Two lectures on matching

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Two invited lectures given at the University of Toronto
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Outline

Lecture 1: Matching models with transferable utility:

Matching models as an optimization problem / regularized optimal transport / generalized linear models

See code here: https://www.math-econ-code.org/mec-optim

Lecture 2: Matching models with imperfectly transferable / nontransferable utility:

Matching models as an equilibrium problem with substiutes

See code here: https://www.math-econ-code.org/mec-equil

These lectures are based on:

[GS] Galichon and Salanie (2022). Cupids invisible hands: Social Surplus and Identification in Matching Models. *Review of Economic Studies.*

[GKW] Galichon, Kominers and Weber (2019). Costly Concessions: An Empirical Framework for Matching with Imperfectly Transferable Utility (2019). *Journal of Political Economy.*

[MEC-OPTIM] Galichon (2022). 'math+econ+code' masterclass on optimal transport and economic applications. https://www.math-econ-code.org/mec-optim

[MEC-EQUIL] Galichon (2022). 'math+econ+code' masterclass on equilibrium transport and matching models in economics. https://www.math-econ-code.org/mec-equil

1 Lecture 1

The references for this lecture are:

[GS] Galichon and Salanie (2022). Cupids invisible hands: Social Surplus and Identification in Matching Models. *Review of Economic Studies.*

[MEC-OPTIM] Galichon (2022). 'math+econ+code' masterclass on optimal transport and economic applications. https://www.math-econ-code.org/mec-optim

[B] Becker (1973). 'A Theory of Marriage: Part 1.' Journal of Political Economy.

[CS] Choo and Siow (2006). 'Who Marries Whom and Why'. Journal of Political Economy.

1.1 Becker model

Assume that there are n_x men of type $x \in X$

Assume that there are m_y women of type $y \in Y$

If x and y match, then this brings

$\alpha_{xy} + w_{xy}$ utils to x, and

$\gamma_{xy} - w_{xy}$ utils to y.

Whatever w, the joint utility that x and y get together is $\alpha_{xy} + \gamma_{xy} =: \Phi_{xy}$ joint surplus of xy.

n, m, α, γ are exogeneous, while w is endogenous.

If x remains single, then get 0

If y remains single, then get 0

Man x's problem is

$u_x = \max_y \{\alpha_{xy} + w_{xy}, 0\}$

this induces a demand for marriage from men:

let $\mu^M_{xy}(w)$ be the number of unions of type xy that are induced by men's problems.

Woman y's problem is

$v_y = \max_x \left\{\gamma_{xy} - w_{xy}, 0\right\}$

this induces a demand for marriage from women:

let $\mu^W_{xy}(w)$ be the number of unions of type xy that are induced by women's problems.

The equilibrium transfer is determined by

$$\begin{aligned}
\mu^W_{xy}(w) &= \mu^M_{xy}(w) \\
\sum_y \mu^M_{xy}(w) + \mu^M_{x0}(w) &= n_x \\
\sum_x \mu^W_{xy}(w) + \mu^W_{0y}(w) &= m_y
\end{aligned}$$

1.2 Add logit random utility: Choo and Siow model

Consider an individual man i of type x_i and one individual woman j of type y_j.

Assume that i's problem is

$\max_j \left\{\alpha_{x_i y_j} + \varepsilon_{i y_j} + w_{ij}, \varepsilon_{i0}\right\}$

Woman j's problem is

$\max_i \left\{\gamma_{x_i y_j} + \eta_{x_i j} - w_{ij}, \eta_{0j}\right\}$

Assume that the random vector $(\varepsilon_{iy})_y$ and $\left(\eta_{xj}\right)_x$ are drawn from Gumbel distribution.

