Title might include words such as: {approximate, auctions, substitutes, budgets}

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Motivation

- In high-value auctions, bidders are often budget constrained.
- Dynamic designs—e.g., SMRA or CCA—help bidders “manage budgets”, but are not always effective (Janssen et al., 2017; Marsden and Sorensen, 2017; Fookes and McKenzie, 2017).
- Budgets are typically not elicited directly in sealed-bid combinatorial auctions or in the allocation phase of CCAs.
- With indivisible goods, markets might not clear with budget constraints even in trivial settings.

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Contribution

- Derive worst-case-yet-practical bounds on market-clearing for auctions for substitutes with budget constraints.
- Pin down and control the tradeoff between relaxing the supply and the budget constraints.
- Allows introduction of budget constraints to existing bidding languages for substitutes.

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Model

- There is a single seller (can extend to multiple sellers).
- There are m goods $j \in M$.
- The available supply of good j is s_j; cost of supplying a unit of good j is $c_j \geq 0$; so the total cost of supply is $\sum_{j \in M} c_j s_j$.
- There are n buyers $i \in N$.
- Buyer i can buy a bundle $x_i \in \{0, 1\}^m$ (can extend $\{0, 1\}^m$ to $\mathbb{Z}^m_{\geq 0}$).
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Model

- Agent i solves
$$\max_{x_i \in \mathcal{X}_i} V_i(x_i) - p \cdot x_i \quad \text{subject to} \quad p \cdot x_i \leq b_i.$$
- See, e.g., Bhattacharya et al. (2010); Dobzinski et al. (2012); Pai and Vohra (2014); Gul et al. (2019); Jagadeesan and Teytelboym (2023)...
- The demand correspondence of agent i is
$$D_i(p) = \arg\max_{x_i \in \mathcal{X}_i} \{V_i(x_i) - p \cdot x_i \mid p \cdot x_i \leq b_i\}.$$

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Competitive and approximate equilibrium

Definition

A *competitive equilibrium* for the economy $((V_i)_{i \in N}, b, s)$ is a price vector $p \geq c$ and demands $x_i \in D_i(p)$ for all $i \in N$ such that $\sum_{i \in N} x_i \leq s$, holding with equality for each $j \in M$ for which $p_j > c_j$.

Definition

An (α, β)-*competitive equilibrium* for the economy $((V_i)_{i \in N}, b, s)$ is a competitive equilibrium for the economy $((V_i)_{i \in M}, b', s')$ where $|s'_j - s_j| \leq \alpha$ for every $j \in M$ and $|b'_i - b_i| \leq \beta$ for every $i \in N$.

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First general result: additive valuations

Theorem

Suppose that V_i is an additive valuation for all i. Then every economy has a $(0, \max_j p_j)$-competitive equilibrium. Moreover, the total cost of supply does not increase.

Main general result: substitutes

Theorem

Suppose that V_i is a substitutes valuation for all i. Then every economy has a $(1 + \lfloor \frac{2}{t} \rfloor, (2 + \lfloor 2t \rfloor) \max_j p_j)$-competitive equilibrium for any $t > 0$. Moreover, the total cost of supply does not increase.

Examples of the supply-budget constraint relaxation tradeoff:

- $t = 2.01$, we have $(1, 6 \max_j p_j)$-CE
- $t = 1.01$, we have $(2, 4 \max_j p_j)$-CE
- $t = 0.67$, we have $(3, 3 \max_j p_j)$-CE
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Practical consequences of the results

- Can elicit budget constraints directly from bidders in the product-mix auction, assignment message auction, or final round of the CCA.
- Designer can optimize over different approximate equilibria.
- Especially useful for large-ish auctions: many items and reasonable supply.

Proof: Step 1/5

- Convexify the economy by replacing the demand correspondence with its convex hull $\mathbf{conv}(D_i(p))$. Agents consume lotteries over bundles.

