k-Broadcasting in Trees

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We continue the investigation of k-broadcasting, a variant of broadcasting in which an informed vertex can call up to k of its neighbors in each time unit. We focus on k-broadcasting in trees. In particular, we asymptotically determine the maximum number of vertices in any tree with given k-broadcast time and describe the structure of the trees that achieve this maximum. © 2001 John Wiley & Sons, Inc.

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1. INTRODUCTION

Broadcasting is the process of message dissemination in a communication network in which a message, originated by one vertex, is transmitted to all vertices of the network by placing a series of calls over the communication lines of the network. This is to be completed as quickly as possible. Typically, it is assumed that each call involves only one informed vertex and one of its neighbors, each call requires one unit of time, a vertex can participate in only one call per unit of time, and a vertex can only call its neighbors. Here, we consider k-broadcasting in which each call involves a caller who sends the message to k (or fewer) of its neighbors in one time unit. (Note: The same problem is called c-broadcasting by some authors [6, 14, 15, 17].) This model allows for "conference calls" of a limited maximum size k [6]. It is useful in the study of DMA-bound systems [13] and also has application in computing functions in networks [1, 2, 3].

Given a connected graph $G = (V, E)$ and a message originator, vertex u, the *k-broadcast time of vertex u*, $b_k(u)$, is the minimum number of time units required to complete a k-broadcast from vertex u. For any vertex u in a connected graph G with n vertices, $b_k(u) \geq \lceil \log_{k+1} n \rceil$, since the number of informed vertices can, at most, be multiplied by $k + 1$ during each time unit. The *k-broadcast time of a graph G*, denoted $b_k(G)$, is the maximum k-broadcast time of any vertex u in G, that is, $b_k(G) = \max\{b_k(u) | u \in V\}$. We use the term *$k$-broadcast graph* to refer to any graph G on n vertices with $b_k(G) = \lceil \log_{k+1} n \rceil$. The *$k$-broadcast function*, $B_k(n)$, is the minimum number of edges in any k-broadcast graph on n vertices. A *minimum k-broadcast graph* is a k-broadcast graph on n vertices having $B_k(n)$ edges.

Most of the previous work in this area has been for $k=1$. For surveys of results on broadcasting and related problems, see Hedetniemi et al.[8], Fraigniaud and Lazard [4], and Hromkovič et al. [9]. Grigni and Peleg [6] showed that $B_k(n) \in \Theta(kL_k(n)n)$, where $L_k(n)$ denotes the exact number of consecutive leading k's in the $(k + 1)$-ary representation of $n - 1$. Lazard [13] studied minimum k-broadcast graphs giving some values of $B_2(n)$, $B_3(n)$, and $B_4(n)$ for small values of n. König and Lazard [11] generalized some results on 1-broadcasting and found minimum k-broadcast graphs for all n in the range $k + 3 \leq n \leq 2k + 3$. Shastri and Gaur [17] considered k-broadcasting in trees, determining some specific minimum multibroadcast trees for $k = 2, 3$ and small n. More recently, the bounds on $B_k(n)$ were improved by Lee and Ventura [15] and by Harutyunyan and Liestman [7].

In this paper, we consider k-broadcasting in trees. $N_k(t)$ is used to denote the maximum number of vertices in any tree with k-broadcast time t. If T is a tree on $N_k(t)$ vertices with k-broadcast time t, we say that T is a *t-optimal k-broadcast tree*. The "inverse" problem may also be considered. Let $T_k(n)$ be the minimum k-broadcast time of any tree on n vertices. The construction of optimal trees was described and the asymptotic behav-

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ior of these functions was estimated by Khachatrian and Haroutunian [10] and by Labahn [12] for $k = 1$. Here, we describe the construction of t-optimal k-broadcast trees for any k and asymptotically estimate the number of vertices in such trees. Lee [14] independently addressed some of these same issues. In particular, he proved a result equivalent to our Lemma 2 and reported partial results related to our Theorem 1.

2. CONSTRUCTION OF OPTIMAL TREES

The *b-nomial tree* T_b^m of dimension m has b^m vertices and can be constructed recursively. The tree T_b^0 is a single vertex. For $m \geq 1$, the tree T_b^m is obtained from b copies of T_b^{m-1} by connecting the roots of $b-1$ copies of T_b^{m-1} to the root u of the remaining copy of T_b^{m-1}. This vertex u is the root of T_b^m. Note that T_b^m has diameter $2m$ if $b > 2$ and diameter $2m - 1$ if $b = 2$. Figure 1 shows T_3^2 and T_3^3.

Consider the problem of k-broadcasting from the root of T_{k+1}^m, a $(k+1)$-nomial tree on $(k+1)^m$ vertices. The root of this tree, denoted v^m, has km children $v_1^{m-1}, v_2^{m-1}, \dots, v_k^{m-1}, v_1^{m-2}, v_2^{m-2}, \dots, v_k^{m-2}, \dots, v_1^0, v_2^0, \dots, v_k^0$. Each vertex v_j^i, where $0 \leq i \leq m-1$, $1 \leq j \leq k$, is the root of a $(k+1)$-nomial tree T_j^i on $(k+1)^i$ vertices. It is easy to see that the root of T_{k+1}^m can broadcast to $(k+1)^m$ vertices (including itself) in m time units.

