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More broadcast graphs ☆

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Abstract

Given a graph $G=(V,E)$ and a vertex $u\in V$, *broadcasting* is the process of disseminating a piece of information from vertex u to every other vertex in the graph where, in each time unit, any vertex which knows the information can pass the information to at most one of its neighbors. A *broadcast graph* on n vertices is a graph which allows any vertex to broadcast in time $\lceil \log n\rceil$. A *minimum broadcast graph* on n vertices is a broadcast graph with the minimum number of edges over all broadcast graphs on n vertices. This minimum number of edges is denoted by $B(n)$. Several papers have presented techniques to construct broadcast graphs for various n and, hence, upper bounds on $B(n)$. In this paper, we present new techniques to construct broadcast graphs that give improved upper bounds for 5417 values of n in the range $1\leqslant n\leqslant 2^{14}$ and for most values of $n\geqslant 2^{20}$. These graphs can be combined using some of the previous methods to produce further improvements. © 1999 Elsevier Science B.V. All rights reserved.

1. Introduction and definitions

Given a graph $G=(V,E)$ and a vertex $u\in V$, *broadcasting* is the process of disseminating a piece of information from vertex u (called the *originator*) to every other vertex in the graph where, in each time unit, any vertex which knows the information can pass the information to at most one of its neighbors. The set of calls used to disseminate the information is called a *broadcast scheme*.

A *broadcast graph* on n vertices is a graph which allows any vertex to broadcast in time $\lceil \log n\rceil$. A *minimum broadcast graph* on n vertices is a broadcast graph with the minimum number of edges over all broadcast graphs on n vertices. This minimum number of edges is denoted $B(n)$.

The study of minimum broadcast graphs and $B(n)$ has a long history. Currently, exact values of $B(n)$ are known only for $n=2^k$ (where $k\geqslant 1$), $n=2^k-2$ (where

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$k \geqslant 2$), and for several specific values of $n < 64$. For other n, the value is not known. Upper bounds on $B(n)$ are obtained by constructing broadcast graphs. Several papers have presented ad hoc constructions of minimum broadcast graphs for different n [11,18–20,22]. A long sequence of papers have presented techniques to construct broadcast graphs for large n by combining broadcast graphs on fewer vertices [5,6–10, 12,16,4,14,24,2,25]. Some other (non-combining) methods to construct broadcast graphs for large n have appeared [4,13,21]. In addition to these constructions, a few papers have presented lower bounds on $B(n)$ [13,12,23]. See [15] for a survey on this and related problems. See [9] for more details on the historical development of this search.

In Section 2, we show how to construct new broadcast graphs by combining minimum broadcast graphs on $2^k - 2$ vertices with other known broadcast graphs. In Section 3, we present a construction based on combining hypercubes (which are minimum broadcast graphs on 2^k vertices) with other known broadcast graphs. In Section 4, we extend a result of Ahslwede et al. [1] to produce improved broadcast graphs for $2^l(2^k - 3)$ vertices. Finally, in Section 5, we present a refinement of an earlier construction Khachatrian and Haroutunian [16], giving improved broadcast graphs for most $n \geqslant 2^{20}$.

2. Construction using minimum broadcast graphs on $2^k - 2$ vertices

Several recent papers have shown methods to construct broadcast graphs by forming the compound of two graphs (see [2], for example). These methods have proven effective but cannot form broadcast graphs on n vertices for every n. In particular, the compound of graphs on n_1 and n_2 vertices is a graph on $n_1 n_2$ vertices. Thus, it is not possible to construct broadcast graphs for n vertices when n is prime by this method (directly). Broadcast graphs of other sizes can sometimes be formed by adding or deleting vertices from known broadcast graphs (see [4], for example). In this section, we present a new method based on compounding that allows the construction of broadcast graphs for many values of n, including many primes. The basic idea of our method is to construct a compound graph on $n_1 n_2$ vertices with m edges and then to merge $r+1$ of the vertices into a single vertex to form a broadcast graph on $n_1 n_2 - r$ vertices which also has m edges. We can do this for any r, $0 \leqslant r \leqslant n_2 - 1$. The method requires that the graphs being used to form the compound have broadcast schemes with particular properties as shown below.

Let H_k (for $k \geqslant 2$) denote the minimum broadcast graph on $2^k - 2$ vertices [16]. These graphs are known as modified Knödel graphs (see [17,3]). H_k has vertex set $V(H_k) = \{0, \ldots, 2^k - 3\}$ and edge set $E(H_k) = \{(x, y) \mid x + y = 2^s - 1 \pmod{2^k - 2}$ for $1 \leqslant s \leqslant k - 1\}$. $|E(H_k)| = (k-1)(2^{k-1} - 1)$. The edges (i, j) such that $i + j = 2^s - 1 \pmod{2^k - 2}$ may be thought of as edges in dimension s. To broadcast from an arbitrary vertex of H_k, all informed vertices call neighbors in dimension j at time j for $1 \leqslant j \leqslant k - 1$ and neighbors in dimension 1 at time k.

The following two lemmas (see [1,3]) will be useful in proving that the graphs constructed in this section are broadcast graphs.

Lemma 2.1. *Any cyclic shift of the dimensions followed by the initial dimension of that shift gives a valid broadcast scheme for H_k and furthermore, the reverse of any such scheme is also a valid broadcast scheme for H_k.*

We will refer to such a scheme as a *dimensional broadcast scheme* for H_k.

Lemma 2.2. *If $(a,b) \notin E(H_k)$ then there is a broadcast scheme for originator a in H_k such that the message reaches b at time k, that is, at the last time unit.*

We now describe how to construct $G' = (V', E')$ which we will show is a broadcast graph. Let $G = (V, E)$ be a broadcast graph on p vertices with $V = \{1, 2, \dots, p\}$ and $e = |E|$.

To form the vertex set V', we begin with the union of the vertices of p copies of H_k, denoted by $H_k^1, H_k^2, \dots, H_k^p$. Each vertex in this set is denoted by x^i where i indicates that the vertex is from H_k^i and $x \in V(H_k)$. For any r, $0 \leqslant r < p$, identify the vertices $0^1, 0^2, \dots, 0^{r+1}$. That is, we merge the $r+1$ vertices into a single vertex. The resulting set of $p(2^k-2)-r$ vertices is $V' = \{x^i \mid x \in V(H_k) \setminus \{0\}, i \in V\} \cup \{0^i \mid i \in V \setminus \{2, 3, \dots, r+1\}\}$. For convenience, we may use any of the $r+1$ different names for the merged vertex 0^1, that is, $0^1 = 0^2 = \cdots = 0^{r+1}$.

The edges of E' are of two types – the edges of the individual minimum broadcast graphs H_k and the product-like edges connecting these graphs. $E' = E_{\text{local}} \cup E_{\text{product}}$ where $E_{\text{local}} = \{(x^i, y^i) \mid x + y = 2^s - 1 \pmod{2^k - 2}$ for $1 \leqslant s \leqslant k-1$, and $1 \leqslant i \leqslant p\}$ and $E_{\text{product}} = \{(x^i, x^j) \mid x$ is odd, $(i, j) \in E\}$. Note that the edges defined for $0^1, 0^2, \dots, 0^{r+1}$ are all incident on the merged vertex 0^1. Thus, for example, 0^1 is adjacent to $1^1, 1^2, \dots,$ and 1^{r+1} where, by contrast, 0^{r+2} is adjacent to 1^{r+2} and to no other vertex of the form 1^i.

To illustrate this construction, if we let $k = 3$, $p = 4$ and $r = 2$, we begin with four copies of H_3 and then merge the vertices 0^1, 0^2, and 0^3 into a single vertex. Within each copy of H_3, the local edges connect the vertices in a cycle of length six. Letting $G = C_4$ (the cycle on four vertices), the product edges connect vertices of each of the forms 1^i, 3^i, and 5^i in three cycles of length four. This construction is shown in Fig. 1. The result is a broadcast graph on 22 vertices with 36 edges. Using $r = 1$ or $r = 3$, the construction produces similar 36 edge broadcast graphs on 23 or 21 vertices, respectively.

