1 Thursday, June 6

1. Matching models with random utility
2. Estimation using generalized linear models

2 L1. Matching models recalled

Individual man $i \in [I]$ and woman $j \in [J]$ have an observable type $x_i \in [X]$ and $y_j \in [Y]$

Assume n_x men of type x and m_y women of type y
if matched, i gets $\alpha_{x_i y_j} + \varepsilon_{i y_j}$
and j gets $\gamma_{x_i y_j} + \eta_{x_i j}$
and if unmatched, i and j get ε_{i0} and η_{0j}

Transferable utility (Becker model of marriage). x and y may decide on transfers w_{xy} from the woman to the man
$\alpha_{xy} + w_{xy} + \varepsilon_{iy}$
$\gamma_{xy} - w_{xy} + \eta_{xj}$

Individual decision-making problem:
$u_i = \max_y \{\alpha_{xy} + w_{xy} + \varepsilon_{iy}, \varepsilon_{i0}\}$
$v_j = \max_x \{\gamma_{xy} - w_{xy} + \eta_{xj}, \eta_{0j}\}$

(Different from Dagsvik-Menzel, which is not scale invariant)

Monge-Kantorovich duality: at equilibrium, people match in order to maximize the total surplus formed at the level of pairs.

Consider a pair ij. Surplus of the match is $\Phi_{ij} = \alpha_{xy} + \gamma_{xy} + \varepsilon_{iy} + \eta_{xj} = \Phi_{xy} + \varepsilon_{iy} + \eta_{xj}$
(separable structure)
where $\Phi_{xy} = \alpha_{xy} + \gamma_{xy}$

that is, letting $\mu_{ij} = 1$ if i and j are matched, 0 otherwise, and $\mu_{i0} = 1$ iff i is unmatched and μ_{0j} etc

$$\begin{aligned} \max_{\mu} \quad & \sum_{ij} \mu_{ij} \Phi_{ij} + \sum_i \mu_{i0} \varepsilon_{i0} + \sum_j \mu_{0j} \eta_{0j} \\ s.t. \quad & \sum_j \mu_{ij} + \mu_{i0} = 1 \\ & \sum_i \mu_{ij} + \mu_{0j} = 1 \end{aligned}$$

this problem has a dual which is

$$\begin{aligned}\min_{u_i,v_j} \quad & \sum_i u_i + \sum_j v_j \\ s.t. \quad & u_i + v_j \geq \Phi_{ij} \\ & u_i \geq \varepsilon_{i0} \\ & v_j \geq \eta_{0j}\end{aligned}$$

If $u_i + v_j < \Phi_{ij}$ then i and j would form a blocking pair.

3 L2. Matching with random utility

Consider the previous problem with the separable structure

$$\begin{aligned}\min_{u_i,v_j} \quad & \sum_i u_i + \sum_j v_j \\ s.t. \quad & u_i + v_j \geq \Phi_{x_i y_j} + \varepsilon_{iy_j} + \eta_{xj_i} \\ & u_i \geq \varepsilon_{i0} \\ & v_j \geq \eta_{0j}\end{aligned}$$

Assume that there are B individuals per X and per Y.
THere are
- $I + J = B(X + Y)$ variables
- $IJ + I + J = B^2XY + BX + BY$
problem is B^2 term!

Let's focus on the "problematic" constraint:
$u_i - \varepsilon_{iy} + v_j - \eta_{xj} \geq \Phi_{xy}$ $\forall i : x_i = x$ and $\forall j : y_j = y$
rewrite this as
$\min_{i:x_i=x} \{u_i - \varepsilon_{iy}\} + \min_{j:y_j=y} \{v_j - \eta_{xj}\} \geq \Phi_{xy}$ for all x and y

Now introduce $U_{xy} = \min_{i:x_i=x} \{u_i - \varepsilon_{iy}\}$ and $V_{xy} = \min_{j:y_j=y} \{v_j - \eta_{xj}\}$
so that we can rewrite the constraint as
$U_{xy} + V_{xy} \geq \Phi_{xy}$

$u_i \geq U_{xy} + \varepsilon_{iy}$ therefore $u_i \geq \max_y \{U_{xy} + \varepsilon_{iy}, \varepsilon_{i0}\}$ and $v_j \geq \max_x \{V_{xy} + \eta_{xj}, \eta_{0j}\}$

We get to an equivalent formulation:

$$\begin{aligned}\min_{u_i,v_j,U_{xy},V_{xy}} \quad & \sum_i u_i + \sum_j v_j \\ s.t. \quad & U_{xy} + V_{xy} = \Phi_{xy} \\ & u_i \geq U_{xy} + \varepsilon_{iy} \\ & u_i \geq \varepsilon_{i0} \\ & v_j \geq V_{xy} + \eta_{xj} \\ & v_j \geq \eta_{0j}\end{aligned}$$

