Matching models with general transfers

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- These lectures will provide a bridge between optimal transport theory and economics, through matching models, multinonial choice models, gravity equations, in connection (time permitting) with quantile methods, random sets, and option pricing.
- Code in R is provided on the course directory (`https://github.com/alfredgalichon/hausdorff-lectures-2018`). Other language work but the LP solver used will be Gurobi, so the language of choice should have a convenient interface to these.

TEXTS

- Still the best introduction:
  - [TOT] C. Villani, *Topics in Optimal Transportation*, AMS, 2003.
- Mathematical foundations:
  - [MTP] Rachev, Ruschendorf. Mass transportation problems. Springer, 1998.
  - [OTON] C. Villani, *Optimal Transport: Old and New*, AMS, 2008.
  - [OTAM] F. Santambrogio, *Optimal Transport for Applied Mathematicians*, Birkhäuser, 2015.
- Computational focus:
  - [NOT] G. Peyré, M. Cuturi (2018). *Numerical optimal transport*, Arxiv.
- Economics focus:
  - [OTME] A. Galichon. *Optimal Transport Methods in Economics*, Princeton, 2016.
  - [MWT] P.-A. Chiappori. *Matching with Transfers: The Economics of Love and Marriage*, Princeton, 2017.

Lecture 1. The Monge-Kantorovich duality: general overview and linear programming

Refs: [OTME], Chapters 1, 2 and 8.
Complement: [TOT], Chapter 1.

Section 1

The Monge-Kantorovich theorem

- Consider the problem of assigning a possibly infinite number of workers and firms. Each worker should work for one firm, and each firm should hire one worker.
- Workers and firms have heterogenous characteristics; let $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ be the vector of characteristics of workers and firms respectively. In the general theory, these sets are Polish spaces, i.e. complete and separable metric spaces. Later on, we shall investigate the finite dimensional case, when these sets are finite.
- Workers and firms are in equal mass, which is normalized to one. The distribution of worker's types is P, and the distribution of the firm's types is Q, where P and Q are probability measures on $\mathcal{X}$ and $\mathcal{Y}$.

- A *coupling* determines which workers are assigned to which firms. If we had a finite number of workers and firms, we would need to count the number of workers of a given type matched with firms of a given type. More generally, a coupling will be defined as the probability measure π of occurence of worker-firm pairs. If $(X, Y) \sim \pi$ is the joint random pair, then $X \sim P$ and $Y \sim Q$, where $X \sim P$ means "X has distribution P." In other words, the first *margin* of π should be P, while its second margin should be Q.
- This motivates the following definition:

DEFINITION

The set of couplings of probability distributions P and Q is the set of probability distributions over $\mathcal{X} \times \mathcal{Y}$ with first and second margins P and Q. This set is denoted $\mathcal{M}(P, Q)$. That is, a probability measure π over $\mathcal{X} \times \mathcal{Y}$ is in $\mathcal{M}(P, Q)$ if and only if

$$\pi(A \times \mathcal{Y}) = P(A) \text{ and } \pi(\mathcal{X} \times B) = Q(B)$$

holds for every subset A of $\mathcal{X}$ and B of $\mathcal{Y}$. By extension, a random pair $(X, Y) \sim \pi$ where $\pi \in \mathcal{M}(P, Q)$ will also be called a coupling of P and Q.

- *Independent coupling*, a.ka. *random matching*: $\pi(A, B) = P(A)\,Q(B)$, so that, if $(X, Y) \sim \pi$, then $X \sim P$ and $Y \sim Q$ are independent.
- *Pure assignment*, or *Monge coupling*: Y is a deterministic function of X; that is, $Y = T(X)$. In our worker-firm example, this assumes that every workers of type x will get assigned the same type of firm, $T(x) \in \mathcal{Y}$. Then if $X \sim P$, then $T(X) \sim Q$, which we denote by

$$T\#P = Q \tag{1}$$

where $T\#P$ is the distribution of $T(X)$ when $X \sim P$ ("push-forward" of P by map T, sometimes denoted $PT^{-1} = Q$).
- In general, a coupling is associated to a *Markov kernels*, $\pi(dy|x)$ such that

$$\int_{\mathcal{X}} \pi(B|x)\,dP(x) = Q(B)$$

for every subset B of $\mathcal{Y}$.