Theorem 2.3. *If $0 \leqslant r < p$ and $\lceil \log(p(2^k - 2) - r) \rceil = \lceil \log p \rceil + k$ then the graph G' constructed as described above (from minimum broadcast graph H_k and broadcast graph $G = (V, E)$) is a broadcast graph and*

$$B(p(2^k - 2) - r) \leqslant (2^{k-1} - 1)(p(k-1) + e).$$

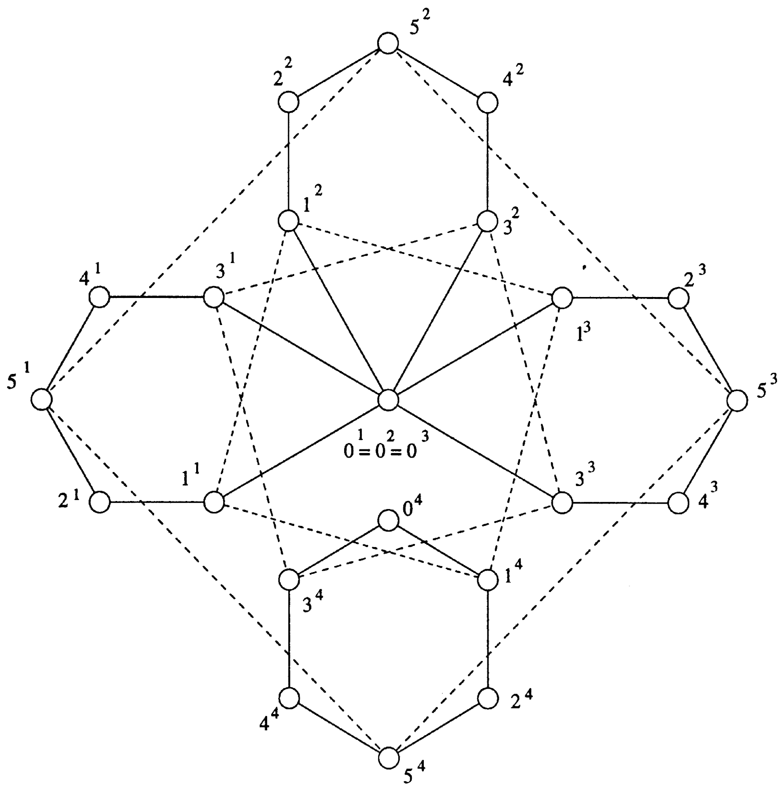

Fig. 1. A broadcast graph G' constructed from H_3 and C_4.

Proof. To prove that G' is a broadcast graph, we will describe broadcast schemes for each originator u in G'.

Let $u = x^i$ where x is odd and x^i is not adjacent to 0^i. Vertex x^i is in a copy of the graph G. Broadcast within this copy of G during time units $1, \ldots, \lceil \log p \rceil$ according to a minimum time broadcasting scheme S in G for originator x. At time $\lceil \log p \rceil$, each copy H_k^j, $1 \leqslant j \leqslant p$, of H_k has one informed vertex, namely x^j. From this point, the broadcast scheme for G' can be completed by broadcasting from x within each copy of H_k such that vertex 0 is informed at the last time unit. Such a scheme exists by Lemma 2.2. Redundant calls to $0^1 = 0^2 = \cdots = 0^{r+1}$ can be omitted from the scheme. An example of such a scheme is shown in Fig. 2 where the originator is indicated by the solid color vertex.

Let $u = x^i$ where x is even and $x \neq 0$. In this situation, a scheme similar to that of case 1 can be used. Choose a broadcast scheme S for originator x in H_k which informs 0 in the last time unit. We first inform all of the neighbors of x^i (that is, vertices $(2^j - 1 - x \pmod{2^k - 2})^i$ for $j = 1, \ldots, k-1$) in the first $k-1$ time units according to S. These vertices are all in distinct copies of G and they initiate broadcasts within their respective copies of G as soon as they are informed by x^i. At the end of

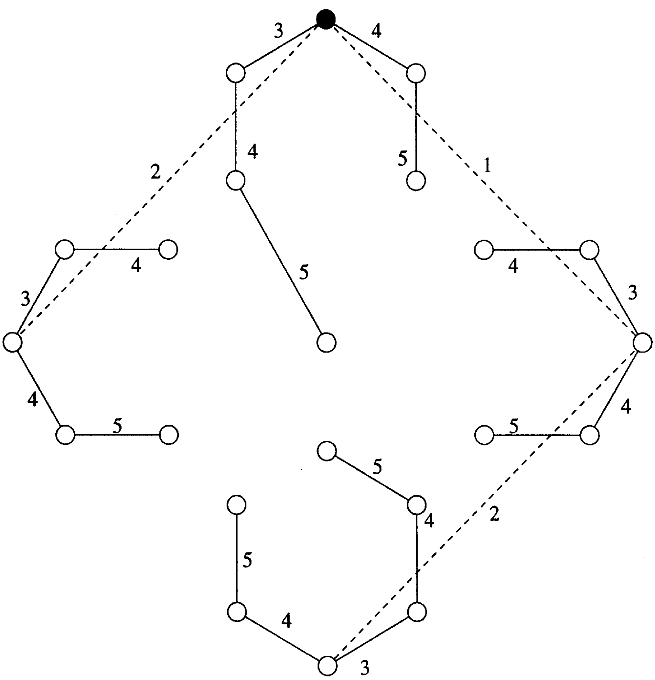

Fig. 2. A broadcast scheme for originator 5^2 in graph G' constructed from H_3 and C_4.

each of these broadcasts within copies of G, there will be several vertices informed in each H_k. The broadcasts within each copy of H_k can be completed from these vertices by time $k + \lceil \log p \rceil$ by having each such vertex "resume" its scheme (S) within H_k as soon as it is finished with its calls within G. Redundant calls can be omitted. The copies x^l of x^i (where $1 \leqslant l \leqslant p$ and $i \neq l$) are yet to be informed. At time $\lceil \log p \rceil + 1$, the neighbors $(1 - x \pmod{2^k - 2})^l$ of the x^l have been informed and have completed their broadcasting within their copy of G. At times $\lceil \log p \rceil + 2, \ldots, \lceil \log p \rceil + k - 1$, these vertices make additional calls according to scheme S. At time $\lceil \log p \rceil + k$, they are idle and can call x^l. An example of such a scheme is shown in Fig. 3 where the originator is indicated by the solid color vertex.

Let $u = x^i$ where $x = 0$ and $1 \leqslant i \leqslant p$. Choose a broadcast scheme S for originator 0 in H_k. We first inform all of the neighbors of 0^i (that is, vertices $(2^j - 1 \pmod{2^k - 2})^i$ for $j = 1, \ldots, k-1$) in the first $k-1$ time units according to S. These vertices are all in distinct copies of G and they initiate broadcasts within their respective copies of G as soon as they are informed by 0^i. The scheme can be completed as in the previous case. An example of such a scheme is shown in Fig. 4 where the originator is one of the vertices indicated by the solid color vertex.

Let $u = x^i$ where x is odd and x^i is adjacent to 0^i. In particular, $x = 2^{j_0} - 1 \pmod{2^k - 2})$ for some j_0, $1 \leqslant j_0 \leqslant k-1$. At time 1, x^i calls 0^i. The vertex 0^i then calls its $k-2$ neighbors $(2^j - 1 \pmod{2^k - 2})^i$ for $j = 1, \ldots, k-1$ and $j \neq j_0$ in time units $2, \ldots, k-1$.

