Direct Complementarity

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Sciences Po
New Perspectives on Substitutes

How should we define complementarity?

- Let preferences $\succeq$ on $\mathbb{R}^n$ (bundle space) be represented by a smooth function $u : \mathbb{R}^n \to \mathbb{R}$. Denote partial derivatives by u_i, u_{ij}, etc.
- Naively, we might try classifying goods i and j as complements or substitutes according to the sign of u_{ij}. (Appears in early work of Edgeworth, Pareto, etc.)

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- Naively, we might try classifying goods i and j as complements or substitutes according to the sign of u_{ij}. (Appears in early work of Edgeworth, Pareto, etc.)
- **Problem**: This is sensitive to the choice of representation: if $u_i u_j \neq 0$, we can make the sign of u_{ij} whatever we want by replacing u with $f \circ u$ for smooth increasing f. (Noticed at least as early as Slutsky (1915).)
- If $v = f \circ u$ then
$$v_{ij} = f' u_{ij} + f'' u_i u_j$$
- Interestingly, if(f) $u_i u_j = 0$, then $\operatorname{sgn}(u_{ij})$ *is* invariant to representation. More on this later.

Demand-Based Definitions

To fill the vacuum, we have:

- **Gross Complementarity** of goods i and j: Negative uncompensated cross-price effect:

$$\frac{\partial x_i}{\partial p_j} < 0$$

with prices p_{-j} and nominal income y fixed.

- **Hicks-Allen Complementarity** of goods i and j (1934): Negative compensated cross-price effect:

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with prices p_{-j} and utility fixed.
- *Samuelson's Complaint* (1974): These definitions don't feel like they are about complementarity, except indirectly.
- Stigler (1950) said it was "difficult to see the purpose" in the Hicks-Allen definition. Harsh.
- If possible, we would like a definition more directly tied to interactions between two goods.

Definition of Direct Complements, Quasilinear Case

Consider quasi-linear utility function

$$u(x) = x_0 + f(x_1, \ldots, x_k)$$

Let H be the Hessian matrix for f; assume H invertible.
Cross-price effects on goods $1, \ldots, k$ are given by the matrix H^{-1}.

- Goods i, j are *direct complements* iff $H_{ij} = u_{ij} > 0$.

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- Goods i, j are Hicks-Allen/gross complements iff $H_{ij}^{-1} < 0$.[1]

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Common Ground – The Three-Good Quasilinear Case

Let

$$u(x_0, x_1, x_2) = x_0 + f(x_1, x_2)$$

Assuming strictly convex preferences, these are equivalent:

- Gross complementarity of Goods 1 and 2
- Hicks-Allen complementarity of Goods 1 and 2
- Direct complementarity of Goods 1 and 2, i.e. $u_{12} > 0$

This family of examples confirms the intuition which motivates the demand-theory definitions of complementarity. But it is very special:

Appearance of indirect demand effects – The Four-Good Quasilinear Case

Let

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- Hicks-Allen complementarity of goods i, j is *not* equivalent to $u_{ij} > 0$
- Intuition: The market for Good 3 allows for “indirect” cross-price effects between Goods 1 and 2

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- Intuition: The market for Good 3 allows for "indirect" cross-price effects between Goods 1 and 2
- If we let

$$H = \begin{pmatrix} -1 & -\varepsilon & \gamma \\ -\varepsilon & -1 & \delta \\ \gamma & \delta & -1 \end{pmatrix}$$

  where $\gamma\delta > \varepsilon > 0$, we find 1,2 are direct substitutes but Hicks-Allen complements.

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- **Idea**: The vector space of possible bundles is fundamental. "Goods" are just one choice of basis for this space.
- There is a unique definition of *direct complementarity* which...

    1. Matches the definition we just made in the quasilinear case.
    2. Is determined by first and second derivatives of utility at a given point.
    3. Is invariant to changes of basis.

- Also, it has other appealing equivalent definitions.

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- Restaurant M: Three goods: drinks, fries, burgers. Prices $p = (p_1, p_2, p_3)$.
- Restauarant M': Three goods: drinks, fries, "meal deal". Prices $q = (p_1, p_2, p_1 + p_2 + p_3)$. Negative quantities allowed. Identical set of available bundles, identical pricing for each bundle.

