Equilibrium with Gross Substitutes: new results for an old problem

Alfred Galichon (NYU), joint work with Larry Samuelson (Yale) and
Lucas Vernet (Banque de France)
Supported by ERC grant 'EQUIPRICE'

Sir John Hicks lecture
SAET conference, June 15, 2021

- The theory of monotone comparative statics (MCS) under gross substitutes is well developed for single-agent optimization problems, but not for equilibrium problems where decisions are aggregated.
- At the same time, ever since the seminal work of Arrow et al., traditional general equilibrium theory has arguably regarded gross substitutes as a mere curiosity.
- However, gross substitutes appear naturally in a class of price equilibrium problems on a network that generalize matching, hedonic models, routing problems and dynamic programming problem.
- We formulate this problem and call it **equilibirum flow problem**, and we build a monotone comparative static theory for it: the theory of **unified gross substitutes**.

Agenda:

1. The equilibrium flow problem
2. Unified gross substitutes
3. Monotone comparative statics

Section 1

THE EQUILIBRIUM FLOW PROBLEM

- Optimal transport is a framework of choice to handle a number of matching problems, with economic applications ranging from marriage to labor market.
  - A few of these applications are reviewed in *Optimal Transport Method in Economics*.
  - Optimal transport is a framework two-sided matching with transferable utility (TU), see Chiappori, McCann and Nesheim (ET 2010).
- However, the framework is not without limitations.
  - Utilities are imposed to be quasilinear: does not allow nontransferable utility (NTU) or imperfectly transferable utility (ITU)
  - The market is two-sided: does not allow supply chains, trading networks, trees, etc.
- This talk is about building a framework for relaxing both of these restrictions into what we call *equilibrium flow problems*, and exploring the structure of this problem, *gross substitutes*.

- Consider a network $(\mathcal{Z}, \mathcal{A})$ where $\mathcal{Z}$ are the nodes, and $\mathcal{A} \subseteq \mathcal{Z} \times \mathcal{Z}$ are the arcs. For $xy \in \mathcal{A}$, μ_{xy} is the flow of commodity transiting through arc xy. For node $z \in \mathcal{Z}$, q_z is the flow of commodity exiting network at z (< 0 if consumed, > 0 if produced).
- The (local) mass balance equation is

$$\sum_{x:xz\in\mathcal{A}} \mu_{xz} - \sum_{y:zy\in\mathcal{A}} \mu_{zy} = q_z$$

  that is

$$\nabla^\top \mu = q$$

  where ∇ is the matrix of term $\nabla_{(xy),z} = 1\{z = y\} - 1\{z = x\}$, called *difference matrix*, or *arc-node incidence matrix*. For $f \in \mathbb{R}^{\mathcal{Z}}$, we have

$$(\nabla f)_{xy} = f_y - f_x.$$

- Note that local mass balance implies $\sum_{z\in\mathcal{Z}} q_z = 0$ (global mass balance).

- Call p_z the price of the commodity at node z.
- Assume that each arc xy is open to carry trade. A *carry trade* on arc xy consists of purchasing at x shipping to y, and selling at y. Call $G_{xy}(p_y)$ the price at which the good needs to be purchased at x for the trade to break even. G_{xy} is increasing in p_y.
  - Example: transferable utility case. $G_{xy}(p_y) = p_y - c_{xy}$, where c_{xy} is the unit shipping cost. A price increase is fully transfered from origin to destination.
- The trading network in one interpretation, but there are many others, depending on the situation:
  - Two-sided matching: splitting a joint surplus (à-la Becker)
  - Scheduling problem: passage times
  - Etc.

- One says that prices $p \in \mathbb{R}^{\mathcal{Z}}$ and quantities supplied to the exterior $q \in \mathbb{R}^{\mathcal{Z}}$ are in correspondence if there is an *equilibrium flow* $\mu \in \mathbb{R}^{\mathcal{A}}$ such that:
  1. **Mass balance** holds:
  $$\mu \geq 0 \text{ and } \nabla^{\top}\mu = q$$
  2. There is **no arbitrage**, i.e. no positive rent associated with a carry trade:
  $$p_x \geq G_{xy}\left(p_y\right) \text{ for all arcs } xy.$$
  3. There is **no forced entry**, i.e. carry trades that are effectively executed break even:
  $$\mu_{xy} > 0 \implies p_x = G_{xy}\left(p_y\right).$$
- We introduce the *equilibrium flow (EQF) problem* as the problem of searching for μ satisfying conditions (1), (2) and (3) above.
- Define $\mathbf{Q}\left(p\right)$ as the set of q for which there exists μ such that conditions (1) to (3) above hold given p. Interpret $\mathbf{Q}\left(p\right)$ as an **excess supply correspondence**.

- The EQF problem is the problem of, given q, finding p (and implicitely, finding μ) such that
$$q \in \mathbf{Q}(p)$$
- It embeds:
  - Optimal transport / matching models with transferable utility (TU) (bipartite network, quasilinear G)
  - Two-sided matching with general transfers (ITU) (bipartite network, general G)
  - Hedonic models with or without quasilinear utilities (three-layer network)
  - Shortest path problems, min-cost flows problem (general network, quasilinear G)
  - Supply chain problems, scheduling problems, dynamic programming problems (general network and G)
- In general, $\mathbf{Q}(p)$ may be empty; however, we proved an **existence result** under topological conditions on the network (not the focus today).

