Thursday, June 6

1. Matching models with random utility
2. Estimation using generalized linear models

1 L1. Matching models recalled

Individual man $i \in [I]$ and woman $j \in [J]$ have an observable type $x_i \in [X]$ and $y_j \in [Y]$

Assume n_x men of type x and m_y women of type y
if matched, i gets $\alpha_{x_i y_j} + \varepsilon_{i y_j}$
and j gets $\gamma_{x_i y_j} + \eta_{x_i j}$
and if unmatched, i and j get ε_{i0} and η_{0j}

Transferable utility (Becker model of marriage). x and y may decide on transfers w_{xy} from the woman to the man
$\alpha_{xy} + w_{xy} + \varepsilon_{iy}$
$\gamma_{xy} - w_{xy} + \eta_{xj}$

Individual decision-making problem:
$u_i = \max_y \{\alpha_{xy} + w_{xy} + \varepsilon_{iy}, \varepsilon_{i0}\}$
$v_j = \max_x \{\gamma_{xy} - w_{xy} + \eta_{xj}, \eta_{0j}\}$

(Different from Dagsvik-Menzel, which is not scale invariant)

Monge-Kantorovich duality: at equilibrium, people match in order to maximize the total surplus formed at the level of pairs.

Consider a pair ij. Surplus of the match is $\Phi_{ij} = \alpha_{xy} + \gamma_{xy} + \varepsilon_{iy} + \eta_{xj} = \Phi_{xy} + \varepsilon_{iy} + \eta_{xj}$
(separable structure)
where $\Phi_{xy} = \alpha_{xy} + \gamma_{xy}$

that is, letting $\mu_{ij} = 1$ if i and j are matched, 0 otherwise, and $\mu_{i0} = 1$ iff i is unmatched and μ_{0j} etc

$$\begin{aligned}
\max_{\mu} \quad & \sum_{ij} \mu_{ij}\Phi_{ij} + \sum_i \mu_{i0}\varepsilon_{i0} + \sum_j \mu_{0j}\eta_{0j} \\
s.t. \quad & \sum_j \mu_{ij} + \mu_{i0} = 1 \\
& \sum_i \mu_{ij} + \mu_{0j} = 1
\end{aligned}$$

this problem has a dual which is

$$\begin{aligned}\min_{u_i,v_j} \quad & \sum_i u_i + \sum_j v_j \\ s.t. \quad & u_i + v_j \geq \Phi_{ij} \\ & u_i \geq \varepsilon_{i0} \\ & v_j \geq \eta_{0j}\end{aligned}$$

If $u_i + v_j < \Phi_{ij}$ then i and j would form a blocking pair.

2 L2. Matching with random utility

Consider the previous problem with the separable structure

$$\begin{aligned}\min_{u_i,v_j} \quad & \sum_i u_i + \sum_j v_j \\ s.t. \quad & u_i + v_j \geq \Phi_{x_i y_j} + \varepsilon_{i y_j} + \eta_{x j_i} \\ & u_i \geq \varepsilon_{i0} \\ & v_j \geq \eta_{0j}\end{aligned}$$

Assume that there are B individuals per X and per Y.
THere are
- $I + J = B\,(X + Y)$ variables
- $IJ + I + J = B^2 XY + BX + BY$
problem is B^2 term!

Let's focus on the "problematic" constraint:
$u_i - \varepsilon_{iy} + v_j - \eta_{xj} \geq \Phi_{xy}$ $\forall i : x_i = x$ and $\forall j : y_j = y$
rewrite this as
$\min_{i:x_i=x} \{u_i - \varepsilon_{iy}\} + \min_{j:y_j=y} \{v_j - \eta_{xj}\} \geq \Phi_{xy}$ for all x and y

Now introduce $U_{xy} = \min_{i:x_i=x} \{u_i - \varepsilon_{iy}\}$ and $V_{xy} = \min_{j:y_j=y} \{v_j - \eta_{xj}\}$
so that we can rewrite the constraint as
$U_{xy} + V_{xy} \geq \Phi_{xy}$

