It's all about parallelograms

Comparative statics in quasilinear settings

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Paris, June 2023

Basic notation

- $\geq$ is the product order on $\mathbb{R}^\ell$;
- $x_K = (x_i)_{i \in K}$, for any $x \in \mathbb{R}^\ell$ and $K \subseteq \{1, \ldots, \ell\}$;
- $X \subseteq \mathbb{R}^\ell$ is the space of inputs; e.g., $X = \{0,1\}^\ell$, $X = \mathbb{Z}^\ell_+$, $X = \mathbb{R}^\ell_+$
- $f : X \to \mathbb{R}$ is a production function;
- demand correspondence $D : \mathbb{R}^\ell \to X$ is given by

$$D(p) \ := \ \operatorname{argmax} \Big\{ f(x) - p \cdot x : x \in X \Big\};$$

- profit function $\pi : \mathbb{R}^\ell \to \mathbb{R}$ is given by

$$\pi(p) \ := \ \max \Big\{ f(x) - p \cdot x : x \in X \Big\}.$$

Section 1

Introduction

Comparative statics in quasilinear settings

Let $i = 1, \ldots, \ell$ and $K \subseteq \{1, \ldots, \ell\}$.

When are goods in K gross complements of good i?

That is, the demand for inputs in K increases as the price of good i falls.

Formally, for any $p'_i \leq p_i$, p_{-i} and $x \in D(p_i, p_{-i})$, $x' \in D(p'_i, p_{-i})$, there is $y \in D(p_i, p_{-i})$, $y' \in D(p'_i, p_{-i})$ such that $y'_K \geq x_K$ and $x'_K \geq y_K$.

Equivalently, $D(p'_i, p_{-i})$ dominates $D(p_i, p_{-i})$ in K by the *weak set order*.

Section 2

Parallelogram order

Comparative statics with linear objectives

Dziewulski & Quah (2022) consider the following problem:

$$\begin{aligned} \text{minimise} \quad & p \cdot x, \\ \text{subject to} \quad & x \in \Gamma(t). \end{aligned}$$

Under what conditions on Γ the optimal solution increases in t?

Parallelogram order

Let $K \subseteq \{1, \ldots, \ell\}$.

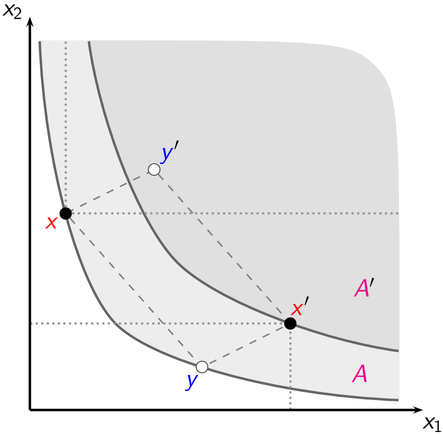
Definition (Parallelogram order)

For any $A, A' \subseteq \mathbb{R}^\ell$, the set A' dominates A in K by the parallelogram order if, for any $x \in A$, $x' \in A'$, there is $y \in A$, $y' \in A'$ satisfying

$$x'_K \geq y_K, \; y'_K \geq x_K, \; \text{and } x + x' = y + y'.$$

A correspondence $\Gamma : T \to X$ *increases in K by the parallelogram order* if $\Gamma(t')$ dominates $\Gamma(t)$ in K by the parallelogram order, for any $t' \geq_T t$.

Parallelogram property



A' dominates A in $K = \{1, 2\}$ by parallelogram order.

Comparative statics with linear objectives

Let $K \subseteq \{1, \ldots, \ell\}$ and $\Gamma : T \to X$ be a compact-valued correspondence.

Theorem (Dziewulski & Quah, 2022)

If Γ *increases in* K *by the parallelogram order, then:*

(i) *For any* $p \in \mathbb{R}^\ell$, *correspondence* $\Phi(t) := \operatorname{argmin}\left\{p \cdot x : x \in \Gamma(t)\right\}$ *increases in* K *by the parallelogram order.*

(ii) *For any* $p \in \mathbb{R}^\ell$, Φ *increases in* K *by the weak set order.*

(iii) *The value function* $f(p, t) := \min\left\{p \cdot x : x \in \Gamma(t)\right\}$ *has increasing differences in* (p_K, t), *i.e., for any* $p'_K \geq p_K$ *and* $t' \geq_T t$,

$$f\left((p'_K, p_{-K}), t'\right) - f\left((p_K, p_{-K}), t'\right) \;\geq\; f\left((p'_K, p_{-K}), t\right) - f\left((p_K, p_{-K}), t\right).$$

If Γ is convex-valued, then statements (i)–(iii) are equivalent to each other, and equivalent Γ being increasing in K by parallelogram order.