- Assume that if worker x works for firm y, this generates a quantity of output $\Phi(x, y)$, measured in some monetary unit. A social planner decides which workers to assign to which firms and seeks to maximize the total output. The theory Optimal Transport studies how to do this.
- The *Monge Problem* (posed at the end of the 18th century) consists of looking among all the *pure* assignments, that is,

$$\max_{T(.)} \mathbb{E}_P \left[\Phi\left(X, T\left(X\right)\right)\right]. \tag{2}$$
$$s.t.\ T\#P = Q$$

- In general, the Monge problem is difficult:
  - It is nonlinear. In the discrete case, assuming P and Q have N equally weighted sample points, this amounts to looking for T within the permutations of $\{1, ..., N\}$: $N!$ possibilities.
  - It may not have a solution. Think e.g. of the discrete case when P has less sample points than Q.
  - In fact, it remained unsolved for more than a century!

- Kantorovich in the 1940s came up with the idea of *linear programming Relaxation*: instead of looking among Monge couplings, look among all the couplings. Hence, instead of maximizing $\mathbb{E}_P [\Phi (X, T (X))]$ s.t. $T\#P = Q$, simply maximize $\mathbb{E}_\pi [\Phi (X, Y)]$ s.t. $X \sim P$ and $Y \sim Q$.
- This leads to the *Kantorovich problem*

$$\max_{\pi \in \mathcal{M}(P,Q)} \mathbb{E}_\pi [\Phi (X, Y)] \tag{3}$$

- This formulation has many advantages:
  - it is a linear programming problem, albeit an infinite-dimensional one. The dual is very informative.
  - It has a solution π under very weak assumptions.
  - In a number of relevant cases, the solution to the Monge and Kantorovich problems will coincide.

THEOREM

Let $\mathcal{X}$ and $\mathcal{Y}$ be two Banach spaces, and let P and Q be two probability measures on $\mathcal{X}$ and $\mathcal{Y}$ respectively. Let $\Phi : \mathcal{X} \times \mathcal{Y} \rightarrow \mathbb{R} \cup \{-\infty\}$ be an upper semicontinuous surplus function bounded above by $\overline{a}(x) + \overline{b}(y)$ where $\overline{a}$ and $\overline{b}$ are respectively integrable with respect to P and Q. Then:
(i) The value of the primal Monge-Kantorovich problem

$$\sup_{\pi \in \mathcal{M}(P,Q)} \mathbb{E}_{\pi}\left[\Phi\left(X, Y\right)\right] \tag{4}$$

coincides with the value of the dual

$$\inf_{u,v} \mathbb{E}_{P}\left[u\left(X\right)\right] + \mathbb{E}_{Q}\left[v\left(Y\right)\right] \tag{5}$$
$$s.t.\ u\left(x\right) + v\left(y\right) \geq \Phi\left(x, y\right)$$

where the infimum is over measurable and integrable functions u and v, and the inequality constraint should be satisfied for almost every x and almost every y (all these statements are respective to measures P and Q).
(ii) An optimal solution π to problem (4) exists.

Theorem

(iii) Assume further Φ *is bounded below by* $\underline{a}(x) + \underline{b}(y)$ *where* $\underline{a}$ *and* $\underline{b}$ *are respectively integrable with respect to* P *and* Q*. Then the dual problem (5) also has solutions.*

- As noted above, this theorem is a result on infinite-dimensional linear programming. Part (i) of this result implies that a strong duality holds: the value of the primal and the dual problems coincide. Part (ii) ensures that the primal problem has an optimal solution.
- The dual variables $u(x)$ and $v(y)$, a.k.a. *Kantorovich potentials*, will find an interpretation as the equilibrium payoffs that worker x and firm y get at equilibrium.
- Problem (4) provides a breakdown of the surplus by pairs, while Problem (5) offers a breakdown of the total surplus at the individual level.

