Lindahl Equilibrium as a Collective Choice Rule

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Room Allocation with Externalities

3 students must be assigned to 2 rooms; a double and a single room.

every student prefers the single room

students 1 and 2 do not want to be matched with each other

student 3 would much rather be matched with student 1 than student 2.

Collective Choice

Utilities depend on the *allocation* and not just on private consumption

3 possible allocations: allocation $k \in \{1, 2, 3\}$ assigns student k to the single room.

$$\begin{aligned} u_1 &= (10, 2, 0) \\ u_2 &= (5, 10, 0) \\ u_3 &= (0, 9, 10) \end{aligned}$$

every student prefers the single room: $u_i^i = 10$

students 1 and 2 prefer not to be matched: $u_1^3 = u_2^3 = 0$

student 3 prefers to be matched with student 1: $u_3^1 = 0, u_3^2 = 9$

Collective Choice Markets

1. Each group member is given a budget of fiat money
2. Each member confronts a price for each of the relevant alternatives under consideration.
3. Each member chooses a lottery over alternatives that maximizes utility subject to the budget constraint.
4. The organization acts as an auctioneer and chooses a lottery over alternatives that maximizes revenue.
5. An allocation is feasible if all members and the auctioneer choose it

Collective Choice Market in Room Example

1. Student i has $\omega_i = 1$ units of fiat money
2. Student i confronts a price $p_i = (p_i^1, p_i^2, p_i^3)$ for the three room allocations. Prices differ across students
3. Each student chooses a room allocation lottery q_i that maximizes $q_i \cdot u_i$ subject to the budget constraint $\{q_i | p_i \cdot q_i \leq \omega_i\}$
4. The auctioneer chooses the revenue maximizing lottery $q \in \arg\max_{q'} q' \cdot \sum_{i=1}^{3} p_i$
5. Feasiblity: $q_1 = q_2 = q_3 = q$

Lindahl Equilibria

$$u_1 = (10, 2, 0)$$
$$u_2 = (5, 10, 0)$$
$$u_3 = (0, 9, 10)$$

Equilibrium 1

$$\begin{aligned}
\text{single} &= 1 && 2 && 3\\
p_1 &= 2.42 && 0.49 && 0.00\\
p_2 &= 0.58 && 1.15 && 0.00\\
p_3 &= 0.00 && 1.36 && 1.51\\
\sum p_i &= 3.00 && 3.00 && 1.51\\
q &= 0.27 && 0.73 && 0
\end{aligned}$$

Equilibrium 2

$$\begin{aligned}
\text{single} &= 1 && 2 && 3\\
p'_1 &= 2.37 && 0.00 && 0.00\\
p'_2 &= 0.63 && 1.27 && 0.00\\
p'_3 &= 0.00 && 1.73 && 1.93\\
\sum p'_i &= 3.00 && 3.00 && 1.93\\
q' &= 0.42 && 0.58 && 0
\end{aligned}$$

Why this Mechanism?

Generalization of market-based mechanisms (Hylland and Zeckhauser, 1979) for allocation problems.

Handles externalities or complementarities.

Yields ex ante Pareto efficient outcomes, unlike deterministic mechanisms.

Main Question and Result

What notion of equity/fairness is implied by equilibria of collective choice markets?

Map collective choice problem to an $n-$person bargaining problem

Define the *weighted Nash Bargaining set*

Main Result

1. Every equilibrium of a collective choice market corresponds to an element of the weighted Nash Bargaining set

2. Every element of the weighted Nash Bargaining set corresponds to an equilibrium of the collective choice market

Collective Choice Markets

Collective Choice Problem

n agents

k outcomes plus a disagreement outcome that gives zero utility to everyone

social outcome: a lottery q over the k-outcomes

$u_i = (u_i^1, \ldots, u_i^k)$ is i's utility index

$u_i \cdot q$ is i's utility if the social outcome is q

u_i is non-negative and not identically zero

The utility profile $u = (u_1, \ldots, u_n)$ defines a collective choice problem

Collective Choice Market: Consumers

$e = (1, \dots 1), q = (q^1, \dots, q^k), p_i = (p_i^1, \dots, p_i^k)$

Consumer i has one unit of fiat money and purchases probability q^j of outcome j at price p_i^j to maximize utility:

$$
\begin{aligned}
\underset{q}{\text{maximize}} \quad & u_i \cdot q \\
\text{subject to} \quad & p_i \cdot q \leq \omega_i, \text{ (Budget constraint)}, \\
& e \cdot q \leq 1. \quad \text{(Probability constraint)}
\end{aligned} \tag{1}
$$

