Direct Complementarity

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Abstract

I propose a notion of complementarity in consumer theory, called *direct complementarity*, for which I provide four equivalent definitions including an axiomatization. I point out a novel critique of the leading definition of complementarity, which is based on cross-price effects: such effects are sensitive to changes in the basis used to describe the space of available bundles. Direct complementarity, on the other hand, is defined for preferences over an abstract vector space of bundles, without reference to a particular basis (or list of "goods"), and provides a consistent definition across all pairs of *composite* goods, i.e. linear combinations of standard goods. Direct complementarity better captures the intuitive notion of one good's effect on the value of another, hence the term *direct*; cross-price effects are best understood as an *indirect* consequence of such relationships.

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1 Introduction

The notions of complementarity and substitutability are central to our understanding of preference and choice in multidimensional environments. The simplest, most intuitive definition of complementarity between goods says that an increase in the quantity of Good i increases the marginal value of Good j. When utility is twice differentiable, this is simply the statement that $u_{ij} > 0$. Under the reverse condition, of course, we would be inclined to call the goods substitutes. In the earliest days of consumer theory, a positive cross-partial derivative was indeed used to define complementarity of goods. Hicks and Allen, in their foundational work of 1934, referred to it as "the established definition...due...to Edgeworth and Pareto." However, as they noted (and as had been observed by Slutsky as early as 1915), the Edgeworth-Pareto definition has a serious flaw: Consumer preferences are customarily taken to be *ordinal*, with many interchangeable cardinal representations, and the sign of second derivatives is not preserved by monotone transformations.

As customary, assume that the consumer's preferences on $\mathbb{R}^n$ (the space of bundles of n goods) are represented by a twice-differentiable function $u : \mathbb{R}^n \to \mathbb{R}$. Denote partial derivatives by u_i, u_{ij}, etc. Assume, as usual, that $u_i(x) > 0$ for all x and i, i.e. all coordinates represent goods. If $f : \mathbb{R} \to \mathbb{R}$ is any strictly increasing smooth function, then $v = f \circ u$ is another equally valid representation. Then, as did Slutsky and Hicks-Allen, we can apply the chain rule twice, and obtain for any i, j:

$$v_{ij} = f' u_{ij} + f'' u_i u_j \tag{1}$$

so that clearly if f'' is large enough and positive, all goods will appear to be complements under representation v, while the opposite will be true if f'' is extremely negative.[1]

[1]Chambers & Echenique (2009) extended this observation to show that, under modest

Accordingly, it has been customary in consumer theory since 1934 to define complements and substitutes by looking at demand functions rather than at utility. If the compensated cross-price effect is negative – that is, if a price increase in good i, along with a change in income which keeps utility constant, decreases the demand for good j – we say that goods i and j are *Hicksian complements*.[2] This definition has come to predominate over one using the uncompensated (gross) cross-price effects, largely because it is symmetric.[3]

Samuelson (1974) gives an excellent survey of the history of complementarity. He complains at length that the Hicksian definition does not match natural intuitions about complementarity. After acknowledging the ambiguity in cross-partials of utility created by (1), he nonetheless argues eloquently that complementarity between, say, sugar and tea "ought" to mean something like the Edgeworth-Pareto concept, i.e. the marginal utility of sugar being increasing in the quantity of tea. The direction of compensated cross-price effects, he points out, is hard to explain to a "plain man" as defining complementarity; surely many of us have felt the same when explaining it to students. I call this "Samuelson's complaint." He proposes defining complementarity by fixing a distinguished utility representation, "money-metric" utility, and then using the straightforward criterion based on the cross-partials of this utility. He calls this "money-metric complementarity." This proposal, though, has gained little attention.[4]

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assumptions, not only local supermodularity but even global supermodularity of utility has very weak consequences: Assuming monotonicity, no finite set of consumer data can falsify supermodularity.

[2]Samuelson (1974) calls these "SHAS complements," to jointly honor Slutsky, Hicks, Allen and Schultz, but the term does not seem to have caught on. Standard graduate textbooks such as Mas-Colell, Whinston & Green (1995) and Kreps (2013) use the unmodified noun "complements" for this relation. I believe the term "Hicksian complements" is clear, as it is defined by second derivatives of the Hicksian demand.

[3]The symmetry was first shown in (Slutsky 1915); the relation was first proposed as a definition of complementarity in Hicks & Allen (1934).

[4]Samuelson (1974) has over 600 citations, but based on a substantial sample, these all, or almost all, reference the survey component of the paper, or the definition of money-

In this paper, I hope to make two important contributions to the understanding of complementarity in consumer theory. One concerns the popular Hicksian definition. To complement Samuelson's intuitive complaints about this definition, I make the following, more formal complaint: the definition is not invariant to changes in coordinates on the space of bundles, and is not, properly speaking, a well-defined relation on *goods*, but rather a relation on *price changes*. My meaning here will be elucidated shortly, with the example and discussion in Section 2.

With these critiques in mind, the next contribution is to give a collection of equivalent definitions for a concept I call *direct complementarity*. One way to define direct complementarity is, given a reference point, to distinguish a representation tailored to that reference point, with a property called *local quasilinearity*, and look at its cross-partials. In the case of quasi-linear preferences, this representation is the standard one, and the definition is equivalent to Samuelson's.[5] Perhaps more compellingly, direct complementarity can be uniquely characterized by a few simple principles, or axioms. One meta-principle is that complementarity should be defined between any two vectors in bundle-space, with the usual complementarity between two goods corresponding to the case of the two corresponding basis vectors. Once we expand our horizons in this way, it is natural to require that complementarity should not depend on the coordinate system chosen for bundle-space; as mentioned, this criterion is failed by Hicksian complementarity, but it is satisfied by direct complementarity. This "basis-free" principle is enforced implicitly, by simply giving definitions which refer to an abstract vector space of bundles without a distinguished basis. The explicit axioms which pin down the definition can be described informally as follows:

1. **Symmetry**: Complementarity (and substitutability) are symmetric relations.

metric utility, but not the definition of complementarity.

[5]There is further discussion of the relationship to Samuelson's definition, including a key distinction, in Section 4.1.

2. **Linearity**: If v is a complement of both w and z, it also complements all non-negative linear combinations of w and z.

3. **Unanimity**: If, for vectors v and w, the cross-partial of utility in directions v, w has the same sign for *all* smooth utility representations, we should classify v and w as complements or substitutes accordingly.

4. **Second-Order Neutrality**: If moving in the v^* direction does not alter any marginal rates of substitution, then v^* is neutral (neither complement nor substitute) to all vectors.

The usefulness of Axiom 3 is revealed by inspecting equation (1). As mentioned earlier, *if* i and j are both *goods*, with positive marginal utility, so that $u_i u_j > 0$, (1) tells us that their cross-partial will vary arbitrarily as we change representations. However, if at least one of them is a *first-order neutral*, with zero marginal utility, their cross-partial is consistent in sign across representations. Slutsky, Hicks-Allen, and Samuelson all discuss equation (1) and observe that the sign of u_{ij} is unambiguous when $u_i u_j = 0$, but quickly dismiss the importance of this observation. For instance, in a footnote in Hicks & Allen (1934), "This possibility does not concern us in our consideration of the individual under market conditions." This dismissal, which is natural if considering only complementarity of coordinate vectors, misses an opportunity: When we expand complementarity to a relation on any pair of vectors, the fact that the Unanimity axiom applies to any pair including a first-order neutral is a powerful observation. The neutrals have codimension 1, so given any non-neutral v^*, any vector is a linear combination of v^* and a neutral. This may give the reader a glimmer of why, under the proper regularity conditions, the axioms pin down a unique definition of complementarity. Indeed, another equivalent definition, Definition B, proceeds by projecting an arbitrary pair of vectors onto the neutral plane and then evaluating complementarity via Unanimity.

The effects of price changes are a cornerstone of demand theory, of course. However, it is quite unlikely that these effects would have come to define the term "complementarity," supplanting the earlier Edgeworth-Pareto notion, if not for the ambiguity created by (1). The thesis here is that "direct complementarity" is a suitable resolution to this ambiguity, and is the natural concept for understanding interactions between goods in their effect on preference. Hicksian complementarity is best understood as an indirect consequence of these interactions.

