香港科技大學
THE HONG KONG UNIVERSITY OF SCIENCE AND TECHNOLOGY

MFIT5008 Decision Analytics

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Group 5 presentation

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How IRR – what it is and how it helps investment decisioning

Internal Rate of Return (IRR) = the discount rate that translates the present value of all cashflow = 0

$$cashflow^0 + \frac{cashflow^1}{1+r} + \frac{cashflow^2}{(1+r)^2} + \cdots = 0$$

IRR

- Higher IRR = more desirable investment
- Minimum IRR = minimum acceptable rate of return
- In case of investment options with similar characteristics, invest in the projects with highest IRRs

How IRR – what it is and how it helps investment decisioning

Multiple IRRs happens when sign of cashflow changes more than once (e.g., cash outflow → cash inflow → cash outflow)

IRR = 0% and 10%

$$-10+\frac{21}{1+r}+\frac{-11}{(1+r)^2}=0$$

Conventional argument

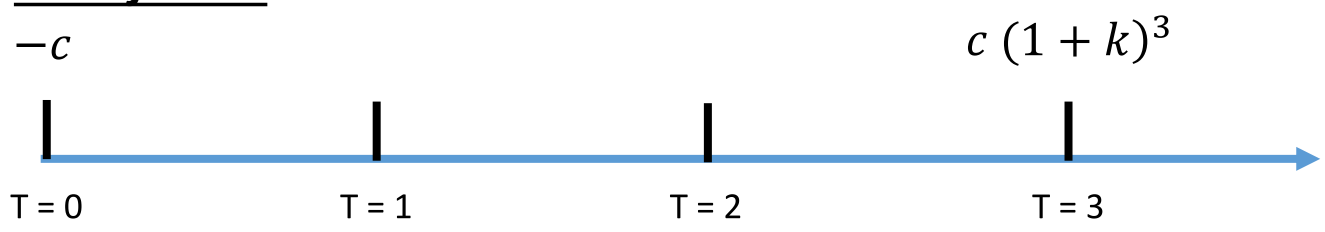
- Drop IRR and use other decision criteria instead e.g., Net Present Value

Author's view

- as long as we can determine if the underlying investment stream is net investment or net borrowing, investment decisioning would be same as NPV no matter which IRR we use

Constant per-period return on investment

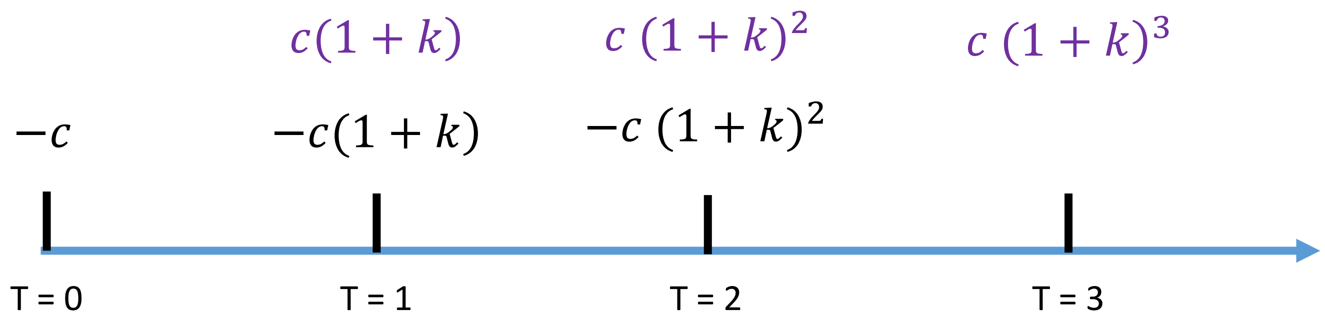
Cash flow 1



$$\mathrm{PV(x|r)} = \mathrm{c}\left(-1 + \frac{(1+k)^3}{(1+r)^3}\right)$$

- Rate of return for cash flow 1 is **constant value k per period**
- $\mathrm{PV(x|r)}$ is zero when r = k

Cash flow 2



- Consider cash flow 2 which has the **same net cash flow** as 1, it's more apparent that investors receive **constant rate of return k** on the prior period's investment

Constant per-period rate of return on an investment stream is k $\Longleftrightarrow$ k is the **internal rate of return** for the corresponding cash flow stream

