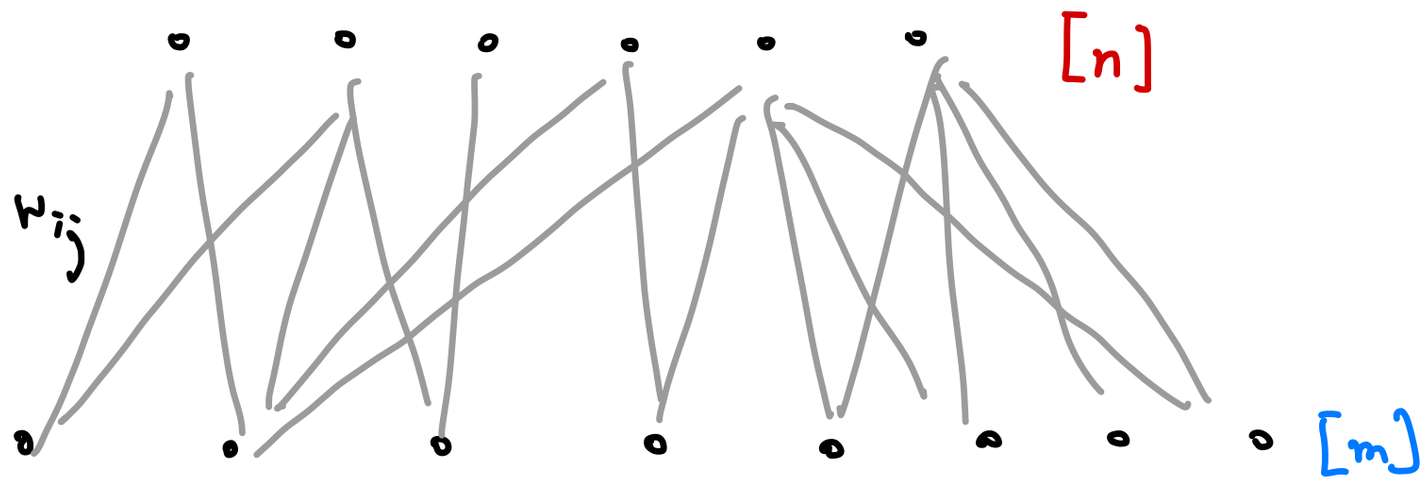

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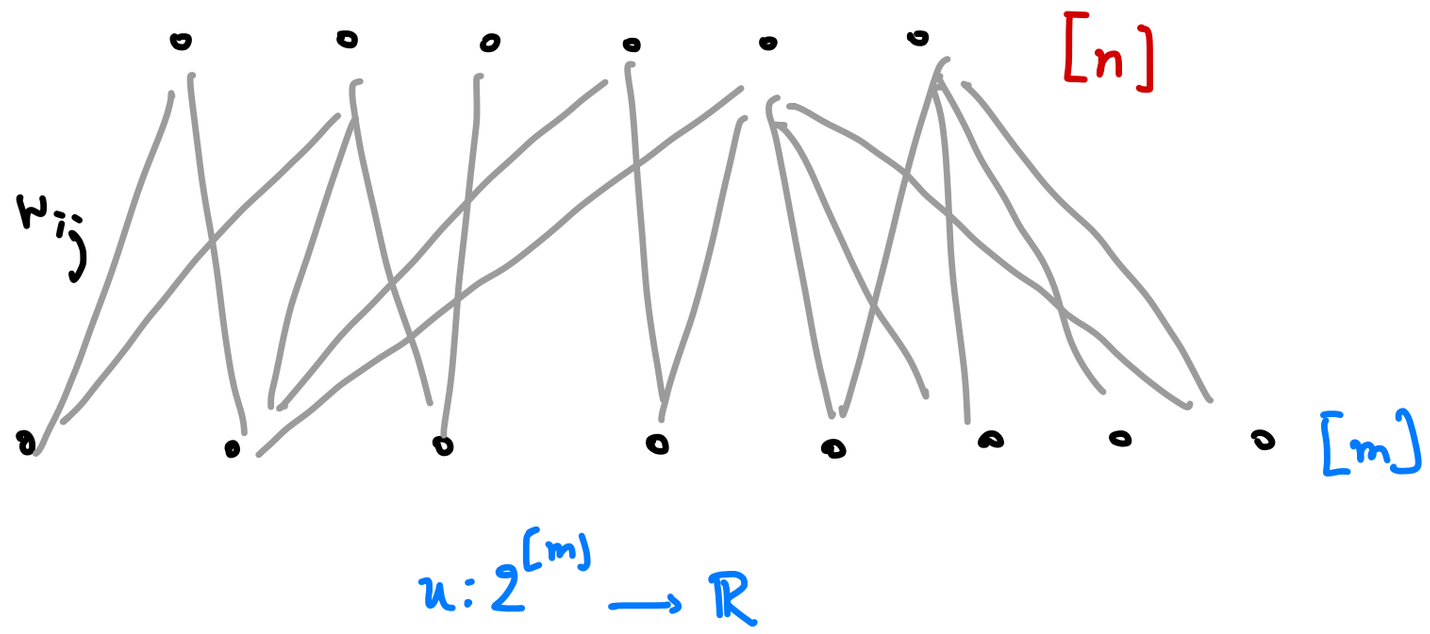

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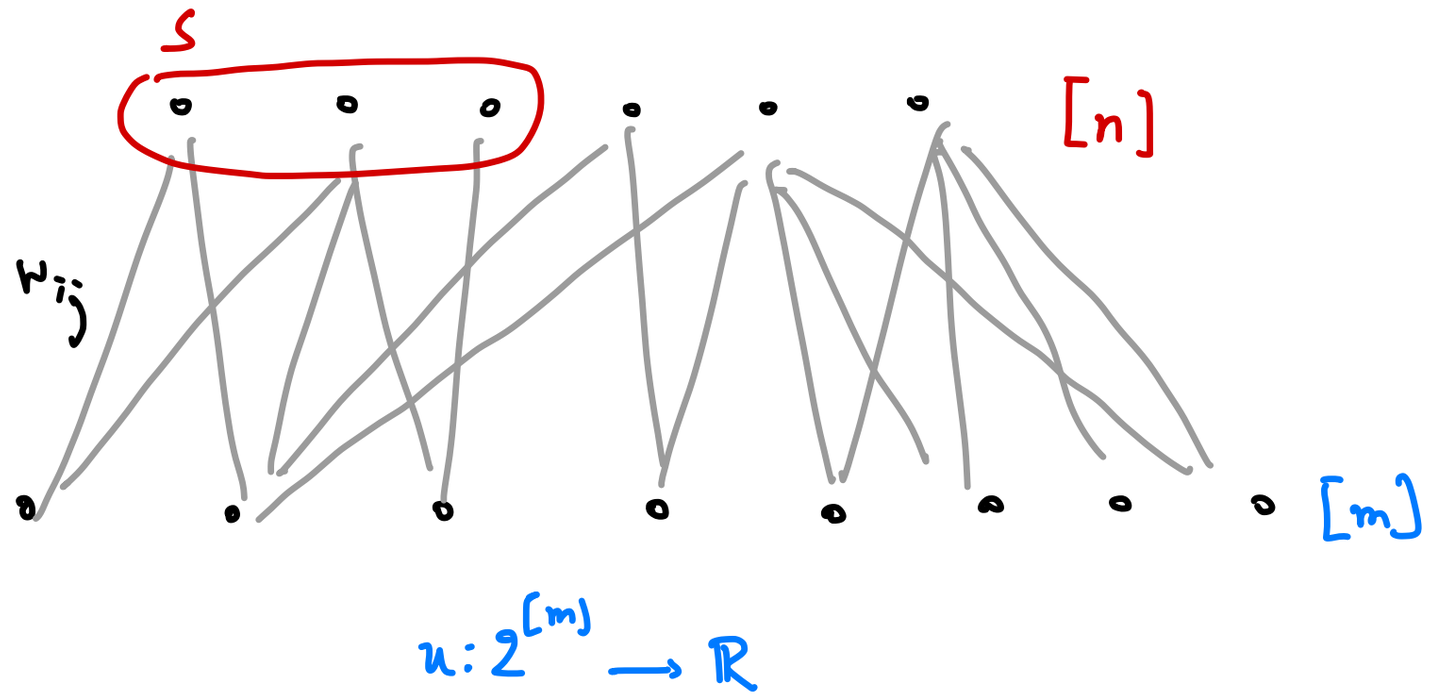

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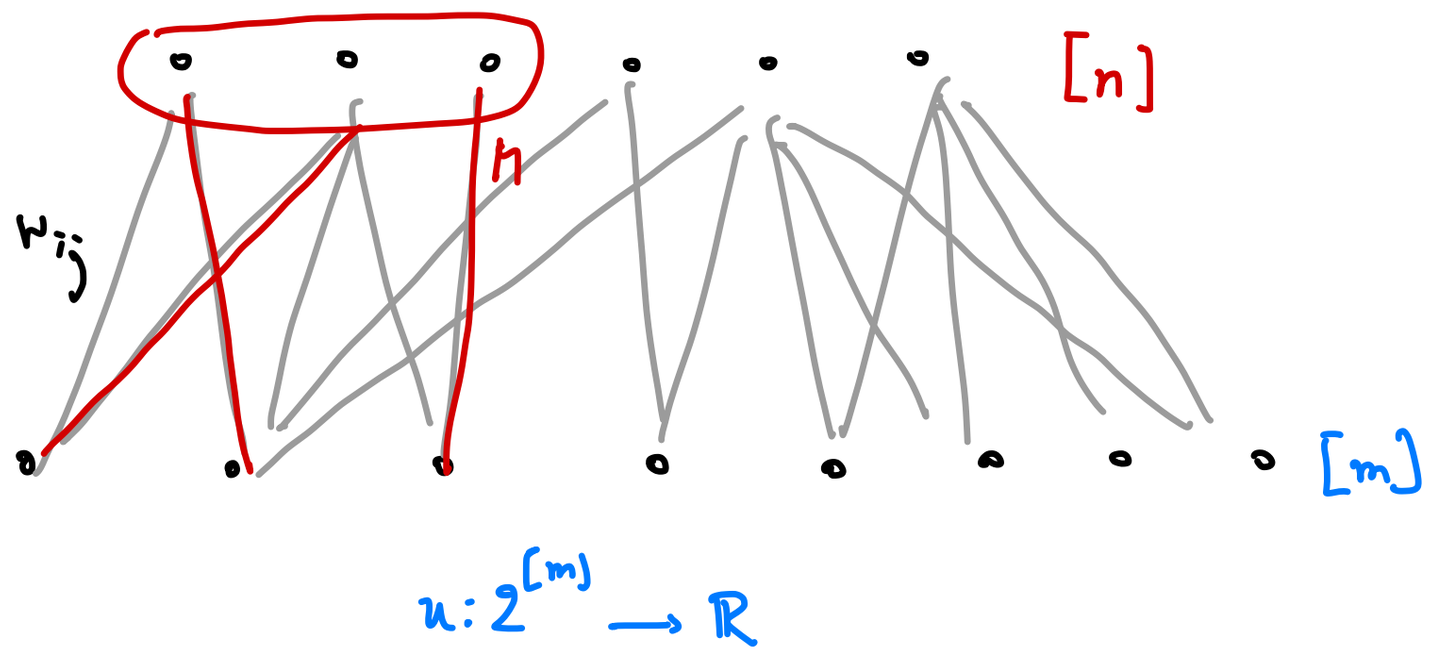
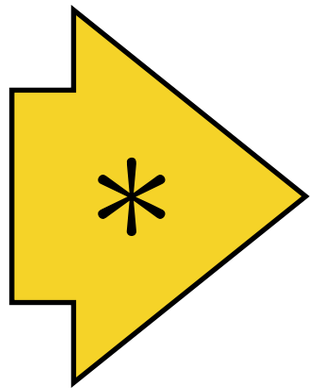


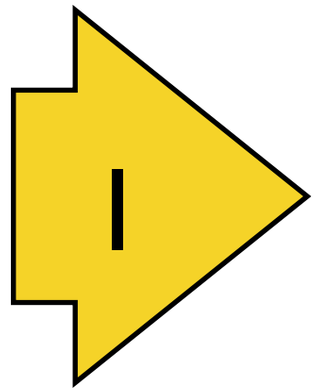
What should be the building block?

