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Methods

S-Convexity and Gross Substitutability

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Abstract. We propose a new concept of S-convex functions (and its variant, semistrictly quasi-S- (SSQS)-convex functions) to study substitute structures in economics and operations models with continuous variables. We develop a host of fundamental properties and characterizations of S-convex functions, including various preservation properties, conjugate relationships with submodular and convex functions, and characterizations using Hessians. For a divisible market, we show that the utility function satisfies gross substitutability if and only if it is S-concave under mild regularity conditions. In a parametric maximization model with a box constraint, we show that the set of optimal solutions is nonincreasing in the parameters if the objective function is (SSQS-) S-concave. Furthermore, we prove that S-convexity is necessary for the property of nonincreasing optimal solutions under some conditions. Our monotonicity result is applied to analyze two notable inventory models: a single-product inventory model with multiple unreliable suppliers and a classic multiproduct dynamic inventory model with lost sales.

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Keywords: S-convexity • gross substitutability • nonincreasing optimal solutions • inventory models

1. Introduction

Many operations models in inventory and revenue management are embedded with substitute structures. For example, products of higher quality can be used to fulfill the demand of lower quality products (Bassok et al. 1999, Yu et al. 2015, Chen and Gao 2019), the same product from one facility can be used to fulfill the demand from another facility (Yang and Qin 2007, Hu et al. 2008, Chen et al. 2015), the same product procured from different suppliers can be regarded as substitutable (Federgruen and Yang 2009, 2011; Chen et al. 2018), and resources consumed by a product can be used by other products consuming the same resources (Talluri and van Ryzin 1998, 2004; Chen et al. 2018).

However, it is challenging to analyze operations models with general substitute structures. As an illustration, consider a company managing inventories of multiple products that can be used to fulfill their own demands and, interchangeably, demands of other products with a cost. Assuming the company has some initial inventories of these products, a fundamental question is whether an increase in the initial inventory of one product implies fewer orders of all products. Such monotone comparative statics is desired in many important operations models in the literature. For instance, Ignall and Veinott (1969) prove that, for a classic multiproduct dynamic stochastic inventory model, a myopic policy is optimal if the optimal ordering quantity of each product is nonincreasing in the initial inventory vector. For the same model, the decreasing property of certain inventory thresholds allows Song and Xue (2007) to propose an efficient algorithm to compute the set of products that should not be ordered.

Unfortunately, there is no general and handy tool to derive nonincreasing optimal solutions in parametric optimization problems. Recently, Chen and Li (2021b) show that, if the objective function is $M^{\natural}$-convex (see Murota 2003 and Chen and Li 2021a) in a parametric minimization problem, then the nonincreasingness property holds. However, $M^{\natural}$-convexity can be too strong to obtain the nonincreasingness property (see Example 3). To seek a tight condition, we introduce a new function class referred to as S-convex functions, a generalization of $M^{\natural}$-convex functions, and a quasi version, referred to as semistrictly quasi-S- (SSQS)-convex functions. We show that S/SSQS-convexity suffices to obtain the nonincreasingness property in general parametric optimization problems and S-convexity is necessary under some conditions.

Our notion of S-convexity is also motivated by an important concept of gross substitutability (GS) in economics for which the letter "S" stands. GS is widely used to analyze market dynamics in both divisible and indivisible markets. The definition of GS roughly says that, if the prices of some goods in the market increase and the others remain unchanged, then the demand for each of the latter goods does not decrease. Although this concept is important, it is difficult to identify utility functions satisfying GS (Beviá et al. 1999). For indivisible markets, there are some characterizations of GS and its variants using $M^\natural$-concavity for utility functions defined on the unit hypercube (Fujishige and Yang 2003) and utility functions defined on integer spaces (Murota and Tamura 2003, Murota et al. 2013). For divisible markets, however, only a few utility functions are known to satisfy GS (e.g., linear utility functions, some constant elasticity of substitution utility functions, Cobb–Douglas utility functions), and there is no sufficient and necessary characterization. Interestingly, the S-concavity proposed here can be used to characterize GS in divisible markets under some regularity conditions.

We now summarize the main technical results. In this paper, we introduce the concept of S-convexity on continuous spaces. We first provide various preservation properties of S-convexity, including operations of parametric minimization, variable scaling and translation, projection, and summation of two variable vectors. Based on these preservation properties, we show that S-convex functions are supermodular, and under mild regularity conditions, they have a one-to-one correspondence with submodular and convex functions through the Legendre transformation. This result extends a core theorem in discrete convex analysis, which states that a function on a continuous space is $M^\natural$-convex if and only if its conjugate is $L^\natural$-convex (see theorem 1.1 of Murota and Shioura 2004a).

The conjugacy result of S-convexity allows us to derive a characterization of twice continuously differentiable S-convex functions using inverse M-matrices, which generalizes that for $M^\natural$-convex functions by Chen and Li (2021b). For quadratic functions, we further prove that S-convex functions are exactly variable scalings of quadratic $M^\natural$-convex functions.

As we mention earlier, S-concavity can characterize GS in divisible markets. Specifically, we prove that a utility function on a continuous space satisfies GS if it is coercive S-concave, and the converse also holds under mild regularity conditions. These results together establish an equivalence between GS and S-concavity for divisible markets. As a by-product of our analysis, we show that the concave conjugate of a function satisfying GS is supermodular, which is a nontrivial generalization of the one proved for polyhedral functions satisfying GS in Danilov et al. (2003).

Finally, we consider a general parametric maximization model in which the variable vector is in a box. It is shown that, for any (SSQS-) S-concave objective function, the set of optimal solutions is nonincreasing in the parameters. Moreover, we prove that, under some regularity conditions, the converse also holds. This result not only generalizes that in Chen and Li (2021b) based on (SSQM$^\natural$-) $M^\natural$-concavity, but also provides a tight condition of the nonincreasingness property, which is not enjoyed by (SSQM$^\natural$-) $M^\natural$-concavity.

The organization of this paper is as follows. Section 2 reviews related literature on discrete convex analysis, GS, and inventory models. Section 3 introduces the definitions of S-convex functions and its semistrictly quasi variant, followed by some basic facts and examples. Section 4 provides theoretical results on the preservation properties, supermodularity and conjugate of S-convexity, the equivalence between GS and S-concavity, the characterization of the twice continuously differentiable S-convex function, and the monotonicity of optimal solutions. Section 5 consists of the application of S-convexity to the random yield inventory model and the classic multiproduct inventory model. We conclude this paper and propose several open problems in Section 6. All the proofs except those in Section 5 are presented in the appendix and online companion.

2. Literature Review

This paper is related to three streams of literature: discrete convex analysis, gross substitutability in economics, and inventory models.

2.1. Discrete Convex Analysis

$M^\natural$-convexity, a key concept in discrete convex analysis, is particularly relevant. This concept is initially defined on the discrete space as an analog of convexity and then extended to continuous spaces in Murota and Shioura (2000, 2002) by adding an additional exchange condition ($M^\natural$-EXC) on convexity. Our definition of S-convexity assumes a more general exchange condition (S-EXC). As we show in Sections 4.2 and 4.4, this generalization leads to some fundamentally different properties.

Murota and Shioura (2002, 2004a) show that continuous $M^\natural$-convexity is conjugate to continuous $L^\natural$-convexity under the Legendre transformation, which extends the discrete counterpart. We generalize the continuous version by proving that, under mild regularity conditions, S-convexity is conjugate to submodularity and convexity under the Legendre transformation. This conjugacy result, interesting in itself, also serves as the foundation of our other results, including characterizations of GS and necessary conditions of nonincreasing optimal solutions.

Our definition of an S-convex set is the same as the "2-axis exchangeable set" proposed in Kashiwabara and Takabatake (2003). The main result of Kashiwabara and Takabatake (2003) is to show that, for a down-monotone polyhedron, the conjugate of its indicator function is submodular and convex if and only if it is 2-axis exchangeable. Our conjugacy result is derived for continuously differentiable functions, which requires a fundamentally different analysis. More importantly, our conjugacy result aims to characterize the conjugate function of submodular and convex functions, whereas the result in Kashiwabara and Takabatake (2003) aims to characterize a generalized class of submodular polyhedra and is a pure geometric result.

The class of S-convex polyhedra (or "2-axis exchangeable polyhedra") belongs to the more general class of polybasic polyhedra studied in Fujishige et al. (2004). Similar to Kashiwabara and Takabatake (2003), the main result of Fujishige et al. (2004) is to show that, for a pointed polyhedron, it is a polybasic polyhedron if and only if its support function is "submodular" on each orthant. Note that the submodularity in Fujishige et al. (2004) is defined with respect to different partial orders for different orthants. Thus, the support function of a polybasic polyhedron is not even a submodular function.

Hirai and Murota (2004) characterize $M^\natural$-convex functions on discrete spaces using a discrete version of Hessians. For quadratic functions over discrete spaces, they show that $M^\natural$-convexity is equivalent to a tree structure of the coefficients. Murota and Shioura (2004b) show that a quadratic function with a nonsingular Hessian is $M^\natural$-convex over the continuous space if and only if the inverse of the Hessian is a diagonally dominant M-matrix. This result is extended to twice continuously differentiable $M^\natural$-convex functions by Chen and Li (2021b). Our paper further extends it to twice continuously differentiable S-convex functions. It is worth mentioning that the Hessian of an S-convex function does not require the diagonal dominance of its inverse.

Chen and Li (2021b) show that, in a parametric maximization model with a box constraint, the optimal solution set is nonincreasing in the parameter if the objective function is $M^\natural$/SSQM$^\natural$-concave. We further extend it to S/SSQS-concave objectives. In addition, we show that, under some conditions, S-concavity is necessary for nonincreasing optimal solutions. Necessary conditions with similar flavor are established in section 2.8.2 of Topkis (1998) for nondecreasing optimal solutions.

2.2. Gross Substitutability

This paper is related to GS in divisible and indivisible markets. For indivisible markets, GS is often used to prove the existence of a competitive equilibrium (e.g., Kelso and Crawford 1982, Beviá et al. 1999, Gul and Stacchetti 1999). Gul and Stacchetti (1999) show that GS is also necessary for the existence of an equilibrium when there is only one unit of each product in the market. To characterize GS in indivisible markets, Fujishige and Yang (2003) show that, for functions with domain $\{0,1\}^n$ for some natural number n, GS is equivalent to $M^\natural$-concavity. For functions defined on integer spaces, two stronger versions of GS, projected GS and GS&LAD, are shown to be equivalent to $M^\natural$-concavity in Murota and Tamura (2003) and Murota et al. (2013), respectively. However, similar equivalent characterizations to the class of functions satisfying GS over integer spaces are still missing.

