# Lindahl Equilibrium as a Collective Choice Rule[†]

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**Abstract**

A collective choice problem specifies a finite set of alternatives from which a group of expected utility maximizers must choose. We associate a public goods economy with every collective choice problem and establish the existence and efficiency of Lindahl equilibrium allocations for that economy. We also associate a cooperative bargaining problem with every collective choice problem and define a set-valued solution concept, the *weighted Nash bargaining set*. We provide axioms that characterize the weighted Nash bargaining set. Our main result shows that weighted Nash bargaining set payoffs with welfare weights $\omega$ are also the Lindahl equilibrium payoffs of the corresponding economy with the same utility functions and incomes $\omega$. Finally, we consider a general class of matching problems and show that the set of Lindahl equilibrium payoffs and the set of Walrasian equilibrium payoffs is the same. More generally, we show that in any discrete-goods economy, the set of Walrasian equilibrium allocations is a subset set of the set of Lindahl equilibrium allocations.

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## 1. Introduction

An organization must decide among several alternatives that affect the welfare of its members. The goal is to find an equitable and efficient solution to the problem when no transfers among members are possible. Examples include a community's decision on how to allocate infrastructure investments among neighborhoods, the allocation of office space among groups within an organization, or the assignment of college roommates.

In this paper, we analyze the following mechanism: each group member is given a budget of fiat money and confronts a price for each of the relevant alternatives under consideration. As in standard consumer theory, members choose an alternative that maximizes their utility subject to the budget constraint. The organization acts as an auctioneer and implements an alternative that maximizes revenue. We allow choices to be stochastic; that is, agents choose lotteries over social outcomes. We also allow the prices for the various social outcomes to be individual-specific. Thus, our mechanism is analogous to a Lindahl mechanism (Lindahl (1919)) in a public goods setting. It differs from the standard Lindahl mechanism in that the objects of choice are social outcomes (allocations) rather than private or public goods. We refer to this mechanism as a *collective choice market* and to its equilibria as Lindahl equilibria.

As an example, consider an organization that must reallocate office space to its members. Utilities depend not only on members' own offices but also on who occupies nearby offices. Thus, utilities depend on the office *allocation*. The organization must choose among several plans to allocate offices. In the collective choice market, every member, $i$, has $\omega_i$ units of fiat money and faces price $p_i^j$ for plan $j$; she must choose an optimal lottery over plans subject to her budget constraint. In equilibrium, the organization chooses a lottery that maximizes revenue and that choice must coincide with every member's optimal choice.

The organization's decision problem can also be described as an $n$-person bargaining problem in which the attainable utility profiles correspond to the members' utilities for lotteries over outcomes. We define and characterize a set-valued and weighted version of the Nash bargaining solution for $n$-person bargaining problems called the weighted-Nash bargaining set (wNBS).[1]

[1] The wNBS is a multivalued solution concept even for a fixed set of weights.

The simplest bargaining problem is one of pure conflict where a single prize must be allocated via a lottery. In this situation, the weighted Nash bargaining set has a unique outcome that assigns the prize to agent $i$ with a probability proportional to their bargaining weight $\omega_i$. We can interpret the pure-conflict case as a benchmark that reveals the organization's attitude to equity among its members. If every agent has the same bargaining weight, then the organization desires to treat each agent equally, while for other weights the organization favors some members over others. For example, if the agents represent neighborhoods of a city, the bargaining weight might be proportional to the neighborhood's population. For bargaining games that are not of the pure-conflict form, the wNBS specifies all the outcomes that can be reconciled with the organization's normative judgment in the benchmark case.

Our main result (Theorem 3) relates Lindahl equilibria to the wNBS. It shows that the set of equilibrium outcomes of a collective choice market with fiat money endowments $(\omega_i)_{i=1}^n$ is the same as the wNBS with weights $(\omega_i)_{i=1}^n$ in the associated bargaining game. By relating Lindahl equilibria to wNBS outcomes, Theorem 3 clarifies the organization's normative judgment implied by Lindahl equilibrium: each person's budget share reflects the probability with which this person would be assigned the prize in a situation of pure conflict.

We provide two axiomatic foundations for the wNBS. The first, Theorem 1, is similar to the characterization of the Nash bargaining solution except that we allow the solution to be set-valued. As in Nash's theorem (Nash (1950)), the substantive axioms are efficiency and a version of IIA. In the second, we introduce a new axiom, *implementability*, which asserts that $x \in B$ is a solution if and only if it is the unique solution of a bargaining problem $A$ that dominates $B$ in the weak set order. The bargaining problem $A$ dominates $B$ in the weak set order[2] if there is an onto function, $g : B \to A$ that maps every $x \in B$ to a better outcome in $A$; that is, $x \leq g(x)$ for all $x$. Theorem 2 shows that scale invariance, efficiency, consistency (i.e., our version of IIA) and implementability yield the wNBS.

The rationale for implementability is as follows: consider the bargaining game $B$ and suppose that a mediator can credibly promise to replace any outcome $x \in B$ with a

[2] Our formal definition follows the literature: $A$ dominates $B$ in the weak set order if for every utility profile $x \in A$ there is a utility profile $y \in B$ such that $x \leq y$ and, conversely, for every utility profile $x \in B$ there is a utility profile $y \in A$ with $y \leq x$. The definition given in the text above is equivalent.

weakly better outcome, $g(x)$. Such promises transform the bargaining problem from $B$ to $A = g(B)$. Furthermore, suppose that an outcome $g(x) \in A$ such that $x = g(x)$ is the obvious choice from $A$. Then, the mediator can implement $x$ *without incurring any cost* simply by restating it as a better problem in which the choice, $x$, is obvious.

### 1.1 Example

An organization with three employees must choose between two new locations for its office space. The offices in the current location, the status quo, yield the utility profile $x^s = (0,0,0)$. If location $a$ is chosen, utilities will be $x^a = (1,0,1/2)$ and if location $b$ is chosen, utilities will be $x^b = (0,1,1)$. Thus, agents 1 and 2 benefit only if their preferred locations – $a$ for agent 1 , $b$ for agent 2 – are chosen while agent 3 benefits in both cases but prefers location $b$. The organization weighs the welfare of its members equally.

In this example, the convex hull of $x^s, x^a$, and $x^b$ comprises the bargaining problem and its Pareto frontier consists of the line segment connecting $x^a$ and $x^b$. The *symmetric* Nash bargaining set, i.e., the weighted Nash bargaining set with equal weights, corresponds to the subset of that segment in which $a$ is chosen with probability $\alpha \in [1/3, 1 - 1/\sqrt{3}]$. One extreme point, $\alpha = 1 - 1/\sqrt{3} \approx .42$, is the standard Nash bargaining solution and yields the highest utility for agent 1. The other extreme point, $\alpha = 1/3$, yields the highest utility for players 2 and 3.

Every allocation $\alpha \in [1/3, 1 - 1/\sqrt{3}]$ can be implemented. To render a particular allocation in this interval the unique choice, a mediator can improve unchosen alternatives or introduce new alternatives that will not be chosen. For example, $\alpha = 1 - \sqrt{3}$ emerges as the unique solution if the mediator promises to introduce a new location (location $c$) with payoffs $x^c = (3 - \sqrt{3}, 0, 0)$, if needed. Location $c$ will never be chosen but it constrains the Nash-bargaining set to contain only the desired solution.[3] Similarly, to implement $\alpha = 1/3$, the mediator can improve the status quo utility of agents 3 from 0 to 1/2. Following this improvement, $\alpha = 1/3$ is the unique solution.[4] Again, the mediator's intervention is

[3] In a symmetric solution with $n$ agents, each agent must receive at least $1/n$ times their bliss point. Alternative $c$, therefore, constrains agent 1's utility to be at least the target level.

[4] The improved status quo point means that agents 2 and 3 have identical utilities (up to re-scaling) and the bargaining game is one of pure conflict between agent 1 on one side and agents 2 and 3 on the other. The unique solution is to give the "prize" to agent 1 with probability 1/3.

costless because the status quo will not be chosen. By contrast, utility profiles outside the indicated range cannot be implemented without expending resources.

Theorem 3 shows that each utility profile in the weighted Nash bargaining set corresponds to a Lindahl equilibrium payoff when budgets are proportional to the agents' bargaining weights. In the example above, each agent has the same weight and, therefore, we can obtain each of the bargaining solutions as a Lindahl equilibrium with equal budgets. Let $\omega_i = 1$ for all $i$ and let $\alpha \in [1/3, 1 - 1/\sqrt{3}]$. Then, $(p, \alpha)$ is a Lindahl equilibrium for $p$ such that:

$$p_1 = \left(\frac{1}{\alpha}, 0\right)$$
$$p_2 = \left(0, \frac{1}{1-\alpha}\right)$$
$$p_3 = \left(\frac{3\alpha - 1}{\alpha}, \frac{2 - 3\alpha}{1 - \alpha}\right)$$

where $p_i$ represents agent $i$'s prices for the two alternatives. It is straightforward to verify that the lottery $(\alpha, 1 - \alpha)$ maximizes utility for each of the three agents at those prices. The range of possible outcomes is restricted by two constraints: first, prices must be non-negative and, therefore, $\alpha \geq 1/3$. Second, agent 3 must prefer the lottery $(\alpha, 1 - \alpha)$ over a lottery that yields the status quo and outcome $b$. This second constraint implies that $\alpha \leq 1 - 1/\sqrt{3}$.

