Gross substitutes, optimal transport, and matching models

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Introduction

Practical information

- Material available on:
  https://github.com/alfredgalichon/seattle-2022

- Schedule: Monday 10:30am-12pm, Tuesday 2pm-3:30pm, Wednesday 2pm-3:30pm

About these lectures

These lectures are about *substitutability*. Substitutability is a fundamental property of optimal transport models, although it is less recognized than convexity.

Consider the problem of computing an economic equilibrium

$$Q(p) = 0$$

where $Q : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is an excess supply function (supply minus demand for each of the n goods), and p is the price vector of each goods.

Grossly speaking, there are only two types of economic models that can be conveniently computed:

- models that rely on convexity, i.e. models where $Q = \nabla U(p)$ and U is a convex function. The problem reformulated as an optimization problem (minimization of U) and *descent methods*, such as the standard gradient descent, will work:

$$p_z^{t+1} = p_z^t - \epsilon Q_z \left(p^t\right).$$

- models that rely on substitutability – essentially the idea that Q_z is increasing in p_z and decreasing in p_x for $x \neq y$. In this case, *coordinate update methods*, such as nonlinear Jacobi, where we set the price of good z to clear the market for good z will work:

$$Q_z \left(p_z^{t+1}, p_{-z}^t\right) = 0.$$

By an extraordinary coincidence, optimal transport inherits both structures. For this reason, one can compute the OT problem and its regularizations both by descent methods and by coordinate update methods (which is called Sinkhorn's algorithm).

Some matching models can be computed using optimal transport. These are called transferable utility models, and assume that everyone's valuations are expressed in the same monetary unit. In that case, equilibrium in matching problems is the solution to an optimization problem. However, in many other cases, this assumption cannot be made. This includes:

- labor economics: one dollar of the firm not not have the same value as one dollar for the employee (pre-tax or post-tax; decreasing marginal utility, etc)

- family economics: partners may transfer utility by allocating public and private expenditures that can be inefficient

- school choice problems: utility is cardinal and cannot be transfered

We will see mathematically that these models *cannot* be recast as an optimization problem, and so the convex optimization structure is lost. However, we will also see that these models can be reformulated as models with substitutability, and therefore a lot of structure and computational methods can be deduced.

This leads us to start with a rather detailed study of substitutability and its mathematical counterpart, *Z- and M-functions*. There was a vibrant literature on the topic in the 1970s (Birkhoff, Rheinboldt, Ortega, More, Porsching, and some others), see in particular Rheinboldt and Ortega's 1970 book – but this literature has somehow fallen out of attention. In our first lecture, we will unearth some of the main results from that literature, and provide some new ones that we will need.

In the second lecture, we will connect with models of matching with transfers, departing from the assumption of perfectly transferable utility and we will appeal to the machinery developped in the first lecture.

In the third lecture, we will revisit the famous theory by Gale and Shapley of "stable marriages", which is the theory of matchings without transfers, and we will reinterpret the famous deferred acceptance algorithm as a damped version of Jacobi's algorithm. We will revisit results on the lattice structure of stable matchings.

We hope these three lectures will provide a solid mathematical introduction to matching models in economics and econometrics.

A roadmap

Lecture 1. Introduction to gross substitutes

M-matrices and M-maps, nonlinear Perron-Froebenius theory, convergence of Jacobi algorithm. A toy hedonic model.

Lecture 2. Models of matching with transfers

Problem formulation, regularized and unregularized case. IPFP and its convergence. Existence and uniqueness of an equilibrium. Lattice structure.

Lecture 3. Models of matching without transfers

Gale and Shapley's stable matchings. Adachi's formulation. Kelso-Craford. Hatfield-Milgrom.

1 Lecture 1: M-maps

Reference

Reference for this lecture include the following published material:

- Ortega, Rheinboldt (1970). *Iterative Solution of Nonlinear Equations in Several Variables*. SIAM.
- Berry, Gandhi and Haile (2013). Connected substitutes and the invertibility of demand. *Econometrica*.
- Galichon, Kominers, Weber (2019). Costly Concessions: An Empirical Framework for Matching with Imperfectly Transferable Utility. *Journal of Political Economy*.

This lecture also significantly draws on current research with co-authors, in particular the following work in progress:

- [GSV] Galichon, Samuelson and Vernet (2022). Monotone comparative statics for equilibrium problems.

- [GL] Galichon and Léger (2022). Z-mappings.

- [CGLW] Choo, Galichon, Liang, and Weber, Existence and Uniqueness in Matching Function Equilibria with Full Assignment

1.1 Sinkhorn's algorithm as Jacobi's algorithm

Consider the regularized optimal transport problem

$$\begin{aligned}\max_{\mu \geq 0} \quad & \sum_{xy} \mu_{xy} \Phi_{xy} - \sum_{xy} \mu_{xy} \ln \mu_{xy} \\ & \sum_{y \in Y} \mu_{xy} = n_x \\ & \sum_{x \in X} \mu_{xy} = m_y\end{aligned}$$

which has solutions of form

$$\mu_{xy} = \exp\left(\Phi_{xy} - a_x - b_y\right).$$

Note that Sinkhorn's algorithm consists of coming up with an initial guess of a^0 and b^0 and setting

$$\begin{aligned}a_x^{t+1} &: \ n_x = \sum_y \exp\left(\Phi_{xy} - a_x^{t+1} - b_y^t\right) \\ b_y^{t+1} &: \ m_y = \sum_y \exp\left(\Phi_{xy} - a_x^t - b_y^{t+1}\right)\end{aligned}$$

and iterating until approximate convergence is reached. As we have written things, a^{2t} and b^{2t+1} depend on the starting point a^0, while a^{2t+1} and b^{2t} depend on the starting point b^0.

Note that in the first part of the update, a_x is an antitone (=weakly monotone with respect to the coordinatewise order) function of (b_y), while in the second part of the update, b_y is an antitone function of (a_x).

Let us change the sign on one side and define

$$p_x = -a_x, p_y = b_y$$

and define

$$\begin{aligned} Q_x(p) &= \sum_y \exp(\Phi_{xy} + p_x - p_y) - n_x \\ Q_y(p) &= -\sum_x \exp(\Phi_{xy} + p_x - p_y) + m_y \end{aligned}$$

so that Sinkhorn's algorithm rewrites as *Jacobi's algorithm*

$$\begin{aligned} p_x^{t+1} &: Q_x\left(p_x^{t+1}, p_{-x}^t\right) = 0 \\ p_y^{t+1} &: Q_y\left(p_y^{t+1}, p_{-y}^t\right) = 0 \end{aligned}$$

where by p_{-z} we denote the vector of all entries but z, and by (a, p_{-z}) the vector p where the zth entry has been replaced by a.

Our goal in this lecture is the study of Jacobi's algorithm in order to solve

$$Q(p) = 0.$$

Note that the $X + Y$ equations are dependent, so we may drop one – say the one corresponding to y_0, and normalize the corresponding entry of $p_{y_0} = 0$.

1.2 Jacobi's algorithm

We consider quite generally a system of nonlinear equations of the form

$$Q(p) = 0$$

where $Q : \mathbb{R}^Z \to \mathbb{R}^Z$, with $|Z| = n$.

Economic interpretation. There are n goods, and p_z is the price for good z. The demand for good z is a function $D_z(p)$ which depends on all the prices. The supply for good z is similarly a function $S_z(p)$. We have

$$Q_z(p) = S_z(p) - D_z(p)$$

which is interpreted as the excess supply for good z.

Assume p^t is a subsolution of $Q = 0$, where we have defined:

Definition 1.1. *We say* p *is a subsolution of* $Q = 0$ *when*

$$Q(p) \leq 0;$$

we say it is a supersolution when

$$Q(p) \geq 0$$

and a solution when it is both a super and a subsolution, i.e. when $Q(p) = 0$.

Broadly speaking, we expect to intepret a subsolution as “prices are too low” and a supersolution as “prices are too high”. More on this later.