2 A key example

The following example will illustrate the basis-dependence of Hicksian complementarity (my "complaint") and the contrasting properties of direct complementarity.[6]

Alice is forming a portfolio from one safe asset (asset 0) and three risky assets (1, 2 and 3.) There are no short-sale constraints. Her preferences are given by the quasi-linear function

$$u(x_0, x_1, x_2, x_3) = x_0 + x_1 + x_2 + x_3 - \frac{x_1^2}{2} - \frac{x_2^2}{2} - \frac{x_3^2}{2}$$

For motivation, note that preferences of this form will arise if the risky assets are identically, independently and normally distributed, and Alice has CARA preferences. Then, the certainty equivalent of her portfolio will be of the same form as u, a quadratic with no cross-terms. Of course, in the reduced form above, this utility could also apply to consumer goods.

For convenience, we fix $p_0 = 1$, i.e. there is a risk-free interest rate of 0. Then, at prices $(1, p_1, p_2, p_3)$, demand for each risky asset is given by

[6]Some early readers have objected that I use a financial example here, while consumer theory is not ordinarily applied to such examples. Other readers like the choice, for as we will see the financial context is natural in other ways. I am open to feedback on changing the story in the example. In talks I have often used the story that goods 1, 2 and 3 are items at a fast-food restaurant, and the "mutual fund" becomes a "meal deal."

$x_i^* = 1 - p_i$, with remaining wealth invested in the safe asset. The three risky assets hence have no cross-price effects. Notice also that because utility is quasi-linear in Asset 0, income effects apply only to Asset 0, so that we do not need to know Alice's wealth, or worry about compensated vs. uncompensated cross-price effects, when evaluating Hicksian complementarity of Assets 1-3.

Now suppose Asset 3 is replaced by a mutual fund, M, containing one-third of a share of each of assets 1, 2, and 3, so Alice will be forming her portfolio from assets 0, 1, 2, and M. Prices $q_0 = 1, q_1, q_2, q_M$ will be such that Alice faces the very same optimization problem; the change of coordinates is given by

$$q_1 = p_1, q_2 = p_2, q_M = \frac{p_1 + p_2 + p_3}{3}$$

The quantities z_i Alice must purchase in order to form an equivalent portfolio are given by the linear transformation

$$z_0 = x_0, z_1 = x_1 - x_3, z_2 = x_2 - x_3, z_M = 3x_3$$

Rewriting Alice's demand in the new coordinate system, i.e. using z and q, gives

$$z_1 = -2q_1 - q_2 + 3q_M, z_2 = -q_1 - 2q_2 + 3q_M, z_M = 3q_1 + 3q_2 - 9q_M + 3$$

So, in the new coordinate system, Assets 1 and 2 have a negative cross-price effect and are Hicksian complements, though they are defined identically as before, the full set of available bundles is unchanged, and the price of any bundle is unchanged. Why does this happen? In the new coordinate system, the meaning of two key concepts has subtly changed. An increase in the price of Asset 1 or 2 has changed meaning, and so has an increase in demand for Asset 1 or 2. When we talk about a price increase, it is always implicit that the price of other goods is held fixed. The change in the definition of Asset 3, to Asset M, changes the meaning of a price increase in Asset 1, because a different *ceteris* is *paribus*. When we increase the price of Asset 1 with Assets

2 and M fixed, the meaning in the original coordinates is that p_1 increases, p_2 is fixed, and p_3 decreases. Still in terms of the original coordinates, this causes a decrease in x_1 and increase in x_3, with x_2 held fixed. So why do we see a cross-price effect on Asset 2? Because though Asset 2 is the same asset, $z_2 \neq x_2$, and holding x_2 fixed is not the same as holding z_2 fixed. To increase x_3 with x_2 fixed, in the z-coordinates we must increase z_M while decreasing z_2.

To help gain further insight, let's rewrite Alice's utility in terms of the z_i:

$$v(z_0, z_1, z_2, z_3) = z_0 + z_1 + z_2 + z_M - \frac{z_1^2}{2} - \frac{z_2^2}{2} - \frac{z_M^2}{6} - \frac{z_1 z_M}{3} - \frac{z_2 z_M}{3}$$

In this formulation, it is intuitive that Asset 1 and Asset 2 are each substitutes for Asset M, since the corresponding cross-partials v_{1M}, v_{2M} are negative – and of course from a risk-management point of view, this makes perfect sense, because Asset M is positively correlated with each of Assets 1 and 2. The cross-price effect, in the $z - q$ coordinate system, between Assets 1 and 2 should be thought of as "indirect." An increase in the price q_1 of Asset 1 causes an increase in demand for its substitute, Asset M, which is in turn a substitute for Asset 2, so there is a decrease in demand for Asset 2. On the other hand, $v_{12} = 0$, just as $u_{12} = 0$, corresponding to the independent returns of Assets 1 and 2, so under a definition based on this cross-partial, Assets 1 and 2 are neither complements nor substitutes, in contrast with their status as Hicksian complements. In fact, the cross-partial $u_{12} = 0$ is *preserved* by any redefinition of assets other than 1 and 2. Their stochastic independence, and the ensuing property that $u_{12} = 0$ for this natural choice of utility function, is a fundamental property of just these two assets, while cross-price effects can be an artifact of the way other assets are defined.

To further explicate the basis-sensitivity of Hicksian complements, recall the textbook characterization of compensated cross-price effects as cross-partials of the expenditure function:

$$\frac{\partial h_i(p,u)}{\partial p_j} = \frac{\partial^2 E(p,u)}{\partial p_i \partial p_j}$$

where h_i is Hicksian demand and E is the expenditure function (the minimum cost of achieving utility u at prices p.) In this formulation, it is clear that Hicksian complementarity is a binary relation on *price* vectors, *not* on goods – more specifically, on the infinitesimal changes in price represented by ∂p_i and ∂p_j. Crucially, price vectors p do *not* lie in the space of bundles of goods; rather, the set of price vectors is the *dual* space to the set of bundles, i.e. the set of linear functions from bundles to $\mathbb{R}$. When, as customary, we speak of the relation of Hicksian complementarity as if it is a relation on goods, we are implicitly making use of an isomorphism between bundle-space and its dual, price-space. As should be familiar from linear algebra, this isomorphism is non-canonical, i.e. basis-dependent. The change ∂p_i in price-space is defined by increasing the price of i with the price of other goods *in the basis* held fixed. In the example, when Asset 3 is replaced by Asset M, changing *one* basis element in bundle-space, *all* basis vectors in price-space change meanings: As discussed earlier, ∂q_1 represents an increase in the price of Asset 1 with the prices of Assets 2 and M held fixed, while ∂p_1 of course rather assumed the prices of Assets 2 and 3 were held fixed. On the other hand, for any fixed utility function u, the value of u_{12} depends only on the definitions of Assets 1 and 2.

To sum up: Hicksian complementarity is sensitive to the entire basis used to span bundle space, and not just the two goods being considered and the larger space of available bundles. This reinforces Samuelson's complaint that Hicksian complementarity doesn't match our intuitions about how two goods enhance each other's consumption; in fact, if we allow the coordinate system to vary, it is not really a well-defined relation on the two goods at all! The vital point here is that, fundamentally, Hicksian complementarity is a symmetric relation between *price changes*, which lie in the dual space to bundles. Our standard textbook language, by treating Hicksian complementarity as a relation between *goods*, conflates a vector space with its dual.

As an aside, it is a common observation, found for instance in Kreps (2013), that we have no good intuition for why Hicksian cross-price effects are symmetric. We have only a proof based on symmetry of cross-partials of the expenditure function. If we accept that the Hicksian effects are *fundamentally* a relationship between price changes, the paradox vanishes. We have exactly as much intuition for this as for cross-partials of any function being symmetric. Our intuition is clouded when we try to think about the symmetry in bundle space, and I believe this is because we are implicitly relying on a non-canonical isomorphism between bundle space and price space in order to even define the cross-price effects.