$u_i \geq U_{xy} + \varepsilon_{iy}$ therefore $u_i \geq \max_y \{U_{xy} + \varepsilon_{iy}, \varepsilon_{i0}\}$ and $v_j \geq \max_x \{V_{xy} + \eta_{xj}, \eta_{0j}\}$

We get to an equivalent formulation:

$$\begin{aligned}\min_{u_i,v_j,U_{xy},V_{xy}} \quad & \sum_i u_i + \sum_j v_j \\ s.t. \quad & U_{xy} + V_{xy} = \Phi_{xy} \\ & u_i \geq U_{xy} + \varepsilon_{iy} \\ & u_i \geq \varepsilon_{i0} \\ & v_j \geq V_{xy} + \eta_{xj} \\ & v_j \geq \eta_{0j}\end{aligned}$$

This problem has
* $I + J + 2XY = (X + Y)\,B + 2XY$ variables
* $XY+I(Y+1)+J\,(X+1) = XY+BX(Y+1)+BY\,(X+1) = B\,(2XY+X+Y)+XY$

We have

$$\begin{array}{ll} \min_{U_{xy},V_{xy}} & \sum_i \max_y \{U_{xy}+\varepsilon_{iy},\varepsilon_{i0}\} + \sum_j \max \left\{V_{xy}+\eta_{xj},\eta_{0j}\right\} \\ s.t. & U_{xy}+V_{xy}=\Phi_{xy} \end{array}$$

therefore

$$\begin{array}{ll} \min_{U_{xy},V_{xy}} & \sum_x n_x \frac{1}{n_x} \sum_{i:x_i=x} \max_y \{U_{xy}+\varepsilon_{iy},\varepsilon_{i0}\} + \sum_y m_y \frac{1}{m_y} \sum_{j:y_j=y} \max_x \left\{V_{xy}+\eta_{xj},\eta_{0j}\right\} \\ s.t. & U_{xy}+V_{xy}=\Phi_{xy} \end{array}$$

Logit model. Assume $B\to+\infty$. Then as $B\to+\infty$,

$$\begin{array}{rcl} \frac{1}{n_x}\sum_{i:x_i=x}\max_y \{U_{xy}+\varepsilon_{iy},\varepsilon_{i0}\} & \to & E\left[\max_y \{U_{xy}+\varepsilon_y,\varepsilon_0\}\right] \\ \frac{1}{m_y}\sum_{j:y_j=y}\max\left\{V_{xy}+\eta_{xj},\eta_{0j}\right\} & \to & E\left[\max_x \{V_{xy}+\eta_x,\eta_0\}\right] \end{array}$$

3 L3. The Choo and Siow model

Assume further $(\varepsilon_{iy})_y$ is iid Gumbel distributed, as well as $\left(\eta_{xj}\right)_x$. Then

$$E\left[\max_y \{U_{xy}+\varepsilon_y,\varepsilon_0\}\right] = \log\left(1+\sum_y \exp U_{xy}\right)$$

so for the logit model, we have that the equilibrium solves

$$\begin{array}{ll} \min_{U_{xy},V_{xy}} & \sum_{x\in[X]} n_x \log\left(1+\sum_y \exp U_{xy}\right) + \sum_{y\in[Y]} m_y \log\left(1+\sum_x \exp V_{xy}\right) \\ s.t. & U_{xy}+V_{xy}=\Phi_{xy}. \end{array}$$

rewrite as

$$\min_{U_{xy}} \sum_{x\in[X]} n_x \log\left(1+\sum_y \exp U_{xy}\right) + \sum_{y\in[Y]} m_y \log\left(1+\sum_x \exp\left(\Phi_{xy}-U_{xy}\right)\right)$$

by foc wrt U_{xy} we have

$$n_x \exp\left(U_{xy}-u_x\right) = m_y \exp\left(V_{xy}-v_y\right)$$

where $u_x = \log\left(1+\sum_y \exp U_{xy}\right)$ and $v_y = \log\left(1+\sum_x \exp\left(\Phi_{xy}-U_{xy}\right)\right)$