Recall Jacobi's algorithm: take an initial guess p^0 and set

$$\begin{aligned} p_x^{t+1} &: \; Q_x\left(p_x^{t+1}, p_{-x}^t\right) = 0 \\ p_y^{t+1} &: \; Q_y\left(p_y^{t+1}, p_{-y}^t\right) = 0 \end{aligned}$$

The relationship between p^t and p^{t+1} is called the coordinate update operator and is denoted

$$p^{t+1} = T\left(p^t\right).$$

Economic interpretation. Each good has an auctioneer. At each period, the auctioneer in charge of good z sets the price for good z in order to clear that market as if the other prices did not change. Re-iterate at every period.

Several questions arise about the next iterate p^{t+1} and the sequence $\left(p^t\right)$:

* is p^{t+1} larger than p^t?

* is p^{t+1} a subsolution?

* does the sequence $\left(p^t\right)$ converge towards a solution?

We will introduce the relevant assumptions on Q to answer these questions affirmatively.

Throughout we shall assume:

Continuity assumption: Q is continuous.

1.2.1 Increasing iterates

Assume p^t is a subsolution, that is

$$Q_z\left(p_z^t, p_{-z}^t\right) \leq 0$$

and assume p^{t+1} exists so that

$$Q_z\left(p_z^{t+1}, p_{-z}^t\right) = 0.$$

In order to have $p_z^t \leq p_z^{t+1}$, one needs to have:

Diagonal isotonicity assumption: $Q_z\left(., p_{-z}\right)$ increasing.

This allows us to redefine Jacobi as

$$p_z^t = \inf\left\{\pi : Q_z\left(\pi, p_{-z}^t\right) \geq 0\right\}$$

where the maximum is taken in $\mathbb{R} \cup \{+\infty\}$, with the convention $\inf \emptyset = +\infty$.

Proposition 1.1. *Assume Q is continuous and diagonal isotone, if p^t is a subsolution, then $p^t \leq p^{t+1}$.*

1.2.2 Stable subsolution

We have

$$\begin{aligned} Q_z\left(p_z^t, p_{-z}^t\right) &\leq 0 \\ Q_z\left(p_z^{t+1}, p_{-z}^t\right) &= 0 \end{aligned}$$

therefore we see that if we want that p^{t+1} should be a subsolution, then one should have:

Definition 1.2. *One says that Q is a Z-function when*

$$Q_z\left(p_z, p_{-z}\right) \text{ is antitone with respect to } \left(p_{-z}\right).$$

Alternatively, one says Q has the gross susbstitutes property.

Proposition 1.2. *Assume Q is a continuous and diagonal isotone Z-function. Then if p^t is a subsolution, $p^t \leq p^{t+1}$ and p^{t+1} is also a subsolution.*

Is this sufficient for a solution to exist? No, as the Jacobi sequence may diverge. Consider the following example.

Example 1.1. *Consider a linear $Q(p)$ given by*

$$Q(p) = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix} p$$

We see that the Jacobi sequence can be written explicitely as

$$\begin{aligned} p_1^t &= 2^t p_2 \\ p_2^t &= 2^t p_1 \end{aligned}$$

and $p^0 = (1, 1)$ is a subsolution but the Jacobi sequence does not converge to the solution.

In fact, any Jacobi sequence starting from a subsolution other than the solution $(0, 0)$. will diverge – or converge to $(+\infty, +\infty)$.

1.2.3 Existence of solution, part 1

Proposition 1.3. *Assume Q is a continuous and diagonal isotone Z-function. Consider a Jacobi sequence p^t starting from a subsolution p^0. Then if p^t is bounded above,* *it converges toward a solution p^* of $Q = 0$. Moreover, the set of solutions of $Q = 0$ is a lattice and p^* is its lattice lower bound.*

Proof. If p^t is bounded above, then it converges towards p^*. We have $Q_z\left(p_z^{t+1}, p_{-z}^t\right) = 0$ so by continuity, $Q(p^*) = 0$.

Assume $Q(p) = Q(p') = 0$. Note that $Q(p \wedge p') \geq 0 \geq Q(p \vee p')$. We need to show that there is a $p \sqcap p'$ such that $Q(p \sqcap p') = 0$ and for any p'' such that $Q(p'') = 0$, and $p'' \leq p$ and $p'' \leq p'$, we have $p'' \leq p \sqcap p'$. $p \sqcap p'$ is obtained as the limit of the Jacobi sequence starting from $p \wedge p'$, thus $p \sqcap p' \leq p \wedge p'$ in general. Similarly, $p \sqcup p'$ is obtained as the limit of the Jacobi sequence starting from $p \vee p'$. ■

1.2.4 M0- and M-functions

The previous example shows us what went wrong: in that example, the mapping $Q(.)$ allowed for an inversion by allowing the subsolution $(1,1)$ to be above the solution $(0,0)$.Recall, we would expect subsolution to mean "prices are too low" and supersolutions "prices are too high". Economically, we would not want the substitution effect to be stronger than the price effect. We should thus rule out such inversions. The following notions are defined in [GSV].

Definition 1.3. *Consider $Q : \mathbb{R}^n \to \mathbb{R}^n$. We introduce the following notions:*

(i) Q is weakly point-nonreversing if

$$\left\{ \begin{array}{c} Q(p) \leq Q(p') \\ p \geq p' \end{array} \right. \quad \textit{implies } Q(p) = Q\left(p'\right).$$

(ii) Q is strongly point-nonreversing if

$$\left\{ \begin{array}{c} Q(p) \leq Q(p') \\ p \geq p' \end{array} \right. \quad \textit{implies } p = p'.$$

Example 1.2. *Consider* $Q(p) = Qp$. *If* Q *is strictly diagonally dominant, then* Q *is strongly point-nonreversing.If it is weakly diagonally dominant, then* Q *is weakly point-nonreversing. Indeed, strict (resp. weak) diagonal dominance implies that* $1^\top Q \geq 0$ *(resp.* $1Q >> 0$*).*

Example 1.3. *Consider* Q *such that* $1^\top Q(p)$ *is weakly isotone in* p. *Then it is weakly point-nonreversing. In particular, this is the case of* $Q : \mathbb{R}^{X \cup Y} \to \mathbb{R}^{X \cup Y}$ *given by*

$$\begin{aligned} Q_x(p) &= \sum_y \exp(\Phi_{xy} + p_x - p_y) - n_x, x \in X \\ Q_y(p) &= -\sum_x \exp(\Phi_{xy} + p_x - p_y) + m_y, y \in Y. \end{aligned}$$

Example 1.4. *Consider* Q *such that* $1^\top Q(p) \leq 1^\top Q(p')$ *and* $p \geq p'$ *implies* $p = p'$. *Then it is strongly point-nonreversing. In particular, this is the case of*

$$Q : \mathbb{R}^{X \cup Y \backslash \{y_0\}} \to \mathbb{R}^{X \cup Y \backslash \{y_0\}}$$

given by

$$\begin{aligned} Q_x(p) &= \sum_y \exp(\Phi_{xy} + p_x - p_y) - n_x, x \in X \\ Q_y(p) &= -\sum_x \exp(\Phi_{xy} + p_x - p_y) + m_y, y \in Y \backslash \{y_0\}. \end{aligned}$$

We next move on to the definition of M- and M0-functions:

Definition 1.4. *A M-function is a Z-function which is strongly point-nonreversing. A M0-function is a Z-function which is weakly point nonreversing.*

We are able to state the following result obtained in [GSV]:

Theorem 1.1 (Inverse isotonicity theorems)**.** *Consider Q a Z-function. Then*

(i) Q is a M-function if and only if it is inverse isotone, i.e.

$$Q(p) \leq Q\left(p'\right) \text{ implies } p \leq p'.$$

(ii) Q is a M0 function if and only if the set valued map Q^{-1} is isotone in the strong set order, i.e.

$$Q(p) \leq Q\left(p'\right) \text{ implies } \begin{cases} Q(p) = Q(p \wedge p') \\ \text{and} \\ Q(p') = Q(p \vee p') \end{cases}$$

Proof. (i) To show the direct implication, assume Q is a M-function and $Q(p) \leq Q(p')$. Then we have

$$\begin{aligned} & Q_z\left(p \vee p'\right) \\ \leq\ & \mathbf{1}\left\{p_z \leq p'_z\right\} Q\left(p'\right) + \mathbf{1}\left\{p_z > p'_z\right\} Q(p) \\ \leq\ & Q\left(p'\right) \end{aligned}$$

and therefore we conclude that $p' = p \vee p'$, and thus $p \leq p'$. Conversely, assume Q is inverse isotone, and show it is strongly point-nonreversing. Assume $Q(p) \leq Q(p')$ and $p \geq p'$. By inverse isotonicity, one has $p \leq p'$, and hence $p = p'$.