- Recall that the transferable utility case specifies $G_{xy}(p_y) = p_y - c_{xy}$, and so the equilibrium conditions are

$$\begin{cases} \mu \geq 0 \text{ and } \nabla^\top \mu = q \\ p_x \geq p_y - c_{xy} \; \forall xy \in \mathcal{A} \\ \mu_{xy} > 0 \implies p_x = p_y - c_{xy} \end{cases}$$

- These are the optimality conditions (complementary slackness) of a linear optimization problem ("min-cost flow")

$$\min_{\mu \geq 0} \sum_{xy \in \mathcal{A}} \mu_{xy} c_{xy}$$

$$s.t. \; \nabla^\top \mu = q$$

whose dual is

$$\max \sum_{z \in \mathcal{Z}} p_z q_z$$

$$s.t. \; \nabla p \leq c$$

- Back to the general case. Consider replacing

$$\begin{cases} p_x \geq G_{xy}\left(p_y\right) \\ \mu_{xy} > 0 \implies p_x = G_{xy}\left(p_y\right) \end{cases}$$

by the following ansatz

$$\mu_{xy} = M_{xy}\left(p\right) := \exp\left(\frac{G_{xy}\left(p_y\right) - p_x}{T}\right)$$

where $T > 0$ is a parameter.

- The problem then becomes

$$Q_z\left(p\right) = 0 \; \forall z \in \mathcal{Z}$$

where

$$Q_z\left(p\right) = \sum_{x:xz\in\mathcal{A}} M_{xz}\left(p\right) - \sum_{y:zy\in\mathcal{A}} M_{zy}\left(p\right).$$

- When $T \to 0$, approximates a solution to the EQF problem (if any)
- In the TU case $G_{xy}(p_y) = p_y - c_{xy}$, regularized problem solves

$$\max_p \sum_{z \in \mathcal{Z}} p_z q_z - T \sum_{xy \in \mathcal{A}} \exp\left(\frac{p_y - p_x - c_{xy}}{T} \right)$$

- Outside of the TU case, Jacobian of the system DQ is generally not symmetric, and problem can no longer be interpreted as FOC of an optimization problem.
- In particular, **matching problems are not optimization problems in general.**

- However, notice that

$$Q(p) = 0$$

has structure, as $Q(p)$ satisfies *gross substitutes*: Q_z increasing in p_z, and weakly decreasing in p_{-z}.
- Further, Q is stochastic in the sense that

$$1^\top Q(p) = \sum_{z \in \mathcal{Z}} Q_z(p) = 0$$

for all p. Hence, we need a normalization.
  - Take some node $0 \in \mathcal{Z}$ and normalize $p_0 = 0$, and restrict attention to the remaining entries of p and $Q(.)$.

- By a result of Berry, Gandhi and Haile (2013), if the network is connected (in an undirected way), then Q is inverse isotone in the sense that
$$Q_z(p) \leq Q_z\left(p'\right) \ \forall z \neq 0$$
implies $p_z \leq p'_z \ \forall z \neq 0$.
- Q is an M-function in the language of Rheinboldt (1970). Useful result as it establishes uniqueness of equilibrium prices as well as the convergence of certain iterative algorithms (Jacobi and Gauss-Seidel).
- Now, what about the unregularized/correspondence case?

Section 2

Unified gross substitutes

- In the unregularized case, recall that $\mathbf{Q}(p)$ is the set of q's that can be written $q = \nabla^{\top}\mu$ where there exists $p \in \mathbb{R}^{\mathcal{Z}}$ with $p_x \geq G_{xy}(p_y)$ and $\mu_{xy} > 0 \implies p_x = G_{xy}(p_y)$.
- Clearly, $\mathbf{Q}(p)$ is a correspondence: $q \in \mathbf{Q}(p)$ implies $\lambda q \in \mathbf{Q}(p)$ for $\lambda > 0$.
- We expect $\mathbf{Q}(p)$ to exhibit a form of *gross substitutes* and *inverse isotonicity*. **How to define these for correspondences**?

- Assume $\mathbf{Q}(p)$ solves the following optimization problem (as in the TU case)
$$\mathbf{Q}(p) = \arg\max_q \left\{ \sum_{z \in \mathcal{Z}} p_z q_z - c(q) \right\}$$
then it is classical (since Ausubel and Milgrom) to define gross substitutes by the submodularity of the indirect cost function $c^*(p)$ defined by
$$c^*(p) = \max_q \left\{ \sum_{z \in \mathcal{Z}} p_z q_z - c(q) \right\}$$
- In this case, inverse isotonicity of $\mathbf{Q}(p)$ follows from the theory of Veinott and Topkis – indeed, by convex duality
$$\mathbf{Q}^{-1}(q) = \arg\max_p \left\{ \sum_{z \in \mathcal{Z}} p_z q_z - c^*(p) \right\}.$$
However, existing MCS results (Topkis, Milgrom-Shannon, Quah...) not longer apply as soon as $\mathbf{Q}^{-1}(q)$ not the outcome of an optimization problem.

