

Efficient construction of broadcast graphs



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ABSTRACT

A broadcast graph is a connected graph, $G = (V, E)$, $|V| = n$, in which each vertex can complete broadcasting of one message within at most $t = \lceil \log n \rceil$ time units. A minimum broadcast graph on n vertices is a broadcast graph with the minimum number of edges over all broadcast graphs on n vertices. The cardinality of the edge set of such a graph is denoted by $B(n)$. In this paper we construct a new broadcast graph with $B(n) \leq (k+1)N - (t - \frac{k}{2} + 2)2^k + t - k + 2$, for $n = N = (2^k - 1)2^{t+1-k}$ and $B(n) \leq (k+1-p)n - (t - \frac{k}{2} + p + 2)2^k + t - k - (p-2)2^p$, for $2^t < n < (2^k - 1)2^{t+1-k}$, where $t \geq 7$, $2 \leq k \leq \lfloor t/2 \rfloor - 1$ for even n and $2 \leq k \leq \lfloor t/2 \rfloor - 1$ for odd n , $d = N - n$, $x = \lfloor \frac{d}{2^{t+1-k}} \rfloor$ and $p = \lfloor \log_2(x+1) \rfloor$ if $x > 0$ and $p = 0$ if $x = 0$.

The new bound is an improvement upon the bounds appeared in Bermond et al. (1995), Bermond et al. (1997), Fertin and Raspaud (2004) and Harutyunyan and Leistman (1999) and the recent bound presented by Harutyunyan and Liestman (2012) for odd values of n .

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1. Introduction

Broadcasting is an information distribution problem in a connected graph, in which one vertex, called the *originator*, has to distribute a message to all other vertices by placing a series of calls among the communication lines of the graph. Once informed, the informed vertices aid the originator in distributing the message. This is assumed to take place in discrete time units. The broadcasting has to be completed within a minimal number of time units subjected to the following constraints:

1. Each call involves only one informed vertex and one of its uninformed neighbors.
2. Each call requires one time unit.
3. A vertex can participate in at most one call at each time unit.
4. At each time unit many calls can be performed in parallel.

Formally, any network can be modeled as a simple connected graph $G = (V, E)$, $|V| = n$, where V is the set of vertices and E is the set of edges (the communication lines). For a given originator vertex, u , the *broadcast time* of u , $b(u)$, is defined as the minimum number of time units needed to complete broadcasting from u . Note that for any vertex $u \in V$, $b(u) \geq \lceil \log n \rceil$ (to the sequel the base of logs is always 2), since at each time unit the number of informed vertices can at most double. The broadcast time $b(G)$ of the graph G is defined as $\max\{b(u) | u \in G\}$ and G is called a broadcast graph if $b(G) = \lceil \log n \rceil$.

The *broadcast number* $B(n)$ is the minimum number of edges in any broadcast graph on n vertices. A *minimum broadcast graph* (*mbg*) is a broadcast graph on n vertices with $B(n)$ edges. Currently, the exact values of $B(n)$ are known only for

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$n = 2^p$, $n = 2^p - 2$, $n = 127$, and for several values of $n \leq 63$, as detailed below. Farley et al. [7] determined the values of $B(n)$ for $n \leq 15$ and showed that hypercubes are *mbgs* such that $B(2^p) = p2^{p-1}$ for any $p \geq 2$. Mitchell and Hedetniemi [16] determined the value of $B(17)$, while Bermond, Hell, Liestman and Peters [3] determined the values of $B(n)$ for $n = 18, 19, 30, 31$. Khachatryan and Haroutunian [15] and independently Dinnen, Fellows and Faber [4] proved that $B(2^p - 2) = (p - 1)(2^{p-1} - 1)$ for all $p \geq 2$.

Since *mbg*'s seem to be difficult to find, many authors have devised methods to construct broadcast graphs. The number of edges in any broadcast graph on n vertices gives an upper bound on $B(n)$. Several papers have shown methods to construct broadcast graphs by forming the compound of two known broadcast graphs (see [1,5,11,15]). These methods have proven effective for graphs on $n_1 n_2$ vertices from two known broadcast graphs on n_1 and n_2 vertices. Thus, compounding produces good upper bound on $B(n)$ for many values of n . In particular, a very tight upper bound was obtained for $n = 2^p - 2^k$ by compounding *mbg*'s on 2^{k-1} and $2^{p-k+1} - 2$ vertices: $B(2^p - 2^k) \leq \frac{2^p - 2^k}{2} (p - \frac{k+1}{2})$ (see [1,15]).

Broadcast graphs on other sizes can sometimes be formed by adding or deleting vertices from known broadcast graphs (see [3] for example). An efficient vertex addition method is suggested in [10]. The authors in [11] presented a method based on compounding and then merging several vertices into one that allows the construction of the best broadcast graphs for almost all values of n , including many prime numbers. In particular, a very tight upper bound on $B(n)$ is $B(2^p - 2^k + 1) \leq 2^{p-1} (p - \frac{k}{2})$ (again by compounding *mbg*'s on 2^k and 2^{p-k} vertices and then merging 2^k vertices into one).