### 1.2 Lindahl vs Walras

The final section of the paper discusses the relationship between Lindahl equilibria and Walrasian equilibria in the context of two applications: a matching market and a discrete goods exchange economy. The matching market involves a group of individuals who must choose partners. Examples include the roommate problem or the classic two-sided matching problem (Gale and Shapley (1962)). We show that in matching problems without transfers, the Lindahl equilibrium outcomes of a collective choice market coincide with Walrasian equilibrium outcomes of the market for partners (Theorem 4). In the collective choice market, consumers express their demands for social outcomes (that is, allocations) while in a Walrasian equilibrium, consumers express their demand for potential partners. In both cases, utility is non-transferable and consumers only have fiat money. Theorem

3 and Theorem 4 together imply that the Walrasian equilibria of matching markets with non-transferable utility correspond to wNBS payoffs with weights proportional to endowments.

We also consider general discrete-good exchange economies. We show that every Walrasian equilibrium of such an economy is a Lindahl equilibrium of the corresponding collective choice market. However, the set of Walrasian equilibria is sensitive to the exact specification of the traded goods (that is, how property rights are defined) while Lindahl equilibria are not. More precisely, the set of Lindahl equilibrium payoffs for two collective choice markets is the same whenever the two choice problems lead to the same bargaining set (i.e., the same set of feasible utility vectors). In contrast, as Sertel and Yildiz (2003) show in a different context, the set of Walrasian equilibrium payoffs may be different for two different exchange economies that yield the same bargaining set. Since every Walrasian equilibrium outcome is a Lindahl equilibrium, our main result implies that all Walrasian equilibria in an economy with fiat money endowments $(\omega_1, \ldots, \omega_n)$ yields wNBS payoffs with welfare weights equal to the money endowments.

### 1.3 Related Literature

Our paper is related to the extensive literature on axiomatic bargaining theory (see Thomson (1994) for a survey). Theorem 1 is related to Nash (1950) and we discuss this relationship in detail below. For 2-person bargaining, the symmetric NBS includes the Kalai-Smorodinsky solution (Kalai and Smorodinsky (1975)) and the Perles-Maschler solution (Perles and Maschler (1981)).

Hylland and Zeckhauser (1979) were the first to propose Walrasian equilibria with randomization as solutions to allocation problems in situations with indivisibilities and without transfers. Gul, Pesendorfer and Zhang (2020) extend Hylland and Zeckhauser from unit demand preferences to general gross-substitutes preferences. Collective choice markets allow for arbitrary preferences, public goods and externalities and hence provide a further generalization of the environment considered in these papers.

Eisenberg and Gale (1959) show that competitive equilibria of a Fisher market with equal budgets yield the Nash bargaining solution. Fain, Goel and Munagala (2016) consider an economy in which the planner must allocate a fixed budget among a finite collection of public goods with linear costs and utilities. They show that the Nash bargaining solution yields a Lindahl equilibrium allocation of an economy in which each agent controls an equal share of the budget. We neither assume equal budgets nor linear costs.[5] Without linear costs, typically there are multiple Lindahl equilibria even with equal budgets. Nonetheless, we show that each Lindahl equilibrium implements a suitable set-valued generalization of the Nash bargaining solution. Brandl, Brandt, Peters, Stricker and Suksompong (2020) examine the incentive properties of the Nash bargaining solution and show that it satisfies a weak form of incentive compatibility.

Foley (1967), Schmeidler and Vind (1972) and Varian (1974) associate fairness with envy-freeness. Walrasian equilibria with equal budgets are envy free and, thus, these authors establish a connection between competitive outcomes and fairness. In a public goods setting, two agents may contribute different amounts to the same public good, and hence adapting the notion of envy freeness to Lindahl equilibria is not straightforward. Moreover, as we show in section 5.2, there are multiple ways to commodify each collective choice market. The same utility profile may be envy free for one specification but fail envy-freeness for another. Thus, collective choice markets do not lend themselves to a coherent definition of envy-freeness. We offer the wNBS as an alternative criterion of fairness and show its equivalence to Lindahl equilibrium.[6]

## 2. Collective Choice Markets and Lindahl Equilibrium

Consider the following collective choice problem: $n$ agents, $i \in \{1, \dots, n\}$, must decide on one of $k$ social outcomes, $j \in K = \{1, \dots, k\}$. A random outcome, $q \in Q := \{\hat{q} \in \mathbb{R}^k_+ \mid \sum_{j \in K} \hat{q}^j = 1\}$, is a probability distribution over the social outcomes. Agents are expected utility maximizers; $i$'s utility from outcome $j \in K$ is $u_i^j \geq 0$ and $u_i = (u_i^1, \dots, u_i^k)$ denotes $i$'s von Neumann-Morgenstern utility index. In addition to the $k$ outcomes described above, there is a disagreement outcome that yields zero utility to every agent. We dismiss all agents who have no stake in the collective decision; that is, we assume that for

[5] Our model allows an arbitrary, but finite, set of feasible allocations.

[6] Sato (1987) adapts the notion of envy-freeness to Lindahl equilibria by assuming that agent $i$ converts $j$'s actual consumption of the public good into a *virtual quantity* based on $j$'s utility of that good. In effect, Sato identifies a commodity space and associated utility functions for which envy freeness of Lindahl equilibria with equal budgets is satisfied.

every utility $u_i$, there is some $j$ such that $u_i^j > 0$. The vector $u = (u_1, \ldots, u_n)$ denotes a profile of utilities.

We will define a social choice rule by identifying the Lindahl equilibria of a corresponding market economy, the *collective choice market*. This market has $n+1$ agents, the $n$ described above, now called consumers, and one firm. Consumer $i$ has $\omega_i$ units of fiat money and can purchase quantity $q^j \geq 0$ of each "public good" $j \in \{1, \ldots, k\}$ at price $p_i^j \geq 0$. Outcome 0 is identified with not purchasing any good (and therefore has price 0). The vector $\omega = (\omega_1, \ldots, \omega_n)$ such that $\omega_i > 0$ denotes a profile of fiat money endowments.[7]

Consumer $i$ faces prices $p_i = (p_i^1, \ldots, p_i^k)$ and solves the following maximization problem:

$$U_i(p, \omega_i) = \max_{q \in Q} u_i \cdot q \text{ subject to } p_i \cdot q \leq \omega_i \tag{1}$$

We say that $q$ is an minimal-cost solution to the maximization problem above if $q$ solves that problem and $p \cdot \hat{q} \leq p \cdot q$ for every other solution $\hat{q}$. The price paid for outcome $j$ depends on the identity of the consumer. We let $p_i = (p_i^1, \ldots, p_i^k) \in I\!R_+^k$ be consumer $i$'s prices and $p = (p_1, \ldots, p_n) \in I\!R_+^{kn}$ be a price profile. The firm chooses $q$ to maximize profit, that is, to solve

$$R(p) = \max_{q \in Q} \sum_{i=1}^{n} p_i \cdot q \tag{2}$$

**Definition:** *The pair $(p, q)$ is a Lindahl equilibrium (LE) for the collective choice market $(u, \omega)$ if $q$ is a minimal-cost solution to every consumer's maximization problem and solves the firm's maximization problem at prices $p$.*

We offer no existence result here since Theorem 3 will ensure the existence of Lindahl equilibria. A broad range of applications including all discrete allocation and matching problems can be modeled as collective choice markets. For example, if the aggregate endowment consists of a collection of indivisible goods, then the set of outcomes is simply the set of all allocations. To map a two-sided matching problem into a collective choice market, we partition "consumers" into two groups, $N_1, N_2$ and let the outcomes be the set of all matchings; that is, one-to-one functions $j : N_1 \cup N_2 \to N_1 \cup N_2$ such that $j(j(i)) = i$

[7] In both the collective choice market and the bargaining problem, we rule out $\omega_i = 0$ since it is equivalent to excluding agent $i$ and dealing with the resulting $n-1$-person problem.

and [$i \in N_l$ implies $j(i) = i$ or $j(i) \notin N_l$]. Hence, a member of group $l$ is either unmatched ($j(i) = i$) or is matched with someone from the other group ($j(i) \notin N_l$). In section 4, we analyze these applications and relate Lindahl equilibrium outcomes to standard Walrasian outcomes of these economies.

