MCS: Dualizing Topkis' theorem

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Prices and Quantities at equilibrium

In the quasilinear setting if f is the indirect utility function then the set of equilibrium prices given a production q is

$$P(q) = \arg\max_{p} p^\top q - f(p)$$

Similarly if c is the cost of production function the set of equilibrium quantities at a price p is

$$Q(p) = \arg\max_{q} q^\top p - c(q)$$

Do P and Q satisfy a kind of MCS in the gross substituability setting?

The strong set order on Prices

Submodularity

A function $f : \mathbb{R}^d \to \mathbb{R}$ is said to be submodular if

$$f(p \vee p') + f(p \wedge p') \leq f(p) + f(p')$$

If f is C^2 this is equivalent to $\frac{\partial^2 f}{\partial p_i \partial p_j} \leq 0$ for $i \neq j$

Veinott's strong set order (1989)

Let $P, P' \subset \mathbb{R}^d$, we say that $P \leq_{P-order} P'$ if for any $p \in P, p' \in P'$ we have $p \vee p' \in P'$ and $p \wedge p' \in P$.
We say that P is a lattice if $P \leq_{P-order} P$.

Remark on ordering functions

If the sets P, P' are convex then $P \leq_{P-order} P'$ is equivalent to

$$\iota_P(p \wedge p') + \iota_{P'}(p \vee p') \leq \iota_P(p) + \iota_{P'}(p')$$

for any p, p'. To unify submodularity and Veinott's strong set order we propose the following P-order on functions

P-order

Let f, f' two functions. We say that f is smaller than f' in the P-order, $f \leq_{P-order} f'$, if for any p', p we have

$$f(p \wedge p') + f'(p \vee p') \leq f(p) + f'(p')$$

Note that $f \leq_{P-order} f$ is equivalent to the submodularity of f.

A first monotone comparative statics

Theorem [Top98]

Let f a function, $q \in \mathbb{R}$ and S a lattice, then

$$P(q, S) = \arg\max_{p \in S} p^\top q - f(p)$$

is increasing in the $P - order$ with respect to (q, S) if and only if f is submodular.

An order on the quantities?

Can Topkis' theorem be used to derive an MCS on quantities?
Set $Q : p \rightrightarrows P^{-1}(p)$. If f is convex and $S = \mathbb{R}^d$ note that $P(q) = \partial f^*(q)$. Then since $\partial f = (\partial f^*)^{-1}$ we have $Q(p) = \partial f(p)$. And we need to study

$$Q(p) = \arg\max_q q^\top p - f^*(q)$$

where f^* is supermodular.

Questions

- How does $Q(p)$, the set of maximizers of a submodular (maybe more) function, behave as p increases?
- Is there a characterization of submodularity involving only the subdifferentials of the function?
- Is there a characterization of submodularity using Legendre-Fenchel duality?

Exchangeability for quantities

Exchangeability

Q-order

Given two functions c, c' we say that c is smaller than c' in the Q-order, $c \leq_{Q-order} c'$, if if for any $q, q' \in \mathbb{R}^d$ and any $\delta_1 \in [0, (q - q')^+]$ there is $\delta_2 \in [0, (q - q')^-]$ such that

$$c(q - (\delta_1 - \delta_2)) + c'(q' + (\delta_1 - \delta_2)) \leq c(q) + c'(q')$$

we say that c is exchangeable if $c \leq_{Q-order} c$.

Using the connection between convex sets and indicator functions we have the following order on convex sets. $Q \leq_{Q-order} Q'$ if $q - (\delta_1 - \delta_2) \in Q$ and $q' + (\delta_1 - \delta_2) \in Q'$.
In particular we say that Q is a matron if $Q \leq_{Q-order} Q$.

S/M - convexity

$M^\natural$-convexity [MS04]

f is $M^\natural$-convex if for any q, q' and $i \in \mathrm{supp}^+(q - q')$ there is $j \in \mathrm{supp}^-(q - q') \cup \{0\}$ and $\alpha > 0$ such that

$$f(q) + f(q') \geq f(q - \alpha(e_i - e_j)) + f(q' + \alpha(e_i - e_j))$$

S-convexity[CL23]

f is S-convex if for any q, q' and $i \in \mathrm{supp}^+(q - q')$ there is $j \in \mathrm{supp}^-(q - q') \cup \{0\}$ and $\alpha, \beta > 0$ such that

$$f(q) + f(q') \geq f(q - \alpha(e_i - \beta e_j)) + f(q' + \alpha(e_i - \beta e_j))$$

A first duality result

Theorem

Let c, c' two convex (closed, proper, lsc) functions the following assertions are equivalent

