An Algorithm for Constructing Minimal *c*-Broadcast Networks

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c-Broadcasting is a special process to disseminate a single message, originated at any node in a network, to all other nodes of the network by letting each informed node transmit the message to at most c neighbors simultaneously. A minimal c-broadcast network (c-mbn) is a communication network in which c-broadcasting can be completed in minimum time from any node. An optimal c-broadcast network (c-obn) is a c-mbn with the smallest number of edges. Previous studies showed that c-obn's are extremely difficult to find. A network compounding algorithm is proposed to construct sparse c-mbn's with $n_1n_2 - i$ nodes by connecting a subset of nodes from several copies of a c-mbn with n_1 nodes using the structure of another c-mbn with n_2 nodes, such that $n_1 \geq 1$ and $0 \leq i < n_2$, satisfying $\lceil \log_{c+1}(n_1n_2 - i)\rceil = \lceil \log_{c+1} n_1\rceil + \lceil \log_{c+1} n_2\rceil$. Computational results are also provided.
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1. INTRODUCTION

Communication in networks is an information dissemination process in which a set of messages, generated by a set of originators, is transferred to a set of receivers by placing calls over the communication lines. The nodes of the network are originators, receivers, or both, and the edges are the communication lines that allow the direct transmission of messages between certain pairs of nodes. There have been many studies on network-design problems in communication networks. These network-design problems can be classified by placing constraints on the number of messages, the numbers of originators and receivers, the characteristics of edges, the network topology, and the transmission characteristics of communication lines [1, 2, 4–7, 9–18]. A communication network can be modeled as a connected graph $G = (V(G), E(G))$ without loops or parallel edges, in which $V(G)$ is a set of nodes with cardinality $n(G)$ and $E(G)$ is a set of undirected or directed edges, connecting certain pairs of nodes in $V(G)$, with cardinality $e(G)$.

c-Broadcasting (also known as *k-broadcasting*) is a special process to disseminate a single message, originated at any node in an undirected network, to all other nodes under the following constraints:

(i) Each message transmission takes one unit of time.
(ii) A node can make at most c simultaneous transmissions in one time unit.
(iii) A node can only transmit the message to its neighbors (two nodes are neighbors if they are adjacent).

Applications of c-broadcasting are found easily in parallel and distributed computing. In parallel and distributed computing, parallel architectures related to a set of processors and an interconnection network have been studied extensively. The current state of the art is such that the communication time (between processors) is much more expensive than is the computation time (inside a processor). Frequently, researchers consider only the communication time for the overall time complexity. In reality, a direct communication line from each processor to every other processor may be desired. However, it is clear that the number of interconnections grows quadratically with the number of processors in the system. From a technological viewpoint, this is impractical. Therefore, other interconnection topologies such as the mesh and the hypercube are more common. The network may also be irregular and the number of neighbors of each processor may be constant or variable. The

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broadcasting processes differ depending on whether a processor may send the message over all links simultaneously (all-port communication model), whether it is restricted to use one link at a time (one-port communication model), or whether it is allowed to send simultaneously a message over c links (c-port communication model) [3,19]. Situations rely on physical restrictions of the system. The c-port communication model is the most general since it includes the other two as special cases. This paper describes the c-port communication model as c-broadcasting.

The *c-broadcast time of a node* $b_c(v, G)$ is defined as the minimum number of time units required to perform c-broadcasting from node v in graph G and the *c-broadcast time of a graph* $b_c(G)$ is defined as the minimum number of time units in which c-broadcasting can be completed in G regardless of the originator, that is, $b_c(G) = \max\{b_c(v, G) | v \in V(G)\}$. In c-broadcasting, the number of informed nodes can at most be multiplied by $c + 1$ in each time unit. This implies that after m time units the number of informed nodes is at most $(c + 1)^m$. Thus, $b_c(G) \geq \lceil \log_{c+1} n \rceil$. A *minimal c-broadcast network* (*c-mbn*) is defined as a communication network G with n nodes such that $b_c(G) = \lceil \log_{c+1} n \rceil$. An *optimal c-broadcast network* (*c-obn*) is a c-mbn with the minimum number of edges. If G is a c-obn, then $b_c(G^*) > \lceil \log_{c+1} n \rceil$ for every proper spanning subgraph $G^* \subset G$, that is, $V(G^*) = V(G)$ and $E(G^*) \subset E(G)$. The *c-broadcast function* $B_c(n)$ is defined as the number of edges of every c-obn with n nodes.

From an application perspective, c-obn's represent the cheapest possible communication networks (having the fewest communication lines) in which c-broadcasting can be accomplished, from any node, as quickly as theoretically possible.

Farley et al. [5] showed that 1-broadcasting from an originator v determines a rooted spanning tree of G. Similarly, c-broadcasting also determines a spanning tree rooted at the originator v. A *c-broadcast protocol* (or *c-broadcast tree*) $P_c(v, G)$ is a rooted spanning tree with n nodes in which the originator v is the root and all nodes are labeled by their receiving times, which are equal to or less than $b_c(G)$. In a c-broadcast protocol, each edge is used exactly once and the message is always transmitted from parent to child. For a network G to be a c-mbn, each node in the network must have a c-broadcast protocol that can be completed in $\lceil \log_{c+1} n \rceil$ time units.

Let v be a node having the message in a communication network. If node v does not have any uninformed neighbor at time t, then it is *idle* at time $t + i$, for $i \geq 1$. If node v has at least one but less than c uninformed neighbors at time t, then it is *partially idle* at time $t + 1$ and idle at time $t + i$, for $i \geq 2$.

A 3-mbn with 10 nodes and its broadcast protocol are given in Figure 1. The 3-mbn with 10 nodes is shown in Figure 1(a). All nodes are topologically equivalent. Thus, the only 3-broadcast protocol is illustrated in Figure 1(b). The graph shown in Figure 1(a) is the well-known Petersen graph, which is of great interest. Lee and Ventura [14] showed that this graph is the only 3-obn with 10 nodes.

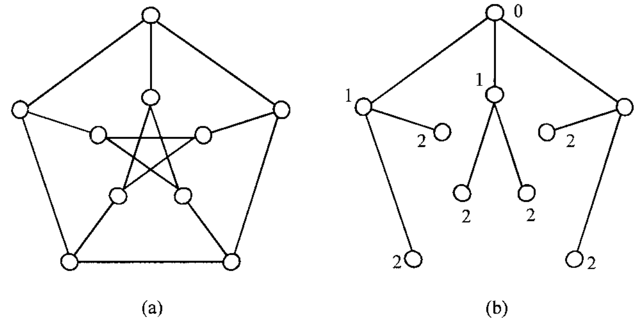

FIG. 1. 3-obn with 10 nodes (the Petersen graph) and its 3-broadcast protocol.

For the complete graph K_n with $n \geq 2$ nodes, $b_c(K_n) = \lceil \log_{c+1} n \rceil$. Thus, K_n is a c-mbn, not necessarily a c-obn, since some edges can be removed from K_n and the resulting graph G still satisfies the condition $b_c(G) = \lceil \log_{c+1} n \rceil$.

In [9], Grigni and Peleg introduced c-broadcasting by using the concept of conference-call, through which a node is allowed to send a message simultaneously to up to c neighbors at a time, for some constant $c \geq 1$. They showed that $B_c(n) \in O(cnL_c(n))$, where $L_c(n)$ denotes the exact number of consecutive leading c's in the $(c+1)$-ary representation of $n - 1$. Another version of conference-call broadcasting was investigated by Richards and Liestman [15]. They studied c-broadcasting in complete $(c + 1)$-uniform hypergraphs, in which a conference call involves the nodes of some hyperedges. Lazard [13] studied c-broadcasting in bounded-degree graphs and provided some c-mbn's for small values of n and c. Some of his results are used as initial data in the proposed construction method. Farley and Proskurowski studied bounded-call broadcasting where the maximum number of calls from a node is a predetermined constant [7]. They also researched whispering ($c = 1$) and shouting ($c = \infty$) [6].

Farley et al. suspected that the problem of determining $B_1(n)$ for an arbitrary n is NP-complete [5]. Thus, it is natural to conjecture that the problem of determining $B_c(n)$ for an arbitrary n is also NP-complete.

c-Broadcasting can be particularized by limiting each conference call to only two participants. This special case of c-broadcasting, called 1-broadcasting or just "broadcasting," where a node can call at most one neighbor in one time unit, has been studied extensively [1, 2, 4, 5, 9–11, 16–18]. Since many studies showed that even 1-obn's are extremely difficult to find, several researchers have proposed methods to construct networks with small numbers of edges, which allow 1-broadcasting from any node. The main idea is to produce large 1-mbn's by combining small known 1-mbn's using as few edges as possible without violating the broadcast time constraint from any node.

There is no known feasible method for determining $B_c(n)$ for an arbitrary number of nodes n. Therefore, in this paper, a network compounding algorithm is proposed to construct sparse c-mbn's with $n_1n_2 - i$ nodes by connecting a subset of nodes from several copies of a c-mbn with n_1 nodes using the structure of another c-mbn with n_2 nodes, such that $n_1 \geq 1$ and $0 \leq i < n_2$, satisfying $\lceil \log_{c+1}(n_1n_2 - i) \rceil = \lceil \log_{c+1} n_1 \rceil + \lceil \log_{c+1} n_2 \rceil$. One node from (each of) i copies of the c-mbn with n_1 nodes is deleted before compounding.

The proposed method is an extension to c-broadcasting from the compounding algorithm of Dinneen et al. [4] for 1-broadcasting. Section 2 introduces the concepts of official c-broadcasting, c-center node, and c-center node set. Section 3 presents the network compounding algorithm for c-broadcasting with some examples. Section 4 describes issues concerning the implementations of the proposed iterative algorithm and the improvement over the compounding algorithm of Dinneen et al. [4]. It provides computational results that show the best known c-mbn's with n nodes for $2 \leq c \leq 4$ and $1 \leq n \leq 128$.

For $n = (c + 1)^p$ and $p \geq 1$, $B_c(n) = cnp/2 = (cp/2)(c+1)^p$ [13] and the c-obn with $(c+1)^p$ nodes is a cp-regular graph [9]. For $n \leq c+1, B_c(n) = (n/2)(n-1)$, and the c-obn is a K_n [13]. For $n = c + 2, B_c(n) = n - 1$ and the c-obn with $c+2$ nodes is a graph consisting of a node with degree $n - 1$ and all other nodes with degree one [13]. Konig and Lazard [12] identified c-obn's for $c + 3 \leq n \leq 2c + 3$. These results only provide c-obn's for a narrow range of nodes. In [14], Lee and Ventura provided a lower bound for the size in the family of c-obn's with n nodes such that $n = (c + 1)^m - k$, where $0 < k < c$ and $m > 0$. Note that for $n \geq c + 3$, if G is a $(c - 1)$-mbn with n nodes and $\lceil \log_c n \rceil = \lceil \log_{c+1} n \rceil$, then G is also a c-mbn.