In this paper, an important desideratum for complementarity is that it should depend only on the two goods under consideration and on preferences over bundle space, not on the coordinate system for bundle space – we call this "basis-free". Now, once a utility function is fixed, its cross-partials automatically satisfy the condition of being basis-free, but as noted we then must reckon with the problem that the choice of utility function matters. Ultimately, one of our definitions for the general case will give, for any preference, a distinguished utility function, chosen locally around each point. Our first definition will focus on the case of quasi-linear preferences, where, as in this example, a particular representation naturally stands out.

2.1 Direct Complementarity: The Quasi-Linear Case

In the special case of quasi-linear preferences, our new concept can be defined very simply, in terms of the standard utility representation:

Definition 1. *Let preferences be quasi-linear, represented by* $u(x) = x_0 + f(x_1, \dots, x_k)$*. Then, at a bundle* x*, we say that two goods* i, j *are* direct complements *if* $u_{ij}(x) > 0$ *and* direct substitutes *if* $u_{ij}(x) < 0$.

Note that any other representation of the same preferences which takes *this functional form* would differ from u only by a constant, so there is no ambiguity in the definition.

In the financial example, we obtain the natural result that any pair of assets such as Asset 1 and Asset 2, which are stochastically independent, are neither complements nor substitutes. Assets which are positively correlated are direct substitutes, and if negatively correlated they are direct complements. This property holds regardless of what other assets are available or how they are defined.

In Section 4, we will define direct complements and substitutes for general preferences. Our Definition A is a natural extension of the quasi-linear case. We will show that at a given point, *any* smooth utility function can be made "locally quasi-linear" (up to a second-order approximation) under an appropriate change of *both* coordinates and utility representation. We extend the definition accordingly. Other equivalent definitions, which don't refer to the quasi-linear case or to any representation, will make the case that the concept of direct complementarity is inevitable.

3 Preliminary Definitions and Results

3.1 Ordinal C^∞ functions and their derivatives

One often gains clarity by working in greater generality and abstraction. In this paper, rather than think about complementarity only for pairs of goods from $\{1, 2, ..., n\}$, we will define it simultaneously for all pairs of vectors (v, w) in bundle-space, with the special case of basis vectors (e_i, e_j) corresponding to the usual complementarity between goods i and j. Though not strictly necessary, we actually gain greater elegance by extending the concept of complementarity to any (single) element of the tensor product $V \otimes V$, with complementarity of v and w corresponding to the complementarity of $v \otimes w$. It will then turn out that the set of complementary tensors is a linear half-space in $V \otimes V$. Please note, though, that all of the main results, in Section

4, are stated entirely without reference to tensors.[7] Furthermore, instead of working with one representation for a preference, it is natural to work with the set of all smooth functions representing the same preference, as we will below.

Let V be a real vector space of finite dimension n; we write V and not $\mathbb{R}^n$ so as not to designate a favored basis. As with existing definitions, we will always treat complementarity as a local property around a given point $x \in V$. The point x will be fixed throughout this section, and we will sometimes omit x from notation when it is convenient and causes no confusion. We use the standard notation V^* for the dual space, i.e. the set of linear functions from V to $\mathbb{R}$. When $u : V \to \mathbb{R}$ is differentiable at x, we write $Du(x) \in V^*$ for the "Fréchet derivative" of u at x, meaning that $Du(x)(v)$ is the first-order approximation around x of $u(x+v) - u(x)$. (If we fixed a basis of V and wrote $Du(x)$ in terms of the dual basis of V^*, it would be the usual gradient of u at x.) We will need preferences to be well-behaved near x:[8]

Assumption 1 (Smooth Representation)**.** *Preferences can be represented by a function u which is C^∞ on a neighborhood around the point $x \in V$ of interest, and satisfies $Du(x) \neq 0$.*

All representations u referenced hereafter will be assumed to satisfy Assumption 1. The two halves of the assumption could be called smoothness and non-degeneracy. Sometimes, for emphasis, I call such a u a *smooth* representation; this refers to both parts of the assumption.

Write $u \sim \hat{u}$ if $\hat{u} = f \circ u$ for a C^∞ function $f : \mathbb{R} \to \mathbb{R}$ with $f' > 0$ everywhere. It is routine to verify that $\sim$ is an equivalence relation.

Definition 2. *For any C^∞ utility function $u : V \to \mathbb{R}$, the equivalence class $[u] = \{\hat{u} : \hat{u} \sim u\}$ is an* ordinal C^∞ function*.*

[7] I have included a quick primer on the relevant definitions and properties concerning $V \otimes V$ in Appendix E. The paper, though, is intended to be readable even if skimming over the references to the tensor product.

[8] It would suffice for all of our purposes for the representation to be merely C^2, but I stick with the common assumption of C^∞.

The derivative of $[u]$, $D[u]$, is defined as the set of derivatives $\{D\hat{u} : \hat{u} \in [u]\}$. The chain rule says that $D\hat{u}(x) = f'(u(x))Du(x)$, so

$$D[u](x) = \{\alpha Du(x) : \alpha \in (0, \infty)\}$$

i.e. the derivative is defined up to positive scalar. Then $D[u](x)$ corresponds, canonically, to a half-space in V, namely the directions corresponding to increased utility, or "goods," $G(x) := \{v \in V : Du(x)(v) > 0\}$, bounded by the "indifference plane," $I(x) := \mathrm{Ker}(Du(x))$. By standard linear algebra, all functionals which are positive on the half-space G differ only by a positive scalar, hence are in $D[u](x)$. So G carries all the first-order information available from the ordinal function $[u]$. Note that, given a choice of coordinates, the hyperplane I tells us all marginal rates of substitution (MRS) between any two goods i and j; the MRS is the λ such that $e_i - \lambda e_j \in I$.

At each $x \in V$, the total second derivative $D^2u(x)$ can be viewed as a (symmetric) bilinear form $D^2u(x) : V \times V \to \mathbb{R}$, where in a coordinate system, $D^2u(x)(e_i, e_j)$ would be the usual cross-partial in coordinates (i, j). An equivalent formulation is to write $D^2u(x) \in (V \otimes V)^*$, i.e. a linear functional on 2-tensors over V. Again, let $\hat{u} = f \circ u$, then

$$D^2\hat{u}(x)(v, w) = f'(u(x))D^2u(x)(v, w) + f''(u(x))Du(x)(v)Du(x)(w) \quad (2)$$

$$D^2\hat{u}(x) = f'(u(x))D^2u(x) + f''(u(x))[Du(x) \otimes Du(x)]$$

where the second line is equivalent to the first by definition of the form $[Du(x) \otimes Du(x)]$. This is simply the abstract form of (1) from the introduction. Then, omitting the x, we can write

$$D^2[u] = \{\alpha D^2u + \beta(Du \otimes Du) : \alpha \in (0, \infty), \beta \in \mathbb{R}\} \quad (3)$$

The second term in (3), parametrized by β which can have either sign, is what challenges our ability to define complementarity, by making the sign of

$D^2u(v,w)$ vary across the equivalence class $[u]$. This motivates us to define the "first-order-indifferent tensors," on which this term vanishes:

$$I_2 := \mathrm{Ker}(Du \otimes Du) \subseteq (V \otimes V)$$

Note that I_2 is a space of codimension one (i.e, dimension $n^2 - 1$) in $V \otimes V$. An alternate characterization is:

$$I_2 = \mathrm{Span}(I \otimes V \cup V \otimes I)$$

where the inclusion $\supseteq$ is immediate, and the reverse holds by counting dimensions. That is, I_2 is the span of tensors $v \otimes w$ where either v or w is a first-order neutral. This extends slightly the earlier observation that the sign of $D^2u(v,w)$ is invariant when v or w is first-order neutral. When restricted to I_2, $D^2[u]$ is well-defined up to a positive scalar. Conversely, any form which, when restricted to I_2, is a positive scalar multiple of D^2u, is an element of $D^2[u]$, i.e.

$$D^2[u] = \{\nu \in (V \otimes V)^* : \exists \alpha \in \mathbb{R}^+ : \forall t \in I_2 : \nu(t) = \alpha D^2u(t)\}$$

The reverse inclusion holds because of the standard fact that $Du(x) \otimes Du(x)$ spans the forms which annihilate its kernel, I_2.