Let $L(u, \omega) \subset \mathbb{R}^n_+$ be the set of utility profiles that can be supported as a Lindahl equilibrium of the collective choice market with utility functions $u$ and endowments $\omega$; that is, $L(u, \omega) = \{(u_1 \cdot q, \ldots, u_n \cdot q) \,|\, (p, q) \text{ is a LE of } (u, \omega)\}$.

## 3. The Bargaining Problem

In this section, we define and characterize a cooperative solution concept, the wNBS. For any $x, y \in \mathbb{R}^n$, we write $x \leq y$ to mean $x_i \leq y_i$ for all $i$. For any bounded set $X$, we let $d(X)$ (the *disagreement point* of $X$) be the greatest lower bound of $X$ and we let $b(X)$ (the *bliss point* $X$) be the least upper bound of $X$.

For any finite, bounded set $X \subset \mathbb{R}^n$, let $\operatorname{conv} X$ denote its convex hull of $X$. A set, $X$, is comprehensive if $d(X) \in X$. Let $\operatorname{comp} X$ denote the comprehensive hull of $X$; that is, $\operatorname{comp} X$ is the smallest comprehensive set that contains $X$. Clearly,

$$\operatorname{comp} X := \{x \in \mathbb{R}^n \,|\, d(X) \leq x \leq y \text{ for some } y \in X\}$$

Then, the set $\operatorname{coco} X := \operatorname{comp} \operatorname{conv} X$ is its convex comprehensive hull; that is, the smallest (in terms of set inclusion) convex and comprehensive set that contains $X$.

For any $a, x \in \mathbb{R}^n$, let $a \otimes x = (a_1 \cdot x_1, \ldots, a_n \cdot x_n)$, $a \otimes B = \{a \otimes x \,|\, x \in B\}$ and $B + z = \{x + z \,|\, x \in B\}$. Let $e^i$ denote unit vector with zeros in every coordinate except $i$, $o := (0, \ldots, 0) \in \mathbb{R}^n$, $e := (1, 1, \ldots, 1) \in \mathbb{R}^n$ and $\Delta = \operatorname{conv}\{o, e^1, \ldots, e^n\}$.

We write $B \leq A$ if $A$ dominates $B$ in the *weak set order*, that is, for every utility vector in $B$, there is a corresponding utility vector in $A$ that dominates it and, conversely, for every utility vector in $A$, there is a corresponding utility vector in $B$ that is dominated by it: for $A, B \in \mathcal{B}$, $B \leq A$ if for every $x \in B, y \in A$, there exist $x' \in B$, $y' \in A$ such that $x' \leq y$ and $x \leq y'$. It is not difficult to see that $B \leq A$ if and only if there is an onto function $g : B \to A$ such that $x \leq g(x)$ for all $x$.

We consider bargaining problems in which the set of attainable utility profiles is a full dimensional, comprehensive polytope $B$ such that $o \leq d(B)$. Let $\mathcal{B}$ denote the set of all such polytopes and let $\mathcal{B}_o = \{B \in \mathcal{B} \,|\, o = d(B)\}$ denote the set of all polytopes with disagreement point $o$. The bargaining problem $B \in \mathcal{B}$ is a simplex if $B = a \otimes \Delta + d$ for some $a$ such that $a_i > 0$ for all $i$. A simplex describes a *pure conflict* situation: there is a single indivisible prize and a lottery is to determine which agent gets it.

A (set-valued) bargaining solution is a mapping $S : \mathcal{B} \to 2^{\mathbb{R}^n} \backslash \emptyset$ such that $S(B) \subset B$. Hence, a bargaining solution chooses a nonempty set of alternatives from every bargaining problem. Below, we define a set-valued bargaining solution, which we call the *weighted* Nash bargaining set.

Recall that the Nash bargaining solution is the unique solution to the following optimization problem: for any $B \in \mathcal{B}$, let

$$f(B, x) := \sum_i \log(x_i - d_i(B))$$

Then, the Nash bargaining solution, $\eta(B)$, of $B$ is the unique element of $B$ that maximizes $f(B, \cdot)$. To define the weighted Nash bargaining solution, let $\omega = (\omega_1, \ldots, \omega_n)$ be a set of weights; that is $\omega_i > 0$ for all $i$ and let

$$f_\omega(B, x) := \sum_i \omega_i \log(x_i - d_i(B))$$

Then, the $\omega$-weighted Nash bargaining solution, $\eta_\omega(B)$, of $B$ is the unique element of $B$ that maximizes $f_\omega(B, \cdot)$.

We define the set-valued analogue of the $\omega$-weighted Nash bargaining solution, the $\omega$-weighted Nash bargaining *set* as follows:

$$N_\omega(B) = \{\eta_\omega(A) \in B \,|\, B \leq A\}$$

When $\omega = \lambda e$ for some $\lambda > 0$, we call $N_\omega(B)$ the symmetric Nash bargaining set. Theorem 1 below establishes that abandoning single-valuedness and symmetry together with minor modifications of Nash's axioms yields the weighted Nash bargaining set.

The first axiom, scale-invariance, is shared by most bargaining solutions including the Nash bargaining solution and the Kalai-Smorodinsky solution. It asserts that positive affine transformations of utilities do not change the set of chosen (physical) outcomes.

**Scale Invariance:** *$S(a \otimes B + z) = a \otimes S(B) + z$ whenever $a_i > 0$ for all $i$.*

Our efficiency axiom applies only to the bargaining problem $\Delta$. It ensures that a unique outcome is chosen from $\Delta$, that this outcome is Pareto-efficient and yields positive utility to all agents.

**Efficiency:** *$S(\Delta) = \{x\}$ for some $x$ such that $x \cdot e = 1$ and $x_i > 0$ for all $i$.*

The following axiom is similar to independence of irrelevant alternatives (IIA) of the Nash bargaining solution.

**Consistency:** *$B \leq A$ implies $S(A) \cap B \subset S(B)$.*

The only difference between Consistency and IIA is that the former is applicable even if $d(A) \neq d(B)$. Requiring $d(A) = d(B)$ would render consistency equivalent to IIA. The logic underlying consistency is the following: if $x \in S(A)$, then $x$ has been deemed a reasonable outcome in $A$; that is, no agent has a compelling objection to $x$. If we also have $x \in B \leq A$, then there should be no compelling objection to $x$ in $B$ either since $B$ does not offer alternatives better than the ones available in $A$ and each agent has a lower disagreement utility in $B$ than in $A$.

The next axiom, minimality, ensures that no outcomes other than those necessitated by the preceding three axioms are included in the set of solutions.

**Minimality:** *If $\hat{S}$ satisfies the three axioms above and $\hat{S}(B) \subset S(B)$ for all $B$, then $\hat{S}(B) = S(B)$ for all $B$.*

Theorem 1 below establishes that the preceding four axioms characterize the weighted Nash bargaining set. Its proof and all other proofs are in the appendix.

**Theorem 1:** *A bargaining solution, $S$, satisfies the four axioms above if and only if there are weights, $\omega$, such that $S(B) = N_\omega(B)$ for all $B$.*

The theorem above reveals that the Nash bargaining set is a set-valued solution concept with parameter $\omega$. This parameter is the planner's vector of welfare weights and is pinned down by a single decision of the planner: the probability that each agent gets the prize in the pure conflict situation $\Delta$. If we replace Efficiency with *Symmetric Efficiency*, below, then the bargaining weights are equal.

**Symmetric Efficiency:** $S(\Delta) = \{\frac{1}{n}e\}$.

To see how our axioms relate to the axioms for the Nash bargaining solution (Nash 1950), assume that $S(B)$ is a singleton for all $B$. In that case, we can dispense with the minimality axiom. Then, restricting consistency to $A, B$ such that $d(A) = d(B)$ and efficiency with symmetric efficiency yields the Nash bargaining solution.

Theorem 2, below, provides an alternative axiomatic foundation of the weighted Nash bargaining set. In it, we replace Consistency and Minimality with Implementability:

**Implementability:** $x \in S(B)$ *if and only if* $\{x\} = S(A)$ *for some* $A$ *such that* $B \leq A$.

We have discussed the rationale for implementability in the introduction. There, we argued that we can interpret $B \leq A$ as the result of a mediator's intervention. The mediator maps each original outcome $x \in B$ to a new better alternative $g(y)$. The target outcome, $x$, is mapped to itself. The mapping $g$ yields a new bargaining problem, $A = g(B)$ in which the choice $x = g(x)$ is obvious. Thus, the mediator can guarantee that $x$ is chosen without incurring a cost. Of course, this argument requires a credible mediator who can commit to transfers; that is, the mapping $g$. In our motivating examples, the organization that seeks a solution to the allocation problem may fit that role.