For divisible markets, GS is widely used to design and analyze efficient and practical algorithms or market dynamics. See Arrow et al. (1959), Codenotti et al. (2004), Garg et al. (2004, 2020), Garg and Kapoor (2006, 2007), Cole and Fleischer (2008), Avigdor-Elgrabli et al. (2015), and the references therein. However, unlike GS in indivisible markets, there is no characterization of GS or its variants in divisible markets.

2.3. Inventory Models

Our application to the multiproduct stochastic inventory model is most related to Ignall and Veinott (1969), Song and Xue (2007), and Chen and Li (2021b). See also Veinott (1965), Pasternack and Drezner (1991), Bassok et al. (1999), and Nagarajan and Rajagopalan (2008) for more on multiproduct inventory models. The application to the multisupplier random yield model is most related to Federgruen and Yang (2009). More studies of random yield inventory models can be found in Agrawal and Nahmias (1997), Swaminathan and Shanthikumar (1999), Dada et al. (2007), Federgruen and Yang (2011), and Chen et al. (2018).

2.4. Notations and Terminologies

We list the notations and terminologies used throughout this paper. Denote $\mathbb{R}$ the real space, $\mathbb{R}_+ = \{x \in \mathbb{R} | x \geq 0\}$ and $\mathbb{R}_{++} = \{x \in \mathbb{R} | x > 0\}$. For a positive integer n, we denote $[n]$ the set $\{1, 2, \ldots, n\}$. For a vector $x \in \mathbb{R}^n$ and $S \subseteq [n]$, x_S is the vector $(x_i)_{i \in S}$ and x_{-i} is the vector obtained by removing the ith coordinate of x. For any subset $S \subseteq \mathbb{R}^n$, we denote δ_S its indicator function, that is, $\delta_S(x) = 0$ if $x \in S$, and $\delta_S(x) = +\infty$ otherwise. This definition of indicator function is commonly used in the convex analysis literature (see Rockafellar 1970). We caution the readers that this definition is different from another commonly used one under the same name that specifies $\delta_S(x) = 1$ if $x \in S$ and $\delta_S(x) = 0$ otherwise. Denote $\mathbf{e}_i$ as the vector with one in its ith coordinate and zero otherwise, $\mathbf{e}_0$ as the zero vector, and $\mathbf{e}$ as the vector with one in each coordinate. Whereas we abuse the same notation $\mathbf{e}$ for different dimensions, it should be clear from the context. For a set $S \subseteq \mathbb{R}^n$, its convex hull, closure, and interior are denoted by $\text{conv}(S)$, $\overline{S}$, and $\text{int}(S)$, respectively. The component-wise minimum and maximum of two vectors $x, y \in \mathbb{R}^n$ are denoted by $x \wedge y = (\min\{x_1, y_1\}, \ldots, \min\{x_n, y_n\})$ and $x \vee y = (\max\{x_1, y_1\}, \ldots, \max\{x_n, y_n\})$, respectively. For $x \in \mathbb{R}^n$, denote $x^+ = \max\{x, 0\}$ and $x^- = (-x)^+$. The positive and negative index set of a vector x are denoted by $\text{supp}^+(x) = \{i | x_i > 0\}$ and $\text{supp}^-(x) = \{i | x_i < 0\}$, respectively. Denote $\text{supp}(x) = \text{supp}^+(x) \cup \text{supp}^-(x)$. In this paper, we work with extended real-valued functions, and the arithmetic operations follow from (Rockafellar 1970). The feasible domain of a function $f : \mathbb{R}^n \to [-\infty, \infty]$ is denoted by $\text{dom}(f) = \{x \in \mathbb{R}^n | -\infty < f(x) < \infty\}$. A function is proper if $\text{dom}(f) \neq \emptyset$. A function $f : \mathbb{R}^n \to [-\infty, \infty]$ is lower semicontinuous (lsc) if $\liminf_{y \to x} f(y) \geq f(x)$ for all $x \in \mathbb{R}^n$. A function f is upper semicontinuous (usc) if $-f$ is lsc. A convex (concave) function is closed if it is lsc (usc). Denote the summation of a function $f(x)$ and a linear function p^{Tx} by $f[p](x) = f(x) + p^T x$. A function $f : \mathbb{R}^n \to [-\infty, \infty]$ is supermodular if $f(x) + f(y) \leq f(x \wedge y) + f(x \vee y)$ for any $x, y \in \mathbb{R}^n$. A function is submodular if its negative is supermodular. The gradient and Hessian of a function f at x are denoted by $\nabla f(x)$ and $\nabla^2 f(x)$, respectively.

3. S-Convex Function

In this section, we introduce a new function class referred to as S-convex function (and its semistrictly quasi version), followed by some basic facts and examples.

Definition 1. A function $f : \mathbb{R}^n \to (-\infty, \infty]$ is called an *S-convex function* if f is convex and satisfies the following exchange property (S-EXC): for any $x, y \in \text{dom}(f)$ and $i \in \text{supp}^+(x-y)$, there exists $j \in \text{supp}^-(x-y) \cup \{0\}$ and $\alpha_0, \beta \in \mathbb{R}_{++}$ such that, for any $\alpha \in [0, \alpha_0]$,

$$f(x) + f(y) \geq f(x - \alpha(\mathbf{e}_i - \beta \mathbf{e}_j)) + f(y + \alpha(\mathbf{e}_i - \beta \mathbf{e}_j)).$$

A function f is called *S-concave* if $-f$ is S-convex. Here, "S" stands for "substitute" given its connection with the important economic concept of GS (established in Section 4.2). Let $Y \subseteq \mathbb{R}^n$ be a set. A function f is called S-convex on Y if $\tilde{f}(x)$ is S-convex, where $\tilde{f}(x) = f(x)$ if $x \in Y$ and $\tilde{f}(x) = +\infty$ if $x \notin Y$ or, equivalently, $\tilde{f} = f + \delta_Y$.

A set $B \subseteq \mathbb{R}^n$ is called S-convex if its indicator function $\delta_B(x)$ is S-convex. That is, an S-convex set B is a convex set that satisfies, for $x, y \in B$, $i \in \text{supp}^+(x-y)$, there exists $j \in \text{supp}^-(x-y) \cup \{0\}$ and $\alpha_0, \beta \in \mathbb{R}_{++}$ such that $x - \alpha(\mathbf{e}_i - \beta \mathbf{e}_j)$ and $y + \alpha(\mathbf{e}_i - \beta \mathbf{e}_j)$ belong to B for any $\alpha \in [0, \alpha_0]$. Our definition of an S-convex function implicitly implies that its feasible domain is S-convex. If B is an open set, then B is S-convex if and only if B is convex because we can choose $j = 0$ and α_0 sufficiently small. However, when B is a closed set, S-convexity requires more structures than convexity (see Example 1(a)).

As we mention in Section 2, the definition of S-convex set appears in Kashiwabara and Takabatake (2003) with a different name, 2-axis exchangeable set, and the class of S-convex polyhedra belongs to the more general class of polybasic polyhedra studied in Fujishige et al. (2004). However, results in this paper are less related to those in Kashiwabara and Takabatake (2003) and Fujishige et al. (2004) as we study S-convex functions and their quantitative properties instead of geometric properties.

Remark 1 (Murota and Shioura 2004a). A function $f : \mathbb{R}^n \to (-\infty, \infty]$ is M$^\natural$-convex if it satisfies the definition of S-convexity with $\beta = 1$.

The following result shows that, on a continuous space, an lsc function satisfying the exchange property is convex. This result allows us to get rid of the convexity assumption in the definition of S-convexity for lsc functions.

Theorem 1. *If $f : \mathbb{R}^n \to (-\infty, \infty]$ is lsc and satisfies property (S-EXC), then f is convex.*

An implication of Theorem 1 is that lsc M$^\natural$-convex functions on continuous spaces can be defined without explicitly assuming the convexity required in the literature (e.g., Murota and Shioura 2004a). We present some examples of S-convex functions.

Example 1.

a. A closed convex set $B \subseteq \mathbb{R}^2$ is an S-convex set if and only if, for any $x \leq y$, the box $[x, y] \subseteq B$.

b. If $f(x)$ is an M$^\natural$-convex function on $\mathbb{R}^n$, then $f(b_1 x_1, \ldots, b_n x_n)$ is S-convex on $\mathbb{R}^n$ for any $b \in \mathbb{R}^n_+$. In particular, as pointed out in Remark 1, M$^\natural$-convex functions are S-convex.

c. If $f : \mathbb{R}^2 \to (-\infty, \infty]$ is supermodular and convex, then f is S-convex on $\mathbb{R}^2$.

Example 1(a) shows that closed S-convex sets are quite general and maybe not polyhedra. For example, the lower left of a unit disc $\{(x_1, x_2) | x_1^2 + x_2^2 \leq 1, x_1 \leq 0, x_2 \leq 0\}$ is S-convex. Its subclass closed M$^\natural$-convex sets are shown to be g-polymatroids by section 4.7 of Murota (2003). Example 1(b) is another example of S-convex functions that are not M$^\natural$-convex because M$^\natural$-convex functions are not preserved under variable scaling (Murota 2003). The converse of Example 1(c) is implied by Proposition 3.

Similar to the definition of semistrictly quasi M$^\natural$-convexity (see Murota and Shioura 2003), we can define semistrictly quasi S-convexity as follows.

Definition 2. A function $f : \mathbb{R}^n \to (-\infty, \infty]$ is SSQS-convex if, for any $x, y \in \text{dom}(f)$ and $i \in \text{supp}^+(x-y)$, there exists $j \in \text{supp}^-(x-y) \cup \{0\}$ and $\alpha_0, \beta \in \mathbb{R}_{++}$ such that, for any $\alpha \in [0, \alpha_0]$,

$$\begin{aligned} & f(x - \alpha(\mathbf{e}_i - \beta\mathbf{e}_j)) < f(x) \\ \text{or} \quad & f(y + \alpha(\mathbf{e}_i - \beta\mathbf{e}_j)) < f(y) \\ \text{or} \quad & f(x - \alpha(\mathbf{e}_i - \beta\mathbf{e}_j)) - f(x) = f(y + \alpha(\mathbf{e}_i - \beta\mathbf{e}_j)) \\ & \qquad - f(y) = 0. \end{aligned}$$

SSQS-convexity is defined by relaxing the exchange property of S-convexity and, thus, is a generalization of S-convexity. The following proposition provides an important method of constructing SSQS-convex functions. The proof is straightforward and, thus, omitted.