A minimal cost solution to the consumer's problem is a solution to the above problem that minimizes the expenditure of fiat money

Collective Choice Market: Firm

The firm chooses the social outcome q to maximize profit:

$$\begin{aligned} \underset{q}{\text{maximize}} \quad & \sum_{i=1}^{n} p_i \cdot q \\ \text{subject to} \quad & e \cdot q \leq 1. \quad \text{(Probability constraint)} \end{aligned} \tag{2}$$

Collective Choice Market: Equilibrium

The pair (p, q) is a *Lindahl equilibrium* (LE) of (u, ω) if q is a minimal-cost solution to every consumer's maximization problem at prices p_i and budget ω_i and solves the firm's maximization problem at prices p.

Lemma 1 Every collective choice market (u, ω) has a Lindahl equilibrium; all Lindahl equilibria are Pareto efficient.

Bargaining

The Bargaining Problem

Bargaining Problem: full dimensional and comprehensive polytope $B \subset \mathbb{R}^n$

Disagreement point: $d(B)$ is the component-wise minimum of B

Comprehensive: if $y \in B$ then B contains all points $d(B) \le x \le y$

unit simplex: Δ is the $n-$dimensional unit simplex

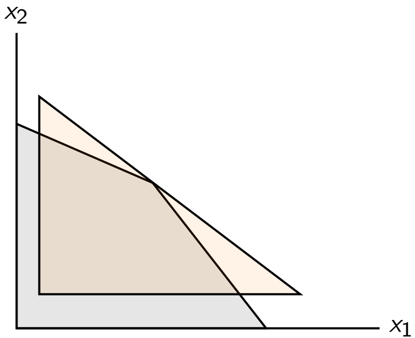
Simplex: $B = a \odot \Delta + z$ affine transformation of the unit simplex where $a \odot x = (a_1 x_1, \dots, a_n x_n)$ and $a_i > 0$ for all i.

Ordering Bargaining Problems

$A \leq B$ if B dominates A in the weak set order:

for every $x \in A, y \in B$ there exist $x' \in A$, $y' \in B$ such that

$$\begin{aligned} x' &\leq y \\ x &\leq y' \end{aligned}$$



Weighted Nash Bargaining Solution

for weights $\omega_i > 0$, define

$$f_\omega(B, x) := \sum_i \omega_i \log(x_i - d_i(B))$$

The weighted Nash Bargaining Solution, $\eta_\omega(B)$, is the unique outcome that maximizes f_ω in B

weighted Nash Bargaining Solution in simplices

Let $\sum \omega_i = 1$. Then:

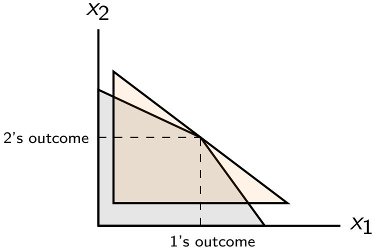
$\eta_\omega(B) = \omega$ if B is the unit simplex Δ

$\eta_\omega(B) = \omega \odot a + z$ if $B = a \odot \Delta + z$

Weighted Nash Bargaining *Set*

All outcomes of B that are the weighted Nash bargaining solution of a game A such that $B \leq A$:

$$N_\omega(B) := \{x \in B \cap \{\eta_\omega(A)\} \mid B \leq A\}$$



Definition stays unchanged if A must be a simplex.

Characterization

S is a set valued solution to the bargaining problem if $S(B) \subset B$

Scale Invariance: $S(a \otimes B + z) = a \otimes S(B) + z$ whenever $a >> 0$.