**Theorem 2:** *A bargaining solution, $S$, satisfies Scale Invariance, Efficiency and Implementability if and only if there are weights, $\omega$, such that $S(B) = N_\omega(B)$ for all $B$.*

In the two-player setting, the symmetric (i.e., equal weights) Nash Bargaining set contains the standard bargaining solutions such as the Nash Bargaining solution, the Kalai-Smorodinsky solution (Kalai and Smorodinsky (1975)) and the Perles-Maschler solution (Perles and Maschler (1981)). Thus, the Nash bargaining set is a permissive solution concept; it deems reasonable all the standard arguments that researchers have provided for various solutions in the two-person case. The following example illustrates this point.

### 3.1 Relation to other Bargaining Solutions

Ann and Bob would like to divide two cakes, a peanut butter cake and a chocolate cake in a reasonable manner. Utilities are linear but Bob is allergic to peanuts while Ann likes the peanut butter cake just as much as the chocolate cake. One implementable division is to give the peanut butter cake to Ann and the chocolate cake to Bob. To obtain this outcome, the mediator could offer to replace the peanut butter cake with a second chocolate cake (if necessary). The exchange leaves Ann's payoff unchanged but increases Bob's bliss point. In the game with two chocolate cakes, the symmetric Nash bargaining set has a unique element: one cake each for Bob and Ann. In utility terms, the same outcome can be replicated in the original game by giving the peanut butter cake to Ann and the chocolate cake to Bob, which is, therefore, an implementable outcome. This solution corresponds to the Nash bargaining solution.

Another implementable outcome is to give the peanut butter cake to Ann and divide the chocolate cake equally between Bob and Ann. In this case, the mediator offers to bake another peanut butter cake (if necessary) to give to Ann in case no agreement is reached. If Ann is guaranteed to receive a peanut butter cake, then the unique element of the symmetric Nash bargaining set is to divide the chocolate cake equally between the two players and give the peanut butter cake to Ann. This solution corresponds to the Perles-Maschler solution (Perles and Maschler (1981)) of the game.

The two solutions described above are the extreme points of the symmetric Nash bargaining set which consists of all solutions in which Ann receives at most 1/2 of the chocolate cake and the whole peanut butter cake. This set contains all standard bargaining solutions that satisfy scale invariance (see Thomson (1994) for a discussion of bargaining solutions). For example, the Kalai-Smorodinsky solution would give 2/3 of the chocolate cake to Bob and 1/3 to Ann.

We interpret the weighted Nash bargaining set as those outcomes that a mediator could bring about. As the example above shows, the mediator can have considerable leeway to favor one or the other agent. The set valued-solution concept takes no position on which of those outcomes is normatively best or fairest; in effect, it concedes that both the Perles-Maschler axioms and Nash's axioms make a valid case.

## 4. The weighted-Nash Bargaining Set and Lindahl Equilibria

We will identify any collective choice problem, $(u, \omega)$ with the bargaining set

$$B_u := \text{coco}\left(\{(u_1^j, \ldots, u_n^j) \mid j \in K\} \cup \{o\}\right)$$

and bargaining weights $\omega$. Hence, $B_u$ is the convex and comprehensive hull of the utilities in the collective choice problem. Our main result, Theorem 3 below, shows that the $\omega$-weighted Nash bargaining outcomes of $B_u$ coincide with the Lindahl equilibrium payoffs of the collective choice problem $(u, \omega)$.

**Theorem 3:** *The set of Lindahl equilibrium payoffs of $(u, \omega)$ is the same as the wNBS of the corresponding bargaining problem with weights $\omega$: $L(u, \omega) = N_\omega(B_u)$ for all $u, \omega$.*

To see how every element of the wNBS can be made into a Lindahl equilibrium payoff, assume, for simplicity, that there are only two agents. Take any $x \in N_\omega(B_u)$ and note that $x$ must be on the Pareto-frontier of $B_u$. The definition of $N_\omega$ implies that there is a bargaining set $A$ such that $B_u \leq A$ and $x = \eta_\omega(A)$. An argument familiar from the characterization of the Nash bargaining solution[8] reveals that we can find a simplex $A^*$ such that $A \leq A^*$ and $x = \eta_\omega(A^*)$. Therefore, assume, without loss of generality, that $A$ itself is a simplex; that is, $A = a \otimes \Delta + d$ for some $a = (a_1, a_2) > 0$ and some $(d_1, d_2) \geq 0$.

In general, $d(A)$ need not be the origin but in the special case in which it is, we have $B_u \subset A$ and $a_i = x_i \frac{\omega_1 + \omega_2}{\omega_i}$. Consider that special case. Since $x$ is Pareto efficient, there is a distribution $q$ over Pareto efficient outcomes that delivers the utility vector $x$. To establish that $q$ is a Lindahl equilibrium allocation we will compute the corresponding Lindahl equilibrium prices. Let $j$ be any outcome. Since $B_u \subset A$, the vector $(u_1^j, u_2^j)$ must be a convex combination of the extreme points of $A$. Hence, there are $z_1, z_2 \geq 0$ such that $z_1 + z_2 \leq 1$ and $(a_1 z_1, a_2 z_2) = (u_1^j, u_2^j)$.[9] Then, let $p_i^j = (\omega_1 + \omega_2) z_i$ be the price agent $i$ pays for outcome $j$. It is not difficult to verify that $(p, q)$ is a Lindahl equilibrium. Our proof that $N_\omega(B_u) \subset L(u, \omega)$ for all $u$ generalizes this argument to cover the cases in which $d(A) \neq o$.

[8] See Lemma A3 in the appendix.

[9] Since $x$ is on the efficient frontier of $A$, $z_1 + z_2 < 1$ implies $q_j = 0$.

For the converse; that is, to see that every Lindahl equilibrium payoff of $(u, \omega)$ is an element of $N_\omega(B_u)$, let $(p, q)$ be a Lindahl equilibrium and recall that $q$ solves the linear program below for every consumer $i$:

$$U_i(p, \omega_i) = \max_{\hat{q} \in Q} u_i \cdot \hat{q} \text{ subject to } p_i \cdot \hat{q} \leq \omega_i \tag{1}$$

Let $\alpha_i$ be the shadow price of the constraint $p_i \cdot q' \leq \omega_i$ and let $c_i$ be the shadow price of the constraint $e \cdot q' \leq 1$. Then, the constraint of the dual of the above linear program requires

$$\alpha_i p_i^j \geq u_i^j - c_i \tag{3}$$

for all $i, j$. Moreover, the optimality of $q$ implies that inequality (3) holds with equality if $q^j > 0$. Suppose $\alpha_i$, the shadow price of the budget constraint, is strictly positive for every consumer. Then, $p_i \cdot q = \omega_i$ and, since $\alpha_i p_i^j = u_i^j - c_i$ if $q^j > 0$, we have $\alpha_i + c_i = u_i \cdot q$ for all $i$; that is, $\alpha + c$, where $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $c = (c_1, \ldots, c_n)$, is the payoff vector associated with the equilibrium $(p, q)$ and therefore, $\alpha + c \in B_u$. Note also that $\alpha + c = \eta_\omega((\omega \cdot e)\alpha \otimes \Delta + c)$. The key step of the proof establishes that $B_u \leq (\omega \cdot e)\alpha \otimes \Delta + c$ and hence $\alpha + c \in N_\omega(B_u)$.

In the next section, we show that in a general class of one-to-one matching problems, the set of Lindahl equilibrium payoffs and the set of Walrasian equilibrium payoffs are the same. In these situations, Theorem 3 ensures the existence of Walrasian equilibrium.

## 5. Walrasian Equilibria and Lindahl Equilibria

In this section, we show that the Walrasian equilibrium payoff allocations are also Lindahl allocations and analyze a setting in which the two are the same. First, we consider one-to-one matching problems; then, we consider general allocation problems with indivisible goods and no transfers.

### 5.1 Matching

A group of agents must decide who matches with whom, for example, they might be choosing roommates or partners. A matching is a bijection $j$ from the set of all agents to itself such that $j(j(i)) = i$ for all $i$. If $j(i) = i$, then $i$ is said to be unmatched. Some matchings may be infeasible. For example, agents could be workers and firms and firms matching with other firms may be disallowed. Let $K = \{1, \ldots, k\}$ be the set of feasible matchings, let $N_i = \{m | \, j(i) = m, \text{ for some } j \in K\}$ be the feasible matches of agent $i$ and assume that $N_i \neq \emptyset$ for all $i$.

Let $v_i^m$ be the utility of agent $i$ when she matches with agent $m$. We normalize agents' utilities so that being unmatched yields 0 utility for every agent and assume that $v_i^m > 0$ for some $m \in N_i$. We eliminate all matchings that are not individually rational and therefore, we assume $v_i^{j(i)} \geq 0$ for all $i$ and $j$. For notational convenience, we set $v_i^m = 0$ if $m \notin N_i$.