Investment stream – definition & concept

- Define **investment stream c =** $(\boldsymbol{c_0}, \boldsymbol{c_1}, \dots, \boldsymbol{c_{T-1}})$ yielding **cash flow stream x =** $(\boldsymbol{x_0}, \boldsymbol{x_1}, \dots, \boldsymbol{x_T})$ at constant per-period rate of return k if the following holds for t = 1, 2, … , T-1

$$x_0 = -c_0$$

$$x_t = (1+k)c_{t-1} - c_t$$

$$x_T = (1+k)c_{T-1}$$

- ***<u>Theorem 1:</u>*** Relationship between $\mathrm{PV}(\mathrm{x}|\mathrm{r})$ and $\mathrm{PV}(\mathrm{c}|\mathrm{r})$ is given by $\mathbf{PV}(\mathbf{x}|\mathbf{r}) = \frac{\boldsymbol{k-r}}{\boldsymbol{1+r}}\,\mathbf{PV}(\mathbf{c}|\mathbf{r})$ where $\mathrm{r} \neq -1$
  - ➢ PV of a cash flow stream is the net investment times rate of return in excess of market rate (i.e. $k - r$), all discounted to the present

- ***<u>Theorem 2:</u>*** k is the internal rate of return for cash flow stream **x** if and only if there exists an investment stream **c** which yields **x** at constant per-period rate of return k
  - ➢ c = $(c_0, c_1, \dots, c_{T-1})$ can be easily solved one by one based on the defining equations as

$$
\begin{aligned}
c_0 &= -x_0\\
c_1 &= -\left((1+k)\cdot x_0 + x_1\right)\\
c_2 &= -\left((1+k)^2\, x_0 + (1+k)x_1 + x_2\right)\\
&\ \vdots\\
c_{T-1} &= -\left((1+k)^{T-1}\cdot x_0 + (1+k)^{T-2}\cdot x_1 + \cdots + (1+k)\cdot x_{T-2} + x_{T-1}\right)\\
c_{T-1} &= (1+k)^{-1}\cdot x_T.
\end{aligned}
$$

Investment stream – pure investment / borrowing, mixed investment & acceptance rule

- **Investment stream c =** $(\boldsymbol{c_0}, \boldsymbol{c_1}, \dots, \boldsymbol{c_{T-1}})$ is
  - **pure investment** if **c** ≥ 0 with at least one positive component
  - **pure borrowing** if **c** ≤ 0 with at least one negative component
  - **mixed investment** if **c** contains both positive & negative components

- ***Theorem 3:*** Suppose **x** is the yield of a pure investment / borrowing stream **c** at a proper internal rate of return k, then k is the only proper internal rate of return for **x**, and
  1) If **x** is a pure investment stream, then $\mathrm{PV}(\mathrm{x}|\mathrm{r}) \geq 0$ if and only if $\mathrm{k} \geq \mathrm{r}$
  2) If **x** is a borrowing investment stream, then $\mathrm{PV}(\mathrm{x}|\mathrm{r}) \geq 0$ if and only if $\mathrm{k} \leq \mathrm{r}$

- ***Theorem 4:*** Suppose k is an internal rate of return for cash flow stream **x**, and let **c** be investment stream yielding **x** at constant per-period rate of return k,
  1) If $\mathrm{PV}(\mathrm{c}|\mathrm{r}) > 0$ (i.e. c is net investment), then $\mathrm{PV}(\mathrm{x}|\mathrm{r}) \geq 0$ if and only if $\mathrm{k} \geq \mathrm{r}$
  2) If $\mathrm{PV}(\mathrm{c}|\mathrm{r}) < 0$ (i.e. c is net borrowing), then $\mathrm{PV}(\mathrm{x}|\mathrm{r}) \geq 0$ if and only if $\mathrm{k} \leq \mathrm{r}$
  3) If $\mathrm{PV}(\mathrm{c}|\mathrm{r}) = 0$, then $\mathrm{PV}(\mathrm{x}|\mathrm{r}) = 0$

- Theorem 4 is the acceptance rule which would never conflict for different internal rates k
  - not matter which internal rate k we use to determine whether we should accept / reject the project, as long as we examine whether **c** is net investment / borrowing and then compare k to r