The value of the primal problem $\sup_{\pi\in\mathcal{M}(P,Q)} \mathbb{E}_\pi \left[\Phi\left(X,Y\right)\right]$ can be rewritten

$$\sup_{\pi\in\mathcal{M}^+} \int \Phi\left(x,y\right) d\pi\left(x,y\right) + A_{P,Q}\left(\pi\right),$$

where $\mathcal{M}^+$ is the set of positive measures over $\mathcal{X}\times\mathcal{Y}$ (not necessarily of total mass one, and not necessarily with fixed marginals), and $A_{P,Q}$ should be such that

$$A_{P,Q}\left(\pi\right) = \left\{ \begin{array}{l} 0 \text{ if } \pi\in\mathcal{M}\left(P,Q\right) \\ = -\infty \text{ else.} \end{array} \right.$$

One can take

$$A_{P,Q}\left(\pi\right) = \inf_{u,v} \int u\left(x\right) dP\left(x\right) + \int v\left(y\right) dQ\left(y\right) - \int \left(u\left(x\right)+v\left(y\right)\right) d\pi\left(x,y\right)$$

so that the value of the primal problem becomes

$$\sup_{\pi\in\mathcal{M}^+} \inf_{u,v} \left\{ \begin{array}{c} \int \Phi\left(x,y\right) - \left(u\left(x\right)+v\left(y\right)\right) d\pi\left(x,y\right) \\ + \int u\left(x\right) dP\left(x\right) + \int v\left(y\right) dQ\left(y\right) \end{array} \right\}.$$

It is the case here that sup inf = inf sup (this fact will be admitted without a proof), which yields

$$\inf_{u,v} \int u(x)\, dP(x) + \int v(y)\, dQ(y) + B_\Phi(u,v)$$

where $B_\Phi(u,v) = \sup_{\pi \in \mathcal{M}^+} \int \Phi(x,y) - (u(x) + v(y))\, d\pi(x,y)$, so that

$$B_\Phi(u,v) = \begin{cases} 0 \text{ if } u(x) + v(y) \geq \Phi(x,y) \text{ for all } x \text{ and } y \\ = +\infty \text{ else.} \end{cases}$$

thus the value of the problem rewrites as (5).This argument is only a rough sketch; the minmax principle which we invoked when inverting the sup and the inf needs to be carefully established, and the spaces in which the functions u and v and the measure μ live need to be made precise. See Villani's [TOT], Ch. 1 for a rigorous argument.

We argue that $u(x)$ can be interpreted as the equilibrium wage of worker x, while $v(y)$ can be interpreted as the equilibrium profit of firm y.

PROPOSITION

If (u, v) is a solution to the dual of the Kantorovich problem, then we can always redefine u and v so that they take value $+\infty$ outside of the supports of P and Q, respectively. In this case,

$$u(x) = \sup_y (\Phi(x, y) - v(y)) \tag{6}$$

$$v(y) = \sup_x (\Phi(x, y) - u(x)) \tag{7}$$

should hold almost surely with respect to the probabilities P and Q, respectively.

PROOF.