To recap,

$$\begin{aligned}
&\mu_{xy} = n_x \exp\left(U_{xy}-u_x\right) = m_y \exp\left(V_{xy}-v_y\right)\\
&U_{xy}+V_{xy} = \Phi_{xy}
\end{aligned}$$

so we have

$$\begin{aligned}
\mu_{xy} &= \sqrt{n_x \exp\left(U_{xy}-u_x\right) m_y \exp\left(V_{xy}-v_y\right)}\\
&= \sqrt{n_x m_y \exp\left(\Phi_{xy}-u_x-v_y\right)}
\end{aligned}$$

in other words,

$$\mu_{xy} = \sqrt{n_x m_y} \exp\left(\frac{\Phi_{xy}-u_x-v_y}{2}\right)$$

and we have

$$\mu_{x0} = n_x \frac{\exp 0}{\exp 0+\sum_y \exp U_{xy}} = n_x \exp\left(-u_x\right)$$

and similarly,

$$\mu_{0y} = m_y \exp\left(-v_y\right).$$

So we have Choo and Siow's formula

$$\mu_{xy} = \sqrt{\mu_{x0}\mu_{0y}} \exp\left(\frac{\Phi_{xy}}{2}\right).$$

WHich allows us to Φ_{xy} by

$$\Phi_{xy} = \ln \frac{\mu_{xy}^2}{\mu_{x0}\mu_{0y}}.$$

Computation of μ.

We have

$$\begin{aligned}
\mu_{xy} &= \sqrt{n_x m_y} \exp\left(\frac{\Phi_{xy}-u_x-v_y}{2}\right)\\
\mu_{x0} &= n_x \exp\left(-u_x\right)\\
\mu_{0y} &= m_y \exp\left(-v_y\right)
\end{aligned}$$

and μ should satisfy,

$$\begin{aligned}
n_x &= \mu_{x0}+\sum_y \mu_{xy}\\
m_y &= \mu_{0y}+\sum_x \mu_{xy}
\end{aligned}$$

thus

$$
\begin{aligned}
n_x &= n_x \exp\left(-u_x\right) + \sum_y \sqrt{n_x m_y} \exp\left(\frac{\Phi_{xy} - u_x - v_y}{2}\right) \\
m_y &= m_y \exp\left(-v_y\right) + \sum_x \sqrt{n_x m_y} \exp\left(\frac{\Phi_{xy} - u_x - v_y}{2}\right)
\end{aligned}
$$

introduce $a_x = -u_x + \ln n_x$ and $b_y = -v_y + \ln m_y$, and $K_{xy} = \exp\left(\Phi_{xy}/2\right)$, so that we can rewrite the equations of the model as

$$
\begin{aligned}
n_x &= \exp\left(a_x\right) + \sum_y K_{xy} \exp\left(\frac{a_x + b_y}{2}\right) \\
m_y &= \exp\left(b_y\right) + \sum_x K_{xy} \exp\left(\frac{a_x + b_y}{2}\right)
\end{aligned}
$$

Computation by gradient descent.
Exercise. Can I interpret this as the FOC associated with

$$
\min_{(a,b)\in R^X \times R^Y} F\left(a,b\right)?
$$

$$
\begin{aligned}
\frac{\partial F\left(a,b\right)}{\partial a_x} &= \exp\left(a_x\right) + \sum_y K_{xy} \exp\left(\frac{a_x + b_y}{2}\right) - n_x \\
\frac{\partial F\left(a,b\right)}{\partial b_y} &= \exp\left(b_y\right) + \sum_x K_{xy} \exp\left(\frac{a_x + b_y}{2}\right) - m_y
\end{aligned}
$$

