On the properties of mollified functions with applications to trajectory tracking and path following

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1. Introduction and problem statement

Most trajectory tracking or path following algorithms require the desired path to be at least twice continuously differentiable to guarantee global convergence. As a result, these methods typically exclude piecewise continuous linear paths or, more generally, continuous but non-differentiable paths. We present a method to regularize such non-differentiable paths via mollification.

Problem 1 (Regularization problem). *Let $f : \mathbb{R} \to \mathbb{R}^n$ be a (Borel) measurable function (trajectory) and $\{\varepsilon_i\}_{i=1}^n$ be a collection of positive real numbers called parameters. The problem is to find a functional T_{ε_i} that acts on each component of f and that satisfies*

1. *For $p \in \mathbb{N}$ with $p \geq 2$, $T_{\varepsilon_i}(f_i) \in C^p(\mathbb{R}, \mathbb{R})$ for $i \in \{1, \ldots, n\}$.*
2. *$T_{\varepsilon_i}(f_i) \to f_i$ as $\varepsilon_i \to 0$ in some sense of convergence.*
3. *$T_{\varepsilon_i}(f_i)$ is computationally simple to compute.*

2. Mollifiers

Definition 1 (Mollifier). *Let $\varphi \in C^\infty(\mathbb{R}^n, \mathbb{R})$ and for $\varepsilon > 0$ define $\varphi_\varepsilon := \frac{1}{\varepsilon}\varphi \circ \frac{\mathrm{id}}{\varepsilon}$. If φ satisfies the following properties it is called a* ***mollifier****.*

1. *$\operatorname{supp} \varphi$ is compact.*
2. *$\int_{\mathbb{R}^n} \varphi d\lambda_n = 1$, where λ_n is the Lebesgue measure.*
3. *For any bounded $f \in C(\mathbb{R}^n, \mathbb{R})$, $\lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} f \varphi_\varepsilon d\lambda_n = f(0)$.*

A typical example is the one dimensional bump function, where $c_1 > 0$ is a normalization constant.

$$\varphi(x) = \begin{cases} c_1 \exp\left(\frac{-1}{1-x^2}\right), & |x| < 1, \\ 0, & |x| \geq 1 \end{cases}, \tag{1}$$

Mollifiers solve problem 1 in each dimension via convolution.

Theorem 1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ such that $f \in L^p_{loc}(\mathbb{R}^n, \mathbb{R})$, then*

- *$(f * \varphi_\varepsilon) \in C^\infty(\mathbb{R}^n, \mathbb{R})$, and $(f * \varphi)^{(n)} = f * \varphi^{(n)}$.*
- *$f * \varphi_\varepsilon \to f$ pointwise a.e., as $\varepsilon \to 0$.*
- *If f is uniformly continuous, then $f * \varphi_\varepsilon \to f$ uniformly as $\varepsilon \to 0$.*
- *If f is continuous, then $f * \varphi_\varepsilon \to f$ as $\varepsilon \to 0$ on compact subsets of $\mathbb{R}^n$.*

What analytical and geometrical properties of $f \in L^p_{loc}(\mathbb{R}^n, \mathbb{R})$ are invariant under mollification? Our analytical contributions are presented in the next section.

3. Analytical results

Let $||\cdot|| : \mathbb{R}^n \to \mathbb{R}$ be a norm in $\mathbb{R}^n$, $B(x, r) = \{y \in \mathbb{R}^n \mid ||x - y|| < r\}$ and φ be a positive mollifier.

Proposition 1. *Let $f \in L^p_{loc}(\mathbb{R}^n, \mathbb{R})$ and suppose $\operatorname{supp} \varphi = \overline{B}(0, 1)$. Then, given $\varepsilon > 0$*

- *If f is convex, concave or quasiconvex, then $f * \varphi_\varepsilon$ is respectively, convex, concave or quasiconvex.*
- *If f is convex (resp. concave) then $f * \varphi_\varepsilon \geq f$ (resp. $f * \varphi_\varepsilon \leq f$).*

What happens if that properties only hold locally? Or $f : \mathbb{R} \to \mathbb{R}^n$?

Proposition 2.

