MONOTONE COMPARATIVE STATICS FOR SUBMODULAR FUNCTIONS, WITH AN APPLICATION TO AGGREGATED DEFERRED ACCEPTANCE

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ABSTRACT. We propose monotone comparative statics results for maximizers of submodular functions, as opposed to maximizers of supermodular functions as in the classical theory put forth by Veinott, Topkis, Milgrom, and Shannon among others. We introduce matrons, a natural structure that is dual to sublattices of  $\mathbb{R}^n$  that generalizes existing structures such as matroids and polymatroids in combinatorial optimization and  $M^{\natural}$ -sets in discrete convex analysis. Our monotone comparative statics result is based on a natural order on matrons, which is dual in some sense to Veinott's strong set order on sublattices. As an application, we propose a deferred acceptance algorithm that operates in the case of divisible goods, and we study its convergence properties.

**Keywords**: two-sided matching, non-transferable utility matching, money burning, rationing by waiting, non-price rationing, aggregate matching, matching function, disequilibrium, discrete choice, optimal transport. This paper has benefited from insightful suggestions by Federico Echenique and John Quah.

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#### 1. Introduction

The theory of substitutability studies what happens to the demand or supply of alternatives when some options become more or less available. The theory is generally presented in the indivisible case when the set of all goods is a finite set  $\mathcal{Z}$ . Assume that a set  $B \subseteq \mathcal{Z}$  costs c(B) to produce. If  $p_z$  is the price of good z, and  $B \subseteq \mathcal{Z}$  is the set of available goods, then the set of goods that are produced is determined by the firm's problem

$$Q(p,B) \in \arg\max_{Q \subset B} \left\{ p(Q) - c(Q) \right\}, \tag{1.1}$$

where  $p(Q) = \sum_{z \in Q} p_z$  and c is any function. A well-established theory exists to formalize two characteristic properties of substitutability:

First, when the set of available options B increases, then the set of options that are not chosen increases too; that is

$$R(p,B) = B \backslash Q(p,B)$$
 increases with B

which is proven in Hatfield and Milgrom (2005).

Second, if all the prices of the goods in B weakly increase, then an option whose price has not changed, and that was not produced previously cannot be produced after the change in price; that is, for x and y in  $\mathbb{Z}$ ,

$$1\{x \in R(p,B)\}\$$
is increasing in  $p_y$  for  $y \neq x.$ 

These facts are proven in Gul and Stacchetti (1999), Hatfield and Milgrom (2005), Paes Leme (2017).

Motivated by matching problems in a large population, we would like to seek a continuous analog of these ideas, when the goods are divisible. The set  $\mathcal{Z}$  is now no longer a set of unique products but of generic commodity types, and  $q \in \mathbb{R}^{\mathcal{Z}}$  is a bundle of commodity of various types with mass  $q_z$  of commodity z. We assume that c(q) is the cost to produce a bundle q, and we introduce the continuous analog of (1.1), namely

$$q(p,\bar{q}) \in \arg\max_{q < \bar{q}} \{pq - c(q)\}.$$
 (1.2)

Note that by forcing  $q_z \in \{0, 1\}$ , we would recover the indivisible case, as in Gul and Stacchetti (1999). The continuous analog of R(p, B), the set of options, that are not chosen

as defined above is

$$r(p,\bar{q}) = \bar{q} - q(p,\bar{q}), \qquad (1.3)$$

which is the vector of quantities that are not chosen.

Just as in the indivisible case, we are interested in the monotone comparative statics of what happens to q's when p's and  $\bar{q}$ 's vary, that is, we would like to argue that  $r_x(p,\bar{q})$  is nondecreasing in  $p_y$ ,  $y \neq x$  and in the full vector  $\bar{q}$ .

A relatively straightforward proof of these facts can be provided in the differentiable case, as seen in section 2 below. However, the theory of monotone comparative statics (MCS) allows us to dispense away from assuming differentiability. We provide a novel theory to show that the set-valued function  $(p, \bar{q}) \Rightarrow r(p, \bar{q})$  is isotone, under a set-monotonicity to be defined later.

Most of the MCS theories developed to this day, e.g., Topkis (1998) and Milgrom and Shannon (1994), does not apply here. Indeed, broadly speaking, the classical theory applies to sets that maximize supermodular functions. In that case the sets are sublattices and the goal is to order these sublattices. This is the purpose of Veinott's strong set order. We complement the literature by providing new results on minimizing supermodular functions, given that c(q) is supermodular—a problem for which Topkis' theorem and its ordinal generalizations remain silent.

We thus have to craft our own tools—namely, a monotone comparative statics result in the vein of Milgrom and Shannon (1994), Echenique (2002) and Quah and Strulovici (2009). The key remark here is that c(q) has a property that implies supermodularity but is stronger—it is the convex conjugate of a submodular function, which implies but is not equivalent to, the fact that it is supermodular. This property is the gross substitutes property, see Paes Leme (2017) and Galichon, Samuelson, and Vernet (2022). The concept of substituability is often necessary for the existence of stable matchings as described in Hatfield and Kominers (2015). Some works have studied equilibrium in the context of substituable goods on specific markets, such as Fleiner, Jagadeesan, Jankó, and Teytelboym (2019) or Kelso and Crawford (1982). We will see that this property of substituability is in fact sufficient for the existence of stable outcomes in a general setting.

This paper studies the class of such functions—the convex conjugates of submodular functions—which we call exchangeable functions. We provide a direct characterization for exchangeable functions and offer a monotone comparative statics theory for them. This theory is "dual" in a very precise sense, to Topkis' theory, and allows us to show that  $r(p, \bar{q})$  is weakly increasing in p and  $\bar{q}$ . This gives insights into the geometry of the preference sets  $q(p, \bar{q})$ . Our results generalize the indivisible-goods cases studied in Baldwin and Klemperer (2019) and Gul, Pesendorfer, and Zhang (2020).

As an application of our theory, we formalize a general version of the deferred acceptance algorithm which works both in small and large markets and with deterministic and random utility. Taking a matching problem between passengers and taxis as an illustration, we propose an algorithm, where instead of keeping track of unique offers made by individual passengers to individual taxis, we cluster both passengers and taxis into observable categories, and we keep track of the number of offers available to be made by one passenger category to one taxi category. At each round, a share of these offers is effectively made from each passenger category to taxi category, and a fraction of these are turned down. The mass of available offers from one category to the other is then decreased by the mass of rejected offers. While this algorithm boils down to the standard deferred acceptance of Gale and Shapley (1962) when there is one individual per category, it provides a version of the algorithm which operates in aggregate markets; i.e., with multiple individuals per category, without resorting to breaking ties at random.

Notations First, we introduce concepts from convex analysis and duality. Let  $f: \mathbb{R}^n \to \mathbb{R}$  be a convex function that satisfies the common regularity hypothesis (closed, proper, lower semi-continuous). We define the Legendre transform of f as  $f^* = \sup_p p^\top s - f(s)$ . This allows us to define the subdifferential of f as  $\partial f(s) = \{p \in \mathbb{R}^n \mid f^*(p) + f(s) = p^\top s\}$ . In a less formal way, this is the set of slopes for which there exist a tangent plane to the graph of f, and  $-f^*$  is interpreted as the intercept. The set  $\arg\max_p p^\top s - f^*(p)$  will be useful in the sequel. Note that the subdifferentials of f and  $f^*$  are the inverse of each other:  $p \in \partial f(s)$  is equivalent to  $s \in \partial f^*(p)$ . This duality relationship is more extensively described and generalized in many convex analysis textbooks Rockafellar (1970). Our main result extends the existing results on submodularity and supermodularity. A function  $f: \mathbb{R}^d \to \mathbb{R}$  is said

to be submodular if for any p, p' we have  $f(p \lor p') + f(p \land p') \le f(p) + f(p')$ . A function is supermodular if -f is supermodular.

#### 2. Monotone comparative statics

2.1. Exchangeable functions. The indirect profit function  $c^*$  of the unconstrained optimization problem is defined as

$$c^{*}\left(p\right) = \max_{q} \left\{ p^{\top}q - c\left(q\right) \right\}. \tag{2.1}$$

We next introduce the indirect profit function  $\bar{c}$  of the constrained optimization problem:

$$\bar{c}\left(p,\bar{q}\right) = \max_{q \leq \bar{q}} \left\{ p^{\top}q - c\left(q\right) \right\} = \max_{r \geq 0} \left\{ p^{\top}\left(\bar{q} - r\right) - c\left(\bar{q} - r\right) \right\},\tag{2.2}$$

where  $r = \bar{q} - q$  is the quantity vector which is *not* chosen. Note that the optimal value of r in (2.2) corresponds to  $r(p,\bar{q})$  in (1.3). As explained in the introduction, we are aiming at showing that  $r(p,\bar{q})$  is monotone with respect to  $\bar{q}$ . Formally, this leads us to investigate  $\partial r(p,\bar{q})/\partial \bar{q}$ . In order to study this quantity, it is useful to note that r is the gradient with respect to  $\bar{q}$  of a certain potential function. Indeed, assuming smoothness, we have

$$r(p,\bar{q}) = \frac{\partial h}{\partial p}(p,\bar{q}), \text{ where } h(p,\bar{q}) := p^{\top}\bar{q} - \bar{c}(p,\bar{q}),$$

$$(2.3)$$

and (still assuming smoothness), we have

$$\frac{\partial r}{\partial \bar{q}}(p,\bar{q}) = \frac{\partial^2 h}{\partial \bar{q}\partial p}(p,\bar{q}) = \frac{\partial^2 h}{\partial p\partial \bar{q}}(p,\bar{q}) = \frac{\partial \pi}{\partial p}(p,\bar{q}), \text{ where } \pi(p,\bar{q}) := \frac{\partial h}{\partial \bar{q}}(p,\bar{q}). \tag{2.4}$$

It is straightforward to see that  $p - \pi$  can be interpreted as the Lagrange multiplier of the constraint  $q \leq \bar{q}$  in the problem at the middle of (2.2).