The above k-broadcast from the root of T_{k+1}^m could be represented as a labeling of the edges of T_{k+1}^m so that the edge from one vertex u to its child v is labeled with t if the call from u to v was made at time t. In fact, in any graph, a k-broadcast from a particular originator corresponds to a labeling of the edges of a spanning subtree rooted at u with edges directed from the root toward the leaves. In such a labeling, all times must be positive, the originator can have at most k outgoing edges with any given label, and any vertex with an incoming edge labeled t can have no outgoing edges labeled less than t and at most k outgoing edges labeled with any particular value $t' > t$. Thus, there will be a (unique) path from the root to each vertex v along which the labels of the edges are increasing.

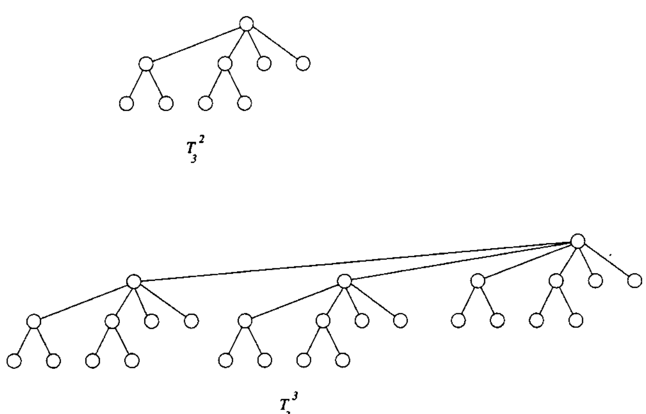

FIG. 1. Sample 3-nomial trees.

Lemma 1. $b_k(T_{k+1}^m) = 2m$.

Proof. As the diameter of T_{k+1}^m is $2m$, $b_k(T_{k+1}^m) \geq 2m$.

To show that $b_k(T_{k+1}^m) \leq 2m$, consider any vertex u in T_{k+1}^m. There is a path from u to the root of T_{k+1}^m of length at most m. The originator u can send the message to the root along that path. Subsequently, the root can k-broadcast the message using the above scheme, completing the k-broadcast no later than time $2m$. ∎

It is useful to know the number of vertices at any given distance $d \leq m$ from the root of T_{k+1}^m. These vertices correspond to the paths with increasing labels of length d starting at the root. There are $\binom{m}{d}$ possible increasing sequences of length d with labels chosen from $\{1, 2, \dots, m\}$. Each of these sequences occurs (overlapping at the beginning) k^d times in the above labeling. Thus, there are $k^d \binom{m}{d}$ vertices at distance $d \leq m$ from the root of T_{k+1}^m.

In our constructions, we will often use subtrees of $(k+1)$-nomial trees. In particular, we let $T(t, d, k)$ denote the subtree of T_{k+1}^t such that all vertices at distance greater than d from the root are deleted. Figure 2 shows $T(3, 2, 2)$.

Lemma 2. *In any graph G, the maximum number of vertices which can be informed in a k-broadcast by time $t \geq 0$ along paths of length at most d is at most $\sum_{i=0}^{d} k^i \binom{t}{i}$ if $d \leq t$ and $(k+1)^t$ if $d > t$.*

Proof. Consider the edge labeling corresponding to a k-broadcast from a particular originator in G. In this edge labeling, there can be at most $k^i \binom{t}{i}$ distinct paths of length i from the originator. Therefore, we can inform at most $k^i \binom{t}{i}$ vertices at each distance i from the originator u in t time units. Thus, we can inform a total of at most $\sum_{i=0}^{d} k^i \binom{t}{i}$ vertices in t time units. ∎

Note that in t time units it is not possible to inform vertices at distance greater than t. In this case, the maximum number of vertices that can be informed is at most $\sum_{i=0}^{t} k^i \binom{t}{i} = (k+1)^t$. This can, in fact, be accomplished from the root of the $(k+1)$-nomial tree T_{k+1}^t.

Although it is difficult to count the number of vertices in a t-optimal k-broadcast tree for large k and t, we can describe the structure of these trees. The structure of these trees is described in the proof of the following theorem:

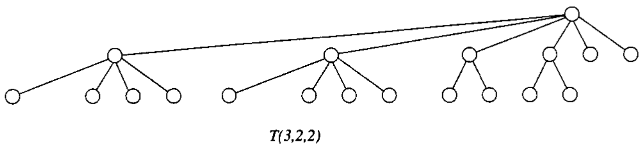

FIG. 2. $T(3,2,2)$.