Theorem. At equilibrium, $w_{ij} = w_{x_i y_j}$. Thus we can restate our choice problems as:

$\max_y \left\{ \alpha_{x_i y} + w_{x_i y} + \varepsilon_{iy}, \varepsilon_{i0} \right\}$
$\max_x \left\{ \gamma_{xy} - w_{xy} + \eta_{xj}, \eta_{0j} \right\}$

The equilibrium is determined by supply/balance equilibrium

$$\begin{aligned} \mu^W_{xy}(w) &= \mu^M_{xy}(w) \\ \sum_y \mu^M_{xy}(w) + \mu^M_{x0}(w) &= n_x \\ \sum_x \mu^W_{xy}(w) + \mu^W_{0y}(w) &= m_y \end{aligned}$$

In a logit model, the probability that x chooses y is

$$\begin{aligned} \frac{\mu^M_{xy}}{n_x} &= \frac{\exp(\alpha_{xy} + w_{xy})}{1 + \sum_y \exp(\alpha_{xy} + w_{xy})} \\ \frac{\mu^M_{x0}}{n_x} &= \frac{1}{1 + \sum_y \exp(\alpha_{xy} + w_{xy})} \end{aligned}$$

Similarly

$$\begin{aligned} \frac{\mu^W_{xy}}{m_y} &= \frac{\exp\left(\gamma_{xy} - w_{xy}\right)}{1 + \sum_x \exp\left(\gamma_{xy} - w_{xy}\right)} \\ \frac{\mu^W_{x0}}{m_y} &= \frac{1}{1 + \sum_x \exp\left(\gamma_{xy} - w_{xy}\right)} \end{aligned}$$

We have

$$\begin{aligned} \mu^M_{xy} &= \mu^M_{x0} \exp(\alpha_{xy} + w_{xy}) \\ \mu^W_{xy} &= \mu^W_{0y} \exp\left(\gamma_{xy} - w_{xy}\right) \end{aligned}$$

Equilibrium condition:

$$\mu_{xy} = \mu^M_{x0} \exp(\alpha_{xy} + w_{xy}) = \mu^W_{0y} \exp\left(\gamma_{xy} - w_{xy}\right)$$

multiplying term by term, we get Choo and Siow's formula

$$\mu^2_{xy} = \mu_{x0}\mu_{0y} \exp\left(\alpha_{xy} + \gamma_{xy}\right),$$

that is

$$\mu_{xy} = \sqrt{\mu_{x0}\mu_{0y}} \exp\left(\frac{\Phi_{xy}}{2}\right),$$

which we solve using the other 2 sytems

$$\begin{aligned} \sum_y \sqrt{\mu_{x0}\mu_{0y}} \exp\left(\frac{\Phi_{xy}}{2}\right) + \mu_{x0} &= n_x \\ \sum_x \sqrt{\mu_{x0}\mu_{0y}} \exp\left(\frac{\Phi_{xy}}{2}\right) + \mu_{0y} &= m_y. \end{aligned}$$

1.3 Reformulation as an optimization problem

Back to the Becker model. We know from Becker, Shapley-Shubik in the 1970 that μ in the Becker model solves

$$\begin{aligned}
\max_{\mu \geq 0} \quad & \sum_{xy} \mu_{xy} \Phi_{xy} \\
s.t. \quad & \sum_{y} \mu_{xy} + \mu_{x0} = n_x \ \ [u_x] \\
& \sum_{x} \mu_{xy} + \mu_{0y} = m_y \ \ [v_y]
\end{aligned}$$

which is a linear programming problem, whose dual is

$$\begin{aligned}
\min_{u \geq 0, v \geq 0} \quad & \sum_{x} u_x n_x + \sum_{y} v_y m_y \\
s.t. \quad & u_x + v_y \geq \Phi_{xy}
\end{aligned}$$

Let's write this down at the individual level, ie the level of i's and j's. We have Becker model solves

$$\begin{aligned}
\max_{\mu_{ij} \geq 0} \quad & \sum_{ij} \mu_{ij} \left(\Phi_{x_i y_j} + \varepsilon_{i y_j} + \eta_{x_i j} \right) + \sum_{i} \mu_{i0} \varepsilon_{i0} + \sum_{j} \mu_{0j} \eta_{0j} \\
s.t. \quad & \sum_{j} \mu_{ij} + \mu_{i0} = 1 \ \ [u_i] \\
& \sum_{i} \mu_{ij} + \mu_{0j} = 1 \ \ [v_y]
\end{aligned}$$

which is a linear programming problem, whose dual is

$$\begin{aligned}
\min_{u_i, v_j} \quad & \sum_{i} u_i + \sum_{j} v_j \\
s.t. \quad & u_i + v_j \geq \Phi_{x_i y_j} + \varepsilon_{i y_j} + \eta_{x_i j} \\
& u_i \geq \varepsilon_{i0}, \\
& v_j \geq \eta_{0j}
\end{aligned}$$

