Matching models with general transfers

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Lecture 3, July 23. Equilibrium transport

Learning objectives: lecture 3

- Matching with nonlinear taxes
- Equilibrium transport

- [KC] Kelso, A. and Crawford, V. (1982). Job matching, coalition formation and gross substitutes. Econometrica.
- [CGJK] Dupuy, A., AG, Jaffe, S., and Kominers, S. On the incidence of taxation in matching markets.
- [GKW] AG, Kominers, S. and Weber, S. (2015). Costly Concessions: An Empirical Framework for Matching with Imperfectly Transferable Utility.
- [GH] Galichon, A., Hsieh, Y.-W. A theory of decentralized matching markets without transfers, with an application to surge pricing.
- [NS] Noeldeke, G. and Samuelson, L. (2018). The implementation duality. Econometrica, forthcoming.

- Consider a model of the labor market. Assume that a population of *workers* is characterized by their type $x \in \mathcal{X}$, where $\mathcal{X} = \mathbb{R}^d$ for simplicity. There is a distribution P over the workers, which is assumed to sum to one.
- A population of *firms* is characterized by their types $y \in \mathcal{Y}$ (say $\mathcal{Y} = \mathbb{R}^d$), and their distribution Q. It is assumed that there is the same total mass of workers and firms, so Q sums to one.
- Each worker must work for one firm; each firm must hire one worker. Let $\pi(x, y)$ be the probability of observing a matched (x, y) pair. π should have marginal P and Q, which is denoted

$$\pi \in \mathcal{M}(P, Q).$$

- In the simplest case, the utility of a worker x working for a firm y at wage $w(x, y)$ will be
$$\alpha(x, y) + w(x, y)$$
while the corresponding profit of firm y is
$$\gamma(x, y) - w(x, y).$$
- In this case, the total surplus generated by a pair (x, y) is
$$\alpha(x, y) + w + \gamma(x, y) - w = \alpha(x, y) + \gamma(x, y) =: \Phi(x, y)$$
which does not depend on w (no transfer frictions). A central planner may thus like to choose assignment $\pi \in \mathcal{M}(P, Q)$ so to
$$\max_{\pi \in \mathcal{M}(P,Q)} \int \Phi(x, y)\, d\pi(x, y).$$
But as it turns out, this is also **the equilibrium solution**.

- The equilibrium assignment is determined by an important quantity: the **wages**. Let $w(x,y)$ be the wage of employee x working for firm of type y.
- Let the indirect surpluses of worker x and firm y be respectively

$$u(x) = \max_y \{\alpha(x,y) + w(x,y)\}$$
$$v(y) = \max_x \{\gamma(x,y) - w(x,y)\}$$

so that (π, w) is an equilibrium when

$$u(x) \geq \alpha(x,y) + w(x,y) \text{ with equality if } (x,y) \in Supp(\pi)$$
$$v(y) \geq \gamma(x,y) - w(x,y) \text{ with equality if } (x,y) \in Supp(\pi)$$

- By summation,

$$u(x) + v(y) \geq \Phi(x,y) \text{ with equality if } (x,y) \in Supp(\pi).$$

- One can show that the equilibrium outcome (π, u, v) is such that π is solution to the primal Monge-Kantorovich Optimal Transportation problem

$$\max_{\pi \in \mathcal{M}(P,Q)} \int \Phi(x, y)\, d\pi(x, y)$$

and (u, v) is solution to the dual OT problem

$$\begin{aligned} &\min_{u,v} \int u(x)\, dP(x) + \int v(y)\, dQ(y) \\ &s.t.\ u(x) + v(y) \geq \Phi(x, y) \end{aligned}$$

- Feasibility+Complementary slackness yield the desired equilibrium conditions

$$\begin{aligned} &\pi \in \mathcal{M}(P, Q) \\ &u(x) + v(y) \geq \Phi(x, y) \\ &(x, y) \in Supp(\pi) \Longrightarrow u(x) + v(y) = \Phi(x, y) \end{aligned}$$

Here, **optimum=equilibrium**. "Second welfare theorem", "invisible hand", etc.

- Is equilibrium always the solution to an optimization problem?
- **It is not**. This is why this talk is about "Equilibrium Transportation," which contains, but is strictly more general than "Optimal Transportation".

- Is equilibrium always the solution to an optimization problem?
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- Consider the same setting as above, but introduce (nonlinear) taxes.
- Instead of assuming that workers' and firm's payoffs are linear in wages, assume

$$u(x) = \max_y \{\alpha_{xy} + N(w(x,y))\}$$
$$v(y) = \max_x \{\gamma_{xy} - w(x,y)\}$$

where $N(w)$ is indecreasing and continuous, interpreted as the net wage if w if the gross wage.

- Of course, OT is recovered when $N(w) = w$ (no tax).