The constraint in (5) implies that $u(x) + v(y) \geq \Phi(x, y)$, thus

$$v(y) \geq \sup_x (\Phi(x, y) - u(x))$$

but if the latter inequality was to hold strictly on a set with positive Q measure, one could strictly improve on the dual objective function $\mathbb{E}_P[u(X)] + \mathbb{E}_Q[v(Y)]$, and contradict optimality of (u, v). Hence the inequality is actually an equality, and (6) holds. □

- Expression (7) stands therefore for the problem of a firm choosing optimally the worker it will hire.
- As a result, Theorem 2 can be interpreted as a welfare theorem: the solution of the central planner (4), coincides with the solution of the decentralized equilibrium, given by (5).
- Of course, an important indeterminacy remains in the dual problem: again, if (u, v) is a solution of the dual, and if $c \in \mathbb{R}$, then $(u - c, v + c)$ is also a solution. In other words, if $u(x)$ is an equilibrium wage curve, then so is $u(x) + c$.

Section 2

Discrete case and linear programming

- Assume that the type spaces $\mathcal{X}$ and $\mathcal{Y}$ are finite, so $\mathcal{X} = \{1, ..., N\}$, and $\mathcal{Y} = \{1, ..., M\}$.
- The total mass of workers and jobs is normalized to one. The mass of workers of type x is p_x; the mass of jobs of type y is q_y, with $\sum_x p_x = \sum_y q_y = 1$.
- Let π_{xy} be the mass of workers of type x assigned to jobs of type y. Every worker is occupied and every job is filled, thus

$$\sum_{y=1}^{M} \pi_{xy} = p_x \text{ and } \sum_{x=1}^{N} \pi_{xy} = q_y. \tag{8}$$

(Note that this formulation does not restrict to Monge couplings, i.e. it allows for $\pi_{xy} > 0$ and $\pi_{xy'} > 0$ to hold simultaneously with $y \neq y'$.)

- Assume the economic output created when assigning worker x to job y is Φ_{xy}. Hence, under assignment π, the total output is $\sum_{xy} \pi_{xy} \Phi_{xy}$.
- Thus, the optimal assignment is

$$\begin{aligned} &\max_{\pi \geq 0} \sum_{xy} \pi_{xy} \Phi_{xy} \\ s.t. \ &\sum_{y=1}^{M} \pi_{xy} = p_x \ \ [u_x] \\ &\sum_{x=1}^{N} \pi_{xy} = q_y \ \ [v_y] \end{aligned} \tag{9}$$

  and it is now a finite-dimensional linear programming problem.
- Note that it is nothing else than the Monge-Kantorovich problem when P and Q are discrete probability measures on $\mathcal{X} = \{1, ..., N\}$, and $\mathcal{Y} = \{1, ..., M\}$.

THEOREM (M-K, FINITE-DIMENSIONAL CASE)

(i) The value of the primal problem (9) coincides with the value of the dual problem

$$\min_{u,v} \sum_{x=1}^{N} p_x u_x + \sum_{y=1}^{M} q_y v_y. \tag{10}$$

$$s.t. u_x + v_y \geq \Phi_{xy} \ \ [\pi_{xy} \geq 0]$$

(ii) Both the primal and the dual problems have optimal solutions. If π is a solution to the primal problem and (u, v) a solution to the dual problem, then by complementary slackness,

$$\pi_{xy} > 0 \text{ implies } u_x + v_y = \Phi_{xy}. \tag{11}$$

(iii) If (u, v) is a solution to the dual problem, then

$$u_x = \max_{y \in \{1,\ldots,M\}} \{\Phi_{xy} - v_y\} \text{ and } v_y = \max_{x \in \{1,\ldots,N\}} \{\Phi_{xy} - u_x\}. \tag{12}$$

- Assume that if a worker of type x works for a firm of type y for wage w_{xy}, then gets $\alpha_{xy} + w_{xy}$, where α_{xy} is the nonmonetary payoff associated with working with a firm of type y. The firm's profit is $\gamma_{xy} - w_{xy}$, where γ_{xy} is the economic output.
- If an employee of type x matches with a firm of type y, they generate joint surplus

$$\Phi_{xy} = \underbrace{\alpha_{xy} + w_{xy}}_{\text{employee's payoff}} + \underbrace{\gamma_{xy} - w_{xy}}_{\text{firm's payoff}} = \alpha_{xy} + \gamma_{xy}$$

which is independent from w.
- Workers choose firms which maximize their utility, i.e. solve

$$u_x = \max_y \{\alpha_{xy} + w_{xy}\} \tag{13}$$

and $u_x = \alpha_{xy} + w_{xy}$ if x and y are matched. Similarly, the indirect payoff vector of firms is

$$v_y = \max_x \{\gamma_{xy} - w_{xy}\} \tag{14}$$

and, again, $v_y = \gamma_{xy} - w_{xy}$ if x and y are matched.

