S-Convexity and Gross Substitutability

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Abstract: We propose a new concept of S-convex functions (and its variant SSQS-convex functions) to study substitute structures in economics and operations models with continuous variables. We develop a host of fundamental properties and characterizations of S-convex functions including various preservation properties, conjugate relationships with submodular and convex functions, and characterizations using Hessians. For a divisible market, we show that the utility function satisﬁes gross substitutability if and only if it is S-concave under some regularity conditions. In a parametric maximization model with a box constraint, we show that the set of optimal solutions is nonincreasing in the parameters if the objective function is (SSQS)S-concave. Furthermore, we prove that S-convexity is necessary for the property of nonincreasing optimal solutions under some conditions. The monotonicity result is applied to a classical multi-product dynamic inventory model.