Theorem 1. *For $t \geq 1$ and $k \geq 1$,*

$$N_k(t) = \max\left\{\begin{array}{l} \max_{0\leq m\leq (t-1)/2}\{2\sum_{i=0}^{m} k^i\binom{t-(m+1)}{i}\}, \\ \max_{0\leq m\leq t/2}\{(k+2)\sum_{i=0}^{m-1} k^i\binom{t-(m+1)}{i}\}, \\ \max_{0\leq m\leq (t-1)/2}\{k\sum_{i=0}^{m-1} k^i\binom{t-(m+1)}{i} \\ \quad + \sum_{i=0}^{m} k^i\binom{t-(m+1)}{i} + (k+1)^{m-1}\} \end{array}\right\}.$$

Proof. Consider a t-optimal k-broadcast tree T on $N_k(t)$ vertices with odd diameter $2m+1$. Denote the vertices on a diameter path by $v_1, v_2, \ldots, v_{2m+2}$. If we remove the edge (v_{m+1}, v_{m+2}) from the tree T, we get two trees, T_a and T_b, with roots v_{m+1} and v_{m+2}, respectively, as shown in Figure 3.

Consider a k-broadcast in T from vertex $v_1 \in T_a$. Let t_1 be the time that vertex v_{m+2} receives the message. As v_{m+2} is at distance $m+1$ from v_1, $t_1 \geq m+1$. To complete the k-broadcast within the subtree T_b we have $t - t_1 \leq t - (m+1)$ time units. From Lemma 2, we see that the number of vertices in T_b, denoted $|T_b|$, is $|T_b| \leq \sum_{i=0}^{m} k^i\binom{t-(m+1)}{i}$. In a similar fashion, by considering vertex v_{2m+2} as the originator, we get $|T_a| \leq \sum_{i=0}^{m} k^i\binom{t-(m+1)}{i}$. Thus, when the diameter of T is $2m+1$, we have $|T| \leq 2\sum_{i=0}^{m} k^i\binom{t-(m+1)}{i}$. Since $t \geq 2m+1$, we obtain $t-(m+1) \geq m$, and all of the binomial coefficients $\binom{t-(m+1)}{i}$ are defined for $i \in \{0, 1, \ldots, m\}$.

We can construct a tree T of odd diameter $2m+1$ by adding an edge joining the roots of two copies of $T(t-m-1, m, k)$ as shown in Figure 4.

Thus, if a t-optimal k-broadcast tree on $N_k(t)$ vertices has odd diameter $2m+1$, then $N_k(t) = 2\sum_{i=0}^{m} k^i\binom{t-(m+1)}{i}$.

Consider a t-optimal k-broadcast tree with even diameter $2m$ and a diameter path $v_1, v_2, \ldots, v_{2m+1}$. Consider a k-broadcast scheme from v_1. In this scheme, v_{m+1} (the midpoint of the diameter path from v_1 to v_{2m+1}) is informed at some time $t_0 \geq m$. At each time unit after t_0, v_{m+1} can forward the message to at most k neighbors. Let $T_1, T_2, \ldots, T_k$ denote the trees rooted at the k neighbors that v_{m+1} calls at time $t_0+1 \geq m+1$. From Lemma 2, these trees can each contain at most $\sum_{i=0}^{m-1} k^i\binom{t-(m+1)}{i}$ vertices.

Now, consider an alternate view of the same t-optimal k-broadcast tree. If we delete the edge between v_m and v_{m+1}, and the edges from v_{m+1} to the roots of $T_1, T_2, \ldots, T_k$, we break the tree into $k+2$ trees as shown in Figure 5. We denote the tree containing v_m and v_1 as T_0 and the remaining tree containing v_{m+1} as T_{k+1}.

By Lemma 2, T_{k+1} contains at most $\sum_{i=0}^{m} k^i\binom{t-(m+1)}{i}$ vertices if its height is m and at most $\sum_{i=0}^{m-1} k^i\binom{t-(m+1)}{i}$ vertices if its height is less than m. By considering a k-broadcast scheme from v_{2m+1} (v_{2m+1} is in some T_i where $i \neq 0$), we can similarly show that T_0 contains at most $\sum_{i=0}^{m-1} k^i\binom{t-(m+1)}{i}$ vertices. Thus, the number of vertices in T is at most $k\sum_{i=0}^{m-1} k^i\binom{t-(m+1)}{i}$ plus the number of vertices in T_{k+1} and in T_0.

