On the properties of mollified functions with applications to trajectory tracking and path following

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1. Introduction and problem statement

Most trajectory tracking or path following algorithms require the desired path to be at least twice continuously differentiable to guarantee global convergence. As a result, these methods typically exclude piecewise continuous linear paths or, more generally, continuous but non-differentiable paths. We present a method to regularize such non-differentiable paths via mollification.

Problem 1 (Regularization problem)**.** *Let $f : \mathbb{R} \to \mathbb{R}^n$ be a (Borel) measurable function (trajectory) and $\{\varepsilon_i\}_{i=1}^n$ be a collection of positive real numbers called parameters. The problem is to find a functional T_{ε_i} that acts on each component of f and that satisfies*

1. *For $p \in \mathbb{N}$ with $p \geq 2$, $T_{\varepsilon_i}(f_i) \in C^p(\mathbb{R}, \mathbb{R})$ for $i \in \{1, \dots, n\}$.*
2. *$T_{\varepsilon_i}(f_i) \to f_i$ as $\varepsilon_i \to 0$ in some sense of convergence.*
3. *$T_{\varepsilon_i}(f_i)$ is computationally simple to compute.*

2. Mollifiers

Definition 1 (Mollifier)**.** *Let $\varphi \in C^\infty(\mathbb{R}^n, \mathbb{R})$ and for $\varepsilon > 0$ define $\varphi_\varepsilon := \frac{1}{\varepsilon}\varphi \circ \frac{\mathrm{id}}{\varepsilon}$. If φ satisfies the following properties it is called a* ***mollifier****.*

1. *$\operatorname{supp} \varphi$ is compact.*
2. *$\int_{\mathbb{R}^n} \varphi d\lambda_n = 1$, where λ_n is the Lebesgue measure.*
3. *For any bounded $f \in C(\mathbb{R}^n, \mathbb{R})$, $\lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} f \varphi_\varepsilon d\lambda_n = f(0)$.*

A typical example is the one dimensional bump function, where $c_1 > 0$ is a normalization constant.

$$\varphi(x) = \begin{cases} c_1 \exp\left(\frac{-1}{1-x^2}\right), & |x| < 1, \\ 0, & |x| \geq 1 \end{cases}, \tag{1}$$

Mollifiers solve problem 1 in each dimension via convolution.

Theorem 1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ such that $f \in L^p_{loc}(\mathbb{R}^n, \mathbb{R})$, then*

- *$(f * \varphi_\varepsilon) \in C^\infty(\mathbb{R}^n, \mathbb{R})$, and $(f * \varphi)^{(n)} = f * \varphi^{(n)}$.*
- *$f * \varphi_\varepsilon \to f$ pointwise a.e., as $\varepsilon \to 0$.*
- *If f is uniformly continuous, then $f * \varphi_\varepsilon \to f$ uniformly as $\varepsilon \to 0$.*
- *If f is continuous, then $f * \varphi_\varepsilon \to f$ as $\varepsilon \to 0$ on compact subsets of $\mathbb{R}^n$.*

What analytical and geometrical properties of $f \in L^p_{loc}(\mathbb{R}^n, \mathbb{R})$ are invariant under mollification? Our analytical contributions are presented in the next section.

3. Analytical results

Let $||\cdot|| : \mathbb{R}^n \to \mathbb{R}$ be a norm in $\mathbb{R}^n$, and $B(x, r) = \{y \in \mathbb{R}^n \mid ||x - y|| < r\}$.

Proposition 1. *Let $f \in L^p_{loc}(\mathbb{R}^n, \mathbb{R})$ and φ be a positive mollifier with support $\overline{B}(0, 1)$. Then, given $\varepsilon > 0$*

- *If f is convex, concave or quasiconvex, then $f * \varphi_\varepsilon$ is respectively, convex, concave or quasiconvex.*
- *If f is affine linear, then $f * \varphi_\varepsilon = f$.*
- *If f is convex (resp. concave) then $f * \varphi_\varepsilon \geq f$ (resp. $f * \varphi_\varepsilon \leq f$).*

What happens if that properties only hold locally?