We identify the following competitive market with this matching problem: each agent has $\omega_i$ units of fiat money and agent $i$ must pay price $\pi_i^m$ for matching with agent $m$. The price of remaining unmatched, $\pi_i^i$, is zero for all agents. Let $\pi_i = (\pi_i^1, \ldots, \pi_i^n)$ be the corresponding price vector and let $\pi = (\pi_1, \ldots, \pi_n)$. A commodity, in this setting, is an ordered pair $(i, m)$ denoting $i$'s match with $m$. Since we allow randomization, $i$'s consumption of $(i, m)$ can vary between 0 and 1. Thus, $i$'s budget given prices $\pi$ and budget $\omega_i$ is:

$$B(\pi, \omega_i) = \left\{ \xi_i \in I\!R_+^n \, \middle| \, \xi_i \cdot e \leq 1, \pi_i \cdot \xi_i \leq \omega_i \right\}$$

Each agent solves the following maximization problem:

$$V_i = \max_{\xi_i \in B(\pi, \omega_i)} v_i \cdot \xi_i \tag{M1}$$

The consumption $\xi_i$ is a minimum cost solution to the consumer's problem if it solves the above maximization problem and no other solution costs less than $\xi_i$ at prices $\pi_i$.

The firm's revenue from matching $j$ is the sum of the prices of the individual matches; that is, $\sum_{i=1}^n \pi_i^{j(i)}$. Therefore, the firm solves the maximization problem:

$$R(\pi) = \max_{q \in Q} \sum_{j \in K} \sum_{i=1}^n \pi_i^{j(i)} \cdot q^j \tag{M2}$$

The pair $(\pi, q)$ is a (strong) Walrasian equilibrium (WE) of the matching market $(v, \omega)$ if $q$ solves the firm's maximization problem (M2) and $\xi_i(q)$ is a minimum cost solution to consumer $i$'s problem (M1), where

$$\xi_i^m(q) = \sum_{\{j | \, j(i) = m\}} q^j \tag{M3}$$

for all $i, m$. Equation (M3) is the market clearing condition; that is, the requirement that consumer $i$'s demand for matches with $m$ is met by the firm's supply of such matches. Let $W(u, \omega)$ be the set of all Walrasian equilibrium payoffs for the competitive economy $(u, \omega)$; that is,

$$W(v, \omega) := \{(v_1 \cdot \xi_1(q), \ldots, v_n \cdot \xi_n(q)) \,|\, (p, q) \text{ is a WE of } (u, \omega)\}$$

The competitive economy differs from the collective choice market in the way consumers express their demands. In the competitive economy, consumers specify demands for private goods (partners $m \in N_i$) while in a collective choice market, consumers specify their desired collective goods (matches $j \in K$). When mapping matching market into a collective choice market, we define $u_i$ such that $u_i^j = v_i^{j(i)}$. Then, without risk of confusion, we identify this $u_i$ with $v_i$ and $u = (u_1, \ldots, u_n)$ with $v = (v_1, \ldots, v_n)$.

**Theorem 4:** *The set of Lindahl equilibrium payoffs of a matching market is the same as its set of Walrasian equilibrium payoffs: $L(v, \omega) = W(v, \omega)$ for all $v, \omega$.*

Theorem 4 together with Theorem 3 establishes that Walrasian equilibrium exists in any matching problem and that the $\omega$-weighted Nash bargaining solution is a Walrasian equilibrium of the matching market with endowments $\omega$. In particular, the Nash bargaining solution is a Walrasian equilibrium of the matching market with equal endowments.

To see why every Walrasian equilibrium payoff vector is also a Lindahl equilibrium payoff vector, note that every agent's optimization problem in the collective choice market is equivalent to their corresponding optimization problem in the Walrasian setting. Condition (M3) implies that $q$ achieves the desired distribution of partners. Moreover, since $\xi_i$ is a least cost solution for consumer $i$ in the competitive setting, it is also a least cost solution in the collective choice setting.

To see why every Lindahl equilibrium payoff vector is also a Walrasian equilibrium payoff vector, let $\pi_i^m = \min\{p_i^j \,|\, j(i) = m\}$ for all $i, m$. Then, note that in a Lindahl equilibrium consumers choose minimum cost solutions to their optimization problems. Therefore, $j(i) = j'(i)$ and $p_i^j > p_i^{j'}$ implies that $q^j = 0$. It follows that the random consumption induced by $q$ must be a least cost solution to the $i$'s optimization problem at prices $\pi$. For the same reason, the firm revenue of any matching $j$ such that $q^j > 0$

is the same as in the collective choice market. For any matching $j$ such that $q^j = 0$, the firm revenue is no greater than in the collective choice market and, therefore, $q$ must be optimal for the firm.

### 5.2 Discrete Exchange Economies

In the general discrete good allocation problem, there is a finite set of goods $H = \{1, \ldots, r\}$ to be distributed to the $n$ agents. There is no divisible good. Agent's $i$ utility for bundle $M$ is $v_i(M)$; we assume that $v_i(\emptyset) = 0$ and $v_i(M) \leq v_i(\hat{M})$ whenever $M \subset \hat{M}$. We write $v_i(h)$ instead of $v_i(\{h\})$. Let $G(H)$ denote this general set of all such utility functions.

The discrete allocation problem can be transformed into an exchange economy by endowing each agent with one unit of fiat money and permitting random consumption. Let $\Theta_i$ be the set of all possible random consumptions; that is, probability distributions over $2^H$ and let

$$\mathbf{P} = I\!R_+^{|H|}$$

be the set of all prices. We write $\mathbf{p}(M)$ to mean $\sum_{h \in M} \mathbf{p}(h)$.

Then, given any price $\mathbf{p}$ and endowment $\omega_i$, a consumer's budget is

$$\mathbf{B}(\mathbf{p}, \omega_i) := \left\{ \theta_i \in \Theta_i \,\middle|\, \sum_M \mathbf{p}(M)\theta_i(M) \leq 1 \right\}$$

and her utility maximization problem is:

$$V_i = \max_{\theta_i \in \mathbf{B}(\mathbf{p}, \omega_i)} \sum_M v_i(M)\theta_i(M)$$

A random consumption $\theta_i$ is a minimal-cost solution to the utility maximization problem if it is a solution to the maximization problem above and no other solution costs less.

A (feasible) allocation is $\mathbf{a} = (M_1, \ldots, M_n) \subset H^n$ such that (i) $M_i \cap M_l = \emptyset$ whenever $i \neq l$ and (ii) $\bigcup_i M_i = H$; a random allocation is a probability distribution over allocations. For any allocation, $\mathbf{a} = (M_1, \ldots, M_n)$, $\mathbf{a}_i$ denotes the $i$'th entry of $\mathbf{a}$; that is, $M_i$. Let $K$

be the set of all allocations and $\Theta$ be the set of all random allocations. For any random allocation $\theta \in \Theta$, let $\theta_i$ denote the $i$'th marginal of $\theta$; that is,

$$\theta_i(M) = \sum_{\mathbf{a}:\mathbf{a}_i=M} \theta(\mathbf{a})$$

The pair $(\mathbf{p}, \theta)$ is a (strong) Walrasian equilibrium if for all $i$, $\theta_i$ is a minimal-cost solution to consumer $i$'s utility maximization problem and $\theta$ maximizes the firm's revenue, $R$, over all allocations in $\Theta$, where

$$R(\mathbf{p}, \theta) = \sum_i \mathbf{p}(M)\theta_i(M)$$

Let $\mathrm{W}(v, \omega)$ be the set of Walrasian equilibrium utility vectors given a utility profile $v$ and endowments $\omega$. Without further assumptions, there may be no Walrasian equilibrium and thus $\mathrm{W}(v, \omega)$ may be empty. Gul, Pesendorfer and Zhang (2020) provide examples of economies without Walrasian equilibria and show that Walrasian equilibria exist if all utilities satisfy the gross substitutes condition.

The bargaining problem associated with the discrete allocation economy $v$ is $B_v := \mathrm{coco}\left(\{(v_1(\mathbf{a}), \ldots, v_n(\mathbf{a})) \,|\, \mathbf{a} \in K\} \cup \{o\}\right)$. Clearly, each allocation problem yields a unique bargaining problem; however, the converse is not true. Distinct allocation problems may yield the same bargaining problem. Specifically, we may alter the way property rights are structured; that is, the specification of the goods to be traded, without affecting the bargaining problem. In all our formulations, the entire aggregate endowment initially belongs to the fictitious firm and the $n$ agents are all endowed with nothing other than quantities of fiat money. Nevertheless, the set of Walrasian equilibrium payoffs depends on how the goods are defined. The two economies below illustrate this point.