So $D^2[u]$ determines, and is determined by, a half-space in I_2 which we call the "complementary tensors," $C_I(x) := \{t \in I_2 : D^2u(x)(t) > 0\}$, bounded by the "neutral tensors," $N_I(x) := I_2 \cap \mathrm{Ker}(D^2u)$, and we call the other half of I_2 the "substitutive tensors" $S_I = -C_I$. Note that $D^2[u]$ provides no information in its behavior outside of I_2 – such behavior is constrained only by linearity and is otherwise completely representation-dependent. In particular, if we are working with a basis where every coordinate vector is a "good," i.e. $e_i \in G$, then, by (2), for large positive f'' we have $D^2\hat{u}(e_i, e_j) > 0$ for all i, j, and for very negative f'' the reverse inequality. That is, as asserted in the introduction, $[u]$ contains both supermodular and submodular functions. The intuition here is that we can "see" second-order effects only when first-order effects are zero. The lack of cardinal utility confounds our ability to distinguish the second-order phenomenon of complementarity, except when first-order effects vanish.

To summarize: Around a point x, the first-order information about an ordinal function $[u]$ can be characterized by the half-space of goods, G, bounded by the indifference plane I. The first derivative also determines the first-order-indifferent tensors I_2. Then, the second-order effects partition I_2 into C_I, N_I and S_I. To make the interpretation of this partition more clear, let $v_1 \in I$, $v_2 \in G$, so $v_1 \otimes v_2 \in I_2$. Then, what does it mean to have $v_1 \otimes v_2 \in C_I$? It means that a small movement in the v_2 direction changes v_1 from a neutral to a good, i.e. $Du(x + \varepsilon v_2)(v_1) > 0$ for small enough positive ε. To be even more concrete, let e_1, e_2, e_3 be basis vectors which are all in G, and let

$$v_{12} := \frac{e_1}{Du(x)(e_1)} - \frac{e_2}{Du(x)(e_2)} \in I \tag{4}$$

be the vector, unique up to a scalar, which is in both the indifference plane and the plane spanned by e_1 and e_2. Then $v_{12} \otimes e_3 \in C_I$ means that motion in the e_3 direction makes v_{12} a good – which is the same as saying that it increases the relative value of Good 1 to Good 2, i.e. the marginal rate of substitution between the goods. We interpret this to mean that Good 3 complements Good 1 *better* than it complements Good 2. This *relative* complementarity – equivalent to complementarity between Good 3 and the "neutral" v_{12} – is robust to changes in representation.

3.1.1 Convex preferences and a geometric interpretation of complementarity

Hicks and Allen (1934) introduced the principle of "increasing marginal rate of substitution", now a standard assumption, which says that the marginal rate of substitution of Good 1 for Good 2 increases (has positive derivative, more precisely) as we substitute Good 1 for 2 along an indifference curve. This condition is slightly stronger than strict convexity of preferences, and is introduced so that demand is well-defined and differentiable. The condition has an easy adaptation to our formalism. Given our discussion in the last paragraph above, the generalized goods v_1 and v_2 satisfy increasing marginal rate of substitution exactly when $D^2u(v_{12}, v_{12}) < 0$, with $v_{12} \in I$ defined as in (4). This means that *all* pairs of goods satisfy this principle exactly when the restriction of D^2u to I is negative definite. Call this condition *strong convexity*.[9]

Under strong convexity, then, the restriction of $-D^2u$ to I is positive definite, i.e. an inner product on I, and it is unique up to scalar as we let u vary over $[u]$. This inner product leads to a nice geometric representation of complementarity on I. A pair $(v_1, v_2) \in I$ are complements in our sense if $-D^2u(v_1, v_2) < 0$, i.e. if the angle between them according to the inner product $-D^2u$ is greater than $\pi/2$, neutrals if they are orthogonal, and substitutes if the angle is less than $\pi/2$. That is, in an appropriate coordinate system, namely any orthonormal basis for the inner product $-D^2u$ on I, complementarity or substitutability depends entirely on the geometric relationship between two vectors: Nearby vectors (those forming an acute angle) are substitutes, and those forming an obtuse angle are complements. That is, being substitutes corresponds to a geometric notion of similarity. Later, when we extend the definition of complementarity beyond I to all of V, one equivalent definition will be to project the two given goods onto I and then examine their complementarity in this way.

Note that strong convexity is equivalent to every element of I being a substitute for itself, which will extend later to show that under strong convexity,

[9] Strictly convex preferences fail to be strongly convex when an indifference curve has zero curvature, i.e. if the slope of the curve is strictly increasing but with zero derivative at some point. This case is commonly excluded, as it was by Hicks-Allen, since it leads to infinite elasticity of substitution, i.e. non-differentiable demand.

any *good* is a substitute for itself, as is natural. A slightly embarrassing point for Hicksian complementarity is that, if one applies the definition naively, any good is a complement for itself.

3.2 The "Money" Good

As in the last section, we will be focusing on complementarity around a fixed point $x \in V$. To recap, we found that when at least one of w, z is a neutral, $D^2u(w, z)$ has the same sign for all smooth representations u, and we can use this sign to define complementarity. The remaining question is how best to extend this definition to any pair of goods in V. The key will be to designate one vector, not in the indifference plane, which is neutral (not a complement or substitute) with respect to all other goods. In the quasi-linear case of Section 2.1, this was Good 0 (money). What can fill in for money in the general case? Such a vector must be neutral with respect to all goods in the indifference plane. Fortunately, such a vector always exists; generically there will be only one choice with this property, up to scalar. In the standard terminology of demand theory, this vector, which we call v^*, is simply the direction of the income effect in bundle space, i.e. the direction with no first-order effect on any marginal rate of substitution, as shown below in Proposition 1.[10] Note that as usual, the marginal rate of substitution between two goods $w, z \in V$ is defined by

$$MRS_{w,z} = \frac{Du(w)}{Du(z)}$$

for any smooth representation u.

Proposition 1. *The following are equivalent properties of a vector $v^* \in V$:*

1. *For each $w \in I$, $v^* \otimes w \in N_I$, i.e. for every (equivalently, any) smooth representation u, $D^2u(v^*, w) = 0$.*

[10]Like much else in our analysis, v^* depends on x and could be called v^*_x, but I believe it causes no confusion to drop the subscript when x is fixed.

2. *v^* satisfies $D(MRS_{w,z})(v^*) = 0$ for all w, z with $z \notin I$.*

The proof is in Appendix A.

Let M (for **m**oney, or nu**m**eraire) be the set of vectors satisfying the conditions in Proposition 1. There is a dimension-counting argument that M is non-trivial: For any smooth u, D^2u induces a map from V to I^*, defined by $v \mapsto (w \mapsto D^2u(v, w))$. The kernel of this map is precisely M, and since V has dimension n while I^* has dimension $n-1$, M is non-trivial. It turns out to be annoying if M has non-trivial intersection with I, so we introduce the following regularity assumption, which holds generically in the space of possible first and second derivatives of u:

Assumption 2. *The following equivalent conditions hold:*

1. $M \cap I = \{0\}$

2. $\dim M = 1$ *and* $M \cup I$ *spans* V.

The equivalence is by simple dimension-counting, along with the observation above that M is non-trivial. We will retain this assumption in all subsequent results. We note that in the standard case of demand theory, when preferences are strongly convex and there is a unique solution to the first-order conditions for utility maximization, which is differentiable in price, Assumption 2 must hold.

It is convenient, of course, for a numeraire to have positive marginal utility as well as the second-order properties defining M. Accordingly, we define

Definition 3. *An element v^* is a* numeraire *if $v^* \in G \cap M$.*

Assumption 2 is equivalent to the statement that a numeraire exists and is unique up to positive scalar. The numeraire v^* plays a key role in all of our coming definitions.