Hatfield-Milgrom observed that many interesting examples can be built starting from unit demand:

Unit Demand Valuations



Assignment Valuations ('matching')



Endowed Assignment Valuations

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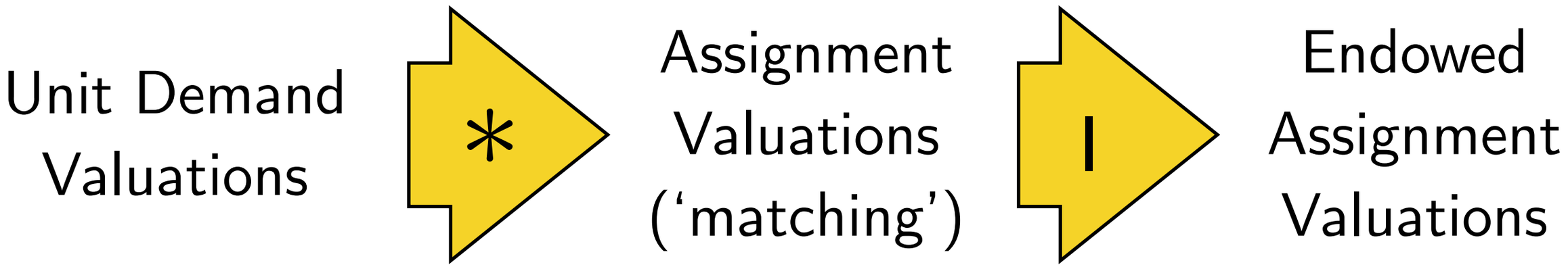

[Ostrovsky, PL]: some matroid rank functions are not endowed assignment valuations.

The natural place to start is with matroids.

Matroids are the natural building blocks

Consider functions $v : 2^{[n]} \to \mathbb{R} \cap \{-\infty\}$ and consider two special cases:

- If v is GS and $v(S) \in \{0, -\infty\}$ then v_v is the indicator function of a matroid: $v(S) = 0, S \in \mathcal{M}$ and $-\infty$ o.w.
- If v is GS with $v(S) = 0$ and $\partial_i v(S) \in \{0, 1\}$ then v is a matroid rank function: $v(S) = \max_{U \in \mathcal{M}} |S \cap U|$

Many connections to matroids established in Gul-Stachetti, Murota-Shioura, Baldwin-Klemperer,...

Potential Constructions

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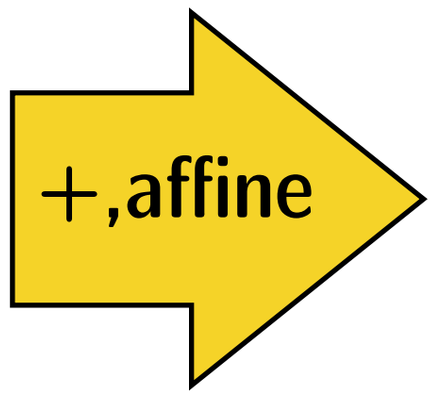
Matroids
Rank
Functions



Weighted
Combinations
MRF

Potential Constructions

Matroids
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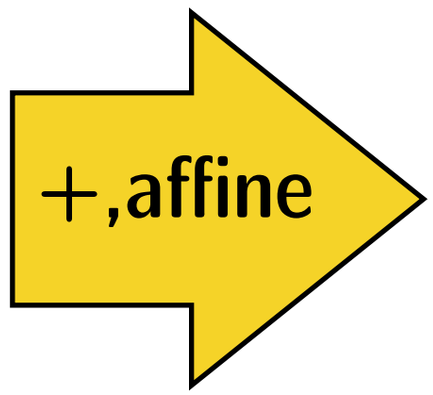
Shioura



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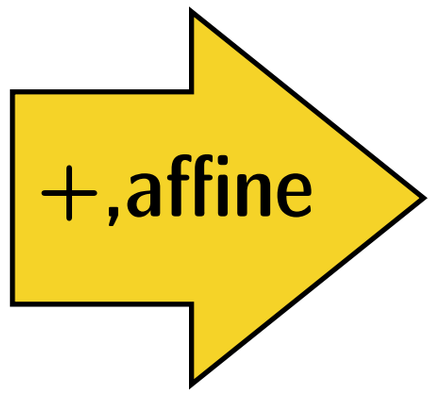
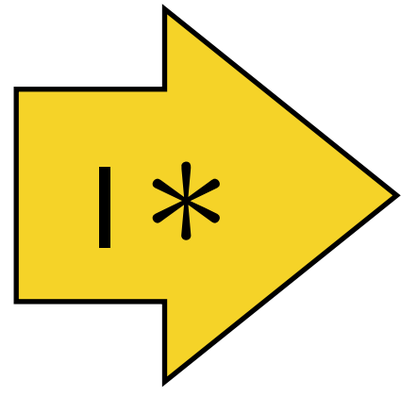


Weighted
Combinations
MRF

Yes, for n=3,4
No for n>4
[Balkanski, PL]

Potential Constructions

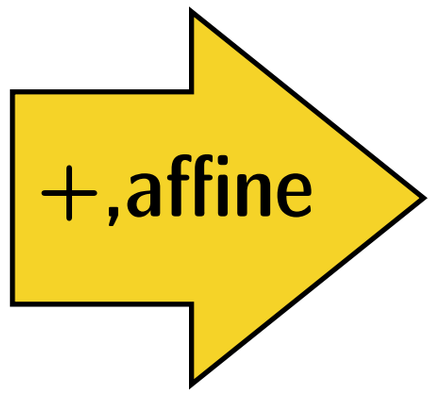
Matroid Rank Functions



Matroid Based Valuations

Matroids Rank Functions

Shioura



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[Balkanski, PL]

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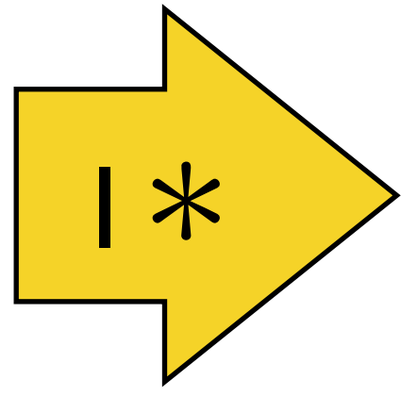
Matroid Rank Functions

PL-Ostrovsky



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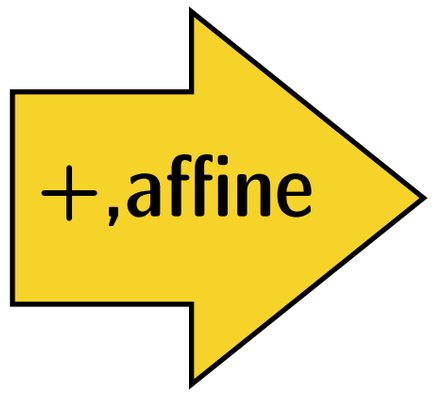

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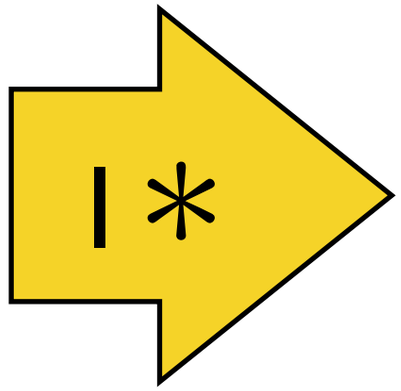

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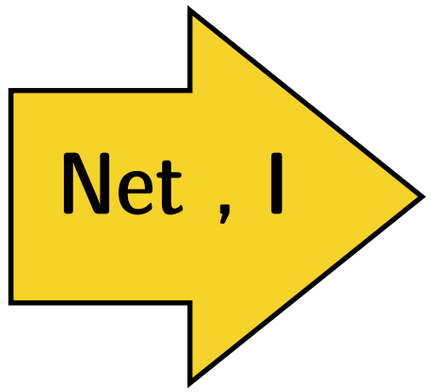

Rado Minor

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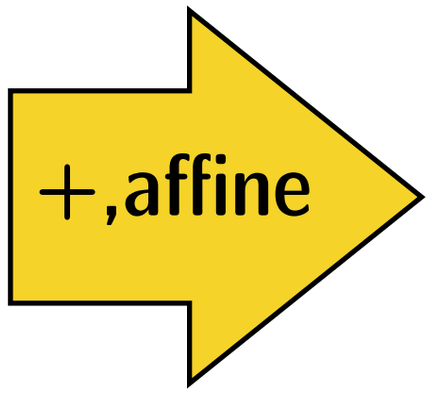

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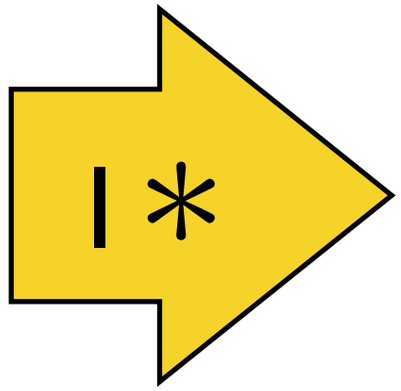

Matroid Based Valuations

Matroid Rank Functions

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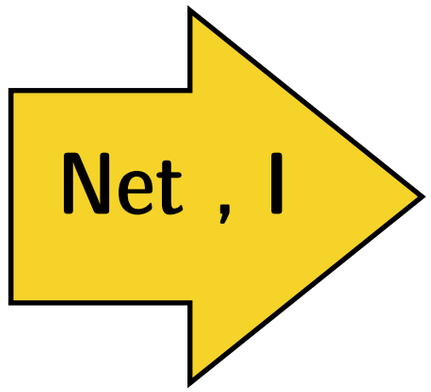

Rado Minor

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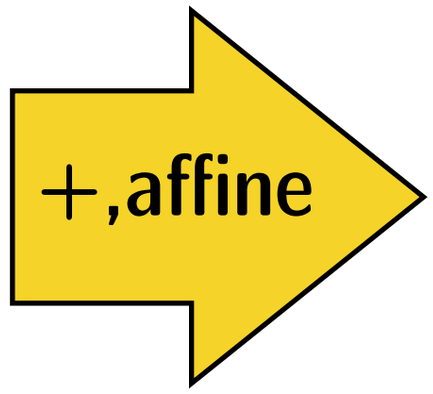

