Dynamic models of matching

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Slides available from http://alfredgalichon.com/

- Dynamic aspects are crucial for matching problems
  - In labor economics (human capital formulation)
  - In family economics (fertility decisions)
  - In mergers and acquisitions
  - In school choice
  - Etc.
- We offer a framework for these dynamic matching problems:
  - with or without unobserved heterogeneity
  - with finite or infinite (stationary) horizon
  - with equilibrium prediction, structural estimation, comparative statics and welfare

- Large current literature on the estimation of **static transferable utility (TU) two-sided** (matching) models in the static case:
  - Choo and Siow (2006), Fox (2010), Galichon and Salanié (2011), Dupuy and Galichon (2014), Chiappori, Salanié and Weiss (2019), Fox et al. (2018)
- Dynamic discrete choice literature on one-sided models since Rust (1987) assumes the decision maker's type evolves stochastically depending on the choice made at the previous period.
- Today's goal: investigate the dynamic aspect of static matching models by assuming that the match has an effect on types *on both sides of the market*. And show how to take models to data on **changing relationships over time.**

- NTU case when matches are forever (e.g. kidney)
  - Unver (2010), Bloch and Cantala (2017), Doval (2021)
- Search and matching: the matching has no effect on partners, but match opportunities are scarse
  - NTU case: Burdett and Coles (1997); Eeckhout (1999), Peski (2021)
  - TU case: Shimer and Smith (2000) .
- TU case:
  - Erlinger, McCann, Shi, Siow and Wolthoff (2015), McCann, Shi, Siow and Wolthoff (2015) – 2 period sequential matching, with universities in a first period, then with firms.
  - Choo (2015) studies a dynamic matching problem with a focus on the age of marriage

Populations:

- $z \in \mathcal{Z}$ agents to be matched, $z = x$ (worker) or $z = y$ (firms)
- q_z=mass of agents of type z (fixed for now)

Matches:

- $a \in \mathcal{A}$ matches; $a = xy$ or $a = x$ (unassigned worker) or $a = y$ (unassigned firm)
- w_a= cardinality of the match (2 for pair, 1 for unassigned)
- $\tilde{S}_a$=joint (random utility) surplus of match a
  - Choo-Siow's separable random utility assumption: $\tilde{S}_a = S_a + \sum_{z \in a} \varepsilon_z$, where (ε_z) vector of random payoff shifters (Gumbel for simplicity)

Equilibrium quantities:

- p_z=payoff of z
- μ_a=mass of match a

Static TU matching with random utility: equilibrium insights (1)

Result 1 (Choo-Siow): μ_a and p_z are related by
$\mu_a = \exp\left(w_a^{-1}\left(S_a + \sum_{z\in a}\left(\log q_z - p_z\right)\right)\right)$ and p solves
$\sum_{a\ni z} \exp\left(w_a^{-1}\left(S_a + \sum_{z\in a}\left(\log q_z - p_z\right)\right)\right) = q_z$ for each z.

(Proof in the appendix at the end of these slides).

Note that at equilibrium, $\sum_{a\in\mathcal{A}} w_a \mu_a = \sum_{z\in\mathcal{Z}} q_z$. Hence, define

$$Z\left(q,p,S\right)=\sum_{a\in\mathcal{A}} w_a \exp\left(w_a^{-1}\left(S_a+\sum_{z\in a}\left(\log q_z - p_z\right)\right)\right)-\sum_{z\in\mathcal{Z}} q_z.$$

We have $\frac{\partial Z(p,q,S)}{\partial p_z} = \sum_{a\ni z}\mu_a$, with
$\mu_a = \exp\left(w_a^{-1}\left(S_a+\sum_{z\in a}\left(\log q_z - p_z\right)\right)\right)$.

Therefore:
Result 2 (Galichon-Salanié): The equilibirum p solves

$$\min_p \sum_{z\in\mathcal{Z}} q_z p_z + Z(p,q,S).$$

We now consider a two-sided Rust-type dynamic matching model with TU. Assume that individuals' types vary across periods, and that the transition depend on current period match.

Consider

$$\mathbb{R}_{za}$$

the mass of individuals z induced forward at next period by one unit of match a.
For instance, if $a = xy$, worker x's type will transition to x' with proba. $\mathbb{P}_{x'|xy}$, and firm y's type will transition to y' with proba. $\mathbb{Q}_{y'|xy}$. In that case,
$\mathbb{R}_{za} = \sum_{x'} 1\{z = x'\} \mathbb{P}_{x'|xy} + \sum_{y'} 1\{z = y'\} \mathbb{Q}_{y'|xy}$.

Note that (as in Rust) the transition are Markovian:
(x chooses $a = xy$ w.p. μ_a / q_x) and then (transitions to x' w.p. $\mathbb{R}_{x'|xy}$).
Hence, conditional transition probability $x \to x'$ equals to $\sum_y \mu_{xy} \mathbb{R}_{x'|xy} / q_x$.

In that case, S_a needs to accrue for future-period payoffs p', in addition to short-term joint payoff Φ_a, and
$S_a = \Phi_a + \beta \sum_z \mathbb{R}_{za} p'_z = \left(\Phi + \beta \mathbb{R}^\top p'\right)_a$.