- Let $\mathcal{F}_{xy}$ be the set of feasible utilities that x and y can achieve through some wage w. One has

$$\mathcal{F}_{xy} = \{(u, v) : \exists w \in \mathbb{R}, u \leq \alpha_{xy} + N(w), v \leq \gamma_{xy} - w\},$$

which rewrites

$$\mathcal{F}_{xy} = \{(u, v) : u - \alpha_{xy} \leq N(\gamma_{xy} - v)\}.$$

- The interior of this set, denoted $\mathcal{F}^0_{xy}$, is the set such that this ineqality holds true.
- In the case of OT,

$$\mathcal{F}_{xy} = \{(u, v) : u + v \leq \alpha_{xy} + \gamma_{xy}\}.$$

- We have therefore that (π, u, v) is an equilibrium outcome when

$$\begin{cases} (PF) : \pi \in \mathcal{M}(P, Q) \\ (DF) : (u(x), v(y)) \notin \mathcal{F}^0_{xy} \\ (NC) : (x, y) \in Supp(\pi) \Longrightarrow (u(x), v(y)) \in \mathcal{F}_{xy}. \end{cases}$$

- This is a **Nonlinear Complementarity Problem** (NCP) with much structure.
- Problem: existence of an equilibrium outcome? yes in the discrete case ($\mathcal{X}$ and $\mathcal{Y}$ finite): Kelso-Crawford, Alkan and Gale.

- Link with Galois connection, see Noeldeke and Samuelson (2015) and Larry's talk. Let

$$G_{xy}(v) = \alpha_{xy} + N(\gamma_{xy} - v).$$

One has $(u_x, v_y) \notin \mathcal{F}^0_{xy}$ if and only if $u_x \geq G_{xy}(v_y)$ which is equivalent to $v_y \geq G^{-1}_{xy}(u_x)$.

- By condition (DF) and (NC), we get that if (π, u, v) is the solution to an Equilibrium transportation problem

$$u(x) = \max_{y \in \mathcal{Y}} G_{xy}(v(y)) \text{ and } v(y) = \max_{x \in \mathcal{X}} G^{-1}_{xy}(u(x))$$

- OT is a special case (Φ-conjugacy, see Villani's book):

$$u(x) = \max_{y \in \mathcal{Y}} \{\Phi_{xy} - v(y)\} \text{ and } v(y) = \max_{x \in \mathcal{X}} \{\Phi_{xy} - u(x)\}$$

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- Assuming everything is smooth, and letting f_P and f_Q be the densities of P and Q we have under some conditions that the equilibrium transportation plan is given by $y = T(x)$, where mass balance yields

$$\left|\det DT(x)\right| = \frac{f_P(x)}{f_Q(T(x))}$$

and optimality yieds

$$\partial_x G^{-1}_{xT(x)}(u(x)) + \partial_u G^{-1}_{xT(x)}(u(x)) \nabla u(x) = 0$$

which thus inverts into

$$T(x) = e(x, u(x), \nabla u(x)).$$

- Trudinger (2014) studies Monge-Ampere equations of the form

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- However, in contrast, the ET problem is symmetric in the role x and y play. Is there a way of restoring symmetry between x and y?

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- [GKW] introduce distance function: for $(u, v) \in \mathbb{R}^2$, let

$$\Psi_{xy}(u, v) = \min\left\{t \in \mathbb{R} : (u - t, v - t) \in \mathcal{F}_{xy}\right\}$$

which is the distance along the diagonal between (u, v) and the frontier of $\mathcal{F}_{xy}$, with a minus sign if (u, v) is in the set.

- Economic intepretation: what is the quantity of utility that we can give or remove to x and y *in the same amount* such that they reach the efficient frontier?
- This object has nice properties:
  - $\Psi_{xy}(u, v) \leq 0$ iff $(u, v) \in \mathcal{F}_{xy}$
  - $\Psi_{xy}(u, v) < 0$ iff $(u, v) \in \mathcal{F}_{xy}^0$
  - $\Psi_{xy}(u + t, v + t) = \Psi_{xy}(u, v) + t$
- Note that in the case of OT,

$$\Psi_{xy}(u, v) = \frac{u + v - (\alpha_{xy} + \gamma_{xy})}{2}.$$

- More generally, the following operations on distance functions correspond to geometric operations on feasible sets:
  - $\max\left\{\Psi^1, \Psi^2\right\}$: intersection
  - $\min\left\{\Psi^1, \Psi^2\right\}$: union
  - $\Psi\left(u-\alpha, v-\gamma\right)$: translation
  - $T\Psi\left(u/T, v/T\right)$: homothety
  - $\lambda\Psi^1 + \left(1-\lambda\right)\Psi^2$: intepolation

- (π, u, v) is an equilibrium outcome when

$$\left\{\begin{array}{l}(PF) : \pi \in \mathcal{M}\left(P, Q\right) \\ (DF) : \Psi_{xy}\left(u\left(x\right), v\left(y\right)\right) \geq 0 \\ (NC) : \left(x, y\right) \in Supp\left(\pi\right) \Longrightarrow \Psi_{xy}\left(u\left(x\right), v\left(y\right)\right) = 0.\end{array}\right.$$

- In the rest of the talk, I will argue that these objects can be useful in the study of the Equilibrium Transport problem.