This problem has
* $I+J+2XY=(X+Y)\,B+2XY$ variables
* $XY+I(Y+1)+J\,(X+1)=XY+BX(Y+1)+BY\,(X+1)=B\,(2XY+X+Y)+XY$

We have

$$\begin{aligned} \min_{U_{xy},V_{xy}} \quad & \sum_i \max_y \{U_{xy}+\varepsilon_{iy},\varepsilon_{i0}\} + \sum_j \max\left\{V_{xy}+\eta_{xj},\eta_{0j}\right\} \\ s.t. \quad & U_{xy}+V_{xy}=\Phi_{xy} \end{aligned}$$

therefore

$$\begin{aligned} \min_{U_{xy},V_{xy}} \quad & \sum_x n_x \frac{1}{n_x} \sum_{i:x_i=x} \max_y \{U_{xy}+\varepsilon_{iy},\varepsilon_{i0}\} + \sum_y m_y \frac{1}{m_y} \sum_{j:y_j=y} \max_x \left\{V_{xy}+\eta_{xj},\eta_{0j}\right\} \\ s.t. \quad & U_{xy}+V_{xy}=\Phi_{xy} \end{aligned}$$

Logit model. Assume $B\to+\infty$. Then as $B\to+\infty$,

$$\begin{aligned} \frac{1}{n_x}\sum_{i:x_i=x} \max_y \{U_{xy}+\varepsilon_{iy},\varepsilon_{i0}\} \quad &\to \quad E\left[\max_y \{U_{xy}+\varepsilon_y,\varepsilon_0\}\right] \\ \frac{1}{m_y}\sum_{j:y_j=y} \max\left\{V_{xy}+\eta_{xj},\eta_{0j}\right\} \quad &\to \quad E\left[\max_x \{V_{xy}+\eta_x,\eta_0\}\right] \end{aligned}$$

Assume further $(\varepsilon_{iy})_y$ is iid Gumbel distributed, as well as $\left(\eta_{xj}\right)_x$. Then

$$E\left[\max_y \{U_{xy}+\varepsilon_y,\varepsilon_0\}\right] = \log\left(1+\sum_y \exp U_{xy}\right)$$

so for the logit model, we have that the equilibrium solves

$$\begin{aligned} \min_{U_{xy},V_{xy}} \quad & \sum_{x\in[X]} n_x \log\left(1+\sum_y \exp U_{xy}\right) + \sum_{y\in[Y]} m_y \log\left(1+\sum_x \exp V_{xy}\right) \\ s.t. \quad & U_{xy}+V_{xy}=\Phi_{xy}. \end{aligned}$$

rewrite as

$$\min_{U_{xy}} \sum_{x\in[X]} n_x \log\left(1+\sum_y \exp U_{xy}\right) + \sum_{y\in[Y]} m_y \log\left(1+\sum_x \exp\left(\Phi_{xy}-U_{xy}\right)\right)$$

by foc wrt U_{xy} we have

$$n_x \exp\left(U_{xy}-u_x\right) = m_y \exp\left(V_{xy}-v_y\right)$$

where $u_x=\log\left(1+\sum_y \exp U_{xy}\right)$ and $v_y=\log\left(1+\sum_x \exp\left(\Phi_{xy}-U_{xy}\right)\right)$

To recap,

$$\mu_{xy} = n_x \exp\left(U_{xy} - u_x\right) = m_y \exp\left(V_{xy} - v_y\right)$$
$$U_{xy} + V_{xy} = \Phi_{xy}$$

so we have

$$\begin{aligned}\mu_{xy} &= \sqrt{n_x \exp\left(U_{xy} - u_x\right) m_y \exp\left(V_{xy} - v_y\right)} \\ &= \sqrt{n_x m_y \exp\left(\Phi_{xy} - u_x - v_y\right)}\end{aligned}$$

in other words,

$$\mu_{xy} = \sqrt{n_x m_y} \exp\left(\frac{\Phi_{xy} - u_x - v_y}{2}\right)$$

and we have

$$\mu_{x0} = n_x \frac{\exp 0}{\exp 0 + \sum_y \exp U_{xy}} = n_x \exp\left(-u_x\right)$$

and similarly,

$$\mu_{0y} = m_y \exp\left(-v_y\right).$$

So we have Choo and Siow's formula

$$\mu_{xy} = \sqrt{\mu_{x0}\mu_{0y}} \exp\left(\frac{\Phi_{xy}}{2}\right).$$

WHich allows us to Φ_{xy} by

$$\Phi_{xy} = \ln \frac{\mu_{xy}^2}{\mu_{x0}\mu_{0y}}.$$

Computation of μ.