- As a result,
$$u_x + v_y \geq \alpha_{xy} + \gamma_{xy} = \Phi_{xy}$$
and equality holds if x and y are matched. Thus, if w_{xy} is an equilibrium wage, then the triple (π, u, v) where π is the corresponding matching, and u_x and v_y are defined by (13) and (14) defines a stable outcome.
- Conversely, let (π, u, v) be a stable outcome. Then let $\bar{w}_{xx}$ and $\underline{w}_{xy}$ be defined by
$$\bar{w}_{xy} = u_x - \alpha_{xy} \text{ and } \underline{w}_{xy} = \gamma_{xy} - v_y.$$
- One has $\bar{w}_{xy} \geq \underline{w}_{xy}$. Any w_{xy} such that
$$\bar{w}_{xy} \geq w_{xy} \geq \underline{w}_{xy}$$
is an equilibrium wage. Indeed, $\pi_{xy} > 0$ implies $\bar{w}_{xy} = \underline{w}_{xy}$, thus (13) and (14) hold.
- One sees that given u and v, w_{xy} is uniquely defined on the equilibrium path (ie. when x and y are such that $\pi_{xy} > 0$), but there are multiple choices of w outside the equilibrium path.

- This problem of computation of the Optimal Assignment Problem, more specifically of (π, u, v), is arguably the most studied problem in Computer Science, and dozens, if not hundreds of algorithms exist, whose running time is polynomial in $\max(n, m)$, typically a power less than three of the latter.
- Famous algorithms include: the Hungarian algorithm (Kuhn-Munkres); Bertsekas' auction algorithm; Goldberg and Kennedy's pseudoflow algorithm. See an introduction to these algorithms in http://www.assignmentproblems.com/doc/LSAPIntroduction.pdf.
- Here, we will show how to solve the problem with the help of a linear programming solver used as a black box; our challenge here will be to carefully set up the constraint matrix as a sparse matrix in order to let a large scale linear programming solvers such as Gurobi recognize and exploit the sparsity of the problem.

- Let Π and Φ be the matrices with typical elements (π_{xy}) and (Φ_{xy}). We let p, q, u, v, and 1 the column vectors with entries (p_x), (q_y), (u_x), (v_y), and 1, respectively. Problem (9) rewrites using matrix algebra as

$$\max_{\Pi \geq 0} Tr\left(\Pi' \Phi\right) \tag{15}$$
$$\Pi 1_M = p$$
$$1_N' \Pi = q'.$$

- We need to convert matrices into vectors; this can be done for instance by stacking the columns of Π into a single column vector (typical in R or Matlab). This operation is called *vectorization*, which we will denote

$$vec\left(A\right),$$

which reshapes a $N \times M$ matrix into a $nm \times 1$ vector. In R, this is implemented by `c(A)`; in Matlab, by `reshape(A,[n*m,1])`.