- We define a notion of uniform gross substitutes for correspondences which generalizes previous ones.
- $\mathbf{Q}$ satisfies *unified gross substitutes* if for $q \in \mathbf{Q}(p)$ and $q' \in \mathbf{Q}(p')$, there exists $q^{\wedge} \in \mathbf{Q}(p \wedge p')$ and $q^{\vee} \in \mathbf{Q}(p \vee p')$ such that:

$$\begin{cases} p_z \leq p'_z \implies q_z \leq q^{\wedge}_z \text{ and } q'_z \geq q^{\vee}_z \\ p_z > p'_z \implies q'_z \leq q^{\wedge}_z \text{ and } q_z \geq q^{\vee}_z \end{cases}.$$

  Equivalently, we say that $\mathbf{Q}$ is a *Z-correspondence*.
- Remarks:
  - MGS is stronger than the definition in Kelso and Crawford (1981). Indeed, Kelso and Crawford do not require $Q(p \wedge p')$ to be nonempty.
  - Notion appears incidentally in Polterovich and Spivak (1984).
- Next, we show that unified gross substitutes generalizes existing notions.

- In the point-valued case $\mathbf{Q}(p) = \{Q(p)\}$, one recovers classical weak gross substitutes. Indeed, if $p \geq p'$, then $p \wedge p' = p'$ and $q^{\wedge} = Q(p') = q'$, and therefore

$$p_z = p'_z \implies p_z \leq p'_z \implies q_z \leq q'_z.$$

- **Theorem**. In argmax case, we have that

$$\mathbf{Q}(p) = \arg\max_q \left\{ p^\top q - c(q) \right\}$$

satisfies UGS if and only if the indirect utility function

$$c^*(p) = \max_q \left\{ p^\top q - c(q) \right\}$$

is submodular.

- Even in the linear case, gross substitute (Z-matrix) is not enough for inverse isotonicity (which requires in addition P-matrix). We therefore need to impose a additional assumption.
- **Definition**. $\mathbf{Q}(p)$ is *nonreversing* if $q \in \mathbf{Q}(p)$, $q' \in \mathbf{Q}(p')$, $p \leq p'$ and $q \geq q'$ imply $q' \in \mathbf{Q}(p)$ and $q \in \mathbf{Q}(p')$.
- In the linear case $\mathbf{Q}(p) = \{Qp\}$, Q is a P-matrix implies $\mathbf{Q}(p)$ is nonreversing.
- Leading cases:
  - Stochastic correspondences: $q \in \mathbf{Q}(p) \implies q^\top 1 = 0$
  - argmax case: $\mathbf{Q}(p) = \partial c^* (p)$

Section 3

Monotone comparative statics

- **Definition**: $\mathbf{Q}$ is an M-correspondence if and only if it is a Z-correspondence and nonreversing.
- **Theorem**: Consider $\mathbf{Q}$ a Z-correspondence. Then the following two statements are equivalent:
  (i) $\mathbf{Q}$ is nonreversing (i.e., $\mathbf{Q}$ is an M-correspondence),
  and
  (ii) $\mathbf{Q}$ is inverse isotone: for $q \in \mathbf{Q}(p)$ and $q' \in \mathbf{Q}(p')$ such that $\sum_z 1\{q_z > q'_z\} 1\{p_z > p'_z\} = 0$, then $q \in \mathbf{Q}(p \wedge p')$ and $q' \in \mathbf{Q}(p \vee p')$

- Point-valued case: recovers Berry, Gandhi and Haile (2013) and the theory of M-functions (Rheinboldt, 1970).
- Argmax case: recovers Veinott and Topkis' theory of monotone comparative statics.
- However, UGS also allows us to prove new results, such as for the EQF problem.

- **Theorem**. The correspondence $\mathbf{Q}(p)$ that appears in the equilibrium flow problem is a M-correspondence.
- It is clearly nonreversing as $\mathbf{Q}$ is stochastic: $q \in \mathbf{Q}(p) \implies 1^{\top} q = 0$.
- To show that $\mathbf{Q}$ is a Z-correspondence, write it as an aggregate supply correspondence

$$\mathbf{Q}(p) = \sum_{a \in \mathcal{A}} \mathbf{Q}^{a}(p)$$

  where each $\mathbf{Q}^{a}(p)$ is the contribution to the flow by traders on arc a, and show that each of the $\mathbf{Q}^{a}(p)$ have UGS.

- Existence: Jacobi algorithm for correspondences.
- Extension to proper NTU case using Adachi's formulation.
- Extension to one-to-many matching problems (Kelso-Crawford, Hatfield-Milgrom).
- Connection with discrete theory (indivisibilities) and Gul-Stacchetti's results.
- More on these topics in the math+econ+code masterclasses: https://www.math-econ-code.org/