By duality we have

$$h(p, \bar{q}) = \max_{\pi \le p} \left\{ \bar{q}^{\top} \pi - c^*(\pi) \right\} = \min_{r \ge 0} \left\{ p^{\top} r + c(\bar{q} - r) \right\}.$$
 (2.5)

Because of the identity  $\partial r/\partial \bar{q} = \partial \pi/\partial p$ , showing that r is monotone with respect to  $\bar{q}$  is equivalent to showing that  $\pi$  is monotone with respect to p. This is (still very formally) obtained by noting  $\pi$   $(p, \bar{q})$  is a maximizer of  $\bar{q}^{\top}\pi - c^*(\pi)$  over  $\pi \leq p$ . It is straightforward to verify that the assumptions of Topkis theorem are met. Recall that Topkis' theorem asserts<sup>1</sup>

<sup>&</sup>lt;sup>1</sup>In the literature on submodularity, Topkis' theorem is usually presented as a result for the argmax of a supermodular function. We prefer to switch to the equivalent point of view of the argmin of a submodular function, which will provide more consistency with convex analysis.

that if  $\varphi(p,\theta)$  is submodular in p and satisfies decreasing differences i.e.  $\varphi(p',\theta) - \varphi(p,\theta) \ge \varphi(p',\theta') - \varphi(p,\theta')$  for  $p' \ge p$  and  $\theta' \ge \theta$ , then we have  $\arg\min\varphi(.,\theta) \le \arg\min\varphi(.,\theta')$ . Here,  $\pi(p,\bar{q})$  arises as the minimizer of  $\varphi = c^*(\pi) - \bar{q}^{\pi} + \iota_{\{\pi \le p\}}$ , where  $\iota_{\{\pi \le p\}}$  is equal to 0 if  $\pi \le p$ , and to  $+\infty$  otherwise. This function  $\varphi$  is submodular and satisfies decreasing differences in  $(\pi,p)$ , so  $\pi$  is increasing in p, and (again in the differentiable case), we have  $\partial_p \pi(p,\bar{q}) \ge 0$  and thus  $\partial_{\bar{q}} r(p,\bar{q}) \ge 0$ .

However, the derivation above is only heuristic. Indeed, it kept requiring that h should be smooth, or more precisely twice continuously differentiable, which has no reason to be the case. Instead, we should find a theory to obtain comparative statics results directly for the original problem, namely

$$\arg\min_{r\geq 0} \left\{ p^{\top} r + c \left( \bar{q} - r \right) \right\} \tag{2.6}$$

where c is supermodular. In fact, c is more than supermodular: it is the convex conjugate of a submodular function, which implies (but is not equivalent with) that it is supermodular. This property is the key one; as we will see, we can characterize it in terms of *exchangeability*, as introduced later on.

It is not possible to apply Topkis' theorem, as this result would give us results for the arg max of c. This is our main contribution, a version of Topkis' theorem for the arg min of c. Still, we can take a look into Topkis to understand the main idea. Under the assumptions of (a common variant of) this theorem, we assume that  $\varphi(p)$  and  $\varphi'(p)$  are two functions which verify: (i) submodularity: both  $\varphi$  and  $\varphi'$  are submodular, and (ii) decreasing differences:  $\varphi(p) - \varphi(p') \leq \varphi'(p) - \varphi'(p')$  for  $p \leq p'$ . These assumptions together imply that for any two price vectors p and p', one has  $\varphi(p \wedge p') - \varphi(p) \leq \varphi'(p') - \varphi'(p \vee p')$ , which we denote

$$\varphi \le_P \varphi', \tag{2.7}$$

which can easily be shown to imply  $\iota_{\{\arg\min\varphi\}} \leq_P \iota_{\{\arg\min\varphi'\}}$ , where for any set  $B \subseteq \mathbb{R}^{\mathcal{Z}}$ ,  $\iota_B$  is the *indicator function* of the set B, defined by  $\iota_B(p) = 0$  if  $p \in B$ , and  $+\infty$  otherwise. Lastly, one sees that  $\iota_B \leq_P \iota_{B'}$  means that for  $p \in B, p' \in B'$ , one has  $p \wedge p' \in B$  and  $p \vee p' \in B'$ , which is classically expressed by saying that B is dominated by B' in Veinott's

strong set order, which we denote by an abuse of notations as

$$B \leq_P B'$$
 if and only if  $\iota_B \leq_P \iota_{B'}$ . (2.8)

To summarize, Topkis' theorem consists in saying that if  $\varphi$  and  $\varphi'$  satisfy supermodularity and increasing differences, then  $\arg \max \varphi \leq_P \arg \max \varphi'$ , that is, the corresponding  $\arg \max$  are ordered by Veinott's strong set order.

Coming back to our problem of finding monotone comparative statics in problem (2.6), we notice that the latter problem consists of *minimizing* (instead of maximizing) a supermodular function. Actually, the function to minimize  $\psi$  is more than supermodular: it is such that its convex conjugate  $\psi^*$  is submodular. Thus, a natural order on the  $\psi$ 's is induced by the P-order on the  $\psi^*$ 's. We define

$$\psi \leq_{Q} \psi' \iff def\psi^* \leq_{P} \psi'^*. \tag{2.9}$$

This order is equivalently characterized by a notion that involves exchangeability:

**Theorem 1.** For two functions  $\psi$  and  $\psi'$ , the following two notions are equivalent:

- (1)  $\psi \leq_Q \psi'$
- (2) For all  $q \in dom \ \psi$ ,  $q' \in dom \ \psi'$  and any  $\delta_1 \in [0, (q q')^+]$ , there exists  $\delta_2 \in [0, (q q')^-]$  such that  $\psi(q \delta_1 + \delta_2) + \psi'(q' + \delta_1 \delta_2) \leq \psi(q) + \psi(q')$ .

With the same logic as above, this partial order on functions induces a partial order on sets. One can show that  $\psi \leq_Q \psi'$  implies  $\iota_{\{\arg\min\psi\}} \leq_Q \iota_{\{\arg\min\psi'\}}$ . It is therefore of particular interest to study a "dual order" on subsets of  $\mathbb{R}^{\mathcal{Z}}$ , denoted  $\leq_Q$ , called the Q-set order, and defined by

$$B \leq_Q B'$$
 if and only if  $\iota_B \leq_Q \iota_{B'}$  that is, if and only if  $\iota_B^* \leq_P \iota_{B'}^*$ . (2.10)

Notice that when  $B = \{b\}$  and  $B = \{b'\}$ ,  $B \leq_Q B'$  implies  $b \leq b'$ , so our Q-set order is an extension of the order on  $\mathbb{R}^{\mathcal{Z}}$ .

Back to our problem, which was to show that

$$\arg\min_{r>0} \left\{ p^{\top} r + c \left( \bar{q} - r \right) \right\}$$

is increasing in some sense, we can now show that the right notion is the Q-set order. Set  $\psi(r,\theta) = p^{\top}r + c(\bar{q} - r)$  where  $\theta = (p,\bar{q})$  and we can show that  $\psi(.,\theta)$  is increasing in  $\theta$  with respect to the  $\leq_Q$  order. As a result, the argmin is ordered int the Q-set order. Formally, we have the following result:

**Theorem 2.** The following statements are equivalent whenever c is a convex lower semi continuous proper function

- (1) c is exchangeable, that is  $c \leq_Q c$ ,
- (2)  $c^*$  is submodular, that is  $c^* \leq_P c^*$ ,
- (3) if  $p \le p'$  then  $r_{\{p=p'\}}(p', \bar{q}) \le_Q r_{\{p=p'\}}(p, \bar{q})$  for all  $\bar{q}$ .

Moreover any of the property above imply: if  $\bar{q} \leq \bar{q}'$  then  $r\left(p, \bar{q}\right) \leq_{Q} r\left(p, \bar{q}'\right)$ .

The above MCS result has an important application for the general study of deferred acceptance algorithms. In this type of bipartite matching algorithms, the stable outcome is selected by a process that consists in having both sides of the market select their favorite contracts among a set available to them. The proposing side selects their favorites picks among contracts that have not yet been rejected by the disposing side. The latter side of the market picks among the contracts that have been proposed to them by the proposing side. The process iterates until no more contract is rejected. The fact that the mass of contracts that have been rejected can only increase provides a dose of monotonicity which guarantees the convergence of the algorithm. More formally, we introduce  $\bar{q}^A(t)$  the mass of available contracts, that is, contracts not yet rejected at time t, and if the payoffs of the proposing side are denoted by  $\alpha$ , the masses of contracts proposed is  $\bar{q}^A(t) - r(\alpha, \bar{q}^A(t))$ , and the masses of contracts rejected by the disposing side is  $r(\gamma, \bar{q}^A(t) - r(\alpha, \bar{q}^A(t)))$ . Therefore, we update the number of contracts available by

$$\bar{q}^A(t+1) = \bar{q}^A(t) - r(\gamma, \bar{q}^A(t) - r(\alpha, \bar{q}^A(t))).$$

2.2. Monotone comparative statics for the dual set order. Let us now define more formally our MCS result. In order to be coherent with convex analysis, we will state the properties of exchangeability and submodularity for convex functions. Let  $\psi, \psi'$  two convex, proper, closed, lsc functions on  $\mathbb{R}^n$ .

**Definition 1.** We say that  $\psi'$  is greater than  $\psi$  in the Q-order,  $\psi \leq_Q \psi'$  if for  $q \in dom(\psi), q' \in dom(\psi')$  and  $\delta_1 \in [0, (x-y)^+]$  there is  $\delta_2 \in [0, (x-y)^-]$ 

$$\psi(q - (\delta_1 - \delta_2)) + \psi'(q' + (\delta_1 - \delta_2)) \le \psi(q) + \psi'(q') \tag{2.11}$$

If  $\psi \leq_Q \psi$  we say that  $\psi$  is exchangeable.