4 Direct Complementarity

In this section, we define what it means for two vectors $v, w \in V$ to be *direct complements*. To strengthen the intuition and motivation for this definition, we provide four equivalent definitions. The preliminary results in the previous section provide a foundation for understanding these definitions and why they are equivalent.

4.1 Definition A: Via a locally quasilinear representation

The defining property of quasi-linear *preferences* is that there is a good ("money") with no impact on any marginal rates of substitution (MRS) between any two goods. In the standard *representation* of quasi-linear preferences, this good appears additively and hence has zero cross-partial with all other goods. In Section 3.2 we showed that at a given point, there is a "numeraire" with no marginal impact on any MRS. Of course, if preferences are not truly quasilinear, this numeraire will vary as we move our reference point, but from this "local quasilinearity" of preferences, we can construct a "locally quasilinear" representation, as defined here:

Definition 4. *A representation u^q is* locally quasilinear *at x if there is a $v^* \in V - \{0\}$ such that $D^2u^q(v^*, v) = 0$ for all $v \in V$.*

For clarity, please note that locally quasilinear does not mean that there is a neighborhood around x where u^q is exactly quasilinear, but only that its first and second derivatives at x are the same as those of a quasilinear utility function, i.e. it is quasilinear up to second-order. Such a representation always exists:

Proposition 2. *For any preferences with a smooth representation u in a neighborhood around x, there is a locally quasilinear representation u^q at x.*

Proof. Once we identify v^* as described in Proposition 1, construct u^q by letting $u^q = f \circ u$ for any f satisfying:

$$\frac{f''(u(x))}{f'(u(x))} = -\frac{D^2u(x)(v^*, v^*)}{(Du(x)(v^*))^2}$$

To prove the defining property of u^q, first note that it holds for $v = v^*$ by application of (2). Also, for $v \in I$, it holds by definition of v^*. Finally, v^* and I together span V (by the last clause in the definition of v^*), and the property extends linearly. ■

Then, by analogy with the quasilinear case discussed earlier, we say:

Definition A. Let u^q be a locally quasilinear representation at x. Then w, z are *direct complements* at x if

$$D^2u^q(x)(w, z) > 0$$

They are neutrals if this derivative is 0, and substitutes if it is negative.

Note that there is apparent ambiguity in this definition, due to the multiplicity of locally quasilinear representations, but it will be shown to be only apparent. Theorem 2 will show Definition A to be equivalent to other definitions regardless of the choice among such representations, so that the choice does not matter. Note that while we did not use Assumption 2 directly in this section, it is needed for Theorem 2 and for its consequence that Definition A is unambiguous.

If preferences are strongly convex, then one locally quasi-linear representation is "money-metric" utility as described in Samuelson (1974). Specifically, given arbitary representation u, we would use $p_x = Du(x) \in V^*$ as our price vector, so that x is an optimal bundle with prices p_x and budget $p_x \cdot x$. Then we would define $u_m^x(x')$ as the minimum expenditure at prices p which can achieve indifference with x'. Formally, we have

Proposition 3. *Let u be a smooth representation of strongly convex preferences, satisfying Assumption 2. For a given reference point x, fix prices $p = Du(x)$ and define* money-metric utility *by*

$$u_x^m(x') = \min_{x'':u(x'')=u(x')} p \cdot x''$$

Then u_x^m is locally quasi-linear at x.

Note that the definition of u_x^m only has the desired properties if x is optimal for some price vector p. Our four equivalent definitions in this paper apply regardless of whether preferences are convex, or whether the reference bundle x is optimal for any vector of prices. There is no reason convexity and optimization should be required to make sense of complementarity.

The proof of Proposition 3 is deferred to the Appendix, Section D, and is more involved than the other equivalence proofs, but the intuition is very simple: If preferences were exactly quasilinear, money-metric utility would be identical to the quasilinear utility representation. Here, we know that preferences are smooth enough to imply a representation u^q which is locally approximated, up to a second-order, by a quasilinear function. The proposition is true, then, because the third-order "error term" in u^q causes only third-order changes to the money-metric utility, so that u^q and u_x^m are the same up to second order near x.

In light of Proposition 3, direct complementarity might appear equivalent to the "money-metric complementarity" of Samuelson (1974), but there are key differences. Samuelson does not stipulate that in evaluating complementarity at point x, we use the prices $p = Du(x)$ which make x the optimal bundle, as is necessary for Proposition 3 to hold. On the contrary, he notes as a shortcoming of his definition that "it does depend upon P^0, one arbitrary set of recently ruling prices." (p.1273) That is, (1) he proposes to use a single representation for every x, while Proposition 3 requires that we use a representation u_x^m specifically tailored to each x, and (2) since his prices are not derived from preferences, he does not actually map preferences unambiguously to a complementarity relation. These key differences would have prevented him from finding the equivalent characterizations given here, reducing the appeal of the definition. In fact, if one simply began with Samuelson's idea one would be unlikely to consider changing prices, and hence utility function, for each x. Fortunately, I came at things from a different route, actually developing the other definitions before I noticed that Samuelson had written more than a survey paper and proposed a new definition of his own. Once I noticed that our definitions coincided in the special case that preferences are quasilinear and $p = Du(x)$, I was led to the modification of his definition which makes them equivalent. I presented the paper often before I noticed this aspect of his work, and interestingly, his definition was sufficiently absent from collective memory, or the connection sufficiently obscure, that no one at any of my presentations was reminded of it.

4.2 Definition B: Decomposition into Nutrients and Flavor

The next definition is created as follows: Note that Assumption 2 tells us that I, of dimension $n-1$, and $v^* \notin I$ together span V. Therefore, any bundle w can be decomposed, uniquely, in the form

$$w = \lambda_w v^* + w^n \tag{5}$$

where $w^n \in I$ is first-order neutral. Conceptually, I think of the first term in this decomposition as **nutrients**: a nutritious but flavorless gruel, utility-rich at first-order, but second-order-neutral, i.e. having no interactions with any other good. The second term is **flavor**: first-order-neutral, but with second-order impact, i.e. having interactions with other flavors. Given any two vectors (w, z), their direct complementarity is determined by their flavors:

Definition B. w, z are *direct complements* at x if, when w and z are decomposed as in (5), $D^2u(x)(w^n, z^n) > 0$ for *all* smooth representations u, or, equivalently, for *any* smooth representation u. They are neutrals if this derivative is 0, and substitutes if it is negative.

The equivalence between "all" and "any" in the definition follows from our earlier observation that the sign of $D^2u(x)(w, z)$ is invariant to representation when w or z is first-order neutral.

Direct complementarity of (w, z) is thus equivalent to complementarity of the "flavors," (w^n, z^n), a condition which is representation-invariant. We also observe that the definition would remain equivalent if we applied our projection to just one of the goods, replacing (w^n, z^n) by (w^n, z) or (w, z^n); this is immediate from bilinearity of D^2u and the defining property of v^*. Recalling the geometric interpretation of complementarity in the neutral plane from the end of Section 3.1, we can recast complementarity here geometrically as (1) projecting both goods to the neutral plane (2) examining the angle between them with respect to an appropriate coordinate system, with an obtuse angle signifying complements and an acute angle signifying substitutes.

Please note that, of course, there are many linear projections from V to I. If we used an arbitrary such projection in this definition, rather than the particular one which takes v^* to 0, we would get a definition satisfying all of the axioms except Neutrality. We would not have the property, as here, that it suffices to project just one of the goods.

This is a good place to say a few words about the $n = 2$ case. As has often been noted, when $n = 2$, the two goods are always Hicksian substitutes. For direct complements, the situation is somewhat less trivial; for the usual strongly convex preferences, two goods are complements whenever both are normal, or substitutes when one is income-inferior. The easiest route to verifying this is through Definition B. The neutral "plane," in this case, is the one-dimensional tangent to the indifference curve. If both goods are normal, then v^*, the vector of income effects, has two positive coordinates, and we find that the projections of the two goods onto the indifference curve will point in opposite directions and hence be direct complements. If one good is inferior, so that v^* has a positive and a negative coordinate, both projections point in the same direction (towards less of the normal good and more of the inferior), and hence the goods are substitutes.