Proposition 1. *Let $f : \mathbb{R}^n \to (-\infty, \infty]$ be an SSQS-convex function and $h : \mathbb{R} \to (-\infty, \infty]$ be a strictly increasing function. Then,*

$$\tilde{f}(x) = \begin{cases} h(f(x)) & x \in \mathrm{dom} f \\ +\infty & x \notin \mathrm{dom} f \end{cases}$$

is SSQS-convex.

4. Properties and Characterizations

In this section, we establish fundamental properties and characterizations of S-convexity.

4.1. Preservation Properties

It is clear from the definition of S-convexity that, if $f : \mathbb{R}^n \to (-\infty, \infty]$ is S-convex, then $\lambda f(x)$ and $f(x + a)$ are S-convex for all $\lambda > 0$ and $a \in \mathbb{R}^n$. The function obtained by fixing the values of some coordinates of f remains S-convex. The restriction of an S-convex function on a box is S-convex, and S-convexity is preserved under variable permutation. More preservation properties are provided as follows.

Proposition 2. *Let $f : \mathbb{R}^n \to (-\infty, \infty]$ be an S-convex function. Then, the following statements hold:*

a. *Function $f(\lambda_1 x_1, \ldots, \lambda_n x_n)$ is S-convex in x for $\lambda_1, \ldots, \lambda_n \geq 0$.*

b. *Let $g : \mathbb{R}^m \to (-\infty, \infty]$ be an S-convex function. Then, $f(x) + g(y)$ is S-convex in (x, y).*

c. *Function $f(x) + f_1(x_1) + \cdots + f_n(x_n)$ is S-convex for univariate convex functions $f_1, \ldots, f_n$.*

d. *Function $f(x + y)$ is S-convex in (x, y).*

e. *Function $f(\sum_{j=1}^{m_1} y_{1j}, \ldots, \sum_{j=1}^{m_n} y_{nj})$ is S-convex in $(y_{ij})_{i \in [n], j \in [m_i]}$, where $m_1, \ldots, m_n$ are positive integers.*

f. *Let $f : \mathbb{R}^n \times \mathbb{R}^m \to (-\infty, \infty]$ be an S-convex function and $g(x) = \min_y f(x, y)$. If $\min_y f(x, y)$ has a minimizer for any feasible $x \in \mathrm{dom}(g)$, then g is S-convex.*

Remark 2. Proposition 2 together with Example 1 can be used to construct a broad class of S-convex functions that are not M$^\natural$-convex functions or their variable scalings.

4.2. Supermodularity, Conjugacy, and GS

In this section, we show that any lsc S-convex function is a supermodular function and establish the conjugacy relationship between S-convexity and joint submodularity and convexity in a continuous space. We then use this conjugacy result to prove the equivalence between S-concavity and GS.

Proposition 3. *If $f : \mathbb{R}^n \to (-\infty, \infty]$ is an lsc S-convex function, then f is supermodular on $\mathbb{R}^n$.*

For a function $f : \mathbb{R}^n \to [-\infty, \infty]$ with $\mathrm{dom} f \neq \emptyset$, its convex conjugate function and concave conjugate function are defined, respectively, by

$$f^*(p) = \sup\{p^T x - f(x) \mid x \in \mathrm{dom}(f)\},\ p \in \mathbb{R}^n,$$

$$f_*(p) = \inf\{p^T x - f(x) \mid x \in \mathrm{dom}(f)\},\ p \in \mathbb{R}^n.$$

The M$^\natural$-convex function is shown to be conjugate with the L$^\natural$-convex function on continuous spaces (see Murota and Shioura 2004a). The following two results extend this to S-convex functions.

Theorem 2. *If $f : \mathbb{R}^n \to (-\infty, \infty]$ is an lsc S-convex function, then its conjugate f^* is submodular and convex.*

Theorem 2 implies that any lsc S-convex function can be written as the conjugate of a submodular and convex function. The converse of Theorem 2 is challenging to establish. In fact, the main result of Kashiwabara and Takabatake (2003) is to prove a special case that, if δ_B^* is submodular and convex, then δ_B is S-convex for a down-monotone polyhedron B. By imposing mild conditions on the function, we are able to prove the following converse direction of Theorem 2.

Theorem 3. *Let $f : \mathbb{R}^n \to (-\infty, \infty]$ be an lsc, continuously differentiable, strictly convex function with an open $\mathrm{dom}(f)$. If f^* is submodular, then f is S-convex.*

It is known from lattice programming that, even though the conjugate of a submodular function is supermodular, the conjugate of a general supermodular function is not submodular except that the space is two-dimensional (see theorem 2.7.6 in Topkis 1998 and proposition 3.6 in Murota and Shioura 2004a). That is, the image of the conjugate operator of the class of submodular functions is a strict subset of the class of supermodular functions. In the literature, the conjugate operator is only shown to be a bijection between a subclass of submodular functions, L$^\natural$-convex functions, and a subclass of supermodular functions, M$^\natural$-convex functions (Murota 1998, Murota and Shioura 2004a). Interestingly, Theorems 2 and 3 provide another bijection between S-convex functions and jointly convex and submodular functions under a mild regularity condition. Although we only prove Theorem 3 for continuously differentiable, strictly convex functions, we conjecture Theorem 3 holds for general lsc convex functions.

We now show that S-concavity is closely related to GS defined on continuous spaces. The following definition of GS is equivalent to the one in Danilov et al. (2003).

Definition 3. A function $f:\mathbb{R}^n \to [-\infty,\infty)$ is said to satisfy GS if, for any $p,q\in\mathbb{R}^n$ with $p\le q$ and $x\in \arg\max f[-p]$, there exists $y\in\arg\max f[-q]$ such that $y_i \ge x_i$ whenever $q_i = p_i$.

For convenience, we call a function satisfying GS of Definition 3 a *GS-function*. A GS-function f is called *proper* if there exists at least one point p such that $\arg\max f[-p]\neq\emptyset$. If this condition is not satisfied, GS does not provide any information on the functions. The following theorem shows that any coercive S-concave function is a GS-function. A concave function is coercive if $\lim_{\|x\|_2\to+\infty}\frac{f(x)}{\|x\|_2}=-\infty$.

Proposition 4. *If $f:\mathbb{R}^n\to[-\infty,\infty)$ is a usc, coercive S-concave function, then f is a GS-function. In particular, f is a GS-function if f is a usc S-concave function with a bounded* dom(f).

Remark 3. Proposition 4 implies that $f(x)-\frac{\lambda}{2}\|x\|_2^2$ is a GS-function for any usc S-concave function f as strongly concave functions are coercive (see, e.g., lemma 2.25 of Planiden and Wang 2016).

To establish the converse of Proposition 4, we first prove the following intermediate result of supermodularity of the conjugate of GS-functions. This result is a nontrivial generalization of the one established for polyhedral GS-functions in Danilov et al. (2003). Indeed, the proof in Danilov et al. (2003) relies heavily on the properties of polyhedral concavity, and there is no known approximation of general GS-functions using polyhedral GS-functions. Our proof directly proves that the conjugate of a GS-function satisfies the supermodularity inequality. A challenge in the proof is to show that the feasible domain of its conjugate is a sublattice of $\mathbb{R}^n$, which is presented in Lemma EC.2 of Online Appendix EC.7. One may consider subtracting a GS-function $f(x)$ by $\frac{\lambda}{2}\|x\|_2^2$, that is, $f_\lambda(x)=f(x)-\frac{\lambda}{2}\|x\|_2^2$, to bypass the discussion of the feasible domain of f_*. If f_λ preserves GS, then the conjugate of f_λ is differentiable on $\mathbb{R}^n$ and GS readily implies its supermodularity. It then follows from theorems 1.25 and 2.26 of Rockafellar and Wets (2009) that f_* as their point-wise limit is supermodular. Unfortunately, it remains a challenge to prove that GS is preserved when subtracting $\frac{\lambda}{2}\|x\|_2^2$.

Proposition 5. *If $f:\mathbb{R}^n\to[-\infty,\infty)$ is a usc proper GS-function, then its concave conjugate f_* is supermodular.*

By Theorem 3 and Propositions 4 and 5, it follows that GS is equivalent to S-concavity under some regularity conditions. A similar equivalence result for divisible markets can be established between M$^\natural$-concavity and a stronger version of GS referred to as projected GS (see corollary 4.3 of Li 2021).

4.3. Twice Continuously Differentiable S-Convex Functions

We now provide a characterization of twice continuously differentiable S-convex functions using their Hessian matrices. This characterization is useful in the analysis of our applications in Section 5.

Recall that a nonsingular square matrix A is an M-matrix if it has nonpositive off-diagonal entries and an entry-wise nonnegative inverse. A matrix is called an *inverse M-matrix* if it is the inverse of an M-matrix.

Theorem 4. *Let $a,b\in[-\infty,\infty]^n$, $a\le b$, and $f:(a,b)\to\mathbb{R}$ be a twice continuously differentiable, lsc, strictly convex function. Then, f is S-convex if and only if $\nabla^2 f(x)+\lambda I$ is an inverse M-matrix for any $x\in(a,b)$ and $\lambda>0$.*

Theorem 4 allows us to make an interesting comparison with the characterization of twice continuously differentiable M$^\natural$-convex functions established in Chen and Li (2021b), which states that f is M$^\natural$-convex on (a, b) if and only if, for any $x\in(a,b)$, $\nabla^2 f(x)+\lambda I$ is nonsingular and its inverse is a diagonally dominant M-matrix for any $\lambda>0$. Note that, in Theorem 4, S-convexity does not have the diagonal dominance property appearing in the characterization of M$^\natural$-convexity. This leads to a discrepancy between Theorem 6 and the monotonicity result established in Chen and Li (2021b), which is explained by Example 3 and Theorem 7.

Applying Theorem 4 to quadratic convex functions immediately yields the following corollary.

Corollary 1. *Let A be a nonsingular symmetric matrix and $f(x)=\frac{1}{2}x^TAx$ be a quadratic function on $\mathbb{R}^n$. Then, f is S-convex if and only if A is an inverse M-matrix.*

The following result extends Corollary 1 to singular matrix A.