If T_{k+1} has height m, then consider a k-broadcast originated at some leaf l of T_{k+1} at distance m from v_{m+1}. In general, there are many such leaves. For each potential leaf l, there is a unique path from l to v_{m+1} through a neighbor u_l of v_{m+1}. We choose a particular l such that the subtree of T_{k+1} rooted at u_l is the smallest such subtree. Since the height of this subtree is $m-1$, then, by Lemma 2, it can have at most $(k+1)^{m-1}$ vertices. In this k-broadcast from originator l, v_{m+1} learns the message at some time $m' \geq m$ and can send the message to at most k of its neighbors at time $m'+1$. Each of these neighbors is either a root of one of the subtrees $T_0, T_1, \ldots, T_k$ or a vertex in T_{k+1}. By Lemma 2, each of these k vertices can inform at most $\sum_{i=0}^{m-1} k^i\binom{t-(m+1)}{i}$ vertices in the remaining time. As we are trying to show an upper bound on the size of T, we can assume that the neighbors called at time $m'+1$ are the roots of $T_1, T_2, \ldots, T_k$. (By assuming that some of the vertices of T_{k+1} are called at time $m'+1$, we would obtain a smaller upper bound as we have already established an upper bound on the size of T_{k+1}.) Thus, we assume that v_m, the root of T_0, is informed no earlier than at time $m'+2$. In the remaining time units [no more than $t-(m'+1)$], v_{m+1} must call

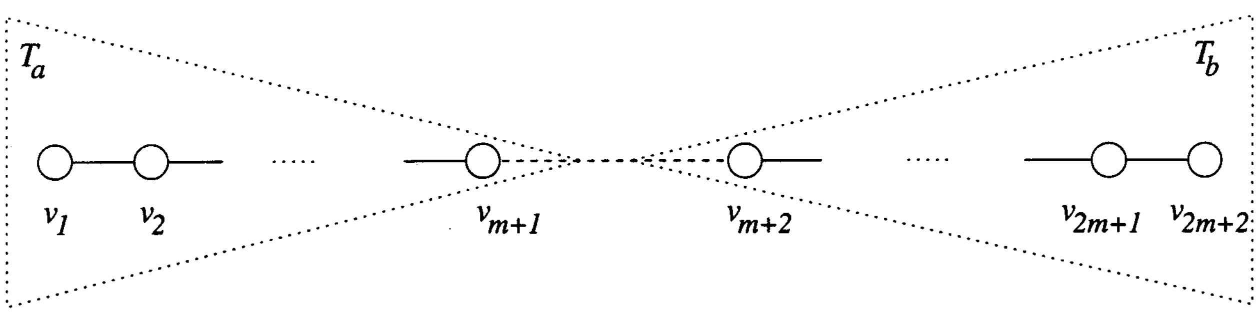

FIG. 3. Two trees obtained by deleting a middle edge from a diameter path.

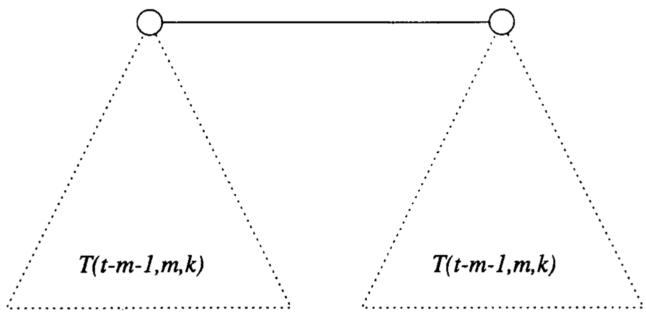

FIG. 4. Tree T of odd diameter $2m + 1$.

all of its uninformed neighbors within T_{k+1} as well as v_m. The number of vertices that can be informed in this manner is at most $\sum_{i=0}^{m} k^i \binom{t-(m+1)}{i}$.

Consider the two situations: In the k-broadcast scheme for originator v_1, v_{m+1} learns the message at time m, calls the roots of $T_1, T_2, ..., T_k$ at time $m+1$, and then initiates a broadcast to its descendants in T_{k+1} beginning at time $m + 2$. From this, we know that T_{k+1} contains, at most, $\sum_{i=0}^{m} k^i \binom{t-(m+1)}{i}$ vertices. In the k-broadcast scheme for originator l, v_{m+1} learns the message at time $m' \geq m$, calls the roots of $T_1, T_2, ..., T_k$ at time $m' + 1$, and then, beginning at time $m' + 2$, initiates a broadcast to the vertices of T_0 plus the vertices of T_{k+1} except for those in the subtree rooted at u_l. Thus, the sum of the sizes of T_0 and T_{k+1} minus the size of the subtree of T_{k+1} rooted at u_l is at most $\sum_{i=0}^{m} k^i \binom{t-(m+1)}{i}$. As the size of the subtree rooted at u_l is at most $(k + 1)^{m-1}$, we conclude that the combined size of T_0 and $T_k + 1$ is at most $\sum_{i=0}^{m} k^i \binom{t-(m+1)}{i} + (k + 1)^{m-1}$. Thus, if T_{k+1} has height m, the total number of vertices in T is at most $n_a = k \sum_{i=0}^{m-1} k^i \binom{t-(m+1)}{i} + \sum_{i=0}^{m} k^i \binom{t-(m+1)}{i} + (k+1)^{m-1}$.