- $c \leq_{Q-order} c'$
- $c'^{*} \leq_{P-order} c^{*}$

Sketch of proof

The Q-order, $c \leq_{Q-order} c'$, amounts to, by replacing quantizers with $\sup, \inf$

$$\sup_{\delta_1 \in [0,(x-y)^+]} \inf_{\delta_2 \in [0,(x-y)^-]} c(x - \delta_1 + \delta_2) + c'(y + \delta_1 - \delta_2) \leq c(x) + c'(y)$$

for every x, y. Then by Lagrangian duality we have equivalently

$$\sup_{\substack{\lambda \in \mathbb{R}^d \\ \mu \in \mathbb{R}^d}} \mu(x + y) + \lambda^+ x - \lambda^- y - c^*(\mu + \lambda) - c'^*(\mu) \leq c(x) + c'(y)$$

for every x, y. Thus we can rewrite it as

$$c^*(\mu + \lambda^+) + c'^*(\mu - \lambda^-) \leq c^*(\mu + \lambda) + c'^*(\mu)$$

which is equivalent to $c'^* \geq_{P-order} c^*$ (set $\lambda = p - p'$ and $\mu = p'$)

MCS for exchangeable functions

Theorem

Let c a convex (closed, proper, lsc) function. Then the following assertions are equivalent.

- c is exchangeable
- c^* is submodular
- For $p \leq p'$, $\pi_{p=p'} \partial c^*(p') \leq_{Q-order} \pi_{p=p'} \partial c^*(p)$

If any one them is satsified then

$$\bar{q} - \arg\max_{q \leq \bar{q}} p^\top q - c(q)$$

is increasing in the Q-order in $\bar{q}$

Regular case

A remark, in case c^* is differentiable. The subdifferential is reduced to a point and by theorem we have $\pi_{p=p'}\{\nabla c^*(p')\} \leq_{Q-order} \pi_{p=p'}\{\nabla c^*(p)\}$ for $p \leq p'$ equivalent to c^* submodular. However $\pi_{p=p'}\{\nabla c^*(p')\} \leq_{Q-order} \pi_{p=p'}\{\nabla c^*(p)\}$ is equivalent to

$$\forall \delta_1 \in [0, (\nabla c^*(p') - \nabla c^*(p))^+ 1_{p=p'}], \exists \delta_2, \delta_2 1_{p=p'} \in [0, (\nabla c^*(p') - \nabla c^*(p))^- 1_{p=p'}],$$
$$\nabla c^*(p') - (\delta_1 - \delta_2) = \nabla c^*(p'), \quad \nabla c^*(p) + (\delta_1 - \delta_2) = \nabla c^*(p)$$

which enforces $(\nabla c^*(p') - \nabla c^*(p))^+ 1_{p=p'} = 0$ or equivalently $\nabla c^*(p')_z \leq \nabla c^*(p)_z$ for all z such that $p_z = p'_z$. The converse is also true.

Remark

Under the assumption of differentiability of c^*, submodularity is equivalent to [KC82] GS.

Recap on Duality results

Theorem [MS04]

The $L^\natural$-convexity and $M^\natural$-convexity properties are in duality with respect to the Legendre-Fenchel conjugacy.

Theorem [CL23]

Under some regularity assumptions of f we have the following equivalence: f is S-convex if and only if f^* is submodular.

$$
\begin{array}{lcccc}
f & M^\natural\text{-convexity} & \Rightarrow S\text{-convexity} \Rightarrow & \text{Exchangeability} & \Rightarrow \text{Supermodularity} \\
 & \Updownarrow & \Updownarrow + \text{Regular} & \Updownarrow & \\
f^* & L^\natural\text{-convexity} & \text{Submodularity} & \text{Submodularity} &
\end{array}
$$

Rejection set is non decreasing

Recall that

$$\bar{q} - \arg\max_{q \leq \bar{q}} p^\top q - c(q)$$

This is exactly the definition of GS found in [HM05]. Let $C : 2^X \to \mathbb{R}$ and $R(.) = X \setminus C(.)$ then C satisfies GS if for $A \subset B$ we have $R(A) \subset R(B)$. This property is essential to the convergence of their generalization of Gale and Shapley algorithm.

An application to NTU matching

Equilibrium matching

Let $n_x, m_y \geq 0$ be two vectors of population and α_{xy}, γ_{xy} two vectors of preferences. A matching outcome $(\mu, u, v) \in \mathbb{R}_+^{(\{0\}\cup\mathcal{X})\times(\{0\}\cup\mathcal{Y})} \times \mathbb{R}^{\mathcal{X}} \times \mathbb{R}^{\mathcal{Y}}$ is an *equilibrium matching* if:

(i) μ is a feasible matching if $\sum_{y\in\mathcal{Y}\cup\{0\}} \mu_{xy} = n_x$ and $\sum_{x\in\mathcal{X}\cup\{0\}} \mu_{xy} = m_y$.