**Economy 1:** There are two consumers, 1, 2, and two goods, $a, b$. Both agents have unit demand preferences. In particular, $v_1(a) = v_1(\{a, b\}) = 3$, $v_1(b) = 2$ and $v_2(a) = v_1(\{a, b\}) = 2$, $v_1(b) = 1$. Let $\omega = (1, 1)$. Note that the bargaining set associated with this economy is the convex, comprehensive hull of the set $\{o, (3, 1), (2, 2)\}$. It is easy to verify that this economy has a unique Walrasian equilibrium in which both consumers receive each good with probability $1/2$. The associated equilibrium payoff vector is $(5/2, 3/2)$.

**Economy 2:** There are two consumers, 1, 2, and four goods, $a, b, c, d$. The four goods are perfect substitutes. Both consumers have additive utility functions but consumer 1 can derive utility from at most three goods while consumer 2 can derive utility from at most two goods. Specifically, $v_1(M) = \min\{|M|, 3\}$ and $v_2(M) = \min\{|M|, 2\}$ where $|M|$ is the cardinality of the set $M$. This new economy yields the same bargaining set as the one above: the convex, comprehensive hull of the set $\{o, (3,1)(2,2)\}$. Let $\omega = (1,1)$. Then, in every Walrasian equilibrium, each consumer receives two goods and, therefore, $(2,2)$ is the unique equilibrium payoff vector.

It is easy to verify that both $(5/2, 3/2)$ and $(2,2)$ are Lindahl equilibrium payoff vectors for this economy. Proposition 1, below, reveals that this is true in general. It shows that the set of Walrasian equilibrium payoffs of any exchange economy is contained in the set of Lindahl equilibrium payoffs of the corresponding collective choice market.

**Proposition 1:** *Every Walrasian equilibrium payoff vector of a discrete exchange economy is also a Lindahl equilibrium payoff vector:* $W(u, \omega) \subset L(u, \omega)$.

Sertel and Yildiz (2003) provide a general statement and proof of the existence of distinct standard $n$-person exchange economies that yield the same set of feasible payoffs but have disjoint sets of Walrasian equilibrium payoffs. The example above shows that the same is true in our setting. Hence, Theorem 3 and Proposition 1 above reveal an advantage of Lindahl equilibria over Walrasian equilibria: the set of Lindahl equilibrium payoffs depends only on the implied bargaining problem whereas the set of Walrasian equilibria depends on how commodities are defined. On the other hand, Walrasian equilibria are simpler than Lindahl equilibria because the former involve many fewer prices. This is so because the number of allocations typically exceeds the number of goods and because Lindahl prices are personal while Walrasian prices are not. However, if the commodity space is rich enough, as in matching problems, the distinction between Lindahl equilibrium and Walrasian equilibrium disappears.

## 6. Conclusion

To achieve compromise when there are indivisibilities it is often necessary to randomize. In practice, this randomization often occurs ex ante to assign priorities to the agents while the mechanism remains deterministic. For example, when allocating offices, an organization may randomly determine a priority order and then ask members to sequentially choose their preferred office. As Hylland and Zeckhauser (1979) point out, such mechanisms lead to ex ante inefficiency. As an alternative, they propose a market mechanism in which agents are given a budget of fiat money and choose lotteries over the available offices. The Walrasian mechanism proposed by Hylland and Zeckhauser is efficient but limited in its applicability to unit demand preferences. Gul, Pesendorfer and Zhang (2019) extend Hylland and Zeckhauser's approach from unit demand to multi-unit demand with gross substitutes utilities. As is shown in that paper, demand complementarities create existence problems for the standard market mechanism. Moreover, externalities and public goods render Walrasian equilibria inefficient. In contrast, the collective choice markets proposed in the current paper are broadly applicable to all discrete allocation problems and are always efficient.

In a collective choice market, each agent expresses her demand for social alternatives rather than private outcomes. This allows us to deal with a much broader range of applications. However, the number of social alternatives can be large and, therefore, the collective choice market may be too unwieldy to implement in practice. For matching markets, we have shown that Lindahl equilibria coincide with standard competitive equilibria and, therefore, each agent need only consider the set of possible partners and not the set of possible matchings (allocations) when formulating her demand. An important direction for future research is to examine other circumstances in which a smaller set of markets (and prices) suffices to implement Lindahl equilibria.

# 7. Appendix

## 7.1 Preliminaries

Let $p$ be a price. In a collective choice market, the consumer solves

$$U_i(p, \omega_i) = \max_q u_i \cdot q \text{ subject to } p_i \cdot q \leq \omega_i, e \cdot q \leq 1 \tag{P}$$

The dual of the maximization problem (P) is

$$\min_{\mu^0, \mu^1 \geq 0} \mu^0 + \mu^1 \omega_i \text{ subject to } \mu^0 e + \mu^1 p_i \geq u_i \tag{D}$$

The vector $(q, \mu_0, \mu_1)$ is feasible if $q$ satisfies the constraints of (P) and $(\mu_1, \mu_2)$ satisfies the constraints of (D). A feasible vector $(q, \mu^0, \mu^1)$ is optimal (that is, $q$ solves (P) and $\mu^0, \mu^1$ solves (D)) if and only if

$$\begin{aligned} \mu^0(e \cdot q - 1) &= 0 \\ \mu^1(p_i \cdot q - \omega_i) &= 0 \\ \text{for all } j, \quad q^j(\mu^0 + \mu^1 p_i^j - u_i^j) &= 0 \end{aligned} \tag{CS}$$

Let $J(q) = \{j \mid q^j > 0\}$. For any utility $u_i$ and $c_i < \max_j u_i^j$, let $\bar{u}_i^j(c_i) = \max\{0, u_i^j - c_i\}$. Since $c_i < \max_j u_i^j$ for some $j$, $\bar{u}_i(c_i)$ is a utility; that is $\bar{u}_i^j(c_i) > 0$ for some $j$.

**Lemma A1:** *Let $q \in Q$ be a minimal cost solution to consumer $i$'s maximization problem for utility $u_i$ and prices $p$. If $p_i \cdot q = \omega_i$, then there are $c \geq 0$ and $\alpha > 0$ such that $\alpha p_i^j \geq u_i^j - c$ for all $j$ and $\alpha p_i^j = u_i^j - c$ for $j \in J(q)$. Moreover, $q$ is a minimal cost solution to consumer $i$'s maximization problem for utility $\bar{u}_i(c)$ and prices $p$.*

**Proof:** Let $\mu^0, \mu^1$ be the associated solution of the dual (D). First, consider the case in which $u_i^j \neq u_i^m$ for some $j, m \in J(q)$. Then, (CS) implies $\mu^1 > 0$. Set $c = \mu^0, \alpha = \mu^1$. Feasibility and (CS) imply $\alpha p_i^j \geq u_i^j - c$ with equality if $j \in J(q)$, as desired.

To conclude this case, we will show that $q$ is a minimal cost solution to the consumer's maximization problem given utility $\bar{u}_i(c)$. Since $(q, \mu^0, \mu^1)$ is feasible (for utility $u_i$), we have $\alpha p_i^j \geq u_i^j - c$ for all $j$. We also have $\alpha p_i^j \geq \bar{u}_i^j(c)$ for all $j$, since the left-hand side is non-negative. The first part of the Lemma implies $u_i^j - c = \bar{u}_i^j(c)$ for all $j \in J(q)$ and, therefore, $q^j(\alpha p_i^j - \bar{u}_i(c)) = 0$ for all $j$. Hence, $(q, \mu^0, \mu^1)$ such that $\mu^0 = 0, \mu^1 = \alpha$ is a feasible solution for (P) and (D) that satisfies (CS) for utility $\bar{u}(c)$; that is, $q$ solves consumer $i$'s maximization problem given utility $\bar{u}_i^j(c)$. Since $u_i^j \neq u_i^m$ for some $j, m \in J(q)$, we have $\bar{u}_i^j(c) = u_i^j - c \neq u_i^m - c = \bar{u}_i^m(c)$ and therefore, this solution must be minimal cost.

Second, consider the case in which $\beta = u_i^j = u_i^m$ for all $j, m \in J(q)$ and $\mu^1 > 0$. Since $q$ is a minimal cost solution and $q \cdot p_i = \omega_i$, we must have $p_i^j = p_i^m = \omega_i$ for all $j, m \in J(q)$. Let $\mu^0, \mu^1$ be the associated solution of the dual (D). Set $\alpha = \mu_1, c = \mu_0$ and repeat the argument above to conclude that $q$ is a solution to (P) given utility $\bar{u}_i(c)$ and $\bar{u}_i^j(c) = u_i^j - c = \alpha p_i^j = \alpha\omega_i > 0$ for all $j \in J(q)$. Then, since $q$ is a minimal cost solution for $u_i$, it is also a minimal cost solution for $\bar{u}_i(c)$.

