Inverse isotonicity for equilibrium problems

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Outline of the talk

1. Motivation
2. Unified gross substitutes and the inverse isotonicity theorem
3. Applications

Section 1

Motivation:

inverse isotonicity for equilibrium problems

Research question

- Consider a simple application of monotone comparative statics results.
- Consider a firm with a production function $f : \mathbb{R}^N \to \mathbb{R}$. The output price is p and the input price vector is r. The firm produces

$$\mathtt{Q}\left(p, r\right) = \arg\max_{q \in \mathbb{R}^N} pf\left(q\right) - r^\top q$$

  - Assume f is non-decreasing and supermodular.
  - Therefore $pf\left(q\right) - r^\top q$ has increasing differences in $\left(q, \left(p, -r\right)\right)$.
  - Topkis theorem: if f has increasing differences in $\left(x, \theta\right)$, and L is a lattice, then $x^*(\theta) = \arg\max_{x \in L} f\left(x, \theta\right)$ is isotone in θ (in Veinott strong set order).
  - Topkis theorem applies and $\mathtt{Q}$ is isotone: if the price of the firm's output increases and/or the price of any of its inputs decreases, then the firm increases the usage of all of its inputs (law of supply).
- *Research question*: what if $\mathtt{Q}$ is not defined as an optimization problem (e.g. general equilibrum, matching with ITU, ...)?

Objectives and results

- In optimization problems as in the previous example, we are interested in the isotonicity of the supply correspondence $\mathtt{Q} : p \mapsto \mathtt{Q}(p)$ (because in producer theory, prices are exogenously set by the market and we study how the firm's output reacts).
- However in an equilibrium problem, endowment is usually exogenous and we study equilibrium prices.
- Therefore in this paper we study the conditions to get the *inverse isotonicity* of $\mathtt{Q}$ i.e. "how the set of equilibrium prices changes when the initial endowment changes".
- Results:
  - We introduce an inverse isotonicity result for equilibrium problems.
  - We introduce a new notion of gross substituability for correspondences and discuss its relation with existing definitions.
  - We discuss applications and introduce a new class of problems - equilibrium flow problems - that nest several classical economic problems.
  - What we don't do: show existence (we didn't assume any form of continuity as we were interested in minimal conditions to reach our conclusions).

Section 2

Unified gross substitutes and the inverse isotonicity theorem

Nonreversingness

Let $P \subseteq \mathbb{R}^N$ be a sublattice, $Q \subseteq \mathbb{R}^N$, $\mathtt{Q} : P \rightrightarrows Q$ be a supply correspondence.

Definition

$\mathtt{Q} : P \rightrightarrows Q$ is *nonreversing* if

$$\left\{ \begin{array}{c} q \in \mathtt{Q}(p) \\ q' \in \mathtt{Q}(p') \\ q \leq q' \\ p \geq p' \end{array} \right. \implies \left\{ \begin{array}{c} q \in \mathtt{Q}(p') \\ q' \in \mathtt{Q}(p) \end{array} \right.$$

Properties

- *Constant aggregate output [there exists $l \in \mathbb{R}^N_{++}$ such that $\sum_{z=1}^N l_z q_z = 0$ holds for all $p \in P$ and $q \in \mathtt{Q}(p)$] $\implies$ nonreversingness.*
- *Monotone total output [for $q \in \mathtt{Q}(p)$ and $q' \in \mathtt{Q}(p')$, $p \geq p'$ implies $\sum_{z=1}^N q_z \geq \sum_{z=1}^N q'_z$] $\implies$ nonreversingness.*
- *Aggregate monotonicity [for $q \in \mathtt{Q}(p)$ and $q' \in \mathtt{Q}(p')$, both $p \geq p'$ and $q < q'$ cannot hold simultaneously] $\implies$ nonreversingness.*
- *Walras law $\implies$ nonreversingness of the supply correspondence measured in monetary terms [$\mathtt{Q}^{\$}(p) = \left\{ (p_z q_z)_{z \in \{1,\ldots,N\}} \text{ with } q \in \mathtt{Q}(p) \right\}$].*