(ii) To show the direct implication, assume Q is a M0-function and $Q(p) \leq Q(p')$. Then we have

$$\begin{aligned} & Q_z\left(p \vee p'\right) \\ \leq\ & \mathbf{1}\left\{p_z \leq p'_z\right\} Q\left(p'\right) + \mathbf{1}\left\{p_z > p'_z\right\} Q(p) \\ \leq\ & Q\left(p'\right) \end{aligned}$$

and therefore we conclude that $Q(p \vee p') = Q(p')$. Similarly, we have

$$\begin{aligned} & Q(p) \\ \leq\ & 1\left\{p_z \leq p'_z\right\} Q(p) + 1\left\{p_z > p'_z\right\} Q\left(p'\right) \\ \leq\ & Q\left(p'\right) \leq Q_z\left(p \wedge p'\right) \end{aligned}$$

and thus $Q(p \wedge p') = Q(p)$. Conversely, assume Q^{-1} is isotone in the strong set order, and show it is weakly point-nonreversing. Assume $Q(p) \leq Q(p')$ and $p \geq p'$. By inverse isotonicity, one has $Q(p) = Q(p \wedge p') = Q(p')$. ■

1.2.5 Zeros of M0-functions

The following result is obtained in [GL]:

Theorem 1.2. *Assume that Q is a continuous M0-map, that a subsolution exists and that a supersolution exists. The set of solutions is a nonempty sublattice of $\mathbb{R}^n$.*

First, let's show that if the set of solutions exists, then it is a sublattice of $\mathbb{R}^n$. We have $Q(p) = Q(p') = 0$, which implies $Q(p \wedge p') \geq 0$ and $Q(p \vee p') \leq 0$, but by weak point-nonreversingness one has $Q(p \wedge p') = Q(p \vee p') = 0$.

If a subsolution $\check{p}$ and a supersolution $\hat{p}$ exist, then because Q is a M0-map and $Q(\check{p}) \leq Q(\hat{p})$, one has $Q(\check{p} \wedge \hat{p}) = Q(\check{p})$ and $Q(\check{p} \vee \hat{p}) = Q(\hat{p})$. Consider $\underline{\mathrm{p}}^t$ and $\bar{p}^t$ the Jacobi sequences starting respectively from $\underline{\mathrm{p}}^0 = \check{p} \wedge \hat{p}$ and $\bar{p}^0 = \check{p} \vee \hat{p}$. As one has $\underline{\mathrm{p}}^0 \leq \bar{p}^0$, it follows that $\underline{\mathrm{p}}^t \leq \bar{p}^t$ and therefore both sequences are bounded. Hence a solution exists.

1.2.6 Zeros of M-functions

The following result appears in Ortega and Rheinboldt.

Theorem 1.3. *Assume that Q is a continuous M-map, that a subsolution exists and that a supersolution exists. A solution exists and is unique.*

Proof. The proof consists in considering two Jacobi sequences $\hat{p}^t$ and $\check{p}^t$ starting from respectively a sub and a supersolution. $\hat{p}^t$ is a increasing subsolution, $\check{p}^t$ is a decreasing supersolution, and by inverse isotonicty, one has $\hat{p}^t \leq \check{p}^t$, and therefore they both converge to respectively p and p'. $Q(p) = Q(p') = 0$ and inverse isotonicity applied twice yields $p = p'$. ∎

The following result appears in Ortega and Rheinboldt.

Theorem 1.4. *Assume that Q is a surjective continuous M-map. Then any Jacobi sequence converges to the unique solution to $Q = 0$.*

Proof. Let p^t be the Jacobi sequence starting from p^0 and consider $\hat{p}^0 = Q^{-1}\left(Q\left(p^0\right) \wedge 0\right)$ and $\check{p}^0 = Q^{-1}\left(Q\left(p^0\right) \vee 0\right)$. Then $\hat{p}^0$ is a subsolution, $\check{p}^0$ is a supersolution, and the Jacobi sequences $\hat{p}^t$ and $\check{p}^t$ starting from these respective vectors are such that

$$\hat{p}^t \leq p^t \leq \check{p}^t$$

and as a result, the limit of these three sequences coincide. ∎

1.3 Constant aggregate

Definition 1.5. *Q has constant aggregates if $1^\top Q(p) = \sum_z Q_z(p)$ is constant for all p.*

Note that this holds in the case of regularized optimal transport. Indeed, in that

$$\begin{aligned}
Q_x(p) &= \sum_y \exp(\Phi_{xy} + p_x - p_y) - n_x \\
Q_y(p) &= -\sum_x \exp(\Phi_{xy} + p_x - p_y) + m_y
\end{aligned}$$

and $\sum_z Q_z(p) = \sum_y m_y - \sum_x n_x$.

Proposition 1.4. *Assume Q is a Z-function such that $1^\top Q(p) = 0$, and that there is a $0 \in Z$ such that the restriction and corestriction of Q to $\mathbb{R}^{Z\backslash\{0\}}$ with the normalization $p_0 = \pi$ is a surjective M-function. Then, denoting $p(\pi)$ the solution to $Q = 0$, one has $\pi \leq \pi'$ implies $p(\pi) \leq p(\pi')$.*

Consider $Q(p) = \mathbf{\Delta} p - Ap$ where $\mathbf{\Delta}$ is a diagonal matrix with positive coefficients A has a zero diagonal and nonnegative coefficients, and

$$1^\top Q = 0.$$

Letting $\delta = \mathbf{\Delta} 1$, this implies $\delta^\top = \delta^\top \mathbf{\Delta}^{-1} A$, that is δ is the left Perron eigenvector associated to Perron eigenvalue 1. Thus there is a right Perron eigenvector $\lambda \geq 0$ such that $Q\lambda = 0$.

1.4 Existence of the regularized optimal transport problem

The reference for this section is [CGLW]. Consider solving

$$Q(p) = 0$$

where

$$Q : \mathbb{R}^{X\cup Y\backslash\{y_0\}} \to \mathbb{R}^{X\cup Y\backslash\{y_0\}}$$

given by

$$\begin{aligned} Q_x(p) &= \sum_y \exp\left(\Phi_{xy} + p_x - p_y\right) - n_x, x \in X \\ Q_y(p) &= -\sum_x \exp\left(\Phi_{xy} + p_x - p_y\right) + m_y, y \in Y\backslash\{y_0\}, \end{aligned}$$

and where we have normalized

$$p_{y_0} = \pi.$$

Step 1. Look for a supersolution. We have

$$\begin{aligned} Q_x(p) &= \sum_y \exp\left(\Phi_{xy} + p_x - p_y\right) - n_x \\ &\geq \exp\left(\Phi_{x0} + p_x - \pi\right) - n_x \end{aligned}$$

so for $\bar{p}_x$ such that

$$\exp\left(\Phi_{x0} + \bar{p}_x - \pi\right) = n_x$$

we have $Q_x(\bar{p}_x, p_y) \geq 0$ for all (p_y). Next, set $\bar{p}_y$ such that

$$\sum_x \exp\left(\Phi_{xy} + \bar{p}_x - \bar{p}_y\right) = m_y$$

and therefore we have

$$\begin{aligned} Q_x\left(\bar{p}\right) &\geq 0 \\ Q_y\left(\bar{p}\right) &= 0 \end{aligned}$$

Step 2. Q is a M-map, and thus the Jacobi sequence starting from $p^0 = \bar{p}$ is a decreasing supersolution. Hence, if is either convergent to a solution, or unbounded. However, because p^t is a supersolution, one has

$$-Q_{y_0}\left(p\right) = \sum_z Q\left(p^t\right) \geq 0$$

and thus

$$\sum_x \exp\left(\Phi_{xy} + p_x^t - \pi\right) \geq m_y$$

and thus all the p_x^t cannot go to $-\infty$. Denote x^* one such element such that

$$p_{x^*}^t \to p_{x^*}^\infty > -\infty.$$

We have

$$\exp\left(\Phi_{x^*y} + p_{x^*} - p_y\right) \leq \sum_x \exp\left(\Phi_{xy} + p_x - p_y\right) \leq m_y$$

thus all p_y remain bounded. Finally, we have

$$\sum_y \exp\left(\Phi_{xy} + p_x - p_y\right) \geq n_x$$

and thus all the p_x remain bounded too.