Efficiency: $S(\Delta) = \{x\}$ for some x such that $x \cdot e = 1$ and $x >> 0$.

Consistency: $B \leq A$ implies $S(A) \cap B \subset S(B)$.

Implementability: $x \in S(B)$ implies $\{x\} = S(A)$ for some $A \geq B$.

Characterization

THEOREM 1: S satisfies the four axioms above if and only if $S = N_\omega$ for some $\omega >> 0$.

Replace efficiency with:

Symmetry: $S(\Delta) = \{\frac{1}{n}, \cdots, \frac{1}{n}\}$.

COROLLARY: S satisfies the symmetry, scale invariance, consistency and implementability if and only if $S = N_\omega$ with $\omega = (\frac{1}{n}, \cdots, \frac{1}{n})$.

Why a set valued solution?

Ann and Bob must divide a peanut butter cake and a chocolate cake.
Bob is allergic to peanuts while Ann likes both cakes equally.

Equal weights

Solution 1: Perles Maschler

pb-cake to Ann; ch-cake is shared equally between Bob and Ann.

Ann: *Since Bob doesn't care about pb-cake, from his perspective, it is as if we only had the chocolate cake.*

Perles and Maschler (1981) provide an axiomatic foundation for this solution.

Mediator could implement this solution by guaranteeing that Ann receives pb cake if negotiations break down. Improves Ann's range of outcomes without affecting Bob's range of outcomes and yields solution 1 as the unique outcome.

Solution 2: Nash

pb-cake to Ann; ch-cake to Bob.

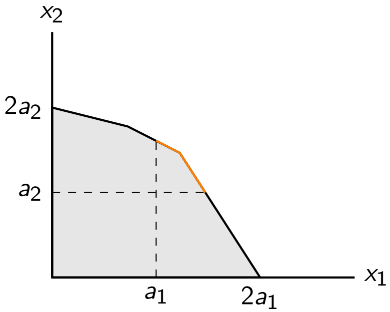
Bob: *My allergy should not affect the solution.*

Nash (1951) provides an axiomatic foundation for Bob's argument.

Mediator could implement solution 2 by offering to replace pb cake with chocolate cake, if necessary. Improves Bob's range of outcomes without affecting Ann's range of outcomes and yields solution 2 as the unique outcome.

Relationship to other Bargaining Solutions

Nash Bargaining Set contains the Nash Bargaining solution



With two players, the (equal weight) Nash Bargaining Set contains all standard bargaining solutions that satisfy scale invariance (Nash, Kalai-Smorodinsky, Perles-Maschler, the discrete Raiffa solution, ...)

Main Result

Collective Choice Problems to Bargaining Problems

A collective choice problem is defined by the utility profiles $u^j = (u_1^j, \ldots, u_n^j)$, $j = 1, \ldots, k$ of the k outcomes.

The convex and comprehensive hull of these utility profiles and the origin forms the bargaining problem B_u

Therefore, **for each** u **we get a bargaining problem** B_u

Bargaining Problems to Collective Choice Problems

Let B be any bargaining problem with the origin as disagreement point

Identify outcomes with the extreme points of B

utility profile at an extreme point is the utility profile of the corresponding outcome

Therefore, **for each B we get a collective choice problem u**

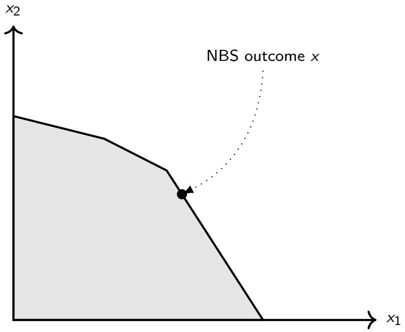
Main Result

THEOREM 2 The set of Lindahl equilibrium payoffs of (u, ω) is the same as the wNBS of the corresponding bargaining problem with weights ω:

$$L(u, \omega) = N_{\omega}(B_u)$$

for all u, ω.

