S-Convexity and Gross Substitutability

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1. Monotone comparative statics: two motivating examples

2. $\mathrm{M}^{\natural}$-convexity and S-convexity

3. Key results

4. Motivating example II revisit

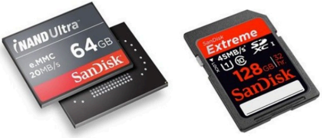
Outline

1. Monotone comparative statics: two motivating examples
2. $\mathrm{M}^\natural$-convexity and S-convexity
3. Key results
4. Motivating example II revisit

Examples of substitution

different expiration dates

high memory → low memory



transshipment

different colors

Motivation

- Parametric optimization problem:

$$\max_{z} \quad G(x,z)$$
$$\text{s.t.} \quad (x,z) \in S$$

Monotonicity of the optimal solution $z^*(x)$ in the parameter x?

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Motivating example I: gross substitutability (GS)

- Let $f : \mathcal{F}^n \to \mathbb{R} \cup \{-\infty\}$ be a valuation function of n types of items.
- $G(z, p) = f(z) - p^T z$, surplus function
- $z^*(p) = \arg\max_z G(z, p)$, demand set, assume singleton

Definition 1 (Gross Substitutability)

A valuation function f satisfies GS if:

$$\forall p, q \in \mathbb{R}^n \text{ with } p \le q, \text{ we have } z_i^*(p) \le z_i^*(q) \text{ if } q_i = p_i.$$

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Importance of GS

- For indivisible items, GS is often used to prove the existence of a competitive equilibrium
  - Kelso and Crawford (1982), Beviá et al. (1999), Gul and Stacchetti (1999), Teytelboym (2014), Baldwin and Klemperer (2019), Baldwin et al. (2023), Gul and Pesendorfer (2022a,b)
- For divisible items, GS is widely used to design and analyze efficient and practical algorithms or market dynamics
  - Arrow et al. (1959), Codenotti et al. (2004), Garg et al. (2004, 2020), Garg and Kapoor (2006, 2007), Cole and Fleischer (2008), Avigdor-Elgrabli et al. (2015)
- Characterizations of GS:
  - Indivisible items: Fujishige and Yang (2003) for set functions, Murota et al. (2013) for general discrete functions, Balkanski and Paes Leme (2020) construction of GS, Dobzinski et al. (2022) approximation
  - Divisible items: unknown

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Motivating example II: capacitated multi-product inventory model

- A company sells n products during a selling season
- Initial inventory $x = (x_1, ..., x_n)$
- Ordering quantity $z = (z_1, ..., z_n)$, s.t., $\mathbf{e}^T z + \mathbf{e}^T x \leq C$
- The ordering quantity arrives immediately
- Linear ordering cost $c^T z$
- Demand $D = (D_1, ..., D_n)$ is realized, unsatisfied demand is lost
- Linear holding and lost-sales cost
- Total expected cost

  $G(x, z) = \mathbb{E}_D[c^T z + h_+^T (z + x - D)^+ + h_-^T (z + x - D)^-]$

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Substitute Property

Company's objective:

$$\min_{z} \; G(x,z) \text{ s.t. } \mathbf{e}^T z + \mathbf{e}^T x \le C, \; z \ge 0$$

$z^*(x)$ is the optimal order quantity

Question: does $x \le \hat{x}$ imply $z^*(x) \ge z^*(\hat{x})$?

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Importance of decreasing optimal solutions

- Substitute property $\Longrightarrow$ a myopic policy is optimal in the multi-period setting

  — Extension to more general inventory models Ignall and Veinott (1969)
- A similar property of decreasing optimal solutions is proposed in Song and Xue (2007) to facilitate the computation of the myopic policy

  — Extension to joint pricing and inventory control models Song and Xue (2007)

Challenge

- Does lattice programming work?

Topkis (1998)

If the objective function G is supermodular in (x, z) and the constraint is $(x, z) \in S$ with S being a sublattice, then the optimal solution is increasing in the parameter x

- A function f is supermodular if $f(x) + f(y) \leq f(\min\{x, y\}) + f(\max\{x, y\})$ for any x, y
- S is a sublattice if for any $x, y \in S$, $\min\{x, y\}, \max\{x, y\} \in S$
- $\mathbf{e}^T z + \mathbf{e}^T x \leq C$ is NOT a sublattice constraint! Even after transformation $x \to -x$.

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Equivalent formulation

$$\min_{z\geq 0} G(x,z) + \delta_{\mathbf{e}^T z + \mathbf{e}^T x \leq C}(x,z)$$

$\delta_S(x) = +\infty$ if $x \in S$ and $\delta_S(x) = 0$ if $x \notin S$
$G(x,z) = \mathbb{E}_D[c^T z + h_+^T (z+x-D)^+ + h_-^T (z+x-D)^-]$

- Minimization of a supermodular objective function!