Section 3

Gross complements

Comparative statics in quasilinear settings

Let $i = 1, \ldots, \ell$ and $K \subseteq \{1, \ldots, \ell\}$.

When are goods in K gross complements of good i?

That is, the demand for inputs in K increases as the price of good i falls.

Formally, for any $p'_i \leq p_i$, p_{-i} and $x \in D(p_i, p_{-i})$, $x' \in D(p'_i, p_{-i})$, there is $y \in D(p_i, p_{-i})$, $y' \in D(p'_i, p_{-i})$ such that $y'_K \geq x_K$ and $x'_K \geq y_K$.

Equivalently, $D(p'_i, p_{-i})$ dominates $D(p_i, p_{-i})$ in K by the *weak set order*.

Complements in a quasilinear setting

Define correspondence $\Gamma^i : T \to X \times \mathbb{R}$,

$$\Gamma^i(t) \;:=\; \Big\{(y, w) : w \geq -f(y) - t y_i, \text{ for } y \in X\Big\},$$

and notice that $x \in D(p_i - t, p_{-i})$ if, and only if,

$$(x, v) \;\in\; \operatorname{argmin}\Big\{(p, 1) \cdot (y, w) : (y, w) \in \Gamma^i(t)\Big\}.$$

It suffices to show that Γ^i increases in K by parallelogram order.

Parallelogram property

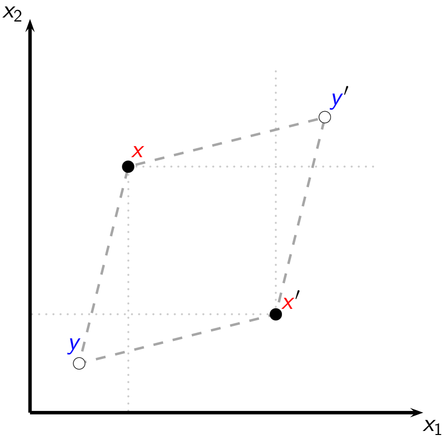
Under what conditions Γ^i increases in K by parallelogram order?

Let $i = 1, \ldots, \ell$ and $K \subseteq \{1, \ldots, \ell\}$.

Definition (Parallelogram property)

Function $f : X \to \mathbb{R}$ obeys parallelogram property for (i, K) if, for any $x, x' \in X$ such that $x'_i > x_i$, there is $y, y' \in X$ satisfying $x_i \geq y_i$, $x'_K \geq y_K$, $x + x' = y + y'$, and $f(x) + f(x') \leq f(y) + f(y')$.

Parallelogram property



For $i = 1$, $K = \{2\}$: $x_1 \geq y_1$, $x_2' \geq y_2$, and $x + x' = y + y'$.

Complements in a quasilinear setting

Proposition

If function $f : X \to \mathbb{R}$ satisfies parallelogram property for (i, K) then inputs in K are complements of input i. Whenever X is convex and f is concave, the converse is also true and the parallelogram property is equivalent to the profit function π having increasing differences in (p_i, p_K).

Parallelogram property is sufficient (and necessary) for gross complements.

Direct proof that PP implies gross complements

Take any $p'_i \leq p_i$, p_{-i} and $x \in D(p_i, p_{-i})$, $x' \in D(p'_i, p_{-i})$.

Claim 1: There is $y, y' \in X$ such that $x_i \geq y_i$, $x'_K \geq y_K$, $y + y' = x + x'$, and $f(y) + f(y') \geq f(x) + f(x')$. If $x_i \geq x'_i$, let $y = x'$, $y' = x$. If $x'_i > x_i$, this holds by parallelogram property.

Claim 2: $y \in D(p_i, p_{-i})$ and $y' \in D(p'_i, p_{-i})$. Take any such y, y'. The conditions above imply $p_i(x_i - y_i) \;\geq\; p'_i(x_i - y_i) \;=\; p'_i(y'_i - x'_i)$, thus

$$p_i x_i + p'_i x'_i \;\geq\; p_i y_i + p'_i y'_i.$$

Moreover, we have $x_{-i} + x'_{-i} = y_{-i} + y'_{-i}$.

Direct proof that PP implies gross complements

Since $x \in D(p_i, p_{-i})$, $x' \in D(p'_i, p_{-i})$,

$$
\begin{aligned}
&\left[f(y) - p_i y_i - p_{-i} \cdot y_{-i}\right] + \left[f(y') - p'_i \cdot y'_i - p_{-i} \cdot y'_{-i}\right] \\
&\quad \leq \ \left[f(x) - p_i x_i - p_{-i} \cdot x_{-i}\right] + \left[f(x') - p'_i \cdot x'_i - p_{-i} \cdot x'_{-i}\right] \\
&\quad = \ \left[f(x) + f(x')\right] - \left[p_i x_i + p'_i \cdot x'_i\right] - p_{-i} \cdot \left[x_{-i} + x'_{-i}\right] \\
&\quad \leq \ \left[f(y) + f(y')\right] - \left[p_i y_i + p'_i \cdot y'_i\right] - p_{-i} \cdot \left[y_{-i} + y'_{-i}\right] \\
&\quad \leq \ \left[f(y) - p_i y_i - p_{-i} \cdot y_{-i}\right] + \left[f(y') - p'_i \cdot y'_i - p_{-i} \cdot y'_{-i}\right],
\end{aligned}
$$

which can be satisfied only if $y \in D(p_i, p_{-i})$ and $y' \in D(p'_i, p_{-i})$.