$$
F\left(a,b\right) = \sum_x \exp\left(a_x\right) + \sum_y \exp\left(b_y\right) + 2\sum_{xy} K_{xy} \exp\left(\frac{a_x + b_y}{2}\right) - \sum_x n_x a_x - \sum_y m_y b_y.
$$

$$
\begin{aligned}
a_x^{t+1} &= a_x^t - \epsilon \frac{\partial F\left(a,b\right)}{\partial a_x} \\
b_y^{t+1} &= b_y^t - \epsilon \frac{\partial F\left(a,b\right)}{\partial b_y}
\end{aligned}
$$

$\nabla F\left(a,b\right) = 0$
$\min F\left(a,b\right)$

$\min \left\|\nabla F\left(a,b\right)\right\|$

Computation by coordiate descent.
Recall the expression of F to be minimized

$$F\left(a,b\right)=\sum_{x}\exp\left(a_{x}\right)+\sum_{y}\exp\left(b_{y}\right)+2\sum_{xy}K_{xy}\exp\left(\frac{a_{x}+b_{y}}{2}\right)-\sum_{x}n_{x}a_{x}-\sum_{y}m_{y}b_{y}.$$

Idea:
Initialize b^{0}
For all $t\geq 0$, set
$\left(a_{x}^{t+1}\right)=\arg\min_{a}F\left(a,b^{t}\right)$
$\left(b_{y}^{t+1}\right)=\arg\min_{b}F\left(a^{t+1},b\right)$

The minimization of F wrt to a yields $\frac{\partial F\left(a,b^{t}\right)}{\partial a_{x}}=0$, that is

$$\exp\left(a_{x}\right)+\sum_{y}K_{xy}\exp\left(\frac{a_{x}+b_{y}^{t}}{2}\right)=n_{x}$$

Introduce $A_{x}^{t}=\exp\left(a_{x}^{t}/2\right)$ and $B_{x}^{t}=\exp\left(b_{y}^{t}/2\right)$ so that the equation becomes

$$A_{x}^{2}+2A_{x}\left(\frac{1}{2}\sum_{y}K_{xy}B_{y}^{t}\right)+\left(\frac{1}{2}\sum_{y}K_{xy}B_{y}^{t}\right)^{2}=n_{x}+\left(\frac{1}{2}\sum_{y}K_{xy}B_{y}^{t}\right)^{2}$$

that is

$$\begin{aligned}A_{x}^{t+1} &= \sqrt{n_{x}+\left(\frac{1}{2}\sum_{y}K_{xy}B_{y}^{t}\right)^{2}}-\frac{1}{2}\sum_{y}K_{xy}B_{y}^{t}\\ B_{y}^{t+1} &= \sqrt{m_{y}+\left(\frac{1}{2}\sum_{x}K_{xy}A_{x}^{t+1}\right)^{2}}-\frac{1}{2}\sum_{x}K_{xy}A_{x}^{t+1}\end{aligned}$$

(generalization of Sinkhorn's algorithm)

Friday, June 7

Friday morning: various exercises.
Same-sex marriage.
$x\in X$ n_{x} individuals of type x
If 2 individuals of types $x,y\in X$ match, this creates joint utility
Φ_{xy}

μ_{xy} mass of pairs of type xy formed at equilibrium
$\mu_{xy}=\mu_{yx}$

$n_x = \mu_{x0} + \sum_{y \neq x} \mu_{xy} + 2\mu_{xx}$

The total surplus is
$\frac{1}{2} \sum_{x \neq y} \mu_{xy} \Phi_{xy} + \sum_x \mu_{xx} \Phi_{xx}$

Therefore, the equilibrium will solve

$$\begin{aligned} & \max_{\mu} \frac{1}{2} \sum_{x \neq y} \mu_{xy} \Phi_{xy} + \sum_x \mu_{xx} \Phi_{xx} \\ s.t. \qquad & n_x = \mu_{x0} + \sum_{y \neq x} \mu_{xy} + 2\mu_{xx} \\ & \mu_{xy} = \mu_{yx} \end{aligned}$$

One-to-many matching. Pauline Corblet's job market paper; see also weak optimal transport

X is set workers types and Y is set of firm's types

Assume that one firm $y \in Y$ hires a bundle of workers $b \in \mathbb{N}^X$ – i.e., b_x is the number of workers of type x in the coalition.