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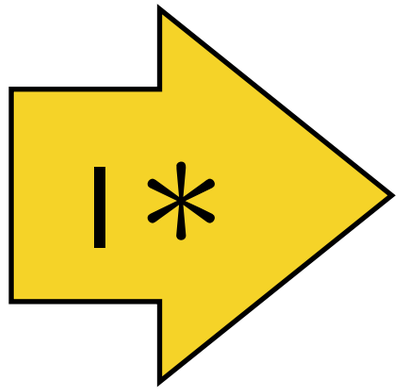

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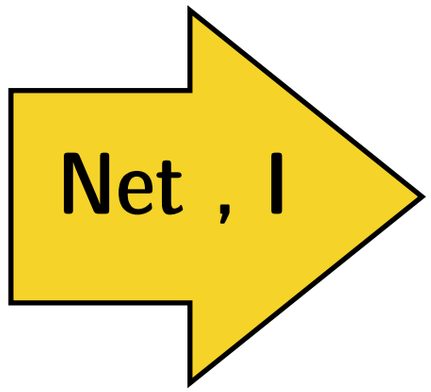

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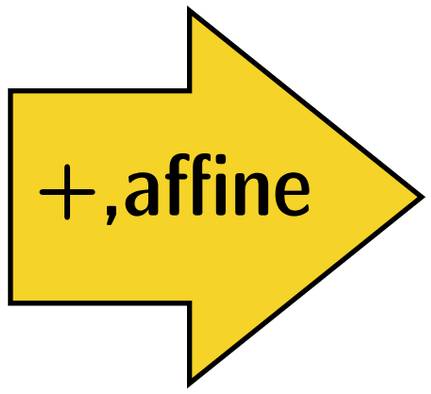

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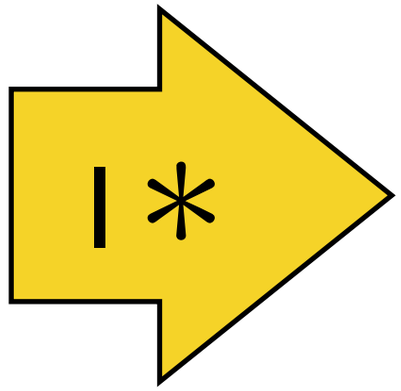

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On complete classes of valuated matroids

[Husic, Loho, Smith, Vegh, 2021]

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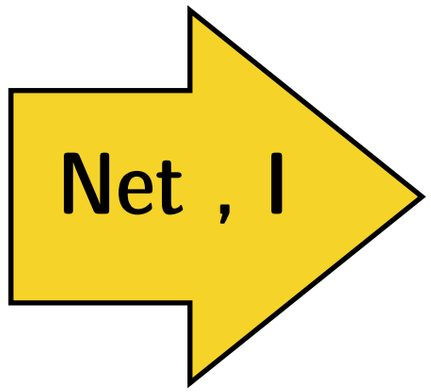

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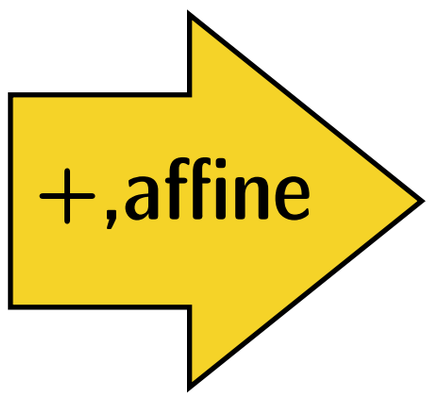

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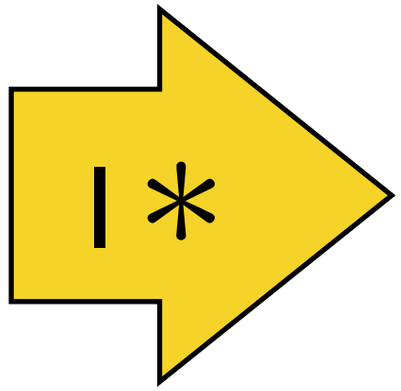

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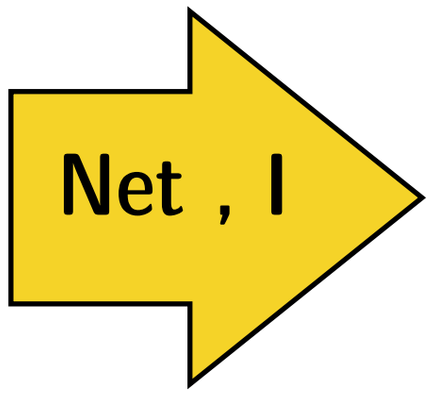

$\neq$?

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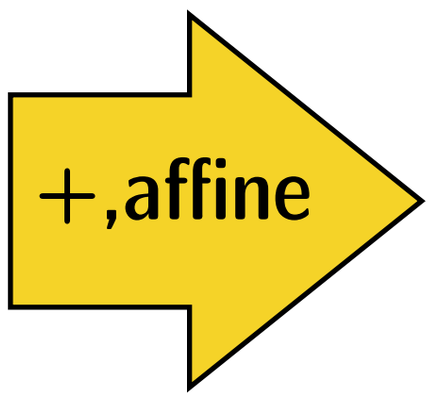

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Sparse Paving Matroids

Let $\binom{[n]}{k}$ be all subset of [n] with k elements. Let $I \subseteq \binom{[n]}{k}$ such that $X, Y \in I, |X \cap Y| \leq k-2$. Now, define a matroid $\mathcal{M}$ where a set S is independent iff $S \subseteq B \in \binom{[n]}{k} \setminus I$.

Construction in Husic et al: take nested SP-matroids

$$\mathcal{M}_1 \subseteq \mathcal{M}_2 \subseteq \ldots \subseteq \mathcal{M}_m$$

and define:

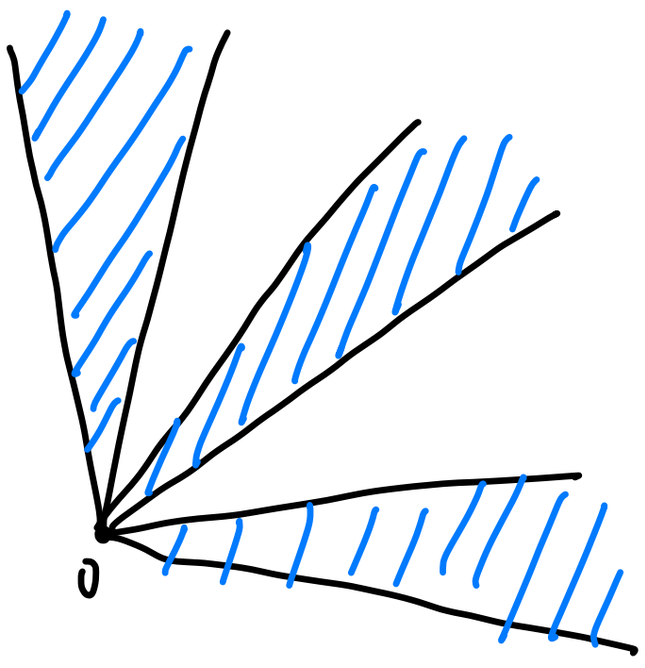
$$v(S) = \sum_{i=1}^{m} w_i \cdot \mathrm{rank}_{\mathcal{M}_i}(S)$$

Show it can't be constructed from N, |, *, ...

However, they are still a weighted combination of matroids.

Operations Preserving GS

We can view **GS** as a (non-convex) cone in $\mathbb{R}^{2^n}$:



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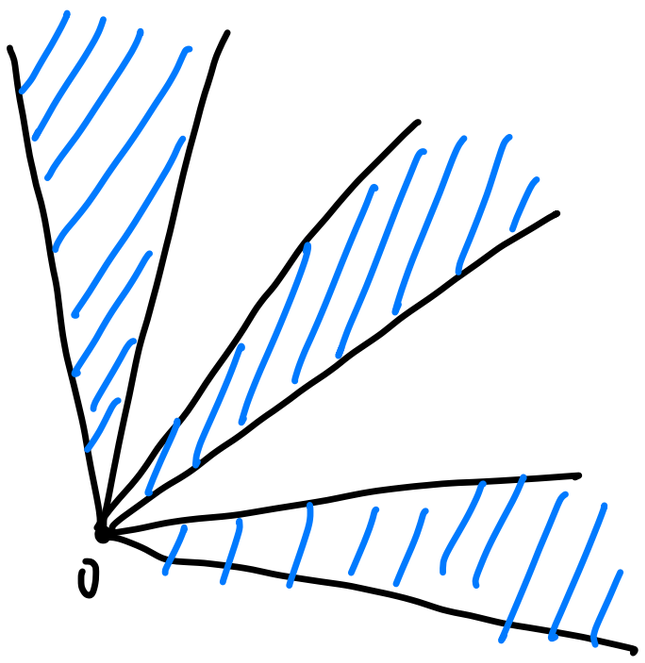

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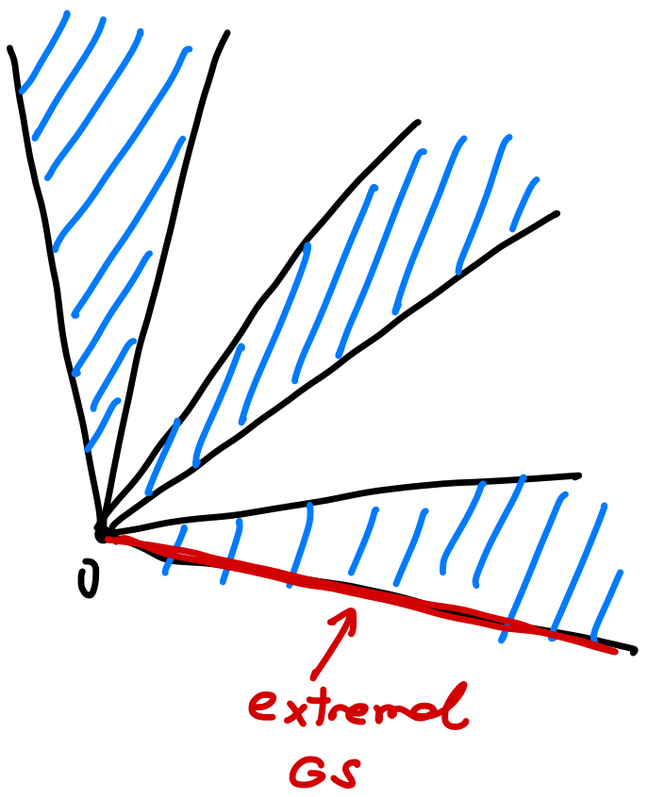

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Part II: Approximating GS

Valuation Function Approximation

From now one, we will assume that valuations $v : 2^{[n]} \to \mathbb{R}$ are normalized $v(\emptyset) = 0$ and monotone.