Rewrite this as

$$\begin{aligned}
\min_{u_i, v_j} \quad & \sum_{i} u_i + \sum_{j} v_j \\
s.t. \quad & u_i - \varepsilon_{i y_j} + v_j - \eta_{x_i j} \geq \Phi_{x_i y_j} \\
& u_i \geq \varepsilon_{i0}, \\
& v_j \geq \eta_{0j}
\end{aligned}$$

Rewrite the constraint

$$u_i - \varepsilon_{i y_j} + v_j - \eta_{x_i j} \geq \Phi_{x_i y_j}$$

as

$$\min_{i:x_i=x}\{u_i-\varepsilon_{iy}\}+\min_{j:y_j=y}\left\{v_j-\eta_{xj}\right\}\geq\Phi_{xy}$$

Introduce

$$\begin{aligned}U_{xy} &= \min_{i:x_i=x}\{u_i-\varepsilon_{iy}\}\\ V_{xy} &= \min_{j:y_j=y}\left\{v_j-\eta_{xj}\right\}\end{aligned}$$

and we have $U_{xy}\leq u_i-\varepsilon_{iy}$, thus $u_i\geq U_{xy}+\varepsilon_{iy}$, thus

$$u_i\geq\max_y\{U_{xy}+\varepsilon_{iy},\varepsilon_{i0}\}$$

I can rewrite my optimization problem as

$$\begin{aligned}\min_{u_i,v_j,U_{xy},V_{xy}} \quad & \sum_i u_i+\sum_j v_j\\ s.t. \quad & U_{xy}+V_{xy}\geq\Phi_{xy}\\ & u_i\geq\max_y\{U_{xy}+\varepsilon_{iy},\varepsilon_{i0}\}\\ & v_j\geq\max_x\left\{V_{xy}+\eta_{xj},\eta_{0j}\right\}\end{aligned}$$

which is simply

$$\begin{aligned}\min_{U_{xy},V_{xy}} \quad & \sum_i\max_y\{U_{xy}+\varepsilon_{iy},\varepsilon_{i0}\}+\sum_j\max_x\left\{V_{xy}+\eta_{xj},\eta_{0j}\right\}\\ s.t. \quad & U_{xy}+V_{xy}\geq\Phi_{xy}\end{aligned}$$

Rewrite this as

$$\begin{aligned}\min_{U_{xy},V_{xy}} \quad & \sum_x n_x\sum_{i:x_i=x}\frac{1}{n_x}\max_y\{U_{xy}+\varepsilon_{iy},\varepsilon_{i0}\}+\sum_y m_y\sum_{j:y_j=y}\frac{1}{m_y}\max_x\left\{V_{xy}+\eta_{xj},\eta_{0j}\right\}\\ s.t. \quad & U_{xy}+V_{xy}\geq\Phi_{xy}\end{aligned}$$

And we have in the large market limit

$$\begin{aligned}\frac{1}{n_x}\sum_{i:x_i=x}\max_y\{U_{xy}+\varepsilon_{iy},\varepsilon_{i0}\} &\to E\left[\max_y\{U_{xy}+\varepsilon_y,\varepsilon_0\}\right]=G_x(U)\\ \frac{1}{m_y}\sum_{j:y_j=y}\max_x\left\{V_{xy}+\eta_{xj},\eta_{0j}\right\} &\to E\left[\max_x\{V_{xy}+\eta_x,\eta_0\}\right]=G_y(V)\end{aligned}$$

Now assume that ε and η are both drawn from a Gumbel distribution. Then we have an explicit form for G_x and G_y, which is

$$\begin{aligned}G_x(U) &= \log\left(1+\sum_y e^{U_{xy}}\right)\\ G_y(V) &= \log\left(1+\sum_x e^{V_{xy}}\right)\end{aligned}$$