Theorem 5. *Let A be a symmetric matrix and $f(x)=\frac{1}{2}x^TAx$ be a quadratic function on $\mathbb{R}^n$. The following three statements are equivalent:*

a. *f is S-convex on $\mathbb{R}^n$.*

b. *$f(x)=g(v_1x_1,\ldots,v_nx_n)$ for some positive numbers $v_1,\ldots,v_n$ and some quadratic M$^\natural$-convex function g.*

c. *$A+\lambda I$ is an inverse M-matrix for any $\lambda>0$.*

Theorem 5 shows that quadratic S-convex functions are exactly the variable scaling of quadratic M$^\natural$-convex functions. The following example shows that this is not true for general twice continuously differentiable S-convex functions.

Example 2. Consider $f(x)=\exp(x_1+x_2)+\exp(2x_1+x_2)$, $x\in\mathbb{R}^2$. It is easy to see that $f(x)$ is convex and supermodular and, thus, S-convex by Example 1. Suppose there exists an $M^\natural$-convex function $g(x)$ and positive numbers u_1, u_2 such that $f(x) = g(u_1^{-1}x_1, u_2^{-1}x_2)$. Then, $g(x) = f(u_1x_1, u_2x_2) = \exp(u_1x_1 + u_2x_2) + \exp(2u_1x_1 + u_2x_2)$. It is clear that $g(x)$ is strictly convex, and thus, $\nabla^2 g(x)$ is positive definite for all $x \in \mathbb{R}^2$. By the characterization of twice continuously differentiable $M^\natural$-convex functions mentioned earlier (or see theorem 1 in Chen and Li 2021b), $g(x)$ is $M^\natural$-convex if and only if $(\nabla^2 g(x))^{-1}$ is an diagonally dominant M-matrix for all $x \in \mathbb{R}^2$. That is,

$$u_1^2(\exp(u_1x_1 + u_2x_2) + 4\exp(2u_1x_1 + u_2x_2))$$
$$\geq u_1u_2(\exp(u_1x_1 + u_2x_2) + 2\exp(2u_1x_1 + u_2x_2)),$$
$$u_1u_2(\exp(u_1x_1 + u_2x_2) + 2\exp(2u_1x_1 + u_2x_2))$$
$$\leq u_2^2(\exp(u_1x_1 + u_2x_2) + \exp(2u_1x_1 + u_2x_2)),$$

or, equivalently,

$$u_1(1 + 4\exp(u_1x_1)) \geq u_2(1 + 2\exp(u_1x_1)),$$
$$u_1(1 + 2\exp(u_1x_1)) \leq u_2(1 + \exp(u_1x_1)). \tag{1}$$

By letting x_1 go to $-\infty$, we have $u_1 = u_2$, which contradicts Inequality (1). Therefore, f is not a variable scaling of $M^\natural$-convex functions.

It is worth pointing out that theorem 3.2 of Fujishige et al. (2004) shows a similar type of result that a pointed polyhedron is a polybasic polyhedron if and only if its certain faces are obtained from a base polyhedron by a reflection and scalings along axes. However, this result is a pure geometric characterization of polybasic polyhedra, whereas Theorem 5 provides a quantitative characterization of quadratic S-convex functions using $M^\natural$-convexity.

4.4. Nonincreasing Optimal Solutions

Consider the following parametric optimization problem:

$$\max G(s, a)$$
$$\text{s.t. } a \in B, \tag{2}$$

where $a \in \mathbb{R}^n, s \in \mathbb{R}^m$, B is a box $[l, u]$ with $l, u \in [-\infty, \infty]^n, l \leq u$, and $G : \mathbb{R}^m \times \mathbb{R}^n \to [-\infty, \infty)$. Note that we allow the objective function to take $-\infty$ value, which can be used to incorporate additional constraints (see Rockafellar and Wets 2009). Let $\mathcal{A}_B^*(s)$ be the set of optimal solutions of Problem (2) and $S = \{s | \mathcal{A}_B(s) \neq \emptyset\}$. We say $\mathcal{A}_B^*(s)$ is nonincreasing (nondecreasing) in $s \in S$ if for any $s, s' \in S$, $s \leq s'$, $a \in \mathcal{A}_B^*(s)$ implies that there exists $a' \in \mathcal{A}_B^*(s')$ with $a \geq a'$ $(a \leq a')$ and $a' \in \mathcal{A}_B^*(s')$ implies that there exists $a \in \mathcal{A}_B^*(s)$ with $a \geq a'$ $(a \leq a')$. This monotonicity of sets refers to weak induced set ordering (see Topkis 1998), which is used throughout this paper. The following result provides a sufficient condition of nonincreasing optimal solutions.

Theorem 6. *If $G : \mathbb{R}^m \times \mathbb{R}^n \to [-\infty, \infty)$ is a usc SSQS-concave function, then $\mathcal{A}_B^*(s)$ is nonincreasing in $s \in S$.*

Theorem 6 is in sharp contrast with the classic result in lattice programming that states, for a parametric maximization model, if the objective function $G(s, a)$ is supermodular and the feasible domain of (s, a) is a sublattice, then the set of optimal solutions is nonincreasing (for lattice programming) in the parameters with respect to induced set ordering (see Topkis 1998, theorem 2.8.1). First and foremost, the monotonicity of optimal solutions derived from Theorem 6 and from lattice programming are opposite. Second, Theorem 6 studies the maximization problem of S-concave functions (thus, submodular by Proposition 3), which is much harder to analyze than the maximization problem of supermodular functions in lattice programming. As we mention in the introduction, Theorem 6 includes the nonincreasing optimal solution result established in Chen and Li (2021b) for (SSQM$^\natural$-) $M^\natural$-concave objective functions as a special case. Yet, when G is $M^\natural$-concave, Chen and Li (2021b) further show that $\mathbf{e}^T\mathcal{A}_B^*(s) + \mathbf{e}^T s$ is nondecreasing in s, which does not hold for S-concavity as illustrated by the following example.

Example 3. Consider a two-dimensional function defined by $G(s, a) = -(6s^2 + 4sa + a^2)$. The Hessian of G is $-\begin{bmatrix} 6 & 2 \\ 2 & 1 \end{bmatrix}$, which implies that G is supermodular and concave. Because it is not diagonally dominant, G is S-concave but not $M^\natural$-concave. It is clear that the maximizer $a^*(s) = -2s$ is decreasing, whereas $a^*(s) + s = -s$ is not nondecreasing but actually decreasing in s.

The following theorem shows that, under some conditions, S-convexity is necessary for nonincreasing optimal solutions.

Theorem 7. *Let $f : \mathbb{R}^n \to (-\infty, \infty]$ be an lsc, continuously differentiable, strictly convex function with open* dom(f). *If, for any separable convex function $w(x)$ and $\lambda > 0$, the optimal solution set of*

$$\inf_{x \in \mathbb{R}^n} f_\lambda(x + t) + w(x) \tag{3}$$

is nonempty and nonincreasing in $t \in \mathbb{R}^n$, then f is S-convex.

Remark 4. The assumption that the optimal solution set of Problem (3) is nonempty is implied by the strong convexity of f_λ. If we replace strict convexity of f with strong convexity of f in Theorem 7, the necessary condition only needs that the optimal solution set of (3) with $\lambda = 0$ (instead of for any positive λ) is nonincreasing in t.

By Proposition 2(c) and (d), the objective function of (3) is S-convex in (x, t) if f is S-convex. Therefore, Theorems 6 and 7 establish an equivalence between S-convexity and the property of nonincreasing optimal solutions, which is not enjoyed by $M^\natural$-convexity.

5. Applications

In this section, we illustrate how S-convexity can be used to perform comparative statics analysis in two important operations models.

5.1. Random Yield Model

Consider a one-period inventory problem in which a buyer selling a single-product needs to procure from n unreliable suppliers to fulfill its random demand. The buyer's problem can be formulated as

$$\min_{u\geq 0} G(I_0, u), \tag{4}$$

where $u = (u_1,\ldots,u_n)^T$ with u_i being the ordering quantity from supplier i, I_0 is the initial inventory, $G(I_0,u) = c^T u + \mathbb{E}[h(I_0 + u^T\xi - D)]$, $c = (c_1,\ldots,c_n)^T$ with c_i being the unit ordering cost from supplier i, $\xi = (\xi_1,\ldots,\xi_n)^T$ with ξ_i being the random yield of supplier i, D is the random demand, and $h(\cdot)$ is the inventory cost function. We allow ξ_i, $i = 1,\ldots,n$ to be dependent of each other. Denote the optimal ordering quantity set by $u^*(I_0) = \arg\min\{G(I_0,u) | u \geq 0\}$. We are interested in conditions under which $u^*(I_0)$ is nonincreasing in I_0. For convenience, we denote $x = (x_0, x_1,\ldots,x_n)^T$ with $x_0 = I_0$, $x_i = u_i$, and $z = (z_0, z_1,\ldots,z_n)^T$ with $z_0 = 1$, $z_i = \xi_i$, $i = 1,\ldots,n$. Then, the objective can be written as $G(I_0,u) = G(x) = \sum_{i=1}^n c_i x_i + \mathbb{E}[h(x^T z - D)]$.

If $h(\cdot)$ is a quadratic convex function (see Ignall and Veinott 1969 for an example of this inventory handling cost), the Hessian of $G(x)$ is

$$\nabla^2 G(x) = a\nabla^2\mathbb{E}[(x^T z - D)^2] = a\mathbb{E}[zz^T] = a(\Sigma + \bar{z}\bar{z}^T),$$

where $a = \nabla^2 h \geq 0$, $\bar{z} = \mathbb{E}[z]$ and Σ is the covariance matrix of z.

If $h(t) = h^+t^+ + h^-t^-$, h^+, h^- are the unit holding and lost sales costs, respectively. This inventory handling cost is commonly used in the literature (see, e.g., Zipkin 2008). The objective function can be written as

$$\begin{aligned} G(x) &= \sum_{i=1}^n c_i x_i + \mathbb{E}[h^+(x^T z - D)^+ + h^-(x^T z - D)^-] \\ &= \sum_{i=1}^n c_i x_i + (h^+ + h^-)\mathbb{E}[h^+(x^T z - D)^+] \\ &\quad - h^-(x^T\mathbb{E}[z] - \mathbb{E}[D]). \end{aligned}$$

Denote $f(y)$ and $F(y)$ the probability density function and cumulative distribution function of D, respectively. Note that

$$\mathbb{E}[(x^T z - D)^+] = \mathbb{E}_z\mathbb{E}_D[(x^T z - D)^+ | z] = \mathbb{E}_z\left[\int_0^{x^T z} F(t)dt\right]. \tag{5}$$

It is clear that $G(x)$ is twice continuously differentiable if D is a continuous random variable.