1.5 A toy hedonic model

Consider a surge pricing priblem in an uber-like environment. We have partitioned the city in a finite number of locations (say, blocks).

$x \in X=$ location of the driver

$y \in Y=$location of the passenger

Assume $z \in Z$ is the pickup location.

Assume that for a drive at x, the cost of picking up at z is c_{xz}

if the price of the ride at z is p_z the utility if the driver is $p_z - c_{xz} + \sigma\varepsilon_z$, where the vector (ε_z) is random.

if the driver does not pickup anyone, the utility is normalized to ε_0.

Assume that $(\varepsilon_z) \sim Gumbel$ and is iid. Then the probability that a driver at x will demand a ride z is

$$\frac{\exp\left(\frac{p_z - c_{xz}}{\sigma}\right)}{1 + \sum_{z'} \exp\left(\frac{p_{z'} - c_{xz'}}{\sigma}\right)}$$

Now assume that there are n_x drivers in area x, and therefore the supply for rides z is

$$S_z(p) = \sum_{x \in X} n_x \frac{\exp\left(\frac{p_z - c_{xz}}{\sigma}\right)}{1 + \sum_{z' \in Z} \exp\left(\frac{p_{z'} - c_{xz'}}{\sigma}\right)}$$

Let's study the properties of $S(p)$. Do we have gross substitutes? Well, $S_z(p) =$

$$\sum_{x \in X} \frac{n_x}{\exp\left(\frac{-p_z + c_{xz}}{\sigma}\right) + \sum_{z' \neq z} \exp\left(\frac{p_{z'} - p_z + c_{xz} - c_{xz'}}{\sigma}\right) + 1}$$

and we immediately see that $S_z(p)$ is decreasing w.r.t. $p_{z'}$ (for $z' \neq z$), and increasing with respect to p_z.

Now. let's focus on demand. This is the same as before, except for the fact that utility of a passenger at y seeking a ride in a cell z is now

$$u_{yz} - p_z + \eta_z$$

where η is iid Gumbel. The induced demand is

$$D_z(p) = \sum_{y \in Y} \frac{m_y \exp(u_{yz} - p_z)}{1 + \sum_{z'} \exp\left(u_{yz'} - p_{z'}\right)}$$

and we see that $-D_z(p)$ is a Z-matrix, and therefore

$$Q_z(p) = S_z(p) - D_z(p)$$

also has the gross substitute property.

Note that when $p_z = c$ and $c \to +\infty$, $Q_z(p) \geq 0$; while when $p_z = c$, $c \to -\infty$ we have $Q_z(p) \leq 0$. As a result, there is a unique solution.

2 Lecture 2: models of matching with transfers

Reference

Reference for today's lecture include:

- Galichon, Kominers, Weber (2019). Costly Concessions: An Empirical Framework for Matching with Imperfectly Transferable Utility. *Journal of Political Economy*.

2.1 Microfoundation of the matching model

Today we shall see models that formulate as

$$\begin{cases} \sum_{y\in\mathcal{Y}} M_{xy}\left(a_x, b_y\right) = n_x \\ \sum_{y\in\mathcal{Y}} M_{xy}\left(a_x, b_y\right) = m_y \end{cases}$$

where $M_{xy}\left(a_x, b_y\right)$ is continuous and increasing in a_x and b_y and will stand for the equilibrium number of matches between types x and y. Obviously, optimal transport falls into this category. We are actually going to consider a slight variant of the problem where we allow agents to remain unmatched, and consider

$$\begin{cases} \sum_{y\in\mathcal{Y}} M_{xy}\left(a_x, b_y\right) + M_{x0}\left(a_x\right) = n_x \\ \sum_{y\in\mathcal{Y}} M_{xy}\left(a_x, b_y\right) + M_{y0}\left(b_y\right) = m_y \end{cases}$$

where $M_{x0}\left(a_x\right)$ and $M_{y0}\left(b_y\right)$ will stand for the number of unmatched agents of types x and y.

Consider a matching model between workers (CEOs) and firms with (possibly) taxes. There are n_x workers of type

$x \in X$ and m_y firms of type $y \in Y$. Assume w_{xy} is now the gross wage paid by the firm to the worker.

If matched with firm y, a worker x gets utility

$$\alpha_{xy} + N\left(w_{xy}\right) + \sigma\varepsilon_y$$

where ε_y is Gumbel distributed and the ε_y's are independent, and if unmatched, a worker x gets utility

$$\sigma\varepsilon_0$$

As a result, the expected indirect utility of a worker of type x is

$$u_x = E\left[\max_{y \in Y}\left\{\alpha_{xy} + N\left(w_{xy}\right) + \sigma\varepsilon_y, \sigma\varepsilon_0\right\}\right]$$

where $N\left(w\right)$ is the net wage assuming the gross wage is w. Typically, N is

- increasing

- concave
- piecewise affine

In that case, we can represent $N(w)$ as

$$N(w) = \min_{k=1,\ldots,K} (1-\tau_k)(w-w_k)$$

where τ_k is the kth tax bracket: $\tau_1 = 0 < \tau_2 < \ldots < \tau_K$.

If there are no taxes, then we have simply $N(w) = w$.

Worker x's problem: choose a firm y such that

$$\max_{y \in Y} \{\alpha_{xy} + N(w_{xy}) + \sigma\varepsilon_y, \sigma\varepsilon_0\}$$

where α_{xy} is the monetary valuation of the job y's amenities. w_{xy} is the wage of worker x working for firm y;

determined at equilibrium. ε_y is the random utilitiy, assumed logit (Gumbel distributed). As a consequence, the average indirect utility of worker x is

$$u_x = \sigma \log \left(1 + \sum_y \exp\left(\frac{\alpha_{xy} + N\left(w_{xy}\right)}{\sigma}\right)\right)$$

and the probability that worker x picks firm y is

$$\begin{aligned} \frac{\mu_{xy}}{n_x} &= \frac{\exp\left(\frac{\alpha_{xy}+N\left(w_{xy}\right)}{\sigma}\right)}{1 + \sum_{y'} \exp\left(\frac{\alpha_{xy'}+N\left(w'_{xy}\right)}{\sigma}\right)} \\ &= \exp\left(\frac{\alpha_{xy} + N\left(w_{xy}\right) - u_x}{\sigma}\right). \end{aligned}$$

Firm's side:

$$\max_{x \in X} \left\{\gamma_{xy} - w_{xy} + \sigma \eta_x\right\}$$

and

$$v_y = \sigma \log \left(1 + \sum_x \exp\left(\frac{\gamma_{xy} - w_{xy}}{\sigma}\right)\right)$$

and the probability that firm y picks worker x is

$$\frac{\mu_{xy}}{m_y} = \mathsf{exp}\left(\frac{\gamma_{xy} - w_{xy} - v_y}{\sigma}\right)$$

To summarize, the equations of the model at this state are:

$$\begin{aligned}
\sum_{y \in Y} \mu_{xy} + \mu_{x0} &= n_x \\
\sum_{x \in X} \mu_{xy} + \mu_{0y} &= m_y \\
n_x \,\mathsf{exp}\left(\frac{\alpha_{xy} + N\left(w_{xy}\right) - u_x}{\sigma}\right) &= \mu_{xy} \\
m_y \,\mathsf{exp}\left(\frac{\gamma_{xy} - w_{xy} - v_y}{\sigma}\right) &= \mu_{xy} \\
n_x \,\mathsf{exp}\left(-u_x\right) &= \mu_{x0} \\
m_y \,\mathsf{exp}\left(-v_y\right) &= \mu_{0y}
\end{aligned}$$

Our strategy to solve it will consist of:

1. eliminate w_{xy} in order to express μ_{xy} as a function of u_x and v_y

2. Solve for u_x and v_y

3. Deduce μ_{xy} and w_{xy}.

2.2 Bargaining sets and distance function

The wage w_{xy} is a way to transfer systematic utility from one partner to the next. After transfer, the systematic utilities of both partners are

$$\begin{aligned} \text{for } x,\ U_{xy} &\leq \alpha_{xy} + N\left(w_{xy}\right) \\ \text{for } y,\ V_{xy} &\leq \gamma_{xy} - w_{xy} \end{aligned}$$

More generally, we will consider models in which $(U, V) \in \mathcal{F}_{xy}$ belong to a feasible set $\mathcal{F}_{xy}$ – we shall make assumptions on $\mathcal{F}_{xy}$.