Rewrite the model as

$$
\begin{array}{ll}
\min_{U_{xy},V_{xy}} & \sum_x n_x G_x(U) + \sum_y m_y G_y(V) \\
s.t. & U_{xy} + V_{xy} = \Phi_{xy}
\end{array}
$$

hence

$$
\min_{(U_{xy})} \sum_x n_x G_x(U) + \sum_y m_y G_y(\Phi - U)
$$

To compute μ, we need to compute the primal problem associated with this one.

$$
\min_{U_{xy},V_{xy}} \max_{\mu \geq 0} \sum_x n_x G_x(U) + \sum_y m_y G_y(V) + \sum_{xy} \mu_{xy} (\Phi_{xy} - U_{xy} - V_{xy})
$$

therefore

$$
\max_{\mu \geq 0} \sum_{xy} \mu_{xy} \Phi_{xy} + \min_{U_{xy},V_{xy}} \sum_x n_x G_x(U) + \sum_y m_y G_y(V) - \sum_{xy} \mu_{xy} (U_{xy} + V_{xy})
$$

that is

$$
\max_{\mu \geq 0} \sum_{xy} \mu_{xy} \Phi_{xy} - \max_{U_{xy},V_{xy}} \left\{ \sum_{xy} \mu_{xy} (U_{xy} + V_{xy}) - \sum_x n_x G_x(U) - \sum_y m_y G_y(V) \right\}
$$

that is

$$
\max_{\mu \geq 0} \left\{ \begin{array}{c}
\sum_{xy} \mu_{xy} \Phi_{xy} \\
-\sum_x n_x \max_{U_{xy}} \left\{ \sum_y \frac{\mu_{xy}}{n_x} U_{xy} - G_x(U) \right\} \\
-\sum_y m_y \max_{V_{xy}} \left\{ \sum_x \frac{\mu_{xy}}{m_y} V_{xy} - G_y(V) \right\}
\end{array} \right\}
$$

We recognize

$$
\max_{(U_{xy})} \left\{ \sum_y \left(\frac{\mu_{xy}}{n_x} \right) U_{xy} - G_x(U) \right\}
$$

as the convex conjugate (or Legendre-Fenchel transform) of G_x evaluated at $\left(\frac{\mu_{x.}}{n_x}\right)$. THis is denoted

$$
G_x^* \left(\frac{\mu_{x.}}{n_x} \right)
$$

We have

$$
\max_{\mu \geq 0} \left\{ \sum_{xy} \mu_{xy} \Phi_{xy} - \sum_x n_x G_x^* \left(\frac{\mu_{x.}}{n_x} \right) - \sum_y m_y G_y^* \left(\frac{\mu_{y.}}{m_y} \right) . \right\}
$$

which is in the Logit model

$$
\begin{aligned}
\max_{\mu \geq 0} \quad & \left\{ \sum_{xy} \mu_{xy} \Phi_{xy} - 2 \sum_{xy} \mu_{xy} \log \frac{\mu_{xy}}{\sqrt{n_x m_y}} - \sum_x \mu_{x0} \log \frac{\mu_{x0}}{n_x} - \sum_y \mu_{0y} \log \frac{\mu_{0y}}{m_y} \right\} \\
s.t. \quad & \sum_y \mu_{xy} + \mu_{x0} = n_x \\
& \sum_x \mu_{xy} + \mu_{0y} = m_y
\end{aligned}
$$

Proposition. The value of the primal problem in the logit model can be expressed as

$$
\min_{u_x, v_y} \sum_x u_x n_x + \sum_y v_y m_y + 2 \sum_{xy} \exp\left(\frac{\Phi_{xy} - u_x - v_y}{2} \right) + \sum_x \exp(-u_x) + \sum \exp(-v_y)
$$