We then introduce a set order which can be seen as dual to Veinott's order, the Q-order. Let X, Y two convex compact subsets of  $\mathbb{R}^n$ . It is defined by saying that  $\iota_X \leq_Q \iota_Y$ 

**Definition 2.** We say that Y is greater than X in the Q-order,  $X \leq_Q Y$  if for any  $x \in X, y \in Y$  and  $\delta_1 \in [0, (x-y)^+]$  there is  $\delta_2 \in [0, (x-y)^-]$  such that  $x - (\delta_1 - \delta_2) \in X$  and  $y + (\delta_1 - \delta_2) \in Y$ . When  $X \leq_Q X$  we say that X is a matron.

The notion of matrons relates to two previous notions introduced respectively in Murota (1998) and in Chen and Li (2022). Both these authors require the assumption that  $\delta_1$  and  $\delta_2$  in the definition above should proportional to a vector of the canonical basis. On top of that, Murota (1998) imposes that  $\delta_1^{\top} 1 = \delta_2^{\top} 1$ , which leads him to define M<sup>\daggerau</sup>-sets. The latter assumption is dropped in Chen and Li (2022), which leads them to define S-convex sets. It can easily be shown (using results in Chen and Li (2022)) that the M<sup>\daggerau</sup> property implies the S-convex one, which in turn implies the matron one. However, the converse of the first implication is false, and the converse of the second one requires strong additional regularity assumptions (see theorem 3 in Chen and Li (2022)).

In recent years multiple generalizations of  $M^{\natural}$ -sets/convexity have been developed in order to tackle problems such as market design problems as described in the recent work of Hafalir, Kojima, Yenmez, and Yokote (2023). Subsequently, we will show that our generalization—exchangeability—is dual to the notion of submodularity and has an application to the study of generalized equilibria.

**Remark 2.1.** In the case of singletons sets this order is equivalent to the coordinate-wise order on  $\mathbb{R}^n$ .

Recalling the submodularity of a function one can derive an order on functions. Let  $\varphi, \varphi'$  two convex, proper, lsc, closed functions on  $\mathbb{R}^n$ 

**Definition 3.** We say that  $\varphi'$  is greater than  $\varphi$  in the P-order,  $\varphi \leq_P \varphi'$  if for  $p \in dom(\varphi)$  and  $p' \in dom(\varphi')$ 

$$\varphi(p \wedge p') + \varphi'(p \vee p') \le \varphi(p) + \varphi'(p'). \tag{2.12}$$

If  $\varphi \leq_P \varphi$  we say that  $\varphi$  is submodular.

In a similar fashion one can retrieve Veinott's order on sets, which we call here *P-order* for consistency.

**Definition 4.** Let X, Y two convex compact subsets of  $\mathbb{R}^n$ . We say that Y is greater than X in the P-order (a.k.a. strong set order), denoted  $X \leq_P Y$  if for all  $p \in X, p' \in Y$  we have  $p \wedge p' \in X$  and  $p \vee p' \in Y$ . If  $X \leq_P X$ , one says that X is a sublattice of  $\mathbb{R}^n$ .

**Remark 2.2.** Just as for the Q-order, the P-order boils down to the coordinate-wise order on  $\mathbb{R}^n$  in the case of singletons sets.

We are now ready to state formally the theorem introduced in the introduction. For  $I \subseteq \{1,...,n\}$ , we will denote  $\pi_I$  the projection on  $\mathbb{R}^I$ . Also, we denote  $\{p=p'\}$  the set of  $i \in \{1,...,n\}$  such that  $p_i = p'_i$ .

**Theorem 3.** The following four assertions are equivalent whenever c is a l.s.c. convex, proper, closed function on  $\mathbb{R}^n$ :

- (1) c is exchangeable, i.e.  $c \leq_Q c$ ;
- (2)  $c^*$  is submodular, i.e.  $c^* \leq_P c^*$ ;
- (3) for all  $p \leq p'$  we have  $\pi_{\{p=p'\}}(\partial c^*(p)) \geq_Q \pi_{\{p=p'\}}(\partial c^*(p'));$
- (4) setting

$$r(p, \bar{q}) = \bar{q} - \arg\max_{q \le \bar{q}} \left\{ p^{\top} q - c(q) \right\}$$

then for all  $(p, \bar{q}) \leq (p', \bar{q}')$  we have

$$\pi_{\{p=p'\}}(r(p,\bar{q})) \leq_Q \pi_{\{p=p'\}}(r(p',\bar{q}')).$$

**Example 1** (Quadratic utility). Assume c is quadratic and strictly convex so that  $c(q) = q^{T}Aq/2$ . c is exchangeable if and only if  $c^{*}$  is submodular, that is, if  $S = A^{-1}$  is a Stieltjes matrix, where  $c^{*}(p) = p^{T}A^{-1}p/2$ . (A Stieltjes matrix is a matrix that is positive definite and

whose off-diagonal entries are non-positive). We have that  $r = \bar{q} - q$  and  $\tau$  are determined by the following linear complementarity problem

$$\begin{cases} r = S\tau + \rho \\ r \perp \tau \\ r \ge 0, \tau \ge 0 \end{cases}$$

with  $\rho = \bar{q} - Sp$ . Indeed,  $\tau$  is a minimizer of  $\bar{q}^{\top}\tau + c^*(p - \tau) = \bar{q}^{\top}\tau + (p - \tau)^{\top}S(p - \tau)/2$ subject to  $\tau \geq 0$ , and optimality conditions yield  $r_i = \bar{q}_i - (S(p - \tau))_i \geq 0$  that is  $r_i = (S\tau + \rho)_i \geq 0$  for all i, with equality if  $\tau_i > 0$ , which is the case of an interior solution.

While formulations (1), (2) and (4) have been discussed in the introduction, it is maybe useful to comment on formulation (3). In the smooth case, the latter boils down to expressing that for i such that  $p_i = p'_i$ ,  $\partial c^*/\partial p_i$  is decreasing in  $p_j$  for  $j \neq i$ , or equivalently that  $\nabla c^*$  is a Z-function, see Galichon and Léger (2023); equivalently still, that  $\partial^2 c^*/\partial p_i \partial p_j \leq 0$ , which is the well-known characterization of submodularity of  $c^*$  in the smooth case. The proof of (2)  $\iff$  (3) and is pretty straightforward once (1)  $\iff$  (2) has been proven. However, the proofs require the introduction of a purely technical order on functions and sets. The proof is given in appendix A.

# 3. An application to NTU matching

3.1. Equilibrium matchings. We consider a two-sided matching problem between two populations, say workers and firms. The set of observable types of workers is finite and is denoted by  $\mathcal{X}$ ; similarly, the set of observable types of firms, also finite, is denoted by  $\mathcal{Y}$ . We assume that there is a mass  $n_x$  and  $m_y$  of workers of type x and of firms of type y, respectively. An arrangement is a specification of either a match between a worker x and a firm y, denoted by xy; or an unmatched worker of type x, denoted by x0; or an unmatched firm of type y, denoted by 0y. We denote  $\mathcal{A} = (\mathcal{X} \times \mathcal{Y}) \cup (\mathcal{X} \times \{0\}) \cup (\{0\} \times \mathcal{Y})$ . We define equilibrium matchings in the following fashion.

**Definition 5.** Let  $n_x$  and  $m_y$  be two vectors of positive real numbers. A matching outcome  $(\mu, u, v) \in \mathbb{R}_+^{\mathcal{A}} \times \mathbb{R}^{\mathcal{X}} \times \mathbb{R}^{\mathcal{Y}}$  is an equilibrium matching if:

(i)  $\mu$  is a feasible matching:

$$\begin{cases} \sum_{y} \mu_{xy} + \mu_{x0} = n_x \\ \sum_{y} \mu_{xy} + \mu_{0y} = m_y. \end{cases}$$

(ii) stability conditions hold, that is

$$\max (u_x - \alpha_{xy}, v_y - \gamma_{xy}) \ge 0$$

$$u_x \ge \alpha_{x0} \text{ and } v_y \ge \gamma_{0y},$$

(iii) weak complementarity holds, that is

$$\mu_{xy} > 0 \implies \max(u_x - \alpha_{xy}, v_y - \gamma_{xy}) = 0$$

$$\mu_{x0} > 0 \implies u_x = \alpha_{x0}$$

$$\mu_{0y} > 0 \implies v_y = \gamma_{0y}.$$

The interpretation of the variables is straightforward. For  $xy \in \mathcal{A}$ ,  $\mu_{xy}$  is the mass of xy arrangements. The quantities  $u_x$  and  $v_y$  are the payoffs that x and y can expect at equilibrium. Let us comment on the requirements for  $\mu$ , u, and v. The requirement (i) implies that at equilibrium, everyone is either matched, or unmatched, or in other words, that  $\mu$  accounts for the right number of agents of every type. The stability condition (ii) expresses that there are no blocking pairs and that individuals make rational decisions. Indeed, if  $\max(u_x - \alpha_{xy}, v_y - \gamma_{xy}) < 0$  for some  $xy \in \mathcal{X} \times \mathcal{Y}$ , it means that if x and ywere to form a pair, x could get  $\alpha_{xy}$  which is more than  $u_x$ , and y could get  $\gamma_{xy}$  which is more than  $v_y$ , and so the pair xy would be a blocking pair, which we want to rule out at equilibrium. Likewise, if  $u_x < \alpha_{x0}$  or  $v_y < \gamma_{0y}$  were to hold, then x or y would be better off in autarky than in a match, and they would form a blocking coalition on their own. Finally, the feasibility conditions (iii) imply that if x and y are matched, then the utility obtained by x (resp. y), which is  $u_x$  (resp.  $v_y$ ), is at most  $\alpha_{xy}$  (resp.  $\gamma_{xy}$ ), which are the utilities that they can obtain in a matched with y, with the possibility of free disposal. Similarly, if x (resp. y) remain unmatched, then they get at most their reservation payoff  $\alpha_{x0}$  (resp.  $\gamma_{0y}$ ).