Lee and Ventura [14] defined the *maximum c-broadcast tree* with respect to node u, degree d for u, maximum degree Δ, and time t, as the c-broadcast tree rooted at u with the largest order $n_c(t, \Delta, d)$, in which all nodes can be informed in t (additional) time units through c-broadcasting with the restriction of degree d for root u and maximum degree Δ for the other nodes. Using the concept of the maximum c-broadcast tree, the following theorem can be proved:

Theorem 1. For a given c-mbn G with n nodes, if $(c - 1)(c + 1)^{m-1} + 1 < n \leq (c + 1)^m$, then $\min\{d(v)|v \in V(G)\} \geq c$, where $d(v)$ is the degree of v.

Proof. Let v be a node in G with minimum degree. Assume that the degree of v is $c-1$. Let $v_1, v_2, v_3, \dots, v_{c-1}$ be the neighbors of v. In a c-broadcast protocol rooted at v, all v_i's can be informed in the first time unit. In the maximum c-broadcast tree rooted at v, the order of the subtree rooted at any v_i is $(c + 1)^{m-1}$. Therefore, in m units of time, the total number of informed nodes by c-broadcasting is at most $1+(c-1)(c+1)^{m-1}$, which is less than n. Thus, c-broadcasting from node v cannot be completed in m time units. Hence, $\min\{d(v)|v \in V(G)\} \geq c$. ∎

The contraposition of Theorem 1 provides an upper bound for the order of a c-mbn when the minimum degree is given. For a given c-mbn G with n nodes, if $\min\{d(v)|v \in V(G))\} < c$, then $n < (c - 1)(c + 1)^{m-1}$.

2. OFFICIAL c-BROADCASTING, c-CENTER NODE, AND c-CENTER NODE SET

The proposed algorithm constructs sparse c-mbn's with an arbitrary number of nodes n. In this section, so as to ensure the existence of c-broadcast protocols from any node of the constructed networks, the idea of official c-broadcasting is introduced. This leads to the definitions of c-center node and c-center node set, which are influential in determining the number of edges added in the proposed network compounding algorithm. In addition, bounds for the cardinality of a c-center node set as well as bounds for the minimum degree and the order of a c-mbn are presented.

Weng and Ventura [18] introduced the concepts of official broadcasting, center node, and center node set in order to describe the doubling procedure for 1-broadcasting. These concepts are extended to c-broadcasting in order to outline the proposed network compounding algorithm.

While in c-broadcasting there exists only one kind of message, two kinds of message can exist in official c-broadcasting, which are official and unofficial. In official c-broadcasting, a subset of nodes, called *c-center nodes*, is capable of making an unofficial message official. If the originator belongs to this subset of nodes, the message is official from the start of c-broadcasting; otherwise, the message is unofficial at the beginning. In official c-broadcasting, all nodes must receive an official message. An unofficial message, originated at a noncenter node, must be made official by c-center nodes during official c-broadcasting and then transmitted to all nodes of the network. An unofficial message becomes official on its arrival at a center node.

A c-center node always receives one message, but a noncenter node may receive one or two messages. In the first case, the message must be official. In the second case, the first message must be unofficial and the second one official. It is possible for a noncenter node to send an unofficial message to c (or fewer) neighbors and receive an official message from another neighbor in the same time unit. Note that a noncenter node cannot send the unofficial message to and receive its officialized message from the same neighbor in one time unit.

An official c-broadcast protocol for a node v with respect to a set $S_c \subseteq V(G)$ in a graph G, denoted by $P_c(v, G; S_c)$, is a proper spanning subgraph (not necessarily a proper spanning tree) of G containing all of its nodes that are labeled by one or two receiving times, which are at most $b_c(G)$. If the originator $v \in S_c$, each node has a single receiving time and the official c-broadcast protocol for v is a spanning tree rooted at v, that is, just a c-broadcast protocol. Otherwise, when $v \notin S_c$, some nodes have a single receiving time and the others, which are not in S_c, have two receiving times; the first receiving time is for the unofficial message and the second one for the official message.

Given an official protocol $P_c = P_c(v, G; S_c)$, the unofficial part P_c^u of P_c is defined as the tree rooted at v induced by all edges that are involved in the transmission of the unofficial message and their incident nodes. The official part P_c^o of P_c is a forest of rooted trees induced by all edges, which transmit the official message, and their incident nodes. These trees are rooted at nodes in S_c, which receive the unofficial message. The set of all such nodes is denoted by $V_{cu}(P_c)$. Since the message officialized by the nodes $V_{cu}(P_c)$ is sent back to their ascendants, the forest P_c^o spans G.

If each node of a graph G has an official c-broadcast protocol of time $b_c(G)$ with respect to S_c, then S_c is called a *c-center node set* (*c-cns*) of G. Since we are only concerned about c-mbn's, unless specified otherwise, all protocols are assumed to take $\lceil \log_{c+1} n(G) \rceil$ time units.

Due to the usage of c-cns's, the attention goes to the c-cns S_c with the smallest cardinality in the proposed compounding algorithm of G, which is called an *optimal c-center node set* (*c-ocns*), such that any superset $S^* \supseteq S_c$ is a c-cns as well. Given a graph G, the *c-center node number* $cn_c(G)$ is defined to be the cardinality of a c-ocns.

In an official c-broadcast protocol for a node $v \in V(G)$, if any neighbor of v is idle or partially idle prior to $b_c(G)$, node v is not a center node and can receive the official message back from one of its neighbors. There may exist multiple c-ocns's for a given graph G while $cn_c(G)$ remains the same. As in 1-broadcasting [4], c-ocns's of nonisomorphic graphs with n nodes and m edges may be different.

An example of official c-broadcasting is provided in Figure 2. A 2-mbn with 11 nodes is shown in Figure 2(a). In Figure 2(a), the black nodes form a 2-cns and the white nodes are noncenter nodes. The official 2-broadcast protocols for the 2-center nodes are not included. There are two classes of noncenter nodes that are topologically different: nodes with degree 2 and nodes with degree 3. Official 2-broadcast protocols for the two classes of noncenter nodes are shown in Figure 2(b, c), respectively.

As mentioned in Section 1, the proposed network compounding algorithm constructs sparse c-mbn's of larger order by combining copies of two c-mbn's of smaller order. Thus, c-mbn's of small order are essential for the compounding process. Some c-mbn's ($2 \le c \le 4$) for the initial data set are given in Figures 3–5.

There is no known method for computing $cn_c(G)$ for an arbitrary graph G. A c-ocns for an arbitrary network G can be found by the exhaustive-enumeration method, which is very time-consuming. It is conjectured that the problem of finding a c-ocns is NP-complete since it is NP-complete to determine $b_c(v, G)$ for an arbitrary node v, as shown in [16] for $c = 1$.

Lemma 1. Let G be a c-mbn with n nodes, where $(c+1)^{m-1} < n \le (c+1)^m$, and let $S \subseteq V(G)$. If S is a vertex cover for G such that every neighbor of every node not in S has degree at most $cm - 1$, then S is a c-cns for G.

Proof. Let $v \notin S$. In every c-broadcast protocol P for v, one of the neighbors that receive the unofficial message in the first time unit must be idle or partially idle at a time no greater than m and, therefore, can send the official message back to v. All other nodes already have an official message and so this yields an official c-broadcast protocol for v. ∎

It is possible to derive a lower bound for $cn_c(G)$ that is valid under special conditions on $n(G)$. The case of 1-broadcasting was also studied in [4]. This result holds in the case when originator v is neither idle nor partially idle.

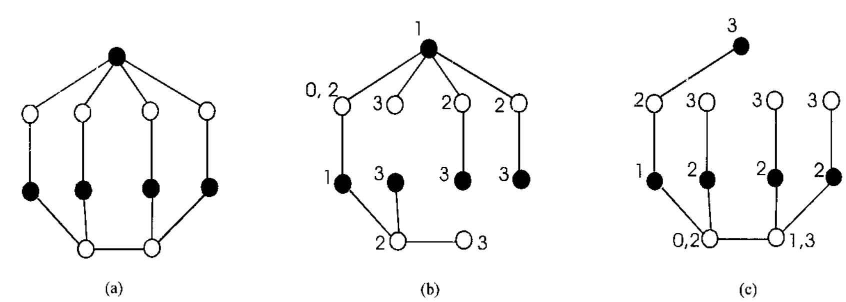

FIG. 2. 2-mbn with 11 nodes and its official protocols for two classes of noncenter nodes.

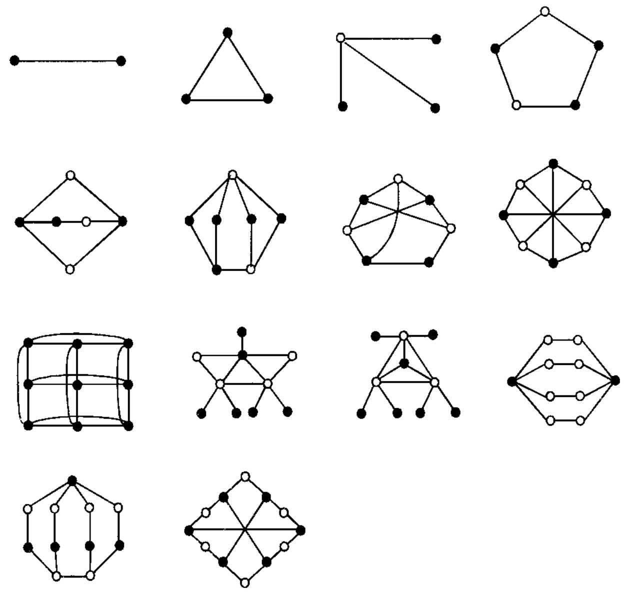

FIG. 3. Small 2-mbn's with n nodes, where $2 \le n \le 12$, and minimal c-cns's.

Lemma 2. Let G be a c-mbn with n nodes, where $(c+1)^{m-1} < n \le (c+1)^m$. Suppose that, in a c-broadcast protocol for $v \in V(G)$, v itself is idle in the last r time unit(s) for some $0 \le r < m$. Node v may be partially idle at time $m-r$. Then,

(i) $n(G) \le (c+1)^m + \{d(v) - c(m-r-1)\}(c+1)^r - (c+1)^{r+1} + 1 \le (c+1)^m - (c+1)^r + 1$.
(ii) The equality occurs if v is idle for the last r time units and no other node is either idle or partially idle. The second equality occurs if in addition the degree of v is a multiple of c.