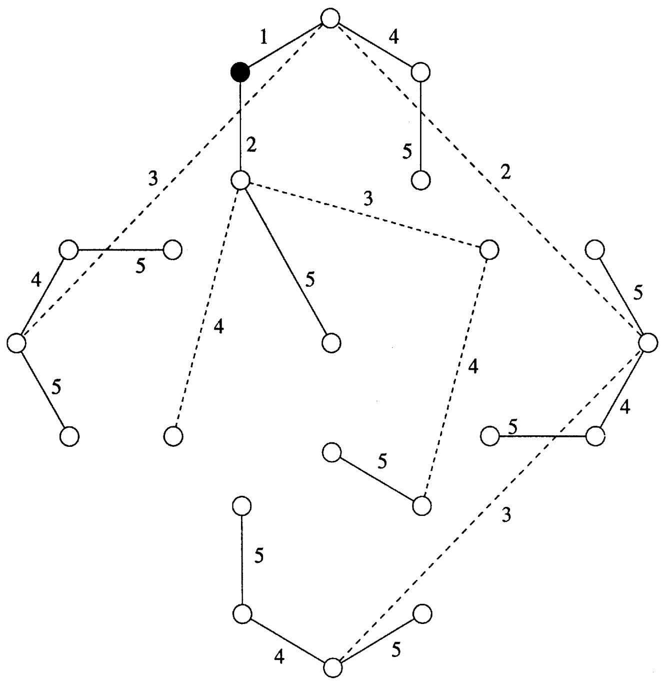

Fig. 3. A broadcast scheme for originator 2^2 in graph G' constructed from H_3 and C_4.

Again, these vertices are all in distinct copies of G and they initiate broadcasts within their respective copies of G as soon as they are informed. The scheme can be completed as in the second case. An example of such a scheme is shown in Fig. 4 where the originator is one of the vertices indicated by the solid color vertex. □

The above construction is particularly effective when the graph chosen for G is a hypercube.

Corollary 2.4. *If* $0 \leqslant r < 2^l$, *then the graph* G' *constructed as described above* (*from minimum broadcast graph* H_k *and hypercube* Q_l) *is a broadcast graph and*

$$B(2^{k+l} - 2^{l+1} - r) \leqslant (2^{k+l-2} - 2^{l-1})(2k + l - 2).$$

The graph G' constructed above can be used with the method of compounding [2,9,16] to produce other broadcast graphs on larger numbers of vertices. In particular, the size of a solid 1-cover of G' (using the terminology from Bermond et al. [2]) is necessary to apply these methods. A solid 1-cover of a graph is a vertex cover of the graph with the additional property that for any vertex u which is not in the cover, there is a broadcast scheme for originator u such that some neighbor of u is idle at some time during the broadcast.

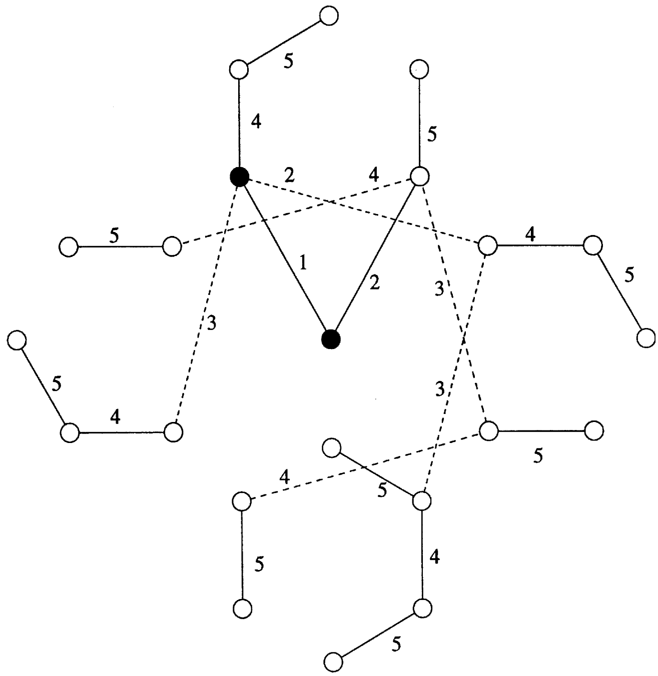

Fig. 4. A broadcast scheme for originators 1^2 or 0^1 in graph G' constructed from H_3 and C_4.

Lemma 2.5. *The minimum size of a solid* 1-*cover of* G' *is at most* $2^{k+l-1} - 2^l$.

Proof. The vertices x^i with x odd and $1 \leqslant i \leqslant 2^l$ form a vertex cover for G'. Let u^i be any vertex not in this cover, that is, u is even. From the broadcast scheme for G', we know that the first call from u^i is to some vertex x^i where x is odd. Since the odd vertices have degree $k+l-1$, this vertex x^i must be idle at some time in the broadcast. Thus, the vertices x^i with x odd and $1 \leqslant i \leqslant 2^l$ form a solid 1-cover for G'. □

3. Construction using hypercubes

The construction of Section 2 makes use of the fact that a vertex in H_k is idle during the last time unit of broadcasting in H_k. Although no vertex in a hypercube has this property, we can, nevertheless, do a similar construction using hypercubes in place of H_k. Again, the basic idea is to construct a compound graph on $n_1 n_2$ vertices with m edges and then to merge $r+1$ of the vertices into one to form a broadcast graph on $n_1 n_2 - r$ vertices which also has m edges. However, when using the hypercube in this construction, we must merge exactly n_2 vertices, resulting in a broadcast graph on $n_1 n_2 - n_1 + 1$ vertices.

In this section, we will construct graphs on vertices which are labeled with k bit binary strings. If v is a vertex, we use $||v||$ to denote the number of ones in the binary string which is v's label. If u and v are vertices, we use $||u \oplus v||$ to denote the number of bits in which the labels of u and v differ.

If the labels of two vertices of the d-dimensional hypercube Q_d differ in only the ith position, we refer to these vertices as neighbors in dimension i (or ith dimensional neighbors). One method to broadcast from any vertex of Q_d is to have every vertex which is informed at time i (where $1 \leqslant i \leqslant d$) call its neighbor in dimension i. In fact, if $\pi_1, \pi_2, \ldots, \pi_d$ is any permutation of the dimensions, one can broadcast from any vertex of Q_d by having every vertex which is informed at time i ($1 \leqslant i \leqslant d$) call its neighbor in dimension π_i (see [3], for example). We will call these schemes *dimensional broadcast schemes* for Q_d.

The following lemmas will be useful in proving that the graphs constructed in this section are broadcast graphs. The first is obvious from the preceding discussion of dimensional broadcast schemes.

Lemma 3.1. *Let u be any vertex of the hypercube Q_d. For any $v \neq u$, there is a minimum time broadcast scheme for originator v in Q_d in which vertex u is informed in the last time unit* (*that is, at time d*).

Lemma 3.2. *Let G_d* (*for even d*) *be the graph on $2^d - 1$ vertices obtained by deleting vertex* $(0,0,\ldots,0)$ *and its incident edges from the hypercube Q_d. Suppose that at time i* ($1 \leqslant i \leqslant d$) *the ith-dimensional neighbor of* $(1,1,\ldots,1)$ *learns the message from some source external to G_d. The message can be broadcast to all vertices of G_d by time d.*

Proof. Let u_i ($1 \leqslant i \leqslant d$) denote the neighbor of $(1,1,\ldots,1)$ in dimension i. We have assumed that u_i is informed at time i.

Each vertex u_i ($2 \leqslant i \leqslant d$) can broadcast the message to all members of G_d whose labels begin with $i-1$ ones followed by 1 zero. The subgraph induced by this vertex set is a $(d-i)$-dimensional hypercube and broadcasting can be accomplished in this subgraph by having all informed vertices call their j-dimensional neighbors at times j for $i+1 \leqslant j \leqslant d$.

The vertices of G_d which are not contained in these $d-1$ small cubes are informed from u_1 beginning at time 2. Let G'_d denote the subgraph induced by these vertices and note that G'_d is not a hypercube. G'_d has vertex set $V' = \{(0, x_2, x_3, \ldots, x_d) \mid \sum_{i=2}^{d} x_i \geqslant 1\} \cup \{(1,1,\ldots,1)\}$. The edges of G'_d are those edges of Q_d induced by V'.