We'll assume free disposal: if $(U, V) \in \mathcal{F}_{xy}$ then $(U', V') \in \mathcal{F}_{xy}$ as soon as $U' \leq U$ and $V' \leq V$.

We describe $\mathcal{F}_{xy}$ by the *distance-to-frontier function*. Introduce

$$D_{xy}(U,V) = \min_{t\in\mathbb{R}} \{t : (U-t, V-t) \in \mathcal{F}_{xy}\}$$

Therefore $D_{xy}(U,V) > 0$ iff $(U,V) \notin \mathcal{F}_{xy}$.

$D_{xy}(U,V) \leq 0$ iff $(U,V) \in \mathcal{F}_{xy}$

$D_{xy}(U,V) < 0$ iff (U,V) is not on the frontier of $\mathcal{F}_{xy}$.

Example 1. Transferable utility case.

$\mathcal{F}_{xy} = \{(U,V) : U+V = \Phi_{xy}\}$ we have

$$D_{xy}(u,v) = \frac{u+v-\Phi_{xy}}{2}$$

indeed, the minimum t such that $(u-t, v-t) \in \mathcal{F}_{xy}$ is such that $(u-t)+(v-t) = \Phi_{xy}$ that is $u+v-2t = \Phi_{xy}$ and thus $t = \frac{u+v-\Phi_{xy}}{2}$.

Example 2. Nonlinear taxes.

In this case $U_{xy} \leq \alpha_{xy} + N\left(w_{xy}\right)$ and $V_{xy} \leq \gamma_{xy} - w_{xy}$. We have

$$(U, V) \in \mathcal{F}_{xy} \text{ if and only } N\left(\gamma_{xy} - V\right) \geq U - \alpha_{xy}$$

that is in this case

$$\mathcal{F}_{xy} = \left\{(U, V) \in \mathbb{R}^2 : N\left(\gamma_{xy} - V\right) \geq U - \alpha_{xy}\right\}.$$

We have

$$\mathcal{F}_{xy} = \left\{(u, v) : N\left(\gamma_{xy} - v\right) \geq u - \alpha_{xy}\right\}.$$

where

$$N\left(w\right) = \min_{k=1,\ldots,K} \left(1-\tau_k\right)\left(w - w_k\right)$$

We have

$$\begin{aligned} \mathcal{F}_{xy} &= \left\{ (u,v) : \min_{k=1,\ldots,K} (1-\tau_k)\left(\gamma_{xy} - v - w_k\right) \geq u - \alpha_{x} \right. \\ &= \cap_k \left\{ (u,v) : (1-\tau_k)\left(\gamma_{xy} - v - w_k\right) \geq u - \alpha_{xy} \right\}. \end{aligned}$$

As the feasible set is an intersection, we need to understand how to compute the distance to the intersection of elementary sets. It turns out that we have

$$D_{F_1 \cap F_2} = \max\left\{ D_{F_1}, D_{F_2} \right\}$$

and

$$D_{F_1 \cup F_2} = \min\left\{ D_{F_1}, D_{F_2} \right\}.$$

Back to our taxation problem

$$D_{F_{xy}}(u,v) = \max_k D^k_{xy}(u,v)$$

where

$$D^k_{xy}(u,v) = \frac{u - \alpha_{xy} + (1-\tau_k)\left(v - \gamma_{xy} + w_k\right)}{2-\tau_k}.$$

Therefore

$$D_{xy}=\max_{k=1..K}\left\{\frac{u-\alpha_{xy}+(1-\tau_k)\left(v-\gamma_{xy}+w_k\right)}{2-\tau_k}\right\}.$$

Note that any distance-to-frontier function D is translation-invariant, that is

$$D(u+t,v+t)=t+D(u,v).$$

2.3 Matching functions

Back to the matching model. Recall that we had

$$\begin{aligned}\frac{\mu_{xy}}{n_x} &= \exp\left(\frac{U_{xy}-u_x}{\sigma}\right)\\ \frac{\mu_{xy}}{m_y} &= \exp\left(\frac{V_{xy}-v_y}{\sigma}\right)\end{aligned}$$

therefore

$$\begin{aligned} U_{xy} &= u_x + \sigma \ln \frac{\mu_{xy}}{n_x} \\ V_{xy} &= v_y + \sigma \ln \frac{\mu_{xy}}{m_y} \end{aligned}$$

and we have

$$D_{xy}\left(U_{xy}, V_{xy}\right) = 0$$

Replacing, we get

$$D_{xy}\left(u_x + \sigma \ln \frac{\mu_{xy}}{n_x}, v_y + \sigma \ln \frac{\mu_{xy}}{m_y}\right) = 0$$

Recalling that

$$D\left(u + t, v + t\right) = t + D\left(u, v\right),$$

this yields

$$\sigma \ln \mu_{xy} + D_{xy}\left(u_x - \sigma \ln n_x, v_y - \sigma \ln m_y\right) = 0$$

that is

$$\begin{aligned}
\mu_{xy} &= \exp\left(-\frac{D_{xy}\left(u_x - \sigma \ln n_x, v_y - \sigma \ln m_y\right)}{\sigma}\right), \\
\mu_{x0} &= \exp\left(-\frac{u_x - \sigma \ln n_x}{\sigma}\right) \\
\mu_{0y} &= \exp\left(-\frac{v_y - \sigma \ln m_y}{\sigma}\right)
\end{aligned}$$

and denoting $a_x = u_x - \sigma \ln n_x$ and $b_y = v_y - \sigma \ln m_y$ we obtain

$$\begin{aligned}
\mu_{xy} &= \exp\left(-\frac{D_{xy}\left(a_x, b_y\right)}{\sigma}\right) \\
\mu_{x0} &= \exp\left(-\frac{a_x}{\sigma}\right) \\
\mu_{0y} &= \exp\left(-\frac{b_y}{\sigma}\right)
\end{aligned}$$

2.4 Solving for the fixed effects

Now we just need to solve for the fixed effects (a_x, b_y), which we do using

$$\begin{aligned} \sum_y \mathsf{exp}\left(-\frac{D_{xy}(a_x, b_y)}{\sigma}\right) + \mathsf{exp}\left(-\frac{a_x}{\sigma}\right) &= n_x \\ \sum_x \mathsf{exp}\left(-\frac{D_{xy}(a_x, b_y)}{\sigma}\right) + \mathsf{exp}\left(-\frac{b_y}{\sigma}\right) &= m_y \end{aligned}$$

Introduce $p_x = a_x$, $p_y = -b_y$ and

$$\begin{aligned} Q_x(p) &= -\sum_y \mathsf{exp}\left(-\frac{D_{xy}(p_x, -p_y)}{\sigma}\right) + \mathsf{exp}\left(-\frac{p_x}{\sigma}\right) + n_x \\ Q_y(p) &= \sum_x \mathsf{exp}\left(-\frac{D_{xy}(p_x, -p_y)}{\sigma}\right) + \mathsf{exp}\left(\frac{p_y}{\sigma}\right) - m_y \end{aligned}$$

A comment about the optimzation structure. If this were to a gradient, one would have $Q_x(p) = \partial V(p) / \partial p_x$

and thus

$$\frac{\partial Q_x(p)}{\partial p_y} = \frac{\partial^2 V}{\partial p_x \partial p_y} = \frac{\partial Q_y(p)}{\partial p_x}$$

But we have

$$\begin{aligned}
\frac{\partial Q_x(p)}{\partial p_y} &= \exp\left(-\frac{D_{xy}(p_x, -p_y)}{\sigma}\right) \frac{\partial_{p_y} D_{xy}(p_x, -p_y)}{\sigma} \\
\frac{\partial Q_y(p)}{\partial p_x} &= \exp\left(-\frac{D_{xy}(p_x, -p_y)}{\sigma}\right) \frac{\partial_{p_x} D_{xy}(p_x, -p_y)}{\sigma}
\end{aligned}$$

which is not symmetric unless

$$\partial_{p_y} D_{xy}(p_x, -p_y) = \partial_{p_x} D_{xy}(p_x, -p_y),$$

which only happens in the optimal transport case, when $D_{xy}(p) = (p_x - p_y - \Phi_{xy})/2$.