(ii) stability conditions hold, that is

$$\max\left(u_x - \alpha_{xy}, v_y - \gamma_{xy}\right) \geq 0$$
$$u_x \geq \alpha_{x0} \text{ and } v_y \geq \gamma_{0y},$$

(iii) weak complementarity holds, that is

$$\begin{aligned}
\mu_{xy} &> 0 \implies \max\left(u_x - \alpha_{xy}, v_y - \gamma_{xy}\right) = 0 \\
\mu_{x0} &> 0 \implies u_x = \alpha_{x0} \\
\mu_{0y} &> 0 \implies v_y = \gamma_{0y}.
\end{aligned}$$

Towards a general framework

We can restate last definition by ensuring that there exists $(\mu, U, V) \in \mathbb{R}_+^{\mathcal{A}} \times \mathbb{R}^{\mathcal{X}\times(\mathcal{Y}\cup\{0\})} \times \mathbb{R}^{(\{0\}\cup\mathcal{X})\times\mathcal{Y}}$ such that

(i) the matching μ is feasible;

(ii) the equality $\max\{U_{xy} - \alpha_{xy}, V_{xy} - \gamma_{xy}\} = 0$ holds for every pair $xy \in \mathcal{X} \times \mathcal{Y}$;

(iii) one has the following implications

$$\begin{cases} \mu_{xy} > 0 \implies U_{xy} = \max_{y' \in \mathcal{Y}}\{U_{xy'}, \alpha_{x0}\} \text{ and } V_{xy} = \max_{x' \in \mathcal{X}}\{V_{x'y}, \gamma_{0y}\} \\ \mu_{x0} > 0 \implies \alpha_{x0} = \max_{y' \in \mathcal{Y}}\{U_{xy'}, \alpha_{x0}\} \\ \mu_{0y} > 0 \implies \gamma_{0y} = \max_{x' \in \mathcal{X}}\{V_{x'y}, \gamma_{0y}\}. \end{cases}$$

You can switch from (μ, U, V) to (μ, u, v) by setting $u_x = \max_y U_{xy}$. And conversely you can switch from (μ, U, V) to (μ, u, v) by setting $U_{xy} = \min(u_x, \alpha_{xy})$

Convex analysis formulation

Note that condition (iii) can be rewritten as

$$\mu \in \partial G(U) \text{ where } G\left(U\right) := \sum_{x\in\mathcal{X}} n_x \max_{y\in\mathcal{Y}}\{U_{xy}, \alpha_{x0}\}$$

and $\mu_{x0} = n_x - \sum_{y\in\mathcal{Y}} \mu_{xy}$. Note that in this case, it is easy to show that $G(U)$ has expression

$$G\left(U\right) = \max_{\mu\geq 0}\left\{ \sum_{xy\in\mathcal{X}\times\mathcal{Y}} \mu_{xy}(U_{xy} - \alpha_{x0}) : \sum_{y\in\mathcal{Y}} \mu_{xy} \leq n_x \right\} + \sum_{x\in\mathcal{X}} n_x \alpha_{x0}.$$

Welfare function

Definition

Let $\underline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$. $G : \underline{\mathbb{R}}^{\mathcal{X}\times\mathcal{Y}} \to \mathbb{R}$ is a welfare function if G is convex (closed proper lsc), $\mathrm{dom}\, G^*$ is compact and $0 = \min\left(\mathrm{dom}\, G^*\right)$.

As before it is possible to write G in the following fashion. Encoding feasability in the Welfare function G^*.

$$G\left(U\right) = \max_{\mu \geq 0} \left\{ \sum_{y\in\mathcal{Y}} \mu_{xy} U_{xy} - G^*\left(\mu\right) \right\}.$$

Equilibrium matching

Definition

A generalized equilibrium matching between two welfare functions G and H, with initial preferences α, γ is a triple $(\mu, U, V) \in \mathbb{R} \times \underline{\mathbb{R}} \times \underline{\mathbb{R}}$ with

(i) $\max \{U - \alpha, V - \gamma\} = 0$

(ii) $\mu \in \partial G(U) \cap \partial H(V)$.

Note that if G, H are of the form

$$G(U) = \max_{\mu \geq 0} \left\{ \sum_{xy \in \mathcal{X} \times \mathcal{Y}} \mu_{xy}(U_{xy} - \alpha_{x0}) : \sum_{y \in \mathcal{Y}} \mu_{xy} \leq n_x \right\} + \sum_{x \in \mathcal{X}} n_x \alpha_{x0}.$$

a generalized equilibrium matching is an equilibrium matching.

Existence of a generalized equilibrium matching

Theorem

Assume G, H are submodular welfare functions then for any initial preferences α, γ a generalized equilibrium matching exists.