A class of functions $\mathbf{C}$ can be α-approximated by class $\mathbf{B}$ if for every function $f \in \mathbf{C}$ there exists $g \in \mathbf{B}$ such that:

$$g(S) \leq f(S) \leq \alpha \cdot g(S), \forall S \subseteq [n]$$

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Example:

Submodular $\subsetneq$ XOS $\subsetneq$ Subadditive

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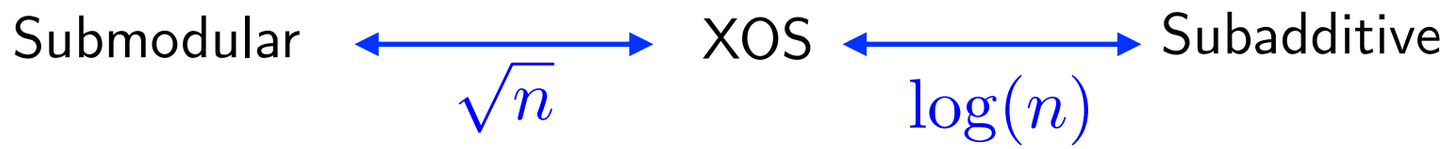
Example:

Why? Another way to measure relative complexity of valuation classes.

Also: small approx means algorithms for one class may work for the other.

Valuation Function Approximation

How does GS compare with other classes?



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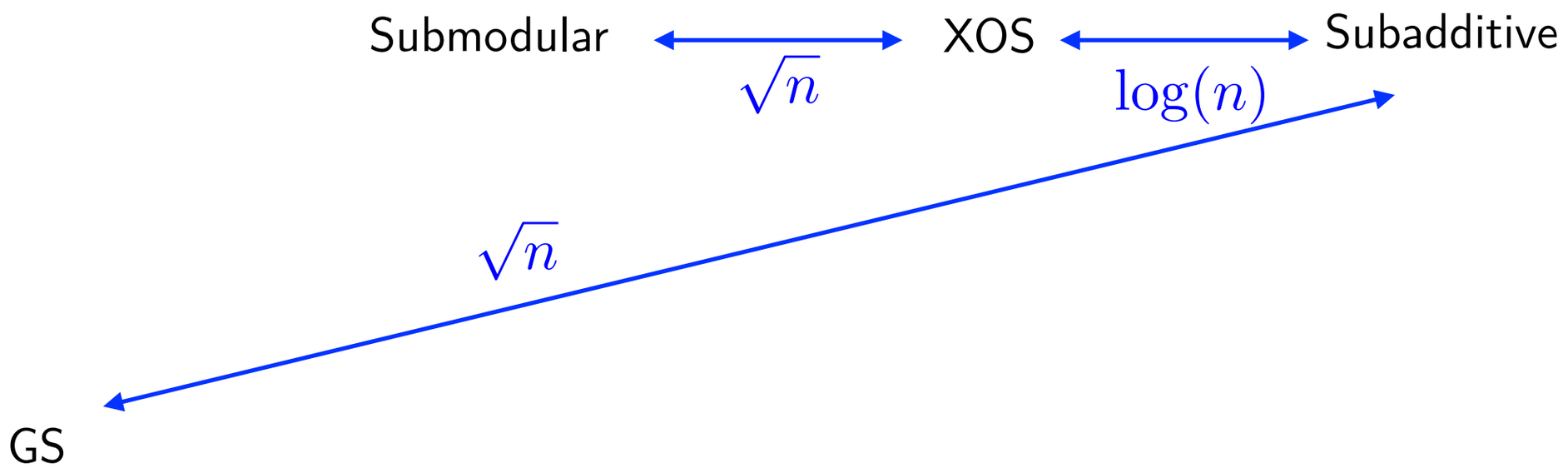

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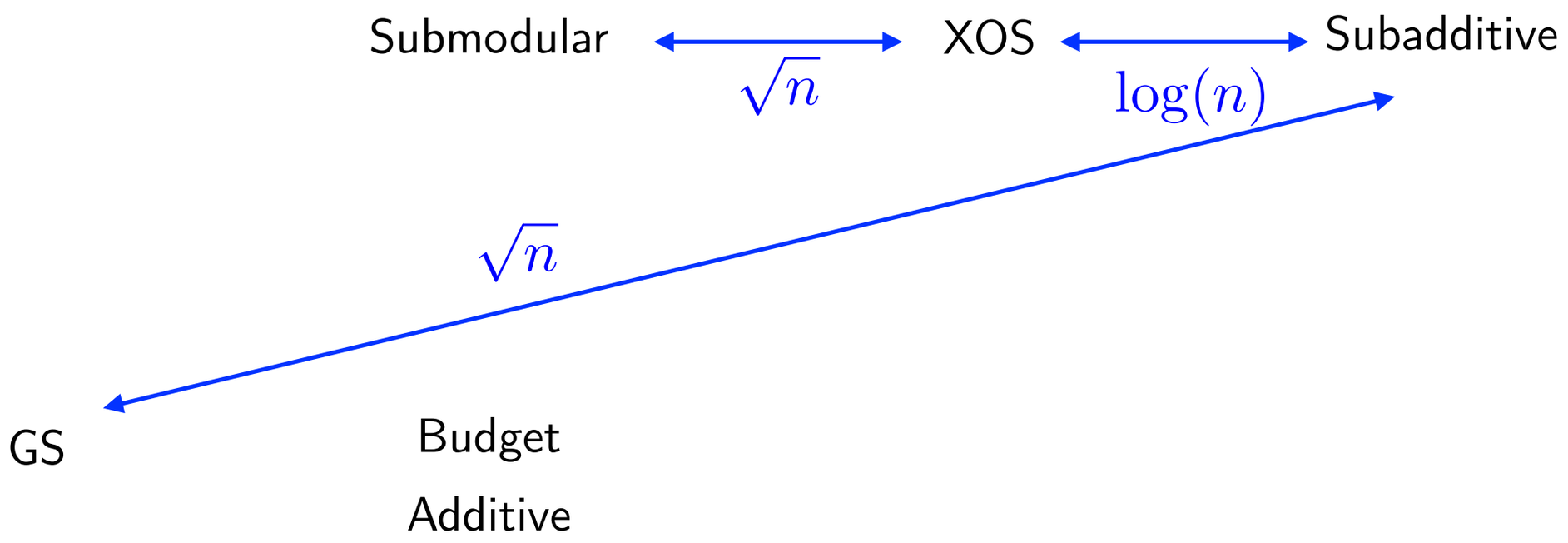

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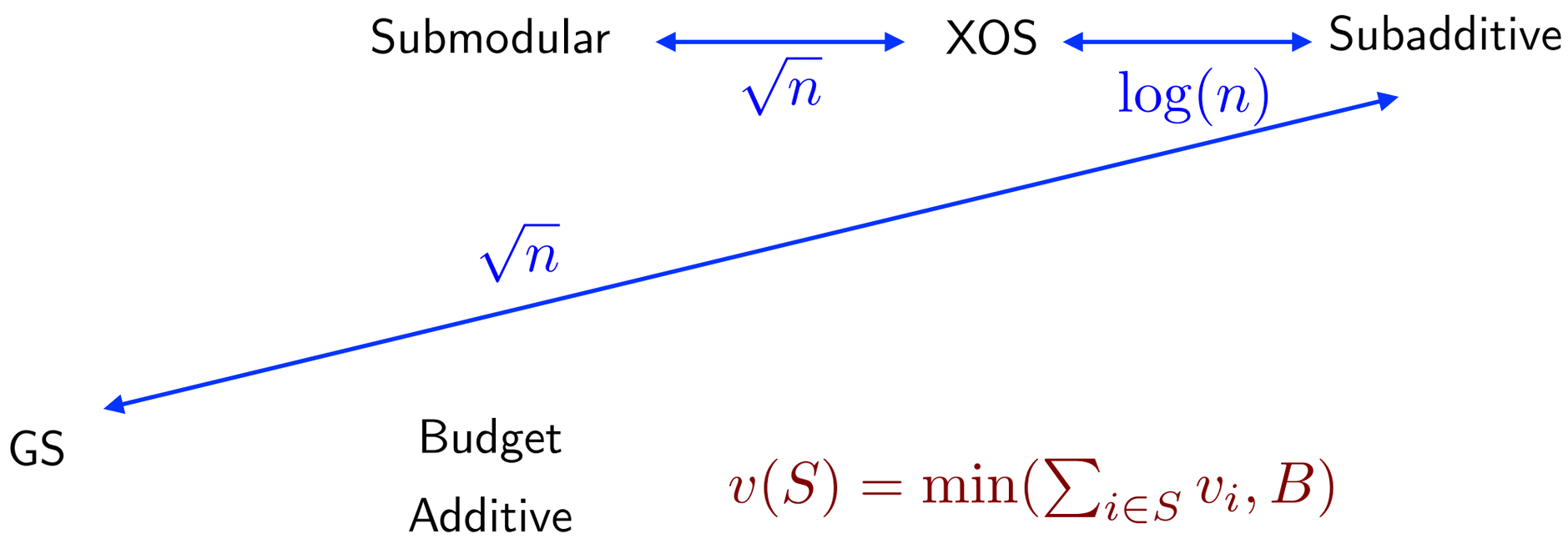

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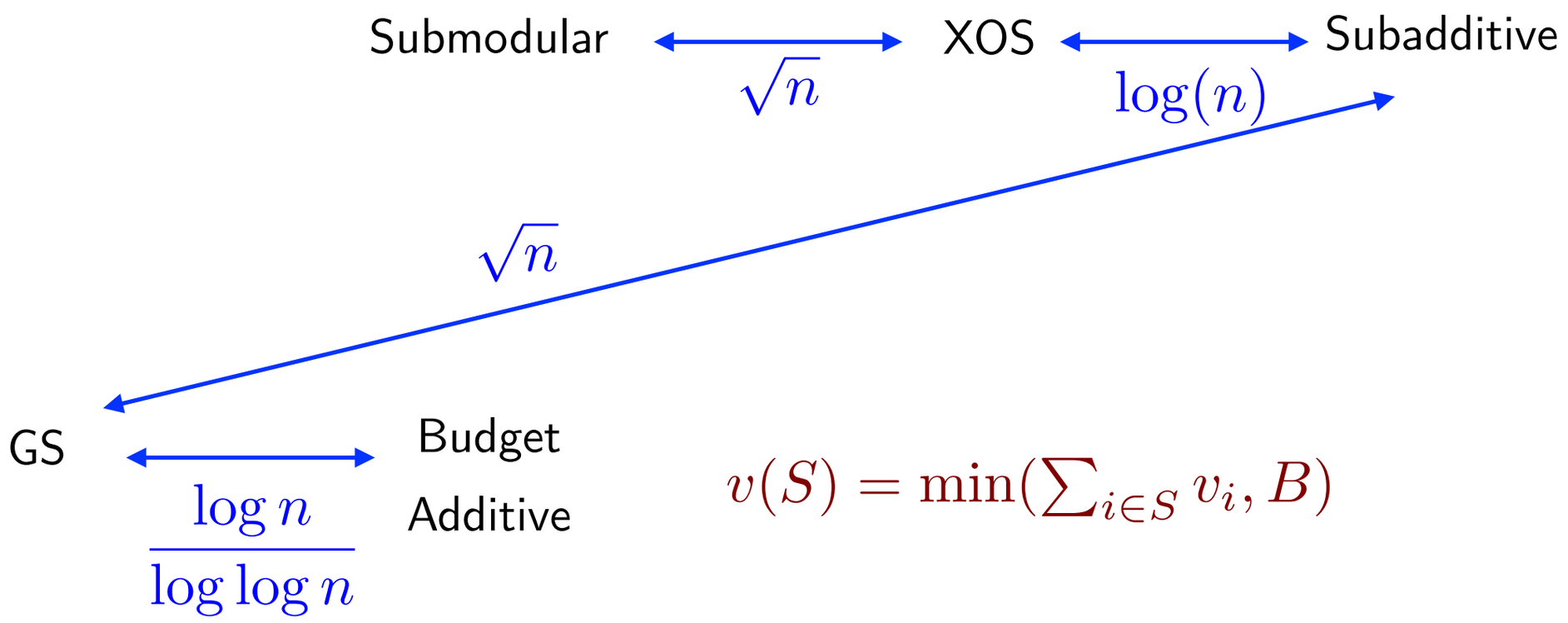

