Efficient Allocation of Indivisible Goods in Pseudo Markets with Constraints

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Randomization and Existence

The task:

Finite set of indivisible goods

Finitely many agents

Allocate the goods to agents efficiently and fairly

Candidate for efficient + fair:

Walrasian equilibrium with equal endowments

Fiat Money

Rather than allocating the indivisible goods as endowments

We allocate fiat money

The designer accepts the money as if it has real value

Prices emerge competitively

Nonconvexities

Indivisibilities create nonconvexities

May lead to nonexistence of Walrasian equilibrium.

Convexifying by allowing trades in lotteries ensures the existence of Walrasian equilibrium in expectation.

But, it may be impossible to implement the equilibrium average consumptions as a probability distribution over allocations.

We call this the implementability problem

implementability problem

Example 1:

Two agents and three indivisible goods

Each agent has one unit of fiat money

Both agents have the following utility function:

$$u_i(A) = \begin{cases} 0 & \text{if } |A| < 2 \\ 2 & \text{if } |A| \geq 2 \end{cases}$$

Example 1 continued

The three goods are perfect substitutes

Standard arguments yield:

All three goods must have the same positive price.

In equilibrium, both agents will want to randomize between consuming 2 units and consuming 0 (.75/.25)

But then it impossible for aggregate consumption to equal 3, the aggregate supply.

Discrete Concavity

The utility function above features two types of nonconvexities

First, the consumption set is not convex

Also, marginal utility of an additional unit increases from 0 to 2

That is, there are complementarities

Our theorems show that the implementability problem always has a solution, and, therefore, Walrasian equilibria exist, if utilities satisfy the discrete analog of convexity called $M^\natural$ concavity.

Randomization and Efficiency

Example 2:

Two indivisible goods (a and b)

Three consumers ($1, 2$ and 3)

	a, b	a	b	$\emptyset$
1	11	10	8	0
2	11	10	2	0
3	11	10	2	0

All agents have the same budget (\$1)
Same ordinal ranking $\{a, b\} \succ \{a\} \succ \{b\} \succ \emptyset$

With equal budgets, here is no equilibrium without randomization

With nearly equal budgets: a to agent i and b to agent j. Choose i, j randomly payoffs: $(6, 4, 4)$

With randomization Walrasian equilibrium payoffs

$$W = \{(8 + 3x/5, 5 - x, 5 - x) \mid x \in [0, 1]\}$$

Equilibrium utility of consumer 1: 8 to 43/5

Equilibrium utilities of the other two consumers in $[4, 5]$.

Every Walrasian equilibrium with randomization

Pareto dominates deterministic eq. with random endowments.

The Pseudo Market

Finite set of agents $\{1, \ldots, N\}$;

Finite set of goods $H = \{1, \ldots, L\}$

Utility functions u_i is the utility function of agent i where

- $A \subset H$ is the set of discrete goods that i consumes
- $u_i : 2^H \to \boldsymbol{R}_+ \cup \{-\infty\}$,
- dom $u_i := \{A \,|\, u_i(A) > -\infty\}$ is the consumption set
- $A \subset B$ implies $u_i(A) \leq u(B)$ (monotone)

Walrasian Equilibrium: Deterministic and Random allocations

Deterministic Walrasian equilibrium is $\omega = (A_1, \ldots A_n)$, $p = (p^1, \ldots, p^L)$ such that

1. (Feasibility) $A_i \subset H$; $A_i \cap A_l \neq \emptyset$ implies $i = l$
2. (Aggregate Feasibility) $H = \bigcup_i A_i$
3. (Optimality) $A \in \mathcal{B}(b_i, p)$ and $u_i(A_i) \geq u_i(B)$ for all $B \in \mathcal{B}(b_i, p)$.

Walrasian Equilibrium with Randomization

A random consumption σ is a probability distribution over the set of goods:

$$\sigma : 2^H \to [0,1]$$

such that $\sum_{A \subset H} \sigma(A) = 1$

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A random allocation is a probability distribution over allocations:

$\alpha \in \Delta(\Omega) \subset \Delta(2^H)^n$

$\Omega = \{\text{Deterministic allocations}\}$ (set of all partitions of H)

Walrasian Equilibrium with Randomization

The pair (p, α) is a Walrasian equilibrium (with randomization) if

The marginals of $\alpha \in \Delta(\Omega)$ are optimal random consumptions at price p.

σ_i is optimal for i if $\sigma_i \in \mathcal{B}(p, b_i)$ and

$$\sum_A u_i(A)\sigma(A) \geq \sum_A u_i(A)\hat{\sigma}_i(A)$$

for all $\hat{\sigma} \in \mathcal{B}(p, b_i)$

where $\mathcal{B}(p, b_i) = \{\tilde{\sigma}_i \mid \sum_B \tilde{\sigma}_i(B)p(B) \leq b_i\}$ and $p(B) = \sum_{j \in B} p^j$

Feasibility

Agents choose optimal random consumptions
$\tau = (\sigma_1, \ldots, \sigma_n) \in (\Delta(2^H))^n$

Equilibrium must specify random allocations $\alpha \in \Delta(\Omega) \subset \Delta(2^H)^n$ such that $\alpha_i = \sigma_i$.