Definition

$\mathtt{Q} : P \rightrightarrows Q$ satisfies *unified gross substitutes* if: given $p \in P, p' \in P$, $q \in \mathtt{Q}(p)$ and $q' \in \mathtt{Q}(p')$, there exists $q^{\wedge} \in \mathtt{Q}(p \wedge p')$ and $q^{\vee} \in \mathtt{Q}(p \vee p')$ such that

$$\begin{cases} p_z \le p'_z & \Longrightarrow \quad q_z \le q_z^{\wedge} \text{ and } q_z^{\vee} \le q'_z \\ p'_z < p_z & \Longrightarrow \quad q'_z \le q_z^{\wedge} \text{ and } q_z^{\vee} \le q_z \end{cases}$$

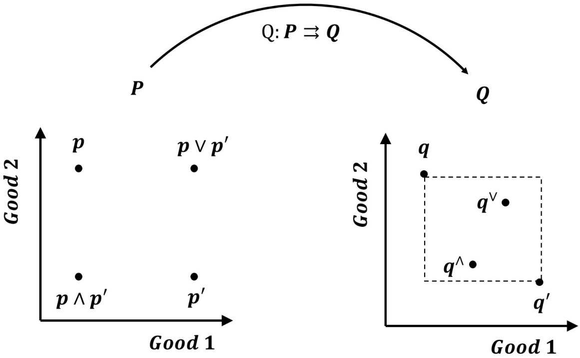

Figure: Illustration of the UGS property

The inverse isotonicity theorem

Definitions

- An M0-correspondence is a correspondence which satisfies unified gross substitutes and is nonreversing.

	$\mathtt{Q}^{-1}$ is point-valued	$\mathtt{Q}^{-1}$ is set-valued
$\mathtt{Q}$ is point-valued	$\mathtt{Q}$ is an M-function	$\mathtt{Q}$ is an M0-function
$\mathtt{Q}$ is set-valued	$\mathtt{Q}$ is an M-correspondence	$\mathtt{Q}$ is an M0-correspondence

- Given the correspondence $\mathtt{Q} : P \rightrightarrows Q$, the inverse correspondence $\mathtt{Q}^{-1}$ mapping from the image set of $\mathtt{Q}$ to P is Veinott isotone if, whenever $q \in \mathtt{Q}(p)$ and $q' \in \mathtt{Q}(p')$ are such that there exists $B \subseteq \mathcal{Z}$ with $p_z \leq p'_z$ for all $z \in B$ and $q_z \leq q'_z$ for all $z \in \mathcal{Z} \backslash B$, we have $q \in \mathtt{Q}(p \wedge p')$ and $q' \in \mathtt{Q}(p \vee p')$.

Theorem

Let $\mathtt{Q} : P \rightrightarrows Q$ satisfy unified gross substitutes, then:
$\mathtt{Q}$ is nonreversing (i.e., $\mathtt{Q}$ is a M0-correspondence) $\Longleftrightarrow$ $\mathtt{Q}^{-1}$ is Veinott isotone.

Corollary: Let $\mathtt{Q}$ be an M0-correspondence. Then the set of prices $\mathtt{Q}^{-1}(q)$ associated with an allocation q is a sublattice of P.

Section 3

Applications

- Consider a network $(\mathcal{Z}, \mathcal{A})$ where $\mathcal{Z}$ is a finite set of nodes and $\mathcal{A} \subseteq \mathcal{X} \times \mathcal{X}$ is the set of directed arcs.
- The arc-node incidence matrix ∇ is defined by for $xy \in \mathcal{A}$ and $z \in Z$, $\nabla_{xy,z} = 1_{\{z=y\}} - 1_{\{z=x\}}$.
- The connection function G_{xy} is such that
  - if $p_x > G_{xy}(p_y)$, the purchase price at node x is excessive, and the trader will not engage in the trade.
  - On the contrary, if $p_x < G_{xy}(p_y)$, positive profit can be made from the trade on the arc xy.