- *If $f \in L^p_{loc}(\mathbb{R}^n, \mathbb{R})$ is convex (resp. concave) on $B(x, \delta)$ with $\delta > 0$, then for each $y \in B(x, \delta)$, $f * \varphi_\varepsilon$ is convex and $f * \varphi_\varepsilon \geq f$ (resp. concave and $f * \varphi_\varepsilon \leq f$) on $B(x, ||x - y||/2)$ for any $\varepsilon \in (0, ||x - y||/2]$.*
- *If $f : \mathbb{R} \to \mathbb{R}^n$ is (Borel) measurable then, given $\varepsilon > 0$:*

$$\{(f * \varphi_\varepsilon)(t) \mid t \in \operatorname{dom} f\} \subset \operatorname{convexHull}\{f(t) \mid t \in \operatorname{dom} f\}.$$

5. Further information

"We have beautiful proofs of our theorems, but there is not enough space in this margin to write them." Nevertheless you can find them, together with additional information in the given QR code or in `https://alfredofbw.github.io/PhDayFisicas2025/`.

4. Numerical and experimental results

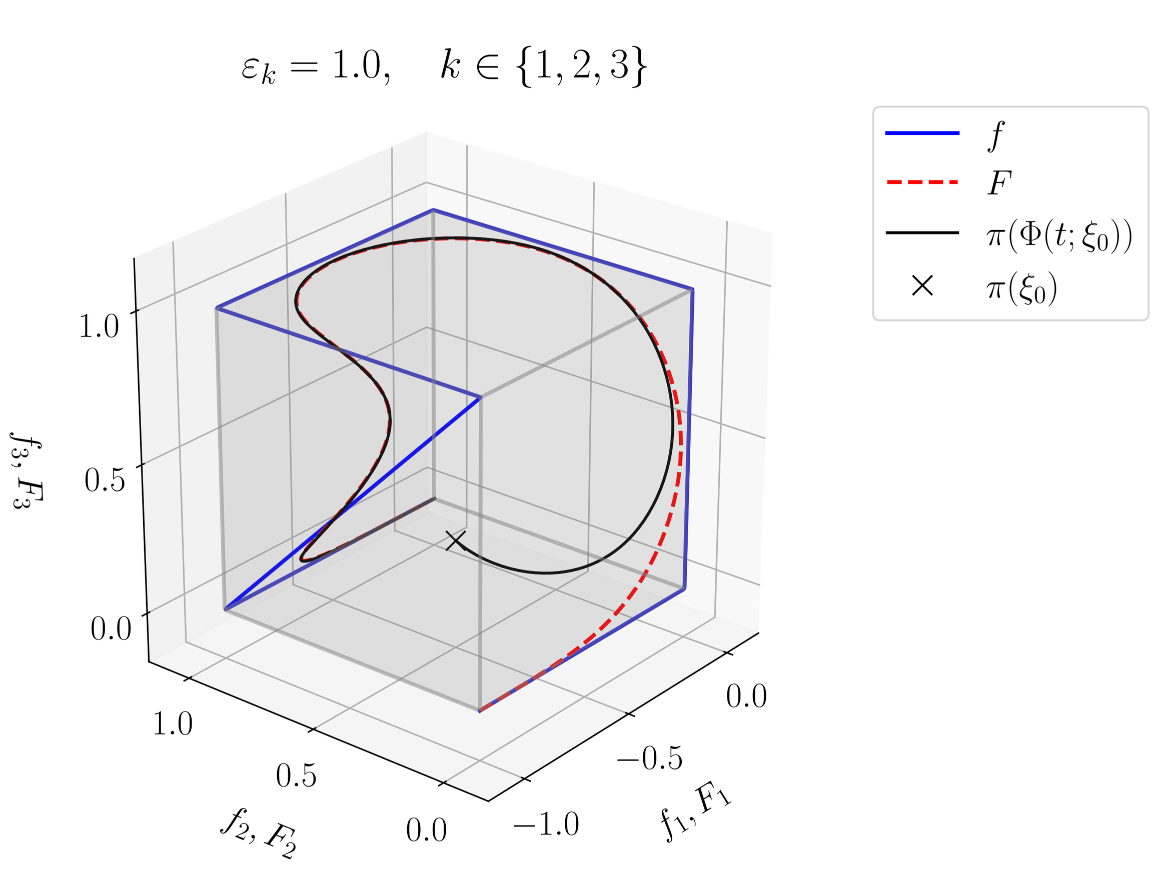

Figure 1: A three dimensional numerical simulation. The original path is shown in blue, the mollified path in red and the simulated dynamical system in black. Each dimension is mollified with parameter $\varepsilon_k = 1.0$.