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- Let x represent quantities chosen at M, z for M'
- Cross-price effects between fries-drinks differ at restaurants M and M':

$$\frac{\partial z_2}{\partial q_1} = \frac{\partial z_2}{\partial p_1} - \frac{\partial z_2}{\partial p_3}$$

$$= \frac{\partial x_2}{\partial p_1} - \frac{\partial x_2}{\partial p_3} - \frac{\partial x_3}{\partial p_1} + \frac{\partial x_3}{\partial p_3} \neq \frac{\partial x_2}{\partial p_1}$$

- ∂q_1 is different from ∂p_1, even though both talk about changes to price of drinks; different things are fixed!
- Similarly, "Effect on z_2" has different meaning from "Effect on x_2".

Basis-Sensitivity: What's going on?

- Recall that cross-price effects are also second derivatives of the expenditure function:

$$\frac{\partial x_2}{\partial p_1} = \frac{\partial x_1}{\partial p_2} = \frac{\partial^2 E}{\partial p_1 \partial p_2}$$

  where $E(p, u)$ is the minimum expenditure to achieve u at prices p.
- Crucially, price vectors do not lie in bundle space; they lie in its *dual*, i.e. price is a linear functional from bundles to $\mathbb{R}$

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- Crucially, price vectors do not lie in bundle space; they lie in its *dual*, i.e. price is a linear functional from bundles to $\mathbb{R}$
- Hicksian complementarity *really* looks at interaction between *dual* vectors (in their effect on E), then relies on an isomorphism between a vector space and its dual...but this isomorphism is *non-canonical*, i.e. basis-dependent.

Basis-Sensitivity: What's going on?

- Intuitively "Increase the price of fries by 1¢" does not have definite meaning, because you need to specify what you hold fixed (the basis).
- Even more obviously, "increase the price of a meal deal" is completely unclear as to what's held fixed. But complementarity should have definite meaning for "composite goods" as well.
- NB the basis-dependence here is not mere dependence on what goods are available (the span of all goods); it is dependence on how available goods are *expressed*. This is *ugly*. (*Weinstein's Complaint*)
- On the other hand, "I'll have another fry" has *basis-free* meaning. To give basis-free meaning to complementarity of a marginal fry with a marginal drink, we must work in bundle-space, not its dual, price-space.

The Advantage of Generality

- Instead of defining complementarity only for pairs of goods i, j, it's cleaner to do so for all pairs of vectors $(v, w) \in V \times V$
- Also, instead of looking at one utility function, we will (sometimes) look simultaneously at the set of all functions representing the same preference

Basis-Free Notation: First Derivatives

- All derivatives are taken at a fixed point x, which is often suppressed in notation
- $Du : V \to \mathbb{R}$ denotes the linear functional for which $Du(v)$ is the directional derivative in direction v
- $Du \in V^*$ is the basis-free analogue of the gradient
- Let $I := \mathrm{Ker}(Du)$ be the set of first-order-neutral goods – the tangent plane to the Indifference set at x
- Write $MRS_{w,z}$ for $Du(w)/Du(z)$, the marginal rate of substitution between w and z – invariant to choice of representation u

The Basis-Free Point of View: Second Derivatives

- The second derivative, $D^2u : V \times V \to \mathbb{R}$ is a (symmetric) bilinear form such that $D^2u(v, w)$ is the cross-partial taken in directions v, w.
- For any given basis, D^2u is represented by the Hessian matrix –

  $$D^2u(v, w) \equiv vHw^T$$

  – but it is fundamentally a basis-free object.
- (On the other hand, the matrix of cross-price effects is the Hessian of the expenditure function, and can be viewed basis-free as a symmetric bilinear form on *price space*, the dual of bundle space.)

Complementarity of Neutrals

$$\operatorname{sgn}(D^2u(x)(v_1, v_2)) \text{ is invariant to choice of representation u}$$

$$\Leftrightarrow$$

$$(Du(x)(v_1))(Du(x)(v_2)) = 0$$

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- That is, the sign of a cross-partial is well-defined iff one of the "goods" is actually a neutral
- Intuition: Taking $Du(x)(v_1) = 0$, $D^2u(x)(v_1, v_2) > 0$ means that heading in direction v_2 converts v_1 from a neutral to a good. Logically, this property refers only to preference, not representation.

A suitable numeraire

Proposition

The following are equivalent properties of a vector $v^* \in V$:

1. *For each $w \in I$, and for every (equivalently, any) smooth representation u, $D^2 u(v^*, w) = 0$.*
2. *v^* satisfies $D(MRS_{w,z})(v^*) = 0$ for all w, z with $z \notin I$.*

Let M be the subspace of vectors satisfying this condition. It is always non-trivial (dimensionality argument.) We make the regularity assumption that M is 1-dimensional and not contained in I. Any element x^* of M which is also a *good*, $Du(x^*) > 0$, is called a *numeraire*. Under our assumption, x^* is unique up to positive scalar.