Definition (Milgrom and Strulovici, 2009)

A *pseudoequilibrium* for the economy $((V_i)_{i\in M}, \boldsymbol{b}, \boldsymbol{s})$ is a price vector $\boldsymbol{p} \geq \boldsymbol{c}$ and demands $\boldsymbol{x}_i \in \mathrm{conv}(D_i(\boldsymbol{p}))$ for all $i \in N$ such that $\sum_{i\in N} \boldsymbol{x}_i \leq \boldsymbol{s}$ with equality for each $j \in M$ for which $p_j > c_j$.

Lemma

For any economy $((V_i)_{i\in M}, \boldsymbol{b}, \boldsymbol{s})$*, a pseudoequilibrium exists.*

- At the pseudoequilibrium, there is a market-clearing price vector p agent i buys a random bundle X_i, expected payment is $E(p \cdot X_i)$ (note that $p \cdot X_i$ can exceed the budget for some X_i), payoff is $E(V(X_i)) - E(p \cdot X_i)$. Expectations are taken over some probability distribution over bundles.

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A *pseudoequilibrium* for the economy $((V_i)_{i\in M}, \boldsymbol{b}, \boldsymbol{s})$ is a price vector $\boldsymbol{p} \geq \boldsymbol{c}$ and demands $\boldsymbol{x}_i \in \text{conv}(D_i(p))$ for all $\boldsymbol{i} \in \boldsymbol{N}$ such that $\sum_{i\in N} x_i \leq \boldsymbol{s}$ with equality for each $j \in M$ for which $\boldsymbol{p}_j > \boldsymbol{c}_j$.

Lemma

For any economy $((V_i)_{i\in M}, \boldsymbol{b}, \boldsymbol{s})$, *a pseudoequilibrium exists.*

- At the pseudoequilibrium, there is a market-clearing price vector $\boldsymbol{p}$ agent $\boldsymbol{i}$ buys a random bundle X_i, expected payment is $E(p \cdot X_i)$ (note that $p \cdot X_i$ can exceed the budget for some X_i), payoff is $E(V(X_i)) - E(p \cdot X_i)$. Expectations are taken over some probability distribution over bundles.

Proof: Step 2/5

Need to show structure of agents' random bundles at equilibrium. Taking k to be the index of a bundle, each agent i chooses z_k to solve

$$\max \sum_k z_k(V_i(x_k) - p \cdot x_k)$$

$$\text{subject to:} \quad z_k \geq 0, \quad \sum z_k = 1, \quad \text{and} \quad \sum_k z_k \cdot p \cdot x_k \leq b_i.$$

Dual variables are $\alpha_i \geq 0, \omega_i \geq 0$, by dual feasibility and complementary slackness

$$\alpha + \omega a_k \leq V_i(x_k) - p \cdot x_k \quad \forall k; \text{ and with equality when } z_k > 0.$$

Bundle x_k with +ve prob. z_k is in $\arg\max_k\{(V_i(x_k) - (p + \omega_i \cdot p) \cdot x_k)\}$ for some $\omega_i \geq 0$.

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Proof: Step 3/5

- A *binary polytope* is the convex hull of a finite set of $(0, 1)$-vectors.
- A binary polytope is *special* if the edges have at most 2 non-zero coordinates and they are of opposite signs.
- Since valuations satisfy the substitutes condition, $\mathbf{conv}(D_i(p))$ is a special polytope $\mathcal{Q}_i$ (Theorem 4.1 in Nguyen and Vohra). Note that with single-copy demand $\mathcal{Q}_i$ is the *demand complex cell* of Baldwin and Klemperer (2019).
- A *face* of a convex polytope is any intersection of the polytope with a halfspace such that none of the interior points of the polytope lie on the boundary of the halfspace.
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- Denoting the "average" bundle is $y_i = E(X_i)$, consider an "expected equilibrium" in which expected budget constraints are met ($p \cdot y_i \leq b_i$) and markets clear in expectation ($\sum_i y_i = s$).
- When $y_i \in \mathcal{Q}_i$, Gul et al. (2019) show that such an "expected" equilibrium can be implemented as a lottery over allocations.
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Proof: Step 4/5

- We need to round each $\mathbf{y}_i$ to a vertex $\mathbf{x}_i$ of $\mathcal{Q}_i$ with minimal violations.