4.3 Definition C: An Axiomatic Approach

The notion of direct complements and substitutes can also be characterized by a list of natural axioms, which were summarized intuitively in the introduction:

Theorem 1. *Given a smooth preference defined on a neighborhood of a point $x \in V$ and satisfying Assumption 2, there is a unique partition of V^2 into three sets C, S, N (complements, substitutes, neutrals) satisfying the following axioms:*

1. ***Symmetry**: For all $(w, z) \in V^2$, (w, z) and (z, w) are in the same category.*
2. ***Linearity**: If $(w, y) \in C$ and $(w, z) \in C \cup N$, then $(w, \alpha y + \beta z) \in C$ for any $\alpha > 0$,$\beta \geq 0$. The same condition holds if S or N replaces C throughout.*
3. ***Unanimity**: If $D^2u(w, z) > 0$ for* all *smooth representations u, then $(w, z) \in C$. Similarly, if $D^2u(w, z) = 0$ for all u then $(w, z) \in N$, and if $D^2u(w, z) < 0$ for all u then $(w, z) \in S$.*
4. ***Neutrality**: If v^* is a numeraire, then $(v^*, v) \in N$ for all $v \in V$.*

where all derivatives are taken at the point x.

The proof of Theorem 1 is in Appendix B. The idea is that the unanimity axiom pins down the definition when at least one of (v, w) is a first-order neutral, the neutrality axiom pins it down when one equals v^*, and linearity and symmetry do the rest, because, as noted, neutrals and v^* span V. Note that rather than making symmetry a separate axiom, we could have written symmetric versions of linearity and neutrality, and the same result would hold. I felt that the axioms were a little easier to read this way. Note that the adjective "smooth" in Axiom 3 is crucial; without it, the axiom is vacuous, because there is always a representation, violating Assumption 1, with $Du = D^2u = 0$.

With this theorem in hand, we naturally make the following definition:

Definition C. w, z are *direct complements* at x if $(w, z) \in C$ where C is as described in Theorem 1. Similarly, they are substitutes if $(w, z) \in S$ and neutrals if $(w, z) \in N$.

4.4 Definition D: Via Marginal Rates of Substitution

The vector v^*, which would be a consumer's marginal consumption under an increase in income, is a natural numeraire with which to judge the value of other goods. That is, we could consider the value of good w to be MRS_{w,v^*}. Accordingly, when we ask whether good z complements good w, it is natural to ask whether moving in direction z increases the value of good w in this sense. This leads to the following definition:

Definition D. We say w, z are *direct complements* at x if

$$D(MRS_{w,v^*})(z) > 0,$$

neutrals if this derivative is 0, and substitutes if negative.

Notice while on its face, this definition does not treat w and z symmetrically, we will prove that it is equivalent to the other definitions, hence *de facto* symmetric. Note also that this symmetry would *not* hold if we used an arbitrary numeraire in place of v^*.

4.5 Equivalence Theorem

The central result about direct complementarity is:

Theorem 2. *Definitions A, B, C, and D are equivalent.*

The full proof is in Appendix C. Much of the groundwork has already been laid. The basic ideas are as follows: Equivalence between Definitions A and B is almost immediate, because for the representation u^q used in A, $D^2u^q(x)(w, z) = D^2u^q(x)(w^n, z^n)$. To show that B is equivalent to C, we only need to verify that Definition B satisfies the axioms. Equivalence between A and D falls out from the definition of MRS via u^q and the defining property of v^*.

Definitions A and C extend easily to a property of tensors in $V \otimes V$ rather than simply pairs in $V \times V$, and in a sense the tensor space $V \otimes V$ is the most natural place to think about complementarity, but I defer a further discussion of this to Appendix F. There is also one further equivalent definition in that Appendix.

4.6 Further Discussion of the Axioms

Once we embark on defining complementarity as a relation between any pair or vectors in bundle-space, not just pairs of basis goods, the axioms become extremely natural. This generalized program appears to be novel to this paper. As discussed in the introduction, Neutrality becomes a powerful axiom because of this generalization, a point missed by earlier work.

The most natural axiom to relax is Neutrality. In fact, if we fix any smooth utility representation and define complementarity by its second derivatives, the other three axioms are satisfied. Neutrality, then, is equivalent to distinguishing a locally quasilinear representation, or, from the point of view of Definition B, fixing a special projection onto I which takes v^* to 0. Without Neutrality, the only pairs whose category is pinned down are those where at least one vector is first-order neutral, as discussed in Section 3, so that

Unanimity applies. The space of tensors where Unanimity applies is somewhat larger, and leads to a notion of "relative complementarity" which does not require Neutrality or a distinguished representation; see Appendix F. Note that Neutrality can equivalently be weakened to the statement that *some* non-zero v^* is neutral with respect to all goods, because in the presence of the other axioms such a v^* would have to be a numeraire anyway. I chose to state the stronger version because, paradoxically, it strikes me as easier to accept it as axiomatic. This a version of Kahneman-Tversky's "conjunction fallacy": it's easier to accept that *some* element should satisfy Neutrality when the axiom carries a reminder that a numeraire exists and is an appealing choice for this element.

5 Relationship between Direct and Hicksian Complementarity

Here, we'll elucidate the relationship between direct and Hicksian complementarity. This is simplest to discuss in the quasi-linear case; keep in mind that our discussion in Section 4.1 tells us that any case can, locally, be transformed into the quasi-linear by a change of basis (which only requires replacing one good with a suitable numeraire.) We'll fix here a basis $e_0, ..., e_k$ for V (letting $k = n - 1$) and a quasi-linear utility function $u(x) = x_0 + f(x_1, \ldots, x_k)$. Let W be the span of $e_1, ..., e_k$. We can view $D^2 f(x)$ as a bilinear form on W, or equivalently as a function from W to W^*. Represent $D^2 f(x)$ as a $k \times k$ matrix H, using the dual basis for W^*. This is simply the ordinary Hessian matrix. Assume that preferences are strongly convex, so that x solves the demand problem with prices $p = Du(x)$ and income $p \cdot x$. Assume also that in a neighborhood around p, there is a unique solution to the demand problem. Then, a little thought shows that cross-price effects are given by the matrix H^{-1}, so

- Goods i, j are direct complements iff $H_{ij} > 0$.
- Goods i, j are Hicksian complements iff $H^{-1}_{ij} < 0$.

This helps us understand the example from the introduction. The Hessian in the original coordinate system is just $H = -I$ where I is the identity, so that both H and $H^{-1} = -I$ are diagonal, and no goods are complements or substitutes in either sense. To translate to the new coordinates, we define a change-of-coordinate matrix

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

and write

$$\hat{H} = CHC^T = \begin{pmatrix} -1 & 0 & -\frac{1}{3} \\ 0 & -1 & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \end{pmatrix}$$

The change in coordinates affects only the third row and column of H, leaving $\hat{H}_{12} = H_{12} = 0$ unchanged; this would be true for any H, exemplifying the fact that the change in coordinates does not affect direct complementarity of Goods 1 and 2. It *does*, though, affect H^{-1}_{12}, which becomes $\hat{H}^{-1}_{12} = -1$, so that as calculated earlier, Goods 1 and 2 are now Hicksian complements. Please note that by setting $H_{12} = -\varepsilon$ instead of 0, we would easily obtain an example where Goods 1 and 2 change from (Hicksian and direct) substitutes to Hicksian complements (but still direct substitutes) under the change of coordinates.

An informative special case is a quasi-linear model with just three goods including money (Good 0), and strongly convex preferences. In this case, direct complementarity between goods 1 and 2 is indeed equivalent to Hicksian complementarity. This supports the intuition by which the Hicksian complementarity got its name in the first place. Mathematically, the matrix

H is 2-by-2 here, and $\det(H) > 0$ by convexity, so $H_{12}^{-1} = -H_{21}/\det(H) = -H_{12}/\det(H)$ has opposite sign from H_{12} implying the claim. A similar result holds whenever all pairs of goods are direct complements. In this case, direct and indirect effects on demand go in the same direction.