**Remark 3.1.** Definition 5 can be restated by ensuring that there exists  $(\mu, U, V) \in \mathbb{R}_+^{\mathcal{A}} \times \mathbb{R}^{(\{0\} \cup \mathcal{X}) \times \mathcal{Y}}$  such that

- (i) the matching  $\mu$  is feasible;
- (ii) the equality  $\max\{U_{xy} \alpha_{xy}, V_{xy} \gamma_{xy}\} = 0$  holds for every pair  $xy \in \mathcal{X} \times \mathcal{Y}$ ;
- (iii) one has the following implications

$$\begin{cases}
\mu_{xy} > 0 \implies U_{xy} = \max_{y' \in \mathcal{Y}} \{U_{xy'}, \alpha_{x0}\} \text{ and } V_{xy} = \max_{x' \in \mathcal{X}} \{V_{x'y}, \gamma_{0y}\} \\
\mu_{x0} > 0 \implies \alpha_{x0} = \max_{y' \in \mathcal{Y}} \{U_{xy'}, \alpha_{x0}\} \\
\mu_{0y} > 0 \implies \gamma_{0y} = \max_{x' \in \mathcal{X}} \{V_{x'y}, \gamma_{0y}\}.
\end{cases} (3.1)$$

Indeed, if  $(\mu, U, V)$  verify the three conditions of the present remark, then setting  $u_x = \max_{y' \in \mathcal{Y}} \{U_{xy'}, 0\}$  and  $v_y = \max_{x' \in \mathcal{X}} \{V_{x'y}, 0\}$  implies the stability conditions (ii) of definition 5, and  $\mu_{xy} > 0$  implies  $u_x = U_{xy}$  and  $v_y = V_{xy}$ , and thus  $\max\{u_x - \alpha_{xy}, v_y - \gamma_{xy}\} = 0$ , which is the main step to show (iii) of definition 5.

Conversely, letting  $(\mu, u, v)$  be an equilibrium matching outcome as in definition 5. Letting  $U_{xy} = \min\{u_x, \alpha_{xy}\}$  and  $V_{xy} = \min\{v_y, \gamma_{xy}\}$ , one has  $U_{xy} - \alpha_{xy} = \min\{u_x - \alpha_{xy}, 0\}$  and  $U_{xy} - \gamma_{xy} = \min\{v_y - \gamma_{xy}, 0\}$ , and therefore condition (ii) of definition 5 implies that one has  $\max\{U_{xy} - \alpha_{xy}, V_{xy} - \gamma_{xy}\} = 0$ . By definition, one has  $u_x \geq U_{xy}$ , but  $\mu_{xy} > 0$  implies that  $u_x \leq \alpha_{xy}$ , which implies in turn equality, which is the main step to show (iii) of the present remark.

**Remark 3.2.** Note that expression 3.1 in condition (iii) of remark 3.1 can be rewritten as

$$\mu \in \partial G(U) \text{ where } G(U) := \sum_{x \in \mathcal{X}} n_x \max_{y \in \mathcal{Y}} \{U_{xy}, \alpha_{x0}\}$$
 (3.2)

and  $\mu_{x0} = n_x - \sum_{y \in \mathcal{Y}} \mu_{xy}$ . This is reminiscent of the Daly-Zachary-Williams in the discrete choice literature. In the next paragraph, we will pursue this connection much further. Note that in this case, it is easy to show that G(U) has expression

$$G(U) = \max_{\mu \ge 0} \left\{ \sum_{xy \in \mathcal{X} \times \mathcal{Y}} \mu_{xy} (U_{xy} - \alpha_{x0}) : \sum_{y \in \mathcal{Y}} \mu_{xy} \le n_x \right\} + \sum_{x \in \mathcal{X}} n_x \alpha_{x0}.$$
 (3.3)

3.2. **Generalized equilibrium matchings.** Remarks 3.1 and 3.2 suggest a generalization of definition 5 in terms of *variational preferences*, which are defined as follows.

**Definition 6.** Let  $\underline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$ .  $G : \underline{\mathbb{R}}^{\mathcal{X} \times \mathcal{Y}} \to \mathbb{R}$  is a welfare function if G is convex (closed proper lsc),  $domG^*$  is compact and  $0 = \min(domG^*)$ .

It follows from the definition of a welfare function that G as a welfare function can be expressed as

$$G(U) = \max_{\mu \ge 0} \left\{ \sum_{y \in \mathcal{Y}} \mu_{xy} U_{xy} - G^*(\mu) \right\}. \tag{3.4}$$

G is interpreted as the maximal total welfare with a penalization  $G^*(\mu)$ , which encodes constraints on  $\mu$ . In that setting  $\partial G(U)$  is interpreted as the set of matchings for which  $\mu_{xy}$  is the demand for the pairing xy. This interpretation motivates the following definition of generalized equilibrium matching.

**Definition 7.** A generalized equilibrium matching between two welfare functions G and H, with initial preferences  $\alpha, \gamma$  is a triple  $(\mu, U, V) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$  with

(i) 
$$\max \{U - \alpha, V - \gamma\} = 0$$

(ii) 
$$\mu \in \partial G(U) \cap \partial H(V)$$
.

The first condition allows for one side of the market to burn utility in order to reach an agreement, which implies that in the pair xy, x will be able to obtain utility  $U_{xy} \leq \alpha_{xy}$ , and y can get utility  $V_{xy} \leq \gamma_{xy}$ . These inequalities are strict when there is money burning, but cannot be both strict at the same time. The second condition is interpreted as the equality of supply and demand for each pair xy given the two endogenous utilities U and V.

In the light of remarks 3.1 and 3.2, definition 7 is an extension of definition 5. Indeed, the function G introduced in expression (3.2) in remark 3.2 is of the form of expression (3.4) with the appropriate choice of  $G^*$ .

3.3. Deferred acceptance welfare algorithm. Gale and Shapley (1962) showed that a stable matching exists and provided the deferred acceptance algorithm to determine one. In this section, we show the existence of a generalized equilibrium matching using a suitable generalization of the deferred acceptance algorithm.

We define  $T^G$  as the vector of the Lagrange multipliers associated with the constraints  $\mu_{xy} \leq \bar{\mu}_{xy}$  in

$$\max_{\mu \ge 0} \left\{ \sum_{xy} \mu_{xy} \alpha_{xy} - G^*(\mu) \right\}$$

$$s.t. \qquad \mu_{xy} \le \bar{\mu}_{xy}$$

$$(3.5)$$

so that  $T^G$  measures the amount by which the cardinal preferences need to be decreased in order to verify the constraint  $\mu \leq \bar{\mu}$ . By duality, it is determined by

$$T^{G}(\alpha, \bar{\mu}) = \underset{\tau \ge 0}{\arg \max} \, \bar{\mu}^{\top}(\alpha - \tau) - G(\alpha - \tau). \tag{3.7}$$

Because  $\text{dom}G^*$  is compact by assumption, there exists a vector  $n^G \in \mathbb{R}^{\mathcal{X} \times \mathcal{Y}}$  such that  $\mu \in \text{dom}G^*$  implies  $\mu \leq n^G$ . Note that we don't require to define  $n^G$  univocally since it will only be used to initialize the algorithm with a non-binding constraint. We use the notation  $T^H(\gamma, \bar{\mu})$  and  $n^H$  for the other side of the market.

The idea of the algorithm is to update the constraint  $\bar{\mu}$  to take into account the discrepancies between demand and offer. The algorithm is comprised of the three classical phases in deferred acceptance: a proposal phase, a disposal phase, and an update phase. The x's make their offers within these that are available to them, initially, the full set of potential offers; the y's pick the offers among the offers that were made to them; and the offers that have not been picked are no longer available to the x's.

**Algorithm 1.** At step 0, we set  $\mu^{A,0} = \min(n^G, n^H)$  and  $\mu^{T,-1} = 0$ . Next, we iterate over  $k \geq 0$  the following three phases:

Proposal phase: The x's make proposals to the y's subject to availability constraint:

$$\mu^{P,k} \in \underset{\mu^{T,k-1} < \mu < \mu^{A,k}}{\arg \max} \mu^{\top} \alpha - G^*(\mu),$$

which is well-defined as shown in the proof of theorem 4.

Disposal phase: The the y's pick their best offers among the proposals:

$$\mu^{T,k} \in \underset{\mu < \mu^{P,k}}{\arg \max} \mu^{\top} \gamma - H^*(\mu)$$

Update phase: The number of available offers is decreased by the number of rejected ones:

$$\mu^{A,k+1} = \mu^{A,k} - (\mu^{P,k} - \mu^{T,k}).$$

The algorithm stops when the norm of  $\mu^{P,k} - \mu^{T,k}$  is below some tolerance level.

Numerically, the proposal and disposal phases require optimizing convex functions, which in itself is not costly (Galichon and Salanié 2015), but we do not provide here a convergence rate for the numerical scheme. However algorithm 1 defines a constructive method to show

the existence of a generalized equilibrium matching. Indeed, under modest assumptions on G and H, the algorithm converges toward a generalized equilibrium matching.

When G, H are submodular welfare functions, our algorithm involves finding the maximizers of submodular functions. In the work of Chade and Smith (2006), a similar kind of greedy algorithm is presented in the case of set submodular functions.