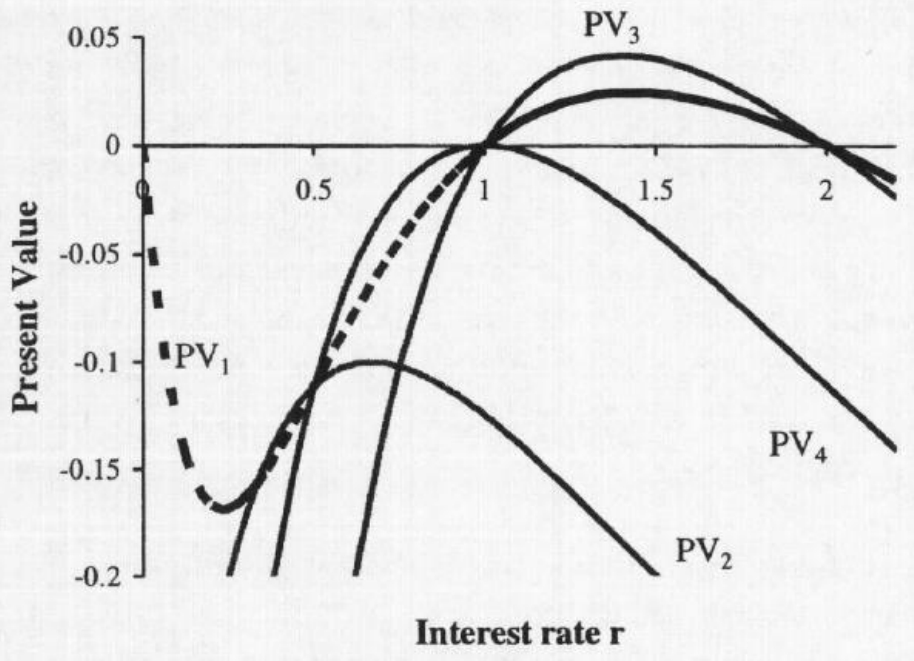
Traditional drawback in IRR evaluation

- Traditional drawback of using IRR to evaluate deterministic cash flow is the possibility of multiple IRR which are conflicting or misleading

➢ Take cash flow streams 1-4 in the right as example,
- 4 cash flow streams have negative present value at r = 10%
  ➢ all cash flows should be undesirable
- However, their internal rates of return are inadequate or misleading

Stream	IRR
1	0%, 100%, 200%
2	NA
3	100%, 200%
4	100%

	x_0	x_1	x_2	x_3	$PV@10\%$
Stream 1	−1	6	−11	6	−0.128
Stream 2	−1	3	−2.5		−0.339
Stream 3	−1	5	−6		−1.413
Stream 4	−1	4	−4		−0.669



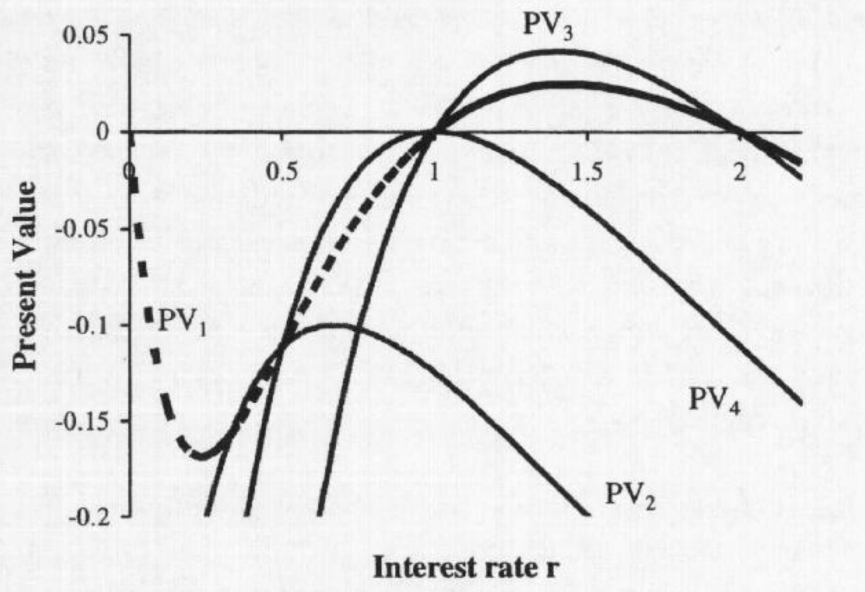
Author’s methodology to derive insight from IRR

1) Compute c_t for different values of k
2) Compute $\mathrm{PV}(\mathrm{c}|\mathrm{r})$ at r = 10%
3) Apply the acceptance rule according to Theorem 4