3 Lecture 3: models of matching without transfers

Reference

Reference for this lecture include the following published material:

- [GS62] David Gale and Lloyd Shapley (1962). College Admissions and the Stability of Marriage. *American Mathematical Monthly*.

- [RS90] Alvin Roth and Marilda Sotomayor (1990). *Two-sided matching*. Econometric Society Monographs, Cambridge University Press.

- [GKW19] Galichon, Kominers, Weber (2019). Costly Concessions: An Empirical Framework for Matching with Imperfectly Transferable Utility. *Journal of Political Economy*.

- [A00] Adachi (2000). On a characterization of stable matchings. *Economics Letters*.

This lecture also significantly draws on current research with co-authors, in particular the following work in progress:

- [GL] Galichon and Léger (2022). Z-mappings.

- [GHS] Galichon, Hshieh and Sylvestre (2022). Monotone comparative statics for submodular functions, with an application to aggregated deferred acceptance.

3.1 Matchings without transfers: an introduction

We consider a labor market with fixed wages. Assume that $\mathcal{I}$ is the set of (individual) workers, and $\mathcal{J}$ is the set of (individual) firms. We model utilities in the following way:

- worker i matched with firm j gets utility α_{ij}
- firm j matched with worker i gets utility γ_{ij}
- unassigned individuals (workers or firms) get utility 0

We make an important assumption: **no indifference**, that is

$$\alpha_{ij} \neq \alpha_{ij'} \text{ for } j \neq j', \text{ and } \gamma_{ij} \neq \gamma_{i'j} \text{ for } i \neq i',$$

and

$$\alpha_{ij} \neq 0 \text{ and } \gamma_{ij} \neq 0.$$

We are looking at a one-to-one model of matching. A matching is denoted by $\mu_{ij} \in \{0, 1\}$ which is equal to 1 if i and j are matched, and zero otherwise. μ_{i0} and μ_{0j} are respectively equal to 1 if i and j are unassigned. The condition on μ to be feasible is therefore

$$\begin{cases} \sum_j \mu_{ij} + \mu_{i0} = 1 \\ \sum_i \mu_{ij} + \mu_{0j} = 1 \end{cases}.$$

A simple example. Consider two workers and one firm, and we assume that agents have a utility 1 if they are matched, and a utility zero if they are matched. We see that both workers want to be matched with the firm; however, the firm can only match with one worker. Unlike the models with transfers seen yesterday, there is no wage to clear the market. As a result, at least one worker will not get their first choice.

As seen in the previous example, we:

- either need to give up on the idea of equilibrium which implies that individuals get their first choice. In this case, we need a central planner to solve the market using an algorithm: this is Gale and Shapley's notion of a *stable matching*, which can be computed by the *deferred acceptance algorithm*. See [GS62] and [RS90].

- or introduce a numéraire to clear the market that cannot be transfered, like waiting times, which will be determined at equilibrium: this is the approach introduced in [GHS], which provides *equilibrium matchings*, which can be computed by the *deferred acceptance with Lagrange multipliers (DALM) algorithm*.

We will review both approaches in that order, and then study their relationships.

3.2 Stable matchings

Introduce u_i and v_j the respective payoffs that worker $i \in \mathcal{I}$ and firm $j \in \mathcal{J}$ obtain in the outcome of the game.

Note that if it turned out to be that there is a pair i and j for which $\alpha_{ij} > u_i$ and $\gamma_{ij} > v_j$, then it would mean that if they match together they can both obtain more than what is guaranteed to them in the outcome of the game: such outcome would then not be stable, and i and j would form a *blocking pair*. Similarly, if $0 > u_i$, then i would be better off remaining unmatched and getting utility 0 rather than taking the outcome payoff u_i. A similar result holds on the other side of the market, and we shall require as a result that

$$\begin{aligned}
&\max\left(u_i - \alpha_{ij}, v_j - \gamma_{ij}\right) \geq 0 \\
&u_i \geq 0, v_j \geq 0,
\end{aligned}$$

which are called *stability conditions*.

Finally, if i and j actually match, we expect u_i to be equal to α_{ij} and v_j to be equal to γ_{ij}. Similarly, if i or j respectively remains unmatched, then u_i or v_j should be zero. We call this a *strong complementarity condition*, for reasons that will become clear later on.

To recapitulate, a matching outcome $\left(\mu_{ij}, u_i, v_j\right)$ is *stable* if:

(i) μ is a feasible matching: $\mu_{ij} \in \{0, 1\}$, and

$$\left\{ \begin{array}{l} \sum_j \mu_{ij} + \mu_{i0} = 1 \\ \sum_i \mu_{ij} + \mu_{0j} = 1 \end{array} \right. .$$

(ii) stability conditions hold, that is

$$\begin{array}{l} \max\left(u_i - \alpha_{ij}, v_j - \gamma_{ij}\right) \geq 0 \\ u_i \geq 0 \text{ and } v_j \geq 0, \end{array}$$

(iii) strong complementarity holds, that is

$$\begin{array}{rcl} \mu_{ij} & > & 0 \implies u_i = \alpha_{ij}, v_j = \gamma_{ij} \\ \mu_{i0} & > & 0 \implies u_i = \alpha_{i0} \text{ and } \mu_{0j} > 0 \implies v_j = \gamma_{0j}. \end{array}$$

3.3 Deferred acceptance algorithm

The idea of Gale and Shapley's deferred acceptance algorithm is as follows. We keep track of a set of "available" offers that can be made by workers to firms; this is initially unconstrained, which means that any worker can initially make an offer to any firm. At each round, each worker makes an offer to their preferred firm among the set of those that are available to them. If a firm receives several offers, it keeps its favorite and rejects the others. Offers that have been rejected are no longer available. The algorithm iterates until no offer is rejected.

Formally, let's define:

$\mathcal{A}^t(i) \subseteq \mathcal{J} =$set of avaible firms to worker i at time t

$\mathcal{P}^t(i) \subseteq \mathcal{J} =$set of proposals made by worker i at time t [either a singleton or the empty set]

$\mathcal{K}^t(j) \subseteq \mathcal{I}$ =set of proposals kept by firm j at the end of round t [either a singleton or the empty set]

$t = 0$ all firms are avaiable to anyone $A^0(i) = \mathcal{J}$

Deferred acceptance algorithm ([GS62]).

Iterate over $t \geq 0$:

$\mathcal{P}^t(i) = \arg\max_j \left\{\alpha_{ij}, 0 : j \in \mathcal{A}^t(i)\right\}$, with the understanding that the $\arg\max$ is empty if 0 is maxizer.

$K^t(j) = \arg\max \left\{\gamma_{ij}, 0 : i \in \left(\mathcal{P}^t\right)^{-1}(j)\right\}$ where $\left(\mathcal{P}^t\right)^{-1}(j)$ is the set of i such that $j \in \mathcal{P}^t(i)$.

$$\mathcal{A}^{t+1}(i) = \mathcal{A}^t(i) \setminus \left\{\mathcal{P}^t(i) \setminus \left(\mathcal{K}^t\right)^{-1}(i)\right\}$$

until $\mathcal{A}^{t+1}(i) = \mathcal{A}^t(i)$.

Theorem. The DA algorithm converges to a stable matching.

Proof: deferred.

3.4 Adachi's reformulation

Adachi's view stems from a simple idea. As argued above, in a stable matching, individuals don't get their first choice, as there is no numéraire to clear the market. However, for each worker, one can define the *consideration set* as the set of firms that is willing to match with that worker; and in a stable matching, each worker is matched with their preferred firm among the consideration set; similarly on the other side of the market, each firm is matched with their preferred worker among its consideration set. Adachi's theorem states that this is actually a necessary and sufficient condition for a stable matching.

