Topics in Equilibrium Transportation

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This talk is based on the following two papers:

- AG, Scott Kominers and Simon Weber (2015a). Costly Concessions: An Empirical Framework for Matching with Imperfectly Transferable Utility.
- AG, Scott Kominers and Simon Weber (2015b). The Nonlinear Bernstein-Schrödinger Equation in Economics, GSI proceedings.

Agenda:

1. Economic motivation
2. The mathematical problem
3. Computation
4. Estimation

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Section 1

Economic motivation

- Consider a very simple model of labour market. Assume that a population of *workers* is characterized by their type $x \in \mathcal{X}$, where $\mathcal{X} = \mathbb{R}^d$ for simplicity. There is a distribution P over the workers, which is assumed to sum to one.
- A population of *firms* is characterized by their types $y \in \mathcal{Y}$ (say $\mathcal{Y} = \mathbb{R}^d$), and their distribution Q. It is assumed that there is the same total mass of workers and firms, so Q sums to one.
- Each worker must work for one firm; each firm must hire one worker. Let $\pi(x, y)$ be the probability of observing a matched (x, y) pair. π should have marginal P and Q, which is denoted

$$\pi \in \mathcal{M}(P, Q).$$

- In the simplest case, the utility of a worker x working for a firm y at wage $w(x,y)$ will be
$$\alpha(x,y) + w(x,y)$$
while the corresponding profit of firm y is
$$\gamma(x,y) - w(x,y).$$
- In this case, the total surplus generated by a pair (x,y) is
$$\alpha(x,y) + w + \gamma(x,y) - w = \alpha(x,y) + \gamma(x,y) =: \Phi(x,y)$$
which does not depend on w (no transfer frictions). A central planner may thus like to choose assignment $\pi \in \mathcal{M}(P,Q)$ so to
$$\max_{\pi \in \mathcal{M}(P,Q)} \int \Phi(x,y)\, d\pi(x,y).$$
But **why would this be the equilibrium solution**?

- The equilibrium assignment is determined by an important quantity: the **wages**. Let $w(x, y)$ be the wage of employee x working for firm of type y.
- Let the indirect surpluses of worker x and firm y be respectively

$$u(x) = \max_y \{\alpha(x, y) + w(x, y)\}$$
$$v(y) = \max_x \{\gamma(x, y) - w(x, y)\}$$

so that (π, w) is an equilibrium when

$$u(x) \geq \alpha(x, y) + w(x, y) \text{ with equality if } (x, y) \in Supp(\pi)$$
$$v(y) \geq \gamma(x, y) - w(x, y) \text{ with equality if } (x, y) \in Supp(\pi)$$

- By summation,

$$u(x) + v(y) \geq \Phi(x, y) \text{ with equality if } (x, y) \in Supp(\pi).$$

- One can show that the equilibrium outcome (π, u, v) is such that π is solution to the primal Monge-Kantorovich Optimal Transportation problem

$$\max_{\pi \in \mathcal{M}(P,Q)} \int \Phi(x,y)\, d\pi(x,y)$$

and (u, v) is solution to the dual OT problem

$$\begin{aligned} &\min_{u,v} \int u(x)\, dP(x) + \int v(y)\, dQ(y) \\ &s.t.\ u(x) + v(y) \geq \Phi(x,y) \end{aligned}$$

- Feasibility+Complementary slackness yield the desired equilibrium conditions

$$\begin{aligned} &\pi \in \mathcal{M}(P,Q) \\ &u(x) + v(y) \geq \Phi(x,y) \\ &(x,y) \in Supp(\pi) \Longrightarrow u(x) + v(y) = \Phi(x,y) \end{aligned}$$

"Second welfare theorem", "invisible hand", etc.

- Is equilibrium always the solution to an optimization problem?
- **It is not**. This is why this talk is about "Equilibrium Transportation," which contains, but is strictly more general than "Optimal Transportation".

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- **It is not**. This is why this talk is about "Equilibrium Transportation," which contains, but is strictly more general than "Optimal Transportation".