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- Minimization of a supermodular objective function!

Research Questions

- When a valuation function on $\mathbb{R}^n$ satisfies GS?
- When the optimal solution $z^*(x)$ of the following problem is decreasing in x?

$$\begin{aligned} \min_z \ & G(x,z) \\ \text{s.t. } & (x,z) \in S \end{aligned} \tag{1}$$

$S \subseteq \mathcal{F}^n \times \mathcal{F}^m$

$G : \mathcal{F}^n \times \mathcal{F}^m \to \mathbb{R} \cup \{+\infty\}$, supermodular

$\mathcal{F} = \mathbb{R}$ or $\mathbb{Z}$.

- Both are addressed by the concept of S-convexity

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Definition of $\mathrm{M}^\natural$-convex function

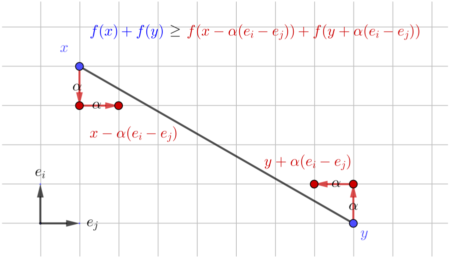
Definition 2 (Murota (2003))

A function $f : \mathcal{F}^n \to \mathbb{R} \cup \{+\infty\}$ is called $\mathrm{M}^\natural$**-convex** if it satisfies the following **exchange condition**:

$\forall x, y \in \mathrm{dom}(f)$, $\forall i \in \mathrm{supp}^+(x-y)$,

$\exists j \in \mathrm{supp}^-(x-y) \cup \{0\}, \alpha_0 \in \mathcal{F}_{++}$ such that for every $\alpha \in [0, \alpha_0] \cap \mathcal{F}$,

$$f(x) + f(y) \geq f(x - \alpha(\mathbf{e}_i - \mathbf{e}_j)) + f(y + \alpha(\mathbf{e}_i - \mathbf{e}_j))$$



Characterization of $\mathrm{M}^\natural$-convex function

Theorem 3 (Murota (2003))

A function $f : \mathcal{F}^n \to \mathbb{R} \cup \{+\infty\}$ is $\mathrm{M}^\natural$-convex if and only if f^* is $\mathrm{L}^\natural$-convex.

$$f^*(p) = \sup\{\langle p, x\rangle - f(x) | x \in \mathcal{F}^n\}$$

Definition 4 (Murota (2003))

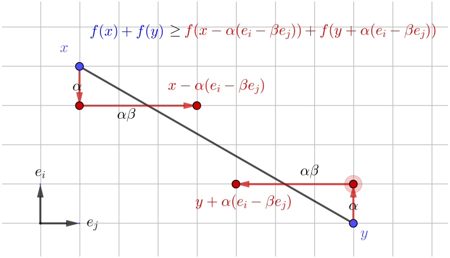
A function $f : \mathcal{F}^n \to \mathbb{R} \cup \{+\infty\}$ is $\mathrm{L}^\natural$-convex if $f(x - \xi \mathbf{e})$ is submodular in (x, ξ).

Definition of S-convex function

Definition 5

A function $f : \mathcal{F}^n \to \mathbb{R} \cup \{+\infty\}$ is called **S-convex** if it satisfies the following **exchange condition**:
$\forall x, y \in \mathrm{dom}(f)$, $\forall i \in \mathrm{supp}^+(x-y)$,
$\exists j \in \mathrm{supp}^-(x-y) \cup \{0\}, \beta, \alpha_0 \in \mathcal{F}_{++}$ such that for every $\alpha \in [0, \alpha_0] \cap \mathcal{F}$

$$f(x) + f(y) \geq f(x - \alpha(\mathbf{e}_i - \beta\mathbf{e}_j)) + f(y + \alpha(\mathbf{e}_i - \beta\mathbf{e}_j))$$



Characterization of S-convex function

Proposition

S-convex functions are supermodular.

Theorem (Conjugacy)

Under some regularity conditions, $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is S-convex if and only if f^* is convex and submodular.