For $\tau = (\sigma_1, \ldots, \sigma_n) \in (\Delta(2^H))^n$ is there a random consumption $\alpha \in \Delta(\Omega)$ such that the i-th marginal $\alpha_i = \sigma_i$ for all i?

Adding up constraint: $\sum_i \sum_{A_i \ni j} \sigma(A_i) = 1$ for all j

Necessary but not sufficient. Recall the implementability problem

$M^\natural$ concavity

$A, B \in \operatorname{dom} u$, $j \in A \backslash B$ implies there is $D \subset B \backslash A$ such that $|D| \leq 1$ and

$$u((A \backslash \{j\}) \cup D) + u((B \backslash D) \cup \{j\}) \geq u(A) + u(B)$$

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$$\begin{array}{ccccc} & A & & B & \\ a_1 & a_2 & c & b_1 & b_2 \end{array}$$

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or

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$$u(a_1 b_2 c) + u(a_2 c\, b_1) \geq u(A) + u(B)$$

Examples of $M^\natural$-Concave Utilities

- unit demand preferences: $u(A) = \max_{j \in A} u(\{j\})$ (one-to-one matching)
- for any concave function f, let $u(A) = f(|A|)$
- H all rows of matrix; for $A \subset H$, $u(A)$ is the rank of A (maximal number of independent rows)
- H all edges of a complete undirected graph, for $A \subset H$ is $u(A)$ is the maximal cardinality of $B \subset A$ that contains no cycles

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Assign fiat money to all agents

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The Pseudo Market $\mathcal{E} = \{(u_i, b_i)_{i \in N}\}$ has fiat money, $b_i > 0$,

$M^\natural$-concave u_i's such that $\emptyset \in \text{dom } u_i$ for all i.

Strong Equilibrium

Random allocation α and prices p are a strong equilibrium if (α, p) is a Walrasian Equilibrium and α delivers, with probability 1, to each i a least expensive consumption among all her optimal consumptions

Fact: Every strong equilibrium is Pareto efficient; other Walrasian equilibria may be inefficient

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Theorem 1: Every pseudo market with $M^\natural$-concave utilities has a strong equilibrium.

How the proof works

Define: quasilinear utility function:

$$U_i(A, m) = u_i(A) + m$$

L indivisible and one divisible good, m; also numeraire

Demand: $D_{U_i}(p) = \{A \,|\, U_i(A) - p(A) \geq U_i(B) - p(B) \, \forall B\}$

Implicit assumption: everyone has plenty of m.

Seller/Designer owns all indivisible goods initially

Step 1

Substitutes condition:

$D_{U_i}(p) = \{j\}$, $D_{U_i}(\hat{p}) = \{l\}$, $\hat{p}^k = p^k$ for all $k \neq j, j^*$
and $\hat{p}^{j^*} \geq p^{j^*}$ implies $j = l$.

An economy of agents with quasilinear substitutes utility functions is a KC (Kelso-Crawford) economy.

Kelso-Crawford prove that every KC economy has a Walrasian equilibrium

First Welfare Theorem holds.

Any efficient (surplus maxizing) allocation (of divisible goods) and any Walrasian price constitute a Walrasian equilibrium (Exchangeability)

The Constrained KS Economy

Quasilinear economy with budget constraints:

Each agent has a limited supply of the divisible good, b_i.

Then, demand is all maximizers of U_i within the set

$\mathcal{B}(p, b) = \{\sigma \mid \sum_B \sigma(B)p(B) \leq b\}$

Economy quasilinear economy with substitutes preferences and budget constraints is a constrained Kelso-Crawford economy CKC.

Step 2

Lemma 1: Every CKC economy has a Walrasian equilibrium
To prove the lemma, for each i choose $\lambda_i \in [0, 1]$
and replace every U_i with

$$\hat{U}_i(A_i, p) = \lambda_i u_i(A) - p(A_i)$$

Ignore constraints, find equilibrium for the transferable utility economy such that

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fixed-point argument to find the λ_i's

Proof uses exchangeability.

Step 3

Fujishige and Yang (2003) show that $U_i = u_i + m$ satisfies substitutes if and only if u_i is $M^\natural$-concave.

Step 4

Take a limit: let the value of the divisible good go to zero so that it becomes fiat money:

Define the CSC economy $\mathcal{E}_n = \{(nu_i, b_i)_{i \in N}\}$

all u_i's have been multiplied by n

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The limit of that subsequence is a strong equilibrium of the pseudo market.