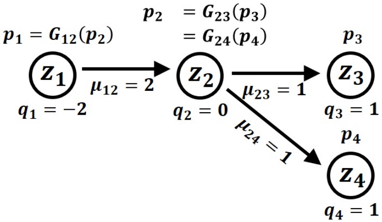

Figure: Example of equilibrium flow

Definition

The triple $(q, \mu, p) \in \mathbb{R}^{\mathcal{Z}} \times \mathbb{R}_{+}^{\mathcal{A}} \times \mathbb{R}^{\mathcal{Z}}$ is an equilibrium flow outcome when the following conditions are met:

1. $\nabla^{\intercal} \mu = q$
2. $p_x \geq G_{xy}(p_y), \forall xy \in \mathcal{A}$
3. $\sum_{xy \in \mathcal{A}} \mu_{xy} (p_x - G_{xy}(p_y)) = 0$

The equilibrium flow correspondence is a correspondence $\mathtt{Q} : P \rightrightarrows \mathbb{R}^{\mathcal{Z}}$ such that for $p \in P$, $\mathtt{Q}(p)$ is the set of $q \in \mathbb{R}^{\mathcal{Z}}$ such that there is a flow μ such that (q, μ, p) is an equilibrium flow outcome.

Theorem

The equilibrium flow correspondence $\mathtt{Q} : \mathbb{R}^{\mathcal{Z}} \rightrightarrows \mathbb{R}^{\mathcal{Z}}$ satisfies unified gross substitutes. Therefore $\mathtt{Q}$ is Veinott inverse isotone and the set of equilibrium prices $\mathtt{Q}^{-1}(q^*)$ is a sublattice of $\mathbb{R}^{\mathcal{Z}}$.

TU and ITU matching

- Let $\mathcal{X}$ be a set of types of workers and $\mathcal{Y}$ a set of types of firms. There are n_x workers of each type $x \in \mathcal{X}$, and m_y firms of each type $y \in \mathcal{Y}$. Assume the total number of workers and firms is the same.
- A match between worker type x and firm type y is characterized by a wage w_{xy}, in which case it gives rise to the utilities $\mathcal{U}_{xy}(w_{xy})$ for the worker and $\mathcal{V}_{xy}(w_{xy})$ for the firm.
- To reformulate the problem as an equilibrium flow problem: let $\mathcal{Z} = \mathcal{X} \cup \mathcal{Y}$, fix the utility vector $u : \mathcal{Z} \to \mathbb{R}$ (resp v) define $p \in \mathbb{R}^{\mathcal{Z}}$ as $p_z = u_z 1_{\{z \in \mathcal{X}\}} - v_z 1_{\{z \in \mathcal{Y}\}}$, $q_z = -n_z 1_{\{z \in \mathcal{X}\}} + m_z 1_{\{z \in \mathcal{Y}\}}$, and $G_{xy}(p_y) = \mathcal{U}_{xy} \circ \mathcal{V}_{xy}^{-1}(-p_y)$.

Theorem

The correspondence that associates the vector of payoffs $(u, -v)$ (up to a change of sign) to the vector of populations $(-n, m)$ (again, upon a change of sign) is a M0-correspondence.

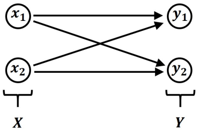

Figure: Reformulation of TU matching as an equilibrium flow problem

Hedonic pricing (1/2)

- $\mathcal{X}$ is the set of types of producers and $\mathcal{Y}$ the set of types of consumers. There are n_x producers of each type $x \in \mathcal{X}$ and m_y consumers of each type $y \in \mathcal{Y}$.
- There is a finite set $\mathcal{W}$ of qualities, also sometimes referred to as contracts or characteristics. Each producer must choose to produce one of the qualities in $\mathcal{W}$, or to remain inactive. Each consumer must choose to consume one quality in $\mathcal{W}$ or remain inactive.
- Let $p : \mathcal{W} \to \mathbb{R}$ be a price vector assigning prices to qualities, with p_w denoting the price of quality w. A producer of type x who produces a quality w that bears price p_w earns the profit $\pi_{xw}(p_w)$. A consumer of type y who consumes a quality w bearing price p_w earns surplus $s_{yw}(p_w)$.