Proof. $\lceil \log_{c+1} n \rceil = m$. Denote by $I(i)$ the number of nodes informed by time i. If v is partially idle at time $m-r$, the $0 < d(v) - (m-r-1)c < c$ neighbor(s) of v do not receive the message originated by v until time $m-r$. Otherwise, $d(v) - (m-r-1)c = c$. Then,

$$\begin{aligned}
I(m-r-1) &\le (c+1)^{m-r-1},\\
I(m-r) &\le I(m-r-1) + c\{I(m-r-1) - 1\}\\
&\quad + d(v) - c(m-r-1)\\
&= (c+1)I(m-r-1)\\
&\quad + d(v) - c(m-r-1) - c\\
&\le (c+1)^{m-r} + d(v) - c(m-r-1) - c,\\
I(m-r+1) &\le I(m-r) + c\{I(m-r) - 1\}\\
&= (c+1)I(m-r) - c\\
&\le (c+1)^{m-r+1} + \{d(v) - c(m-r-1)\}\\
&\quad \times (c+1) - c(c+1) - c,\\
I(m-r+2) &\le I(m-r+1) + c\{I(m-r+1) - 1\}\\
&= (c+1)I(m-r+1) - c\\
&\le (c+1)^{m-r+2} + \{d(v) - c(m-r-1)\}\\
&\quad \times (c+1)^2 - c(c+1)^2 - c(c+1) - c,\\
&\ \vdots\\
I(m) &\le I(m-1) + c\{I(m-1) - 1\}\\
&= (c+1)I(m-1) - c\\
&\le (c+1)^m + \{d(v) - c(m-r-1)\}\\
&\quad \times (c+1)^r - c(c+1)^r - c(c+1)^{r-1}\\
&\quad - \cdots - c\\
&\le (c+1)^m + \{d(v) - c(m-r-1)\}\\
&\quad \times (c+1)^r - (c+1)^{r+1} + 1. \qquad (1)
\end{aligned}$$

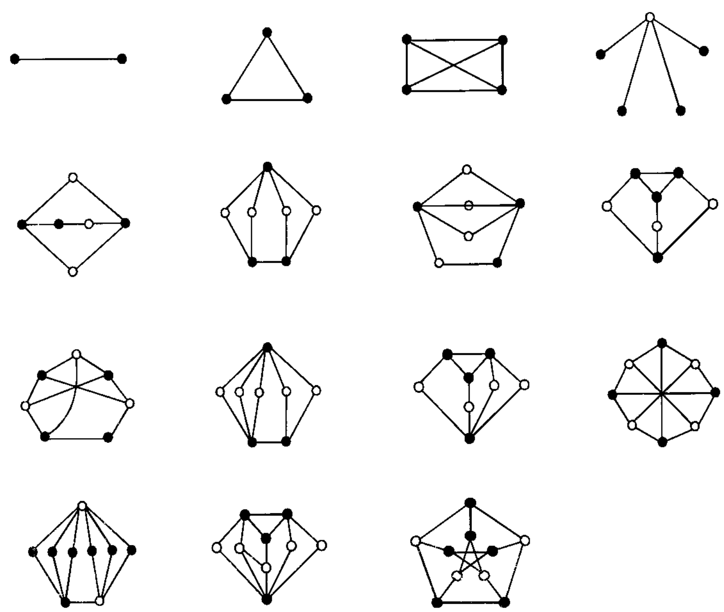

FIG. 4. Small 3-mbn's with n nodes, where $2 \le n \le 10$, and minimal c-cns's.

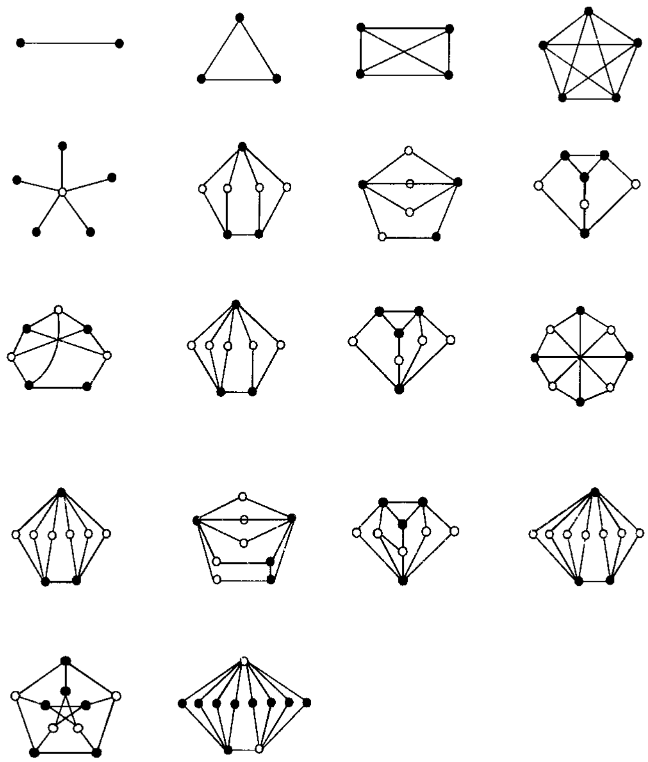

FIG. 5. Small 4-mbn's with n nodes, where $2 \le n \le 11$, and minimal c-cns's.

When $d(v)$ is a multiple of c, that is, v is not partially idle at time $m - r$, (1) can be rewritten as

$$I(m) \le (c+1)^m + \{d(v) - c(m-r-1) - c - 1\}(c+1)^r + 1 = (c+1)^m - (c+1)^r + 1.$$

Therefore, the maximum number of nodes which can be informed in m time units during the c-broadcasting is $(c+1)^m + \{d(v) - c(m-r-1)\}(c+1)^r - (c+1)^{r+1} + 1$ when v is partially idle at time $m - r$ and, otherwise, $(c+1)^m - (c+1)^r + 1$. ∎

This bound matches the bound that is derived from the concept of maximum c-broadcast tree in [14], which is $I(m) \le n_c(t, \Delta, d) = (c+1)^m - (c+1)^{m-\lfloor d(v)/c \rfloor} + (d(v) - c \cdot \lfloor d(v)/c \rfloor)(c+1)^{m-\lceil d(v)/c \rceil} + 1$. The bound in Lemma 2 can be used even though the minimum degree is unknown.

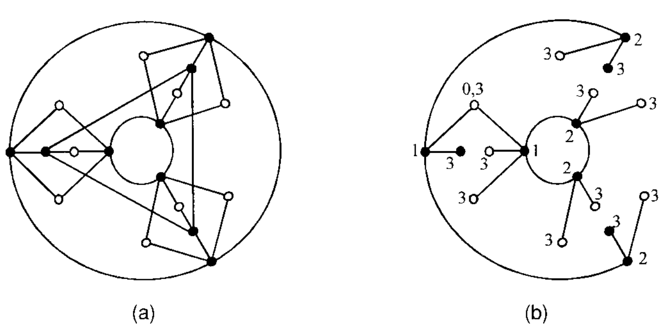

FIG. 6. 2-mbn with 18 nodes and one of its official 2-broadcast protocols.

Using Lemma 2, we can derive a lower bound for the minimum degree in a network with n nodes. Let $\delta(G)$ be the minimum degree of a node in a network G.

Lemma 3. Let G be a c-mbn with n nodes such that $(c+1)^{m-1} < n \le (c+1)^m$. Then,

(i) $\delta(G) \ge c \cdot [m - \lfloor \log_{c+1}\{(c+1)^m + 1 - n\} \rfloor] - c + 1$.
(ii) $B_c(n) \ge (cn/2)[m - \lfloor \log_{c+1}\{(c+1)^m + 1 - n\} \rfloor] - (c-1)n/2$.

Proof. Let $v \in V(G)$ be a node with the minimum degree. The proof is obvious when $d(v) > cm$. Otherwise, $d(v) \le cm$. In this case, node v is either partially idle or idle in the last $m - \lfloor d(v)/c \rfloor$ units. If $d(v)$ is a multiple of c, then v is idle in the last $m - \lceil d(v)/c \rceil$ units. Therefore, $n \le (c+1)^m - (c+1)^{m-\lceil d(v)/c \rceil} + 1$ by Lemma 2. This implies that $\lceil d(v)/c \rceil \ge m - \log_{c+1}\{(c+1)^m + 1 - n\}$. Solving for $d(v)$ yields $d(v) \ge c \cdot [m - \lfloor \log_{c+1}\{(c+1)^m + 1 - n\} \rfloor] - c + 1$. Part (i) is proved. Part (ii) follows immediately by multiplying the lower bound of $\delta(G)$ by $n/2$. ∎

Lemma 3 provides tighter bounds than Lemma 3.2 in [12], which is an extension of Theorem 2.2 in [9]. The idea of the minimum degree leads to the following two lemmas. For $d \in Z$, let $V_d(G)$ be the set of nodes of degree d in c-mbn G.

Lemma 4. Let G be a c-mbn with $(c+1)^m - (c+1)^r + 1$ nodes, where $0 \le r < m$. Let $\delta = c(m - r)$. Then, $cn_c(G) \ge |V_\delta(G)|$. Equality holds when $V_\delta(G)$ is a vertex cover.

Proof. Let v be a node with the degree δ. By Lemma 2, δ is automatically the minimum degree in G. In an official c-broadcast protocol for v with respect to any subset $S \subseteq V(G)$, the only node that can possibly be either partially idle or idle is v itself by Lemma 2. If node v is a noncenter node, no node can send an official message back to v and so, necessarily, $v \in S$. This proves that $cn_c(G) \ge |V_\delta(G)|$. By Lemma 1, the equality is true if $V_\delta(G)$ is a vertex cover. ∎

Lemma 5. Let G be a c-mbn with $n(G) \le (c+1)^m - (c+1)^{m-\lfloor \delta/c \rfloor} + (\delta - c\lfloor \delta/c \rfloor)(c+1)^{m-\lceil \delta/c \rceil} + 1$ nodes, where $0 \le \delta \le cm$. Then, $cn_c(G) \ge |V_\delta(G)|$. Equality holds when $V_\delta(G)$ is a vertex cover.

The proof of Lemma 5 is similar to that of Lemma 4. If the number of edges in a graph is reduced, while keeping the resulting graph still a c-mbn, then $cn_c(G)$ may grow. Finally, general bounds for $cn_c(G)$ are given in Theorem 2.