If we were to broadcast from u_1 (beginning at time 2) following a dimensional broadcast scheme (with an added call to inform $(1,1,\ldots 1)$), the broadcast tree for u_1 would be as illustrated for $d=6$ in Fig. 5. In this scheme, u_1 calls its dimension $2,3,4,\ldots,d,1$ neighbors at times $2,3,4,\ldots,d,d+1$, respectively. Any other vertex that receives the message at time j then sends the message to its dimension $j+1, j+2, \ldots, d$ neighbor (if any) at times $j+1, j+2, \ldots, d$, respectively.

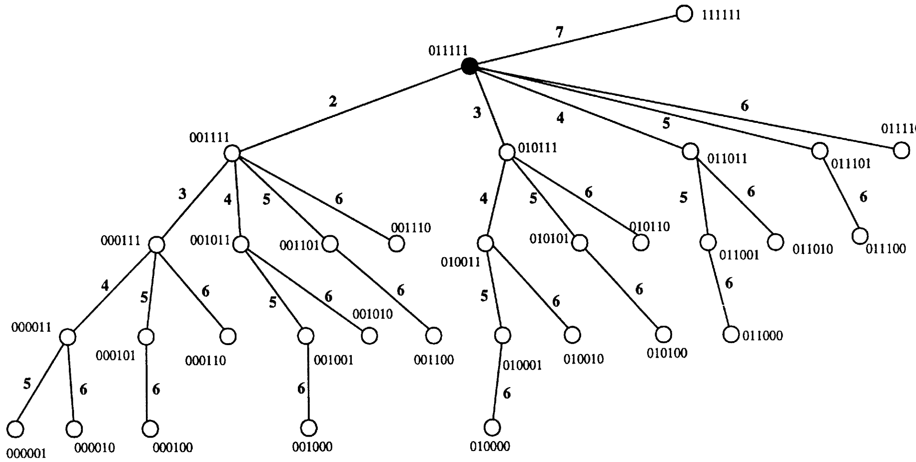

Fig. 5. Dimensional broadcast scheme for u_1 (shaded) in G_6'.

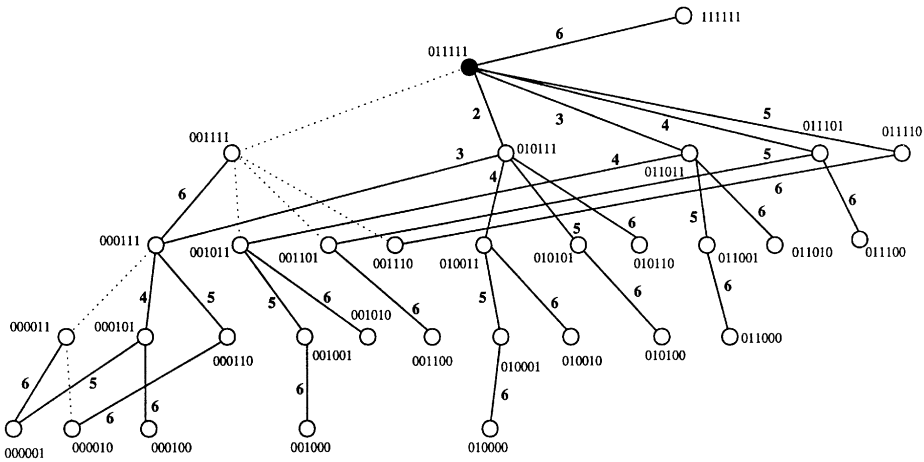

Fig. 6. Modified broadcast scheme for u_1 (shaded) in G_6'.

By modifying the calls made by vertices of the form $(0,0,0,\ldots,0,1,1,\ldots,1)$ with an odd number of leading zeroes and those of their children, as illustrated for $d=6$ in Fig. 6, we can see how to broadcast to all vertices of G_d' beginning at time 2 and completing at time d. To be more specific, vertex $(0,0,0,\ldots,0,1,1,\ldots,1)$ with i leading zeroes (with i odd) learns the message at time i and calls its $j+1$ dimensional neighbors at times j for $i+1 \leqslant j \leqslant d-1$ and its dimension i neighbor at time d. Any vertex of the form $(0,0,0,\ldots,0,1,1,\ldots,1,0,1,1,\ldots,1)$ with i leading zeroes (i odd), followed by $s-1$ ones ($2 \leqslant s \leqslant d-i-1$), followed by 1 zero, followed by $d-i-s$ ones learn the message at time $i+s-1$ and then calls its neighbors in dimensions

$i+1, i+s+1, i+s+2, \ldots, d$ at times $i+s, i+s+1, i+s+2, \ldots, d$, respectively. Vertex $(0,0,0,\ldots,0,1,1,\ldots,1)$ with i leading zeroes (with i even) learns the message at time d and makes no further calls. Other vertices learn the message and make calls as in the dimensional scheme above. Thus, all vertices of G'_d are informed by time d. $\square$

Suppose that we have a broadcast graph $G=(V,E)$. Let $p=|V|, e=|E|$ and $k=\lceil \log|V| \rceil$. Let $Q^1_{m-k}, Q^2_{m-k}, \ldots, Q^p_{m-k}$ be hypercubes of dimension $m-k$ for $m>k$.

We now describe how to construct $G'=(V',E')$ which we will show is a broadcast graph.

To form the vertex set V', we begin with the union of the vertices of the Q^i_{m-k}. Each vertex in this set is denoted x^i where i indicates that the vertex is from Q^i_{m-k} and $x=(x_1,x_2,\ldots,x_{m-k})$ is the $m-k$ bit binary label for the vertex. We then merge the p vertices z^i where $z=(0,0,\ldots,0)$. V' is the resulting set. In particular, $V'=\{(x_1,,x_2,\ldots,x_{m-k})^i \mid \sum_{j=1}^{m-k} x_j > 0, 1 \leqslant i \leqslant p\} \cup (0,0,\ldots,0)$. For convenience, we may refer to the merged vertex as z^i for any $1 \leqslant i \leqslant p$ when $z=(0,0,\ldots,0)$ or, simply, as z.

The edges of E' are of two types: the edges of the individual hypercubes and the product-like edges connecting these hypercubes. $E' = E_{\text{local}} \cup E_{\text{product}}$ where $E_{\text{local}} = \{(x^i,y^i) \mid ||x^i \oplus y^i|| = 1, 1 \leqslant i \leqslant p\}$ and $E_{\text{product}} = \{(x^i,x^j) \mid ||x|| \text{ is odd}, i \neq j, (i,j) \in E\}$. Note that the edges defined for $z^1, z^2, \ldots, z^p$ are all incident on the merged vertex z.

To illustrate this construction, if we let $p=5$, $k=2$ and $m=5$, we begin with five copies of Q_3 and then merge the vertices $(0,0,0)^1$, $(0,0,0)^2, \ldots, (0,0,0)^5$ into a single vertex. Within each copy of Q_3, the local edges connect the vertices as a three-dimensional hypercube. In addition, letting $G=C_5$ (the cycle on five vertices which is an mbg), the produce edges connect the three vertices of each of the forms $(1,0,0)^i$, $(0,1,0)^i$, $(0,0,1)^i$, and $(1,1,1)^i$ in a cycle of length five. This construction is shown in Fig. 7. The result is a broadcast graph on 36 vertices with 80 edges.