Proof. Start from the dual expression

$$
\begin{aligned}
\min_{U_{xy}, u_x, v_y} \quad & \sum_x n_x u_x + \sum_y m_y v_y \\
s.t. \quad & u_x \geq \log \left(1 + \sum_y \exp U_{xy} \right) \\
& v_y \geq \log \left(1 + \sum_x \exp (\Phi_{xy} - U_{xy}) \right)
\end{aligned}
$$

which we rewrite as

$$
\begin{aligned}
\min_{U_{xy}, u_x, v_y} \quad & \sum_x n_x u_x + \sum_y m_y v_y \\
s.t. \quad & 1 \geq e^{-u_x} + \sum_y \exp (U_{xy} - u_x) \\
& 1 \geq e^{-v_y} + \sum_x \exp (\Phi_{xy} - U_{xy} - v_y)
\end{aligned}
$$

and introduce Lagrange multipliers N_x and M_y; we get

$$
\min_{U_{xy}, u_x, v_y} \max_{N_x, M_y} \left\{ \begin{array}{c} \sum_x n_x u_x + \sum_y m_y v_y \\ + \sum_x N_x \left(e^{-u_x} + \sum_y \exp (U_{xy} - u_x) - 1 \right) \\ + \sum_y M_y \left(e^{-v_y} + \sum_x \exp (\Phi_{xy} - U_{xy} - v_y) - 1 \right) \end{array} \right\}
$$

Foc in N and M yield

$$
\begin{aligned}
e^{-u_x} + \sum_y \exp (U_{xy} - u_x) &= 1 \\
e^{-v_y} + \sum_x \exp (\Phi_{xy} - U_{xy} - v_y) &= 1
\end{aligned}
$$

and FOC in u_x yields

$$\begin{aligned}
N_x e^{-u_x} + \sum_y N_x \exp\left(U_{xy} - u_x\right) &= n_x \\
M_y e^{-v_y} + \sum_x M_y \exp\left(\Phi_{xy} - U_{xy} - v_y\right) &= m_y
\end{aligned}$$

thus $N_x = n_x$ and $M_y = m_y$. Foc in U_{xy} yields

$$n_x \exp\left(U_{xy} - u_x\right) = m_y \exp\left(\Phi_{xy} - U_{xy} - v_y\right)$$

thus $n_x \exp\left(U_{xy} - u_x\right) + m_y \exp\left(\Phi_{xy} - U_{xy} - v_y\right) = \sqrt{n_x m_y} \exp\left(\frac{\Phi_{xy} - u_x - v_y}{2}\right)$, thus the problem becomes

$$\min_{u_x, v_y} \left\{ \begin{array}{c} \sum_x n_x u_x + \sum_y m_y v_y \\ + \sum_x n_x e^{-u_x} + \sum_y m_y e^{-v_y} \\ +2 \sum_{xy} \sqrt{n_x m_y} \exp\left(\frac{\Phi_{xy} - u_x - v_y}{2}\right) \\ - \sum_x n_x - \sum_y m_y \end{array} \right\}$$

QED.

Let's write down first order conditions

$$\begin{aligned}
n_x &= \sum_y \exp\left(\frac{\Phi_{xy} - u_x - v_y}{2}\right) + \exp\left(-u_x\right) \\
m_y &= \sum_x \exp\left(\frac{\Phi_{xy} - u_x - v_y}{2}\right) + \exp\left(-v_y\right)
\end{aligned}$$

this is exactly Choo-Siow's equations with $\mu_{x0} = \exp\left(-u_x\right)$, $\mu_{0y} = \exp\left(-v_y\right)$ and

$$\mu_{xy} = \exp\left(\frac{\Phi_{xy} - u_x - v_y}{2}\right) = \sqrt{\mu_{x0}\mu_{0y}} \exp\left(\frac{\Phi_{xy}}{2}\right)$$

Which yields Choo-Siow's identification formula

$$\Phi_{xy} = \ln \frac{\mu_{xy}^2}{\mu_{x0}\mu_{0y}}.$$

Formulation as a generalized linear model / Poisson regression.

Generalized linear model:

Dependent variable $E\left[\mu_a | z_a\right] = \exp\left(z_a^\top \theta\right)$ where θ is a parameter to be estimated.

$E\left[\mu_a | z_a\right] = g^{-1}\left(z_a^\top \theta\right)$ where g is the link function (here, $g = \log$, log link function).

Estimate θ using

$$\max_{(\theta^k)} \sum_a \mu_a \left(z_a^\top \theta\right) - \sum_a \exp\left(z_a^\top \theta\right)$$

First order conditions yield

$$\sum_a \mu_a z_a^k = \sum_a \exp\left(z_a^\top \theta\right) z_a^k$$

in other words,

$$\sum_a \left(\mu_a - \exp\left(z_a^\top \theta\right)\right) z_a^k = 0$$

the K predicted moments match with the K observed moments.