To prove the convergence of the algorithm, we shall use some particular Lagrange multipliers associated with the availability constraints. We define them by

$$\tau^{P,k} = \inf T^G(\alpha, \mu^{A,k}) \text{ and } \tau^{T,k} = \inf T^H(\gamma, \mu^{P,k})$$
(3.8)

and they are well-defined because these set are sublattice of  $\mathbb{R}^{\mathcal{X} \times \mathcal{Y}}$  bounded below by 0. This allows us to formulate the following result, whose proof can be found in appendix C:

**Theorem 4.** If G, H are convex (proper, lsc, closed) submodular welfare functions. Then the algorithm is well defined and up to an extraction, the sequence

$$(\mu^{P,k}, \tau^{P,k}, \tau^{T,k}) \tag{3.9}$$

converges. Moreover, let  $(\mu, \tau^P, \tau^T)$  be the limit of a converging subsequence, and define  $\tau^{\alpha} := \tau^P 1_{\mu < n^G}$  and  $\tau^{\gamma} := \tau^T 1_{\mu < n^H}$ . Then  $(\mu, \alpha - \tau^{\alpha}, \gamma - \tau^{\gamma})$  is a generalized equilibrium matching.

3.4. Links with the literature. The deferred acceptance algorithm that we have presented here can be seen as a continuous extension of the original deferred acceptance algorithm of Gale and Shapley (1962). However our extension is not the first one tackling the question of stable matching with continuous quantities  $\mu \in \mathbb{R}^{\mathcal{X} \times \mathcal{Y}}$ . Indeed Alkan and Gale (2003) presented a continuous deferred acceptance algorithm using the formalism of choice functions. In this paragraph, we detail the links between their work and the ours. Given a vector  $\bar{\mu}$  of available quantities, they introduce an oracle choice function C which is continuous and point valued such that  $C(\bar{\mu}) \leq \bar{\mu}$  represents the preferred vector of choices that falls below the capacity vector  $\bar{\mu}$ . In the setting of generalized stable matchings introduced here, C is the set of maximizers of a welfare function under the capacity constraint  $\bar{\mu}$ . Indeed in our setting we have  $C(\bar{\mu}) = \arg\max_{\mu \leq \bar{\mu}} p\mu - G^*(\mu)$ . We thus allow for a choice correspondence instead of restricting ourselves to a choice function; but we impose that that correspondence arises as the solution to a welfare maximization problem.

Given a choice function  $C_X$  for the x's and  $C_Y$  a choice function for the y's the goal is to find a matching  $\mu$  which is acceptable and stable. Acceptability states that  $\mu$  can actually be chosen by the agents, that is  $\mu \in \text{Im}C_X \cap \text{Im}C_Y$ . In our case this amounts to saying that  $\mu \in \text{range}\partial G \cap \text{range}\partial H$ . Stability is essentially the fact there are no blocking pairs, which as we have seen earlier is linked to the fact that  $\min(\tau^P, \tau^T) = 0$ . Under the assumption that the choice functions are consistent and persistent, a notion to which we will come back, Alkan and Gale (2003) propose an algorithm which converge towards a stable matching:

**Algorithm 2.** At step 0, we initialize  $\mu^{A,0}$  at a non constraining value. Next, we iterate over  $k \geq 0$  the following three phases:

Proposal phase: The x's make proposals to the y's subject to availability constraint:

$$\mu^{X,k} = C_X(\mu^{A,k})$$

Disposal phase: The the y's pick their best offers among the proposals:

$$\mu^{Y,k} = C_Y(\mu^{X,k})$$

<u>Update phase</u>: The number of available offers is adjusted to take into account the rejections

$$\mu_{xy}^{A,k+1} = \begin{cases} \mu_{xy}^{A,k} & \text{if } \mu_{xy}^{Y,k} = \mu_{xy}^{X,k} \\ \mu_{xy}^{Y,k} & \text{if } \mu_{xy}^{Y,k} < \mu_{xy}^{X,k} \end{cases}$$

Note that the key difference with algorithm 1 is the update phase in the case  $\mu_{xy}^{Y,k} < \mu_{xy}^{X,k}$ . In that case the capacity constraint  $\mu^{A,k}$  is reduced by  $\mu^{A,k} - \mu^{Y,k}$  and not  $\mu^{X,k} - \mu^{Y,k}$ . This difference is tied to the property of *persistence* of the choice function. Note that persistence implies the property that we found for our choice functions which is  $\bar{\mu} \mapsto \bar{\mu} - C(\bar{\mu})$  is increasing in the Q-order. As we have seen both persistence and the Q-order are closely linked with substituability. More precisely theorem 3 shows that the property of growth in the Q-order is implied by substituability.

3.5. Application to matching with random utility. We apply this procedure in the random utility setting, which as we shall see, provides instances of welfare functions. Let us describe that random utility setting, also called the discrete choice framework, see McFadden

(1976). We shall adopt the language of passengers and taxi drivers for concreteness. There are  $n_x$  passengers of type  $x_i$  an individual passenger i of type  $x_i = x$  enjoys a total utility which is the sum of two terms: a deterministic component  $\alpha_{xy}$  called "systematic utility," plus a random utility term  $(\varepsilon_{iy})_y$ . The distribution  $\mathbf{P}_x$  of the random utility term may depend on x – including the random utility  $\varepsilon_{i0}$  associated with exiting the market. Similarly, there are  $m_y$  drivers of type y, and driver j of a type-y car also enjoys a total utility which is the sum of a deterministic (systematic) term plus a random term:  $\gamma_{xy} + (\eta_{xj})_x$  if driver j matches with a passenger of type x, and  $\eta_{0j}$  if driver j remains unmatched. Again, the distribution  $\mathbf{Q}_y$  may depend on y.

If passengers faced an unconstrained choice problem, the indirect utility of a passenger i of type x would be  $\max_{y \in \mathcal{Y}} \{\alpha_{xy} + \varepsilon_{iy}, \varepsilon_{i0}\}$ . As a result, the total indirect utility among drivers is  $G(\alpha) = \sum_{x \in \mathcal{X}} n_x \mathbb{E} \left[\max_{y \in \mathcal{Y}} \{\alpha_{xy} + \varepsilon_y, \varepsilon_0\}\right]$ . This leads to a demand vectors  $\mu$  where  $\mu_{xy}$  stands for the mass of passengers of type x demanding a car of type y. When the probability of being indifferent between two types of cars is zero, we have  $\mu_{xy} = \sum_x n_x \Pr\left(y \in \arg\max_{y \in \mathcal{Y}} \{\alpha_{xy} + \varepsilon_y, \varepsilon_0\}\right)$ , and in this case  $\Pr\left(y \in \arg\max_{y \in \mathcal{Y}} \{\alpha_{xy} + \varepsilon_y, \varepsilon_0\}\right)$  is the gradient of  $\mathbb{E} \left[\max_{y \in \mathcal{Y}} \{\alpha_{xy} + \varepsilon_y, \varepsilon_0\}\right]$  with respect to  $\alpha_{xy}$ , and therefore

$$\mu = \nabla G(\alpha), \tag{3.10}$$

a result known in discrete choice theory as the Daly-Zachary-Williams theorem, see McFadden (1976). More generally, as seen in Bonnet, Galichon, Hsieh, O'Hara, and Shum (2021), if we don't assume any restrictions on the distribution of  $\varepsilon$ , we may have several ways to break the potential ties that may arise, and the total set of resulting demand vectors is given by an extension of relation (3.10) to a set inclusion, namely

$$\mu \in \partial G(\alpha)$$
, (3.11)

where  $\partial G(\alpha)$  is the subdifferential of G.

Along algorithm 1, the set of available possibilities will shrink, due to the fact that the mass of turned down offers was deduced from the mass of available offers, and if  $\bar{\mu}_{xy}$  is the mass of available xy matches, the constraint  $\mu_{xy} \leq \bar{\mu}_{xy}$  may start to become binding. The shadow price  $\tau_{xy} \geq 0$  associated with this constraint can be interpreted as a waiting time  $\tau_{xy} \geq 0$  which will regulate the demand of y's by x's so that the surplus obtained by a x

matched with a y will now become  $\alpha_{xy} - \tau_{xy} + \varepsilon_y$ . Thus  $\alpha$  needs to be replaced by  $\alpha - \tau$  in the expression of demand for matches (3.11), and  $\tau$  is determined by the complementary conditions

(i) 
$$\mu \in \partial G(\alpha - \tau)$$
; (ii)  $\tau_{xy} \ge 0, \mu_{xy} \le \bar{\mu}_{xy}$ ; (iii)  $\tau_{xy} > 0 \implies \mu_{xy} = \bar{\mu}_{xy}$ 

which are the optimality conditions of primal problem (in  $\mu$ ) and dual problem (in  $\tau$ )

$$\max_{\mu \leq \bar{\mu}} \sum_{xy} \mu_{xy} \alpha_{xy} - G^* \left( \mu \right) = \min_{\tau \geq 0} \sum_{xy} \bar{\mu}_{xy} \tau_{xy} + G \left( \alpha - \tau \right), \tag{3.12}$$

which recover the expressions (3.5) seen above. It is easy to see that the functions given by  $G(\alpha) = \sum_{x \in \mathcal{X}} n_x \mathbb{E}_{\mathbf{P}_x} \left[ \max_{y \in \mathcal{Y}} \left\{ \alpha_{xy} + \varepsilon_y, \varepsilon_0 \right\} \right] \text{ and } H(\gamma) = \sum_{y \in \mathcal{Y}} m_y \mathbb{E}_{\mathbf{Q}_y} \left[ \max_{x \in \mathcal{X}} \left\{ \gamma_{xy} + \eta_x, \eta_0 \right\} \right]$  are submodular welfare functions.

As a corollary of theorem 4, it follows that a generalized equilibrium matching exists in the case of models of random utility.