6 Concluding Remarks

The initial motivation for this project was one which has struck many observers: the oddity of the Hicksian definition of complementarity. When I explain this definition, or that of gross complementarity, to students, invariably I fall back on a loose argument that if one good enhances the consumption of the other, it seems likely their demands would have the appropriate relationship. This argument is only valid in a quasi-linear model with only three goods, the two in question and money. Nonetheless, absent a distinguished form of cardinal utility, this definition appeared to be the best we could do – a situation I have tried to remedy.

The example in Section 2 seems to me a powerful companion to existing complaints about the definition. Not only does the effect of p_j on x_i include indirect effects, but such effects can be changed by a "change of basis." i.e. a redefinition of another good. Calling a demand effect by the name "complementarity," evoking only the simplest of its possible causes, is like declaring any two particles accelerating towards each other to be "attracted," when of course they could both be driven by attraction to a third particle. I hope the principles underlying direct complementarity are compelling enough to provide a good alternative.

A natural empirical conjecture about direct complementarity is that it is more likely than Hicksian complementarity to be robust across data sets, since it represents a more fundamental relationship between a pair of goods. As discussed in Section 5, direct complementarity can be recovered from the matrix of demand relationships, so it is realistic to test this conjecture.

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A Proof of Proposition 1

The equivalence of "every" with "any" follows from the earlier results stating that D^2u is unique up to positive scalar on pairs including an element of I.

In condition 2, note that $Du(z) \neq 0$ by assumption. Using the quotient rule and cancelling the denominator, $D(MRS_{w,z})(v^*) = 0$ is equivalent to

$$Du(z)D^2u(v^*, w) - Du(w)D^2u(v^*, z) = 0$$

By bilinearity of D^2u, this is equivalent to:

$$D^2u(v^*, Du(z)w - Du(w)z) = 0 \tag{6}$$

and by linearity of Du, $Du(Du(z)w - Du(w)z) = 0$, i.e. $Du(z)w - Du(w)z \in I$. Therefore, condition 1 implies that equation (6) holds for every w, z and hence implies condition 2.

To prove $2 \Rightarrow 1$, start with any $w \in I$, and apply equation (6) for any $z \notin I$ (such z exists by Assumption 1). The equation reduces to $D^2u(v^*, Du(z)w) = 0$, and since $Du(z) \neq 0$ this gives the desired result.

B Proof of Theorem 1

Existence follows from the fact that the partition into C, S, N given by Definition A satisfies the axioms. In fact, the first three axioms would hold if any representation were used. Symmetry and linearity come from the fact that second derivatives are always symmetric bilinear forms, and unanimity is trivial. The neutrality axiom is clear from the definition of local quasilinearity, along with Assumption 2.

As for uniqueness: Let U_C, U_S, U_N be the sets of pairs which are in the same category (C, S, N respectively) for all partitions satisfying the axioms, and $U = U_C \cup U_S \cup U_N$; we must show that $U = V^2$. From our preliminary results and the unanimity axiom, we know that $(w, z) \in U$ if $w \in I$ or $z \in I$. From the neutrality and symmetry axioms, we know that for all w, $(v^*, w) \in U_N$ and $(w, v^*) \in U_N$, where v^* is the unique element described in Assumption 2. Also, recall that by this assumption, $I \cup \{v^*\}$ spans V. So, given arbitrary (w, z), write

$$z = \lambda_z v^* + z^n$$

for a scalar λ_z and $z^n \in I$. Now, as just mentioned, $(w, v^*) \in U_N$ and

$(w, z_n) \in U$. Hence, the linearity axiom can be applied to (w, z), showing that it is in U, specifically in the same category as (w, z_n). This concludes the proof.

C Proof of Theorem 2

$A \Leftrightarrow B$: Let the representation u^q be as in Section 4.1, and given any $w, z \in V$, let each be decomposed as in Section 4.2. Then

$$\begin{aligned} D^2u^q(x)(w,z) &= D^2u^q(x)(\lambda_w v^* + w^n, \lambda_z v^* + z^n) \\ &= D^2u^q(x)(w^n, z^n) \end{aligned}$$

by bilinearity of D^2u^q and the defining property of u^q, and the result follows. Please note that as promised, this implies that Definition A cannot be sensitive to the choice of u^q among locally quasi-linear representations.

$A \Leftrightarrow C$: In the proof of Theorem 1, we showed that Definition A satisfies the four axioms. Along with the uniqueness portion of that theorem, this suffices to show equivalence here.

$A \Leftrightarrow D$: Let the representation u^q be as in Section 4.1. Applying the definition of MRS and the quotient rule, $D(MRS_{w,v^*})(z)$ has the same sign as

$$Du_q(v^*)D^2u_q(w,z) - Du_q(w)D^2u_q(v^*,z)$$

and, by the defining properties of u^q and v^*, the second term is zero and the first term has the same sign as $D^2u_q(w,z)$, proving the result.

D Proof of Theorem 3

Without loss of generality, let the reference point be $x = 0$. By Proposition 2, we can let u be locally quasilinear with numeraire v^*, and by using an affine transformation on u it is without loss to let $u(0) = 0$ and $Du(0)(v^*) = 1$ (the latter is non-zero by Assumption 2). Establish a coordinate system where the first basis vector is v^* and the other $n-1$ basis vectors span the space of first-order neutrals, i.e. each e_i for $2 \leq i \leq n$ satisfies $Du(0)(e_i) = 0$ – again by Assumption 2, this gives a basis for $\mathbb{R}^n$. With respect to the dual basis, our price vector $p = Du(0)$ is $p = (1, 0, \ldots, 0)$.

Let f be the transformation taking the locally quasilinear utility function u to money-metric utility, i.e. $u^m = f \circ u$. By Equation 2, it will be sufficient for our proposition (also necessary, actually) to show that $f''(0) = 0$; then the local quasi-linearity of u carries over to u^m.

By convexity, the cheapest bundle corresponding to prices p and utility level a (the Hicksian demand) can be calculated by solving the first-order conditions, which say that each good $2, ..., n$ with price zero must have zero marginal utility, so we define a map $\delta : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\delta(x) = (u(x), \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_n})$$

where derivatives and taken at x, and observe that Hicksian demand is given by

$$h(p, \bar{u}) \equiv \operatorname*{argmin}_{x:u(x)=\bar{u}} p \cdot x = \delta^{-1}(\bar{u}, 0, \ldots, 0)$$

Then, the transformation f is given by

$$f(\bar{u}) = p \cdot h(p, \bar{u}) = (\delta^{-1}(\bar{u}, 0, \ldots, 0))_1$$

Let $J(\bar{u})$ be the Jacobian of δ at $h(p, \bar{u})$. Now, $f'(\bar{u})$ is given by the upper-left entry of the Jacobian of δ^{-1} at $(\bar{u}, 0, \ldots, 0)$, which is the inverse matrix of $J(\bar{u})$. Also, from the definition of δ, $J(\bar{u})$ has first column equal to the gradient of u at $h(p, \bar{u})$, with remaining columns equal to those of the Hessian of u at $h(p, \bar{u})$. Let $c(\bar{u})$ be the (1,1) cofactor of $J(\bar{u})$. Then by Cramer's Rule,

$$f'(\bar{u}) = \frac{c(\bar{u})}{\det(J(\bar{u}))} \tag{7}$$

Now, our goal is to show that $f''(0) = 0$. Note that by local quasi-linearity, our choice of basis, and our normalization of u, $J(0)_{11} = 1$ while $J(0)_{i1} = J(0)_{1j} = 0$ for any $i, j \neq i$. This immediately gives $c(0) = \det(J(0))$, so $f'(0) = 1$. Also, applying the definition of determinant, we can rewrite (7) as