Take $a = xy$ and $x0$ and $0y$'s.
$z_a = (\Phi_a, -1_{x\in a}, -1_{y\in a})$ and $\theta = (1, u_x, v_y)$
so that $z_a^\top \theta = \Phi_{xy} - u_x - v_y$.
We have

$$\max_{(\theta^k)} \sum_a \mu_a \left(z_a^\top \theta\right) - \sum_a \exp\left(z_a^\top \theta\right)$$

which rewrites as

$$\max_{u_x, v_y} \sum_{xy} \hat{\mu}_{xy} \left(\Phi_{xy} - u_x - v_y\right) - \sum_a \exp\left(\Phi_{xy} - u_x - v_y\right)$$

or in other words

$$\min_{u_x, v_y} \sum_{xy} \hat{\mu}_{xy} \left(u_x + v_y - \Phi_{xy}\right) + \sum_a \exp\left(\Phi_{xy} - u_x - v_y\right)$$

1.4 Estimation

Take a linear parameterization of Φ, that is

$$\Phi_{xy}^\beta = \sum_k \beta_k \phi_{xy}^k$$

$z_a = \left(\phi_a^k, -1_{x\in a}, -1_{y\in a}\right)$ and $\theta = \left(\beta^k, u_x, v_y\right)$
so that $z_a^\top \theta = \Phi_{xy}^\beta - u_x - v_y$.
We have

$$\max_{(\theta^k)} \sum_a \mu_a \left(z_a^\top \theta\right) - \sum_a \exp\left(z_a^\top \theta\right)$$

which rewrites as

$$\max_{u_x, v_y, \beta^k} \sum_{xy} \hat{\mu}_{xy} \left(\Phi_{xy}^\beta - u_x - v_y\right) - \sum_{xy} \exp\left(\Phi_{xy}^\beta - u_x - v_y\right)$$

or in other words

$$\min_{u_x,v_y,\beta^k} \sum_{xy} \hat{\mu}_{xy} \left(u_x + v_y - \Phi^{\beta}_{xy}\right) + \sum_{xy} \exp\left(\Phi^{\beta}_{xy} - u_x - v_y\right)$$

FOC with respect to u_x and v_y

$$\begin{aligned}
\sum_y \hat{\mu}_{xy} &= \sum_y \exp\left(\Phi^{\beta}_{xy} - u_x - v_y\right) \\
\sum_x \hat{\mu}_{xy} &= \sum_x \exp\left(\Phi^{\beta}_{xy} - u_x - v_y\right)
\end{aligned}$$

FOC with respect to β^k

$$\sum_{xy} \hat{\mu}_{xy} \phi^k_{xy} = \sum_{xy} \exp\left(\Phi^{\beta}_{xy} - u_x - v_y\right) \phi^k_{xy}.$$

2 Lecture 2: matching with imperfectly transferable utility

The reference for this lecture is:

[GKW] Galichon, Kominers and Weber (2019). Costly Concessions: An Empirical Framework for Matching with Imperfectly Transferable Utility (2019). *Journal of Political Economy.*

[MEC-EQUIL] Galichon (2022). 'math+econ+code' masterclass on equilibrium transport and matching models in economics. https://www.math-econ-code.org/mec-equil

[BCW] Browning,Chiappori, Weiss (2014). *Economics of the Family*. Cambridge.

2.1 More general transfers

Consider a marriage model with marital valuation and private consumption.

If x and y match, then x gets

$$u_x = \tilde{\alpha}_{xy} + \tau \log c_x$$

and y gets

$$v_y = \tilde{\gamma}_{xy} + \tau \log c_y$$

where c_x and c_y are the private consumptions of x and y respectively. We assume that c_x and c_y are endogenously allocated from the joint income $I_x + I_y$.