Let Φ_{by} be the surplus of a firm of type y hiring bundle b of workers.

E.g. $\Phi_{by} = \sum_x b_x e_x + f_y$ – this is additive case, uninteresting as rules out complementarities.

More interestingly – complementarities between firm and workers of the sort $\Phi_{by} = \left(\sum_x b_x e_x \right) f_y$.

μ_{by} yields output $\sum_{b,y} \mu_{by} \Phi_{by}$
$m_y = \sum_b \mu_{by}$
$n_x = \sum_b \sum_y \mu_{by} b_x$

Therefore, the optimal welfare solves
$\max \sum_{b,y} \mu_{by} \Phi_{by}$
s.t.
$m_y = \sum_b \mu_{by}$
$n_x = \sum_b \sum_y \mu_{by} b_x$

Simulating from the Gumbel distribution
$\varepsilon \sim \mathcal{G}$
then
$\exp\left(-\exp\left(-\varepsilon\right)\right) = U \sim U\left([0,1]\right)$
$\varepsilon = -\ln\left(-\ln U\right)$

Compute

$$\min_{U_{xy}} F(U)$$

where

$$F(U) = \sum_{x \in [X]} n_x \log \left(1 + \sum_y \exp U_{xy}\right) + \sum_{y \in [Y]} m_y \log \left(1 + \sum_x \exp\left(\Phi_{xy} - U_{xy}\right)\right)$$

$$\frac{\partial F}{\partial U_{xy}} = n_x \frac{\exp U_{xy}}{1 + \sum_y \exp U_{xy}} - m_y \frac{\exp\left(\Phi_{xy} - U_{xy}\right)}{1 + \sum_x \exp\left(\Phi_{xy} - U_{xy}\right)}$$

4 L4. Estimation via generalized linear models

Linear model $E\left[\mu_a|\rho_a\right] = \rho_a^\top \theta$ – estimate by OLS

ρ_a is a vector of regressors whcih is R-dimensional

"Generalized" linear models $f\left(E\left[\mu_a|\rho_a\right]\right) = \rho_a^\top \theta$ where $f : R \to R$ increasing and continuous and invertible is called a "link function"

OLS – $f(z) = z$

$E\left[\mu_a|\rho_a\right] = f^{-1}\left(\rho_a^\top \theta\right)$

Assume we have observations μ_i, ρ_i. How to estimate the model?

I can view the relation

$$E\left[\mu_a|\rho_a\right] = f^{-1}\left(\rho_a^\top \theta\right)$$

as the first order conditions associated with a convex optimization problem. f^{-1} is increasing, so I can call F^* is primitive (i.e. $F^{*\prime} = f^{-1}$), and $E\left[\mu_a|\rho_a\right] = f^{-1}\left(\rho_a^\top \theta\right)$ imply

$$\rho_a E\left[\mu_a|\rho_a\right] - \rho_a f^{-1}\left(\rho_a^\top \theta\right) = 0$$

that is

$$E\left[\rho_a E\left[\mu_a|\rho_a\right] - \rho_a f^{-1}\left(\rho_a^\top \theta\right)\right] = 0$$

that is

$$E\left[\rho_a \mu_a - \rho_a f^{-1}\left(\rho_a^\top \theta\right)\right] = 0.$$

We can reformulate this as a convex optimization problem

$$\max_{\theta \in \mathbb{R}^R} \left\{E\left[\mu_a\left(\rho_a^\top \theta\right) - F^*\left(\rho_a^\top \theta\right)\right]\right\}$$

and the estimator of θ is given by

$$\max_{\theta \in \mathbb{R}^R} \left\{\sum_i \left(\mu_i\left(\rho_i^\top \theta\right) - F^*\left(\rho_i^\top \theta\right)\right)\right\}$$

Poisson regression. In a Poisson regression, we assume that $\mu_a|\rho_a$ follows a Poisson distribution of parameter $\exp\left(\rho_a^\top \theta\right)$.