Theorem 3.3. *If* $(2^{m-1}-1)/(2^{m-k}-1) < p \leqslant 2^k$, *that is, if* $p \leqslant 2^k$ *and* $\lceil \log(p(2^{m-k}-1)+1) \rceil = m$, *then the graph* G' *constructed as described above* (*from broadcast graph* $G=(V,E)$ *with* $k=\lceil \log p \rceil$ *and hypercube* Q_{m-k}) *is a broadcast graph and*

$$B(p(2^{m-k}-1)+1) \leqslant 2^{m-k-1}(p(m-k)+e).$$

Proof. In general, if we can ensure that each hypercube Q^i_{m-k} has one informed vertex (other than z) at time k, then all of the hypercubes can complete the broadcast scheme by time m. To do this, each of the hypercubes can simply broadcast internally such that vertex z is informed at the last time unit as described in Lemma 3.1 In fact, if all but one of the hypercubes Q^i_{m-k}, say Q^1_{m-k}, have one informed vertex at time k (other than z) and the vertex z is also informed at time k, then we can still complete broadcast by time m. In this case, the hypercube Q^1_{m-k} can broadcast internally from z while the other hypercubes broadcast internally such that vertex z is informed at the last time unit as described above. (Those calls to z in the last time unit can be omitted.)

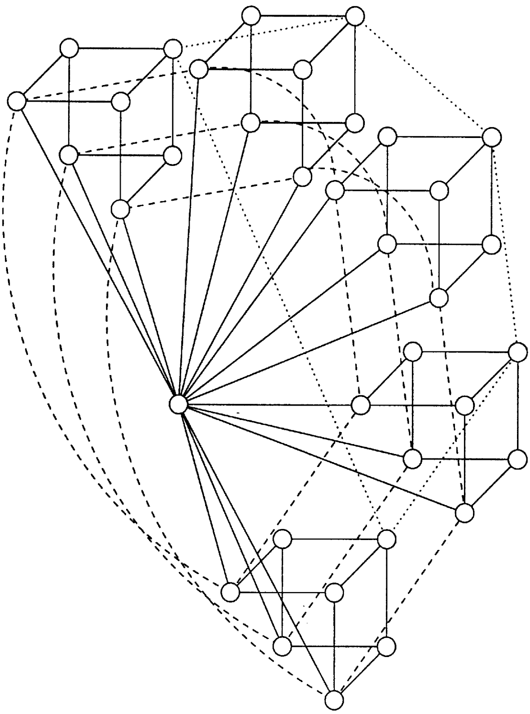

Fig. 7. A broadcast graph G' constructed from Q_3 and C_5.

Consider the case that the originator is x^i where $||x||$ is odd. Then x^i is a vertex in a copy of G. In k time units, all vertices x^i in that copy of G can be informed by using a minimum time broadcast scheme for G. This ensures that every hypercube Q^i_{m-k} has one informed vertex at time k. From this point, the broadcast scheme for G' can be completed as described above. An example of such a scheme is shown in Fig. 8 where the originator is indicated by the solid color vertex.

Consider the case that the originator is $z=(0,0,\ldots,0)$. The vertices $x^i=(1,0,0,\ldots,0)^i$, $1\leqslant i\leqslant p$, are all adjacent to z by local (hypercube) edges. Consider a minimum time broadcasting scheme S in G with originator 1. In this scheme, without loss of generality, vertex 1 calls vertex p at time 1 and vertex p calls vertices $2,3,4,\ldots,l$ at times $2,3,4,\ldots,l$ where $l\leqslant k$. Now, in graph G', we want to ensure that all but one of the vertices x^i in the copy of G with vertices labeled $(1,0,\ldots,0)$ are informed at time k. To do this, originator z mimics the calls involving p in the scheme S. At time 1, z calls x^1 (where, in S, 1 calls p). Subsequently x^1 can call x^j whenever 1 calls j in S. At times $i=2,3,\ldots,l$, z calls x^i (where, in S, p calls i). After x^i receives the message, for $i=2,3,\ldots,l$, x^i can call x^j whenever i calls j in S. Thus, all vertices $x^i=(1,0,\ldots,0)^i$ are informed at time k except for x^p. This means that all but one of the hypercubes Q^i_{m-k}, in particular Q^p_{m-k}, has one informed vertex (other than z) at time k and the vertex z is also informed at time k. As described above, these hypercubes can complete

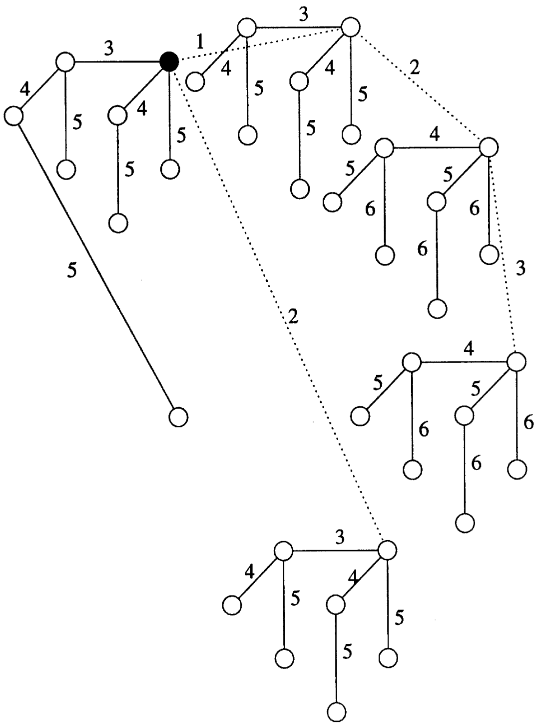

Fig. 8. A broadcast scheme in graph G' constructed from H_3 and C_4.

broadcasting internally by time m. An example of such a scheme is shown in Fig. 9 where the originator is indicated by the solid color vertex.

Consider the case that the originator is x^i where $||x||$ is even and $x^i \neq z$. From this originator, we will inform all of the neighbors y^i of x^i such that $||y||$ is odd. These vertices y^i are all in distinct copies of G and they initiate broadcasts within their respective copies of G as soon as they are informed by x^i. At the end of each of these broadcasts within copies of G, there will be several vertices informed in each Q^i_{m-k}. Without loss of generality, let $x^i = (0,0,\ldots,0,1,1,\ldots,1)$ where $||x^i|| = r$. Let y^i_j be the neighbor of x^i whose label differs from x^i in the jth bit position. (For example, $y^i_2 = (0,1,0,0,\ldots,0,1,1,\ldots,1)$.) The broadcast scheme begins with originator x^i informing y^i_j at time j where $1 \leqslant j \leqslant m-k$. Each of these vertices y^i_j begins a broadcast within its copy of G at time $j+1$ and this broadcast (within G) is complete at time $j+k$.

Consider the hypercube Q^i_{m-k} which contains originator x^i and all of its neighbors y^i_j. One broadcast scheme for initiator x^i within Q^i_{m-k} corresponds to having each informed vertex of Q^i_{m-k} call its neighbor in dimension d at time d for $1 \leqslant d \leqslant m-k$. The scheme which we are developing for G' has x^i calling its neighbors in this order at these times. However, these neighbors y^i_d are subsequently busy with calls in their

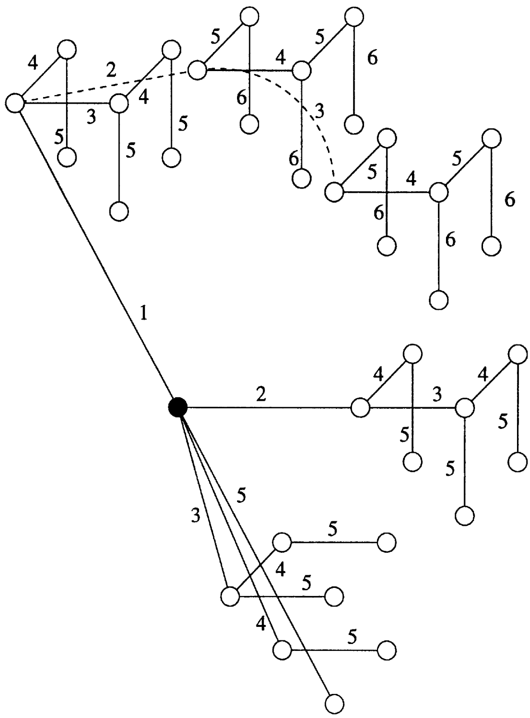

Fig. 9. A broadcast scheme in graph G' constructed from H_3 and C_4.

respective copies of G at times $d+1,\ldots,d+k$. To complete broadcasting within Q^i_{m-k}, these neighbors (and any other informed vertices in Q^i_{m-k} other than x^i) can call their dimension d neighbors at time $d+k$ to complete broadcast within Q^i_{m-k} by time $(m-k)+k=m$.

