Project 1: Cosmological models

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Analysis of cosmological models

The relationship that our dataset follows is

$$d_L(z) = c(1+z)\int_0^z \frac{dz'}{H(z')} \tag{1}$$

where d_L is the distance to an exploding star in Mpc, z is the measured redshift and $H(z)$ is the Hubble parameter.

$$H(z) = H_0\sqrt{\Omega_{M,0}(1+z)^3 + \Omega_{k,0}(1+z)^2 + \Omega_{\Lambda,0}} \equiv H_0 E(z)^{1/2} \tag{2}$$

For $z << 1$ a Taylor approximation can be used to simplify the relationship $d_L(z)$

$$E(z) \approx \Omega_{M,0}(1+3z) + \Omega_{k,0}(1+2z) + \Omega_{\Lambda,0} = 1 + 2z(q_0+1), \tag{3}$$

$$d_L(z) = c(1+z)\int_0^z \frac{dz'}{H(z')} \approx \frac{c}{H_0}\left(z + \frac{1}{2}(1-q_0)z^2 + \dots\right). \tag{4}$$

Task 1

The likelihood function used for the models is a multivariate normal distribution with a diagonal weight matrix to account for the heteroscedastic errors with $w_{ii} = 1/\sigma_i^2$ then normalized to $\sum_i w_{ii} = N_d$,

$$\begin{aligned} p(\mathcal{D}|\theta,\sigma^2) &= \mathcal{N}(\mathcal{D}|\theta,\sigma^2\mathbf{W}) \\ &= \frac{1}{(2\pi\sigma^2)^{N_d/2}|\mathbf{W}|^{-1/2}}\exp\left\{-\frac{1}{2}\frac{(\mathcal{D}-\mu(\theta))^T\mathbf{W}(\mathcal{D}-\mu(\theta))}{\sigma^2}\right\}. \end{aligned} \tag{5}$$

Here $\mu(\theta)$ represents our model with parameters $\theta = [H_0, q_0]$ based on Eq. 4 with two terms considered. The priors used are independent of each other and defined as an inverse gamma prior for σ^2, and uniform priors for H_0, q_0.

$$\pi(\sigma^2) = J\mathcal{G}(\sigma^2|\alpha_0,\beta_0) = \frac{\beta_0^{\alpha_0}}{\Gamma(\alpha_0)}\left(\frac{1}{\sigma^2}\right)^{\alpha_0+1}\exp\left\{-\frac{\beta_0}{\sigma^2}\right\}. \tag{6}$$

$$\pi(H_0) = \mathcal{U}(1\,000, 100\,000). \tag{7}$$

$$\pi(q_0) = \mathcal{U}(-10, 10). \tag{8}$$

Which means that our posterior is, neglecting the marginal distribution since it will not be relevant when sampling the posterior or when estimating the maximum a posteriori,

$$p(H_0, q_0, \sigma^2 \mid \mathcal{D}) = \mathcal{N}(\mathcal{D}|\Phi\theta, \sigma^2\mathbf{W})\pi(\sigma^2)\pi(H_0)\pi(q_0). \tag{9}$$

With a multivariate normal distribution for our likelihood function, uniform priors for H_0, q_0 and inverse gamma for σ^2 we can easily let `pymc` sample our posterior distribution. Due to the Taylor expansion used in Eq. 3 only data points with $z < 0.5$ will be regarded. In fig. 1 both posterior distributions and the joint posterior is shown, as expected from the $d_L(z)$ relationship H_0, q_0 are highly correlated which is visible in the joint posterior distribution. The most probable values of H_0, q_0 can also be extracted from the posterior distributions along with their respective variance. From the q_0 distribution we find that the expansion of the Universe is accelerating ($q_0 < 0$) because the probability of q_0 being positive would be very unlikely ($< 10^-9$) given that $q_0 = 0$ is > 6 standard deviations away from the mean assuming an approximate normal distribution for q_0.