For both cases, Theorems 4–6 imply that $u^*(I_0)$ is nonincreasing in I_0 if $\nabla^2 G(x) + \lambda I$ is an inverse M-matrix for any $\lambda > 0$. The following result provides conditions under which this is satisfied.

Theorem 8. *The optimal ordering quantity set $u^*(I_0)$ is nonincreasing in I_0 if one of the following conditions is satisfied:*

1. *h is quadratic convex, $\Sigma + \lambda I$ is an inverse M-matrix, and $(\Sigma + \lambda I)^{-1}\bar{z} \geq 0$ for any $\lambda > 0$. In particular, this is satisfied if ξ_i, $i = 1,\ldots,n$ are independent.*
2. *$h(t) = h^+t^+ + h^-t^-$, ξ_i, $i = 1,\ldots,n$ are independent, and D is exponentially distributed.*

Proof.

1. By theorem 9.1 of Johnson and Smith (2011), $\nabla^2 G(x) = a\bar{z}\bar{z}^T + a\Sigma + \lambda I$ is an inverse M-matrix for any $\lambda > 0$ as $(a\Sigma + \lambda I)^{-1}\bar{z} \geq 0$. If ξ_i, $i = 1,\ldots,n$ are independent, Σ is a diagonal matrix with diagonal entries being the variances of $z_0, z_1,\ldots,z_n$. It is clear that $(\Sigma + \lambda I)^{-1}\bar{z} \geq 0$.
2. Let $f(t) = \gamma\exp(-\gamma t)$, $t \geq 0$ for some $\gamma > 0$. It follows from Equation (5) that

$$\begin{aligned} \nabla^2 G(x) &= \gamma(h^+ + h^-)\mathbb{E}[\exp(-\gamma x^T z)zz^T] \\ &= \gamma(h^+ + h^-)\mathbb{E}[\exp(-\gamma x^T z)](\Lambda + \eta\eta^T), \end{aligned}$$

where Λ is a diagonal matrix with diagonal entries $\lambda_0,\ldots,\lambda_n$,

$$\lambda_i = \frac{\mathbb{E}[z_i^2\exp(-\gamma x_i z_i)]\mathbb{E}[\exp(-\gamma x_i z_i)] - \mathbb{E}[z_i\exp(-\gamma x_i z_i)]^2}{\mathbb{E}[\exp(-\gamma x_i z_i)]^2} \geq 0,$$

$$\eta_i = \frac{\mathbb{E}[z_i\exp(-\gamma x_i z_i)]}{\mathbb{E}[\exp(-\gamma x_i z_i)]} \geq 0.$$

The nonnegativity of λ_i is from the Cauchy–Schwarz inequality. Again, by theorem 9.1 of Johnson and Smith (2011), $\nabla^2 G(x) + \lambda I$ is an inverse M-matrix for any $\lambda > 0$. □

5.2. Multiproduct Inventory Model

Consider a classic stationary multiproduct dynamic inventory model in which a company sells n products over N periods. At the beginning of period t, after observing the initial inventory $x_t = (x_{t1},\ldots,x_{tn})$, the manager decides the order-up-to level $y_t = (y_{t1},\ldots,y_{tn})$ from a feasible domain Y. Assume that the ordering cost is $c^T(y_t - x_t)$, where $c = (c_1,\ldots,c_n)^T$ is the unit ordering cost vector. After receiving the ordered quantity, the random demand $D_t = (D_{t1},\ldots,D_{tn})$ is realized and fulfilled by the current inventory. D_t may have dependent components but is independent and identically distributed over time. Assume that unsatisfied demand is lost (our following analysis also holds for the backlogging setting), and the remaining inventory is carried over to the next period. At the end of this period, the company incurs a holding and lost sales cost $H(y_t - D_t)$. At period $N+1$, the remaining inventory x_{N+1} is returned with a revenue $c^T x_{N+1}$. Let $\alpha \in (0,1]$ be a discount factor. The objective is to sequentially determine the ordering quantity of each period to minimize the expected overall discounted cost. It is well-known that the objective function can be written as $-c^T x_1 + \sum_{t=1}^N \alpha^{t-1}\mathbb{E}[G(y_t)]$, where

$$G(y) = \mathbb{E}_{D_t}[c^T y + H(y - D_t) - \alpha c^T (y - D_t)^+].$$

Assume that G is strictly convex. Define a myopic policy $\overline{y}(\cdot)$ by $\overline{y}(x) = \arg\min\{G(y) | y \geq x, y \in Y\}$. It is shown by Ignall and Veinott (1969) that, if $\overline{y}(x) - x$ is nonincreasing in x, which is referred to as a substitute property, then the myopic policy is optimal for any x_1. Moreover, this myopic policy can be computed by an efficient pooling algorithm (see Song and Xue 2007 for more on the pooling algorithm) if the following decreasing property holds. Let S^+ be any subset of $[n]$ and $S^- = [n] \setminus S^+$. Define the inventory threshold

$$y^*(x_{S^+}) = \arg\min\{G(y) | y \in Y, y_{S^+} = x_{S^+}\}.$$

The inventory thresholds satisfy the decreasing property if $x_{S^+} \leq x'_{S^+}$ implies $(y^*(x_{S^+}))_{S^-} \geq (y^*(x'_{S^+}))_{S^-}$ for any $S \subseteq [n]$. The following result shows that these two properties hold if G is SSQS-convex on Y.

Theorem 9. *If G is SSQS-convex on Y, then the substitute property and the decreasing property hold.*

Proof. Let $\tilde{G}(y) = G(y) + \delta_Y(y)$ and $z = y - x$. The original problem $\min\{G(y) | y \geq x, y \in Y\}$ is equivalent to

$$\min\{\tilde{G}(z + x) | z \geq 0\}. \tag{6}$$

Let $z^*(x)$ be the optimal solution of (6). By assumption and Proposition 2(d), $\tilde{G}(z + x)$ is SSQS-convex. Then, Theorem 6 implies that $z^*(x) = \overline{y}(x) - x$ is nonincreasing in x.

Let S^+ be a subset of $[n]$. It is clear that $(y^*(x_{S^+}))_{S^-} = \arg\min\{\tilde{G}(x_{S^+}, y_{S^-}) | y_{S^-} \in \mathbb{R}^n\}$. By assumption and Theorem 6, $(y^*(x_{S^+}))_{S^-}$ is nonincreasing in x_{S^+}. □

Theorem 9 implies that, when G is SSQS-convex on Y, the myopic policy is optimal and can be computed efficiently using the pooling algorithm proposed in Song and Xue (2007). It includes several results in the literature as special cases. Ignall and Veinott (1969) show that the substitute property holds if Y is a box and $\nabla^2 G(y)$ is a substitute matrix (or, equivalently, an inverse M-matrix by Chen and Li 2021b) for any $y \in Y$, which can be directly obtained by Theorems 4, 6, and 9. Chen and Li (2021b) show that the substitute property holds if G is SSQM$^\natural$-convex on Y, which is a special case of Theorem 9.

Song and Xue (2007) derive the decreasing property when $\nabla^2 G(y)$ is an inverse M-matrix for any $y \in Y$, and Y can be any compact convex set. However, their proof only works when optimal solutions are in the interior of Y. Instead, our approach works even if optimal solutions are on the boundary.

We provide two examples in which G is S-convex.

Example 4. Consider the inventory handling cost $H(y_t - D_t) = h^T(y_t - D_t)^+ + p^T(y_t - D_t)^-$, where h, p are the unit holding and lost sales cost vectors, respectively. Assume that all the products are stored in a warehouse, and product i consumes v_i units of spaces. Let $Y = \{y | \sum_{i=1}^n v_i y_i \leq C, y \geq 0\}$ for some warehouse capacity C. The one-period cost function is $G(y) = c^T y + \mathbb{E}_{D_t}[h^T(y - D_t)^+ + p^T(y - D_t)^- - \alpha c^T (y - D_t)^+]$. Let $\tilde{F}(y) = G\left(\frac{y_1}{v_1}, \ldots, \frac{y_n}{v_n}\right) + \sum_{i=1}^n \delta_{\{y_i \geq 0\}}(y_i) + \delta_W(\mathbf{e}^T y)$, where $W = \{t | t \leq C\}$. By definition, $\tilde{F}(y)$ is laminar convex and, thus, M$^\natural$-convex (see Murota 2003). Note that $\tilde{G}(y) = G(y) + \delta_Y(y) = \tilde{F}(v_1 y_1, \ldots, v_n y_n)$. Thus, $\tilde{G}(y)$ is S-convex by Proposition 2(a), which implies that $G(y)$ is S-convex on Y.

Example 5. Suppose there are two products. The inventory handling cost function H can be any supermodular convex function. Again, $Y = \{y | v_1 y_1 + v_2 y_2 \leq C, y \geq 0\}$. It is clear that $\tilde{G}(y) = G(y) + \delta_Y(y)$ is jointly convex and supermodular (see Topkis 1998, lemma 2.6.2) and, thus, S-convex by Example 1. It is worth pointing out that though the substitute property can be obtained using lattice programming by changing one variable to its negative, our monotonicity result does not need this unnatural variable transformation.

6. Conclusion

In this paper, we introduce the concept of S-convexity and establish its various properties and characterizations. We illustrate that it is closely related to the important concept of gross substitutability in the economics literature. Moreover, we show that S-convexity is sufficient and necessary for deriving monotone comparative statics.

There are many interesting issues remaining to be answered. For example, it is likely that Theorem 3 holds under more general conditions. Specifically, we conjecture that S-convexity is conjugate to convexity and submodularity for general lsc convex functions. An affirmative answer of this conjecture would allow us to establish the equivalence between GS and S-concavity for general usc concave functions and the preservation of S-convexity under the infimal convolution operation. We also conjecture that S-convexity is preserved under a point-wise limit, which may allow us to establish Theorem 4 without strict convexity.

We are exploring other applications of S-convexity. For example, as with M$^\natural$-concavity, S-concavity may be useful in game theory models. The adoption of S-convexity to other inventory models is also promising.