Finally, consider the case in which $\beta = u_i^j = u_i^m$ for all $j, m \in J(q)$ and $\mu^1 = 0$. Since $q$ is a minimal cost solution, we have $p_i^j \leq \omega_i$ for all $j \in J(q)$; $u_i^j \leq \beta$ for all $j$. Set $c = \max\{u_i^j \,|\, p_i^j < \omega_i, u_i^j < \beta\}$ if $\{u_i^j \,|\, p_i^j < \omega_i, u_i^j < \beta\} \neq \emptyset$ and $c = 0$ otherwise. Then, set $\alpha = \frac{\beta - c}{\beta}$. Since $\beta > 0$, $\beta - c > 0$.

If $p_i^j < \beta$, we have $c + \alpha p_i^j \geq c \geq u_i^j$; if $p_i^j \geq \beta$ we have $c + \alpha p_i^j \geq \beta \geq u_i^j$. Therefore, $\alpha p_i^j \geq u_i^j - c$ for all $j$ and with equality if $j \in J(q)$. Note that, for $\bar{u}_i(c)$, $(q, c, \alpha)$ is a feasible vector and since $p_i \cdot q = \omega_i$, it satisfies (CS). Hence $q$ is a solution to the consumer's optimization problem. Since $\beta - c > 0$ and $q$ is a minimal cost solution to the optimization problem for $u_i$, it must also be a minimal cost solution for $\bar{u}_i(c)$. □

**Lemma A2:** *Let $o \leq \lambda$ and $v_i = u_i + (\lambda_i, \ldots, \lambda_i)$ for all $i$. Then, $(p, q)$ is a Lindahl equilibrium of $(u, \omega)$ implies $(p, q)$ is a Lindahl equilibrium of $(v, \omega)$.*

**Proof:** Suppose $o \leq \lambda$ and let $(p, q)$ be a Lindahl equilibrium of $(u, \omega)$. Then, for all $i$ and $\hat{q}$ such that $e \cdot \hat{q} \leq 1$, $u_i \cdot \hat{q} \leq u_i \cdot q = v_i \cdot q - \lambda_i$ and $v_i \cdot \hat{q} - \lambda_i \leq u_i \cdot \hat{q}$. It follows that $q$ is a minimal cost solution to consumer $i$'s maximization problem in $v$. Clearly, $q$ is a solution to the firm's maximization problem in the collective choice market $v$ and hence $(p, q)$ is a Lindahl equilibrium. □

Recall that $A \in \mathcal{B}$ a *simplex* if $A = a \otimes \Delta + b$ for some $a$ such that $a_i > 0$ for all $i$ and $b$ such that $b_i \geq 0$ for all $i$. We say that the simplex $A$ supports $B$ at $x$ with (exterior normal) $\theta$ if $x \in B \subset A$, $d(B) = d(A)$ and $\theta \cdot y \leq \theta \cdot x$ for all $y \in A$. Let $\nabla f_\omega(B, x)$ denote the gradient of $f_\omega(B, \cdot)$ at $x$.

**Lemma A3:** *(i) Let $B \in \mathcal{B}_o$ and $x = \eta_\omega(B)$ for some $\omega$. Then, the simplex $A = \text{conv}\,\{o, (\omega \cdot e)(x_1/\omega_1)e^1, \ldots, (\omega \cdot e)(x_n/\omega_n)e^n\}$ supports $B$ at $\eta_\omega(B)$ with $\nabla f(B, \omega)$. (ii) For any simplex $A$, $\eta_\omega(A) = \frac{1}{\omega \cdot e}\omega \otimes (b(A) - d(A)) + d(A)$. (iii) For any $B$, there is a simplex $A$ such that $B \leq A$ and $\eta_\omega(B) = \eta_\omega(A)$. (iv) For any simplex $A$, $N_\omega(A) = \{\eta_\omega(A)\}$.*

**Proof:** Assume $d(B) = o$, let $x = \eta_\omega(B)$ and $\theta = \nabla f_\omega(B, \eta_\omega(B)) = (\omega_1/x_1, \ldots, \omega_n/x_n)$. Then, $\theta \cdot y \leq \theta \cdot x$ for all $y \in B$. Note that $\theta \cdot x = \omega \cdot e$. Hence, the simplex $A = \text{conv}\,\{o, (\omega \cdot e)(x_1/\omega_1)e^1, \ldots, (\omega \cdot e)(x_n/\omega_n)e^n\}$ supports $B$ at $\eta_\omega(B)$ with $\nabla f_\omega(B, \eta_\omega(B))$.

For arbitrary $B$, let $C = B - d(B)$. By the preceding argument, there is a simplex $A$ that supports $C$ at $\eta_\omega(C)$ with $\theta = \nabla f_\omega(C, x)$. Then, note that $\eta_\omega(B) = \eta_\omega(C) + d(B)$, $\nabla f_\omega(B, \eta_\omega(B)) = \nabla f_\omega(C, \eta_\omega(C))$ and therefore, $A + d(B)$ supports $B$ at $\eta_\omega(B)$ with $\nabla f_\omega(B, \eta_\omega(B))$.

The proof of (ii) is straightforward and omitted. For part (iii), take the simplex $A$ constructed in the proof of part (i) and apply part (ii) to conclude that $\eta_\omega(B) = \eta_\omega(A)$.

For part (iv), first note that $\eta_\omega(A) \in N_\omega(A)$ since $A \leq A$. It remains to show that $N_\omega(A)$ is a singleton set. Let $A \leq B, B \neq A$. By part (iii) there is a simplex $A'$ such that $B \leq A'$ and $\eta_\omega(B) = \eta_\omega(A')$. Since $A \leq B, B \neq A$, we have $A \leq A', A' \neq A$. Part (ii) then implies that $\eta_\omega(A') \notin A$. $\square$

**Lemma A4:** *For all $x \in N_\omega(B_u)$, there is a LE, $(p, q)$ and $c \in I\!R^n_+$, such that for all $i$, (1) $\frac{u_i^j - c_i}{p_i^j} = \frac{x_i - c_i}{w_i}$ if and only if $p_i^j > 0$, (2) $u_i^j \leq c_i$ if $p_i^j = 0$ and (3) $u_i^j \geq c_i$ if $q^j > 0$.*

**Proof:** Let $x \in N_\omega(B_u)$. Hence, $x = \eta_\omega(B)$ for some $B$ such that $B_u \leq B$. Then, by Lemma A3, there is a simplex $A$ such that $B \leq A$, $B \subset A = \text{conv}\,\{o, (\omega \cdot e)(x_1/\omega_1)e^1, \ldots, (\omega \cdot e)(x_n/\omega_n)e^n\}$ and $x = \eta_\omega(A)$. Assume for now, that $d(A) = o$. Then, $B_u \subset B \subset A$. For any $j \in K$, let $y^j = (u_1^j, \ldots, u_n^j)$. Since $y^j \in B_u \subset A$, it is a unique convex combination of the extreme points of $A$. Let $z_i^j$ be the weight of $(\omega \cdot e)(x_i/\omega_i)e^i$ in that convex combination and set $p_i^j = (\omega \cdot e)z_i^j$. Note that $u_i^j > 0$ if and only if $p_i^j > 0$ and that $u_i^j/p_i^j = x_i/\omega_i$ if $p_i^j > 0$. Since $x \in B_u$, it is a convex combination of the extreme points of $B_u$. Let $q^j$ be the weight of $y^j$ in one such convex combination. Hence, $u_i \cdot q = x_i$. Note that for any $\hat{q}$ such that $p_i \cdot \hat{q} \leq \omega_i$ we have

$$u_i \cdot \hat{q} = \frac{x_i}{\omega_i} p_i \cdot \hat{q}^j \leq x_i$$

with equality only if $\hat{p}_i \cdot q = \omega_i$. Therefore, $q$ is a minimal cost solution to the consumer's problem.

Since $x = \eta_\omega(A)$ and $x \in B_u \subset A$, $x$ is on the Pareto frontier of $B_u$ and since $x = \sum_j q^j \cdot y^j$, $y^j$ must be on the Pareto frontier of $B_u$ whenever $q^j > 0$. That is, $\sum_i z_i^j = 1$ whenever $q^j > 0$. Therefore, $\sum_i p_i^j = \omega \cdot e$ for all $j$ such that $q_j > 0$, It follows that $\sum_i p_i^j \leq \omega \cdot e$ for all $y^j \in B_u$ proving the optimality of $q$ for the seller. Hence, $(p, q)$ is a Lindahl equilibrium. Setting $c_i = 0$ for all $i$ ensures that $(p, q)$ satisfies (1)-(3).