Proposition 2. *If f is convex (resp. concave) on $B(x, \delta)$ with $\delta > 0$, then for each $y \in B(x, \delta)$, $f * \varphi_\varepsilon$ is convex and $f * \varphi_\varepsilon \geq f$ (resp. concave and $f * \varphi_\varepsilon \leq f$) on $B(x, ||x - y||/2)$ for any $\varepsilon \in (0, ||x - y||/2]$.*

5. Further information

Wola!

4. Numerical and experimental results

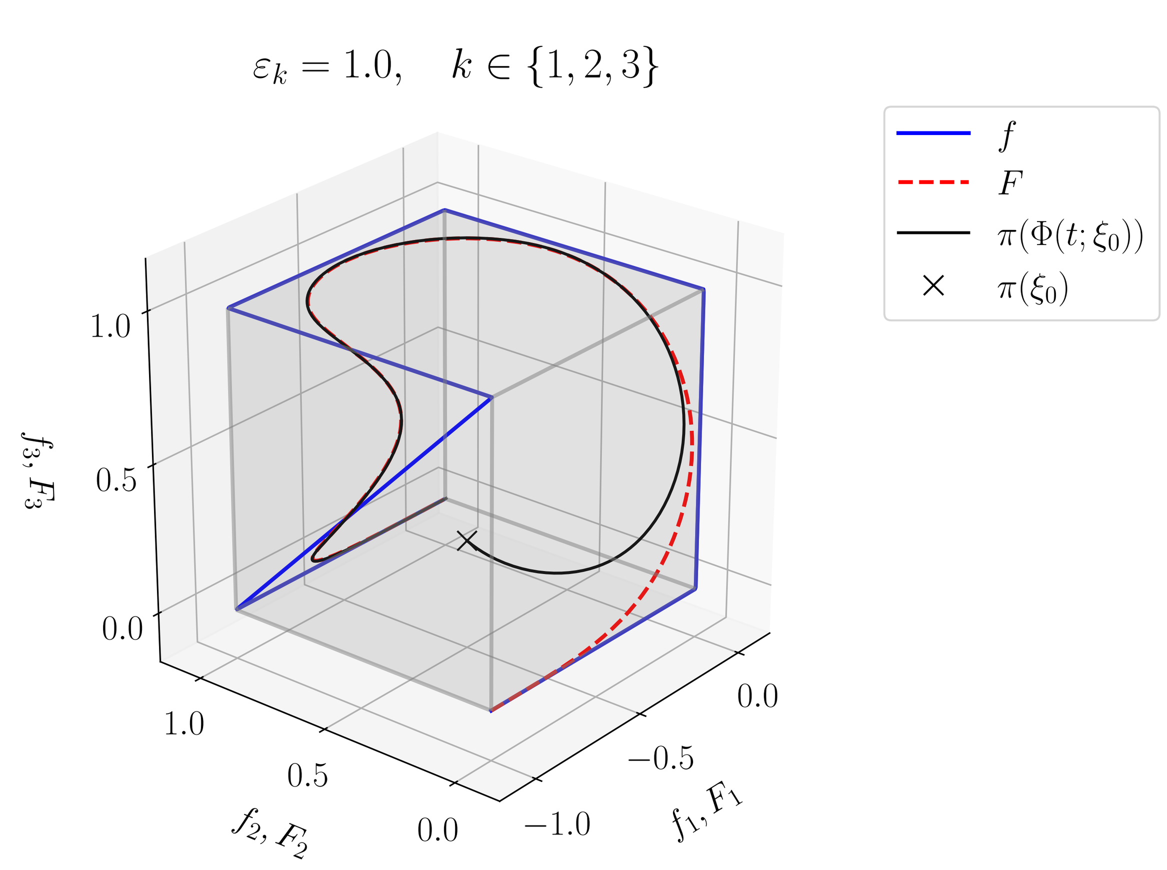

Figure 1: A three dimensional numerical simulation. The original path is shown in blue, the mollified path in red and the simulated dynamical system in black. Each dimension is mollified with parameter $\varepsilon_k = 1.0$.

Figure 2: Capture of an experiment carried in Paraninfo's football field. The original trajectory is shown in green, the mollified one in yellow, and the actual trajectory of the vehicle in orange.