Take cash flow stream 1 with multiple IRR as example, the results are as follows:

k	c_0	c_1	c_2	$\mathrm{PV}(\mathrm{c}\|r)$
0%	1	-5	6	1.41
100%	1	-4	3	-0.157
200%	1	-3	2	-0.0744

	x_0	x_1	x_2	x_3	PV@10%
Stream 1	−1	6	−11	6	−0.128
Stream 2	−1	3	−2.5		−0.339
Stream 3	−1	5	−6		−1.413
Stream 4	−1	4	−4		−0.669



<u>Result & interpretation</u>

i. For k = 0%, $\mathrm{PV}(\mathrm{c}|\mathrm{r})$ = 1.41 > 0 (i.e. net investment) and k < r
ii. For k = 100%, $\mathrm{PV}(\mathrm{c}|\mathrm{r})$ = -0.157 < 0 (i.e. net borrowing) and k > r
iii. For k = 200%, $\mathrm{PV}(\mathrm{c}|\mathrm{r})$ = -0.0744 < 0 (i.e. net borrowing) and k > r

All 3 values of k lead to the same conclusion that $\mathrm{PV}(x|r) \leq 0$, so cash flow stream 1 is undesirable

Example: Mineral Extraction

1) Compute $\boldsymbol{c_t}$ for different values of k
2) Compute $\mathrm{PV}(\mathrm{c}|\mathrm{r})$ at r = 5% & r =12%
3) Apply the acceptance rule according to Theorem 4

	T_0	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8
Cashflow	-4	3	2.25	1.5	0.75	0	-0.75	-1.5	-2.25

k	c_0	c_1	c_2	c_3	c_4	c_5	c_6	c_7	$\mathrm{PV}(\mathrm{c}\|r=5\%)$	$\mathrm{PV}(\mathrm{c}\|r=12\%)$
10.4%	4	1.416	(0.687)	(2.258)	(3.243)	(3.580)	(3.203)	(2.036)	-6.53485	-3.52642
26.3%	4	2.052	0.342	(1.068)	(2.099)	(2.652)	(2.599)	(1.783)	-3.52642	0.382086

All 2 values of k & r = 5% lead to the same conclusion that $PV(x|r) \leq 0$, and k is higher than market rate

=> Cash flow stream is undesirable

All 2 values of k & r = 12%

k_1 = 10.4% : $PV(x|r) \leq 0$ / net borrowing

k_2 = 26.3% : $PV(x|r) \geq 0$ / net investment

k_1 is lower & k_2 is higher than market rate

=> Cash flow stream is desirable

Comparing Competing Projects

1) Compute c_t for 2 investments x & y at k of 28.3% & 16%
2) Compute $\mathrm{PV}(\mathrm{c}|\mathrm{r})$ at r = 10%
3) Compare NPV + Apply acceptance rule based on Theorem 4

t	0	1	2	3	4	5
x	−20	14	10	6	2	-2
y	−20	-6	1.1	8.2	15.3	22.4

	k	c_0	c_1	c_2	c_3	c_4	$\mathrm{PV}(\mathrm{c}\|r = 10\%)$	$\mathrm{PV}(t\|r = 10\%)$
Cx	28.3%	20	11.66	4.95978	0.363398	-1.53376	33.92444	5.623809
Cy	16%	4	2.052	0.342	(1.068)	(2.099)	109.2032	5.97407

Both are net investment & k is higher than market rate r

:both cashflow stream are desirable

Relative desirability is hard to define. Why?

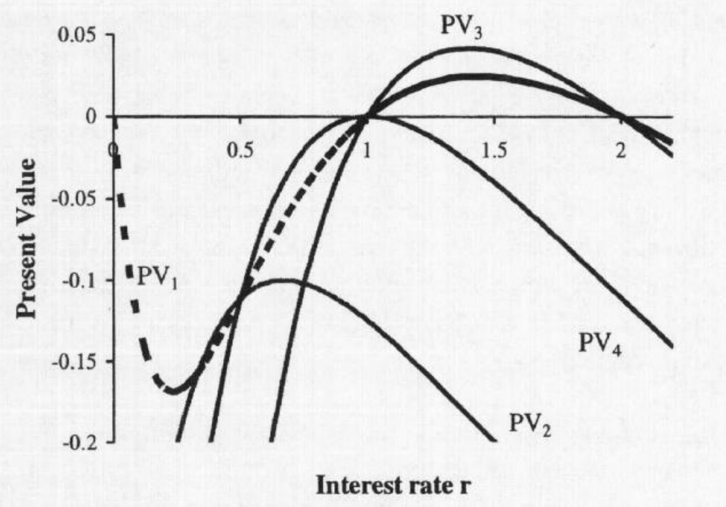
- NPV : CF y is higher
- k: CF x is higher
- Net investment (PV(c|r)) : x's net investment is 1/3 of y