**Example 2.** If we assume that the random utility terms  $(\varepsilon_{xy})_y$  follow i.i.d. Gumbel distributions, then it is well-known that  $G(\alpha) = \sum_{x \in \mathcal{X}} n_x \log(1 + \sum_{y \in \mathcal{Y}} \exp \alpha_{xy})$ . In this case, we have  $M^G(\alpha, \overline{\mu}) = \mu$  which satisfies  $\mu_{xy} = \min(\mu_{x0} \exp(\alpha_{xy}), \overline{\mu}_{xy})$ , where  $\mu_{x0}$  is defined by

$$\mu_{x0} + \sum_{y \in \mathcal{Y}} \min \left( \mu_{x0} \exp \left( \alpha_{xy} \right), \overline{\mu}_{xy} \right) = n_x,$$

and  $T^{G}(\alpha, \overline{\mu}) = \tau$  is deduced by  $\tau_{xy} = \max\left(0, \alpha_{xy} + \log\frac{\mu_{x0}}{\overline{\mu}_{xy}}\right)$ .

As a result, if we assume that on the other side of the market, the random utility terms  $(\eta_{xy})_x$  follow i.i.d. Gumbel distributions, then the generalized equilibrium matching is such that

$$\mu_{xy} = \min \left( \mu_{x0} \exp \left( \alpha_{xy} \right), \mu_{0y} \exp \left( \gamma_{xy} \right) \right). \tag{3.13}$$

where  $\mu_{x0}$  and  $\mu_{0y}$  are the unique solution to the following system of equations:

$$\mu_{x0} + \sum_{y \in \mathcal{Y}} \min \left( \mu_{x0} \exp \left( \alpha_{xy} \right), \mu_{0y} \exp \left( \gamma_{xy} \right) \right) = n_x,$$
  

$$\mu_{0y} + \sum_{x \in \mathcal{X}} \min \left( \mu_{x0} \exp \left( \alpha_{xy} \right), \mu_{0y} \exp \left( \gamma_{xy} \right) \right) = m_y.$$
(3.14)

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### APPENDIX A. PROOF OF THE MCS RESULTS

In order to prove theorem 3 we need to introduce slightly weaker and more elaborate versions of the Q-order and the P-order. Let  $\epsilon > 0$  and  $D \subset [1, n]$ .

**Definition 8.** We say that g is greater than f in the  $(\epsilon, D)$  Q-order,  $f \leq_{(\epsilon, D), Q} g$  if for any x in the domain of f, y in the domain of g and  $\delta_1 \in [0, (x-y)^+] \cap \mathbb{R}^D$  there is a  $\delta_2, (\delta_2)_{|D} \in [0, (x-y)^-]$  such that

$$f(x - (\delta_1 - \delta_2)) + g(y + (\delta_1 - \delta_2)) < f(x) + g(y) + \epsilon.$$

**Definition 9.** We say that g is greater than f in the  $(\epsilon, D)$  P-order,  $f \leq_{(\epsilon, D), P} g$  if for any  $\lambda \in \mathbb{R}^d$  and  $d_f, d_g \in \mathbb{R}^D$ 

$$f(\lambda + d_f \wedge d_g) + g(\lambda + d_f \vee d_g) < f(\lambda + d_f) + g(\lambda + d_g) + \epsilon.$$

We furthermore introduce  $\leq_{D,Q}$  and  $\leq_{D,P}$  which follow the same scheme but the strong inequality is replaced by a weak inequality and  $\epsilon = 0$ . Notice that in the case of an indicator function we have  $X \leq_{D,Q} Y \iff X \leq_{D,Q} Y \iff \pi_D X \leq_Q \pi_D Y$ . These equivalence justify the introduction of these order as they will enable to interpret directly the meaning for two set to have their projection to be in relationship.

**Theorem 5.** Let f, g two convex, closed, proper, lsc functions on  $\mathbb{R}^n$  the following equivalence is satisfied:  $f \leq_{\epsilon,D,Q} g \iff g^* \leq_{\epsilon,D,P} f^*$ 

Before we give the proof of theorem 5, we state an important corollary, the proof of which will be provided after the proof of theorem 5. Given a set  $X \subseteq \mathbb{R}^n$  we denote  $\sigma_X$ 

the support function of X, where  $\sigma_X(y)$  is defined as the supremum of  $x^{\top}y$  over  $x \in X$ . It coincides  $\iota_X^*$ , the Legendre-Fenchel transform of  $\iota_X$ .

Corollary 1.  $X \leq_{D,Q} Y$  is equivalent to  $\sigma_Y \leq_{D,P} \sigma_X$ .

Proof of theorem 5. Let f, g two convex, closed, proper, lsc functions on  $\mathbb{R}^n$ . Let  $x \in \text{dom}(f)$  and  $y \in \text{dom}(g)$ . We study the following quantity

$$\sup_{\delta_1 \in [0,(x-y)^+] \cap \mathbb{R}^D} \inf_{(\delta_2)_{|D} \in \left[0,(x-y)_{|D}^-\right]} f(x - \delta_1 + \delta_2) + g(y + \delta_1 - \delta_2)$$
(A.1)

which will prove the existence of the desired  $\delta_2$  for any given  $\delta_1$  if it is small enough. By duality on the infimum term this quantity is equal to

$$\sup_{\substack{\lambda \in \mathbb{R}^D \\ \mu \in \mathbb{R}^n}} \mu(x+y) + \lambda^+ x - \lambda^- y - f^*(\mu+\lambda) - g^*(\mu). \tag{A.2}$$

Finally  $f \leq_{\epsilon,D,Q} g$  is equivalent to A.1 by definition. This expression is equal to A.2. Which is equivalent to  $g^* \leq_{\epsilon,D,P} f^*$  by taking  $\lambda = d_f - d_g$  and  $\mu = \alpha + d_g$ .

Proof of corollary 1. Here  $f = \iota_X$ ,  $g = \iota_Y$  and D is any subset of [1, n]. Since f, g take values in  $\{0, \infty\}$  we have  $\forall \epsilon > 0$ ,  $(f \leq_{\epsilon, D, Q} g \iff f \leq_{D, Q} g)$ . Moreover since the Legendre transforms are the support functions and because  $\forall \epsilon > 0$ ,  $\sigma_Y \leq_{\epsilon, D, P} \sigma_X \iff \sigma_Y \leq_{D, P} \sigma_X$ . We conclude with theorem 5.

We are now able to prove the characterization of submodularity using the subdifferentials. Finally we can prove the last equivalence  $(1) \iff (4)$ . Using methods close to classical MCS, by ranking two functions and seeing the implication on the argmin.

Proof of theorem 3. **Proof** (1)  $\Longrightarrow$  (4). Assume c exchangeable. Let  $(p,\bar{q}) \leq (p',\bar{q}')$ . Notice that  $r(p,\bar{q}) = \arg\min_{u\geq 0} c(\bar{q}-u) + p^{\top}u$ . We then set  $\psi(u) = c(\bar{q}-u) + p^{\top}u$  and  $\psi'(u) = c(\bar{q}'-u) + p'^{\top}u$ . In order to show the wanted result it is sufficient to show that  $\psi \leq_{\{p=p'\},Q} \psi'$  since this imply the wanted order on their argmin (as seen in the introduction).

Let  $u \in \text{dom}(\psi), u' \in \text{dom}(\psi')$  and  $\delta_1 \in [0, (u-u')^+]$  with support in  $\{p = p'\}$ . First notice that  $[0, (u-u')^+] \subset [0, (\bar{q}'-u'-(\bar{q}-u))^+]$  because  $\bar{q}' \geq \bar{q}$  thus by exchangeability of c

there exists  $\delta_2 \in [0, (\bar{q}' - u' - (\bar{q} - u))^-] \subset [0, (u - u')^-]$  (thus  $(\delta_2)_{|\{p=p'\}}$  also satisfy this condition) such that

$$c(\bar{q} - u) + c(\bar{q}' - u') \ge c(\bar{q} - u + (\delta_1 - \delta_2)) + c(\bar{q}' - u' - (\delta_1 - \delta_2))$$

This rewrites as

$$\psi(u) + \psi'(u') \ge \psi(u - (\delta_1 - \delta_2)) + \psi'(u' + (\delta_1 - \delta_2)) + (p - p')^{\top}(\delta_1 - \delta_2)$$

However  $(p-p')^{\top}(\delta_1 - \delta_2) = (p'-p)^{\top}\delta_2 \geq 0$  first because the support of  $\delta_1$  is in  $\{p=p'\}$  and then because  $p' \geq p, \delta_2 \geq 0$ . Finally we do have  $\psi \leq_{\{p=p'\},Q} \psi'$  yielding the wanted result.

**Proof** (4)  $\Longrightarrow$  (3). Let  $p' \geq p$  and  $\tilde{q} \in \pi_{p=p'}\partial c^*(p), \tilde{q}' \in \pi_{p=p'}\partial c^*(p')$ . Let  $\delta_1 \in [0, (\tilde{q}' - \tilde{q})^+]$ . Set  $q \in \partial c^*(p), q' \in \partial c^*(p')$  such that  $\pi_{p=p'}(q) = \tilde{q}, \pi_{p=p'}(q') = \tilde{q}'$ . Let  $\bar{q} = q \vee q'$ , notice that  $\bar{q} - q \in r(p, \bar{q}), \bar{q} - q' \in r(p', \bar{q})$ . We then have  $\pi_{p=p'}(\bar{q}) - \tilde{q} \in \pi_{p=p'}r(p, \bar{q}), \pi_{p=p'}(\bar{q}) - \tilde{q}' \in \pi_{p=p'}r(p', \bar{q})$ . Since  $\pi_{p=p'}r(p, \bar{q}) \leq_Q \pi_{p=p'}r(p', \bar{q})$  there exists  $\delta_2 \in [0, (\pi_{p=p'}(\bar{q}) - \tilde{q} - (\pi_{p=p'}(\bar{q}) - \tilde{q}'))^-] = [0, (\tilde{q}' - \tilde{q})^-]$  such that  $\pi_{p=p'}(\bar{q}) - \tilde{q} - (\delta_1 - \delta_2) \in \pi_{p=p'}r(p, \bar{q})$  thus  $\tilde{q} + (\delta_1 - \delta_2) \in \pi_{p=p'}\partial c^*(p)$  and similarly  $\tilde{q}' - (\delta_1 - \delta_2) \in \pi_{p=p'}\partial c^*(p')$ . This proves that  $\pi_{p=p'}\partial c^*(p') \leq_Q \pi_{p=p'}\partial c^*(p)$ .