Theorem ([A00]). Consider $\left(u_i, v_j\right)$ vectors of payoffs. Then the following statements are equivalent:

(i) (μ, u, v) is stable

(ii) One has

$$\begin{aligned} u_i &= \max_j \left\{\alpha_{ij}, 0 : \gamma_{ij} \geq v_j\right\} \\ v_j &= \max_i \left\{\gamma_{ij}, 0 : \alpha_{ij} \geq u_i\right\} \end{aligned}$$

[where $\max_j \left\{\alpha_{ij}, 0 : \gamma_{ij} \geq v_j\right\}$ is a shortcut notation for $\max\left\{\max_j \left\{\alpha_{ij} : \gamma_{ij} \geq v_j\right\}, 0\right\}$]

Some remarks.

1. Yields a formulation of stable matchings as the fixed point of some operator – more on this shortly.

2. u is antitone with respect to v and v is antinone with respect to u.

3. The mapping $v \rightarrow u : u_i = \max_j \left\{\alpha_{ij}, 0 : \gamma_{ij} \geq v_j\right\}$ is reminiscent of $v_j^* = \max_j \left\{\Phi_{ij} - u_i\right\}$. However, $u = v^*$ and $v = u^*$ is not sufficient for (u, v) to be solution to the dual optimal transport problem.

Proof. For simplicity, we present the proof when $|\mathcal{I}| = |\mathcal{J}|$ and all α and γ are positive. Assume that (μ, u, v) is stable, and let's show (ii). By contradiction, assume (ii) does not hold, wlog

$$u_i \neq \max_j \left\{\alpha_{ij} : \gamma_{ij} \geq v_j\right\}$$

for some i. i is matched with $J(i)$, and $u_i = \alpha_{iJ(i)}$. We know that $v_{J(i)} = \gamma_{iJ(i)}$. As a result

$$\max_j \left\{\alpha_{ij} : \gamma_{ij} \geq v_j\right\} \geq \alpha_{iJ(i)} = u_i$$

thus we have $\max_j \left\{\alpha_{ij} : \gamma_{ij} \geq v_j\right\} > u_i$. Denote j^* the maximizer of $\max_j \left\{\alpha_{ij} : \gamma_{ij} \geq v_j\right\}$, we have $\alpha_{ij^*} = \max_j \left\{\alpha_{ij} : \gamma_{ij} \geq v_j\right\} > u_i$. We have $\gamma_{ij^*} \geq v_{j^*}$ and $\alpha_{ij^*} > u_i = \alpha_{iJ(i)}$ this is going to imply to $j^* \neq J(i)$

and $\gamma_{ij^*} > v_{j^*}$, therefore i, j^* is a blocking pair. Contradition, thus (ii) in fact holds.

Conversely, let's assume

$$
\begin{aligned}
u_i &= \max_j \left\{ \alpha_{ij}, 0 : \gamma_{ij} \geq v_j \right\} \\
v_j &= \max_i \left\{ \gamma_{ij}, 0 : \alpha_{ij} \geq u_i \right\}
\end{aligned}
$$

Define $\mu_{ij} = 1$ iff $u_i = \alpha_{ij}$. If $u_i = \alpha_{ij}$, then i is in the set of i's such that $\alpha_{ij} \geq u_i$, thus $v_j \geq \gamma_{ij}$.

But $u_i = \alpha_{ij}$, then j must be in the feasible set for i, and hence $\gamma_{ij} \geq v_j$. As a result, $\gamma_{ij} = v_j$.

Let us now show that (μ, u, v) is Gale-Shapley stable. Assume ij is a blocking pair. Then $\alpha_{ij} > u_i$ and $\gamma_{ij} > v_j$. But $\gamma_{ij} > v_j$ implies $\gamma_{ij} \geq v_j$, thus $u_i \geq \alpha_{ij}$, which is a contradiction.

3.5 Reformulation as a M0-map

[GL] build on Adachi's formulation in order to reformulate stability as the zeros of a M0-function.

We perform the change-of-sign trick: set $p_i = -u_i$ and $p_j = v_j$ and define Adachi's map T as

$$\begin{aligned} T_i(p) &= \min_j \left\{-\alpha_{ij}, 0 : \gamma_{ij} \geq p_j\right\} \\ T_j(p) &= \max_i \left\{\gamma_{ij}, 0 : p_i \geq -\alpha_{ij}\right\}. \end{aligned}$$

Clearly, T is an isotone map and (μ, u, v) is stable if and only if $p = (-u, v)$ satisfies the fixed point equation

$$p = T(p).$$

However, $p - T(p)$ is a Z-function but not a M0-function in general. Instead, we set

$$\begin{aligned} M_{ij}(p) &= 1\left\{p_i \geq -\alpha_{ij}, \gamma_{ij} \geq p_j\right\} \\ M_{i0}(p) &= 1\{p_i \geq 0\} \text{ and } M_{0j}(p) = 1\left\{0 \geq p_j\right\} \end{aligned}$$

(so that $M_{ij}(p) = 1\left\{\alpha_{ij} \geq u_i, \gamma_{ij} \geq v_j\right\}$), and we set the map Q such that

$$\begin{aligned} Q_i(p) &= \sum_{j \in \mathcal{J}} M_{ij}(p) + M_{i0}(p) - 1 \\ Q_j(p) &= -\sum_{j \in \mathcal{J}} M_{ij}(p) - M_{i0}(p) + 1 \end{aligned}$$

Theorem ([GL]). We have:

(i) The map Q is a M0-map.

(ii) The coordinate update map is Adachi's map T.

As a result, $T(p) = p$ if and only if $Q(p) = 0$, and thus the set of zeros of Q is a sublattice of $\mathbb{R}^n$. Hence:

Corollary (Conway). Given two stable outcomes (μ, u, v) and (μ', u', v'), we define

$$\begin{aligned} \left(\mu \wedge_{\mathcal{I}} \mu'\right)_{ij} &= 1\left\{u_i \leq u_i'\right\} \mu_{ij} + 1\left\{u_i > u_i'\right\} \mu_{ij}' \\ \left(\mu \vee_{\mathcal{I}} \mu'\right)_{ij} &= 1\left\{u_i > u_i'\right\} \mu_{ij} + 1\left\{u_i \leq u_i'\right\} \mu_{ij}' \end{aligned}$$

and $\mu \wedge_{\mathcal{I}} \mu'$ and $\mu \vee_{\mathcal{I}} \mu'$ are stable matching. Further, one has

$$\begin{aligned} \mu \wedge_{\mathcal{I}} \mu' &= \mu \vee_{\mathcal{J}} \mu' \\ \mu \vee_{\mathcal{I}} \mu' &= \mu \wedge_{\mathcal{J}} \mu'. \end{aligned}$$

In other words, i's and j's have *opposite interests*: what is better for i's is worse for j's and conversely.

As a result, the lattice of stable matching has an element which is preferred by all the i's and least liked by the j's; and an element which is preferred by all the j's and least liked by the i's.

3.6 Adachi's algorithm

Adachi's algorithm is nothing else than Gauss-Seidel's algorithm applied to finding the zeros of Q.

Start from p_i^0 small enough ie $p_i^0 = \min_j \left\{-\alpha_{ij}, 0\right\}$ and $p_j^0 = \min_i \left\{\gamma_{ij}, 0\right\}$.

Set

$$\begin{aligned} p_i^{t+1} &= \min_j \left\{-\alpha_{ij}, 0 : \gamma_{ij} \geq p_j^t\right\} \\ p_j^{t+1} &= \max_i \left\{\gamma_{ij}, 0 : p_i^{t+1} \geq -\alpha_{ij}\right\} \end{aligned}$$

until $p^{t+1} = p^t$ and then stop.

Note that if one starts from a subsolution, then Adachi's algorithm will give us the $\mathcal{I}$-preferred stable matching, and if one starts from a supersolution, it will return the $\mathcal{J}$-preferred stable matching.

3.7 Adachi vs deferred acceptance

Note that in deferred acceptance, at each period, u_i^t can only decrease by one step in the scale of the rankings induced by α_{ij}. Indeed, if the offer made by i is kept, then i keeps making the same offer; while if the offer made by i is turned down, then i will turn to their immediately next preferred option. This, however, may be inefficient, as it can induce i to make an offer to a firm j who already has a dominating offer, and which will turn down the offer.

A more efficient version of deferred acceptance consists in modifying Gale and Shapley's algorithm in order to induce i's whose offer have been turned down to move to the next j in their preference list which does not already have an offer better than i's. This is exactly what Adachi's algorithm does.