Theorem 2. Let G be a c-mbn with n nodes such that $(c+1)^{m-1} < n \le (c+1)^m$ for $m \in Z^+$:

(i) $cn_c(G) = n = (c+1)^m$, if $n = (c+1)^m$,
$cn_c(G) \ge m - \lfloor \log_{c+1}\{(c+1)^m - n\} \rfloor$, if $n < (c+1)^m$.

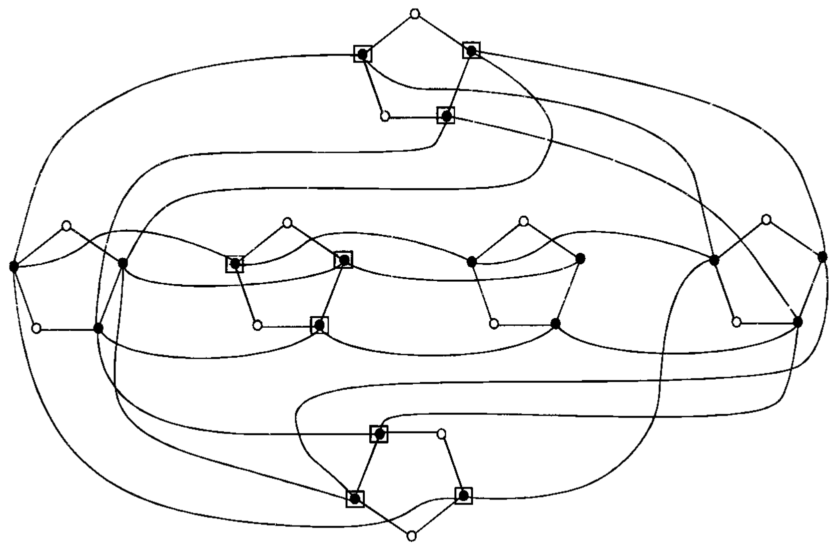

FIG. 7. 2-mbn with 30 nodes by compounding 2-mbn with five nodes into 2-mbn with six nodes.

(ii) If equality occurs in the second equation of (i), then G has a c-ocns, which is a vertex cover.
(iii) If there is a subset $R \subseteq V(G)$ of nonadjacent nodes, whose neighbors have degree at most $cm - 1$, then $cn_c(G) \leq n - |R|$.

Proof. (i) If $n = (c + 1)^m$, then no node can be idle during c-broadcasting. Therefore, all nodes should belong to the c-cns and it is the only c-ocns. If $n < (c+1)^m$, then let p be the cardinality of a c-cns for G. If $p > m$, the result is obvious. Thus, assume that $p \leq m$. Let v be a noncenter node. Then, the maximum number of nodes with the official message at time p is $(c + 1)^p - 1$, and this occurs only if all neighbors of v are c-center nodes. The maximum number of nodes in the official forest at time m is $(c + 1)^{m-p}\{(c + 1)^p - 1\}$. Therefore, $(c + 1)^m - (c + 1)^{m-p} \geq n$. Solving for p yields $p \geq m - \log_{c+1}\{(c+1)^m - n\}$ and (i) follows since $p \in Z$. Part (ii) is immediate since all neighbors of v are c-center nodes. Part (iii) holds by Lemma 1 since $V(G) - R$ is a vertex cover for G. ∎

3. MINIMAL c-BROADCASTING NETWORK-COMPOUNDING ALGORITHM

Similar to that of 1-broadcasting [4,18], the following definition will be used in the rest of the paper. Let $(n_1, n_2, i)_c$ be a triple of integers with $n_1 \geq 1$ and $0 \leq i \leq n_2$. Then,

$$\lceil \log_{c+1}(n_1 n_2 - i) \rceil \leq \lceil \log_{c+1} n_1 \rceil + \lceil \log_{c+1} n_2 \rceil.$$

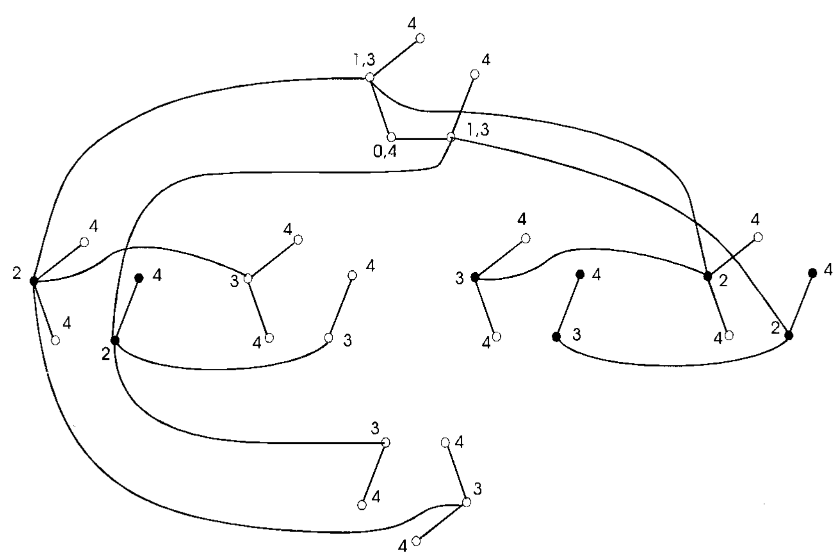

FIG. 8. Official 2-broadcast protocol of 2-mbn in Figure 7.

It is said that the triple $(n_1, n_2, i)_c$ satisfies the *broadcast condition* in c-broadcasting if the equality holds. If $i = 0, (n_1, n_2)_c = (n_1, n_2, 0)_c$.

Compounding a graph G into another graph H, relative to a set $S \subseteq V(G)$, can be performed by replacing each node of H with a copy of G and each edge of H with a matching between the corresponding copies of $s \in S$. The resulting graph i s denoted by $G_s[H]$, where $n(G_s[H]) = n(G) \cdot n(H)$. The algorithm for this compounding will be presented in Section 3.1. Moreover, to construct c-mbn's for an arbitrary number of nodes, a node may be deleted from certain copies of G before compounding, which is called the vertex deletion method. It is incorporated into the network compounding algorithm in Section 3.2.

All the compounding methods have been devised only for 1-broadcasting. In [11], a compounding algorithm for 1-broadcasting was developed in which G is an 1-mbn with the restriction of maximum degree bounded by $\lceil \log_2 n(G) \rceil - 1$ and S is a vertex cover of G and $H = K_2$. In [2], a more general compounding procedure for 1-broadcasting was presented, in which G and H are general 1-mbn's satisfying the broadcast condition, and S is called a *solid h-cover* of G. It is not possible to generate 1-mbn's with an arbitrary number of nodes n by their procedure (e.g., when n is prime).

Weng and Ventura [18] proposed a more general compounding algorithm, called the *doubling procedure*, which includes the procedure of [2] as a special case. The doubling procedure's extra generality arises mainly from the vertex deletion method. The method constructs a large 1-mbn from given 1-mbn's G and H and an integer i with $0 \le i \le n(H) - 1$, such that $(n(G), n(H), i)_1$ satisfies the broadcast condition. The subset $S \subseteq V(G)$ used to connect the copies of G is called a *center-node set*. Dinneen et al. [4] devised a center node reduction method, which is a procedure to change center nodes with certain properties to noncenter nodes after compounding, and implemented their compounding algorithm for 1-broadcasting. However, their implementation allowed a possible deterioration of the results by adding parallel edges, which can be removed, in their vertex deletion method.

3.1. Network Compounding Algorithm with Center-node Reduction

This section describes the network compounding algorithm for a triple of integers $(n_1, n_2, 0)_c$ satisfying the broadcast condition, that is, when a pair of integers $(n_1, n_2)_c$ satisfies the broadcast condition.

Let G and H be c-mbn's, and S_c, a c-cns for G. A network $\Gamma = G_{Sc}[H]$ can be constructed by connecting $n(H)$ copies of s, for each $s \in S_c$, to form a copy of H_s, which is the same as H. The resulting c-cns for Γ contains only the nodes that are c-center nodes in copies of both G and H. The term *center-node reduction* comes from the fact that some center nodes in certain copies of H can be changed to noncenter nodes.

Theorem 3. Let G and H be c-mbn's, and S_c and T_c, c-cns's for G and H, respectively. H_s is the corresponding copy of H with respect to $s \in S_c$. Suppose that $(n(G), n(H))_c$ satisfies the broadcast condition. Then, the resulting graph $\Gamma = G_{Sc}[H]$ is a c-mbn with $n(G) \cdot n(H)$ nodes and the Cartesian product $\sum_c = S_c \times T_c$ is a c-cns for Γ.

Proof. Let v be a node of G. If the originator $v \in \sum_c$, then first broadcast officially in H_s with respect to T_c. After $\lceil \log_{c+1} n(H) \rceil$ time units, all nodes in H_s, which v belongs to, have the official message. Then, from those nodes already informed, broadcast in the appropriate copies of G. All nodes of Γ can be officially informed by no later than $\lceil \log_{c+1} n(G) \rceil + \lceil \log_{c+1} n(H) \rceil$ time units.