Let us see the bargaining set of x and y. We have

$$I_x + I_y = c_x + c_y = \exp\left(\frac{u_x - \tilde{\alpha}_{xy}}{\tau}\right) + \exp\left(\frac{v_y - \tilde{\gamma}_{xy}}{\tau}\right)$$

thus

$$2 = \exp\left(\frac{u_x - \tilde{\alpha}_{xy} - \tau\log\left(I_x + I_y\right)}{\tau}\right) + \exp\left(\frac{v_y - \tilde{\gamma}_{xy} - \tau\log\left(I_x + I_y\right)}{\tau}\right)$$

and we set $\alpha_{xy} = \tilde{\alpha}_{xy} + \tau\log\left(I_x + I_y\right)$ and $\gamma_{xy} = \tilde{\gamma}_{xy} + \tau\log\left(I_x + I_y\right)$, and we get that the bargaining set is the set of (u, v) such that

$$2 = \exp\left(\frac{u_x - \alpha_{xy}}{\tau}\right) + \exp\left(\frac{v_y - \gamma_{xy}}{\tau}\right).$$

More generally, assume that if x and y match, they can split (nonrandom part of) utility into U_{xy} that goes to the man and V_{xy} that goes to the woman such that

$$(U_{xy}, V_{xy}) \in F_{xy}$$

where F_{xy} is the feasible set.

In the previous example,

$$F_{xy} = \left\{(U_{xy}, V_{xy}) : \exp\left(\frac{U_{xy} - \alpha_{xy}}{\tau}\right) + \exp\left(\frac{V_{xy} - \gamma_{xy}}{\tau}\right) \leq 2\right\}$$

but more generally, little needs to be assumed on F_{xy} besides free disposal.

Assume that on top of the nonrandom part of the utility, agents enjoy a random utility shock so that i and j of respective types x and y get

$$U_{xy} + \varepsilon_{iy}$$
$$V_{xy} + \eta_{xj}$$

where ε_{iy} and η_{xj} are iid Gumbel (logit setting). Take $U_{x0} = 0$ and $V_{0y} = 0$.

This is a generalization of Choo and Siow: in Choo and Siow, we have

$$F_{xy} = \left\{(U_{xy}, V_{xy}) : U_{xy} + V_{xy} \leq \Phi_{xy}\right\}.$$

We solve the equilibrium problem

$$\begin{aligned} \frac{\mu_{xy}}{n_x} &= \frac{\exp\left(U_{xy}\right)}{..} \\ \frac{\mu_{x0}}{n_x} &= \frac{1}{...} \end{aligned}$$

therefore I get

$$\frac{\mu_{xy}}{\mu_{x0}} = \exp U_{xy}$$

and similarly on the other side,

$$\frac{\mu_{xy}}{m_y} = \exp V_{xy}$$

We need to express the fact that $(U_{xy}, V_{xy}) \in F_{xy}$.

2.2 Distance function

Introduce a tool to describe F_{xy}: the distance-to-frontier function.

$$D_{xy}(u,v) = \min\{t : (u-t, v-t) \in F_{xy}\}$$

which is the amount of utility that I need to substract to u, v to make it feasible.

We have that

$$D(u+a, v+a) = D(u,v) + a.$$

We can show easily that

$$D_{xy}(u,v) \leq 0$$

iff $(u,v) \in F_{xy}$, and $D_{xy}(u,v) = 0$ if and only if (u,v) is on the frontier of F_{xy}.

In Choo and Siow:

$$D_{xy}(u,v) = \frac{u+v-\Phi_{xy}}{2}$$

In the exponentially transferable utility model

$$D_{xy}(u,v) = \log\left(\frac{\exp\left(\frac{U_{xy}-\alpha_{xy}}{\tau}\right) + \exp\left(\frac{V_{xy}-\gamma_{xy}}{\tau}\right)}{2}\right)$$

For unions of sets,

$$D_{F^1 \cup F^2} = \min\left\{D_{F^1}, D_{F^2}\right\}$$

and for intersections of sets

$$D_{F^1 \cap F^2} = \max\left\{D_{F^1}, D_{F^2}\right\}.$$

Unions are interesting in family economics because of public goods.