Now redefine Z by inserting expression for S, we have

$$Z\left(q, p, p'\right) = \sum_{a \in \mathcal{A}} w_a \exp\left(w_a^{-1}\left(\left(\Phi + \beta \mathbb{R}^\top p'\right)_a + \sum_{z \in a}\left(\log q_z - p_z\right)\right)\right) - \sum_{z \in \mathcal{Z}} q_z$$

Z is all we need to write the equilibrium equations of the model. Indeed,

- $\partial Z / \partial q_z = \sum_{a \ni z} \mu_a / q_z - 1$ excess share of demand for type z
- $-\partial Z / \partial p_z = \sum_{a \ni z} \mu_a =$ mass of z at current period
- $\beta^{-1} \partial Z / \partial p'_z = \sum_{a \in \mathcal{A}} \mathbb{R}_{za} \mu_a =$ mass of z at next period

A stationary equilibrium has

$$p = p' \text{ [rational expectations]}$$

and expresses as

$$\left\{ \begin{array}{l} \frac{\partial Z(q,p,p)}{\partial q_z} = 0 \text{ [market clearing for each type]} \\ \beta \frac{\partial Z(q,p,p)}{\partial p_z} + \frac{\partial Z(q,p,p)}{\partial p'_z} = 0 \text{ [stationarity]} \end{array} \right. .$$

Note that Z is concave in q and jointly convex in (p, p').

When $\beta = 1$, set $F(q,p) = Z(q,p,p)$ is concave-convex and the equations of the model

$$\begin{cases} \partial F(q,p) / \partial q = 0 \\ \partial F(q,p) / \partial p = 0 \end{cases}$$

are obtained as the saddlepoint conditions for the min-max problem

$$\min_{p} \max_{q} F(q,p).$$

Computation using Chambolle-Pock's first order scheme:

$$\begin{cases} q^{t+1} = q^{t} - \epsilon \partial_{q} F\left(q^{t}, 2p^{t} - p^{t-1}\right) \\ p^{t+1} = p^{t} + \epsilon \partial_{p} F\left(q^{t}, p^{t}\right) \end{cases}$$

Surprising fact: algorithm works even for $\beta < 1$ although min-max interpretation is lost.

Now assume we want to solve the inverse problem: based on observed $\hat{\mu}_a$ recover information about Φ.
Parameterize $\Phi_a = \sum_k \phi_{ak}\lambda_k$ and look for λ.

Express

$$
\begin{aligned}
& Z\left(q, p, p', \lambda\right) \\
& = \sum_{a\in\mathcal{A}} w_a \exp\left(w_a^{-1}\left(\left(\sum_k \phi_{ak}\lambda_k + \beta\mathbb{R}^\top p'\right)_a + \sum_{z\in a}\left(\log q_z - p_z\right)\right)\right) \\
& - \sum_{z\in\mathcal{Z}} q_z
\end{aligned}
$$

and note that the partial derivatives of Z with respect to the new variables λ_k also have a natural interpretation. Indeed,

$$\frac{\partial Z}{\partial \lambda_k} = \sum_{a\in\mathcal{A}} \mu_a \phi_{ak}$$

is the predicted k-th moments of ϕ.

Define a function H as

$$H\left(q,p,p',\lambda\right)=Z\left(q,p,p',\lambda\right)-\sum_{a\in\mathcal{A}}\hat{\mu}_a\phi_{ak}\lambda_k$$

which is jointly convex in (p,p',λ), and note that the indentifying equations are now

$$\begin{cases}\frac{\partial H(q,p,p',\lambda)}{\partial q}=0 \text{ [market clearing]}\\ \beta\frac{\partial H(q,p,p',\lambda)}{\partial p}+\frac{\partial H(q,p,p',\lambda)}{\partial p'}=0 \text{ [stationarity]}\\ \frac{\partial H(q,p,p',\lambda)}{\partial \lambda}=0 \text{ [moment matching]}\end{cases}$$

In the case $\beta=1$ this is still a saddlepoint problem, now

$$\min_{p,\lambda}\max_{q}H\left(q,p,p,\lambda\right)$$

for which Chambolle-Pock's first order scheme still applies. It even (mysteriously) still applies when $\beta<1$.

- Indentification issues à la Kalouptsidi, Scott & Souza-Rodrigues (2019) and Kalouptsidi, Kitamura and Lima (2021).
- Theoretical convergence of the first order scheme outisde of $\beta = 1$ (min-max).
- Empirical application: human capital accumulation on the labor market.
- With Dupuy, Ciscato and Weber: application to family economics (divorce, remarriage and the number of kids).
- Extention to imperfectly transferable utility (later).

- The next ‘math+econ+code’ masterclass on equilibrium transport and matching models in economics will take place June 21-25, 2021. More info on

  http://alfredgalichon.com/mec_equil/

- Jules Baudet and I are organizing a **kidney transplant hackaton** for the math+econ+code. More info at

  http://alfredgalichon.com/kindey-transplant-hackaton/

  and email us: ag133@nyu.edu or jules.baudet99@gmail.com if you are interested!

We need to determine

- μ_a= mass of matches of type a is formed so that $\sum_{a \ni z} \mu_a = q_z$
- U_{za}= z's share of surplus in a match a so that $S_a = \sum_{z \in a} U_{za}$ and so that agent z in a match a gets $U_{za} + \varepsilon_a$
- p_z= average payoff of players of type z, so that $p_z = \log \sum_{a \ni z} \exp U_{za}$

Logit model: probability that z chooses match a is
$\mu_a / q_z = \frac{\exp U_{za}}{\sum_{a \ni z} \exp U_{za}} = \exp \left(U_{za} - p_z \right)$
hence
$\log \left(\mu_a / q_z \right) = U_{za} - p_z$

Choo-Siow: summing over $z \in a$ yields
$\mu_a = \exp \left(w_a^{-1} \left(S_a + \sum_{z \in a} \left(\log q_z - p_z \right) \right) \right).$