If T_{k+1} has height $< m$, then T_{k+1} has at most $\sum_{i=0}^{m-1} k^i \binom{t-(m+1)}{i}$ vertices since the k-broadcast scheme for v_1 does not begin to inform vertices in this tree (other than its root, v_{m+1}) prior to time $m + 1$. Thus, the total number of vertices in T is at most $(k + 1) \sum_{i=0}^{m-1} k^i \binom{t-(m+1)}{i} + \sum_{i=0}^{m-1} k^i \binom{t-(m+1)}{i} = (k + 2) \sum_{i=0}^{m-1} k^i \binom{t-(m+1)}{i} = n_b$.

We can construct a tree T of even diameter $2m$ which has n_b vertices and k-broadcast time t as follows: Begin with $k + 2$ copies of $T(t - m - 1, m - 1, k)$ with roots $u_1, u_2, ..., u_{k+2}$. Connect u_{k+2} to each of the other roots u_i, $1 \leq i \leq k + 1$. The resulting tree structure is shown in Figure 6.

The resulting tree has n_b vertices, even diameter $2m$, and k-broadcast time t.

We can construct a tree T of even diameter $2m$ with n_a vertices and k-broadcast time t as follows: Begin with k copies of $T(t-m-1, m-1, k)$ with roots $u_1, u_2, ..., u_k$, respectively, a copy of $T(m-1, m-1, k)$ with root u_{k+1}, and a copy of $T(t - m - 1, m, k)$ with root u_{k+2}. Connect u_{k+2} to each of the other roots $u_1, u_2, ..., u_{k+1}$. The resulting tree structure is shown in Figure 7.

The resulting tree has n_a vertices, diameter $2m$, and k-broadcast time t.

Thus, if a t-optimal k-broadcast tree on $N_k(t)$ vertices has even diameter $2m$, then $N_k(t) = \max\{n_a, n_b\}$.

The result follows. ∎

3. ASYMPTOTIC ESTIMATION OF $N_K(T)$

As there is no known closed-form expression for partial sums of binomial coefficients [5], we have no simple formula for $N_k(t)$. However, by evaluating the above maxima for various values of m, we can obtain different lower bounds on $N_k(t)$ [and, thus, upper bounds on $T_k(n)$]. In particular, by letting $m = \lfloor \frac{t}{2} \rfloor$, we get

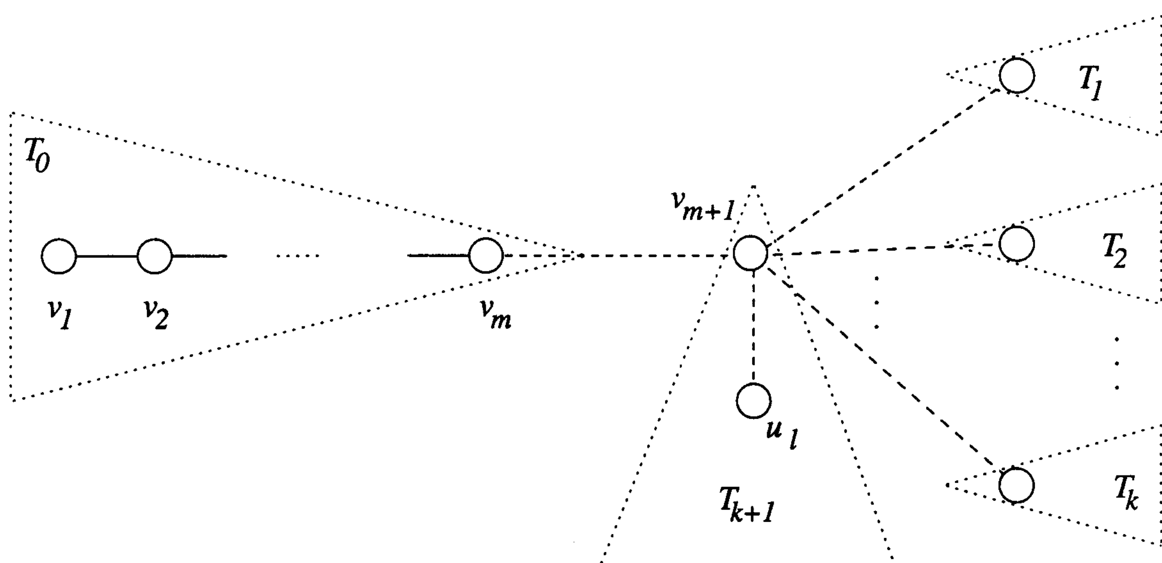

FIG. 5. $k + 2$ subtrees.