$$f'(\bar{u}) = \frac{c(\bar{u})}{c(\bar{u}) + e(\bar{u})} \tag{8}$$

where each term in $e(\bar{u})$ (an "error term") contains factors from both the first row of $J(\bar{u})$ and the first column (away from the corner), which are all zero at $\bar{u} = 0$. The product rule immediately tells us that the derivative of such terms is zero at $\bar{u} = 0$, so $e'(0) = e(0) = 0$, i.e. the error terms in $f'(\bar{u})$ are all second-order. (Note that if u is exactly quasi-linear, these error terms are all exactly zero for all $\bar{u}$, because $J(\bar{u})_{1j} = 0$ for all $\bar{u}$, for $j \neq 1$, so that $f' \equiv 1$ and $u_{mm} \equiv u$.) Then applying the quotient rule to (8) gives

$$f''(\bar{u}) = \frac{c(\bar{u})c'(\bar{u}) + e(\bar{u})c'(\bar{u}) - c(\bar{u})c'(\bar{u}) - c(\bar{u})e'(\bar{u})}{(c(\bar{u}) + e(\bar{u}))^2} = \frac{e(\bar{u})c'(\bar{u}) - c(\bar{u})e'(\bar{u})}{(c(\bar{u}) + e(\bar{u}))^2}$$

whence $f''(0) = 0$ as desired. $\square$

E Key Facts about the Tensor Product

This paper is primarily concerned with categorizing pairs $(v, w) \in V \times V$ as complements, neutrals or substitutes. However, it is natural to extend this categorization to *linear combinations* of such pairs, namely elements of the tensor product $V \otimes V$. We give here the definitions and key facts concerning this space, as can be found, for instance, in Roman (2008). To first summarize verbally: elements of $V \otimes V$ are linear combinations of pairs denoted $v \otimes w$ for any $v, w \in V$. Any bilinear form on V can be extended to such expressions by linearity. Any two expressions for which *every* bilinear form takes the same value are considered equal.

To elaborate: Form the free vector space $F(V \times V)$ on $V \times V$, i.e. a vector space with basis $V \times V$, so that each element is a finite sum with terms of the form $\alpha(v, w)$ for scalar α and $(v, w) \in V$. The space $F(V \times V)$ has uncountable dimension. The tensor product $V \otimes V$ is an n^2-dimensional quotient of $F(V \times V)$. Specifically, we form the subspace K of $F(V \times V)$ consisting of elements which "look like" they ought to equal zero, namely the span of all elements of the forms:

1. $(\alpha v_1 + \beta v_2, w) - \alpha(v_1, w) - \beta(v_2, w)$
2. $(v, \alpha w_1 + \beta w_2) - \alpha(v, w_1) - \beta(v, w_2)$

and let $V \otimes V = F(V \times V)/K$. Then $v \otimes w \in V \otimes V$ denotes the equivalence class containing (v, w). Elements of $V \otimes V$ are called tensors. Key facts about this space:

1. Given a basis $e_1, \ldots, e_n$ for V, the n^2 tensors $e_i \otimes e_j$ form a basis for $V \otimes V$.
2. Let $\mathrm{Bilin}(V, V)$ be the vector space of bilinear mapppings $V \times V \to \mathbb{R}$. Any $f \in \mathrm{Bilin}(V, V)$ can be extended to a linear function $\tilde{f}$ on $V \otimes V$ by letting $\tilde{f}(v \otimes w) = f(v, w)$. The mapping $f \mapsto \tilde{f}$ is a natural isomorphism from $\mathrm{Bilin}(V, V)$ to $(V \otimes V)^*$.
3. Because of the last observation, an equivalent definition of $(V \otimes V)$ is as the dual of $\mathrm{Bilin}(V, V)$.

F Complementarity of Tensors, Relative Complementarity, and Definition E

We noted in Section 3 that for the first-order-indifferent tensors, I_2, the Unanimity axiom applies: for any $t \in I_2$, $D^2u(x)(t)$ will have the same sign for all smooth representations u. By adding versions of the other axioms, we can extend the resulting definition of complementarity uniquely to all of $V \otimes V$. The key to the extension is the Neutrality axiom. The obvious analogue would be to assume that $v^* \otimes w$ is neutral for all w, but it is sufficient to simply assume that the single element $v^* \otimes v^*$ is neutral, with the rest following by linearity. As before, we need Assumption 2 to ensure that $v^* \otimes v^* \notin I_2$. Actually, we could consistently extend the definition by designating any element we chose of $V \otimes V - I_2$ to be neutral, though v^* is the only element which can be neutral with respect to *all* other elements. The axiomatic definition will again be equivalent to a version defined by the second derivatives of a locally quasilinear representation. Here is a formal summary of this discussion:

Definition 5. *Given preferences represented by u^q which is locally quasilinear at x, we define an element $t \in V \otimes V$ to be complementary, substitutive, or neutral at x according as the sign of $D^2u(x)(t)$ is positive, negative, or zero.*

Proposition 4. *Definition 5 describes (for any locally quasilinear u^q) the unique partition of $V \otimes V$ into subsets C, S, N which satisfies (omitting x from our notation):*

1. ***Linearity***: *Each of C, S, N is a convex cone, i.e. closed under positive linear combinations.*

2. ***Unanimity***: *If $D^2u(t) > 0$ for* all *smooth representations u, then $t \in C$. Similarly, if $D^2u(t) = 0$ for all u then $t \in N$, and if $D^2u(t) < 0$ for all u then $t \in S$.*

3. ***Neutrality***: *If v^* is a numeraire, then $v^* \otimes v^* \in N$.*

Note that the partition also satisfies Symmetry, i.e. $v \otimes w$ and $w \otimes v$ are always in the same category, but we don't need this as an assumption.

Without assuming Neutrality, we can, as noted in Section 3, categorize all tensors which are first-order neutral, i.e. in the kernel of the linear functional $Du \otimes Du$ on $V \otimes V$. For these, Unanimity applies. This leads to an economically interesting notion of "relative complementarity". Notice that for any four vectors $v_1, ..., v_4$ with each $Du(v_i) \neq 0$, the tensor

$$t := \left(\frac{v_1 \otimes v_2}{(Du(v_1))(Du(v_2))} - \frac{v_3 \otimes v_4}{(Du(v_3))(Du(v_4))} \right) \tag{9}$$

is first-order neutral. When $D^2u(t)$ is positive for this t, it then makes sense to say that v_1 complements v_2 better than v_3 complements v_4. Essentially, the denominators in (9) normalize the changes in each good to have the same marginal utility. We can then ask whether the consumer would rather have ε utils of both Goods 1 and 2, or of both Goods 3 and 4, and the answer will not depend on the utility representation. It makes sense to define "link strength" between two goods by the expression

$$L_{u,x}[v_1, v_2] = \frac{D^2u(x)(v_1, v_2)}{(Du(x)(v_1))(Du(x)(v_2))}$$

and the sign-invariance of (9) then tells us that order relations between any two L's does not depend on the choice of u. This gives a robust notion of relative complementarity between pairs of goods. In fact, I rediscovered this order-invariance of the L's (also noted by Samuelson and others) very early in my work, and it progressed into my first definition of direct complementarity:

Definition E. Let u be any smooth representation. We say w, z are *direct complements* at x if

$$L_{u,x}[w, z] > L_{u,x}[w, v^*]$$

They are neutrals if these quantities are equal, and substitutes if the comparison is reversed.

This is easily seen to be equivalent to Definition A and hence the others. Definition E has the advantage that it, like Definition B, can be applied to any representation u. I decided, though, that the other definitions are more suitable to the body of the paper.

There is a particularly simple interpretation of $D^2u(t)$ when two of the vectors are equal, say $v_1 = v_3$. Then, if we define n_{24} as the neutral vector $\frac{v_2}{Du(v_2)} - \frac{v_4}{Du(v_4)}$, t is a scalar multiple of $v_1 \otimes n_{24}$. Then $D^2u(t) > 0$ exactly when motion in the v_1 direction changes n_{24} from a neutral to a good, i.e. increases the marginal rate of substitution of v_4 for v_2. It certainly makes sense, in this case, to say that v_1 complements v_2 better than it complements v_4.