Now suppose that the originator $v \notin \sum_c$, in which there are two possible cases: One is the case of $v \notin S_c$ and the other is the case that v is not in T_c but in S_c. Let P_c be an official c-broadcast protocol for v in G with respect to S_c. Recall the notation P_c^u, P_c^o, and $V_{cu}(P_c)$ from Section 2. Broadcast in G, which v belongs to, according to P_c^u, ending at $V_{cu}(P_c)$. On receiving the message, each element of $V_{cu}(P_c)$ then broadcasts within its corresponding copy of H, and this broadcasting is official and takes $\lceil \log_{c+1} n(H) \rceil$ time units. As soon as this is completed in each copy of H, each node in each copy of $V_{cu}(P_c)$ continues according to P_c^o. Since it takes $\lceil \log_{c+1} n(G) \rceil$ time units to perform official c-broadcasting in each copy of G according to P_c^u and P_c^o, the total time needed to broadcast officially in Γ is $\lceil \log_{c+1} n(G) \rceil + \lceil \log_{c+1} n(H) \rceil$. The resulting graph Γ is a c-mbn since c-broadcasting can be completed from any node in Γ when official c-broadcasting is possible. In addition, since official c-broadcasting in Γ is possible with respect to $\sum_c$, the set $\sum_c$ is a c-cns for Γ. ∎

In this network compounding algorithm, only the copies of G have to be isomorphic. The only requirement for the H_s networks is that c-broadcasting be completed in $\lceil \log_{c+1} n(H) \rceil$ time units. The best results are obtained by using multiple copies of the same c-mbn H with the smallest known $e(H)$ value. The following corollary is an immediate result of Theorem 3:

Corollary 1. For a c-mbn Γ that is generated by the network compounding algorithm in Theorem 3, the following results hold:

$$n(\Gamma) = n(G) \cdot n(H),$$
$$e(\Gamma) = e(H) \cdot e(G) + |S_c| \cdot e(H),$$
$$|\sum_c| = |S_c| \cdot |T_c|.$$

When all nodes in H are c-center nodes, center-node reduction cannot be accomplished (Fig. 6). Otherwise, the

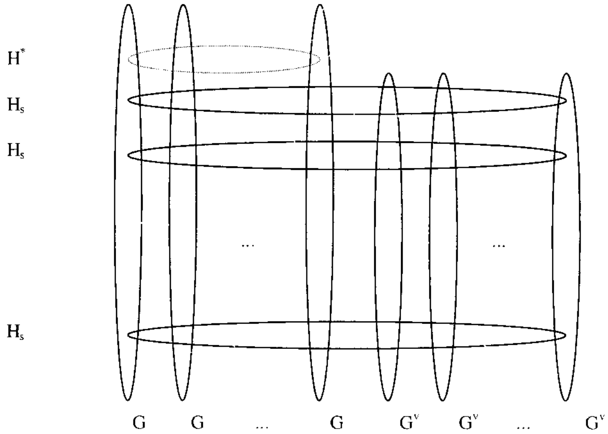

FIG. 9. The c-mbn compounding method.

center-node reduction method can decrease the cardinality of c-cns, which enables the construction of sparser c-mbn's after further compounding (Figs. 7 and 8).

An example of a 2-mbn with 18 nodes, generated by Theorem 3, is illustrated in Figure 6(a) with one of its official 2-broadcast protocols in Figure 6(b). In the example, $(6, 3)_2$ satisfies the broadcast condition, that is, $\lceil \log_3 18 \rceil = \lceil \log_3 6 \rceil + \lceil \log_3 3 \rceil$. No center node reduction is possible in this example.

Figures 7 and 8 provide an example of a 2-mbn with 30 nodes, in which center-node reduction takes place. Graphs G and H are 2-mbn's with five and six nodes, respectively, which are shown in Figure 3. Here, the 2-cns for G is represented by three black nodes in each copy of G. By Theorem 3, the boxed black nodes can be changed to noncenter nodes in Γ. Figure 8 shows an official 2-broadcast protocol, in which the originator is the node labeled with "0, 4." Identification of the remaining official 2-broadcast protocols for the other nodes is left to the reader.

3.2. *Network-compounding Algorithm with Center-node Reduction and Node Deletion*

The method described in Section 3.1 cannot generate c-mbn's whose orders are prime numbers. The node deletion method that is explained later in this section enables the network compounding algorithm to generate sparse c-mbn's, in terms of size, for an arbitrary number of nodes (even for prime numbers).

The following theorem provides an upper bound for the size of c-obn's with $n - 1$ nodes if a c-mbn with n nodes is known. A similar theorem was provided by Wang [17] for the case of 1-broadcasting.

Theorem 4. If a c-mbn with e edges and $n \neq (c+1)^m$ nodes has a node with minimum degree d, then $B_c(n-1) \leq e + \frac{1}{2}d(d-3)$.

Proof. Let G be a c-mbn with e edges and $n \neq (c+1)^m$ nodes, and v, a node in G with minimum degree d. Denote the neighbors of v, which are informed by v in a c-broadcast protocol of G, as v_i for $1 \leq i \leq p < d$, where $i \leq j$ implies that v_i is informed no later than v_j and p is the number of neighbors informed by v. By deleting v and its incident edges from G and then adding at most $\frac{1}{2}d(d-1)$ edges to form a clique among the neighbors of v, the resulting subgraph G^* with $n - 1$ nodes can be constructed.

The claim that G^* is a c-mbn needs to be proved. Because of the condition on n, $\lceil \log_{c+1}(n-1) \rceil = \lceil \log_{c+1} n \rceil$. In any c-broadcast protocol of G, let u be the node that informs node v at time t. Then, v calls its neighbors $(v_1, v_2, \ldots, v_c)$ at time $t+1, (v_{c+1}, v_{c+2}, \ldots, v_{2c})$ at time $t+2, \ldots, (v_{kc+1}, v_{kc+2}, \ldots, v_p)$ at time $t+k$, where $k = \lceil p/c \rceil$. C-broadcast protocols in G^* can be obtained as follows: At time t, u calls v_1 and then v_1 calls its neighbors $(v_2, v_3, \ldots, v_{c+1})$ at time $t+1$. Likewise, v_{ic+1} calls $(v_{ic+2}, v_{ic+3}, \ldots, v_{ic+1})$ at time $t+i$ and, lastly, v_{kc+1} calls $(v_{kc+2}, v_{kc+3}, \ldots, v_p)$ at time $t+k$. If $v_{kc+1} = v_p$, it can be idle at time $t+k$. Each node in $(v_{ic+2}, v_{ic+3}, \ldots, v_{(i+1)c+1})$ or $(v_{kc+2}, v_{kc+3}, \ldots, v_p)$ is ready to continue c-broadcasting from time $t+i+1$. The proof is completed. ∎

The vertex deletion method can be described as follows: Assume that $(n(G), n(H), i)_c$ satisfies the broadcast condition. Let G and H be c-mbn's and S_c a c-cns of G. If $S_c \neq V(G)$, then choose a noncenter node v of G with the minimum degree. Otherwise, let v be any (center) node of S_c with the minimum degree. Construct a new network G^v by deleting v and all its incident edges from G and adding the appropriate edges to form a clique among the neighbors of v. Let $S_c^v = S_c - \{v\}$. Then, $|S_c^v| = n(G) - 1$ if $|S_c| = n(G)$ and $|S_c^v| = |S_c|$ if $|S_c| < n(G)$.

Lemma 6. Official c-broadcasting within G^v can be completed with respect to a c-cns S_c^v in $\lceil \log_{c+1} n(G) \rceil$ time units.

Proof. Note that c-broadcasting within G^v can be completed in $\lceil \log_{c+1} n(G) \rceil$ time units by Theorem 4.

TABLE 1. Initial database for 2-broadcasting.

$n(G)$	$e(G)$	$cn_2(G)$	nG	nH	$d_n(G)$	$d_c(G)$	$p_n(G)$	$p_c(G)$
1	0	1	1	1	0	0	0	0
2	1	2	2	1	0	1	0	0
3	3	3	3	1	0	2	0	1
4	3	3	2	2	3	1	0	0
5	5	3	5	1	2	2	0	0
6	7	3	6	1	2	2	0	0
7	9	5	7	1	3	2	0	0
7	10	4	7	1	3	2	0	0
8	12	4	8	1	3	3	0	0
9	18	9	3	3	0	3	0	2
10	12	2	10	1	2	4	0	0
10	12	6	10	1	2	1	1	0
10	12	7	10	1	5	1	3	0
11	13	5	11	1	2	2	0	0
12	15	6	15	1	2	3	0	0

When $|S_c^v| = n(G^v)$, all nodes in G^v are c-center nodes. Therefore, the lemma holds for this case by Theorem 4.

If $|S_c| < n(G^v)$, node v is a noncenter node in G. Consider any official c-broadcast protocol P for an originator other than v in G. It is now necessary to show that an official c-broadcast protocol for the same originator in G^v, denoted as P^v, can be obtained by updating the message transmissions in P. If v is a node receiving only one message in P, which is official, node v and all its descendant nodes only receive an official message. Then, an official c-broadcast protocol P^v in G^v can be obtained as shown in Theorem 4.

Assume that v is a noncenter node receiving two messages in P: The first message is unofficial, and the second, official. Let v_1 and v_2 be the nodes that send the unofficial and official messages to node v at times t_1 and t_2 ($t_1 < t_2$), respectively. Let $V_u = \{u_i \in V(G^v) | i = 1, 2, \ldots, r_u < d(v), s_i = t_1 + \lceil i/c \rceil\}$ and $V_o = \{o_i \in V(G^v) | i = 1, 2, \ldots, r_o < d(v), w_i = t_2 + \lceil i/c \rceil\}$ be the sets of neighbors of v that receive the unofficial and official messages from v in $P, r_u = |V_n|$ and $r_o = |V_o|$. In addition, s_i and w_i are the receiving times at nodes u_i and o_i, respectively. Note that a neighbor of v in G can be in neither, one, or both sets. Since u_1 is idle at time t_1 in P_u, node v_1 can send an unofficial message to u_1 instead of v in P_u^v. After u_1 receives the unofficial message at time t_1, P_u^v can be obtained by a scheme similar to that of Theorem 4. Note that the receiving times of the unofficial message in P_u^v are less than or equal to the corresponding times in P_u. Contrary to the case of P_u^v, node o_1 can be busy in sending the unofficial message to some of its neighbors at time t_2 in P_o. Note that all nodes in V_o are available to receive the official message at time t_2. Node v_2 can send an official message to o_1 at time t_2 in P_o^v, since a noncenter node can send the unofficial message to at most c neighbors and receive the official message from another neighbor in the same time unit. After o_1 receives the official message from v_2 at time t_2, P_o^v can be obtained by updating the calls from P_o as shown in Theorem 4. Thus, all nodes in G^v are ready to send and receive the messages in P^v no later than they were in P. ∎

A network Γ can be constructed by connecting $n(H) - i$ copies of G and i copies of G^v as follows. For each fixed $s \in S_c^v$, connect all $n(H)$ copies of s to form a copy H_s of H. If v is a c-center node of $G, n(G) - 1$ copies of H are obtained and $n(H) - i$ copies of v are connected to form another c-mbn H^* with $n(H) - i$ nodes. If v is a noncenter node, then $n(G)$ copies of H are obtained and H^* is the empty graph. This procedure is illustrated in Figure 9. As shown in Figure 9, the terms vertical (within G or G^v) or horizontal (within H or H^*) broadcasting will be used in the rest of paper without confusion. This algorithm is referred to as the *c-mbn compounding algorithm*. The formal proof of the proposed algorithm is provided in Theorem 5.

As described in the previous section, the H_s networks do not have to be isomorphic. H^* need not be a c-mbn. It is only necessary that c-broadcasting on H^* be completed in $\lceil \log_{c+1} n(H) \rceil$ time units.

TABLE 2. Initial database for 4-broadcasting.