$M^\natural$ Concavity Preserving Operations

for $M^\natural$-concave v, w

endowment: $u(A) := v(A \cup B) - v(B)$

restriction: $u(A) := v(A \cap B)$

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satiation: $u(A) := \max_{B \subset A : |B| \leq k} v(B)$ for $k \geq 0$.

lower bound: $u(A) := \max_{B \subset A : |B| \geq k} v(B)$ for $k \geq 0$ and
$:= -\infty$ if $|A| < k$.

u is $M^\natural$-concave.

Individual and Aggregate Constraints

In many market design problems,
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(a) two versions of an introductory physics course are to be offered: Phy 101 without calculus; Phy 102 with calculus.
Phy 101, 102 can have at most 60 students each but lab resources limit the total enrollment two courses ≤ 90

Constraints as $M^\natural$ Concavity Preserving Operations

Simplest individual constraints:

Bounds on the number of goods an agent may consume from a given set.

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Student is required to take 4 classes each semester, is barred from enrolling in more than 6.

$$u_4^6(A) = \max_{B \subset A, |B| \leq 6} u_4(B)$$

where

$$u_4(A) = \begin{cases} u(A) & \text{if } |A| \geq 4 \\ -\infty & \text{if } |A| < 4 \end{cases}$$

We can impose multiple constraints even hierarchies of constraints provided constraints and preferences line-up nicely

Imposing Multiple Individual Constraints Simultaneously

a Module of u is a set B such that

$$u(A) = u(A \cap B) + u(A \cap B^c)$$

B is a module if u is separable across B and B^c.

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$C = \{(A_k, l_k, h_k)_{k=1}^K\}$ is a Hierarchical Constraint if $\mathcal{H} = \{A_k\}_{k=1}^K$ is a hierarchy, $l_k, h_k > 0$ are integers for every $k = 1, ..., K$, and the set of plans that satisfy the constraint is non-empty:

$$\mathcal{A}_C = \{A \subset H | \, l_k \leq |A_k \cap A| \leq h_k \text{ for all } k\} \neq \emptyset$$

$C = \{(A_k, l_k, h_k)_{k=1}^K\}$ is Modular if each A_k is a module of u

C-Constrained Utilities

For any u, define $u_C(\cdot)$:

$$u_C(A) = \begin{cases} \max_{\substack{B \in \mathcal{A}_C \\ B \subset A}} u(B) & \text{if } \mathcal{A}_C \cap \{B^* \mid B^* \subset A\} \neq \emptyset \\ -\infty & \text{otherwise} \end{cases}$$

Aggregate Constraints and Production

Aggregate constraints only impose upper bounds.

$$F = \{(A_k, h_k)_{k=1}^K\}$$

for $h_k > 0$,

$\{A_k\}_{k=1}^K$ is a hierarchy.

$$\mathcal{H}_F = \{A \subset H \,|\, |A \cap A_k| \leq h_k \text{ for all } k\}$$

$\mathcal{H}_F$ is the production set; it is the set of feasible plans.

Random Allocation in the Constrained Pseudo Market

$\alpha \in \Delta(H^N \times H)$ is a random allocation iff

whenever $\alpha(A_1, \ldots, A_{N+1}) > 0$

- $A_i \cap A_k \neq \emptyset$ implies $i = k$
- $\bigcup_{i \leq N} A_i = A_{N+1}$

Feasibility Condition

There are pairwise disjoint $A_1, \cdots, A_{N+1}$ such that $A_i \in \mathrm{dom}\ u_k$ for all $k, i \leq N$ and $\sum_{i=1}^{N+1} A_i \in \mathcal{H}_F$.

Feasibility Condition

There are pairwise disjoint $A_1, \cdots, A_{N+1}$ such that $A_i \in \text{dom } u_k$ for all $k, i \leq N$ and $\sum_{i=1}^{N+1} A_i \in \mathcal{H}_F$.

Theorem 2: Every constrained economy with $M^\natural$-concave utilities and equal endowments that satisfies the feasibility condition has a strong equilibrium.

How the Proof works

Step 1: Show that if C is a hierarchical constraint and is modular for u, then u_C is $M^\natural$-concave.

Step 2

Replace production with agent 0 to get an $n+1$ person exchange economy with aggregate endowment H (all possible goods)

Give agent 0 a lot of fiat money and

$$u(A) = \begin{cases} 1 & \text{if } H \backslash A \in \mathcal{H}_F \\ 0 & \text{otherwise} \end{cases}$$

Show that equilibrium of $n+1$ person exchange economy with aggregate endowment H is an equilibrium of the original production economy

Related Literature

Shapley and Shubik (1971) unit demand quasilinear economy show that quilibrium exists, is efficient and equal to the core

Hylland and Zeckhauser (1979) unit demand pseudo market. show some equilibria are inefficient, efficient equilibria exist

Kelso and Crawford (1982) introduce substitutes economy, prove existence and efficiency of equilibrium

Budish, Che, Kojima and Milgrom (2013) pseudo market mechanism smaller class of preferences, no aggregate constraints, no individual constraints on minimal consumption

Nguyen and Vohra (2022) Generalizes Kelso and Crawford and our Theorem 1 and offers an alternative to Theorem 2.