- The objective function rewrites as

$$vec\left(\Phi\right)' vec\left(\Pi\right).$$

- Recall that if A is a $M \times p$ matrix and B a $N \times q$ matrix, then the Kronecker product $A \otimes B$ of A and B is a $mn \times pq$ matrix such that
$$vec\left(BXA'\right) = (A \otimes B)\, vec\,(X)\,. \tag{16}$$
In R, $A \otimes B$ is implemented by `kronecker(A,B)`; in Matlab, by `kron(A,B)`.
- The first constraint $I_N \Pi 1_M = p$, vectorizes therefore as
$$\left(1_M' \otimes I_N\right) vec\,(\Pi) = vec\,(p)\,,$$
and similarly, the second constraint $1_N' \Pi I_M = q'$, vectorizes as
$$\left(I_M \otimes 1_N'\right) vec\,(\Pi) = vec\,(q)\,.$$
- Note that the matrix $1_M' \otimes I_N$ is of size $N \times NM$, and the matrix $I_M \otimes 1_N'$ is of size $M \times NM$; hence the full matrix involved in the left-hand side of the constraints is of size $(N + M) \times NM$. In spite of its large size, this matrix is *sparse*. In R, the identity matrix I_N is coded as `sparseMatrix(1:N,1:N)`, in Matlab as `Speye(N)`.

- Setting $z = vec\left(\Pi\right)$, the linear programming problem then becomes

$$\begin{aligned} &\max_{z \geq 0} vec\left(\Phi\right)' z \\ s.t.\ &\left(1_M' \otimes I_N\right) z = vec\left(p\right) \\ &\left(I_M \otimes 1_N'\right) z = vec\left(q'\right) \end{aligned} \tag{17}$$

  which is ready to be passed on to a linear programming solver such as Gurobi.

- A LP solver typically computes programs of the form

$$\begin{aligned} &\max_{z \geq 0} c' z \\ s.t.\ &Az = d. \end{aligned} \tag{18}$$

  In R, Gurobi is called to compute program (18) by `gurobi(list(A=A,obj=c,modelsense="max",rhs=d,sense="="))`.

See subdirectory `code/OptimalAssignment/` in the Github repository.

Section 3

Extension: minimum cost flows on a network

- Consider a slightly different interpretation of the problem. Assume $\mathcal{X}$ is a set of warehouses (sources), and $\mathcal{Y}$ is a set of stores (target). The quantity of commodity supplied at warehouse x is p_x, and the quantity of commodity needed at store y is q_y. The cost of shipping from warehouse x to store y is c_{xy}.
- The social planner decides on the amount π_{xy} to ship from each x to each y. It does so to minimizes total cost:

$$\begin{aligned} &\min_{\pi \geq 0} \sum_{xy} \pi_{xy} c_{xy} \\ &s.t. \ \sum_{y} \pi_{xy} = p_x \\ &\sum_{x} \pi_{xy} = q_y \end{aligned}$$

which has dual

$$\begin{aligned} &\max_{u,v} \sum_{x \in \mathcal{X}} p_x u_x + \sum_{y \in \mathcal{Y}} q_y v_y \\ &s.t. \ u_x + v_y \leq c_{xy} \end{aligned}$$

- To interpret the dual better, it is worth changing notations. Denote $w_x = -u_x$ if $x \in \mathcal{X}$ and $w_y = v_y$ if $y \in \mathcal{Y}$, so that the dual rewrites

$$\max_w \sum_{y \in \mathcal{Y}} q_y w_y - \sum_{x \in \mathcal{X}} p_x w_x$$
$$s.t.\ w_y - w_x \leq c_{xy}$$

- With these new notations, w_x inteprets as the equilibrium price of the commodity at location x. The price w_y at store y should be less than $c_{xy} + w_x$; otherwise it would be profitable to buy the good at location x, and ship at y to get a better deal.
- Intepretation: suppose the company subcontracts shipping operations. Then value of primal = price paid to the subcontractor; while value of dual = minimal cost to the contractor.
- One then performs one further change of notations. Set $b_x = -p_x$ if $x \in \mathcal{X}$ and $b_y = q_y$ if $y \in \mathcal{Y}$, so that for $z \in \mathcal{X} \cup \mathcal{Y}$, b_z is the algebraic quantity consumed in the network. The dual further rewrites

$$\max_w \sum_{z \in \mathcal{X} \cup \mathcal{Y}} b_z w_z : w_y - w_x \leq c_{xy}\ \forall x \in \mathcal{X}, y \in \mathcal{Y}$$