Consider a different hypercube Q^s_{m-k} (where $s \neq i$). In the worst case, vertex y^s_1 of Q^s_{m-k} is informed and has completed making calls in its copy of G at time $k+1$. Thus, y^s_1 is available to make calls within Q^s_{m-k} at time $k+2$. Similarly, y^s_j is available to make calls within Q^s_{m-k} at time $k+j+1$ for $2 \leqslant j \leqslant m-k$. To complete broadcasting within Q^s_{m-k}, each of these vertices initiates a broadcast within a subcube of Q^s_{m-k}. In particular, all of the y^s_j except for $y^s_{m-k-r+1}$ call their neighbors in dimension d at times $k+j+d$ for $1 \leqslant d \leqslant m-k$ and these neighbors continue the broadcast as in the above scheme for Q^i_{m-k}. Vertex $y^s_{m-k-r+1}$ initiates a broadcast of the type described in Lemma 3.2 at time $m-r+2$ which informs x^s. Thus, all vertices of Q^s_{m-k} are informed by time m, completing the broadcast from x^i. An example of such a scheme is shown in Fig. 10 where the originator is indicated by the solid color vertex. □

An interesting corollary to the above is obtained by choosing Q_k for G:

Corollary 3.4. $B(2^m-2^k+1) \leqslant 2^{m-1}(m-k/2)$ *for any* $1 \leqslant k < m$.

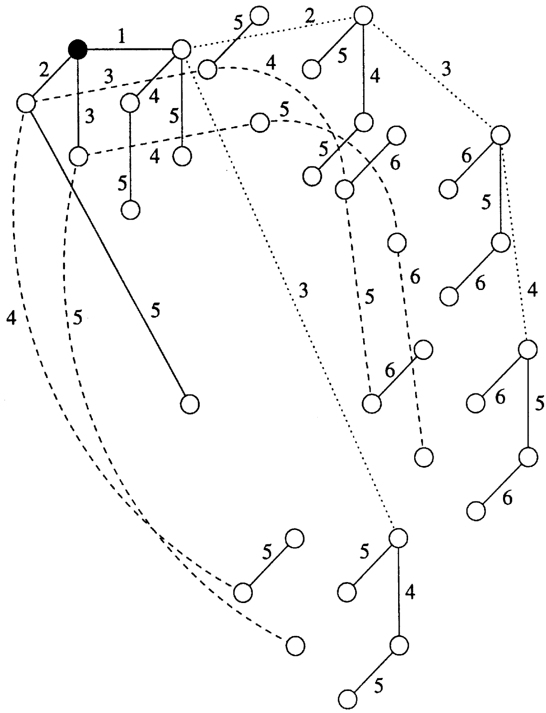

Fig. 10. A broadcast scheme in graph G' constructed from H_3 and C_4.

4. Constructions based on $2^k - 3$ vertices

Ahlswede et al. [1, Property 6.3], showed how to construct a broadcast graph on $2^k - 3$ vertices with $(2^k - 3)(k-1)/2$ edges for odd k. We show how to modify this construction to obtain a broadcast graph on $2^k - 3$ vertices with $((2^k - 3)(k-1)+1)/2$ edges for even k, giving a general bound of $B(2^k - 3) \leqslant \lceil (2^k - 3)(k-1)/2 \rceil$ for all k. This new bound can then be used with the compounding method to produce an upper bound on $B(n)$ for some other values of n.

The construction of [1] is as follows: Let H_k (for odd $k \geqslant 3$) denote the minimum broadcast graph on $2^k - 2$ vertices as described in Section 2. Delete vertex 0 and its incident edges from H_k. Add the edges $(2^i - 1, 2^{i+1} - 1)$ for odd i in the range $1 \leqslant i \leqslant k-2$. The resulting $k-1$ regular graph is a broadcast graph.

To construct a similar graph for the case of even k, we do the following: Begin with H_k for even $k \geqslant 4$. Delete vertex 0 and its incident edges from H_k. Add the edges $(2^i - 1, 2^{i+1} - 1)$ for odd i in the range $1 \leqslant i \leqslant k-2$. In addition, add edge $(2^{k-1} - 1, 1)$. The resulting graph, which we will call H, has $2^k - 4$ vertices of degree $k-1$ and one vertex (namely, 1) of degree k.

Theorem 4.1. *The graph H constructed as described above is a broadcast graph and*

$$B(2^k-3) \leqslant \frac{(2^k-3)(k-1)+1}{2}$$

when k is even and $k \geqslant 4$.

Proof. Consider originator u in H such that $u \neq 2^i - 1$ for any $1 \leqslant i \leqslant k-1$. From Lemma 2.2 and the fact that u is not adjacent to 0, there is an ordering of the dimensions of H_k such that a dimensional broadcast scheme for H_k informs 0 in the last time unit. The calling scheme obtained by deleting this call to 0 is a valid broadcast scheme for originator u in H.

Consider originator $u = 2^i - 1$ in H where i is odd and $i \neq k-1$. Note that u is adjacent to $2^{i+1}-1$ in H. The ordering of the dimensions $i+1, i+2, \ldots, k-1, 1, 1, \ldots, i, i+1$ is a valid dimensional calling scheme in H_k for originator u. In this scheme, vertex 0 is informed at time $k-1$ (by u) and then informs $2^{i+1}-1$ at time k. Except for those two calls, this scheme uses only edges of H. Replacing these two calls with a call from $u = 2^i - 1$ to $2^{i+1}-1$ at time $k-1$, we obtain a valid broadcast scheme for originator u in H. For originator $u = 2^{k-1}-1$, the ordering $1, 2, \ldots, k-1, 1$ behaves similarly.

Consider originator $u = 2^i - 1$ in H where i is even. Note that u is adjacent to $2^{i-1}-1$ in H. The ordering of the dimensions $i-1, i-2, \ldots, 1, k-1, k-2, \ldots, i, i-1$ is a valid dimensional calling scheme in H_k for originator u. In this scheme, vertex 0 is informed at time $k-1$ (by u) and then informs $2^{i-1}-1$ at time k. Replacing these two calls with a call from $u = 2^i - 1$ to $2^{i-1}-1$ at time $k-1$, we obtain a valid broadcast scheme for originator u in H. $\square$

Combining this result with that of [1], we obtain the following:

Corollary 4.2. $B(2^k-3) \leqslant \lceil (2^k-3)(k-1)/2 \rceil$, *for* $k \geqslant 3$.

We can use this result and the compounding method of [2,9,16] to create a new bound. Let G be a broadcast graph on 2^k-3 vertices as described above for even $k \geqslant 4$ or as described by Ahlswede et al. [1] for odd $k \geqslant 3$ with vertices $\{1, 2, 3, \ldots, 2^k-3\}$. Label 2^l copies of G as $G^1, G^2, \ldots, G^{2^l}$ for $l \geqslant 1$ and denote the vertices of G^i by x^i where x is a vertex in G and i indicates that it comes from copy G^i. For each even v, connect the vertices v^i, $1 \leqslant i \leqslant 2^l$, as an l-dimensional hypercube.

This gives the following bound:

Theorem 4.3. $B(2^l(2^k-3)) \leqslant 2^l(\lceil (k-1)(2^k-3)/2 \rceil + l(2^{k-2}-1))$.