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Appendix. Proofs

Proof of Theorem 3. Let $x, y \in \mathrm{dom}(f)$ and denote $p = \nabla f(x)$, $q = \nabla f(y)$. By theorems 23.5 and 26.5 of Rockafellar (1970), $x = \nabla f^*(p)$, $y = \nabla f^*(q)$, and $\{p\} = \arg\min f^*[-x]$, $\{q\} = \arg\min f^*[-y]$. Let $i \in \mathrm{supp}^+(x-y)$. We first prove the following statement: there exists $j \in \mathrm{supp}^-(x-y) \cup \{0\}$ and $\beta > 0$ such that $p_i - q_i > \beta(p_j - q_j)$. We consider three cases:

a. If $p_i - q_i > 0$, then the statement holds for $j = 0$ and any positive β.

b. Suppose $p_i - q_i < 0$. Then, $p \neq p \vee q$ or, equivalently, $q \neq p \wedge q$. Denote $S = \{j \in [n] | p_j < q_j\}$. It is clear that $i \in S$. By the optimality of p, q and supermodularity of f^*,

$$f^*(p) - x^T p < f^*(p \vee q) - x^T(p \vee q)$$
$$f^*(q) - y^T q < f^*(p \wedge q) - y^T(p \wedge q)$$
$$f^*(p) + f^*(q) \geq f^*(p \vee q) + f^*(p \wedge q),$$

where the first two inequalities are from $p \neq p \vee q$ and $q \neq p \wedge q$. It then follows that

$$0 < x^T(p - p \vee q) + y^T(q - p \wedge q) = \sum_{j \in S} x_j(p_j - q_j) + \sum_{j \in S} y_j(q_j - p_j) = \sum_{j \in S}(p_j - q_j)(x_j - y_j),$$

and thus,

$$\sum_{j \in S \cap \mathrm{supp}^-(x-y)} (p_j - q_j)(x_j - y_j) > \sum_{j \in S \cap \mathrm{supp}^+(x-y)} (p_j - q_j)(y_j - x_j) \geq (p_i - q_i)(y_i - x_i). \tag{A.1}$$

Observe that $S \cap \mathrm{supp}^-(x-y) \neq \emptyset$ as $(p_i - q_i)(y_i - x_i) > 0$. Let $\beta_0 = \max\left\{\frac{x_j - y_j}{y_i - x_i} \middle| j \in \mathrm{supp}^-(x-y)\right\} > 0$. Divide Inequality (A.1) by $y_i - x_i$, and we have

$$p_i - q_i > \sum_{j \in S \cap \mathrm{supp}^-(x-y)} (p_j - q_j)\frac{x_j - y_j}{y_i - x_i} \geq \beta_0 \sum_{j \in S \cap \mathrm{supp}^-(x-y)} (p_j - q_j) > \beta_0 n \min_{j \in S \cap \mathrm{supp}^-(x-y)} (p_j - q_j).$$

Therefore, the statement holds with $\beta = n\beta_0$.

c. Suppose $p_i = q_i$. It suffices to prove that there exists $j \in \mathrm{supp}^-(x-y)$ such that $p_j < q_j$. Suppose this is not true, that is, $p_j \geq q_j$ for all $j \in \mathrm{supp}^-(x-y)$. We consider three cases:

i. Suppose that there exists $i' \in \mathrm{supp}^+(x-y)$ such that $p_{i'} < q_{i'}$. Replacing i with i' in case (b) yields

$$0 > p_{i'} - q_{i'} > \sum_{j \in S \cap \mathrm{supp}^-(x-y)} (p_j - q_j)\frac{x_j - y_j}{y_{i'} - x_{i'}} \geq 0,$$

a contradiction. Here, the last inequality is by the assumption that $p_j \geq q_j$ for all $j \in \mathrm{supp}^-(x-y)$.

ii. Suppose $p_k < q_k$ for some k with $x_k = y_k$. Replacing i with k in case (b) yields

$$\sum_{j \in S}(p_j - q_j)(x_j - y_j) > 0.$$

It then follows that there exists $j \in S$ with $x_j < y_j$, contradicting the assumption in (c).

iii. Suppose $p \geq q$. Because f^* is submodular, $\nabla_i f^*$ is nonincreasing in coordinates other than i. As $p_i = q_i$ and $p \geq q$, we have $\nabla_i f^*(p) \leq \nabla_i f^*(q)$, that is, $x_i \leq y_i$, which contradicts $x_i > y_i$.

Therefore, we prove that there exists $j \in \mathrm{supp}^-(x-y) \cup \{0\}$ and $\beta > 0$ such that $p_i - q_i > \beta(p_j - q_j)$, that is, $f'(x; -\mathbf{e} + \beta\mathbf{e}_j) + f'(y; \mathbf{e}_i - \beta\mathbf{e}_j) < 0$. By the continuity of ∇f, there exists $\alpha_0 > 0$ such that, for any $\alpha \in [0, \alpha_0]$,

$$f'(x - \alpha(\mathbf{e}_i - \beta\mathbf{e}_j); \beta\mathbf{e}_j - \mathbf{e}_i) + f'(y + \alpha(\mathbf{e}_i - \beta\mathbf{e}_j); \mathbf{e}_i - \beta\mathbf{e}_j) < 0.$$

By convexity of f,

$$f(x) \geq f(x - \alpha(\mathbf{e}_i - \beta\mathbf{e}_j)) + \alpha f'(x - \alpha(\mathbf{e}_i - \beta\mathbf{e}_j); \mathbf{e}_i - \beta\mathbf{e}_j)$$
$$f(y) \geq f(y + \alpha(\mathbf{e}_i - \beta\mathbf{e}_j)) + \alpha f'(y + \alpha(\mathbf{e}_i - \beta\mathbf{e}_j); \beta\mathbf{e}_j - \mathbf{e}_i).$$

These three inequalities imply that, for any $\alpha \in [0, \alpha_0]$,

$$\begin{aligned} f(x) + f(y) &\geq f(x - \alpha(\mathbf{e}_i - \beta\mathbf{e}_j)) + f(y + \alpha(\mathbf{e}_i - \beta\mathbf{e}_j)) \\ &\quad + \alpha f'(x - \alpha(\mathbf{e}_i - \beta\mathbf{e}_j); \mathbf{e}_i - \beta\mathbf{e}_j) \\ &\quad + \alpha f'(y + \alpha(\mathbf{e}_i - \beta\mathbf{e}_j); \beta\mathbf{e}_j - \mathbf{e}_i) \\ &= f(x - \alpha(\mathbf{e}_i - \beta\mathbf{e}_j)) + f(y + \alpha(\mathbf{e}_i - \beta\mathbf{e}_j)) \\ &\quad - \alpha f'(x - \alpha(\mathbf{e}_i - \beta\mathbf{e}_j); \beta\mathbf{e}_j - \mathbf{e}_i) \\ &\quad - \alpha f'(y + \alpha(\mathbf{e}_i - \beta\mathbf{e}_j); \mathbf{e}_i - \beta\mathbf{e}_j) \\ &\geq f(x - \alpha(\mathbf{e}_i - \beta\mathbf{e}_j)) + f(y + \alpha(\mathbf{e}_i - \beta\mathbf{e}_j)). \end{aligned}$$

Therefore, f is S-convex. □

Proof of Proposition 5. Let $a, b \in \mathrm{dom}(f_*)$. Suppose a, $b \in \mathrm{int}(\mathrm{dom} f_*)$. By Lemma EC.2, the box $[a \wedge b, a \vee b] \subseteq \mathrm{int}(\mathrm{dom} f_*)$. It suffices to prove $f_*(a) + f_*(b) \leq f_*(a \wedge b) + f_*(a \vee b)$ for a, b with only two different coordinates (see Topkis 1998, corollary 2.6.1). Let $a = a \wedge b + \alpha\mathbf{e}_i$, $b = a \wedge b + \beta\mathbf{e}_j$ for some $1 \leq i < j \leq n$ and $\alpha, \beta > 0$. By theorem 23.5 of Rockafellar (1970), $\partial f_*(p) = \arg\max f[-p]$ for any $p \in [a \wedge b, a \vee b]$. By theorem 3.1.8 of Borwein and Lewis (2006), $(f_*)'(a \wedge b + t\mathbf{e}_j; \mathbf{e}_j)$ is the jth coordinate of some supergradient of f_* at $a \wedge b + t\mathbf{e}_j$. By the GS property, there exists a supergradient $h^t \in \partial f_*(a + t\mathbf{e}_j)$ such that its jth coordinate $h^t_j \geq (f_*)'(a \wedge b + t\mathbf{e}_j; \mathbf{e}_j)$. Because theorem

3.1.8 of Borwein and Lewis (2006) implies $h_j^t \le -(f_*)'(a + t\mathbf{e}_j; -\mathbf{e}_j), (f_*)'(a \wedge b + t\mathbf{e}_j; \mathbf{e}_j) \le h_j^t \le -(f_*)'(a + t\mathbf{e}_j; -\mathbf{e}_j)$ By corollary 24.2.1 of Rockafellar (1970),

$$f_*(b) - f_*(a \wedge b) = \int_0^\beta (f_*)'(a \wedge b + t\mathbf{e}_j; \mathbf{e}_j) \le \int_0^\beta -(f_*)'(a + t\mathbf{e}_j; -\mathbf{e}_j) = f_*(a \vee b) - f_*(a).$$

We now consider any $a, b \in \text{dom}(f_*)$. Let $c \in \text{int}(\text{dom} f_*)$, where the existence is by Lemma EC.2. Let $a_\lambda = (1-\lambda)a + \lambda c$, $b_\lambda = (1-\lambda)b + \lambda c$, $(a \wedge b)_\lambda = a_\lambda \wedge b_\lambda$ and $(a \vee b)_\lambda = a_\lambda \vee b_\lambda$. It is clear that they are interior points of $\text{dom}(f_*)$ for any $\lambda \in (0,1)$. Because f_* is a usc concave function, corollary 7.5.1 of Rockafellar (1970) implies $f_*(x) = \lim_{\lambda \to 0} f_*(x_\lambda)$ for $x = a, b, a \vee b, a \wedge b$. Thus,

$$f_*(a) + f_*(b) = \lim_{\lambda \to 0} f_*(a_\lambda) + \lim_{\lambda \to 0} f_*(b_\lambda) \le \lim_{\lambda \to 0} f_*(a_\lambda \wedge b_\lambda) + \lim_{\lambda \to 0} f_*(a_\lambda \vee b_\lambda) = f_*(a \wedge b) + f_*(a \vee b). \quad \square$$

Proof of Theorem 4. Suppose that f is S-convex on (a,b). Denote $f_\lambda(x) = f(x) + \frac{\lambda}{2}\sum_{i=1}^n x_i^2$ for $\lambda > 0$. By Proposition 2(c), f_λ is S-convex and strongly convex. By theorem 26.5 in Rockafellar (1970), $\nabla f_\lambda^*(\nabla f_\lambda(x)) = x$ for any $x \in (a,b)$, which implies $\nabla^2 f_\lambda^*(\nabla f_\lambda(x))\nabla^2 f_\lambda(x) = I$. By Theorem 2, f_λ^* is submodular, and thus, $(\nabla^2 f_\lambda(x))_{ij}^{-1} = \nabla_{ij}^2 f_\lambda^*(\nabla f_\lambda(x)) \le 0$ for any $i \ne j$. By Proposition 3, $\nabla^2 f_\lambda(x) \ge 0$. Therefore, $\nabla^2 f_\lambda(x) = \nabla^2 f(x) + \lambda I$ is an inverse M-matrix.