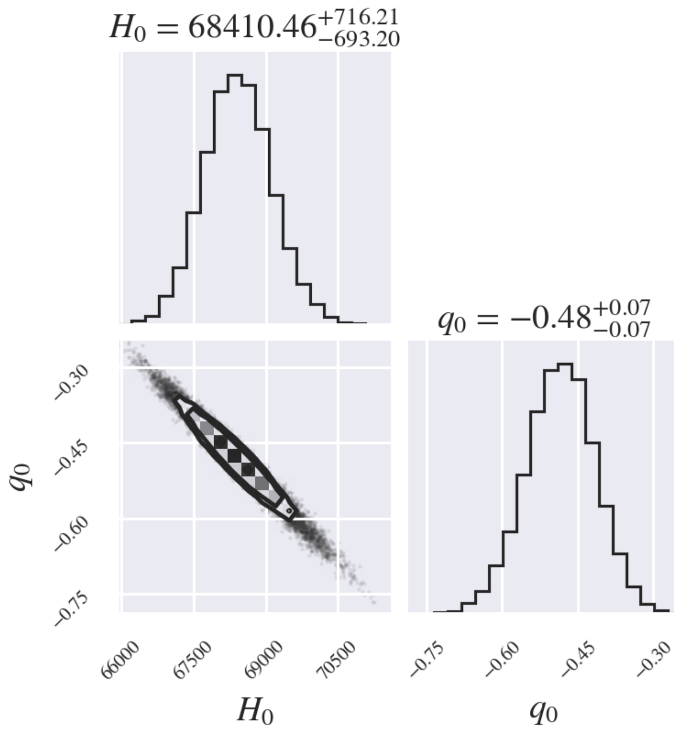

Figure 1: Posterior distribution of H_0, q_0, and their joint probability distribution.

From the MAP estimate of H_0 we find the most probable value is 68 410.5 km/s/Mpc and using a 90% credible interval, H_0 has 90% probability of being inside the range [67 217, 69 572] km/s/Mpc. In fig. 2 a model with θ_{MAP} is shown over entire of the data sets z values along with the data set to show that the model accurately fits to the data.

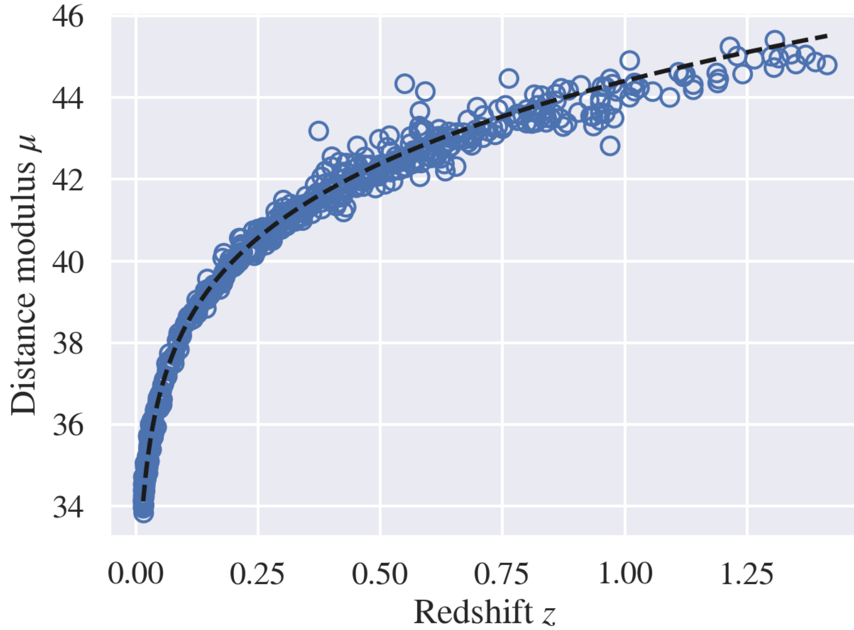

Figure 2: Posterior prediction of the distance modulus versus redshift and data points.

To improve the inference of H_0 and q_0, we could: adapt our model to be more complex with more parameters to achieve a better representation of the real world, e.g. adding more terms in the Taylor expansion. Could also use a more informative prior, changing the likelihood function to one that better represents the probability of observing the data given the model parameters. Finally, a larger dataset with less measurement error could also contribute to better inference of H_0, q_0.