Therefore, we have shown that $D(p'_i, p_{-i})$ dominates $D(p_i, p_{-i})$ in K by parallelogram order. Thus, goods in K are gross complements to i.

Super*modularity

When are <u>all</u> goods mutual complements, i.e., the demand for all goods increase if p_i decreases, for any i?

Definition (Super*modularity)

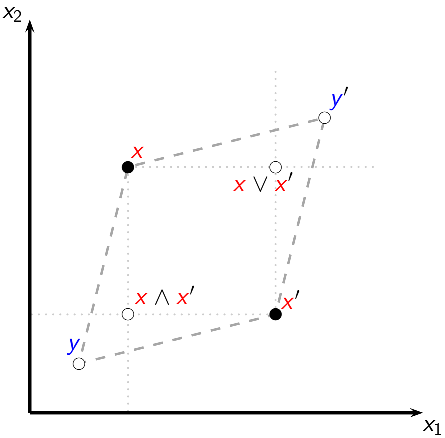
*Function f is super*modular if, for any $x, x' \in X$, there is $y, y' \in X$:*

$$(x \wedge x') \geq y, \; x + x' = y + y', \; \text{and } f(x) + f(x') \leq f(y) + f(y').$$

Proposition (Dziewulski & Quah, 2022)

*If function $f : X \to \mathbb{R}$ is super*modular then all goods are complements. Whenever X is convex and f is concave, the converse is also true and super*modularity is equivalent to supermodularity of the profit function π.*

Super*modularity



Super*modularity in $K = \{1, 2\}$.

Representative agent

Based on Dziewulski & Quah (2022).

Consider a representative firm with production function

$$f(x) \;:=\; \max\left\{ \sum\nolimits_j f_j(y^j) : \sum\nolimits_j y^j = x \right\}.$$

For example, f is the optimal production of a firm that operates in multiple production plants j, endowed with production functions f_j.

- Such aggregation preserves complements, i.e., if all production functions f_j induce gross complementarities, then so does f.
- However, supermodularity is *n*ot preserved, i.e., f may not be supermodular, even if all functions f_j are.
- Super*modularity is preserved, i.e., if all functions f_j are super*modular (e.g., supermodular), then f is super*modular (although it need not be supermodular).

Section 4

Gross substitutes

Gross substitutes

Our approach to gross complements can be applied to study substitutes.

Let $i = 1, \ldots, \ell$ and $K \subseteq \{1, \ldots, \ell\} \setminus \{i\}$.

When are goods in K gross substitutes of good i?

That is, the demand for goods in K increases with the price of good i.

Formally, for any $p'_i \geq p_i$, p_{-i} and $x \in D(p_i, p_{-i})$, $x' \in D(p'_i, p_{-i})$, there is $y \in D(p_i, p_{-i})$, $y' \in D(p'_i, p_{-i})$ such that $y'_K \geq x_K$ and $x'_K \geq y_K$.

Equivalently, $D(p'_i, p_{-i})$ dominates $D(p_i, p_{-i})$ in K by the *weak set order*.

Gross substitutes

When are goods in K gross substitutes of good i?

Similarly to the previous case, $x \in D(p_i + t, p_{-i})$ if, and only if,

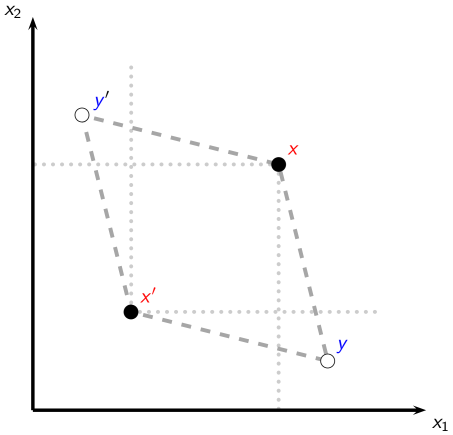
$$(x, v) \ \in \ \text{argmin} \Big\{ (p, 1) \cdot (y, w) : (y, w) \in \tilde{\Gamma}^i(t) \Big\},$$

where

$$\tilde{\Gamma}^i(t) \ := \ \Big\{ (x, v) : v \geq -f(x) - t(-x_i), \ \text{for } x \in X \Big\}.$$

Hence, it suffices for $\tilde{\Gamma}^i$ to be increasing in K by parallelogram order (by the theorem in Dziewulski & Quah, 2022). Therefore, $\tilde{f}(x) := f(-x_i, x_{-i})$ must obey parallelogram property for (i, K) (by modifying the argument from gross complements).