Reminder: A random variable $\tilde{\mu}$ distributed on $\mathbb{N}$ follows a Poisson distribution of parameter λ if

$$\Pr\left(\tilde{\mu} = m\right) = \frac{\lambda^m e^{-\lambda}}{m!}$$

The expectation of $\tilde{\mu}$ is λ, and the variance of $\tilde{\mu}$ is also λ.

Back to our Poisson regression. We have that the conditional likelihood of (μ_i, ρ_i) is

$$\frac{\exp\left(\mu_i \rho_i^\top \theta\right) \exp\left(-\exp\left(\rho_i^\top \theta\right)\right)}{\mu_i!}$$

and therefore the conditional likelihood of the sample is

$$l(\theta) = \sum_{i \in [I]} \mu_i \rho_i^\top \theta - \exp\left(\rho_i^\top \theta\right) - \ln \mu_i!$$

and the maximum likelihood estimator is $\max l(\theta)$ that is

$$\max_{\theta \in \mathbb{R}^R} \sum_{i \in [I]} \mu_i \rho_i^\top \theta - \exp\left(\rho_i^\top \theta\right)$$

which is exactly a GLM

$$\max_{\theta \in \mathbb{R}^R} \left\{ \sum_i \left(\mu_i \left(\rho_i^\top \theta\right) - F^* \left(\rho_i^\top \theta\right)\right) \right\}$$

with $F^* = \exp$ therefore $f^{-1} = \exp$ and $f = \ln$. Hence the Poisson regression is a GLM with log link function.

Logistic regression. Consider the choice problem of an individual i of type $x_i \in [X]$ choosing between options $y \in [Y]$. Assume that the utility of the decision maker if taking option y is

$$\phi_{xy} + \varepsilon_{iy}$$

Assume a linear parameterization $\phi_{xy}^\lambda = \sum_k \Phi_{xyk} \lambda_k$ where $\lambda \in R^L$ is the parameter I want to estimate, and (ε_{iy}) are iid Gumbel vectors. Conditional probability that i chooses y

$$\frac{\exp \phi_{xy}^\lambda}{\sum_{y'} \exp \phi_{xy'}^\lambda}$$

and therefore the log-likelihood associated with the choice y_i of individual i is

$$\phi_{x_i y_i}^\lambda - \log \sum_{y'} \exp \phi_{x_i y'}^\lambda$$

therefore the log-likelihood of the sample

$$l(\lambda) = \sum_{xy} \hat{\mu}_{xy} \phi_{xy}^\lambda - \sum_x n_x \log \sum_{y'} \exp \phi_{xy'}^\lambda$$

the max-likelihood is $\max_\lambda l(\lambda)$ which is

$$\max_\lambda \left\{ \sum_{xy} \hat{\mu}_{xy} \phi^\lambda_{xy} - \sum_x n_x \log \sum_{y'} \exp \phi^\lambda_{xy'} \right\}$$

which is the logistic regression. By first order conditions, we have at the optimal value of λ that

$$\sum_{xy} \hat{\mu}_{xy} \Phi_{xyk} = \sum_{xy} n_x \frac{\Phi_{xyk} \exp \phi^\lambda_{xy}}{\sum_{y'} \exp \phi^\lambda_{xy'}}$$

introduce the predictor $\mu^\lambda_{xy} = n_x \frac{\exp \phi^\lambda_{xy}}{\sum_{y'} \exp \phi^\lambda_{xy'}}$, and we have that at the optimal value of λ

$$\sum_{xy} \hat{\mu}_{xy} \Phi_{xyk} = \sum_{xy} \mu^\lambda_{xy} \Phi_{xyk}.$$

The "Poisson trick". Recall that:
Poisson regression:

$$\max_{\theta \in \mathbb{R}^R} \left\{ \sum_i \hat{\mu}_i \left(\rho_i^\top \theta \right) - \sum_i \exp \left(\rho_i^\top \theta \right) \right\}$$