Elements of the Nash Bargaining Set are Lindahl Equilibria



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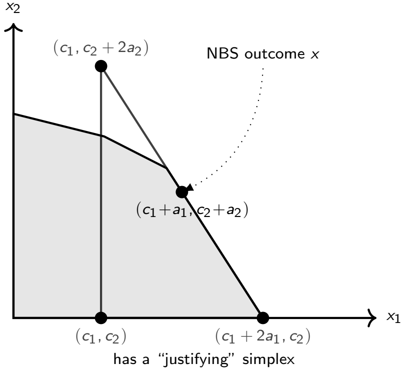


$x_1 = a_1 + c_1, x_2 = a_2 + c_2$ where a and c are parameters of the justifying simplex

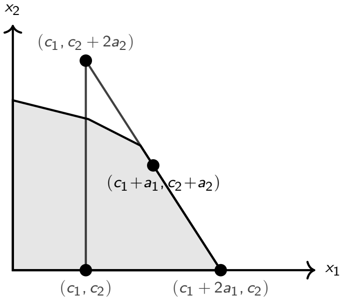
Every weighted NB-Outcome is a Lindahl Equilibrium

Let q yield payoffs x.

Set $p_i^j = \frac{u_i^j - c_i}{a_i}$

Consumer can afford allocations $\hat{q}$ such that

$$
\begin{aligned}
p_i \hat{q} = \textstyle\sum_j \frac{u_i^j - c_i}{a_i} \hat{q}^j &\leq 1 \\
\textstyle\sum_j u_i^j \hat{q}^j &\leq a_i + c_i \textstyle\sum_j \hat{q}^j \\
&\leq a_i + c_i \\
&= u_i q
\end{aligned}
$$



Every Lindahl Equilibrium is a weighted NB outcome

In a Lindahl equilibrium every consumer solves:

$$\begin{aligned} \underset{q}{\text{maximize}} \quad & u_i \cdot q \\ \text{subject to} \quad & e \cdot q \le 1 \quad (c_i), \\ & p_i \cdot q \le 1 \quad (a_i) \end{aligned}$$

$$\begin{aligned} \underset{c_i, a_i \ge 0}{\text{minimize}} \quad & c_i + a_i \\ \text{subject to} \quad & c_i e + a_i p_i \ge u_i \end{aligned}$$

value of the dual, $c_i + a_i$, is equal to value of the primal

(c_i, a_i, q) optimal $\Leftrightarrow$ constraints hold and $q \cdot (c_i e + a_i p_i - u_i) = 0$

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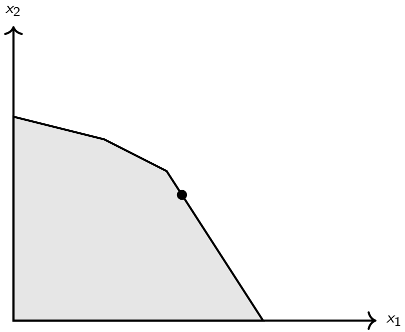
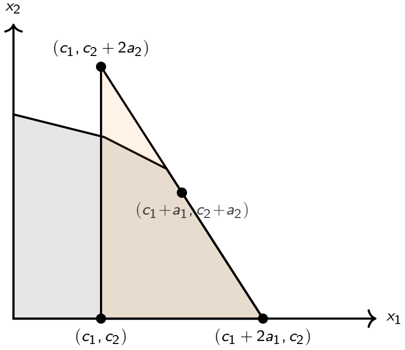
(c_i, a_i, q) optimal $\Leftrightarrow$ constraints hold and $q \cdot (c_i e + a_i p_i - u_i) = 0$

use the values of the dual to construct justifying simplex

- disagreement point: $d = (c_1, \ldots, c_n)$
- extreme points $(c_1, \ldots, c_{i-1}, c_i + na_i, c_{i+1}, \ldots, c_n)$


x_2
x_1



x_2
$(c_1, c_2 + 2a_2)$
(c_1+a_1, c_2+a_2)
x_1
(c_1, c_2)
$(c_1 + 2a_1, c_2)$

Walras vs Lindahl

Matching

A group of agents must decide who matches with whom

A matching is a bijection j from the set of all agents to itself such that $j(j(i)) = i$ for all i. If $j(i) = i$, then i is said to be unmatched.

w_i^m is the utility of agent i when she matches with agent m.