- It is convenient to parameterize the frontier of $\mathcal{F}_{xy}$ by a scalar parameter. To do this, consider a reparameterization by a 45° rotation: if u_x and v_y are such that $\Psi_{xy}(u_x, v_y) = 0$, let $w_{xy} = u_x - v_y$.
- More generally, the set of equations

$$\begin{cases} \Psi_{xy}(u_x, v_y) = 0 \\ u_x - v_y = w_{xy} \end{cases}$$

defines $u_x = \mathcal{U}_{xy}(w_{xy})$ and $v_y = \mathcal{V}_{xy}(w_{xy})$.

- One has a very simple expression of $\mathcal{U}_{xy}$ and $\mathcal{V}_{xy}$ from Ψ:

$$\mathcal{U}_{xy}(w_{xy}) = -\Psi_{xy}(0, -w_{xy})$$
$$\mathcal{V}_{xy}(w_{xy}) = -\Psi_{xy}(w_{xy}, 0)$$

and

$$G_{xy}(v) = \mathcal{U}_{xy} \circ \mathcal{V}_{xy}^{-1}(v) \text{ and } G_{xy}^{-1}(u) = \mathcal{V}_{xy} \circ \mathcal{U}_{xy}^{-1}(u).$$

- Look at the individual rationality conditions:

$$\begin{cases} u_x \geq \mathcal{U}_{xy}\left(v_y\right) \\ \pi_{y|x} > 0 \Longrightarrow u_x \geq \mathcal{U}_{xy}\left(v_y\right) \end{cases}$$

where $\pi_{y|x} = \pi_{xy} / p_x$ is the conditional distribution of Y given X under π.

- A regularized version of these is provided by the Gibbs distribution

$$\pi_{y|x} = \frac{\exp \mathcal{U}_{xy}\left(w_{xy}\right) / T}{\exp u_x / T}$$

where the max has been replaced by the smooth-max. Hence, letting $a_x = u_x - T \ln p_x$, we get

$$T \ln \pi_{xy} + a_x = \mathcal{U}_{xy}\left(w_{xy}\right).$$

- Similarly, individual rationality on the side of firms relaxes into

$$\pi_{x|y} = \frac{\exp \mathcal{V}_{xy}\left(w_{xy}\right) / T}{\exp v_y / T}$$

and thus, after letting $b_y = v_y - T \ln q_y$,

$$T \ln \pi_{xy} + b_y = \mathcal{V}_{xy}\left(w_{xy}\right).$$

- We get
$$\begin{cases} T\ln\pi_{xy}+a_x=\mathcal{U}_{xy}\left(w_{xy}\right)\\ T\ln\pi_{xy}+b_y=\mathcal{V}_{xy}\left(w_{xy}\right)\end{cases}$$
- Thus, applying Ψ term by term yields
$$\Psi_{xy}\left(T\ln\pi_{xy}+a_x,T\ln\pi_{xy}+b_y\right)=0$$
hence
$$\pi_{xy}=\exp\left(-\Psi_{xy}\left(a_x,b_y\right)/T\right)$$
where a_x and b_y solve the system
$$\begin{cases}\sum_y\exp\left(-\Psi_{xy}\left(a_x,b_y\right)/T\right)=p_x\\ \sum_x\exp\left(-\Psi_{xy}\left(a_x,b_y\right)/T\right)=q_y\end{cases}.$$
- By substraction, we have
$$a_x-b_y=w_{xy}.$$

- **Theorem**: when $T \to 0$, u and v tend to the solution of the ET problem.
- **Open question**: when there are multiple solutions at $T = 0$, which solution is selected?
- In the case of OT, it is the maximal entrop solution. Does this result still hold here?

- In OT, a lot of attention recently on the Monge-Kantorovich problem with entropic regularization

$$\max_{\pi \in \mathcal{M}(P,Q)} \int \Phi\left(x, y\right) d\pi\left(x, y\right) - 2T \int \log \pi\left(x, y\right) d\pi\left(x, y\right)$$

which is useful for computation (via Iterated Bregman projections) and econometrics (link w the logit model).

- In the OT case, this boils down to

$$\pi_{xy} = \exp\left(\frac{\Phi_{xy} - a_x - b_y}{2T}\right)$$

where a and b are adjusted to meet the constraint $\pi \in \mathcal{M}\left(P, Q\right)$.