**Proof** (3)  $\Longrightarrow$  (2). We suppose that  $c^*$  is submodular. Notice that for  $p \in \mathbb{R}^n$  the support function of  $\partial c^*(p)$  is the directional derivative of  $c^*$  taken at p. Thus we need to prove The lemma 1 on the characterization via support functions and the remark on projection of sets allow us to only prove that for any  $p \leq p'$  we have  $(c^*)'(p,.) \leq_{\{p=p'\},P} (c^*)'(p',.)$ . Let  $p,p' \in \mathbb{R}^d$ , set  $D = \{p = p'\}$ . Let  $\lambda \in \mathbb{R}^d$  and  $d_1, d_2 \in \mathbb{R}^D$ . Let  $\epsilon > 0$ . Since  $c^*$  is submodular we have

$$c^{*}(p' + \epsilon(\lambda + d_{2})) + c^{*}(p + \epsilon(\lambda + d_{1})) \ge$$

$$c^{*}(p + \epsilon(\lambda + d_{1}) \vee p' + \epsilon(\lambda + d_{2})) + c^{*}(p + \epsilon(\lambda + d_{1}) \wedge p' + \epsilon(\lambda + d_{2})) \ge$$

$$c^{*}(p' + \epsilon(\lambda + d_{1} \vee d_{2})) + c^{*}(p + \epsilon(\lambda + d_{1} \wedge d_{2}))$$

The last inequality holds because  $d_1, d_2 \in \mathbb{R}^D$  and  $D = \{p = p'\}$ . Then by substracting  $c^*(p) + c^*(p')$  and letting  $\epsilon \to 0$ , we have the wanted result.

Now for the reciprocal, thanks to 1 and the remark on projection of sets, the hypothesis is the same as  $(c^*)'(p,.) \leq_{\{p=p'\},P} (c^*)'(p',.)$  for  $p \leq p'$ . Let  $x,y \in \mathbb{R}^d$  and  $t \in [0,1]$  we have

 $\operatorname{supp}(y-x\wedge y)\subset \{x+t(y-x\wedge y)=x\wedge y+t(y-x\wedge y)\}. \text{ Since } (c^*)'(x\wedge y+t(y-x\wedge y),.)\geq_{D(p,p')-sub}(c^*)'(x+t(y-x\wedge y),.) \text{ we have for any } t\in [0,1]$ 

$$(c^*)'(x+t(y-x\wedge y),y-x\wedge y)\leq (c^*)'(x\wedge y+t(y-x\wedge y),y-x\wedge y).$$

Finally by integration we get that  $c^*$  is submodular.

**Proof that (2)**  $\Longrightarrow$  **(1)**. In that case f = g = c and D = [1, n]. Thus by compactness of  $[0, (x - y)^-]$  for all  $x, y \in \mathbb{R}^d$  we have the following equivalence  $(\forall \epsilon > 0, c \leq_{\epsilon,[1,n],Q} c) \iff c \leq_Q c$ . The right inequality means that c is exchangeable. We also have  $(\forall \epsilon > 0, c^* \leq_{\epsilon,[1,d],P} c^*) \iff c^* \leq_P c^*$ . Where the right inequality means that  $c^*$  is submodular. Finally by 5 we have the wanted result.

Though a fairly technical section, the methods deployed here in order to derive MCS are quite effective and shed new light on the work of Milgrom and Shannon (1994).

# APPENDIX B. AN EXTENSION OF MCS RESULTS

Originally the question of monotone comparative statics has been studied by Milgrom and Shannon (1994) using tools developed by Veinott and Topkis which are the strong set order and the characterization of sublattices. MCS results have been useful to study the behaviour of maximizers of quasilinear supermodular problems. As seen in the introduction these sets can be viewed as the subdifferential of the dual perturbating function. This has raised the question of finding the dual notion to the strong set order. In the case of polyhedral perturbating functions the question has been answered and thoroughly treated in the book of Murota (1998) on discrete convex analysis. The goal of this section is to display how Q-order and P-order relate to their discrete counterpart.

The dual of polyhedral lattices are  $M^{\natural}$  sets.

**Definition 10.** Let X a non empty subset of  $\mathbb{R}^n$ . We say that X is an  $M^{\natural}$  set if for any  $x, y \in X$  and  $i \in supp^+(x-y)$  there exist  $\alpha > 0$  and  $j \in supp^-(x-y) \cup \{0\}$  such that  $x - \alpha(e_i - e_j) \in X$  and  $y + \alpha(e_i - e_j) \in X$ .

Notice that by breaking symmetry we obtain an order on sets. This order on sets has been proven Murota (1998) to be the dual notion, in the case of polyhedral sets, to  $L^{\natural}$  order, which is essentially Veinott's order. Notice that a set is a matron whenever the continuous version

of this exchange property is satisfied. Frank (1984) introduced generalized polymatroids which are another representation of  $M^{\natural}$  sets using support functions.

**Definition 11.** P is a generalized polytmatroid if there exist  $h, g : \mathcal{P}([1, n]) \to \mathbb{R}$ , h submodular and g supermodular such that

$$P = \left\{ x \in \mathbb{R}^n \mid \forall B \subset [1, n], g(B) \le x^{\top} \chi_B \le h(B) \right\}$$
 (B.1)

The linear constraints can be extended to positive test vectors which are not indicators of set by taking the Lovasz extension of g and h, which we still denote by g, h. We are then able to recover the support function of P:  $\sigma(d) = h(d^+) - g(d^-)$ . When X is a matron we have a similar representation. Indeed we have

$$X = \left\{ x \in \mathbb{R}^n \mid \forall d \ge 0, g(d) \le x^{\top} d \le h(d) \right\}$$
 (B.2)

with g supermodular, h submodular defined as  $g(d) = -\sigma(-d^-)$  and  $h(d) = \sigma(d^+)$ . Here  $\sigma$  is the support function of the set. This representation of matrons is always possible. And given two functions g, h there is a matron associated to them if they satisfy a compatibility condition which is the paramodularity Frank (1984).

**Definition 12.** Let g supermodular, h submodular. We say that (g,h) is paramodular if for any  $d,b \in \mathbb{R}^n_+$  we have  $h(d) - g(b) \ge h(d \lor b - b) - g(d \lor b - d)$ .

For any paramodular pair the set defined above is a matron. This gives a definition of matrons as continuous generalized polymatroids. Finally in general the Q-order on functions can be seen as the extension of  $M^{\natural}$ -convexity to non polyhedral functions. We recall the definition of  $M^{\natural}$ -convexity

**Definition 13.** Let f, g two convex functions we say that f is smaller than g in the M order,  $g \leq_M g$  if for any x, y and any direction  $i \in supp^+(x - y)$  there exists  $\alpha > 0$  and  $j \in supp^-(x - y) \cup \{0\}$  such that  $f(x) + g(y) \geq f(x - \alpha(e_i - e_j)) + g(y + \alpha(e_i - e_j))$ 

As proven by Murota (1998) this is equivalent to  $f^* \geq_{L^{\natural}} g^*$  where the  $L^{\natural}$ -order is an extension of Veinott's order to functions. Once again the result of duality between Q-order and P-order is the extension of that result to non polyhedral convex functions.

# APPENDIX C. CONVERGENCE OF THE ALGORITHM

Though expensive the algorithm described has the interesting property of converging towards a generalized equilibrium matching. This section will prove the convergence of the algorithm and as a consequence the existence of generalized equilibrium matchings for a wide class of models.

In order to formally prove the convergence of the algorithm we need to extend some definitions. If at some point it is no longer possible for x to match with y then the waiting time for agent to try and match will be infinite. In order to manage this degenerate case we need to extend the domain of G to  $\mathbb{R}^{X\times Y}$  by setting  $G(\alpha) = \max_{\mu} \mu^{\top} \alpha - G^*(\mu) > -\infty$  for  $\alpha \in \mathbb{R}^{X\times Y}$ . Using the convention  $0 \times \infty = 0$ , this is the same as optimizing a continuous functions over a compact. Since  $0 \in \text{dom} G^*$  the right inequality is true. We do the same for H.

We extend naturally the subdifferential of G to  $\underline{\mathbb{R}}^{X \times Y}$ . Similarly we extend the subdifferential of  $G^*$  to have values in  $\underline{\mathbb{R}}^{X \times Y}$ . Note that

$$M^{G}(\alpha, \bar{\mu}) = \underset{\mu \leq \bar{\mu}}{\arg \max} \mu^{\top} \alpha - G^{*}(\mu) = \partial \left(G^{*} + \iota_{\cdot \leq \bar{\mu}}\right)^{*}(\alpha)$$
$$T^{G}(\alpha, \bar{\mu}) = \underset{\tau \geq 0}{\arg \max} \bar{\mu}^{\top}(\alpha - \tau) - G(\alpha - \tau) = \alpha - \partial \left(G + \iota_{\cdot \leq \alpha}\right)^{*}(\bar{\mu})$$

where second argmax is taken in  $\overline{\mathbb{R}}^{X \times Y}$ . We will first prove that the algorithm is well defined

**Lemma 1.** For all  $\alpha$  and  $\bar{\mu} \geq 0$  we have  $M^G(\alpha, \bar{\mu}) \neq \emptyset$  and  $T^G(\alpha, \bar{\mu}) \neq \emptyset$ .