$J \neq \emptyset$ is the set of feasible matches.

Matching Market:

1. Agents specify demands for individual matches (private goods)
2. Each agent has one unit of fiat money, must pay price π_i^m for matching with agent m
3. Agents choose lotteries over partners that maximize their utilities subject to the budget constraint
4. Feasibility: there is a lottery over J that implements all the chosen lotteries

Collective Choice Market:

1. Agents specify demands for matchings (collective goods)
2. Each agent has one unit of fiat money, must pay price p_i^j for matching j
3. Agents choose lotteries q over matchings, J, that maximize their utilities subject to their budget constraint, $p_i \cdot q \leq 1$
4. Auctioneer chooses lottery q that maximizes revenue
5. Feasibility requires that all choices coincide

Equivalence Result

THEOREM 3 Lindahl equilibrium allocations coincide with Walrasian equilibrium allocations

Corollary

(1) Every matching market with equal budgets has a Walrasian equilibrium

(2) A matching is in the (equal weight) NB set if and only if it is a Walrasian outcome of a matching market with equal budgets

Allocation Problems with no Externalities

Walrasian economy 1

2 consumers, 2 goods, equal budgets of fiat money.

$$v_1(\{a\}) = 3, v_1(\{b\}) = 2, v_1(\{a, b\}) = 3$$
$$v_2(\{a\}) = 2, v_2(\{b\}) = 1, v_2(\{a, b\}) = 2$$

Bargaining set $\{o, (3, 1), (2, 2)\}$

Unique Walrasian equilibrium: each consumer receives a or b with probability $1/2$

Walrasian equilibrium payoff $(5/2, 3/2)$ is a Lindahl equilibrium

Walrasian economy 2

2 consumers, 4 goods: $\{a, b, c, d\}$, equal budgets of fiat money

$$\begin{aligned} v_1(M) &= \min\{|M|, 3\} \\ v_2(M) &= \min\{|M|, 2\} \end{aligned}$$

where $M \subset \{a, b, c, d\}$.

Bargaining set $\{o, (3, 1), (2, 2)\}$: same as economy 1

Unique Walrasian equilibrium in which each consumer receives two goods

Equilibrium payoff $(2, 2)$ is a Lindahl equilibrium payoff

Lindahl vs Walrasian equilibria

PROPOSITION 1 Every Walrasian equilibrium payoff of a discrete exchange economy is a Lindahl equilibrium payoff.

Lindahl equilibria depend only on the bargaining game

Two exchange economies that yield the same bargaining game may have distinct Walrasian equilibria. (Sertel and Yildiz (2003))

Related Literature

Bargaining: Axiomatic treatment is closely related to Nash (1950), Kalai and Smorodinsky (1975), Perles and Maschler (1981). Thomson (1994) provides a comprehensive survey.

Lindahl Allocations and Bargaining: Fain, Guel, and Munagala (2016) demonstrate that in a public goods setting with linear costs and transfers, the Nash bargaining solution coincides with the Lindahl equilibrium.

Walrasian Equilibria as Allocation Mechanisms: Hylland and Zeckhauser (1979) propose Walrasian equilibria as solutions to stochastic allocation problems. Gul, Pesendorfer, and Zhang (2019) extend Hylland and Zeckhauser's work from unit demand preferences to general gross-substitutes preferences. Budish (2011) studies approximate Walrasian equilibrium with combinatorial prices, but without randomization. Collective choice markets allow for arbitrary preferences, public goods, and externalities.

Fairness and Equilibrium: Foley (1967), Schmeidler and Vind (1972), and Varian (1974) associate equity with envy-freeness. Walrasian equilibria with equal budgets are envy-free. The Nash Bargaining Set with equal weights is a notion of fairness adapted to collective choice markets.