Complex-valued Internal Rates of Return

Theorem 4 naturally extends to complex-valued IRR k

- ***Theorem 5:*** Suppose k is a (possibly complex-valued) internal rate of return x, and let **c** be the corresponding (possibly complex-valued) investment stream yielding **x** at ar return k,
  a. If $\mathrm{PV}(\mathrm{Re}(\mathrm{c})|\mathrm{r}) > 0$ (Re(c) is net investment), then $\mathrm{PV}(\mathrm{x}|\mathrm{r}) \geq 0$ if and only if $\mathrm{Re}(\mathrm{k}) \geq \mathrm{r}$
  b. If $\mathrm{PV}(\mathrm{Re}(\mathrm{c})|\mathrm{r}) < 0$ (Re(c) is net borrowing), then $\mathrm{PV}(\mathrm{x}|\mathrm{r}) \geq 0$ if and only if $\mathrm{Re}(\mathrm{k}) \leq \mathrm{r}$

	x_0	x_1	x_2	x_3	PV@10%
Stream 1	−1	6	−11	6	−0.128
Stream 2	−1	3	−2.5		−0.339
Stream 3	−1	5	−6		−1.413
Stream 4	−1	4	−4		−0.669



← Recall project 2

- $\mathrm{k} = 0.5 \pm 0.5\mathrm{i}$
- Complex-valued investment are C = (1, -1.5 $\pm$ 0.5i)
  - $\mathbf{PV(Re(c)}|\boldsymbol{r})$ is -0.364 < 0 => real part of k is net borrowing
  - real part of k is 0.5 which is higher than market rate

=> project 2 is undesirable

This is consistent with NPV being negative

Can still conclude and make economic recommendation

Real and Complex Internal Rates

	T_0	T_1	T_2	T_3	T_4	T_5
Cashflow	500	0	-1000	250	250	250

k	c_0	c_1	c_2	c_3	c_4	$PV(\mathrm{Re}(c)\|r)$
−1.618	−500	1309	−809.0	250	−404.5	−67.05
−1.149 + 0.603i	−500	1074 − 301.4i	22.075 + 692.4i	−670.7 − 89.565i	−96.365 − 391.0i	−74.82
−1.149 −0.603i	−500	1074 + 301.4i	22.075 − 692.4i	−670.7 + 89.565i	−96.365 + 391.0i	−74.82
0.618	−500	191.0	309.0	250	154.5	222.4
0.297	−500	351.4	455.8	341.3	192.7	584.3

Cashflow stream with multiple IRR consisting of real and complex IRR

Regardless of which, the real parts are all giving same conclusion

- First 3 are net borrowing
  - + Re(k) < r => desirable
- Bottom 2 are net investment
  - + Re(k) > r => desirable

Interpreting Multiple Internal Rates

k	c_0	c_1	c_2	c_3	c_4	$PV(\mathrm{Re}(c)\mid r)$
−1.618	−500	1309	−809.0	250	−404.5	−67.05
−1.149 + 0.603i	−500	1074 − 301.4i	22.075 + 692.4i	−670.7 − 89.565i	−96.365 − 391.0i	−74.82
−1.149 −0.603i	−500	1074 + 301.4i	22.075 − 692.4i	−670.7 + 89.565i	−96.365 + 391.0i	−74.82
0.618	−500	191.0	309.0	250	154.5	222.4
0.297	−500	351.4	455.8	341.3	192.7	584.3

- Is it meaningful to have different IRR k or Re(k) or compare between them?
- Is it meaningful to just compare IRR between mutually exclusive projects?