Assume that k is the number of kids. Conditional on k, the feasible set of utilities is F_{xy}^k. The overall set of utilities

$$\cup_k F_{xy}^k.$$

2.3 Equilibrium

Recall that

$$U_{xy} = \log\left(\frac{\mu_{xy}}{\mu_{x0}}\right)$$

and similarly on the other side,

$$V_{xy} = \log\left(\frac{\mu_{xy}}{\mu_{0y}}\right)$$

and express that U_{xy}, V_{xy} is on the frontier of F_{xy}, which means

$$D_{xy}(U_{xy}, V_{xy}) = 0$$

Replacing yields

$$
\begin{aligned}
D_{xy}\left(\log\left(\frac{\mu_{xy}}{\mu_{x0}}\right),\log\left(\frac{\mu_{xy}}{\mu_{0y}}\right)\right) &= 0\\
D_{xy}\left(\log\mu_{xy}-\log\mu_{x0},\log\mu_{xy}-\log\mu_{0y}\right) &= 0
\end{aligned}
$$

thus by the translation invariance property we get

$$
\log\mu_{xy}+D_{xy}\left(-\log\mu_{x0},-\log\mu_{0y}\right)=0
$$

that is

$$
\begin{aligned}
\mu_{xy} &= \exp\left(-D_{xy}\left(-\log\mu_{x0},-\log\mu_{0y}\right)\right)\\
&= M_{xy}\left(\mu_{x0},\mu_{0y}\right).
\end{aligned}
$$

μ_{x0} and μ_{0y} are determined

$$
\begin{aligned}
n_x &= \mu_{x0}+\sum_y M_{xy}\left(\mu_{x0},\mu_{0y}\right)\\
m_y &= \mu_{0y}+\sum_x M_{xy}\left(\mu_{x0},\mu_{0y}\right)
\end{aligned}
$$

$\mu_{x0}=n_x\exp\left(-u_x\right)$ $\mu_{0y}=m_y\exp\left(-v_y\right)$, rewrite this (upon redefining M_{xy}) as

$$
\begin{aligned}
n_x &= M_{x0}\left(u_x\right)+\sum_y M_{xy}\left(u_x,v_y\right)\\
m_y &= M_{0y}\left(v_y\right)+\sum_x M_{xy}\left(u_x,v_y\right)
\end{aligned}
$$

This is a system of the form

$$
\begin{aligned}
F_x\left(u,v\right) &= n_x\\
F_y\left(u,v\right) &= m_y
\end{aligned}
$$

2.4 How to compute it?

Claim 1. Matching models beyond transferable utility are not optimization problems.

Indeed, the Jacobian of the system is

$$
\begin{pmatrix}
\partial_x F_x\left(u,v\right) & \partial_y F_x\left(u,v\right)\\
\partial_x F_y\left(u,v\right) & \partial_y F_y\left(u,v\right)
\end{pmatrix}
$$

the blocks on the diagonal are diagonal. But we have htat in general

$$
\left(\partial_x F_y\left(u,v\right)\right)^{\top}\neq\partial_y F_x\left(u,v\right)
$$

as indeed

$$\partial_{v_y} M_{xy}\left(u_x, v_y\right) \neq \partial_{u_x} M_{xy}\left(u_x, v_y\right)$$

unless M_{xy} depends on the sum $u_x + v_y$ – that is the Choo-Siow case.

Claim 2. Two-sided matching models (even in the imperfectly transferable utility case) are equilibrium problems with substitutes.

As a reminder, an equilibrium problem w substitutes is of the form

$$Z\left(p\right) = 0$$

where Z is the excess supply function and $\partial_{p_i} Z_i\left(p_i\right) > 0$ (normal good) and $\partial_{p_j} Z_i\left(p_i\right) \leq 0$.

Recall that our matching problem expresses as

$$\begin{aligned} F_x\left(u, v\right) &= n_x \\ F_y\left(u, v\right) &= m_y \end{aligned}$$

And introduce $p_x = -u_x$ and $p_y = v_y$

$$\begin{aligned} Z_x\left(p\right) &= F_x\left(-p_x, \left(p_y\right)_y\right) - n_x \\ Z_y\left(p\right) &= -F_y\left(-\left(p_x\right)_x, p_y\right) + m_y. \end{aligned}$$

Computation/Existence using Jacobi's algorithm. Book by Rheinboldt and Ortega.

Uniqueness using Berry, Gandhi and Haile.