$n(G)$	$e(G)$	$cn_4(G)$	nG	nH	$d_n(G)$	$d_c(G)$	$p_n(G)$	$p_c(G)$
1	0	1	1	1	0	0	0	0
2	1	2	2	1	0	1	0	0
3	3	3	3	1	0	2	0	1
4	6	4	4	1	0	3	0	3
5	10	5	5	1	0	4	0	6
6	5	5	6	1	5	1	0	0
7	9	3	7	1	2	2	0	0
7	9	3	7	1	2	3	0	0
7	9	4	7	1	2	3	0	1
7	10	4	7	1	3	2	0	0
8	11	3	8	1	2	3	0	0
8	11	4	8	1	2	3	0	1
8	12	4	8	1	3	3	0	0
9	13	3	9	1	2	4	0	0
9	13	4	9	1	2	3	0	1
9	14	4	9	1	2	3	0	1
10	15	3	10	1	2	4	0	0
10	15	6	10	1	3	3	0	0
11	17	9	11	1	5	2	0	1

Theorem 5. Suppose that $(n(G), n(H), i)_c$ satisfies the broadcast condition. The resulting graph Γ constructed by the c-mbn compounding algorithm is a c-mbn with $n(G) \cdot n(H) - i$ nodes. Let T be a c-cns for H_s (or H) and $U \subseteq V(H^*)$ such that official c-broadcasting in H^* with respect to U can be completed in $\lceil \log_{c+1} n(H) \rceil$ time units. Then, $\sum = \cup_{s \in S^v} T_s \cup U$ is a c-cns for Γ.

Proof. Official c-broadcasting within G^v can be completed with respect to S^v in $\lceil \log_{c+1} n(G) \rceil$ time units by Lemma 6. For each (official) protocol P originating at any node of G other than v, there is a modification P^v of P that works for G^v.

It is necessary to show that official c-broadcasting in Γ with respect to $\sum$ can be completed in $\lceil \log_{c+1} n(\Gamma) \rceil$

TABLE 3. Initial database for 3-broadcasting.

$n(G)$	$e(G)$	$cn_3(G)$	nG	nH	$d_n(G)$	$d_c(G)$	$p_n(G)$	$p_c(G)$
1	0	1	1	1	0	0	0	0
2	1	2	2	1	0	1	0	0
3	3	3	3	1	0	2	0	1
4	6	4	4	1	0	3	0	3
5	4	4	5	1	4	1	0	0
6	7	3	6	1	2	2	0	0
7	9	3	7	1	2	2	0	0
7	9	3	7	1	2	3	0	0
7	9	4	7	1	2	3	0	1
7	10	4	7	1	3	2	0	0
8	11	3	8	1	2	3	0	0
8	11	4	8	1	2	3	0	1
8	12	4	8	1	3	3	0	0
9	13	7	9	1	4	2	0	0
9	14	4	9	1	2	3	0	1
10	15	6	10	1	3	3	0	0

```
Main()
Load the initial database
Loop1 : set next large c-mbn to G with respect to Loop1
   Loop2 : set next large c-mbn to H with respect to Loop2
      Check the broadcast condition of (G,H)
      If yes,
          generate a new c-mbn
          if newly generated c-mbn is better or non-dominating, add it to database
          if _VERTEX_DELETION_, call vertex_deletion(G,H)
      otherwise, continue;
      if n(Γ) > _MAX_ORDER_, stop.
      EndLoop1
   EndLoop2
Output the database

vertex_deletion(G,H)
   Loop : set i from 1 to n(H)
      Check the broadcast condition of (G,H,i)
      If yes,
          generate a new c-mbn
          if newly generated c-mbn is better or non-dominating, add it to database
      otherwise, continue;
   EndLoop
return
```

FIG. 10. The structure of the program.

time units. Let u be a node of Γ. Six cases need to be considered:

CASE 1. $u \in H_s, u \in G$.
CASE 2. $u \in H_s, u \in G^v$.
CASE 3. $u \in H^*, u \in U$.
CASE 4. $u \in H^*, u \notin U$.
CASE 5. $u \notin H_s, u \in G$.
CASE 6. $u \notin H_s, u \in G^v$.

CASES 1 and 2. Official c-broadcasting with respect to T_s in the copy of H_s that contains u can be completed horizontally in $\lceil \log_{c+1} n(H) \rceil$ time units. At this stage, all nodes in the copy of H_s have an official message. In the remaining $\lceil \log_{c+1} n(G) \rceil$ time units, each node in H_s broadcasts vertically within its copy of G or G^v. Hence, all nodes in Γ are officially informed, proving the result in these cases.

CASE 3. First, broadcast vertically and officially in the copy of G that u belong to. This takes $\lceil \log_{c+1} n(G) \rceil$ time units. At the end of vertical broadcasting, all nodes in the copy of G have the official message. Since H^* is not the empty graph, all nodes of G are c-center nodes and all nodes in Γ belong to H^* or some copies of H_s. Now broadcast horizontally and officially in the appropriate H_s or H^*. This can be done in $\lceil \log_{c+1} n(H) \rceil$ time units and all nodes are officially informed in the correct time.

CASE 4. As in Case 3, broadcast vertically and officially in the copy of G that includes u, which takes $\lceil \log_{c+1} n(G) \rceil$ time units. At the end of vertical broadcasting, all nodes in the copy of G have the official message except the originator u. All nodes in Γ belong to H^* or some copies of H_s. Now broadcast horizontally and officially in the appropriate H_s or H^* in $\lceil \log_{c+1} n(H) \rceil$ additional time units. Even if the originator does not have an official message at time $\lceil \log_{c+1} n(G) \rceil$, official c-broadcasting within H^* can be completed in $\lceil \log_{c+1} n(H) \rceil$ additional time units.

CASE 5. As in the proof of Theorem 3, broadcast in the copy of G that contains u, according to the unofficial part P_u of an official protocol P, ending at $V_{cu}(P)$. Each c-center node in $V_{cu}(P)$ broadcasts horizontally and officially in its corresponding copy of H. Then, each copy of nodes in $V_{cu}(P)$ continues as in P_o or P_o^v, depending on whether it belongs to a copy of either G or G^v. Thus, official c-broadcasting in Γ can be completed in the required time.

CASE 6. Let P be an official c-broadcast protocol for u in G. Since a noncenter node is deleted, the c-cns of G^v is the same as that of G. Broadcast in G^v according to P^v ending in $V_{cu}(P)$. Let Q be the subprotocol followed so far. Let V_1 (respectively, V_2) be the subset of $V_{cu}(P)$ consisting of nodes that have (respectively, do not have) v as an ancestor. For nodes in V_2, only P_u^v has been followed and this is precisely P_u for such nodes. Now, Q takes the same time as P_u^v for all nodes in V_2. This is the same as the time taken by P_u for such nodes. In addition, Q takes the same time as P_u for nodes in V_1 since all nodes in Q have the receiving times which are less than or equal to those in P_u, as shown in Lemma 6.

Hence, Q takes the same time as P_u for all nodes in $V_{cu}(P)$. Now, broadcast horizontally and officially in the appropriate H_s, which takes $\lceil \log_{c+1} n(H) \rceil$ time units. Each copy of each $u \in V_{cu}(P)$ then broadcasts until the end of P or P^v as appropriate. Since P^v takes $\lceil \log_{c+1} n(G) \rceil$ time units, and P_u and P_o also use this many time units, all nodes in Γ are informed by the required time.

The resulting graph Γ is a c-mbn since c-broadcasting can be completed from any node in Γ when official c-broadcasting is possible. In addition, since official c-broadcasting in Γ is possible with respect to $\sum = \cup_{s \in S^v} T_s \cup U$, the set $\sum$ is a c-cns for Γ. ∎

The condition on U in Theorem 5 is always satisfied if H^* is a c-mbn and U is a c-cns for H^*. To implement the c-mbn compounding algorithm, useful data are collected in the following corollary. Let $\delta_n(G)$ and $\delta_c(G)$ be the minimum degree of a noncenter node and a center node in G, respectively. In addition, $\varphi_n(G)$ and $\varphi_c(G)$ are defined to be the number of edges connecting the neighbors of the noncenter node with degree $\delta_n(G)$ and the center node with degree $\delta_c(G)$, respectively, in G.

Corollary 2. The following results hold for the c-mbn compounding algorithm described in Theorem 5. Note that H^* exists only when $S_c = V(G)$. Therefore, if H^* does not exist, $n(H^*) = e(H^*) = |U| = \delta_n(H^*) = \delta_c(H^*) = 0$.

$$n(\Gamma) = n(G)v(H) - i,$$

$$e(\Gamma) = \begin{cases} e(G)n(H) + \dfrac{d(d-3)}{2} i - \varphi_n(G) \cdot i + \sum\limits_{s \in S_c^v} e(H_s), & \text{if } S_c \neq V(G), \\ e(G)n(H) + \dfrac{d(d-3)}{2} i - \varphi_c(G) \cdot i + \sum\limits_{s \in S_c^v} e(H_s) + e(H^*), & \text{if } S_c = V(G), \end{cases}$$

$$|\textstyle\sum| = \sum\limits_{s \in S_c^v} |T_s| + |U|,$$

$$\delta_n(\Gamma) \leq \begin{cases} \delta_n(G), & \text{if } S_c \neq V(G), \quad T_c = V(H_s), \\ \min\{\delta_n(G), \min\limits_{s \in S_c^v}\{\delta_n(H_s) + \delta_c(G)\}\}, & \text{if } S_c \neq V(G), \quad T_c \neq V(H_s), \\ 0, & \text{if } S_c = V(G), \quad T_c = V(H_s), \quad U = V(H^*), \\ \delta_n(H^*) + \delta_c(G), & \text{if } S_c = V(G), \quad T_c = V(H_s), \quad U \neq V(H^*), \\ \min\limits_{s \in S_c^v}\{\delta_n(H_s) + \delta_c(G)\}, & \text{if } S_c = V(G), \quad T_c \neq V(H_s), \quad U = V(H^*), \\ \min\{\delta_n(H^*) + \delta_c(G), \min\limits_{s \in S_c^v}\{\delta_n(H_s) + \delta_c(G)\}\}, & \text{if } S_c = V(G), \quad T_c \neq V(H_s), \quad U \in V(H^*), \end{cases}$$

$$\delta_c(\Gamma) \leq \begin{cases} \min\limits_{s \in S_c^v}\{\delta_c(G) + \delta_c(H_s)\}, & \text{if } S_c \neq V(G), \\ \min\{\delta_c(G) + \delta_c(H^*), \min\limits_{s \in S_c^v}\{\delta_c(G) + \delta_c(H_s)\}\}, & \text{if } S_c = V(G), \end{cases}$$

$$\varphi_n(\Gamma) \geq \begin{cases} \varphi_n(G), & \text{if } S_c \neq V(G), \quad T_c = V(H_s), \\ \min\{\varphi_n(G), \min\limits_{s \in S_c^v}\{\varphi_n(H_s) + \varphi_c(G)\}\}, & \text{if } S_c \neq V(G), \quad T_c \neq V(H_s), \\ 0, & \text{if } S_c = V(G), \quad T_c = V(H_s), \quad U = V(H^*), \\ \varphi_n(H^*) + \varphi_c(G), & \text{if } S_c = V(G), \quad T_c = V(H_s), \quad U \neq V(H^*), \\ \min\limits_{s \in S_c^v}\{\varphi_n(H_s) + \varphi_c(G)\}, & \text{if } S_c = V(G), \quad T_c \neq V(H_s), \quad U = V(H^*), \\ \min\{\varphi_n(H^*) + \varphi_c(G), \min\limits_{s \in S_c^v}\{\varphi_n(H_s) + \varphi_c(G)\}\}, & \text{if } S_c = V(G), \quad T_c \neq V(H_s), \quad U \neq V(H^*), \end{cases}$$

$$\varphi_c(\Gamma) \geq \begin{cases} \min\limits_{s \in S_c^v}\{\varphi_c(G) + \varphi_c(H_s)\}, & \text{if } S_c \neq V(G), \\ \min\{\varphi_n(G) + \varphi_c(H^*), \min\limits_{s \in S_c^v}\{\varphi_c(G) + \varphi_c(H_s)\}\}, & \text{if } S_c = V(G). \end{cases}$$