Suppose that $\nabla^2 f_\lambda(x)$ is an inverse M-matrix for any $x \in (a,b)$. Because f_λ is strongly convex, $\text{dom}(f_\lambda^*) = \mathbb{R}^n$ (see theorem 11.8 of Rockafellar and Wets 2009 and note that strong convexity implies coerciveness). By theorem 26.5 in Rockafellar (1970), $\nabla f_\lambda(\nabla f_\lambda^*(p)) = p$, and thus, $\nabla^2 f_\lambda(\nabla f_\lambda^*(p))\nabla^2 f_\lambda^*(p) = I$ for any $p \in \mathbb{R}^n$. It then follows from the assumption that $\nabla^2 f_\lambda^*(p)$ is an M-matrix, and thus, f_λ^* is submodular. Denote $g_\lambda(x) = \frac{\lambda}{2}\sum_{i=1}^n x_i^2$. Its conjugate $g_\lambda^*(x) = \frac{1}{2\lambda}\sum_{i=1}^n x_i^2$. Note that $f_\lambda^* = (f + g_\lambda)^* = f^* \square g_\lambda^*$ is the Moreau envelope of f^* (Rockafellar and Wets 2009). By theorems 1.25 and 2.26 of Rockafellar and Wets (2009), $\lim_{\lambda \to 0} f_\lambda^*(p) = f^*(p)$ for any $p \in \mathbb{R}^n$, which implies submodularity of f^*. By Theorem 3, f is S-convex. $\square$

Proof of Corollary 1. If f is S-convex, then it is strongly convex as A is nonsingular. By Theorem 4, A is an inverse M-matrix. Conversely, if A is an inverse M-matrix, then A is positive definite (see, e.g., theorem 1 in Plemmons 1977). Thus, f is strongly convex, which implies S-convexity of f by Theorem 4. $\square$

Proof of Theorem 5. By Proposition 2(a), statement (b) implies statement (a). We show that (a) implies (c). By Proposition 2(c), $f_\lambda(x) = f(x) + \frac{\lambda}{2}\|x\|_2^2$ is S-convex for any $\lambda > 0$. Note that A is positive semidefinite as f is convex. Then, $\nabla^2 f_\lambda(x) = A + \lambda I$ is nonsingular and, thus, is an inverse M-matrix by Corollary 1.

We now show that (c) implies (b). We first prove that, if there exists $v \in \mathbb{R}_{++}^n$ and $r \in \mathbb{R}_{++}$ such that $Av = rv$, then $f(x) = g_0(v_1x_1, \ldots, v_nx_n)$ for some quadratic M$^\natural$-convex function g_0. Let $V = \text{Diag}(v)$ be the diagonal matrix with diagonal entries $v_1, \ldots, v_n$. $V(A+\lambda I)^{-1}V$ is diagonally dominant because $V(A+\lambda I)^{-1}V\mathbf{e} = V(A+\lambda I)^{-1}v = V(r+\lambda)^{-1}v \in \mathbb{R}_{++}^n$. Because $A + \lambda I$ is an inverse M-matrix, $V(A+\lambda I)^{-1}V$ is a diagonally dominant M-matrix. Let $g_\lambda(x) = \frac{1}{2}x^TV^{-1}(A+\lambda I)V^{-1}x$. It follows from Murota (2009) that g_λ is M$^\natural$-convex, and thus, $g_0(x) = \lim_{\lambda\to 0} g_\lambda(x) = \frac{1}{2}x^TV^{-1}AV^{-1}x = f(V^{-1}x)$ is M$^\natural$-convex. Therefore, $f(x) = g_0(Vx)$.

We next prove that there exists a permutation matrix P such that P^TAP can be written as

$$\text{Diag}(A^1, \ldots, A^k, \mathbf{O}) = \begin{bmatrix} A^1 & & & \\ & \ddots & & \\ & & A^k & \\ & & & \mathbf{O} \end{bmatrix},$$

where $\mathbf{O}$ is a square zero matrix, A^i is a symmetric matrix with a positive eigenvalue r^i and a positive eigenvector v^i, that is, $A^iv^i = r^iv^i$, $i = 1, \ldots, k$. Because $A + \lambda I$ is an inverse M-matrix, $A + \lambda I \ge 0$ for any $\lambda > 0$, and thus, $A \ge 0$. If A is not a zero matrix, by the Perron–Frobenius theorem (see Berman and Plemmons 1994), A has a positive eigenvalue r^1 with a nonnegative eigenvector $v = (v_1, \ldots, v_n)^T$, that is, $Av = r^1v$. By some permutation of indices $1, \ldots, n$, we have $v_1, \ldots, v_m > 0$ and $v_{m+1} = \cdots = v_n = 0$ for $1 \le m \le n$. Let $A = \begin{bmatrix} A^1 & B^1 \\ (B^1)^T & C^1 \end{bmatrix}$ (after permutation), where the dimensions of matrices A^1, C^1 are $m, n-m$, respectively. Then, $Av = rv$ implies that B^1 is a zero matrix. Let $v^1 = (v_1, \ldots, v_m)^T$. It is clear that $A^1v^1 = r^1v^1$. Because C^1 is nonnegative, the same argument for C^1 implies that $C^1 = \text{Diag}(A^2, C^2)$ with positive r^2, v^2 such that $A^2v^2 = r^2v^2$, or $C^2 = \text{Diag}(\mathbf{O}, C^2)$. Repeat this procedure, and we get the preceding decomposition.

Because $A + \lambda I$ is an inverse M-matrix, $P^TAP + \lambda I$ remains an inverse M-matrix (see theorem 1.2.1 of Johnson and Smith 2011), which implies that $A^i + \lambda I$ is an inverse M-matrix for any $i = 1, \ldots, k$. Let V^i be the diagonal matrix $\text{Diag}(v^i)$, $x = (x^1, \ldots, x^k, x^{k+1})$ and $g^i(x^i) = \frac{1}{2}(x^i)^T(V^i)^{-1}A^iV^ix^i$, where the vectors $x^1, \ldots, x^k, x^{k+1}$ have the same dimensions of $A^1, \ldots, A^k$, $\mathbf{O}$, respectively. Because $A^iv^i = r^iv^i$ and v^i, r^i are positive, $g^i(x^i)$ is M$^\natural$-convex by the preceding argument. Note that

$$f(Px) = \frac{1}{2}x^TP^TAPx = \frac{1}{2}x^T\begin{bmatrix} A^1 & & & \\ & \ddots & & \\ & & A^k & \\ & & & \mathbf{O} \end{bmatrix}x = \sum_{i=1}^k g^i(V^ix^i) = g(Vx),$$

where $g(x) = \sum_{i=1}^k g^i(x^i)$ and $V = \text{Diag}(V^1, \ldots, V^k, I)$. Note that here $g(x)$ only depends on $(x^1, \ldots, x^k)$. Because $g^i(x^i)$ is M$^\natural$-convex, $g(x)$, and thus, $\tilde{g}(x)$ are M$^\natural$-convex (see theorem 6.50 of Murota 2003), where $\tilde{g}(x) = g(P^{-1}x)$. Denote $U = PVP^{-1}$. It is clear that U is a diagonal matrix. Because $f(x) = g(VP^{-1}x) = g(P^{-1}Ux) = \tilde{g}(Ux)$, we have (b). $\square$

Proof of Theorem 7. Let $f_\lambda = f(x) + g_\lambda(x)$ for $\lambda > 0$, where $g_\lambda(x) = \frac{\lambda}{2}\sum_{i=1}^n x_i^2$. Note that $f_\lambda^* = (f + g_\lambda)^* = f^* \square g_\lambda^*$ is the

Moreau envelope of f^*. By theorems 1.25 and 2.26 of Rockafellar and Wets (2009), $\lim_{\lambda\to 0} f_\lambda^*(p) = f^*(p)$ for any $p \in \mathbb{R}^n$. By Theorem 3, we only need to show that f_λ^* is submodular for any $\lambda > 0$. Note that f_λ is strongly convex, and thus, $\mathrm{dom}(f_\lambda^*) = \mathbb{R}^n$ (see theorem 11.8 of Rockafellar and Wets 2009 and note that strong convexity implies coerciveness). By Lemma EC.4, it suffices to show that, for any $t', t'' \in \mathbb{R}^n$ with $t' \leqslant t''$, and any compact box B of $\mathbb{R}^n$, there exists $p' \in \arg\max_{p\in B} p^T t' - f_\lambda^*(p)$ and $p'' \in \arg\max_{p\in B} p^T t'' - f_\lambda^*(p)$ such that $p' \leqslant p''$. For any $t \in \mathbb{R}^n$,

$$\begin{aligned}
&\max_{p\in B} p^T t - f_\lambda^*(p)\\
&= \max_{p\in B} p^T t - \sup_{x\in\mathbb{R}^n}\{p^T x - f_\lambda(x)\}\\
&= \inf_{x\in\mathbb{R}^n} \max_{p\in B} p^T(t-x) + f_\lambda(x),\\
&= \inf_{z\in\mathbb{R}^n} \max_{p\in B} -p^T z + f_\lambda(t+z),
\end{aligned}$$

where the second equality is from Sion's minimax theorem (see corollary 3.3 of Sion 1958). Let $B = [a, b]$ with $a, b \in \mathbb{R}^n$, $a \le b$. It is clear that $\max_{p\in B} -p^T z = b^T z^- - a^T z^+$ with optimal solution

$$p_i^* = \begin{cases} a_i, & \text{if } z_i > 0,\\ b_i, & \text{if } z_i < 0,\\ \text{any value in } [a_i, b_i], & \text{if } z_i = 0, \end{cases}$$

$i \in [n]$. Hence,

$$\begin{aligned}
\max_{p\in B} p^T t - f_\lambda^*(p) &= \inf_{z\in\mathbb{R}^n} b^T z^- - a^T z^+ + f_\lambda(t+z)\\
&= \inf_{z\in\mathbb{R}^n} (b-a)^T z^- - a^T z + f_\lambda(t+z).
\end{aligned}$$

Denote $Z(t)$ the optimal solution set of the preceding problem. By assumption, there exists $z' \in Z(t'), z'' \in Z(t'')$ such that $z' \ge z''$. By the definition of p_i^*, it is easy to see that there exists $p' \in \arg\max_{p\in B} p^T t' - f_\lambda^*(p)$ and $p'' \in \arg\max_{p\in B} p^T t'' - f_\lambda^*(p)$ such that $p' \leqslant p''$. □

References

Agrawal N, Nahmias S (1997) Rationalization of the supplier base in the presence of yield uncertainty. *Production Oper. Management* 6(3):291–308.