We have

$$\begin{aligned}\mu_{xy} &= \sqrt{n_x m_y} \exp\left(\frac{\Phi_{xy} - u_x - v_y}{2}\right) \\ \mu_{x0} &= n_x \exp\left(-u_x\right) \\ \mu_{0y} &= m_y \exp\left(-v_y\right)\end{aligned}$$

and μ should satisfy,

$$\begin{aligned}n_x &= \mu_{x0} + \sum_y \mu_{xy} \\ m_y &= \mu_{0y} + \sum_x \mu_{xy}\end{aligned}$$

thus

$$
\begin{aligned}
n_x &= n_x \exp\left(-u_x\right) + \sum_y \sqrt{n_x m_y} \exp\left(\frac{\Phi_{xy} - u_x - v_y}{2}\right) \\
m_y &= m_y \exp\left(-v_y\right) + \sum_x \sqrt{n_x m_y} \exp\left(\frac{\Phi_{xy} - u_x - v_y}{2}\right)
\end{aligned}
$$

introduce $a_x = -u_x + \ln n_x$ and $b_y = -v_y + \ln m_y$, and $K_{xy} = \exp\left(\Phi_{xy}/2\right)$, so that we can rewrite the equations of the model as

$$
\begin{aligned}
n_x &= \exp\left(a_x\right) + \sum_y K_{xy} \exp\left(\frac{a_x + b_y}{2}\right) \\
m_y &= \exp\left(b_y\right) + \sum_x K_{xy} \exp\left(\frac{a_x + b_y}{2}\right)
\end{aligned}
$$

Computation by gradient descent.

Exercise. Can I interpret this as the FOC associated with

$$
\min_{(a,b) \in R^X \times R^Y} F\left(a, b\right)?
$$

$$
\begin{aligned}
\frac{\partial F\left(a, b\right)}{\partial a_x} &= \exp\left(a_x\right) + \sum_y K_{xy} \exp\left(\frac{a_x + b_y}{2}\right) - n_x \\
\frac{\partial F\left(a, b\right)}{\partial b_y} &= \exp\left(b_y\right) + \sum_x K_{xy} \exp\left(\frac{a_x + b_y}{2}\right) - m_y
\end{aligned}
$$

$$
F\left(a, b\right) = \sum_x \exp\left(a_x\right) + \sum_y \exp\left(b_y\right) + 2 \sum_{xy} K_{xy} \exp\left(\frac{a_x + b_y}{2}\right) - \sum_x n_x a_x - \sum_y m_y b_y.
$$

$$
\begin{aligned}
a_x^{t+1} &= a_x^t - \epsilon \frac{\partial F\left(a, b\right)}{\partial a_x} \\
b_y^{t+1} &= b_y^t - \epsilon \frac{\partial F\left(a, b\right)}{\partial b_y}
\end{aligned}
$$

$\nabla F\left(a, b\right) = 0$
$\min F\left(a, b\right)$

$\min \left\| \nabla F\left(a, b\right) \right\|$

Computation by coordiate descent.

Recall the expression of F to be minimized

$$F\left(a,b\right)=\sum_{x}\exp\left(a_{x}\right)+\sum_{y}\exp\left(b_{y}\right)+2\sum_{xy}K_{xy}\exp\left(\frac{a_{x}+b_{y}}{2}\right)-\sum_{x}n_{x}a_{x}-\sum_{y}m_{y}b_{y}.$$

Idea:
Initialize b^{0}
For all $t\geq 0$, set
$\left(a_{x}^{t+1}\right)=\arg\min_{a}F\left(a,b^{t}\right)$
$\left(b_{y}^{t+1}\right)=\arg\min_{b}F\left(a^{t+1},b\right)$

The minimization of F wrt to a yields $\frac{\partial F\left(a,b^{t}\right)}{\partial a_{x}}=0$, that is

$$\exp\left(a_{x}\right)+\sum_{y}K_{xy}\exp\left(\frac{a_{x}+b_{y}^{t}}{2}\right)=n_{x}$$

Introduce $A_{x}^{t}=\exp\left(a_{x}^{t}/2\right)$ and $B_{x}^{t}=\exp\left(b_{y}^{t}/2\right)$ so that the equation becomes

$$A_{x}^{2}+2A_{x}\left(\frac{1}{2}\sum_{y}K_{xy}B_{y}^{t}\right)+\left(\frac{1}{2}\sum_{y}K_{xy}B_{y}^{t}\right)^{2}=n_{x}+\left(\frac{1}{2}\sum_{y}K_{xy}B_{y}^{t}\right)^{2}$$

that is

$$\begin{aligned}A_{x}^{t+1}&=\sqrt{n_{x}+\left(\frac{1}{2}\sum_{y}K_{xy}B_{y}^{t}\right)^{2}}-\frac{1}{2}\sum_{y}K_{xy}B_{y}^{t}\\B_{y}^{t+1}&=\sqrt{m_{y}+\left(\frac{1}{2}\sum_{x}K_{xy}A_{x}^{t+1}\right)^{2}}-\frac{1}{2}\sum_{x}K_{xy}A_{x}^{t+1}\end{aligned}$$

(generalization of Sinkhorn's algorithm)