4. COMPUTATIONAL RESULTS

Tables 1–3 contain the initial database for the execution of the proposed c-mbn compounding algorithm for the cases of 2-, 3-, and 4-broadcasting. The c-mbn corresponding to each row in the tables is illustrated in Figures 3–5, where the dark nodes represent c-center nodes. The entries in Tables 1–3 were validated by the authors. Discovery of additional c-obn's (or sparser c-mbn's) for the initial database will contribute to further improvements in the construction of new c-mbn's of large order.

The columns in the tables include $n(G)$, $e(G)$, $cn_c(G)$, number of copies of G, number of copies of H, $d_n(G)$, $d_c(G)$, $p_n(G)$, and $p_c(G)$, where $d_n(G)$, $d_c(G)$, $p_n(G)$, and

TABLE 4. Upper bounds for $B_c(n)$, where $2 \le c \le 4$ and $1 \le n \le 50$.

	c = 2			c = 3			c = 4		
$n(\Gamma)$	$e(\Gamma)$	$cn_2(\Gamma)$	Compound	$e(\Gamma)$	$cn_3(\Gamma)$	Compound	$e(\Gamma)$	$cn_4(\Gamma)$	Compound
1	0	1		0	1		0	1	
2	1	2		1	2		1	2	
3	3	3		3	3		3	3	
4	3	3	(2, 2)	6	4		6	4	
5	5	3		4	4		10	5	
6	7	3		7	3		5	5	
7	9	5		9	3		9	3	
8	12	4		11	3		11	3	
9	18	9	(3, 3)	13	7		13	3	
10	12	2		15	6		15	3	
11	13	5		25	11	(3, 4, 1)	17	9	
12	15	6		30	12	(3, 4)	30	12	(3, 4), (3, 5, 3)
13	22	7	(2, 8, 3), (2, 7, 1)	33	13	(4, 4, 3)	33	13	(4, 4, 3)
14	23	9	(5, 3, 1)	37	14	(4, 4, 2)	37	14	(4, 4, 2)
15	24	9	(5, 3)	42	15	(4, 4, 1)	42	15	(4, 4, 1)
16	28	8	(8, 2), (6, 3)	48	16	(4, 4)	48	16	(4, 4), (4, 5, 4)
17	29	9	(6, 3, 1)	29	9	(6, 3, 1)	52	17	(4, 5, 3)
18	30	9	(6, 3)	30	9	(6, 3)	57	18	(4, 5, 2)
19	40	13	(3, 7, 2)	34	9	(7, 3, 2)	63	19	(4, 5, 1)
20	42	12, 15	(7, 3, 1), (7, 3, 1)	35	9	(7, 3, 1)	70	20	(4, 5)
21	42	12, 15	(7, 3), (7, 3)	36	9	(7, 3)	74	21	(5, 5, 4)
22	48	12	(8, 3, 2)	40	9	(8, 3, 2)	79	22	(5, 5, 3)
23	48	12	(8, 3, 1)	41	9	(8, 3, 1)	85	23	(5, 5, 2)
24	48	12	(8, 3)	42	9	(8, 3)	92	24	(5, 5, 1)
25	68	23	(3, 9, 2)	51	12	(7, 4, 3)	100	25	(5, 5)
26	73	22	(3, 9, 1)	52	12	(7, 4, 2)	47	9	(9, 3, 1)
27	81	27	(3, 9)	53	12	(7, 4, 1)	48	9	(9, 3)
28	40	6	(10, 3, 2)	54	12	(7, 4)	52	9	(10, 3, 2)
29	41	6	(10, 3, 1)	59	12	(8, 4, 3)	53	9	(10, 3, 1)
30	42	6	(10, 3)	60	12	(8, 4, 2)	54	9	(10, 3)
31	52	15	(11, 3, 2)	61	12	(8, 4, 1)	61	12	(8, 4, 1)
32	53	15	(11, 3, 1)	62	12	(8, 4)	62	12	(8, 4)
33	54	15	(11, 3, 0)	77	16	(9, 4, 3)	67	12	(9, 4, 3)
34	61	9	(6, 6, 2), (5, 7, 1)	78	16	(9, 4, 2)	68	12	(9, 4, 2)
35	62	9	(6, 6, 1), (5, 7)	79	16	(9, 4, 1)	69	12	(9, 4, 1)
36	63	9	(6, 6)	80	16	(9, 4)	70	12	(9, 4)
37	71	15	(6, 7, 5)	96	24	(4, 10, 3)	75	12	(10, 4, 3)
38	72	15	(6, 7, 4)	96	24	(4, 10, 2)	76	12	(10, 4, 2)
39	73	15	(6, 7, 3)	96	24	(4, 10, 1)	77	12	(10, 4, 1)
40	74	15	(6, 7, 2)	96	24	(4, 10)	78	12	(10, 4)
41	75	15	(6, 7, 1)	143	36	(4, 11, 3)	91	15	(9, 5, 4)
42	76	15	(6, 7)	147	38	(3, 16, 6)	92	15	(9, 5, 3)
43	87	12	(6, 8, 5)	153	39	(4, 11, 1)	93	15	(9, 5, 2)
44	88	12	(6, 8, 4)	157	43	(4, 13, 8)	94	15	(9, 5, 1)
45	89	12	(6, 8, 3)	163	42	(4, 13, 7)	95	15	(9, 5)
46	90	12	(6, 8, 2)	168	42	(4, 13, 6)	101	15	(10, 5, 4)
47	91	12	(6, 8, 1)	172	46	(4, 14, 9)	102	15	(10, 5, 3)
48	92	12	(6, 8)	178	45	(4, 14, 9), (4, 13, 4)	103	15	(10, 5, 2)
49	106	20	(7, 7)	183	45	(4, 14, 7), (4, 13, 3)	104	15	(10, 5, 1)
50	113	27	(6, 9, 4)	188	45	(4, 14, 6)	105	15	(10, 5)

$p_c(G)$ denote upper bounds for $\delta_n(G)$ and $\delta_c(G)$ and lower bounds for $\varphi_n(G)$ and $\varphi_c(G)$, respectively.

All data records, each of which represents a c-mbn for the given number of nodes, should be arranged in the lexicographical ascending order of $[n(G), e(G), cn_c(G)]$. Domination of G_1 over G_2 can be defined if and only if $n(G_1) = n(G_2), e(G_1) \le e(G_2), cn_c(G_1) \le cn_c(G_2)$. If nondominating c-mbn's are obtained, multiple records for the same number of nodes are stored in the database.

Two versions of the network compounding algorithm were implemented in this paper: One is the network compounding algorithm with center-node reduction, which was explained in Section 3.1. The other is the c-mbn compounding algorithm with center-node reduction and modified node deletion, which was outlined in Section 3.2. Since the second algorithm can be implemented in a modular way by adding the subroutine of vertex deletion to the first, the results of the second naturally dominate those of the first. Both algorithms are necessarily iterative in order to generate c-mbn's for an arbitrary large number of nodes. During the execution of the algorithms, newly generated c-mbn's replace existing c-mbn's in the database if they dominate the old ones. If newly generated c-mbn's are nondominating over the existing c-

TABLE 5. Upper bounds for $B_c(n)$, where $2 \le c \le 4$ and $51 \le n \le 100$.