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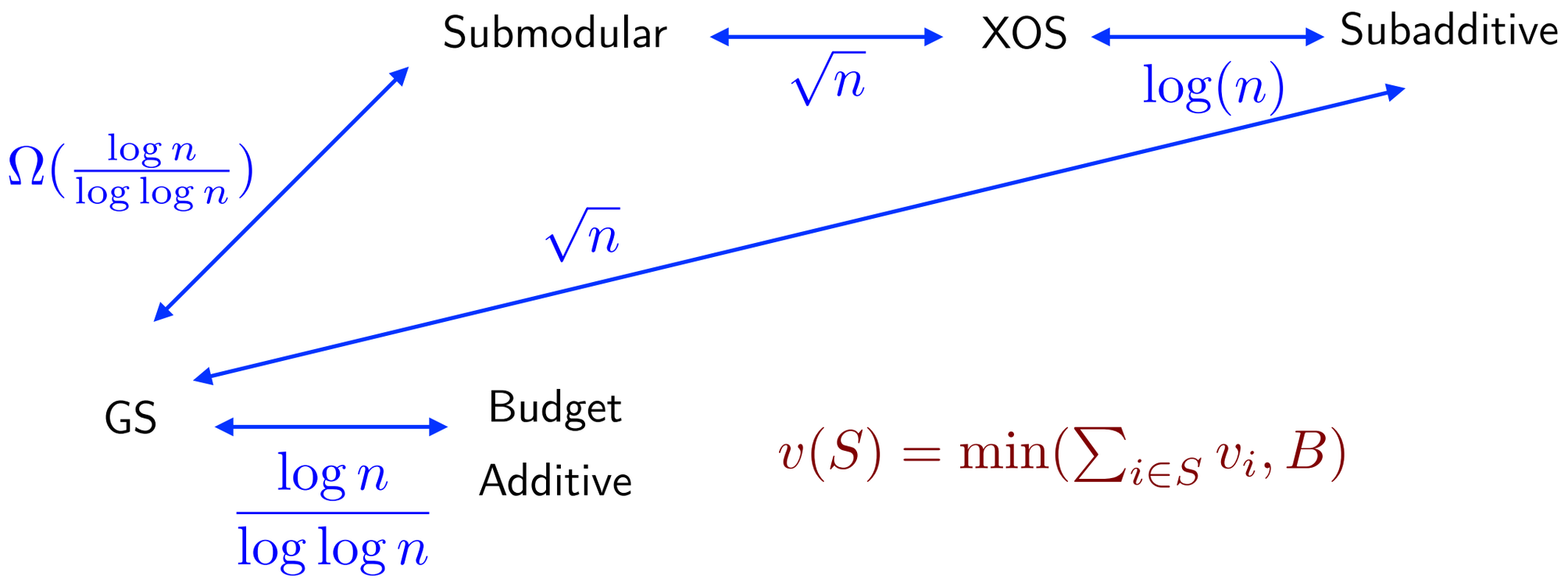

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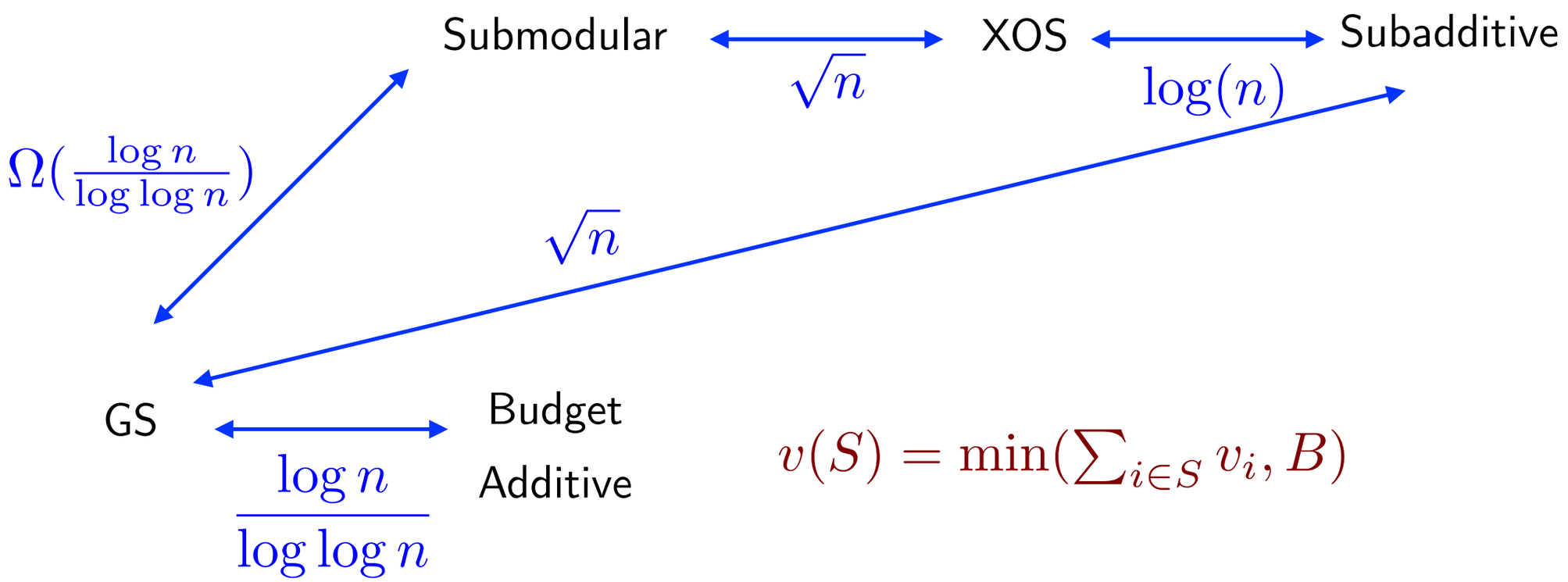

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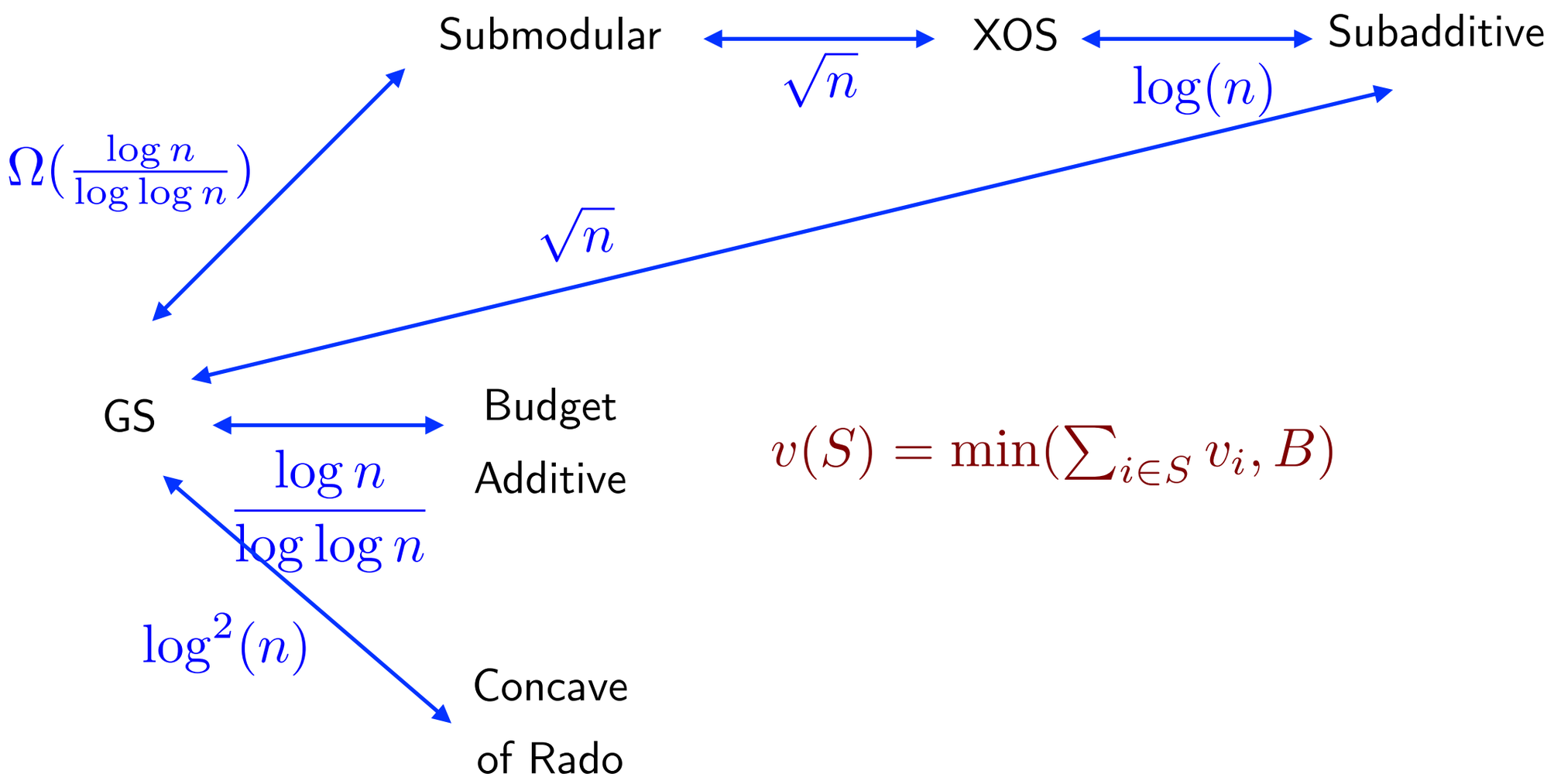