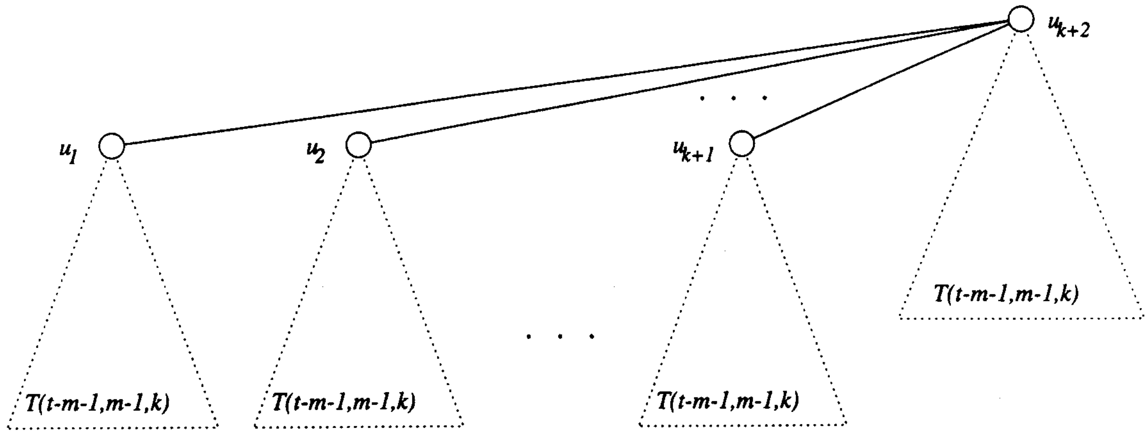

FIG. 6. Tree T of even diameter $2m$ and n_b vertices.

Lemma 3. *$N_k(t) \geq 2(k+1)^{\lfloor \frac{t}{2} \rfloor}$ for $t \geq 1$ and $T_k(n) \leq 2\lceil \log_{k+1} n \rceil$ for $n \geq 1$ and $k \geq 1$.*

The following technical lemmas will be used for the asymptotic estimation of $N_k(t)$: Since $N_k(t)$ is the maximum of three expressions that are sums of the same form, for large t, they will have the same limit. Thus, we will only estimate $M = \max_{0 \leq m \leq (t-1)/2} \{\sum_{i=0}^{m} k^i \binom{t-(m+1)}{i}\}$. We begin by giving bounds on M.

Lemma 4. *For $k \geq 2$, $\max_{0 \leq m \leq (t-1)/2} \{k^m \binom{t-m-1}{m}\} \leq M \leq \max_{0 \leq m \leq t/2} \{k^{m+1} \binom{t-m+1}{m+1}\}$.*

Proof. Suppose that $k^m \binom{t-m-1}{m}$, $\sum_{i=0}^{m} k^i \binom{t-m-1}{i}$ and $\{k^{m+1} \binom{t-m+1}{m+1}\}$ are maximized at m_0, m_1, and m_2, respectively. By the definition of m_0, $\max_{0 \leq m \leq (t-1)/2} \{k^m \binom{t-m-1}{m}\} = k^{m_0} \binom{t-m_0-1}{m_0}$. Adding additional terms, we know that $k^{m_0} \binom{t-m_0-1}{m_0} \leq \sum_{i=0}^{m_0} k^i \binom{t-m_0-1}{i}$ and $\sum_{i=0}^{m_0} k^i \binom{t-m_0-1}{i} \leq \sum_{i=0}^{m_1} k^i \binom{t-m_1-1}{i} = M$ by the definition of m_1.

Since $M = \sum_{i=0}^{m_1} k^i \binom{t-m_1-1}{i}$, then $M \geq \sum_{i=0}^{m_1-1} k^i \binom{t-(m_1-1)-1}{i} = \sum_{i=0}^{m_1-1} k^i \binom{t-m_1}{i}$ by the choice of m_1. Using the equality $\binom{a}{b} = \binom{a-1}{b} + \binom{a-1}{b-1}$ in the last sum, we obtain $M \geq \sum_{i=0}^{m_1-1} k^i \binom{t-m_1-1}{i} + \sum_{i=0}^{m_1-2} k^{i+1} \binom{t-m_1-1}{i}$. By subtracting $\sum_{i=0}^{m_1-1} k^i \binom{t-m_1-1}{i}$ from both sides, we get $k^{m_1} \binom{t-m_1-1}{m_1} \geq k \sum_{i=0}^{m_1-2} k^i \binom{t-m_1-1}{i}$. Dividing by k, we get $k^{m_1-1} \binom{t-m_1-1}{m_1} \geq \sum_{i=0}^{m_1-2} k^i \binom{t-m_1-1}{i}$. Adding two terms to each side and reversing the order of the inequality, we get $\sum_{i=0}^{m_1} k^i \binom{t-m_1-1}{i} \leq k^{m_1-1} \binom{t-m_1-1}{m_1} + k^{m_1-1} \binom{t-m_1-1}{m_1-1} + k^{m_1} \binom{t-m_1-1}{m_1}$. Since $k^{m_1-1} < k^{m_1}$, we know that this sum is less than $k^{m_1} \binom{t-m_1-1}{m_1} + k^{m_1} \binom{t-m_1-1}{m_1-1} + k^{m_1} \binom{t-m_1-1}{m_1}$. Combining the first two terms, we get $k^{m_1} \binom{t-m_1}{m_1} + k^{m_1} \binom{t-m_1-1}{m_1}$. Since $\binom{t-m_1}{m_1} \geq \binom{t-m_1-1}{m_1}$, we see that $M = \sum_{i=0}^{m_1} k^i \binom{t-m_1-1}{i} < 2k^{m_1} \binom{t-m_1}{m_1}$. Since $k \geq 2$, $2k^{m_1} \binom{t-m_1}{m_1} \leq k^{m_1+1} \binom{t-m_1}{m_1}$. Since $m_1 \leq \frac{t-1}{2}$, $t - m_1 + 1 > m_1 + 2$, so $k^{m_1+1} \binom{t-m_1}{m_1} < k^{m_1+1} \binom{t-m_1+1}{m_1+1}$. Thus, $M < k^{m_1+1} \binom{t-m_1+1}{m_1+1} \leq k^{m_2+1} \binom{t-m_2+1}{m_2+1}$, by the definition of m_2. ∎