- Consider the same setting as above, but instead of assuming that workers' and firm's payoffs are linear in surplus, assume

$$u(x) = \max_y \{\mathcal{U}_{xy}(w(x,y))\}$$
$$v(y) = \max_x \{\mathcal{V}_{xy}(w(x,y))\}$$

where $\mathcal{U}_{xy}(w)$ is nondecreasing and continuous, and $\mathcal{V}_{xy}(w)$ is nonincreasing and continuous.

- Motivation: taxes, decreasing marginal returns, risk aversion, etc. Of course, Optimal Transportation case is recovered when

$$\mathcal{U}_{xy}(w) = \alpha_{xy} + w$$
$$\mathcal{V}_{xy}(w) = \gamma_{xy} - w.$$

- For $(u, v) \in \mathbb{R}^2$, let

  $$\Psi_{xy}(u, v) = \min \left\{ t \in \mathbb{R} : \exists w, u - t \leq \mathcal{U}_{xy}(w) \text{ and } v - t \leq \mathcal{V}_{xy}(w) \right\}$$

  so that Ψ is nondecreasing in both variables and $(u, v) = (\mathcal{U}_{xy}(w), \mathcal{V}_{xy}(w))$ for some w if and only if $\Psi_{xy}(u, v) = 0$. Optimal Transportation case is recovered when

  $$\Psi_{xy}(u, v) = (u + v - \Phi_{xy})/2.$$

- As before, (π, w) is an equilibrium when

  $$u(x) \geq \mathcal{U}_{xy}(w(x, y)) \text{ with equality if } (x, y) \in Supp(\pi)$$
  $$v(y) \geq \mathcal{V}_{xy}(w(x, y)) \text{ with equality if } (x, y) \in Supp(\pi)$$

- We have therefore that (π, u, v) is an equilibrium when

  $$\Psi_{xy}(u(x), v(y)) \geq 0 \text{ with equality if } (x, y) \in Supp(\pi).$$

Section 2

THE MATHEMATICAL PROBLEM

- We have therefore that (π, u, v) is an equilibrium outcome when

$$\begin{cases} \pi \in \mathcal{M}(P, Q) \\ \Psi_{xy}(u(x), v(y)) \geq 0 \\ (x, y) \in Supp(\pi) \Longrightarrow \Psi_{xy}(u(x), v(y)) = 0 \end{cases}.$$

- Problem: existence of an equilibrium outcome? This paper: yes in the discrete case ($\mathcal{X}$ and $\mathcal{Y}$ finite), via entropic regularization.

As soon as Ψ_{xy} is strictly increasing in both variables, $\Psi_{xy}(u, v) = 0$ expresses as

$$u = G_{xy}(v) \text{ and } v = G_{xy}^{-1}(u)$$

where the generating functions G_{xy} and G_{xy}^{-1} are decreasing and continuous functions. In this case, relations

$$u(x) = \max_{y \in \mathcal{Y}} G_{xy}(v(y)) \text{ and } v(y) = \max_{x \in \mathcal{X}} G_{xy}^{-1}(u(x))$$

generalize the Legendre-Fenchel conjugacy. This pair of relations form a Galois connection; see Singer (1997) and Noeldeke and Samuelson (2015).

Assuming everything is smooth, and letting f_P and f_Q be the densities of P and Q we have under some conditions that the equilibrium transportation plan is given by $y = T(x)$, where mass balance yields

$$\left|\det DT(x)\right| = \frac{f(x)}{g(T(x))}$$

and optimality yieds

$$\partial_x G^{-1}_{xT(x)}(u(x)) + \partial_u G^{-1}_{xT(x)}(u(x)) \nabla u(x) = 0$$

which thus inverts into

$$T(x) = e(x, u(x), \nabla u(x)).$$

Trudinger (2014) studies Monge-Ampere equations of the form

$$\left|\det De(., u, \nabla u)\right| = \frac{f}{g(e(., u, \nabla u))}.$$

(more general than Optimal Transport where no dependence on u).

