



# On the properties of mollified functions with applications to trajectory tracking and path following

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## 1. Introduction and problem statement

Most trajectory tracking or path following algorithms require the desired path to be at least twice continuously differentiable to guarantee global convergence. As a result, these methods typically exclude piecewise continuous linear paths or, more generally, continuous but non-differentiable paths. We present a method to regularize such non-differentiable paths via mollification.

**Problem 1** (Regularization problem). Let  $f : \mathbb{R} \rightarrow \mathbb{R}^n$  be a (Borel) measurable function (trajectory) and  $\{\varepsilon_i\}_{i=1}^n$  be a collection of positive real numbers called parameters. The problem is to find a functional  $T_{\varepsilon_i}$  that acts on each component of  $f$  and that satisfies

1. For  $p \in \mathbb{N}$  with  $p \geq 2$ ,  $T_{\varepsilon_i}(f_i) \in C^p(\mathbb{R}, \mathbb{R})$  for  $i \in \{1, \dots, n\}$ .
2.  $T_{\varepsilon_i}(f_i) \rightarrow f_i$  as  $\varepsilon_i \rightarrow 0$  in some sense of convergence.
3.  $T_{\varepsilon_i}(f_i)$  is computationally simple to compute.

## 2. Mollifiers

**Definition 1** (Mollifier). Let  $\varphi \in C^\infty(\mathbb{R}^n, \mathbb{R})$  and for  $\varepsilon > 0$  define  $\varphi_\varepsilon := \frac{1}{\varepsilon} \varphi \circ \frac{\text{id}}{\varepsilon}$ . If  $\varphi$  satisfies the following properties it is called a **mollifier**.

1.  $\text{supp } \varphi$  is compact.
2.  $\int_{\mathbb{R}^n} \varphi d\lambda_n = 1$ , where  $\lambda_n$  is the Lebesgue measure.
3. For any bounded  $f \in C(\mathbb{R}^n, \mathbb{R})$ ,  $\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^n} f \varphi_\varepsilon d\lambda_n = f(0)$ .

A typical example is the one dimensional bump function, where  $c_1 > 0$  is a normalization constant.

$$\varphi(x) = \begin{cases} c_1 \exp\left(\frac{-1}{1-x^2}\right), & |x| < 1, \\ 0, & |x| \geq 1 \end{cases}, \quad (1)$$

Mollifiers solve problem 1 in each dimension via convolution.

**Theorem 1.** Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  such that  $f \in L_{loc}^p(\mathbb{R}^n, \mathbb{R})$ , then

- $(f * \varphi_\varepsilon) \in C^\infty(\mathbb{R}^n, \mathbb{R})$ , and  $(f * \varphi)^{(n)} = f * \varphi^{(n)}$ .
- $f * \varphi_\varepsilon \rightarrow f$  pointwise a.e., as  $\varepsilon \rightarrow 0$ .
- If  $f$  is uniformly continuous, then  $f * \varphi_\varepsilon \rightarrow f$  uniformly as  $\varepsilon \rightarrow 0$ .
- If  $f$  is continuous, then  $f * \varphi_\varepsilon \rightarrow f$  as  $\varepsilon \rightarrow 0$  on compact subsets of  $\mathbb{R}^n$ .

What analytical and geometrical properties of  $f \in L_{loc}^p(\mathbb{R}^n, \mathbb{R})$  are invariant under mollification? Our analytical contributions are presented in the next section.

## 3. Analytical results

Let  $\|\cdot\| : \mathbb{R}^n \rightarrow \mathbb{R}$  be a norm in  $\mathbb{R}^n$ , and  $B(x, r) = \{y \in \mathbb{R}^n \mid \|x - y\| < r\}$ .

**Proposition 1.** Let  $f \in L_{loc}^p(\mathbb{R}^n, \mathbb{R})$  and  $\varphi$  be a positive mollifier with support  $\overline{B}(0, 1)$ . Then, given  $\varepsilon > 0$

- If  $f$  is convex, concave or quasiconvex, then  $f * \varphi_\varepsilon$  is respectively, convex, concave or quasiconvex.
- If  $f$  is affine linear, then  $f * \varphi_\varepsilon = f$ .
- If  $f$  is convex (resp. concave) then  $f * \varphi_\varepsilon \geq f$  (resp.  $f * \varphi_\varepsilon \leq f$ ).

What happens if that properties only hold locally?