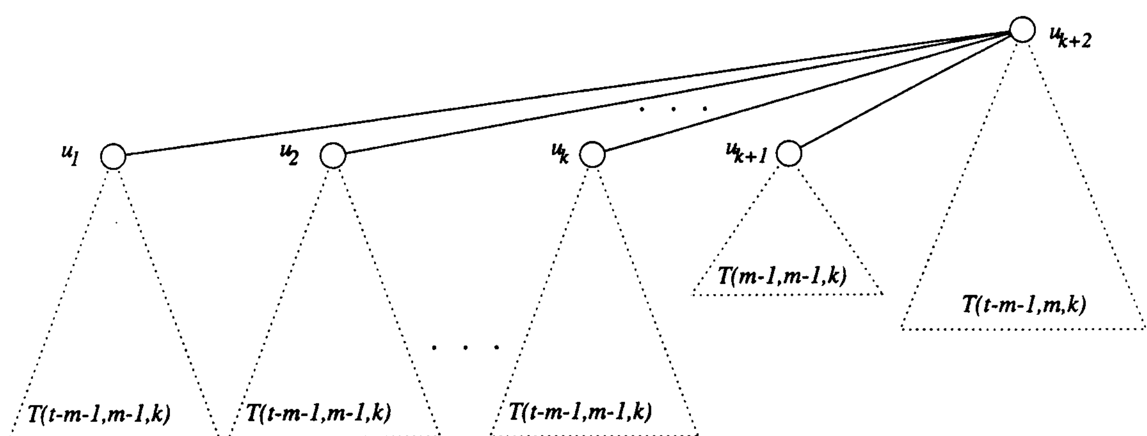

FIG. 7. Tree T of even diameter $2m$ and n_a vertices.

Note that the above upper and lower bounds are similar expressions. We now try to find an asymptotic expression for these bounds:

Lemma 5. *Let $f(t,m) = k^m\binom{t-m}{m}$. For fixed $k \geq 2$ and as $t \to \infty$, $\max_{0 \leq m \leq \lfloor \frac{t}{2} \rfloor}\{f(t,m)\} = f(\frac{\sqrt{4k+1}-1}{2\sqrt{4k+1}}t)$.*

Proof. Let $a = \sqrt{4k+1}$. We will show that the function $f(t,m)$ increases in the interval $0 \leq m \leq \frac{a-1}{2a}t$ and decreases in the interval $\frac{a-1}{2a}t \leq m \leq \lfloor \frac{t}{2} \rfloor$. It follows that the maximum is achieved at $\frac{a-1}{2a}t$.

Consider two consecutive values of this function: If $k^m\binom{t-m}{m} \leq k^{m+1}\binom{t-(m+1)}{m+1}$, then f increases. This can be simplified to $\binom{t-m}{m} \leq k\binom{t-(m+1)}{m+1}$. From this second inequality, after expanding the binomial coefficients and performing some algebraic manipulations, we get the following inequality: $(4k+1)m^2 + (-4kt + 2k - t + 1)m + kt^2 - kt - t \leq 0$. When t is large, we obtain $0 \leq m \leq \frac{a-1}{2a}t + O(k)$. In other words, for large t, the function $f(t,m)$ increases in the interval $0 \leq m \leq \frac{a-1}{2a}t$.

When $k^m\binom{t-m}{m} \geq k^{m+1}\binom{t-(m+1)}{m+1}$, similar manipulations show that $f(t,m)$ is decreasing in the interval $\frac{a-1}{2a}t \leq m \leq \lfloor \frac{t}{2} \rfloor$.