5. Improvement of an earlier construction for large *n*

We now turn our attention to larger n and asymptotic results. Grigni and Peleg [13] showed that $B(n) \in \Theta(nL(n))$ where $L(n)$ is the number of consecutive leading 1's in the binary representation of the integer $n-1$. The original construction of Peleg [21] gave a slightly better upper bound than that given in [13]. Khachatrian and Haroutunian [16] gave a similar construction with an upper bound somewhat better than that of Peleg [21]. In this section, we describe a modification of the construction of Khachatrian and Haroutunian, resulting in a further improvement of the asymptotic upper bound.

Before describing the modification, we will first describe the original construction [16] which produces a broadcast graph on $2^m - 2^k - r$ vertices for $0 \leqslant k \leqslant m-2$ and $0 \leqslant r \leqslant 2^k - 1$. Note that we can construct such graphs for all $n \geqslant 5$.

To understand the construction of Khachatrian and Haroutunian, begin with a binomial tree on $n = 2^m$ vertices. The root of the tree has m children $v_1, v_2, \ldots, v_m$ such that each v_i is the root of a binomial tree T_i on 2^{m-i} vertices. To broadcast from the root of this tree, the root calls v_i at time i for $1 \leqslant i \leqslant m$. Each vertex which learns the message at time i where $2 \leqslant i \leqslant m-1$ calls its $m-i$ children at times $i+1, \ldots, m$. The broadcast schemes for the graphs constructed in this section are based on this broadcast scheme which we will refer to as the binomial tree scheme.

To construct a broadcast graph G on $n = 2^m - 2^k - r$ vertices, Khachatrian and Haroutunian begin with the trees $T_1, T_2, \ldots, T_{m-k}$ from the above binomial tree. The union of these trees contains $2^m - 2^k$ vertices and $2^m - 2^k - (m-k)$ edges. The next step in the construction is to delete r vertices (and r edges) from T_{m-k}. This can be done by repeatedly removing a leaf furthest from the root of T_{m-k} r times. Note that we will call the resulting tree T_{m-k} below in order to simplify the notation. (This should not cause any confusion since we will have no reason to refer to the original T_{m-k} in the text below.) This union of trees $T_1, T_2, \ldots, T_{m-k}$ now contains $2^m - 2^k - r = n$ vertices and $2^m - 2^k - r - (m-k) = n - (m-k)$ edges. These remaining vertices and edges of $T_1, T_2, \ldots, T_{m-k}$ are included in G along with some additional edges.

Each vertex v_i is the root of tree T_i with 2^{m-i} vertices (or fewer, in the case of T_{m-k}). One child of vertex v_i, which we will call u_i, is the root of a subtree of T_i with 2^{m-i-1} vertices (or fewer in the case of u_{m-k}). In particular, u_i is the child of v_i which is the root of the largest subtree among those rooted at children of v_i. Let C_i denote the children of u_i in T_i. Connect every vertex of the union of trees to each root v_j, $1 \leqslant j \leqslant m-k$, except that the vertices in C_i do not connect to v_i.

To count the number of edges we added to the union of trees, note that for every root v_i, where $i \neq m-k$, we added $(n-1) - (m-i) - (m-i-1)$ edges, that is, we added one edge for every other vertex except for the $m-i$ children of v_i and the $m-i-1$ children of u_i. For the root v_{m-k}, we added at most $n-1$ edges. When T_{m-k} is a single vertex, we added exactly $n-1$ edges. Otherwise, if none of the children of v_{m-k} or of u_{m-k} were deleted, we added exactly $(n-1) - k - (k-1)$ edges. In other cases, the number of edges added lies between these two values. To calculate the total number of new edges added, we sum the above and subtract $\binom{m-k}{2}$ to avoid counting the edges between the roots twice. Thus, the number of edges added is at least $\sum_{i=1}^{m-k-1} [(n-1) - (m-i) - (m-i-1)] + [(n-1) - k - (k-1)] - \binom{m-k}{2}$ $= (n-1)(m-k) - \frac{1}{2}(m-k)(3m+k-5)$ and at most $\sum_{i=1}^{m-k-1} [(n-1)-(m-i)-(m-i-$

$1)]+(n-1)-\binom{m-k}{2}=(n-1)(m-k)-\frac{1}{2}(m-k)(3m+k-5)+(2k-1)$. Note that the exact number of added edges depends on which leaves of T_{m-k} are deleted. Adding these edges to the $n-(m-k)$ edges of the union of trees, we see that G has between $n(m-k+1)-\frac{1}{2}(m-k)(3m+k-1)$ and $n(m-k+1)-\frac{1}{2}(m-k)(3m+k-1)+(2k-1)$ edges. (Note that Ref. [16] incorrectly states the number of edges in this graph.) We can remove some of the edges from this graph G to obtain another broadcast graph G' with even fewer edges.

Consider T_i for some i in the range $1\leqslant i<m-k$. Recall that T_i is a binomial tree on 2^{m-i} vertices with root v_i. Consider a broadcast scheme for the originator v_i in T_i. Let S_l denote the set of 2^{l-1} vertices which receive the message at time l in this scheme. Let T_j be another tree (with root v_j) such that $i<j\leqslant m-k$ and let $D_{i,j}=\{(v_j,v)\,|\,v\in S_{j-i-1}\}$. To construct G' from G, delete the edges $D_{i,j}$ for all $1\leqslant i<j\leqslant m-k$. The number of edges deleted from vertices of a particular T_i to other roots v_j is $|\bigcup_{j=i+1}^{m-k}D_{i,j}|=2^0+2^1+\cdots+2^{m-k-i-1}=2^{m-k-i}-1$. Thus, the total number of edges deleted is $\sum_{i=1}^{m-k-1}(2^{m-k-i}-1)=2^{m-k}-(m-k+1)$. So, the constructed graph G' has at most $n(m-k+1)-\frac{1}{2}(m-k)(3m+k-1)+(2k-1)-(2^{m-k}-(m-k+1))=n(m-k+1)-2^{m-k}-\frac{1}{2}(m-k)(3m+k-3)+2k$.

Theorem 5.1. *The graph G' constructed as described above for $n=2^m-2^k-r$ with $0\leqslant k\leqslant m-2$ and $0\leqslant r\leqslant 2^k-1$ is a broadcast graph and*

$$B(n)\leqslant n(m-k+1)-2^{m-k}-\tfrac{1}{2}(m-k)(3m+k-3)+2k.$$

Proof. Consider an originator u in G' which is either a root v_i of some T_i for $1\leqslant i\leqslant m-k$ or is a leaf in some T_i but not an element of C_i. Such vertices are connected to all of the v_i. To broadcast, u calls v_i at time i for $1\leqslant i\leqslant m-k$. Each of these vertices then initiate a broadcast within T_i and all vertices of G' are informed by time m.

Consider an originator $u\in T_i$ $(1\leqslant i\leqslant m-k)$ which is not included in the previous case and which is not an element of C_i. Such a vertex is connected to $m-k-1$ roots and is not connected to one particular root, say v_j for $j>i$. Vertex u would be informed at time j by its parent $p(u)$ in the binomial tree scheme for T_i and would call its children at times $j+1,\dots,m$. To broadcast in G', u calls v_q at time q for $q=1,2,\dots,j-1$. At time j, v_i calls v_{j+1} and, subsequently, u calls its children as in the binomial tree scheme for T_i. All other vertices call as in their binomial tree schemes except that $p(u)$ calls v_j at time j (rather than calling u) and v_q calls v_{q+1} at time q for $j+1\leqslant q\leqslant m-k-1$. Note that these additional calls from v_q to v_{q+1} do not displace calls from the binomial tree schemes. All vertices of G' are informed by time m.