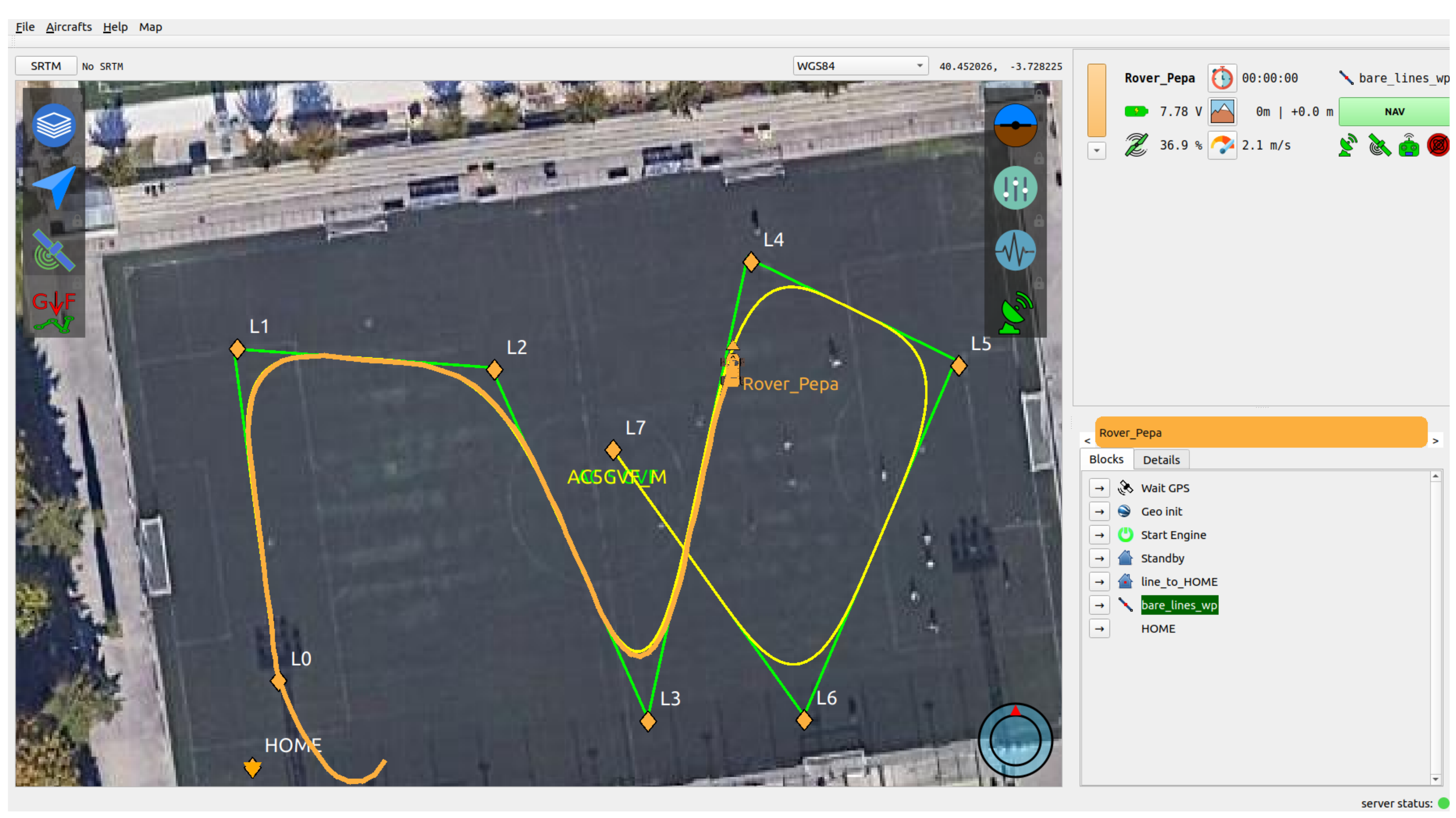

Figure 2: Capture of an experiment carried in Paraninfo's football field. The original trajectory is shown in green, the mollified one in yellow, and the actual trajectory of the vehicle in orange.