Task 2

We will now do a comparison of the wCDM and ΛCDM models. The models will be defined as: for ΛCDM

$$E(z) = \Omega_{M,0}(1+z)^3 + \Omega_{\Lambda,0}, \tag{10}$$

and for w-CDM

$$E(z) = \Omega_{M,0}(1+z)^3 + \Omega_{\Lambda,0}(1+z)^{3(1+w)}. \tag{11}$$

To compare the two models we used AIC and BIC scores to quantify the models ability to fit to the dataset while also penalizing the model's complexity. The AIC score is defined as

$$\mathrm{AIC} = 2\ln(p(\mathcal{D} \mid \theta)) - 2N_d, \tag{12}$$

and BIC as

$$\mathrm{BIC} = 2\ln(p(\mathcal{D} \mid \theta)) - N_d \ln(N_p). \tag{13}$$

For the AIC, BIC scores to be evaluated the models need to be optimized to the dataset. For this we minimized the $-\ln(p(\mathcal{D} \mid \theta))$ using `scipy.optimize.minimize` with the Nelder-Mead solver (this solver mostly due to other solvers not converging), while assuming the scale to be known, not having it as a parameter. In our two models θ corresponds to $[\Omega_{M,0}, \Omega_{\Lambda,0}]$ for the ΛCDM model and for the wCDm model there is an additional parameter w. If the two models are optimized to within an acceptable tolerance the wCDM should at the very least be equivalent to ΛCDM if not better since the only difference between the models is that wCDM has an additional degree of freedom that can be optimized above that of ΛCDM. However, when penalizing the complexity it is no longer a given which model is favourable. From our evaluation, we would choose $wCDM$ model to explain to SCP 2.1 dataset due to it having (slightly) higher AIC, BIC scores.

	ΛCDM	wCDM
AIC	1.0	25.7
BIC	3.6	27.5

Table 1: AIC, BIC scores for ΛCDM, wCDM. The scores are normalized so that the lowest values goes to 1.0 for easier comparison.

From this we can see that the posterior distributions of $\Omega_{M,0}, \Omega_{\Lambda,0}$ in fig. 3 are close to the other Ω parameter approximations, $\Omega_{M,0} = 0.3158$[Boylan-Kolchin(2023)] and $\Omega_{\Lambda,0}$ is approximate to 0.68[(ESA)(2023)].

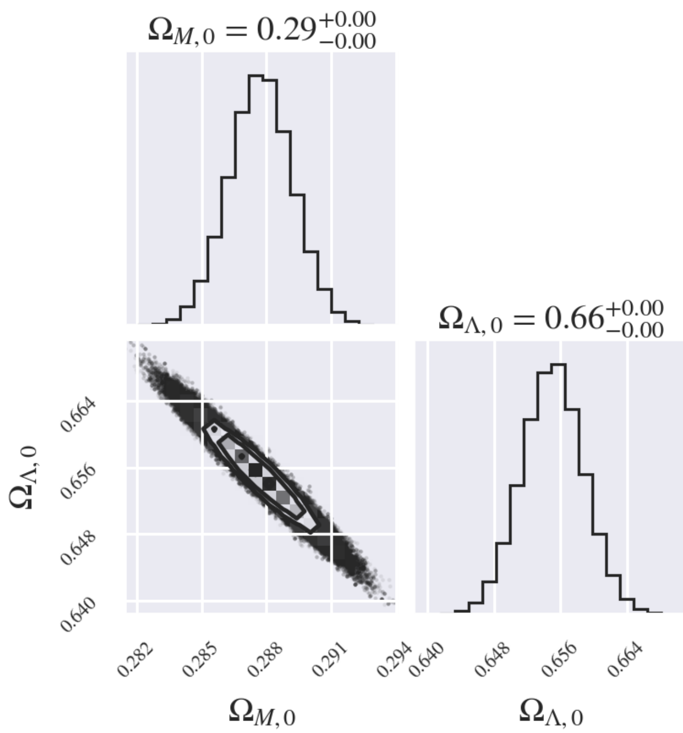

Figure 3: Posterior probability distribution of $\Omega_{M,0}$, $\Omega_{\Lambda,0}$, and the joint distribution

References

[Boylan-Kolchin(2023)] Michael Boylan-Kolchin. Stress testing {*upLambda*CDM with high-redshift galaxy candidates. *Nature Astronomy*, 7(6):731–735, apr 2023. doi: 10.1038/s41550-023-01937-7. URL https://doi.org/10.1038%2Fs41550-023-01937-7.

[(ESA)(2023)] European Space Agency (ESA). What is dark energy?, 2023. URL https://sci.esa.int/web/euclid/-/what-is-dark-energy#:~:text=Dark%20energy%20is%20an%20unidentified,energy%20density%20of%20the%20Universe. 11.13.2023.