Direct Complements, General Case: Definition A

- Let v_x^* be a numeraire as defined previously
- There is then a "locally quasilinear" representation u^x such that $D^2 u^x(v_x^*, v) = 0$ for all v
- By analogy with the quasilinear case, we call w, z *direct complements* at x if

$$D^2 u^x(x)(w, z) > 0$$

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- The choice among such representations doesn't matter
- One choice of such representation is "money-metric" utility for prices $p = Du(x)$ which induce demand x.

Direct Complements: Equivalent Definition B

At point x, there is a unique way of partitioning the set of pairs of bundles, V^2, into three classes C, N, S (complement, neutral, substitute) such that:

1. **Symmetry**: (v, w) and (w, v) are always in the same class.
2. **Unanimity**: If
$$D^2u(x)(v, w) > 0$$
for *all* smooth representations u, then $(v, w) \in C$. Similarly, if $D^2u(x)(v, w) = 0$ for all u then $(v, w) \in N$, and if negative then $(v, w) \in S$.
3. **Linearity**: If $(v, w) \in C$ and $(v, z) \in C \cup N$, then $(v, \alpha w + \beta z) \in C$ for any $\alpha > 0, \beta \geq 0$. Similarly for S.
4. **Neutrality**: If $v^* \in M$, then $(v^*, v) \in N$ for all v.

Direct Complements: Equivalent Definition C

- Any bundle w can be decomposed as

$$w = \lambda_w v_x^* + w^n$$

where v_x^* is the numeraire as before and $w^n \in I$

- Bundles are composed of **nutrients** (utility-rich at first-order, second-order-neutral) and *flavor* (first-order-neutral, with second-order impact).
- Direct complementarity of (w, z) at x is equivalent to

$$D^2u(w^n, z^n) > 0 \Leftrightarrow D^2u(w, z^n) > 0 \Leftrightarrow D^2u(w^n, z) > 0$$

for *any* representation u.[2]

- It is the flavors which are complements (or substitutes.)

[2] Derivatives taken at x as usual.

Direct Complements: Equivalent Definition D

- Direct complementarity of (w, z) is equivalent to

$$D(MRS_{w,v_x^*})(z) > 0 \Leftrightarrow D(MRS_{z,v_x^*})(w) > 0$$

- Note that if we used some other numeraire in place of v_x^*, we would not get a symmetric definition.

Calculus on Ordinal Functions (time permitting)

- Write $u \sim \hat{u}$ if $\hat{u} = f \circ u$ for a C^∞ function $f : \mathbb{R} \to \mathbb{R}$ with $f' > 0$ everywhere. Write $[u]$ for the associated equivalence class (an "ordinal C^∞ function").

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- Chain rule says $D\hat{u}(x) = f'(u(x))Du(x)$. So

$$D[u](x) = \{\alpha Du(x) : \alpha \in \mathbb{R}^+\} \in V^*/\mathbb{R}^+$$

  i.e. the derivative is defined up to positive scalar.
- Specifying $D[u](x)$ is equivalent to specifying a half-space in V, the "goods," bounded by the "indifference plane," $I = \mathrm{Ker}(Du)$

A summary of $D^2[u](x)$ (time permitting)

In generic cases, $D^2[u](x)$ can be summarized by

1. A bilinear form on I, defined up to positive scalar, together with
2. A vector $v_x^* \notin I$, the vector of "income effects," or "numeraire," defined up to scalar, satisfying $D^2[u](x)(v_x^*, w) = 0$ for all $w \in I$

Summarizing $D^2[u](x)$, continued (time permitting)

- Locally, preferences are convex iff $D^2u(x)(v, v) < 0$ for all $v \in I$, i.e. D^2u is negative-definite on I. In this case $-D^2u$ is an inner product on I, unique up to scalar.
- In a corresponding orthonormal basis, any two distinct basis elements are second-order neutral.
- Viewed in this basis, elements of I are direct substitutes if the "angle" between them is less than $\pi/2$, direct complements otherwise.

This suggests a way to derive from preferences the "characteristics" of Lancaster (1966).

Final Comments

- Assuming convexity, each good is a direct substitute for itself, as is logical. (Under the Hicksian definition, every good is a complement for itself, a point normally avoided by refusing to apply the definition in this case.)
- Even if you are comfortable fixing a set of basis goods, and mostly care about cross-price effects, the concept of direct complementarity helps you understand how complementarity in *preferences* leads to cross-price effects.
- After estimating a matrix of cross-price effects, inverting the matrix to find out about direct complementarity should give insight into which goods are “really” complementary.