Deferred acceptance algorithm

At step 0, we set $\mu^{A,0} = \min(n^G, n^H)$ and $\mu^{T,-1} = 0$.

Proposal phase: The x's make proposals to the y's subject to availability constraint:

$$\mu^{P,k} \in \underset{\mu^{T,k-1} \leq \mu \leq \mu^{A,k}}{\arg\max} \mu^\top \alpha - G^*(\mu),$$

Disposal phase: The the y's pick their best offers among the proposals:

$$\mu^{T,k} \in \underset{\mu \leq \mu^{P,k}}{\arg\max} \mu^\top \gamma - H^*(\mu)$$

Update phase: The number of available offers is decreased by the number of rejected ones:

$$\mu^{A,k+1} = \mu^{A,k} - \left(\mu^{P,k} - \mu^{T,k}\right).$$

Sketch of proof

- $\mu^{A,k}$ is decreasing, thus it converges. This implies up to extraction that $\mu^{P,k}, \mu^{T,k} \to \mu$.
- Set $\tau^{P,k} \in \arg\min_{\tau \geq 0} (\mu^{P,k})^\top \alpha + G(\alpha - \tau)$ and similarly for $\tau^{T,k}$. Topkis' theorem ensures that $\tau^{P,k}$ is decreasing and $\tau^{T,k}$ is increasing. Note that we have $\min(\tau^{T,k}, \tau^{P,k}) = 0$, otherwise $\min(n^G, n^H) > \mu = \mu^{A,K} = \mu^{A,0} = \min(n^G, n^H)$. Thus the limit of $\tau^{T,k}, \tau^{P,k}$ satisifies $\min(\tau^P, \tau^T)$.
- Set $U = \alpha - \tau^P, V = \gamma - \tau^T$. By duality $\mu \in \partial G(U) \cap \partial H(V)$.

Random Utility model

All the functions generated by the random utility model are submodular Welfare functions. More precisely for any collection $(\mathbf{P}_x)$ of probability measures on $\mathbb{R}^{\mathcal{Y}}$ the function

$$G(\alpha) = \sum_{x \in \mathcal{X}} n_x \mathbb{E}_{\mathbf{P}_x} \left[\max_{y \in \mathcal{Y}} \{ \alpha_{xy} + \varepsilon_y, \varepsilon_0 \} \right]$$

is a submodular welfare function.

The Logit model i

If we assume that the random utility terms $(\varepsilon_{xy})_y$ follow i.i.d. Gumbel distributions, then it is well-known that

$$G(\alpha) = \sum_{x \in \mathcal{X}} n_x \log(1 + \sum_{y \in \mathcal{Y}} \exp \alpha_{xy}).$$

In this case, we have $M^G(\alpha, \overline{\mu}) = \mu$ which satisfies $\mu_{xy} = \min\left(\mu_{x0} \exp(\alpha_{xy}), \overline{\mu}_{xy}\right)$, where μ_{x0} is defined by

$$\mu_{x0} + \sum_{y \in \mathcal{Y}} \min\left(\mu_{x0} \exp(\alpha_{xy}), \overline{\mu}_{xy}\right) = n_x,$$

and $T^G(\alpha, \overline{\mu}) = \tau$ is deduced by $\tau_{xy} = \max\left(0, \alpha_{xy} + \log \frac{\mu_{x0}}{\overline{\mu}_{xy}}\right)$.

The Logit model ii

As a result, if we assume that on the other side of the market, the random utility terms $(\eta_{xy})_x$ follow i.i.d. Gumbel distributions, then the generalized equilibrium matching is such that

$$\mu_{xy} = \min\left(\mu_{x0}\exp\left(\alpha_{xy}\right), \mu_{0y}\exp\left(\gamma_{xy}\right)\right).$$

where μ_{x0} and μ_{0y} are the unique solution to the following system of equations:

$$\mu_{x0} + \sum_{y\in\mathcal{Y}} \min\left(\mu_{x0}\exp\left(\alpha_{xy}\right), \mu_{0y}\exp\left(\gamma_{xy}\right)\right) = n_x,$$
$$\mu_{0y} + \sum_{x\in\mathcal{X}} \min\left(\mu_{x0}\exp\left(\alpha_{xy}\right), \mu_{0y}\exp\left(\gamma_{xy}\right)\right) = m_y.$$

Perspectives

We have shown that in the convex optimization framework submodularity is equivalent to exchangeability as well as a generalized form of [KC82] GS. This result has an application to a generalized Gale and Shapley algorithm. We now ask ourselves the following two questions to go beyond the convex framework

- How relevant is the Q-order for general correspondences?
- What are the conditions for Gale and Shapley to converge ?

Thank you!

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