4a Solve this linear program to obtain a corner solution

$$\min c \cdot \left(\sum_{i \in N} z_i\right) \text{ subject to:} \tag{1}$$

$$z_1, .., z_n \in \mathcal{Q}_1 \times .. \times \mathcal{Q}_n; \tag{2}$$

$$\left(\sum_{i \in N} z_i\right)_j = s_j \text{ for every good } j \tag{3}$$

$$p \cdot z_i \leq b_i \text{ for every agent } i \tag{4}$$

4b Let $\mathcal{Q}' = \mathcal{Q}'_1 \times .. \times \mathcal{Q}'_n$ be the minimal faces that contain that solution.

- If $\mathcal{Q}'$ is a vertex then done.
- Else, fixing a $t > 0$, drop a binding budget constraint with at most $2 + \lfloor 2t \rfloor$ coordinates with fractional values or supply constraints that contains at most $1 + \lfloor \frac{2}{t} \rfloor$, coordinates with fractional values. Return to step 4a with $\mathcal{Q}'$.

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Such a constraint in step 4b exists.

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Proof of this lemma: some facts

Definition

Let $\mathcal{Q}$ be a binary polytope. A coordinate is *free* w.r.t $\mathcal{Q}$ if there are 2 vectors in $\mathcal{Q}$ with different values on this coordinate.

Fact 1 For $x \in \mathcal{Q}$, let $\mathcal{Q}'$ be the minimal face of $\mathcal{Q}$ containing x. A coordinate i is free w.r.t $\mathcal{Q}'$ iff $0 < x_i < 1$.

Fact 2 Let x be corner point of $\mathcal{Q} \cap \{Ax = b\}$. Let $\mathcal{Q}'$ be the minimal face of $\mathcal{Q}$ containing x, then the dimension of $\mathcal{Q}'$ is at most the number of constraints in $\{Ax = b\}$.

Fact 3 Let $\mathcal{Q}$ be a binary polytope with edges having at most 2 non-zero coordinates, then the number of free coordinates w.r.t $\mathcal{Q}$ is at most $2\dim(\mathcal{Q})$.

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Proof of this lemma and the final step

- Using Fact 3. Each edge of $\mathcal{Q}'$ has at most 2 non-zero coordinates. If the dimension of $\mathcal{Q}'$ is $\boldsymbol{d}$, then the number of free coordinates w.r.t $\mathcal{Q}'$ is at most $\boldsymbol{2d}$.
- Using Fact 2, the dimension of $\mathcal{Q}'$ is at most the number of binding budget (n_2) and supply constraints (n_1) that has not been dropped.
  Therefore, #free coordinates $\leq 2(n_1 + n_2)$. So, for example, setting $t = 2\ldots$
  If $n_1 > 2n_2$, then #free coordinates $< 2(n_1 + n_1/2) = 3n_1$, by pigeonhole principle, there is 1 supply constraint with at most 2 free coordinates.
  If $n_1 \leq 2n_2$, then #free coordinates $\leq 2(2n_2 + n_2) = 6n_2$, by pigeonhole principle, there is 1 budget constraint with at most 6 free coordinates.
- Using Fact 1, we convert these constraint violations into size violations.
- More generally, we can compare n_1 with tn_2 to get the trade-off between the violation of capacity and budget constraint and to obtain the general result.

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- Multiple sellers (matching market)
- Multiple units

Ongoing work:

- Tighter result for assignment messages
- General cost of funds for buyers
- Ordinary/Δ substitutes? Complements? Unimodular basis changes of substitutes?

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