- Our work (GKW 2015a and b) focuses on the discrete case, when P and Q have finite support. Call p_x and q_y the mass of $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ respectively.
- In the discrete case, problem boils down to looking for (π, u, v) such that

$$\left\{ \begin{array}{l} \pi_{xy} \geq 0,\ \sum_y \pi_{xy} = p_x,\ \sum_x \pi_{xy} = q_y \\ \Psi_{xy}\left(u_x, v_y\right) \geq 0 \\ \pi_{xy} > 0 \Longrightarrow \Psi_{xy}\left(u_x, v_y\right) = 0 \end{array} \right. .$$

Section 3

COMPUTATION

- Take temperature parameter $T > 0$ and look for π under the form

$$\pi_{xy} = \exp\left(-\frac{\Psi_{xy}\left(u_x, v_y\right)}{T}\right)$$

- Note that when $T \to 0$, the limit of $\Psi_{xy}\left(u_x, v_y\right)$ is nonnegative, and the limit of $\pi_{xy}\Psi_{xy}\left(u_x, v_y\right)$ is zero.

- If $\pi_{xy} = \exp\left(-\Psi_{xy}\left(u_x, v_y\right) / T\right)$, condition $\pi \in \mathcal{M}\left(P, Q\right)$ boils down to set of nonlinear equations in $\left(u, v\right)$

$$\begin{cases} \sum_{y \in \mathcal{Y}} \exp\left(-\frac{\Psi_{xy}(u_x, v_y)}{T}\right) = p_x \\ \sum_{x \in \mathcal{X}} \exp\left(-\frac{\Psi_{xy}(u_x, v_y)}{T}\right) = q_y \end{cases}$$

  which we call the *nonlinear Bernstein-Schrödinger* equation.

- In the optimal transportation case, this becomes the classical B-S equation

$$\begin{cases} \sum_{y \in \mathcal{Y}} \exp\left(\frac{\Phi_{xy} - u_x - v_y}{2T}\right) = p_x \\ \sum_{x \in \mathcal{X}} \exp\left(\frac{\Phi_{xy} - u_x - v_y}{2T}\right) = q_y \end{cases}$$

- Note that $F_x : u_x \to \sum_{y\in\mathcal{Y}} \exp\left(-\frac{\Psi_{xy}(u_x,v_y)}{T}\right)$ is a decreasing and continuous function. Mild conditions on Ψ therefore ensure the existence of u_x so that $F_x\left(u_x\right) = p_x$.
- Our algorithm is thus a nonlinear Jacobi algorithm:
  - Make an initial guess of v_y^0
  - Determine the u_x^{k+1} to fit the p_x margins, based on the v_y^k
  - Update the v_y^{k+1} to fit the q_y margins, based on the u_x^{k+1}.
  - Repeat until v^{k+1} is close enough to v^k.
- One can proof that if v_y^0 is high enough, then the v_y^k decrease to fixed point. Convergence is very fast in practice.

Section 4

Statistical Estimation

- In practice, one observes $\hat{\pi}_{xy}$ and would like to estimate Ψ. Assume that Ψ belongs to a parametric family Ψ^θ, so that $\pi^\theta_{xy} = \exp\left(-\Psi^\theta_{xy}\left(u^\theta_x, v^\theta_y\right)\right) \in \mathcal{M}\left(P, Q\right)$.
- The log-likelihood $l\left(\theta\right)$ associated to observation $\hat{\pi}_{xy}$ is

$$\begin{aligned} l\left(\theta\right) &= \sum_{xy} \hat{\pi}_{xy} \log \pi^\theta_{xy} \\ &= -\sum_{xy} \hat{\pi}_{xy} \Psi^\theta_{xy}\left(u^\theta_x, v^\theta_y\right) / T \end{aligned}$$

and thus the maximum likelihood procedure consists in

$$\min_\theta \sum_{xy} \hat{\pi}_{xy} \Psi^\theta_{xy}\left(u^\theta_x, v^\theta_y\right) / T.$$