Graphical example



For $i = 1$ and $K = \{2\}$: $-x_1 \geq -y_1$, ${x_2}' \geq y_2$, and $x + x' = y + y'$.

Complements in a quasilinear setting

Proposition

If function $\tilde{f} : X \to \mathbb{R}$, given by $\tilde{f}(x) := f(-x_i, x_{-i})$, satisfies parallelogram property for (i, K) then goods in K are gross substitutes to good i. Whenever X is convex and f is concave, the converse is also true and the condition above is equivalent to the profit function π having decreasing differences in (p_i, p_K).

The modified parallelogram property is sufficient (and necessary) for gross complements. If the property holds for all (i, K), where $K = \{1, \ldots, \ell\} \setminus \{i\}$, then all goods are (mutual) gross substitutes and the profit function π is submodular.

Examples

- Whenever $X \subseteq \mathbb{R}^2$ is a lattice, any submodular function $f : X \to \mathbb{R}$ safisfies this condition for $(1, \{2\})$ and $(2, \{1\})$.
- Whenever h, g^i are concave, for all i, function

$$f(x) \;:=\; h\left(\sum_{i=1}^{\ell} g^i(x_i)\right)$$

  obeys our substitutes condition for any (i, K).
- Whenever h is submodular, function

$$f(x) \;:=\; h\left(\sum_{i \in I} x_i, \sum_{i \notin I} x_i\right)$$

  obeys our substitutes condition for any (i, K). In particular, the domain of f may be discrete.

Section 5

Relation to the existing results

Exchange property (Murota, 1996)

Let $X = \{0,1\}^\ell$ or $X = \mathbb{Z}^\ell$.

Definition (Exchange property)

$f : X \to \mathbb{R}$ obeys the exchange property if, for any $x, x' \in X$, either

(i) *there is some $i = 1, \ldots, \ell$ such that $x'_i > x_i$ and*
$f(x) + f(x') \leq f(x + \epsilon_i) + f(x' - \epsilon_i)$, *or*

(ii) *there is some $j \neq i$ such that $x'_i > x_i$, $x'_j < x_j$ and*
$f(x) + f(x') \leq f(x + \epsilon_i - \epsilon_j) + (x' - \epsilon_i + \epsilon_j)$;

where ϵ_k is the vector with the k'th entry equal to 1 and zeros elsewhere.

Since $x + x' = \left(x + \epsilon_i - \epsilon_j\right) + \left(x' - \epsilon_i + \epsilon_j\right)$, exchange property is a parallelogram-like condition.

Exchange property (Murota, 1996)

- For $X = \{0,1\}^\ell$, exchange property is necessary and sufficient for all goods to be gross substitutes.
- For $X = \mathbb{Z}^\ell$, it is sufficient but not necessary.
- We can show directly that exchange property is equivalent to our substitute property for $X = \{0,1\}^\ell$ ...
- ... but strictly stronger for $X = \mathbb{Z}^\ell$, e.g., $f(x) := \min\{2, x_1 + 2x_2\}$ fails the exchange property. Indeed, take $x = (2,0), x' = (0,1)$. Then, $f(x) + f(x') = 4$, but $f(2,1) + f(0,0) = 2$ and $f(1,1) + f(1,0) = 3$. At the same time $f(-x_i, x_j)$ satisfies parallelogram property for $(i, \{j\})$, for any $i, j = 1,2$ where $i \neq j$ (since f is submodular).

Unified substitutes (Galichon, Samuelson, & Vernet, 2022)

Definition (Unified gross substitutes)

For any $p, p' \in \mathbb{R}^\ell$ *and* $x \in D(p)$, $x' \in D(p')$, *there is some* $y \in D(p \wedge p')$, $y' \in D(p \vee p')$ *such that*

- $p_i \leq p'_i$ *implies* $x_i \geq y_i$ *and* $y'_i \geq x_i$;
- $p_i > p'_i$ *implies* $x'_i \geq y_i$ *and* $y_i \geq x_i$.

Galichon, Samuelson, & Vernet, (2022) show that, for convex X and concave f, this condition is equivalent to supermodularity of the profit function π. Thus, it must be equivalent to the demand satisfying our definition of (weak) substitutes, which is equivalent to production function satisfying the modified parallelogram property for all (i, K), where $K = \{1, \ldots, \ell\} \setminus \{i\}$.

Section 6

Summary

Summary

It's all about parallelograms.