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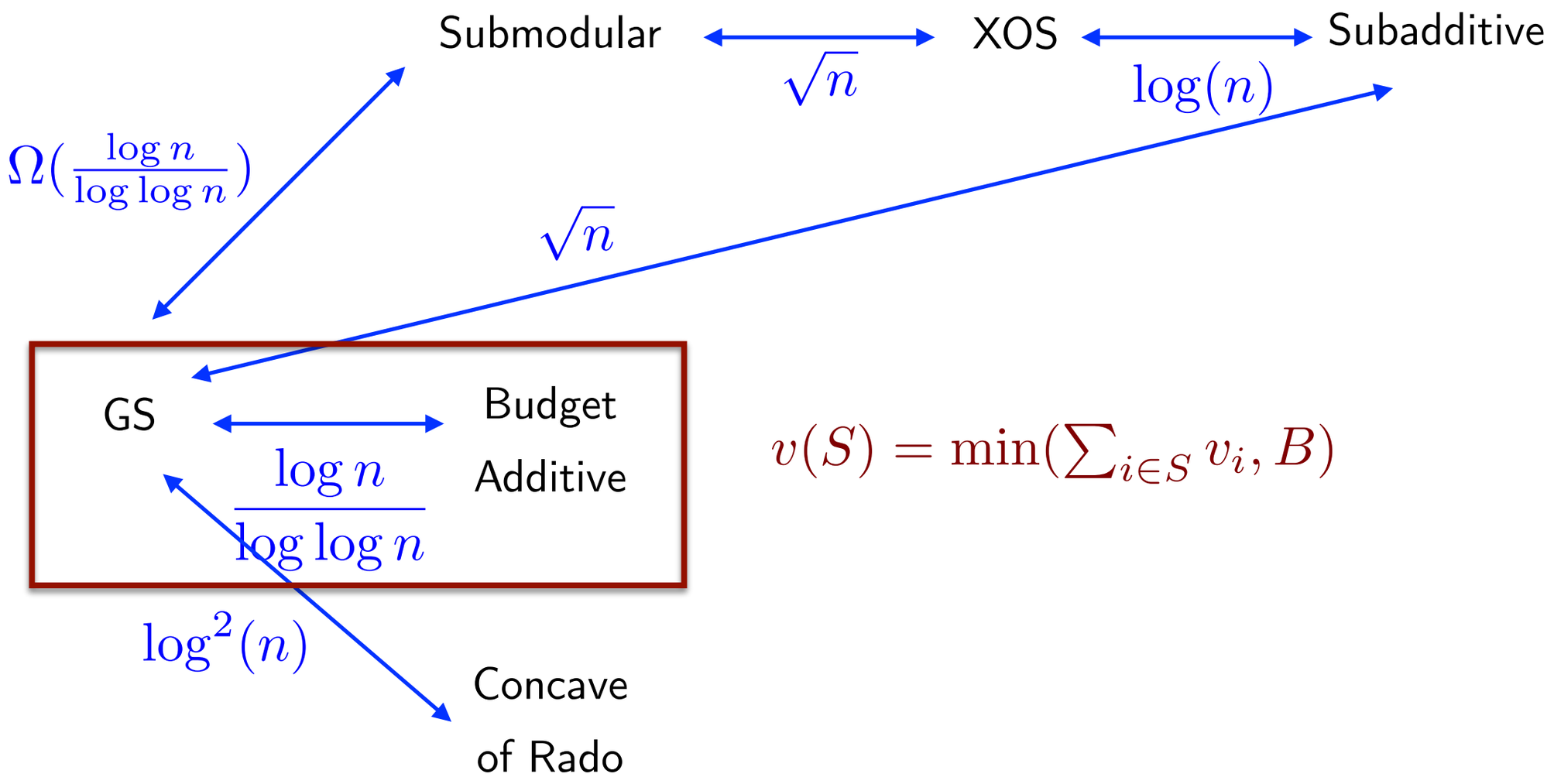

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Valuation Function Approximation

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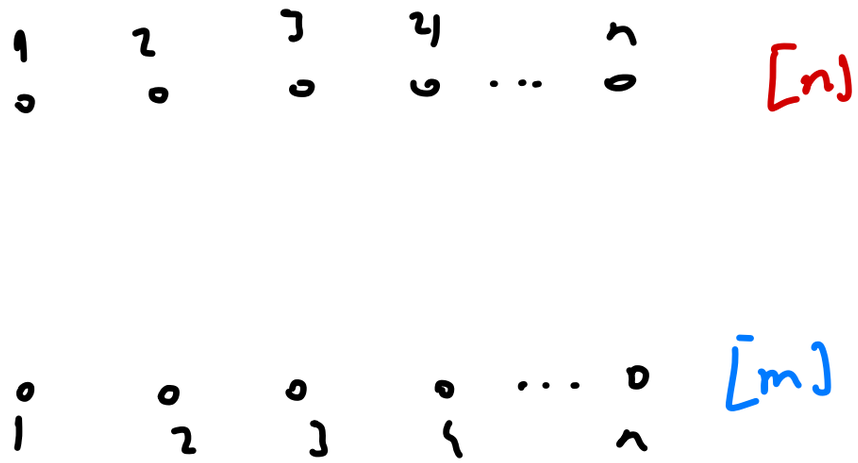

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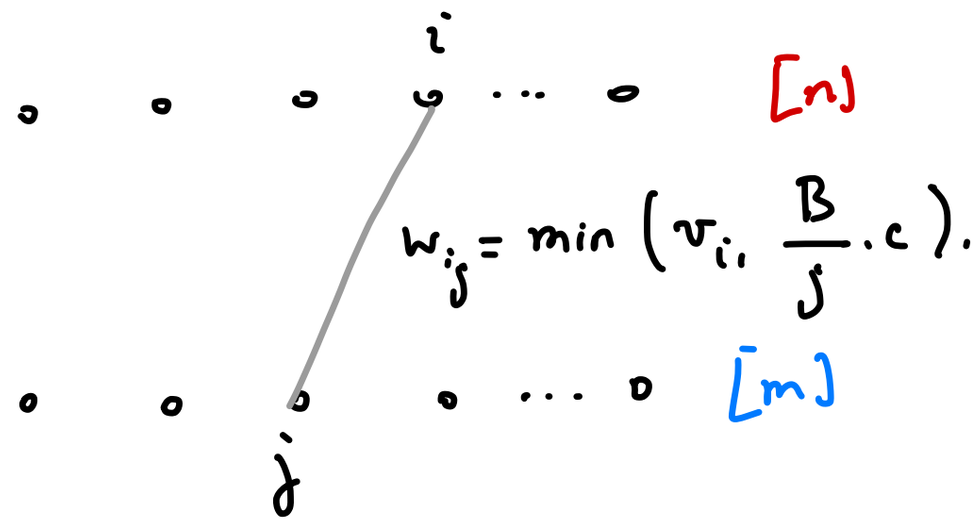

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e.g. $v(S \cup \{i\}) = v(S \cup \{j\}), \forall S \setminus \{i, j\}$

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Max-symmetrization: given f in GS and two items i and j, define the symmetrized version f_{sym} such that for all $S \subseteq [n] \setminus \{i, j\}$:

$$f_{\text{sym}}(S) = f(S) \qquad f_{\text{sym}}(S \cup \{i, j\}) = f(S \cup \{i, j\})$$

$$f_{\text{sym}}(S \cup \{i\}) = f_{\text{sym}}(S \cup \{j\}) = \max[f(S \cup \{i\}), f(S \cup \{j\})]$$

Valuation Function Approximation

Open question: can we approximate any submodular function by a GS function with ratio polylog(n) ?

Conclusion

Two different ways to estimate the complexity of GS functions with respect to other classes:

* Can GS be constructed from simpler functions?
* Can GS approximate richer classes of valuations?

Many open questions in both areas.

Appendix

Discrete Differential Equations

- Given a function $v : 2^{[n]} \to \mathbb{R}$ we define the discrete derivative with respect to $i \in [n]$ as the function $\partial_i v : 2^{[n]\setminus i} \to \mathbb{R}$ which is given by:

$$\partial_i v(S) = v(S \cup i) - v(S)$$

(another name for the marginal)

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- If we apply it twice we get:

$$\partial_{ij} v(S) := \partial_j \partial_i v(S) = v(S \cup ij) - v(S \cup i) - v(S \cup j) + v(S)$$

- Submodularity: $\partial_{ij} v(S) \leq 0$

Discrete Differential Equations

- [Reijnierse, Gellekom, Potters] A function $v : 2^{[n]} \to \mathbb{R}$ is in gross substitutes iff it satisfies:

$$\partial_{ij} v(S) \leq \max(\partial_{ik} v(S), \partial_{kj} v(S)) \leq 0$$

condition on the discrete Hessian.

$$H(S) = [\partial_{ij}(S)]_{i,j \notin S}$$

- Easier notation to work with positive numbers:

$$\Delta_{ij}^S = -\partial_{ij} v(S)$$

Then the condition becomes:

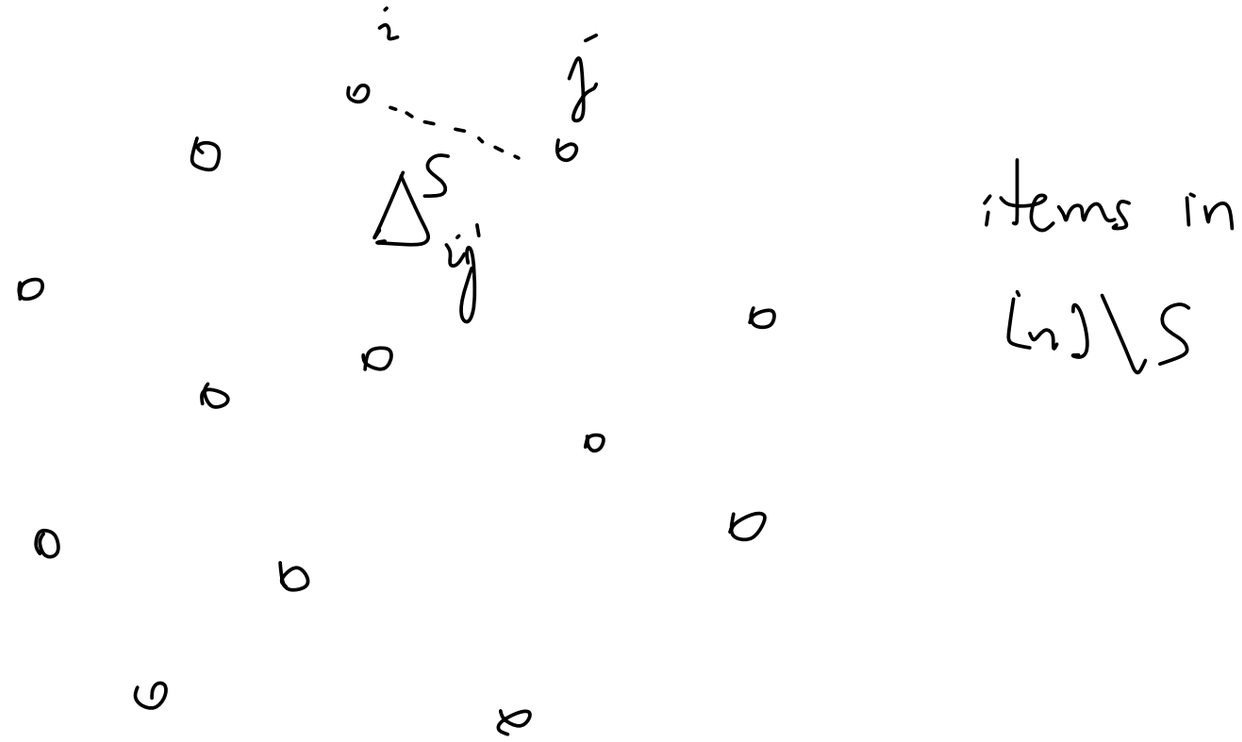
$$\Delta_{ij}^S \geq \min(\Delta_{ik}^S, \Delta_{kj}^S) \geq 0$$