*Proof.* For  $\alpha$  and  $\bar{\mu} \geq 0$  we have that  $M^G(\alpha, \bar{\mu}) \neq \emptyset$  because we try to maximize a continuous function on a non empty compact set since  $0 \in \text{dom}G^*$  by hypothesis. Moreover  $\overline{\mathbb{R}_+}$  is a compact set. Thus  $T^G(\alpha, \bar{\mu}) \neq \emptyset$ 

In the proof we will use the following result that is essentially an extension of a result that is true in the discrete case

**Lemma 2.** Let X, Y two non empty subsets of  $\mathbb{R}^n$  such that  $X \leq_Q Y$ . Then for any  $y \in Y$  there is  $x \in X$  such that  $x \leq y$ . The statement is also true when first choosing an element in X.

*Proof.* Let  $y \in Y$  and  $x \in X$  since  $X \leq_Q Y$  there exists  $\delta \in [0, (x-y)^-]$  such that  $x - (x-y)^+ + \delta \in X$ . Finally notice that  $x - (x-y)^+ + \delta \leq y$ .

## **Proposition 1.** The algorithm is well defined.

Proof. We proceed by induction.  $\mu^{A,0}$  is well defined and positive. Thus by  $1 \mu^{P,0}$  exists since  $\mu^{T,-1}=0=\min \mathrm{dom} G^*$ . Notice that  $\mu^{P,0}\geq 0$ , thus  $\mu^{T,0}$  is well defined and is also positive due to the assumption on  $\mathrm{dom} H^*$ . Let  $k\geq 0$  such that  $\mu^{A,k},\mu^{P,k},\mu^{T,k}$  are well defined and positive. First  $\mu^{A,k+1}=\mu^{A,k}-(\mu^{P,k}-\mu^{T,k})\geq \mu^{A,k}-\mu^{P,k}\geq 0$ . This ensure that  $M^G(\alpha,\mu^{A,k+1})$  is non empty by 1. Since G is submodular, by  $3, \bar{\mu}-M^G(\alpha,\bar{\mu})$  is increasing in the Q-order. Thus since  $\mu^{A,k+1}\leq \mu^{A,k}$  by 2 there exists  $\mu\in M^G(\alpha,\mu^{A,k+1})$  such that  $\mu^{A,k+1}-\mu\leq \mu^{A,k}-\mu^{P,k}$ . By rearranging the terms we see that  $\mu\geq \mu^{T,k}$  thus  $\mu^{P,k+1}$  exists and we notice that  $\mu^{P,k+1}\in M^G(\alpha,\mu^{A,k+1})$  and by definition of that set  $\mu^{P,k+1}\geq 0$ . Similar to the case for k=0 by 1  $\mu^{T,k+1}$  is well defined and positive. The algorithm is well defined by induction.

Now in order to prove the convergence result we will need the following two lemmas:

**Lemma 3.** For any  $\mu \in M^G(\alpha, \bar{\mu})$ 

$$T^{G}(\alpha, \bar{\mu}) = (\gamma - \partial G^{*}(\mu)) \cap \{ \cdot \geq 0 \} \cap \left\{ \cdot^{\top} (\bar{\mu} - \mu) = 0 \right\}$$

*Proof.*  $\mu$  is a kuhn tucker vector for the minimisation problem associated with  $T^G(\gamma, \bar{\mu})$ . Thus using the result presented in Rockafellar (1970), modified to work for the extended definition of subdifferential, we have the result.

**Lemma 4.**  $(\tau^{P,k})$  is weakly increasing,  $(\tau^{T,k})$  is weakly decreasing.

*Proof.*  $(\mu^{A,k})$  is decreasing thus  $T^G$  is increasing by Topkis (1998) so the first result is clear. Thanks to the result before we notice that  $\inf T^H(\gamma,\mu^{P,k}) = \inf T^H(\gamma,\mu^{T,k})$ . Since  $\mu^{T,k} \leq \mu^{P,k+1}$  we have once again that  $T(\gamma,\mu^{P,k+1}) \leq T(\gamma,\mu^{T,k})$  and thus  $\tau^{T,k+1} \leq \tau^{T,k}$ 

We are now ready to prove the convergence result 4.

*Proof.* By construction  $(\mu^{A,k})$  is decreasing and bounded from below by 0 thus it converges to a  $\mu^A$ . Because  $(\mu^{P,k})$  lies in the compact  $[0,\mu^{A,0}]$  up to an extraction it converges to  $\mu$ .

Since  $\mu^{P,k} - \mu^{T,k} = \mu^{A,k} - \mu^{A,k+1} \to 0$  we have that  $\mu^{T,k} \to \mu$ . The two sequences left  $(\tau^{P,k})$ ,  $(\tau^{T,k})$  are in the compact  $\overline{\mathbb{R}}$  and are monotone so they converge to  $\tau^P$  and  $\tau^T$  respectively. We will first show  $\tau^P \in T^G(\alpha, \mu^A)$  and  $\tau^T \in T^H(\gamma, \mu)$ . Note that  $\tau^i_{xy} = +\infty \implies \mu^A_{xy} = 0$  for i = T, P. Indeed we have

$$\mu^{A,k} \tau^{P,k} - \min_{s} G(s) \le \mu^{A,k} \tau^{P,k} + G(\alpha - \tau^{P,k}) = \min_{\tau > 0} \mu^{A,k} \tau + G(\alpha - \tau) \le G(\alpha)$$

By lsc of G on  $\mathbb{R}$  this implies

$$\liminf_{k} \min_{\tau > 0} \mu^{A,k} \tau + G(\alpha - \tau) = \liminf_{k} \mu^{A,k} \tau^{P,k} + G(\alpha - \tau^{P,k}) \ge \mu^A \tau^P + G(\alpha - \tau^P).$$

Moreover by duality

$$\min_{\tau \ge 0} \mu^{A,k} \tau + G(\alpha - \tau) = \max_{\mu \in [0,\mu^{A,k}]} \mu \alpha - G^*(\mu).$$

Taking the limsup by lsc of  $G^*$  we get

$$\limsup_{k} \min_{\tau \ge 0} \mu^{A,k} \tau + G(\alpha - \tau) \le \min_{\tau \ge 0} \mu^{A} \tau + G(\alpha - \tau)$$

because  $\min_{\tau\geq 0} \mu^A \tau + G(\alpha - \tau) = \max_{\mu\in[0,\mu^A]} \mu\alpha - G^*(\mu)$ . Putting the last two inequalities together we get  $\mu^A \tau^P + G(\alpha - \tau^P) = \min_{\tau\geq 0} \mu^A \tau + G(\alpha - \tau)$ . Which is the definition of  $\tau^P \in T^G(\alpha,\mu^A)$ . Similarly we get the result for  $\tau^T$ . The second convergence holds because the minimisation take place in a compact. Similarly since  $\operatorname{dom}(G^*), \operatorname{dom}(H^*)$  are compact we have  $\mu \in M^G(\alpha,\mu^A)$  and  $\mu \in M^H(\gamma,\mu)$ . Thanks to 3 we have  $\mu \in \partial G(\alpha - \tau^P)$  and  $\mu \in \partial H(\gamma - \tau^T)$ . We will now prove that we can replace  $\tau^T$  by  $\tau^\gamma$  in the last equation. This result is also true for  $\tau^P$  and the proof is similar. Let  $\tilde{\mu} \in \operatorname{dom} H^*$ ,

$$\mu(\gamma - \tau^{\gamma}) - H^{*}(\mu) = \mu(\gamma - \tau^{T}) - H^{*}(\mu) + \mu(\tau^{T} 1_{\mu=m})$$

$$= H(\gamma - \tau^{T}) + \mu(\tau^{T} 1_{\mu=m}) \qquad \text{since } \mu \in \partial H(\gamma - \tau^{T})$$

$$= H(\gamma - \tau^{T}) + m(\tau^{T} 1_{\mu=m})$$

$$\geq \tilde{\mu}(\gamma - \tau^{T}) - H^{*}(\tilde{\mu}) + m(\tau^{T} 1_{\mu=m})$$

$$\geq \tilde{\mu}(\gamma - \tau^{\gamma}) - H^{*}(\tilde{\mu}) \qquad \text{because } \tilde{\mu} \leq m$$

Which proves  $\gamma - \tau^{\gamma} \in \partial H^*(\mu)$ . Thus the first condition of generalized equilibrium matchings is satisfied by  $(\mu, \tau^{\alpha}, \tau^{\gamma})$ . We will now prove that  $\max(\alpha - \tau^{\alpha}, \gamma - \tau^{\gamma}) = 0$  or equivalently  $\min(\tau^{\alpha}, \tau^{\gamma}) = 0$ . Let x, y such that  $\min(\tau^{\alpha}, \tau^{\gamma}) > 0$  this implies  $\mu_{xy} < \min(m_{xy}, n_{xy})$  and  $\min(\tau^{P}_{xy}, \tau^{T}_{xy}) > 0$ . Lemma 4 on monotony ensures that for any k  $\tau^{T,k}_{xy} > 0$ . However

 $au_{xy}^{T,k}$  is the Lagrangian multiplier of the problem its positiveness implies  $\mu_{xy}^{T,k} = \mu_{xy}^{P,k}$  thus  $\mu^{A,k+1} = \mu^{A,k}$ . We deduce that  $\mu_{xy}^{A,0} = \mu_{xy}^{A}$ . Once again by positiveness of the Lagrangian multiplier  $\tau_{xy}^{P}$  we get that  $\mu_{xy} = \mu_{xy}^{A} = \mu_{xy}^{A,0} > \min(m_{xy}, n_{xy})$ . Which gives a contradiction because  $\mu_{xy} < \min(m_{xy}, n_{xy})$ . Finally  $\max(\alpha - \tau^{\alpha}, \gamma - \tau^{\gamma}) = 0$ .