Arrow KJ, Block HD, Hurwicz L (1959) On the stability of the competitive equilibrium, II. *Econometrica* 27(1):82–109.

Avigdor-Elgrabli N, Rabani Y, Yadgar G (2015) Convergence of tâtonnement in Fisher markets. Preprint, submitted January 26, 2014, https://arxiv.org/abs/1401.6637.

Bassok Y, Anupindi R, Akella R (1999) Single-period multiproduct inventory models with substitution. *Oper. Res.* 47(4):632–642.

Berman A, Plemmons RJ (1994) *Nonnegative Matrices in the Mathematical Sciences, Classics in Applied Mathematics* (Society for Industrial and Applied Mathematics, Philadelphia).

Beviá C, Quinzii M, Silva JA (1999) Buying several indivisible goods. *Math. Social Sci.* 37(1):1–23.

Borwein JM, Lewis AS (2006) *Convex Analysis and Nonlinear Optimization: Theory and Examples*, CMS Books in Mathematics, 2nd ed. (Springer, New York).

Chen X, Gao X (2019) Stochastic optimization with decisions truncated by positively dependent random variables. *Oper. Res.* 67(5):1321–1327.

Chen X, Li M (2021a) Discrete convex analysis and its applications in operations: A survey. *Production Oper. Management* 30(6): 1904–1926.

Chen X, Li M (2021b) M$^\natural$-convexity and its applications in operations. *Oper. Res.* 69(5):1396–1408.

Chen X, Gao X, Hu Z (2015) A new approach to two-location joint inventory and transshipment control via $L^\natural$-convexity. *Oper. Res. Lett.* 43(1):65–68.

Chen X, Gao X, Pang Z (2018) Preservation of structural properties in optimization with decisions truncated by random variables and its applications. *Oper. Res.* 66(2):340–357.

Codenotti B, Pemmaraju S, Varadarajan K (2004) The computation of market equilibria. *ACM SIGACT News* 35(4):23–37.

Cole R, Fleischer L (2008) Fast-converging tatonnement algorithms for one-time and ongoing market problems. *Proc. 40th Annual ACM Sympos. Theory Comput.* (Association for Computing Machinery, New York), 315–324.

Dada M, Petruzzi NC, Schwarz LB (2007) A newsvendor's procurement problem when suppliers are unreliable. *Manufacturing Service Oper. Management* 9(1):9–32.

Danilov VI, Koshevoy GA, Lang C (2003) Gross substitution, discrete convexity, and submodularity. *Discrete Appl. Math.* 131(2): 283–298.

Federgruen A, Yang N (2009) Optimal supply diversification under general supply risks. *Oper. Res.* 57(6):1451–1468.

Federgruen A, Yang N (2011) Procurement strategies with unreliable suppliers. *Oper. Res.* 59(4):1033–1039.

Fujishige S, Yang Z (2003) A note on Kelso and Crawford's gross substitutes condition. *Math. Oper. Res.* 28(3):463–469.

Fujishige S, Makino K, Takabatake T, Kashiwabara K (2004) Polybasic polyhedra: Structure of polyhedra with edge vectors of support size at most 2. *Discrete Math.* 280(1–3):13–27.

Garg J, Husić E, Végh LA (2020) Auction algorithms for market equilibrium with weak gross substitute demands and their applications. Preprint, submitted August 21, 2019, https://arxiv.org/abs/1908.07948.

Garg R, Kapoor S (2006) Auction algorithms for market equilibrium. *Math. Oper. Res.* 31(4):714–729.

Garg R, Kapoor S (2007) Market equilibrium using auctions for a class of gross-substitute utilities. Deng X, Graham FC, eds. *Internet and Network Economics*, Lecture Notes in Computer Science (Springer, Berlin, Heidelber), 356–361.

Garg R, Kapoor S, Vazirani V (2004) An auction-based market equilibrium algorithm for the separable gross substitutability case. Jansen K, Khanna S, Rolim JDP, Ron D, eds. *Approximation, Randomization, and Combinatorial Optimization: Algorithms and Techniques*, Lecture Notes in Computer Science (Springer, Berlin, Heidelberg), 128–138.

Gul F, Stacchetti E (1999) Walrasian equilibrium with gross substitutes. *J. Econom. Theory* 87(1):95–124.

Hirai H, Murota K (2004) M-convex functions and tree metrics. *Japanese J. Indust. Appl. Math.* 21(3):391–403.

Hu X, Duenyas I, Kapuscinski R (2008) Optimal joint inventory and transshipment control under uncertain capacity. *Oper. Res.* 56(4):881–897.

Ignall E, Veinott AF (1969) Optimality of myopic inventory policies for several substitute products. *Management Sci.* 15(5):284–304.

Johnson CR, Smith RL (2011) Inverse M-matrices, II. *Linear Algebra Appl.* 435(5):953–983.

Kashiwabara K, Takabatake T (2003) Polyhedra with submodular support functions and their unbalanced simultaneous exchangeability. *Discrete Appl. Math.* 131(2):433–448.

Kelso AS, Crawford VP (1982) Job matching, coalition formation, and gross substitutes. *Econometrica* 50(6):1483–1504.

Li M (2021) M-natural-convexity, S-convexity, and their applications in operations. Unpublished PhD thesis, University of Illinois at Urbana-Champaign.

Murota K (1998) Discrete convex analysis. *Math. Programming* 83(1–3): 313–371.

Murota K (2003) *Discrete Convex Analysis*, SIAM Monographs on Discrete Mathematics and Applications (SIAM, Philadelphia).

Murota K (2009) Recent developments in discrete convex analysis. Cook W, Lovász L, Vygen J, eds. *Research Trends in Combinatorial Optimization* (Springer, Berlin, Heidelberg), 219–260.

Murota K, Shioura A (2000) Extension of M-convexity and L-convexity to polyhedral convex functions. *Adv. Appl. Math.* 25(4):352–427.

Murota K, Shioura A (2002) M-convex and L-convex functions over the real space—Two conjugate classes of combinatorial convex functions. Technical report, METR2002-09, Department of Mathematical Engineering and Information Physics, University of Tokyo, Tokyo.

Murota K, Shioura A (2003) Quasi M-convex and L-convex functions—Quasiconvexity in discrete optimization. *Discrete Appl. Math.* 131(2):467–494.

Murota K, Shioura A (2004a) Conjugacy relationship between M-convex and L-convex functions in continuous variables. *Math. Programming* 101(3):415–433.

Murota K, Shioura A (2004b) Quadratic M-convex and L-convex functions. *Adv. Appl. Math.* 33(2):318–341.

Murota K, Tamura A (2003) New characterizations of M-convex functions and their applications to economic equilibrium models with indivisibilities. *Discrete Appl. Math.* 131(2):495–512.

Murota K, Shioura A, Yang Z (2013) Computing a Walrasian equilibrium in iterative auctions with multiple differentiated items. Cai L, Cheng SW, Lam TW, eds. *Algorithms and Computation*, vol. 8283 (Springer, Berlin, Heidelberg), 468–478.

Nagarajan M, Rajagopalan S (2008) Inventory models for substitutable products: Optimal policies and heuristics. *Management Sci.* 54(8):1453–1466.

Pasternack BA, Drezner Z (1991) Optimal inventory policies for substitutable commodities with stochastic demand. *Naval Res. Logist.* 38(2):221–240.

Planiden C, Wang X (2016) Strongly convex functions, Moreau envelopes, and the generic nature of convex functions with strong minimizers. *SIAM J. Optim.* 26(2):1341–1364.

Plemmons RJ (1977) M-matrix characterizations. I—Nonsingular M-matrices. *Linear Algebra Appl.* 18(2):175–188.

Rockafellar RT (1970) *Convex Analysis* (Princeton University Press, Princeton, NJ).

Rockafellar RT, Wets RJB (2009) *Variational Analysis.* Grundlehren Der Mathematischen Wissenschaften, 3rd ed. (Springer, Berlin).

Sion M (1958) On general minimax theorems. *Pacific J. Math.* 8(1):171–176.

Song JS, Xue Z (2007) Demand management and inventory control for substitutable products. Unpublished working paper, Duke University, Durham, NC.

Swaminathan JM, Shanthikumar JG (1999) Supplier diversification: Effect of discrete demand. *Oper. Res. Lett.* 24(5):213–221.

Talluri K, van Ryzin G (1998) An analysis of bid-price controls for network revenue management. *Management Sci.* 44(11):1577–1593.

Talluri K, van Ryzin G (2004) Revenue management under a general discrete choice model of consumer behavior. *Management Sci.* 50(1):15–33.

Topkis DM (1998) *Supermodularity and Complementarity* (Princeton University Press, Princeton, NJ).

Veinott AF (1965) Optimal policy for a multi-product, dynamic, nonstationary inventory problem. *Management Sci.* 12(3):206–222.

Yang J, Qin Z (2007) Capacitated production control with virtual lateral transshipments. *Oper. Res.* 55(6):1104–1119.

Yu Y, Chen X, Zhang F (2015) Dynamic capacity management with general upgrading. *Oper. Res.* 63(6):1372–1389.

Zipkin P (2008) On the structure of lost-sales inventory models. *Oper. Res.* 56(4):937–944.

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