Logistic regression:

$$\max_\lambda \left\{ \sum_{xy} \hat{\mu}_{xy} \phi^\lambda_{xy} - \sum_x n_x \log \sum_{y'} \exp \phi^\lambda_{xy'} \right\}$$

Poisson trick:
observations $i = xy$
dependent variable $\hat{\mu}_i = \hat{\mu}_{xy}$
$\theta^\top = \left(\lambda^\top, u \right)$
$\rho_i^\top \theta = \sum_k \Phi_{xyk} \lambda_k - u_x = \sum_k \Phi_{xyk} \lambda_k - \sum_{x'} 1_{\{x'=x\}} u_{x'}$

Write then the Poisson regression: we have

$$\max_{\lambda, u} \left\{ \sum_{xy} \hat{\mu}_{xy} \left(\phi^\lambda_{xy} - u_x \right) - \sum_{xy} \exp \left(\phi^\lambda_{xy} - u_x \right) \right\}$$

Let's focus on the maximization wrt u_x. We have

$$\max_\lambda \left\{ \sum_{xy} \hat{\mu}_{xy} \phi^\lambda_{xy} - \min_u \left\{ \sum_x n_x u_x + \sum_{xy} \exp \left(\phi^\lambda_{xy} - u_x \right) \right\} \right\}$$

therefore I need to compute $\min_u \left\{ \sum_x n_x u_x + \sum_{xy} \exp\left(\phi_{xy}^\lambda - u_x\right) \right\}$ inside. We have by FOC

$$n_x = \sum_y \exp\left(\phi_{xy}^\lambda - u_x\right)$$

therefore $u_x = \log \sum_y \exp\left(\phi_{xy}^\lambda\right) - \log n_x$

$$\sum_{xy} \exp\left(\phi_{xy}^\lambda - u_x\right) = \sum_x n_x$$

and therefore

$$\begin{aligned} & \min_u \left\{ \sum_x n_x u_x + \sum_{xy} \exp\left(\phi_{xy}^\lambda - u_x\right) \right\} \\ = & \sum_x n_x u_x + \sum_x n_x \\ = & \sum_x n_x \log \sum_y \exp\left(\phi_{xy}^\lambda\right) - \sum_x n_x \log n_x + \sum_x n_x \end{aligned}$$

The Poisson regression plus x-fixed effect becomes

$$\max_\lambda \left\{ \sum_{xy} \hat{\mu}_{xy} \phi_{xy}^\lambda - \left(\sum_x n_x \log \sum_y \exp\left(\phi_{xy}^\lambda\right) - \sum_x n_x \log n_x + \sum_x n_x \right) \right\}$$

and after discarding the constants, one has

$$\max_\lambda \left\{ \sum_{xy} \hat{\mu}_{xy} \phi_{xy}^\lambda - \sum_x n_x \log \sum_y \exp\left(\phi_{xy}^\lambda\right) \right\}$$

the logistic regression!

Therefore, we have seen that the logistic regression = Poisson plus fixed effect.