- To reformulate the problem as an equilibrium flow problem: consider $\mathcal{Z} = \mathcal{X} \cup \mathcal{Y} \cup \mathcal{W}$ and $\mathcal{Z}_0 = \mathcal{Z} \cup \{0\}$ where 0 is an additional node. Denote as well $\mathcal{W}_0 = \mathcal{W} \cup \{0\}$. The set of arcs $\mathcal{A}$ is given by $\mathcal{A} = (\mathcal{X} \times \mathcal{W}_0) \cup (\mathcal{W}_0 \times \mathcal{Y})$.
- Define prices, stocks and connection functions

$$\begin{cases} \text{if} \quad x \in \mathcal{X} & p_x = u_x, \quad q_x = -n_x \\ \text{if} \quad w \in \mathcal{W} & p_w = p_w, \quad q_w = 0 \\ \text{if} \quad y \in \mathcal{Y} & p_y = -v_y, \quad q_y = m_y \\ & p_0 = 0, \quad q_0 = \sum n_x - \sum m_y \end{cases}, \quad \begin{cases} G_{xw}(p_w) = \pi_{xw}(p_w) \\ G_{x0}(p_0) = p_0 \\ G_{wy}(p_y) = s_{yw}^{-1}(-p_y) \\ G_{0y}(p_y) = p_y \end{cases}$$

Theorem

The correspondence that associates the vector $(u, p, -v)$ to the set of vectors $(-n, m)$ such that the allocation $\mu \geq 0$ is an hedonic pricing equilibrium is a M0-correspondence.

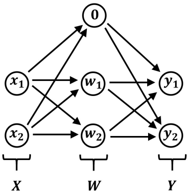

Figure: Reformulation of hedonic pricing as an equilibrium flow problem

Appendix

Additional results

Link with other notions of gross substitutes

Properties

- *Subdifferentials of convex submodular functions* [$\partial c^*(p) = \left\{q : p' \to \left[q^\top p' - c(p')\right] \text{ is maximal at } p\right\}$] *satisfy unified gross substitutes.*
- *Monetary measurement* [$\mathtt{Q}^{\$}(p) = \left\{(p_z q_z)_{z \in \{1,\ldots,N\}} \text{ with } q \in \mathtt{Q}(p)\right\}$] *preserves unified gross substitutes.*
- *Aggregation preserves unified gross substitutes.*

Properties

q *satisfies weak gross substitutes* $\iff \mathtt{q}_i(p)$ *is nonincreasing in* p_j *for* $i \neq j$.

- *For functions, unified gross substitutes and weak gross substitutes are equivalent.*

$\mathtt{Q}$ *satisfies Kelso-Crawford gross substitutes* $\iff$ *given* $p' \leq p$, *for any* $q \in \mathtt{Q}(p)$ *there exists* $q' \in \mathtt{Q}(p')$ *such that* $p_z = p'_z \implies q'_z \geq q_z$.

- *Unified gross substitutes implies Kelso-Crawford gross substitutes.*

$\mathtt{Q}$ *satisfies Polterovich and Spivak's gross substitutes* $\iff$ *for any price vectors* $p \leq p'$ *and any* $q \in \mathtt{Q}(p)$ *and* $q' \in \mathtt{Q}(p')$, *it is not the case that* $q'_z > q_z \; \forall z$ *s.t.* $p_z = p'_z$.

- *Polterovich and Spivak's Gross Substitutes does not imply UGS, nor the converse.*

Profit maximization

- A competitive multiproduct firm faces output price vector $p \in \mathbb{R}^N$ and convex cost function $c : \mathbb{R}^N \to \mathbb{R}$, the set of optimal production vectors $\mathtt{Q}(p)$ is

$$\mathtt{Q}(p) = \arg\max_{q \in \mathbb{R}^N} \left\{ p^\top q - c(q) \right\}$$

- Given the convexity of the cost function c, Shephard's lemma gives $\mathtt{Q}(p) = \partial c^*(p)$, where $\partial c^*(p)$ is the subdifferential of the indirect profit function $c^*(p) = \max_{q \in \mathbb{R}^N} \left\{ p^T q - c(q) \right\}$.

Theorem

The following conditions are equivalent:

- *The indirect profit function c^* is submodular.*
- *The supply correspondence $\mathtt{Q}(p) = \partial c^*(p)$ satisfies unified gross substitutes.*
- *The supply correspondence $\mathtt{Q}(p) = \partial c^*(p)$ is an M0-correspondence.*