Tree Structure

- [Hirai-Murota] Tree structure on the Hessian. Since:

$$\Delta^S_{ij} \geq \min(\Delta^S_{ik}, \Delta^S_{kj}) \geq 0$$

implies that in $\Delta^S_{ij}, \Delta^S_{ik}, \Delta^S_{kj}$ the minimum must appear twice. So, let $m = \min_{i,j \notin S} \Delta^S_{ij}$



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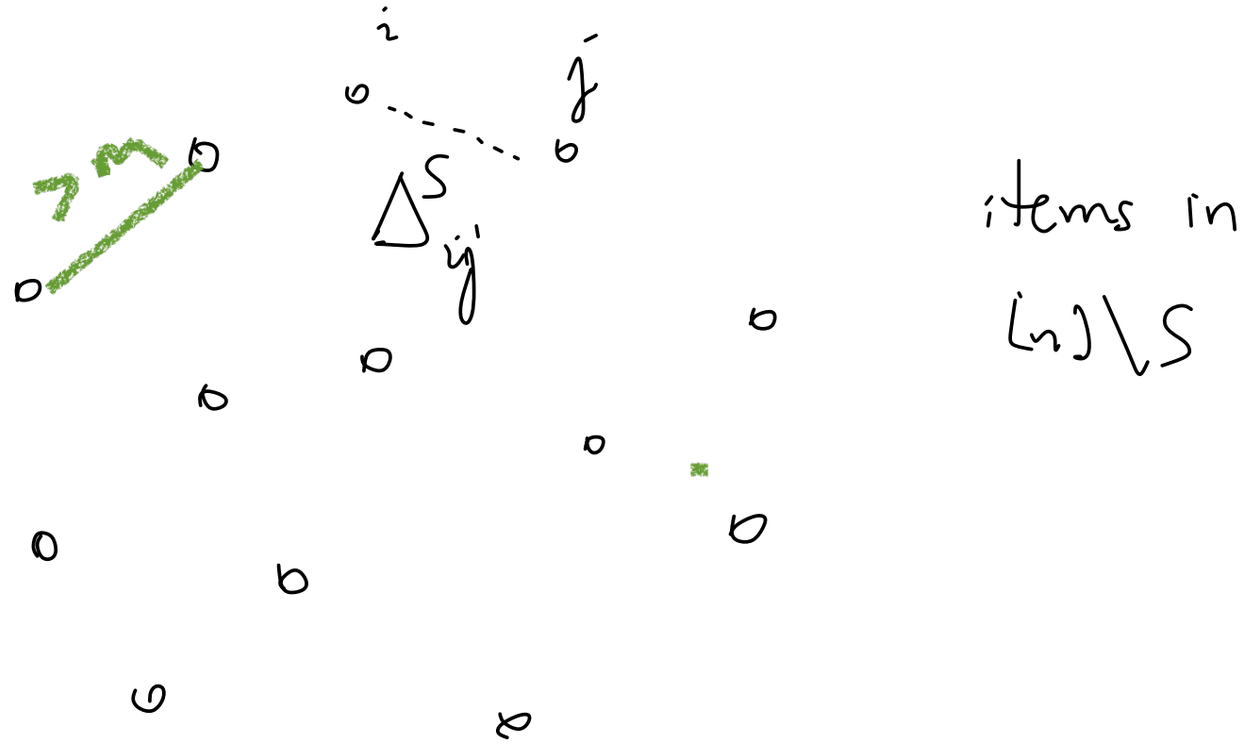

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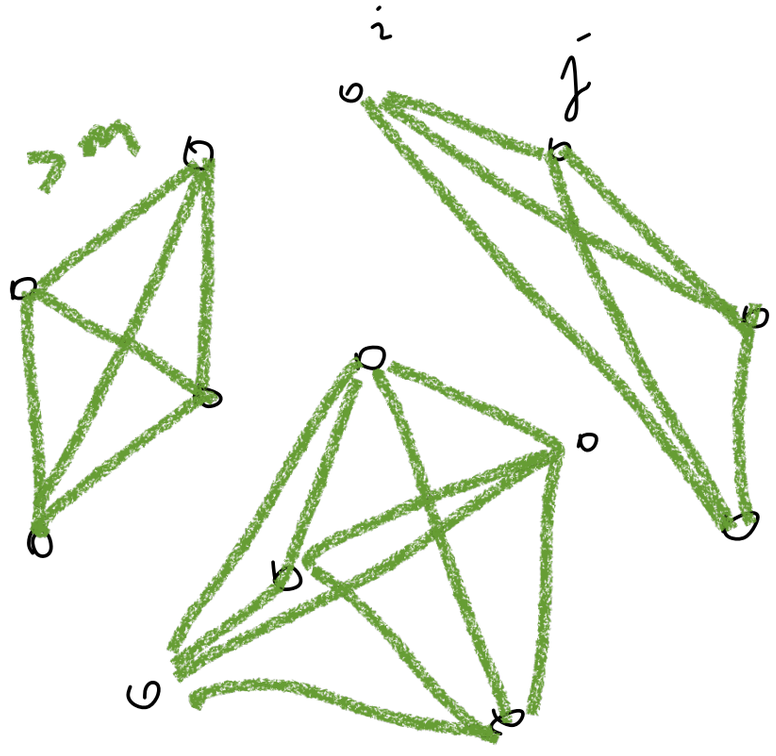

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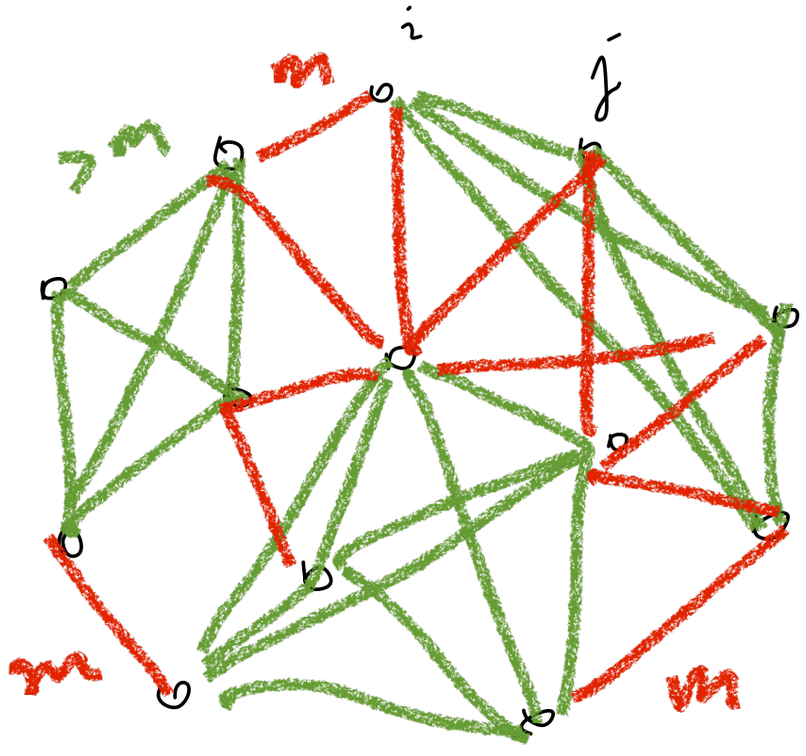

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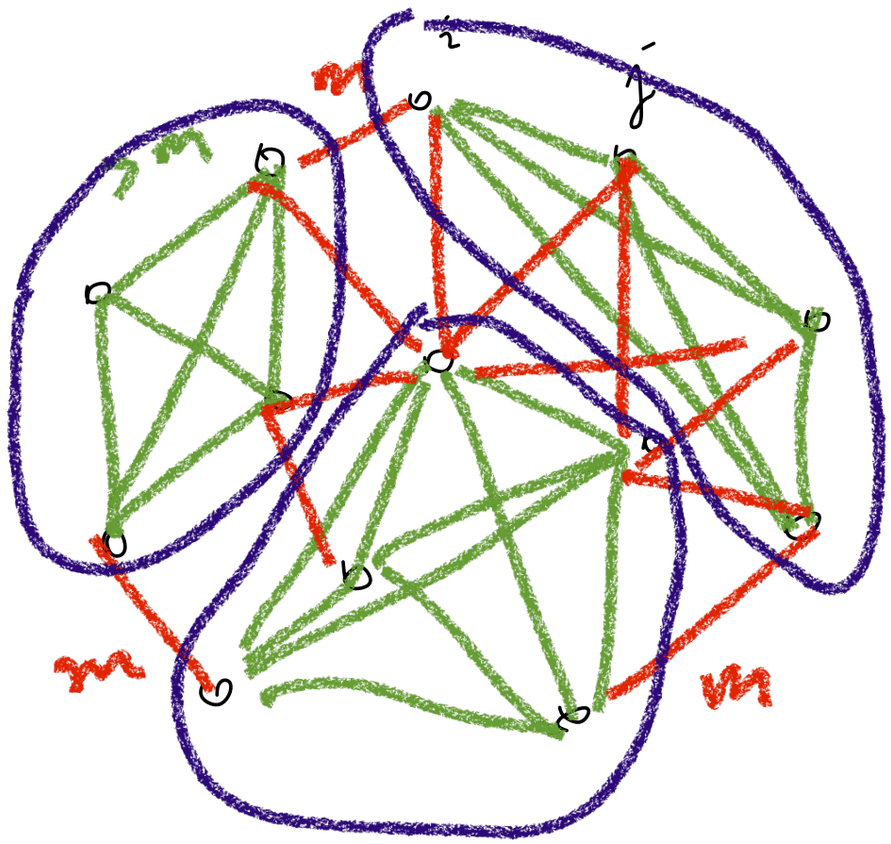