- In the ET case, no optimization formulation... but works just as well. Recall

$$\pi_{xy} = \exp\left(-\Psi_{xy}\left(a_x, b_y\right) / T\right).$$

- Take temperature parameter $T > 0$ and look for π under the form

$$\pi_{xy} = \exp\left(-\frac{\Psi_{xy}\left(a_x, b_y\right)}{T}\right)$$

Note that when $T \to 0$, the limit of $\Psi_{xy}\left(a_x, b_y\right)$ is nonnegative, and the limit of $\pi_{xy}\Psi_{xy}\left(a_x, b_y\right)$ is zero.

- If $\pi_{xy} = \exp\left(-\Psi_{xy}\left(a_x, b_y\right) / T\right)$, condition $\pi \in \mathcal{M}\left(P, Q\right)$ boils down to set of nonlinear equations in $\left(u, v\right)$

$$\left\{\begin{array}{l} \sum_{y\in\mathcal{Y}} \exp\left(-\frac{\Psi_{xy}\left(a_x, b_y\right)}{T}\right) = p_x \\ \sum_{x\in\mathcal{X}} \exp\left(-\frac{\Psi_{xy}\left(a_x, b_y\right)}{T}\right) = q_y \end{array}\right.$$

which we call the *nonlinear Bernstein-Schrödinger* equation.

- In the optimal transportation case, this becomes the classical B-S equation

$$\left\{\begin{array}{l} \sum_{y\in\mathcal{Y}} \exp\left(\frac{\Phi_{xy}-a_x-b_y}{2T}\right) = p_x \\ \sum_{x\in\mathcal{X}} \exp\left(\frac{\Phi_{xy}-a_x-b_y}{2T}\right) = q_y \end{array}\right.$$

- Note that $F_x : a_x \to \sum_{y \in \mathcal{Y}} \exp\left(-\frac{\Psi_{xy}(a_x, b_y)}{T}\right)$ is a decreasing and continuous function. Mild conditions on Ψ therefore ensure the existence of a_x so that $F_x(a_x) = p_x$.
- Our algorithm [GKW] is thus a nonlinear Jacobi algorithm:
  - Make an initial guess of b_y^0
  - Determine the a_x^{k+1} to fit the p_x margins, based on the b_y^k
  - Update the b_y^{k+1} to fit the q_y margins, based on the a_x^{k+1}.
  - Repeat until b^{k+1} is close enough to b^k.
- One can proof that b_y^k decrease to fixed point. Convergence is very fast in practice.

- The distance function formalism allows to make a connection with Gale-Shapley's theory of stable marriages. Introduce

$$\Psi_{xy}\left(u,v\right)=\max\left(u-\alpha_{xy},v-\gamma_{xy}\right).$$

- This interprets as: no matter what happens, x can get at most α_{xy} and y can get at most γ_{xy}.
- Assume $n_x=1$ and $m_y=1$ for all x and y. Then GH show that for any stable matching π in the Gale-Shapley sense, there exist u and v such that (π,u,v) is an equilibrium transport; and conversely, if (π,u,v) is an equilibrium transport, then π is stable in the Gale and Shapley sense.

- Back to the tax example [DGJK], assume that

$$N(w) = \min_{k=1...K} \left\{ n^k + \left(1-\theta^k\right)\left(w - w^k\right) \right\}$$

where $\theta^1 = 0 < \theta^2 < ... < \theta^K$, and $n^0 = 0$, $n^{k+1} = n^k + \left(1-\theta^k\right)\left(w^{k+1} - w^k\right)$.

- Recall $\mathcal{F}_{xy} = \left\{(u,v) : u - \alpha_{xy} \leq N(\gamma_{xy} - v)\right\}$. Because the tax schedule is progressive (N concave), $\mathcal{F}_{xy}$ can be written as the intersection of the $\mathcal{F}^k_{xy}$, where

$$\mathcal{F}^k_{xy} = \left\{(u,v) : u - \alpha_{xy} - n^k \leq \left(1-\theta^k\right)\left(\gamma_{xy} - v - w^k\right)\right\}.$$

- Note that Ψ associated with the intersection of Ψ^k is equal to the $\max_k \Psi^k$. One has

$$\Psi^k(u,v) = \frac{u - \alpha_{xy} - n^k - \left(1-\theta^k\right)\left(\gamma_{xy} - v - w^k\right)}{2-\theta^k},$$

and as a result

$$\Psi(u,v) = \max_{k=1,...,K} \left\{ \frac{u - \alpha_{xy} - n^k - \left(1-\theta^k\right)\left(\gamma_{xy} - v - w^k\right)}{2-\theta^k} \right\}.$$