- Until now, we have implicitely considered a directed network where there can only be arcs from elements of $\mathcal{X}$ to elements of $\mathcal{Y}$. Such a network is called *bipartite*; however, we would like to get rid of this restriction. In particular, we would like to be able to introduce intermediary nodes which are transit nodes only ($b_z = 0$), so that the problem of how to getting from x to y is captured.
- For this, replace $\mathcal{X} \cup \mathcal{Y}$ by a unique set of nodes $\mathcal{Z}$. Source nodes are those for which $b < 0$; intermediate ones are those such that $b = 0$, and target nodes are those such that $b > 0$.
- The *min-cost flow problem* formulates as

$$\max_{u,v} \sum_{z \in \mathcal{Z}} b_z w_z$$
$$s.t.\ w_y - w_x \leq c_{xy}\ \forall x \in \mathcal{Z}, y \in \mathcal{Z}$$

and if $c_{xy} = +\infty$ one says there is no arc from x to y. W.l.o.g. one can set $c_{xx} = +\infty$.

- Its primal formulation is given by

$$\min_{\pi \geq 0} \sum_{(x,y)\in\mathcal{Z}^2} \pi_{xy} c_{xy}$$
$$s.t. \sum_{z\in\mathcal{Z}} \pi_{zx} - \sum_{z\in\mathcal{Z}} \pi_{xz} = b_x \;\forall x \in \mathcal{Z}$$

- We can rewrite the constraints as a linear system as

$$\min_{\pi\geq 0} \pi^{\intercal} c$$
$$s.t.\; A\pi = b$$

where π is a vector of $\mathbb{R}^{X\times Y}$ $A_{z,(xy)} = 1\{z = y\} - 1\{z = x\}$. Looks high-dimensional, but one can eliminate all xy which are no arcs. Therefore A is a matrix whose rows are the nodes, and whose columns are the arcs of the network. A is called a *node-arc incidence matrix*.

- Consider now the particular case with one single source s and one single target t. Then the vector b is given by

$$b_z = 1\{z = t\} - 1\{z = s\}.$$

- The min-cost problem then gives then how to move one unit of mass from s to t through the network at minimal cost. It is the shortest path problem (assuming c_{xy} is the distance of arc xy).

Consider a network with `nbNodes` nodes and `nbArcs` arcs. The typical structure of the data needed is:
(i) `arcs`, an array of integers of size `nbArcs`$\times 2$ whose entries are between 1 and `nbNodes`, so that each line describes an arc, represented by its origin node (first column) and its destination node (second column);
(ii) `n`, a vector of length `nbNodes` such that n_x is the net demand at node x;
(iii) `c`, a vector of length `nbArcs` such that c_a is the transportation cost at arc a.

The dual constraint matrix $A^{\intercal}$ is coded in a sparse way. In R, this is done using the package '`Matrix`', and the relevant instruction is:

```
At=
sparseMatrix(i=1:nbArcs,j=arcs[,1],dims=c(nbArcs,nbNodes),x=-1)
+sparseMatrix(i=1:nbArcs,j=arcs[,2],dims=c(nbArcs,nbNodes),x=1).
```

We will use Gurobi for LP. The solver is called by:

```
sol = gurobi(list(A=t(At),obj=c,modelsense='min',rhs=n,sense='='))
```

We run Gurobi. The optimal flow `pi` and the potential `v` are computed using:

```
pi = sol$x
v = sol$pi.
```

See subdirectory `code/ShortestPath/` in the Github repository.

- More on these techniques in my ‘math+econ+code’ week-long masterclass series

  http://alfredgalichon.com/mec

  offered several times a year on different topics at NYU (in various campuses).
- Next m+e+c classes will take place on the NYU Paris campus.
- Interested students should contact me at ag133@nyu.edu.