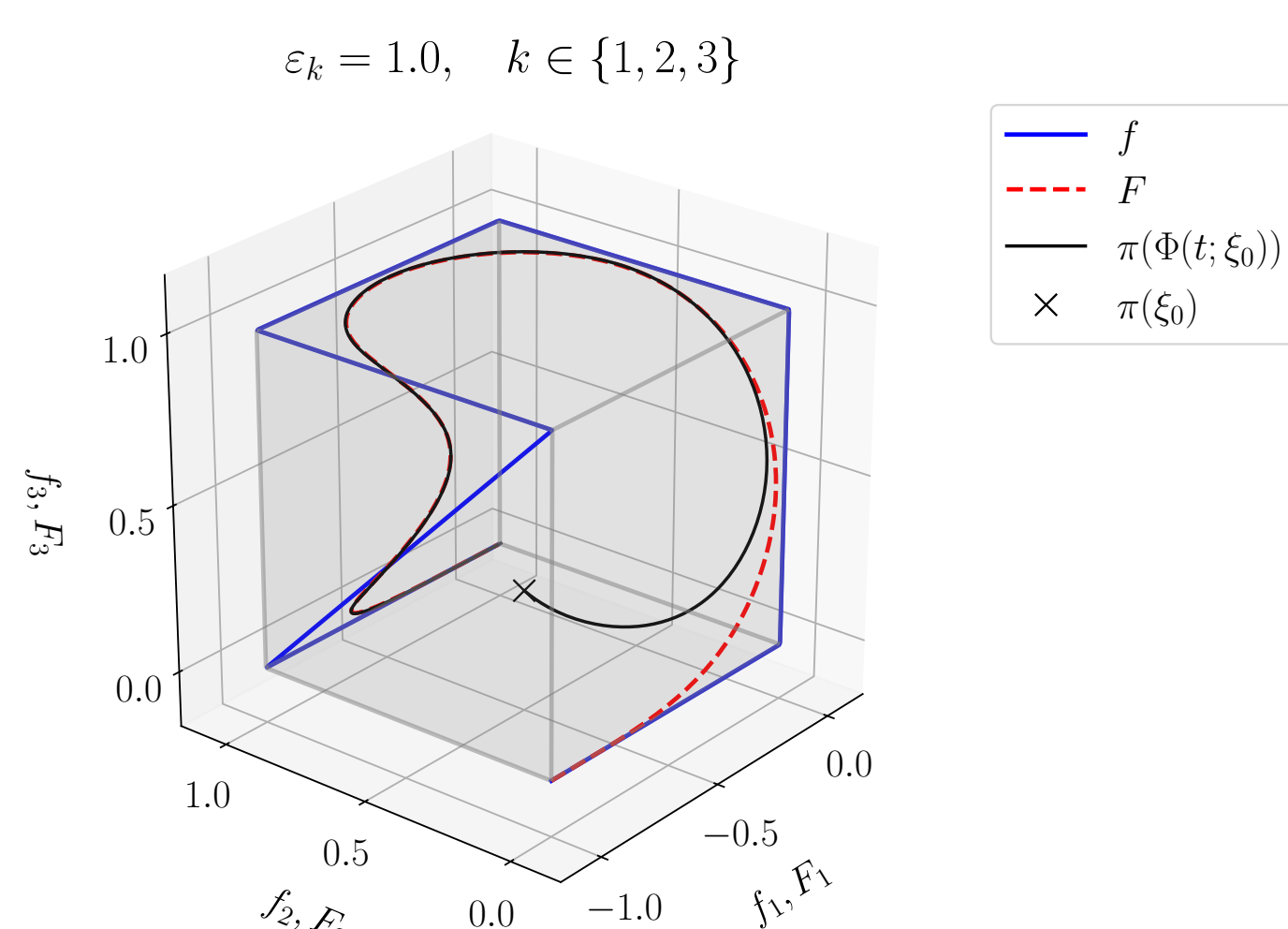
**Proposition 2.** If  $f$  is convex (resp. concave) on  $B(x, \delta)$  with  $\delta > 0$ , then for each  $y \in B(x, \delta)$ ,  $f * \varphi_\varepsilon$  is convex and  $f * \varphi_\varepsilon \geq f$  (resp. concave and  $f * \varphi_\varepsilon \leq f$ ) on  $B(x, \|x - y\|/2)$  for any  $\varepsilon \in (0, \|x - y\|/2]$ .

## 5. Further information



We have beautiful proofs of our theorems, but there is not enough space in this margin to write them. Nevertheless you can find the proofs and additional information in the given QR code.

## 4. Numerical and experimental results



**Figure 1:** A three dimensional numerical simulation. The original path is shown in blue, the mollified path in red and the simulated dynamical system in black. Each dimension is mollified with parameter  $\varepsilon_k = 1.0$ .



**Figure 2:** Capture of an experiment carried in Paraninfo's football field. The original trajectory is shown in green, the mollified one in yellow, and the actual trajectory of the vehicle in orange.