Consider an originator $u\in C_i$ for some i, $1\leqslant i\leqslant m-k$. In the binomial tree scheme for T_i, u is called by u_i at time $i+l$. Vertex u is connected to all of the roots except for v_i and (if $i+l\leqslant m-k$)v_{i+l}. To broadcast in G', u calls $v_1,v_2,\dots,v_{i-q}$ at times $1,2,\dots,i-1$, u_i at time i, $v_{i+1},v_{i+2},\dots,v_{i+l-1}$ at times $i+1,i+2,\dots,i+l-1$, v_{i+l+1} at time $i+l$, and then calls its children as in the binomial tree scheme. Vertex u_i calls

v_i at time $i+1$ and v_{i+l} at time $i+l$ (rather than calling u). Vertex v_q calls v_{q+1} at time q for $i+l+2 \leqslant q \leqslant m-k-1$. All other calls are made as in the binomial tree schemes. All vertices of G' are informed by time m. □

6. Summary of improvements to upper bounds

For $n = 2^m$ and for $n = 2^m - 2$, with $m \geqslant 1$, the exact value of $B(n)$ is known, but for other values of $n \geqslant 65$ the exact value is not known. One powerful method to produce upper bounds on $B(n)$ is compounding [2,9]. In particular, we believe that compounding produces good upper bounds for $n = 2^k - 2^l$. To produce upper bounds for n which are slightly smaller than $2^k - 2^l$, one method that has been proposed is to delete certain vertices and join the neighbors of the deleted vertices with additional edges [4,9,25]. By deleting a vertex and then adding edges, often more edges are added than are removed resulting in pairs of bounds such as $B(n) \leqslant x$ and $B(n-1) \leqslant x + c$. By merging vertices (and keeping the edges), we do not increase the number of edges. This results in bounds such as $B(n-r) \leqslant x$ for several values of $r \geqslant 0$. Theorem 2.3 is a general bound of this type. By specifying that the graph G used in the construction of the proof of Theorem 2.3 is a hypercube, we have produced improved upper bounds for many n of the form $n = 2^k - 2^l - r$ where $r < 2^l$ in Corollary 2.4. Similarly, Theorem 3.3 is another general bound and by specifying that the graph G used in the construction of the proof of Theorem 3.3 is a hypercube, we have produced a new upper bound for n of the form $n = 2^k - 2^l + 1$ in Corollary 3.4. By extending an earlier result [1] and then compounding these graphs with hypercubes, in Section 4 we improved the upper bound for n of the form $n = 2^k - 3 \cdot 2^l$ (see Theorem 4.3).

These bounds provide immediate improvements for particular n. However, these improvements can also be used (with compounding and the methods of Sections 2–4) to create additional improvements for larger n.

To determine the effectiveness of this new method, we wrote a program to calculate the new upper bounds on $B(n)$ for n in the range $1 \leqslant n \leqslant 2^{14}$ which can be obtained directly by the methods in Sections 2–4. We used the best currently known upper bounds for the same n as seed values. On the first run of the program, the upper bounds for 3594 values of n in this range were improved (that is, approximately 22%). The smallest value of n for which the upper bound was improved was $n = 123$. The previous best upper bound was $B(123) \leqslant 346$. Using Theorem 2.3, we improved this to $B(123) \leqslant 341$.

We then ran the program a second time with the 3594 new upper bounds replacing the corresponding old bounds as seed values. On the second run, an additional 3074 values were improved. Over the two runs, upper bounds were improved for 5249 values of n. Some of the bounds that were improved in the first run were further improved in the second run. For example, consider $n = 3541$. The previous best upper bound was $B(3541) \leqslant 11557$. The first run produced $B(3541) \leqslant 11520$ and the second run improved this to $B(3541) \leqslant 11515$.

Table 1
Improvements of bounds for small n

n	Previous	Our bound
123	346	341
125	379	375
127	417	416
245	759	744
246	754	744
247	749	744
249	844	832
250	843	812
251	828	819
253	921	886
255	992	960
449	1262	1260
488	1604	1584
489	1647	1612
490	1642	1612
491	1637	1612
492	1632	1612
493	1627	1612
494	1622	1612
495	1617	1612
497	1823	1792
500	1811	1748
501	1791	1764
502	1782	1764
503	1773	1764
505	1968	1920
506	1960	1898
507	1919	1905
509	2135	2036
511	2270	2176

We then ran the program a third time with the 3074 new upper bounds included as seed values. On the third run, an additional 168 improvements were obtained. After the three runs, upper bounds were improved for 5417 distinct values of n in the range $1 \leqslant n \leqslant 2^{14}$ (or approximately 33%).

A fourth run of the program yielded no further improvements. Thus, using our new methods iteratively, we improved approximately 33% of the upper bounds on $B(n)$ for $1 \leqslant n \leqslant 2^{14}$.

As some indication of the results obtained, we have included two tables. In Table 1, we show all of the improvements obtained for n in the range $123 \leqslant n \leqslant 511$. In Table 2, we show all of the improvements obtained for n in the range $16,321 \leqslant n \leqslant 16,383$. These ranges are not claimed to be particularly significant.

In Section 5, we described a modification of the construction of Khachatrian and Haroutunian [16] which improves the asymptotic upper bound on $B(n)$. Recall that Grigni and Peleg [13] showed that $B(n) \in \Theta(nL(n))$ where $L(n)$ is the number of

Table 2
Improvements of bounds for large n

n	Previous	Our bound
16321	92169	90112
16336	91884	89840
16337	90341	89936
16338	90314	89936
16339	90287	89936
16340	90260	89936
16341	90233	89936
16342	90206	89936
16343	90179	89936
16344	90152	89936
16345	90125	89936
16346	90098	89936
16347	90071	89936
16348	90044	89936
16349	90017	89936
16350	89990	89936
16351	89963	89936
16353	96294	94208
16360	96112	94064
16361	94361	94116
16362	94326	94116
16363	94291	94116
16364	94256	94116
16365	94221	94116
16366	94186	94116
16367	94151	94116
16369	100379	98304
16372	100277	98232
16373	98388	98256
16374	98344	98256
16375	98300	98256
16377	104452	102400
16378	104409	102362
16379	102429	102375
16381	110517	106477
16383	114626	110592

consecutive leading 1's in the binary representation of the integer $n-1$. Let $2^{m-1} < n < 2^m$ where $m \geqslant 8$. It is interesting to consider the bounds on $B(n)$ for the eight subintervals of this range of equal size. Let I_i denote the range of values $2^{m-1} + i \cdot 2^{m-4} < n \leqslant 2^{m-1} + (i+1)2^{m-4}$ for $0 \leqslant i \leqslant 6$ and I_7 denote the range of values $2^{m-1} + 7 \cdot 2^{m-4} < n < 2^m$.

Gargano and Vaccaro [12] and Ventura and Weng [24] present upper-bound formulas which produce good bounds for these subintervals when n is small. These formulas are all at least $\frac{1}{9}(m+8)n$ for $n \in I_i$ with $0 \leqslant i \leqslant 3$. However, Theorem 5.1 shows that $B(n) < 3n$ for $n \in I_i$ with $0 \leqslant i \leqslant 3$. In subintervals I_4 and I_5, the formulas previously reported are at least $\frac{1}{13}(4m+2)n$. Theorem 5.1 shows that $B(n) < 4n$ for $n \in I_i$ with $4 \leqslant i \leqslant 5$. In subinterval I_6, the formulas previously reported are at least $\frac{1}{4}mn$. Theorem 5.1 shows that $B(n) < 5n$ for $n \in I_6$. When $n \geqslant 2^{20}$, our new upper bound on $B(n)$ is better than any previous upper bounds for all n in subintervals $I_0, \ldots, I_6$. Note that in particular, the previous upper bound formulas for subintervals $I_0, \ldots, I_6$ are all $\mathcal{O}(n \log n)$ where our new upper bounds are $\mathcal{O}(n)$.

Finally, in subinterval I_7, although we do not give an explicit upper bound formula, the methods of Sections 2–4 give numerous improvements in this range. For example, when $m = 12$, 195 of 256 values in subinterval I_7 (that is, 76%) are improved, when $m = 13$, 388 of 512 values in subinterval I_7 (76%) are improved, and when $m = 14$, 896 of 1024 values in subinterval I_7 (86%) are improved.

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