Gravity equation. Take the Poisson regression with x and y fixed effects. We have

$$\max_{\lambda,u,v} \left\{ \sum_{xy} \hat{\mu}_{xy} \left(\phi_{xy}^\lambda - u_x - v_y\right) - \sum_{xy} \exp\left(\phi_{xy}^\lambda - u_x - v_y\right) \right\}$$

FOC of this problem:

wrt on u_x, v_y and λ_k

$$\begin{aligned} n_x &= \sum_y \hat{\mu}_{xy} = \sum_{y \in [Y]} \exp\left(\phi_{xy}^\lambda - u_x - v_y\right) \\ m_y &= \sum_x \hat{\mu}_{xy} = \sum_{x \in [X]} \exp\left(\phi_{xy}^\lambda - u_x - v_y\right) \\ \sum_{xy} \hat{\mu}_{xy} \Phi_{xyk} &= \sum_{xy} \exp\left(\phi_{xy}^\lambda - u_x - v_y\right) \Phi_{xyk} \end{aligned}$$

This is the gravity equation in trade (Santos Silva and Tenreyro).

5 L5. Matching estimation

Assume we observe $\hat{\mu}_{xy}$ and we want to estimate λ, which is the parameter of the surplus $\phi^{\lambda}_{xy} = \sum_k \Phi_{xyk} \lambda_k$.

Remember, the equilibrium in matching models expressed as

$$W(\lambda) = \min_{a,b} \left\{ \sum_x \exp(a_x) + \sum_y \exp(b_y) + 2 \sum_{xy} \exp\left(\frac{\phi^{\lambda}_{xy} + a_x + b_y}{2} \right) - \sum_x n_x a_x - \sum_y m_y b_y \right\}.$$

We have

$$\begin{aligned} \frac{\partial W(\lambda)}{\partial \lambda_k} &= \sum_{xy} \exp\left(\frac{\phi^{\lambda}_{xy} + a_x + b_y}{2} \right) \Phi_{xyk} \\ &= \sum_{xy} \mu^{\lambda}_{xy} \Phi_{xyk} \end{aligned}$$

We are looking for λ such that

$$\sum_{xy} \mu^{\lambda}_{xy} \Phi_{xyk} = \sum_{xy} \hat{\mu}_{xy} \Phi_{xyk}$$

which is the first order condition associated with

$$\min_{\lambda} \left\{ W(\lambda) - \sum_{xyk} \hat{\mu}_{xy} \Phi_{xyk} \lambda_k \right\}$$

that is

$$\min_{a,b,\lambda} \left\{ \sum_x \exp(a_x) + \sum_y \exp(b_y) + 2 \sum_{xy} \exp\left(\frac{\phi^{\lambda}_{xy} + a_x + b_y}{2} \right) - \sum_x n_x a_x - \sum_y m_y b_y - \sum_{xyk} \hat{\mu}_{xy} \Phi_{xyk} \lambda_k \right\}$$

in other words

$$\min_{a,b,\lambda} \left\{ \begin{array}{c} \sum_x \exp(a_x) + \sum_y \exp(b_y) + 2 \sum_{xy} \exp\left(\frac{\phi^{\lambda}_{xy} + a_x + b_y}{2} \right) \\ -2 \sum_{xy} \hat{\mu}_{xy} \left(\frac{\phi^{\lambda}_{xy} + a_x + b_y}{2} \right) - \sum_x \hat{\mu}_{x0} a_x - \sum_y \hat{\mu}_{0y} b_y \end{array} \right\}$$

Let's try to set this up as Poisson regression. Observations are of the form $x0$, $0y$ or xy

We set $w_{xy} = 2$, $w_{x0} = 1$ and $w_{0y} = 1$,

Let's setup the design matrix, ie the matrix of regressors

$$R = \begin{pmatrix} \frac{\Phi_{xy,k}}{2} & \frac{I_X \otimes 1_Y}{2} & \frac{1_X \otimes I_Y}{2} \\ 0_{X,K} & I_X & 0_{X \times Y} \\ 0_{Y,K} & 0_{Y,X} & I_Y \end{pmatrix}$$

Choo-Siow (2006)
Dupuy and G (2015)
Chiappori, Salanie and Weiss (2019)
G and Salanie (2022)
Chiappori Fiorio G and Verzillo (2024+)

Corblet (2024)
Chone Gozlan Kramarz (2024)