Thus, the maximum is achieved at $\frac{a-1}{2a}t = \frac{\sqrt{4k+1}-1}{2\sqrt{4k+1}}t$. ∎

Theorem 2. *For fixed $k \geq 2$ and as $t \to \infty$, $N_k(t) \approx (\frac{\sqrt{4k+1}+1}{2})^t$.*

Proof. Recall that $N_k(t) = M$ for large t. Lemma 4 shows that $\max_{0 \leq m \leq (t-1)/2}\{k^m\binom{t-m-1}{m}\} \leq N_k(t) \leq \max_{0 \leq m \leq t/2}\{k^{m+1}\binom{t-m+1}{m+1}\}$. We use m_t to denote $\frac{\sqrt{4k+1}-1}{2\sqrt{4k+1}}t$. From Lemma 5, the lower bound is $k^{m_{t-1}}\binom{(t-1)-m_{t-1}}{m_{t-1}}$ and the upper bound is $k^{m_{t+2}}\binom{(t+2)-m_{t+2}}{m_{t+2}}$. As before, let $a = \sqrt{4k+1}$. After some algebraic manipulation, we get $k^{\frac{a-1}{2a}(t-1)}\binom{\frac{a+1}{2a}(t-1)}{\frac{a-1}{2a}(t-1)} \leq N_k(t) \leq k^{\frac{a-1}{2a}(t+2)}\binom{\frac{a+1}{2a}(t+2)}{\frac{a-1}{2a}(t+2)}$. Since the upper and lower bounds are the same asymptotically, it follows that $N_k(t) \approx k^{\frac{a-1}{2a}t}\binom{\frac{a+1}{2a}t}{\frac{a-1}{2a}t}$.

Applying Stirling's formula (see, e.g., [5]), we get

$$N_k(t) \approx \frac{c(k)}{\sqrt{t}} k^{\frac{a-1}{2a}t} \frac{(\frac{a+1}{2a})^{\frac{a+1}{2a}t}}{(\frac{a-1}{2a})^{\frac{a-1}{2a}t}(\frac{1}{a})^{\frac{1}{a}t}},$$

where $c(k)$ is a function of k. Since $k = \frac{(a-1)}{2}\frac{(a+1)}{2}$, then substituting this value for k and simplifying, we get $N_k(t) \approx (\frac{a+1}{2})^t = (\frac{\sqrt{4k+1}+1}{2})^t$ for large t. ∎

From Theorem 2, the next corollary follows immediately:

Corollary 1. *For fixed $k \geq 2$ and as $t \to \infty$, $T_k(n) \approx \log_{\frac{\sqrt{4k+1}+1}{2}} n$.*

Note that letting $k = 1$ in Theorem 2 and Corollary 1, we get the results obtained by Khachatrian and Haroutunian [10] and by Labahn [12].

REFERENCES

[1] A. Bar-Noy, J. Bruck, C.T. Ho, S. Kipnis, and B. Schieber, Computing global combine operations in the multi-port postal model, IEEE Trans Par Distr Syst 6 (1995), 896–900.

[2] J. Bruck and C.T. Ho, Efficient global combine operations in the multi-port message passing systems, Par Proc Lett 3 (1993), 335–346.

[3] M. Dietzfelbinger, Gossiping and broadcasting versus computing functions in networks, manuscript, Technische Universität Ilmenau, 1999.

[4] P. Fraigniaud and E. Lazard, Methods and problems of communication in usual networks, Discr Appl Math 53 (1994), 79–133.

[5] R. Graham, D. Knuth, and O. Patashnik, Concrete mathematics, 2nd ed., Addison-Wesley, Reading, MA, 1994.

[6] M. Grigni and D. Peleg, Tight bounds on minimum broadcast networks, SIAM J Discr Math 4 (1991), 207–222.

[7] H.A. Harutyunyan and A.L. Liestman, Improved upper and lower bounds for k-broadcasting, Networks 37 (2001), 94–101.

[8] S.T. Hedetniemi, S.M. Hedetniemi, and A.L. Liestman, A survey of broadcasting and gossiping in communication networks, Networks 18 (1988), 319–349.

[9] J. Hromkovič, R. Klasing, B. Monien, and R. Peine, "Dissemination of information in interconnection networks (broadcasting & gossiping)," Combinatorial network theory, Science Press & AMS, 1994, pp. 125–212.

[10] L. Khachatrian and H. Haroutunian, Minimal broadcast trees, XIV Moscow–Minsk All Union School of Computing Networks, 1989, pp. 36–40 (in Russian).

[11] J.-C.König and E. Lazard, Minimum k-broadcast graphs, Discr Appl Math 53 (1994), 199–209.

[12] R. Labahn, Extremal broadcasting problems, Discr Appl Math 23 (1989), 139–155.

[13] E. Lazard, Broadcasting in dma-bound bounded degree graphs, Discr Appl Math 37/38 (1992), 387–400.

[14] S. Lee, Information dissemination theory in communication networks: Design of c Broadcast Networks, Ph.D. Dissertation, Dept. of Industrial and Manufacturing Engineering, Pennsylvania State University, 1999.

[15] S. Lee and J.A. Ventura, An algorithm for constructing minimal c-broadcast networks (to appear).

[16] A.L. Liestman, Fault-tolerant broadcast graphs, Networks 15 (1985), 159–171.

[17] A. Shastri and S. Gaur, Multi-broadcasting in communication networks. I: Trees, Proc Int Symp on Communications (ISCOM97), 1997, pp. 167–170.