If $o = d(B_u) \neq d(B) = d(A)$, then, since $B_u \leq B \leq A$, we must have $d(B_u) \leq d(A)$. Then, let $c = d(A)$. Define a set of outcome $\hat{K}$ as follows: for each $j \in K$, there is an element of $(v_1^{\phi(j)}, \ldots, v_n^{\phi(j)}) \in \hat{K}$ such that $v_i^{\phi(j)} = \max\{0, y_i^j - c_i\}$. Hence, $\phi : K \to \hat{K}$ is an onto but not necessarily one-to-one function. Then repeat the above construction for $A' = A - d(A)$, $x - d(A)$ to get a Lindahl equilibrium $(p, q)$ of the economy $(v, \omega)$ such that $v_i^j > 0$ if and only if $p_i^j > 0$ and $v_i^j / p_i^j = x_i / \omega_i$ if $\hat{p}_i^j > 0$. By Lemma A2, $(p, q)$ is a Lindahl equilibrium of the economy $(\hat{v}, \omega)$ such that $\hat{v}_i = v_i + (c_i, \ldots, c_i)$. It follows that this equilibrium satisfies (1)-(3) of this Lemma. To convert this Lindahl equilibrium into a Lindahl equilibrium of $(u, \omega)$, set $\hat{p}_i^j = p_i^{\phi(j)}$ for all $j \in K$ and note that $(\hat{p}, q)$ is a Lindahl equilibrium of $(u, \omega)$ that satisfies (1)-(3) above. □

### 7.2 Proof Theorems 1 and 2

**Proof of Theorem 1:** First we show that $N_\omega$ satisfies the four axioms above for any $\omega$ such that $\omega_i > 0$ for all $i$. Lemma 3(ii) and Lemma 3(iv) imply that $N_\omega$ satisfies Efficiency. Scale Invariance follows since the Nash bargaining solution satisfies Scale Invariance. Consistency follows since the weak set order is transitive. Finally consider any bargaining solution $S \subset N_\omega$, and let $x \in N_\omega(A)$. Then, $x = \eta_\omega(B)$ for some $A \leq B$. By Lemma 3(iii) there is a simplex $A \leq B \leq A'$ such that $\eta_\omega(B) = \eta_\omega(A')$. Since $S$ is non-empty, Lemma 3(iv) implies that $S(A') = \{\eta_\omega(A')\}$. Consistency then implies $\eta_\omega(A') \in S(A)$, proving Minimality.

For the converse, let $S$ be a bargaining solution that satisfies the axioms and note that Efficiency implies $S(\Delta) = \{\eta_\omega(\Delta)\}$ for some weights $\omega$. By Lemma 3(iv), $N_\omega(\Delta) = \{\eta_\omega(\Delta)\}$. Then, Scale Invariance implies $S(a \otimes \Delta + z) = a \otimes N_\omega(\Delta) + z = \{\eta_\omega(a \otimes \Delta + z)\}$; that is, $S(A) = N_\omega(A)$ for any simplex $A$. Take any bargaining set $B$ and $x \in S(B)$. By Lemma 3(iii), there is a simplex $A$ such that $B \leq A$ and $\eta_\omega(A) = \eta_\omega(B)$. We have already shown that $x = \eta_\omega(A) \in S(A)$. Then, by Consistency, $x \in S(B)$, proving $N_\omega \subset S$. Then, the first part of the theorem and minimality yield $S = N_\omega$. □

**Proof of Theorem 2:** The proof of Theorem 1 establishes that $N_\omega$ satisfies Scale Invariance and Efficiency. Lemma 3(iii) implies that $N_\omega$ satisfies Implementability. For the converse, we proceed as in the proof of Theorem 1 to show that Efficiency and Scale Invariance imply that for some weights $\omega$, $S(A) = N_\omega(A)$ for every simplex $A$. Then, Consistency and Lemma 3(iii) imply $N_\omega(A) \subset S(A)$; Implementability and Lemma 3(iii) imply $S(A) \subset N_\omega(A)$ for all $A$. Hence, $N_\omega(A) = S(A)$ for all $A$. □

### 7.3 Proof Theorem 3

Note that $x \in N_\omega(B_u)$ implies $x \in L(u, \omega)$ follows from Lemma A4. Next, we show that $x \in L(u, \omega)$ implies $x \in N_\omega(B_u)$. Let $x \in L(u, \omega)$ and let $(p, q)$ be the corresponding Lindahl equilibrium outcome. First, consider the case in which $p_i \cdot q = \omega_i$ for all $i$. By Lemma A1, there is, for each $i$, some $c_i \geq 0$ and $\alpha_i > 0$ such that $u_i^j - c_i \geq \alpha_i p_i^j$ for all $j$ with equality for $j \in J(q)$. Lemma A1 also implies that $(p, q)$ is a Lindahl equilibrium for the collective choice market $(\bar{u}_i(c_i))_{i=1}^n$ as well. Since $\bar{u}_i^j(c_i) = \alpha_i p_i^j$ for all $j \in J(q)$ and $p_i \cdot q = \omega_i$, we have $\bar{u}_i(c_i) \cdot q = \alpha_i p_i \cdot q = \alpha_i \omega_i$ and $x_i = c_i + \alpha_i \omega_i$. Let $z \in B_u$ and let $\hat{q}$ be such that $z_i = u_i \cdot \hat{q}$.

Since $\alpha_i p_i^j \geq u_i^j - c_i$ for all $i, j$, we have $\alpha_i p_i \cdot \hat{q} \geq z_i - c_i$ for all $i$ or $p_i \cdot \hat{q} \geq (z_i - c)/\alpha_i$. Note that the firm's profit cannot be greater than $\omega \cdot e$, the aggregate endowment of money. Therefore, firm optimality implies $\omega \cdot e \geq \sum_{i=1}^n p_i \cdot q \geq \sum_{i=1}^n p_i \cdot \hat{q}$. Thus, we obtain

$$\omega \cdot e \geq \sum_{i=1}^n \frac{z_i - c}{\alpha_i} \qquad (A1)$$

for all $z \in B_u$. Let $d = (c_1, \ldots, c_n)$, $a = (\alpha_1, \ldots, \alpha_n)$ and $A = (\omega \cdot e) a \otimes \Delta + d$ and note that $B_u \leq A$ follows from equation (A1). Since $\eta_\omega(A) = \alpha_i \omega_i + d = x$, we have $x \in N_\omega(B_u)$.

Finally, consider the case in which $p_i \cdot q < \omega_i$ for some $i$. Let $I$ be the set of all such agents. Let $J_i^* = \{j | u_i^j \geq u_i^m \, \forall m\}$ be the bliss outcomes for $i$. If $i \in I$, then $J(q) \subset J_i^*$ since otherwise $i$ is not choosing a utility maximizing plan. Furthermore, since $q$ is a minimal cost solution for consumer $i \in I$, $p_i^j = p_i^m$ for all $j, m \in J_i(q)$. Define $\bar{p} = (\bar{p}_1, \ldots, \bar{p}_n)$ as follows: $\bar{p}_i^j = \omega_i$ if $i \in I$ and $j \in J_i$ and $\bar{p}_i^j = p_i^j$ otherwise. It is easy to see that $(\bar{p}, q)$ is also a Lindahl equilibrium. Moreover, every consumer satisfies $\bar{p}_i \cdot q = \omega_i$ for all $i$. Thus, we can apply the argument above to show that $x \in N_\omega(B_u)$. This concludes the proof that $N_\omega(B_u) = L(u, \omega)$ for all $B$. □

### 7.4 Proof of Theorem 4

By Theorem 3, $x \in L(v, \omega)$ implies $x \in N_\omega(B_v)$. By Lemma A4, any element of $N_\omega(B_v)$ can be made into a Lindahl equilibrium payoff for a Lindahl equilibrium $(p, q)$ such that $p_i^j = p_i^m$ whenever $v_i^j = v_i^m$ and $p_i^j = 0$ whenever $v_i^j = 0$. Hence, if the allocation $j, m$ yields the same match for player $i$, the price of these two allocations must be the same for player $i$ and if $j$ yields no match for player $i$, then $p_i^j = 0$. Set $\pi_i^m = p_i^j$ for any $j$ such that $j(i) = m$ and note that $(\pi, q)$ is a WE. Therefore, $x \in W(v, \omega)$.

For the converse, take $x \in W(v, \omega)$. Let $(\pi, q)$ be a WE that yields the payoff vector $x$ and let $p_i^j = \pi_i^{j(m)}$. Then, clearly, $(p, q)$ is a LE and hence, $x \in L(v, \omega)$. □

### 7.5 Proof of Proposition 1

Let $(\mathbf{p}, \theta)$ be a WE and let $K = \{\mathbf{a}(1), \ldots, \mathbf{a}(k)\}$ be the set of all feasible allocations. Then, for every feasible allocation $\mathbf{a}(j)$, let

$$p_i^j = \sum_{h \in \mathbf{a}_i} \mathbf{p}(h)$$

Then, it is easy to verify that $(p, \theta)$ is an LE. □

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