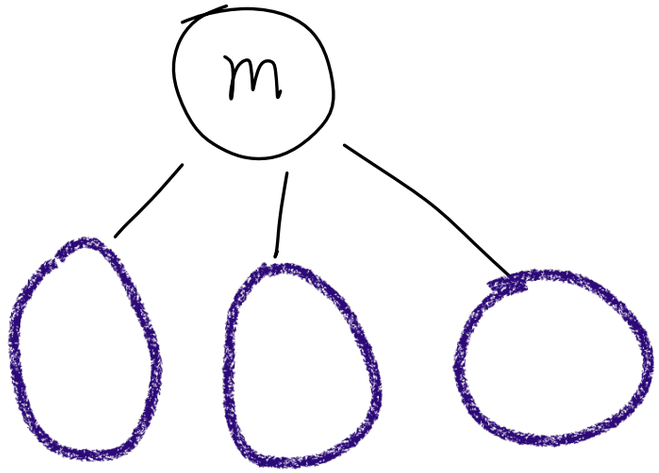

Tree Structure

- For every subset S we can represent the Hessian by a tree with $[n] \setminus S$ in the leaves and labels in the internal nodes such that:

$$\Delta^S_{ij} = \mathrm{label}(\mathrm{LCA}_{ij})$$

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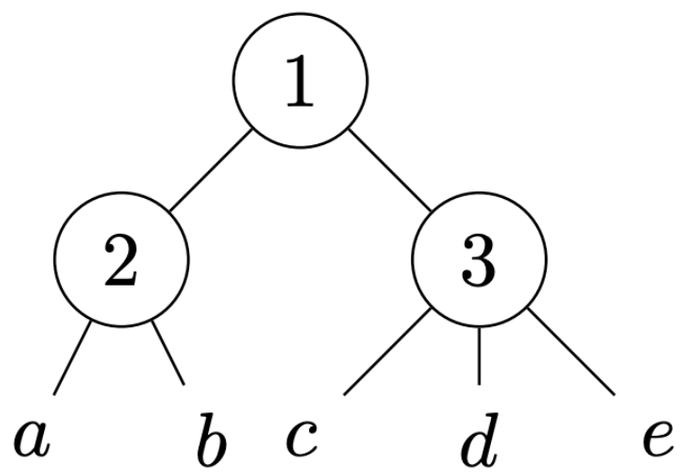
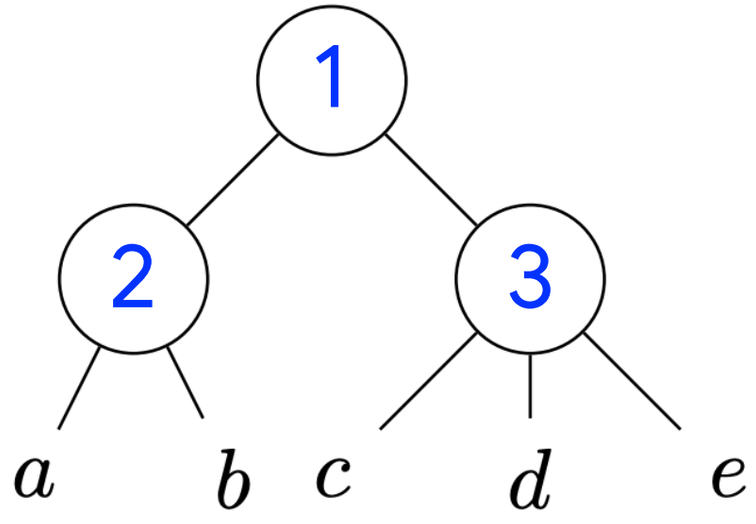
$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} * & 2 & 1 & 1 & 1 \\ 2 & * & 1 & 1 & 1 \\ 1 & 1 & * & 3 & 3 \\ 1 & 1 & 3 & * & 3 \\ 1 & 1 & 3 & 3 & * \end{bmatrix} \end{array}$$



Tree Structure

- This allows us to decompose a GS function in a combinatorial structure and numerical data. For each subset S we associate:



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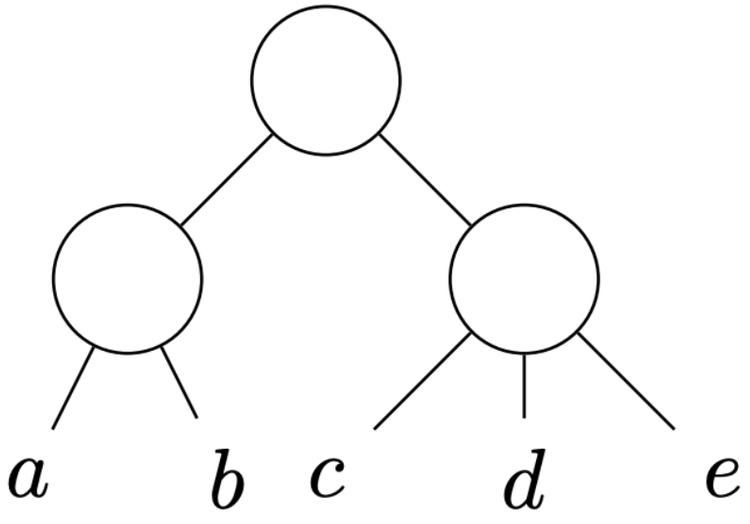

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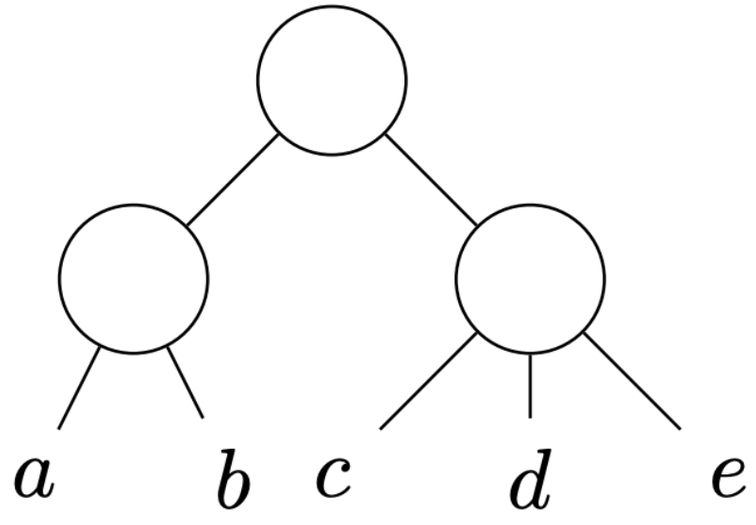

Combinatorial Structure

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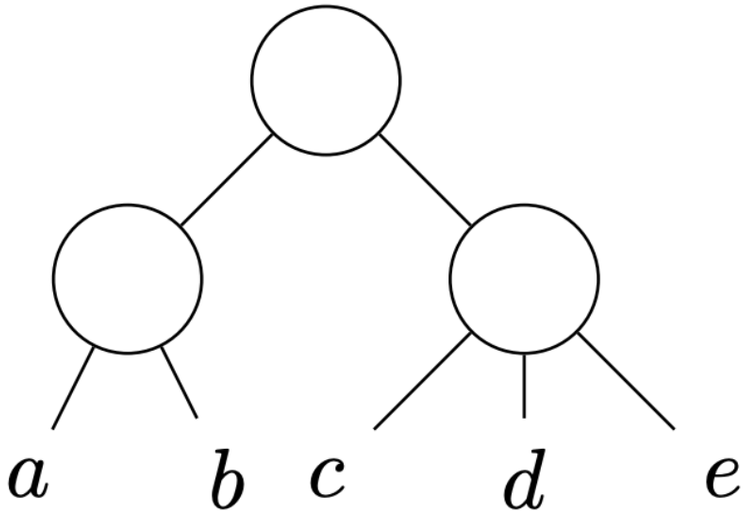

Combinatorial Structure

Numerical Data

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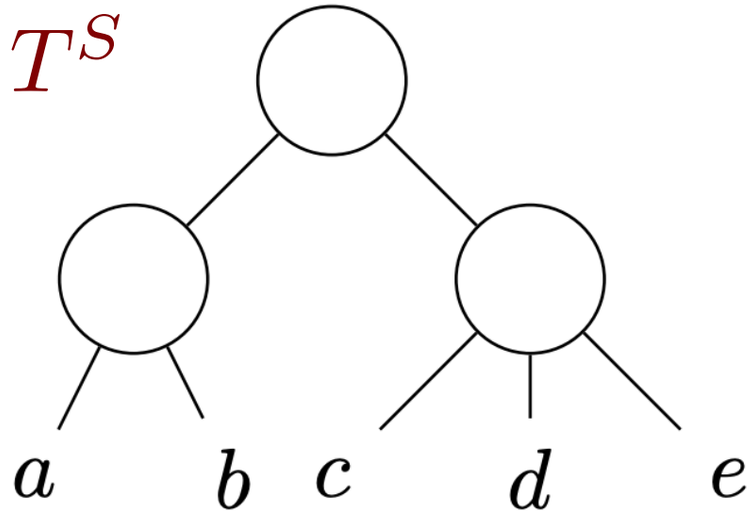

Combinatorial Structure

Numerical Data

Signature $= \{T^S\}_{S \subseteq [n]}$

Tree Structure

- Thm: given $u, v \in \mathrm{GS}$ then $u + v \in \mathrm{GS}$ iff they share the same signature.

- What are the building blocks of GS (under tree-concordant sum) ?

- Let's fix a signature $\{T^S\}_{S \subseteq [n]}$ and look at the GS functions with that signature. This forms a convex set.
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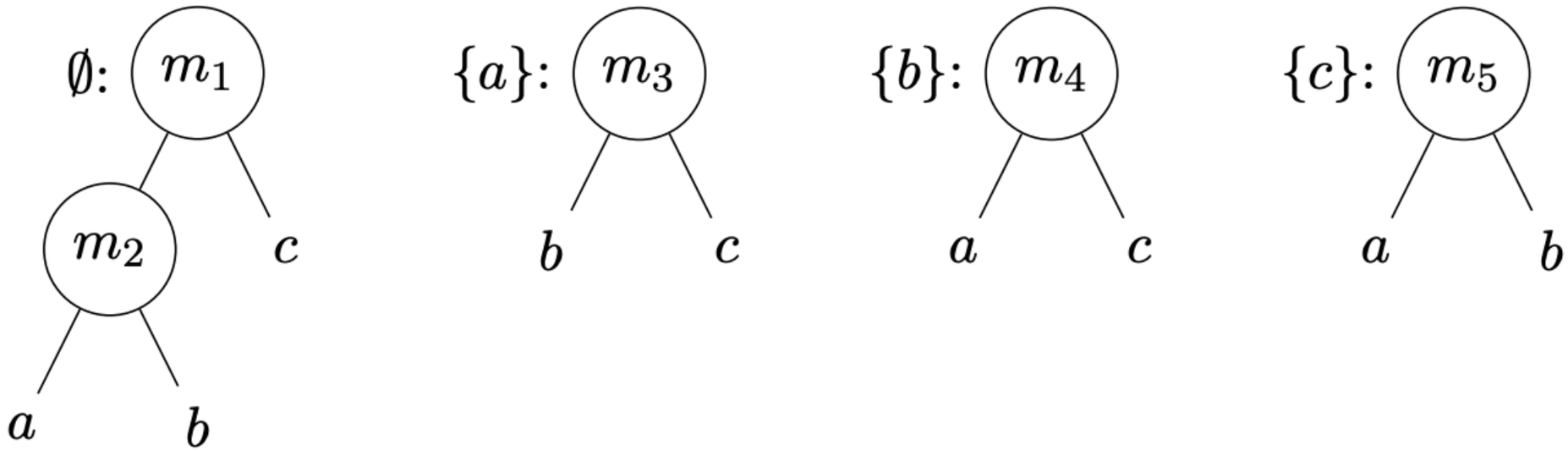
I was hoping they would be all matroid rank functions. But it is not always the case.

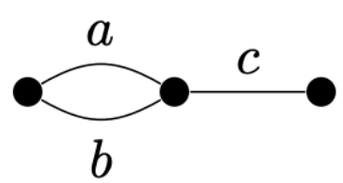
Extremal GS valuations

- Say we have 3 items. Then there is only one signature (modulo symmetries):



- Look at the polytope formed by m_1, m_2, m_3, m_4, m_5

  Monotonicity ($m_1 \leq m_2$) + Integrability

- For n = 3, the extreme points are all matroid rank functions: 

+ uniform matroids of ranks 1 and 2

Extremal GS valuations

- Matroid rank functions correspond to {0,1} labels, i.e. $v \in \mathrm{GS}$ and $\Delta_{ij}^{S} \in \{0,1\}$ iff v is matroid rank function.
- So we want to check if polytopes are integral.
- True for n $=$ 4 but fails for n $>$ 4.

- By product is a complete description of GS up to 4 items.