To formalize things, let us denote

$$
\begin{aligned}
N_i(p) &= \max_j \left\{ -\alpha_{ij} : -\alpha_{ij} < p_i \right\} \\
N_j(p) &= \max_j \left\{ \gamma_{ij} : \gamma_{ij} < p_j \right\}
\end{aligned}
$$

which look for the match just below the match implies by p for the i's, and the match just above the matched implied byu p for the j's.

Gale and Shapley's algorithm can be interpreted as

$$\begin{aligned} p_i^{2t+1} &= \min\left\{T_i\left(p^{2t}\right), N_i\left(p^{2t}\right)\right\} \\ p_j^{2t+2} &= T_i\left(p^{2t+1}\right) \end{aligned}$$

which is a "damped Gauss-Seidel" algorithm.

3.8 Equilibrium matchings

Just as stable matchings, equilibrium matchings require to rule out blocking pairs, so the stability conditons will be the same. The only difference is that we shall not require that if i and j are matched, then $u_i = \alpha_{ij}$ and $v_j = \gamma_{ij}$, but that $u_i \leq \alpha_{ij}$ and $v_j \leq \gamma_{ij}$ with one of these inequalities being strict. That is, we are fine with burning utility on one side of the market provided that we don't burn it on both sides. Hence the strong complementarity condition becomes a weak one, which requires that $\mu_{ij} > 0 \implies \max\left(u_i - \alpha_{ij}, v_j - \gamma_{ij}\right) = 0$.

A matching outcome $\left(\mu_{ij}, u_i, v_j\right)$ is thus an *equilibrium matching* if:

(i) μ is a feasible matching: $\mu_{ij} \in \{0, 1\}$, and

$$\left\{ \begin{array}{l} \sum_j \mu_{ij} + \mu_{i0} = 1 \\ \sum_i \mu_{ij} + \mu_{0j} = 1 \end{array} \right. .$$

(ii) stability conditions holds, that is

$$\begin{array}{l} \max\left(u_i - \alpha_{ij}, v_j - \gamma_{ij}\right) \geq 0 \\ u_i \geq 0 \text{ and } v_j \geq 0, \end{array}$$

(iii) weak complementarity holds, that is

$$\begin{array}{lcl} \mu_{ij} & > & 0 \implies \max\left(u_i - \alpha_{ij}, v_j - \gamma_{ij}\right) = 0 \\ \mu_{i0} & > & 0 \implies u_i = \alpha_{i0} \text{ and } \mu_{0j} > 0 \implies v_j = \gamma_{0j}. \end{array}$$

Of course, if a matching is stable, then it is an equilibrum matching. We have seen that stable matching exist, so why is the notion of an equilibrium matching useful? the reason is that one have a notion of aggregate equilibrium matching, but not of aggregate stable matching.

3.9 Aggregate equilinbrium matchings

Consider n_x identical workers and m_y identical firms.

What can be a stable matching?

Remark: Aggregate stable matchings don't exist. Indeed, assume a very simple example with 1 worker and 2 identical firms. A match between worker and firm is going to generate a utility of 1 both for the worker and the matched firm. Unmatched workers or firms get 0.

Gale-Shapley stable matching: one of the firms is matched with the worker.

Then one firm will get a utility of one, the unmatched firm will get utility zero.

The worker will get utility one.

In constrast, aggregate equilibrium matchings exist.

Definition. A matching outcome $\left(\mu_{xy}, u_x, v_y\right)$ is thus an *aggregate equilibrium matching* if:

(i) μ is a feasible matching: $\mu_{xy} \geq 0$, and

$$\left\{ \begin{array}{l} \sum_y \mu_{xy} + \mu_{x0} = 1 \\ \sum_y \mu_{xy} + \mu_{0y} = 1 \end{array} \right. .$$

(ii) stability conditions holds, that is

$$\begin{array}{l} \max\left(u_x - \alpha_{xy}, v_y - \gamma_{xy}\right) \geq 0 \\ u_x \geq 0 \text{ and } v_y \geq 0, \end{array}$$

(iii) weak complementarity holds, that is

$$\begin{array}{rcl} \mu_{xy} & > & 0 \implies \max\left(u_x - \alpha_{xy}, v_y - \gamma_{xy}\right) = 0 \\ \mu_{x0} & > & 0 \implies u_x = \alpha_{x0} \text{ and } \mu_{0y} > 0 \implies v_y = \gamma_{0y}. \end{array}$$

It is the same notion as (individual) equilibirum matching! We have just replaced the margins. This is part of the theory seen yesterday with distance function

$$D_{xy}(u_x, v_y) = \max\left(u_x - \alpha_{xy}, v_y - \gamma_{xy}\right).$$

3.10 DALM (Lagrangian Deferred Acceptance with Lagrange multipliers)

[GHS] consider an aggregate version of Gale-Shapley.

$\mu_{xy}^{A,t}$ is the number of positions of type y available to workers of type x at time t.

Initially, there is a maximum number of positions y available to workers x.

$\mu_{xy}^{A,0} = \min\{n_x, m_y\}$

At step t

Proposal phase:

$$\begin{aligned} \mu^{P,t} \in \arg\max_{\mu} \quad & \left\{ \sum_{xy} \mu_{xy} \alpha_{xy} \right\} \\ s.t. \quad & \sum_{y} \mu_{xy} \leq n_x \\ & \mu_{xy} \leq \mu_{xy}^{A,t} \end{aligned}$$

Disposal phase:

$$\begin{aligned} \mu^{K,t} \in \arg\max_{\mu} \quad & \left\{ \sum_{xy} \mu_{xy} \gamma_{xy} \right\} \\ s.t. \quad & \sum_{x} \mu_{xy} \leq m_y \\ & \mu_{xy} \leq \mu_{xy}^{P,t} \end{aligned}$$

Adjustment phase:

$$\mu_{xy}^{A,t+1} = \mu_{xy}^{A,t} - \left(\mu_{xy}^{P,t} - \mu_{xy}^{K,t} \right)$$

In the proposal and disposal phase, there are Lagrange multipliers $\tau_{xy}^{\alpha} \geq 0$ and $\tau_{xy}^{\gamma} \geq 0$ which are such that the perceived utility of workers and firms are respectively

$$\alpha_{xy} - \tau_{xy}^{\alpha} \text{ and } \gamma_{xy} - \tau_{xy}^{\gamma}$$

and we can show that τ_{xy}^{α} is increasing and τ_{xy}^{γ} is decreasing.

[GHS] prove the following result:

Theorem ([GHS]). Aggregate equilibrium matchings exist, and DALM returns one of them.

The proof relies on novel comparative statics for *exchangeable functions* – dual to Topkis' theory.

3.11 Aggregate equilibrium matching with logit heterogeneity

As a result, if we incoprorate random utilities $\alpha_{xy} + T\varepsilon_{iy}$ for a worker i of type x matched with a firm of type y and $\gamma_{xy} + T\eta_{xj}$ for a firm j of type y matched with a worker of type x, then the equilibrium in this model will be such that

$$M_{xy}\left(\mu_{x0}, \mu_{0y}\right) = \exp\left(-\frac{1}{T} D_{xy}\left(-T \ln \mu_{x0}, -T \ln \mu_{0y}\right)\right).$$

Theorefore one has

$$M_{xy}\left(\mu_{x0}, \mu_{0y}\right) = \min\left\{\mu_{x0} \exp\left(\frac{\alpha_{xy}}{T}\right), \mu_{0y} \exp\left(\frac{\gamma_{xy}}{T}\right)\right\}$$

and the μ_{x0} and μ_{0y}'s satisfy

$$\begin{aligned}
\mu_{x0} + \sum_y \min\left\{\mu_{x0} \exp\left(\frac{\alpha_{xy}}{T}\right), \mu_{0y} \exp\left(\frac{\gamma_{xy}}{T}\right)\right\} &= n_x \\
\mu_{0y} + \sum_x \min\left\{\mu_{x0} \exp\left(\frac{\alpha_{xy}}{T}\right), \mu_{0y} \exp\left(\frac{\gamma_{xy}}{T}\right)\right\} &= m_y
\end{aligned}$$

The general theory extends: one can reformulate this system as the inversion of a M-function. There exist a sub- and supersolution, and theefore the equilibrium exists and is unique.