NO

For Same project, one IRR is higher just means corresponding net investment is smaller

All that matters is

Re(k) vs r (market rate)

Codes – how to generate multiple IRR and investment stream

- Use polyroot to solve for multiple IRR (Including complex roots)

```
z<-rev(temp)
irr<-polyroot(z)-1
```

- Convert cashflow X to investment stream C as defined

```
for (i in 1:n){
  for (j in 1:i){
      a[i,j]<- -(1+irr)^(i-j)
  }
}
c<-a%*%temp[1:n]
c
```

-1	0	0	0	0		x0		c0
-(1+r)	-1	0	0	0		x1		c1
-(1+r)^2	-(1+r)	-1	0	0	x	x2	=	c2
-(1+r)^3	-(1+r)^2	-(1+r)	-1	0		x3		c3
-(1+r)^4	-(1+r)^3	-(1+r)^2	-(1+r)	-1		x4		c4

- Compute PV of C at market rate and compare it with the sign of (IRR – Market rate)

```
for (i in 1:length(Ra)){
  StreamC[,i]<-ConvertX(case,times,Ra[i])
  PV[i]<-Re(StreamC[1,i])+NPV(0,Re(StreamC[-1,i]),c(1:(length(StreamC[,i])-1)),MR)
}
Rdiff<-Re(Ra)-MR
Final<-sign(Rdiff)*sign(PV)
```

Codes – our codes can generate same result as the papers

Test case 1 (example from paper, p.10)

	x_0	x_1	x_2	x_3	PV@10%
Stream 1	–1	6	–11	6	–0.128

k	c_0	c_1	c_2	PV @ 10%
0%	1	–5	6	1.41
100%	1	4	3	–0.157
200%	1	–3	2	–0.0744

Irr	PV	MR	RateDiff	Decision
0+0i	1.4132231	0.1	-0.1	-1
1+0i	-0.1570248	0.1	0.9	-1
2+0i	-0.0743802	0.1	1.9	-1

-We are able to replicate the result, same conclusion on all three IRR: sign(PV(c))*sign(Irr-MR)<0, not a desirable project

EXAMPLE: REAL AND COMPLEX INTERNAL RATES

The cash flow stream $\boldsymbol{x}$ = (500, 0, –1000, 250, 250, 250) is considered by Sullivan et al. (2000, Example 4-A-1, p. 186). It has five distinct internal rates of return, two of which are complex-valued. These rates k and their corresponding investment streams $\boldsymbol{c}$ are as follows:

r,

k	c_0	c_1	c_2	c_3	c_4	$PV(\mathrm{Re}(c)\|r)$
–1.618	–500	1309	–809.0	250	–404.5	–67.05
–1.149 + 0.603i	–500	1074 – 301.4i	22.075 + 692.4i	–670.7 – 89.565i	–96.365 – 391.0i	–74.82
–1.149 –0.603i	–500	1074 + 301.4i	22.075 – 692.4i	–670.7 + 89.565i	–96.365 + 391.0i	–74.82
0.618	–500	191.0	309.0	250	154.5	222.4
0.297	–500	351.4	455.8	341.3	192.7	584.3

Irr	PV	MR	RateDiff	Decision
-1.2908070+0.4411068i	-134.64536	0.1	-1.3908070	1
-1.2908070-0.4411068i	-134.64536	0.1	-1.3908070	1
0.0422429-0.3249235i	-109.29919	0.1	-0.0577571	1
0.0422429+0.3249235i	-109.29919	0.1	-0.0577571	1
-2.5028716+0.0000000i	-79.18282	0.1	-2.6028716	1

- Same conclusion on all five IRR: sign(PV(Re(c))*sign(Re(Irr)-MR)>0, It is a desirable project
- The table in the paper looks wrong (incorrect IRR)

Code – our codes can be generalized to non-consecutive cashflow

Test case 3

Our tool can access general cashflow stream, they need not to be consecutive.
Again, all IRRs would provide the same conclusion that the project is desirable.

	A	B
1	CF	Time
2	500	0
3	0	1
4	-1000	3
5	250	6
6	250	10
7	250	15
8		
9		

Irr	PV	MR	RateDiff	Decision
-0.5420694+0.7385287i	-33.95214	0.1	-0.6420694	1
-1.8096926+0.4385604i	-25.18955	0.1	-1.9096926	1
-1.4065403-0.8523512i	-25.46358	0.1	-1.5065403	1
-0.2692748-0.4400412i	-56.66830	0.1	-0.3692748	1
-1.0423753+0.9004707i	-27.34165	0.1	-1.1423753	1
-1.8191203+0.0000000i	-26.38775	0.1	-1.9191203	1