	$c = 2$			$c = 3$			$c = 4$		
$n(\Gamma)$	$e(\Gamma)$	$cn_2(\Gamma)$	Compound	$e(\Gamma)$	$cn_3(\Gamma)$	Compound	$e(\Gamma)$	$cn_4(\Gamma)$	Compound
51	114	27	(6, 9, 3)	193	49	(4, 14, 5)	171	39	(5, 11, 4)
52	115	27	(6, 9, 2)	198	48	(4, 14, 4)	177	39	(5, 11, 3)
53	116	27	(6, 9, 1)	206	48	(4, 15, 7)	183	39	(5, 11, 2)
54	117	27	(6, 9)	211	52	(4, 15, 6)	180	45	(11, 5, 1)
55	120	20	(8, 7, 1)	216	51	(4, 15, 5)	175	45	(11, 5)
56	120	20	(8, 7)	227	51	(4, 16, 8)	221	54	(4, 15, 4)
57	144	16	(8, 8, 7)	232	55	(4, 16, 7)	230	56	(4, 17, 11)
58	144	16	(8, 8, 6)	237	54	(4, 16, 6)	237	51	(4, 16, 6)
59	144	16	(8, 8, 5)	250	59	(4, 15, 1)	242	54	(4, 17, 9)
60	144	16	(8, 8, 4)	258	60	(4, 15)	247	54	(4, 17, 8)
61	144	16	(8, 8, 3)	264	61	(4, 16, 3)	252	54	(4, 17, 7)
62	144	16	(8, 8, 2)	271	62	(4, 16, 2)	257	60	(4, 17, 6)
63	144	16	(8, 8, 1)	279	63	(4, 16, 1)	265	57	(4, 18, 9)
64	144	16	(8, 8)	288	64	(4, 16)	270	57	(4, 18, 8)
65	180	36	(8, 9, 7)	130	18	(7, 10, 5)	275	63	(4, 18, 7)
66	180	36	(8, 9, 6)	131	18	(7, 10, 4)	286	60	(4, 19, 10)
67	180	36	(8, 9, 5)	132	18	(7, 10, 3)	291	60	(4, 19, 9)
68	180	36	(8, 9, 4)	133	18	(7, 10, 2)	296	66	(4, 19, 8)
69	180	36	(8, 9, 3)	134	18	(7, 10, 1)	307	63	(5, 15, 6)
70	180	36	(8, 9, 2)	135	18	(7, 10)	313	63	(5, 15, 5)
71	180	36	(8, 9, 1)	137	21	(8, 9, 1)	319	69	(5, 15, 4)
72	180	36	(8, 9)	138	21	(8, 9)	325	66	(4, 21, 12)
73	255	58	(3, 25, 2)	148	18	(8, 10, 7)	330	66	(4, 21, 11)
74	257	58	(3, 25, 1)	149	18	(8, 10, 6)	335	72	(4, 21, 10)
75	266	56	(3, 26, 3)	150	18	(8, 10, 5)	343	69	(4, 22, 13)
76	268	56	(3, 26, 2)	151	18	(8, 10, 4)	348	69	(4, 22, 12)
77	283	66	(3, 26, 1)	152	18	(8, 10, 3)	353	75	(4, 22, 11)
78	285	66	(26, 3)	153	18	(8, 10, 2)	364	72	(4, 23, 14)
79	307	77	(3, 27, 2)	154	18	(8, 10, 1)	369	72	(4, 23, 13)
80	314	76	(3, 27, 1)	155	18	(8, 10)	374	78	(4, 23, 12)
81	324	18	(9, 9), (3, 27)	178	28	(9, 9)	385	75	(5, 18, 9)
82	136	18	(10, 9, 8)	190	33	(8, 11, 6)	391	75	(5, 18, 8)
83	137	18	(10, 9, 7)	191	33	(8, 11, 5)	397	81	(5, 18, 7)
84	138	18	(10, 9, 6)	192	33	(8, 11, 4)	409	79	(5, 19, 11)
85	139	18	(10, 9, 5)	193	33	(8, 11, 3)	415	79	(5, 19, 10)
86	140	18	(10, 9, 4)	194	33	(8, 11, 2)	421	79	(5, 19, 9)
87	141	18	(10, 9, 3)	195	33	(8, 11, 1)	427	85	(5, 19, 8)
88	142	18	(10, 9, 2)	196	33	(8, 11)	443	83	(5, 20, 12)
89	143	18	(10, 9, 1)	199	24	(9, 10, 1)	449	83	(5, 20, 11)
90	144	18	(10, 9, 0)	200	24	(9, 10)	451	89	(5, 21, 15)
91	187	24	(12, 8, 5)	216	39	(7, 13)	459	87	(5, 21, 14)
92	188	24	(12, 8, 4)	218	36	(8, 12, 4)	465	87	(5, 21, 13)
93	189	24	(12, 8, 3)	219	36	(8, 12, 3)	471	87	(5, 21, 12)
94	190	24	(12, 8, 2)	220	36	(8, 12, 2)	477	87	(5, 21, 11)
95	191	24	(12, 8, 1)	221	36	(8, 12, 1)	483	93	(5, 21, 10)
96	192	24	(12, 8)	222	36	(8, 12)	491	91	(5, 22, 12)
97	201	27	(6, 17, 5)	235	39	(8, 13, 7)	497	91	(5, 22, 11)
98	202	27	(6, 17, 4)	236	39	(8, 13, 6)	503	91	(5, 22, 10)
99	203	27	(6, 17, 3)	237	39	(8, 13, 5)	509	97	(5, 22, 9)
100	204	27	(6, 17, 2)	238	39	(8, 13, 4)	521	95	(5, 23, 15)

mbn's, they are inserted into the database instead of replacing the existing ones. This updated database serves for further construction of c-mbn's of larger order. The structure of the program that executes both algorithms is described in Figure 10. The program maintains the structure and ordering of the database. Thus, the output database is used again as the input database to compound the best c-mbn's with the previous results until no further improvement is found.

The network compounding algorithm described in Section 3.1 cannot provide all c-mbn's for an arbitrary number of nodes, especially for the prime numbers and for the numbers $(c+1)^m - k$, where $1 \le k < c$. The c-mbn compounding algorithm proposed in Section 3.2 can generate all c-mbn's for an arbitrary number of nodes, only if the initial database contains at least the c-mbn's with k nodes, where $1 \le k \le c$. This is true because the broadcast condition in c-broadcasting is always satisfied for $(c, n)_c$ and $(c, n, k)_c$, where $1 \le k < c$. The code developed by Dinneen et al. [4] was modified to implement the c-mbn compounding algorithms. In the vertex deletion method of Dinneen et al. [4] for the case of 1-broadcasting, after deleting v and all its incident edges from G, they added $d(v)(d(v)-1)/2$ edges

TABLE 6. Upper bounds for $B_c(n)$, where $2 \le c \le 4$ and $101 \le n \le 128$.

	c = 2			c = 3			c = 4		
$n(\Gamma)$	$e(\Gamma)$	$cn_2(\Gamma)$	Compound	$e(\Gamma)$	$cn_3(\Gamma)$	Compound	$e(\Gamma)$	$cn_4(\Gamma)$	Compound
101	205	27	(6, 17, 1)	239	39	(8, 13, 3)	527	95	(5, 23, 14)
102	206	27	(6, 17)	240	39	(8, 13, 2)	533	95	(5, 23, 13)
103	211	27	(6, 18, 5)	241	39	(8, 13, 1)	539	101	(5, 23, 12)
104	212	27	(6, 18, 4)	242	39	(8, 13)	555	99	(5, 24, 16)
105	213	27	(6, 18, 3)	258	42	(8, 14, 7)	561	99	(5, 24, 15)
106	214	27	(6, 18, 2)	259	42	(8, 14, 6)	567	99	(5, 24, 14)
107	215	27	(6, 18, 1)	260	42	(8, 14, 5)	573	105	(5, 24, 13)
108	216	27	(6, 18)	261	42	(8, 14, 4)	590	108	(5, 23, 7)
109	248	39	(6, 19, 5)	262	42	(8, 14, 3)	597	109	(5, 24, 11)
110	249	39	(6, 19, 4)	263	42	(8, 14, 2)	605	103	(5, 25, 15)
111	250	39	(6, 19, 3)	264	42	(8, 14, 1)	611	109	(5, 25, 14)
112	251	39	(6, 19, 2)	265	42	(8, 14)	624	112	(5, 24, 8)
113	252	39	(6, 19, 1)	284	45	(8, 15, 7)	632	113	(5, 24, 7)
114	253	39	(6, 19)	285	45	(8, 15, 6)	641	114	(5, 24, 6)
115	259	36	(39, 3, 2)	286	45	(8, 15, 5)	651	115	(5, 24, 5)
116	260	36	(39, 3, 1)	287	45	(8, 15, 4)	662	116	(5, 25, 9)
117	261	36	(39, 3)	288	45	(8, 15, 3)	670	117	(5, 24, 3)
118	262	36	(40, 3, 2)	289	45	(8, 15, 2)	679	118	(5, 24, 2)
119	263	36	(40, 3, 1)	290	45	(8, 15, 1)	689	119	(5, 24, 1)
120	264	36	(40, 3)	291	45	(8, 15)	700	120	(5, 24)
121	268	36	(41, 3, 2)	313	48	(8, 16, 7)	708	121	(5, 25, 4)
122	269	36	(41, 3, 1)	314	48	(8, 16, 6)	717	122	(5, 24, 3)
123	270	36	(41, 3)	315	48	(8, 16, 5)	727	123	(5, 24, 2)
124	271	36	(42, 3, 2)	316	48	(8, 16, 4)	738	124	(5, 24, 1)
125	272	36	(42, 3, 1)	317	48	(8, 16, 3)	750	125	(5, 24)
126	273	36	(42, 3)	318	48	(8, 16, 2)	293	42	(9, 14)
127	304	32	(8, 16, 1)	319	48	(8, 16, 1)	319	48	(8, 16, 1)
128	304	32	(8, 16)	320	48	(8, 16)	320	48	(8, 16)

to form a clique among the neighbors of v, according to a theorem by Wang [17]. They mentioned that it is possible to obtain better results by considering the existing edges connecting the neighbors of v and adding only the necessary edges. In this paper, the vertex deletion method is improved by adding $\varphi_n(G)$ and $\varphi_c(G)$ to their database. In this case, the number of edges that are added to form a clique among the neighbors of v is only $d(v)(d(v) - 3)/2 - r$, where r is the number of old edges in the clique and can be either $\varphi_n(G)$ or $\varphi_c(G)$ depending on v.

The proposed c-mbn compounding algorithm with center-node reduction and vertex deletion generates the c-mbn's for $2 \le c \le 4$ and $1 \le n \le 128$, which are summarized in Tables 4–6. Each entry in the column labeled "Compound" provides the triple $(n(G), n(H), i)$ that constructs the c-mbn with the smallest known number of edges for $n(G) * n(H) - i$ nodes. The modified vertex deletion method improved 14.65, 50, and 70.94% of entries for 2-, 3-, and 4-broadcasting with respect to the vertex deletion method in [4]. K_3, K_4, and K_5 are c-obn's which are often used to construct the best known c-mbn's for $c = 2, 3$, and 4, respectively.

In the Compound column of Tables 4 and 5, two entries exist for some cases. For example, when $c = 2$ and $n = 13$, both compounds $(2, 8, 3)_2$ and $(2, 7, 1)_2$ can generate nondominating nonisomorphic 2-mbn's with 13 nodes and seven center nodes. Nonisomorphic c-mbn's can provide the flexibility and alternatives in the communication network design stage.

The justification for keeping nondominating c-mbn's in the database can be found in Table 4. When $c = 2$ and $n = 20$, there are two entries in $cn_2(G)$ and the Compound column. Both c-mbn's with 20 nodes can be compounded by $(7, 3)_2$. This is due to the existence of nondominating c-mbn's in the initial database. In Table 1, there are two entries for $n = 7$. The first 2-mbn with seven nodes has nine edges and five center nodes and the second 10 edges and four center nodes. In Table 4, the 2-mbn with 12 center nodes, which has been compounded using the second 2-mbn with seven nodes and four center nodes in Table 1, dominates the 2-mbn with 15 center nodes, which has been compounded using the first 2-mbn with seven nodes and five center nodes in Table 1. By keeping both 2-mbn's in the initial database, it is observed that nondominating c-mbn's contribute to compounding better than do 2-mbn's.

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