Examples of S-convex functions

- All $\mathrm{M}^\natural$-convex function
- $f(c_1 x_1, ..., c_n x_n)$, where f is $\mathrm{M}^\natural$-convex, $c \geq 0$

  E.g., $f(c^T x)$, f is a univariate convex function
- Two-dimensional jointly convex and supermodular function

Examples of $\mathrm{M}^\natural$-convex functions

Definition 6 (Murota (2003))

A function $f : \mathcal{F}^n \to \mathbb{R} \cup \{+\infty\}$ is called a **laminar convex function** if it can be represented as

$$f(x) = \sum_{S \in \mathcal{L}} f_S(\sum_{i \in S} x_i).$$

f_S: univariate convex functions

$\mathcal{L} \subseteq 2^{[n]}$: $\forall\ A, B \in \mathcal{L} \implies A \cap B = \emptyset$ or $A \subseteq B$ or $B \subseteq A$.

Example:

- $f(x_1, x_2, x_3) = f_1(x_1) + f_2(x_2) + f_3(x_3) + \delta_{x_1+x_2+x_3 \leq 1}(x_1, x_2, x_3)$, $\mathcal{L} = \{\{1\}, \{2\}, \{3\}, \{1, 2, 3\}\}$

Outline

Key result I

Theorem 7

If $f : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ satisfies some regularity conditions, then

f *is S-concave* $\iff$ f *satisfies GS*

Key result II

$$
\begin{aligned}
g(x) = \max_{z} \quad & G(x, z) \\
\text{s.t.} \quad & (x, z) \in S
\end{aligned}
$$

Theorem 8

- *If G is S-convex on S, then $z^*(x)$ is decreasing in x.*
- *$g(x)$ preserves S-convexity.*

Theorem 9

- *If G is $\mathrm{M}^\natural$-convex on S, then $z^*(x)$ is decreasing in x and $\mathbf{e}^T x^T + \mathbf{e}^T z^*(x)$ is increasing in x.*
- *$g(x)$ preserves $\mathrm{M}^\natural$-convexity.*

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Necessity of S-convexity

Theorem 10

Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ satisfy some regularity conditions. If for any separable convex function $w(x)$, the optimal solution of

$$\min_x f(x+t) + w(x)$$

is decreasing in t, then f is S-convex.

Comparison with lattice programming

$$\min_{z:(x,z)\in S} G(x,z)$$

Our results:

- minimizing a **(S-)** $\mathrm{M}^{\natural}$**-convex (supermodular)** function
- can be a **box** constraint or **capacity** constraint or some more general constraints
- **decreasing** optimal solutions
- preserve supermodularity

Results in lattice programming:

- minimizing a **submodular** function
- **sublattice** constraint
- **increasing** optimal solutions
- preserve submodularity

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- $Y = \{y \in \mathbb{R}^n : \mathbf{e}^T y \leq C,\ y \geq 0\}$
- $G(y) = \mathbb{E}_D[c^T y + h_+^T (y - D)^+ + h_-^T (D - y)^+]$
- $\tilde{G}(y) := G(y) + \delta_Y(y)$

$$\min_z \tilde{G}(x + z),\ \text{s.t. } z \geq 0$$

$\tilde{G}(y)$ is laminar convex
$\tilde{G}(z + x)$ is $\mathrm{M}^\natural$-convex in (z, x)
$z^*(x)$ is decreasing in x

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$\tilde{G}(z+x)$ is $\mathrm{M}^\natural$-convex in (z,x)
$z^*(x)$ is decreasing in x

A general multi-product inventory model

- A company sells n products during a selling season
- x: the initial inventory
- $y \in Y$: the order-up-to level. (e.g., $Y = \{y \in \mathbb{R}^n : \mathbf{e}^T y \leq C\}$)
- linear ordering cost $c^T(y - x)$
- m demand classes, $D_i = (D_{i1}, ..., D_{in})$ random demand vector of demand class i, $D = (D_1, ..., D_m)$.
- $h(y, D)$: inventory handling cost

Sufficient condition of decreasing ordering quantity

$$\min G(y) = \mathbb{E}_D[c^T y + h(y, D)], \text{ s.t. } y \geq x,\ y \in Y$$

Theorem 11

If $G(y)$ is ($\mathrm{M}^\natural$) S-convex on Y, then $y^*(x) - x$ is decreasing in x (and $\mathbf{e}^T y^*(x)$ is increasing in x).

Remark: Ignall and Veinott (1969) prove the special case of G being laminar convex on Y.

"The proof is constructive. It consists in applying Fulkersons [2] out-of-kilter algorithm to a sequence of approximating linear minimum cost network flow problems in circulation form....Also since the proof is rather long, we shall omit many tedious details"
(Ignall and Veinott 1969, p. 296)

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Takeaways

S-convexity is a proper concept for substitutes

- S-concavity is equivalent to GS under some regularity conditions
- S-convexity implies decreasing optimal solutions in parametric optimization models, and it is necessary in some cases

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