Farley [6] proposed the recursive method to construct minimal broadcast graphs and proved the general upper bound

$$B(n) \leq \frac{n \lceil \log n \rceil}{2}, \quad 2^{p-1} < n \leq 2^p. \quad (1)$$

Other general upper bounds on $B(n)$ are obtained from a direct construction using binomial trees (see [9,11,15]) for some values of n . The authors in [11,12] have considered combination of hypercubes and binomial trees to construct sparse broadcast graphs.

Direct construction of broadcast graphs is a difficult problem. The best upper bound from a direct construction for any n is

$$B(n) \leq n(p - k + 1) - 2^{p-k} - \frac{1}{2}(p - k)(3p + k - 3) + 2k, \quad (2)$$

where $n = 2^p - 2^k - r$, $0 \leq k \leq p - 2$ and $0 \leq r \leq 2^k - 1$ (see [11]). While this bound is tight for $p - k$ is small for $k < p/2$ it is not as good as the bound from [6], in (1).

The best general upper bound on $B(n)$ for even n , namely,

$$B(n) \leq \frac{n \lfloor \log n \rfloor}{2} \quad (3)$$

obtained from the modified Knödel graph (see [1,2,8]). This bound, is better than the one in (1) for all even $n \neq 2^p$.

In [14], Harutyunyan and Xu presented an upper bound on $B(n)$ for odd n . They proved that for integers n, p , where $n > 65$ is odd, $p \geq 7$ and $n \neq 2^p + 1$, $B(n) \leq \frac{(n+1) \lfloor \log n \rfloor}{2} + 2 \lceil \frac{n-1}{10} \rceil - \lfloor \frac{\lfloor \log n \rfloor + 2}{4} \rfloor$.

However, recently Harutyunyan and Liestman presented in [13] a new upper bound for odd, positive integers, namely,

Theorem 1.1. *Let n be an even integer such that $\lceil \log n \rceil > 2$ is prime, $m = \lceil \log n \rceil \neq 2^j - 1$ for any integer j , m divides n , and for any integer $d \neq m - 1$ which is a divisor of $m - 1$, $2^d \not\equiv 1 \pmod{(m)}$. Then,*

$$B(n + 1) \leq \frac{n \lfloor \log n \rfloor}{2} + \frac{n}{\lceil \log n \rceil} + \lceil \log n \rceil - 2. \quad (4)$$

In this paper we present a new construction of a broadcast graph and thus a new upper bound for $B(n)$. This new upper bound is the best existing upper bound and improves the bounds in (1)–(4). Our main result is,

Theorem 1.2. *Let t, k, n be positive integers. Then, for a given $t \geq 7$ and $2 \leq k \leq \lfloor t/2 \rfloor - 1$,*

1. *If $n = N = (2^k - 1)2^{t+1-k}$,*

$$B(n) \leq (k + 1)N - \left(t - \frac{k}{2} + 2\right)2^k + t - k + 2. \quad (5a)$$

If $2^t < n < (2^k - 1)2^{t+1-k}$,

$$B(n) \leq (k + 1 - p)n - \left(t - \frac{k}{2} + p + 2\right)2^k + t - k - (p - 2)2^p, \quad (5b)$$

where $d = N - n$, $x = \lfloor \frac{d}{2^{t+1-k}} \rfloor$ and $p = \begin{cases} \lfloor \log_2(x + 1) \rfloor & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases}$

2. Proof of Theorem 1.2

2.1. Construction of the broadcast graph

First we construct a broadcast graph and then demonstrate the broadcast scheme. Our construction is composed of binomial trees and hypercubes.

Definition 2.1. A binomial tree of order t , denoted by B^t , is defined recursively as follows:

A binomial tree of order 0 is the trivial tree (a single vertex).

A binomial tree of order t has vertex which is a root vertex whose children are roots of binomial trees of orders $t - 1, t - 2, \dots, 2, 1, 0$ (in this order).

Observation. The Binomial tree B^t has 2^t vertices and height t . Because of its unique structure, a binomial tree of order t can be constructed trivially from two trees of order $t - 1$ by attaching one of them as the rightmost child of the root of the other one.

Lemma 2.1. Let B^n be the binomial tree of order n . Let u be the root of B^n . Then, $b(u) = n$.

Lemma 2.2. Let Q^n be the n -dimensional hypercube. Then, for each vertex $u \in V(Q^n)$, $b(u) = n$.

The proofs of both lemmas are easy and are omitted.

Proof of Theorem 1.2. First we demonstrate the construction of a broadcast graph G giving the upper bound of $B(N)$ declared in (5a), for $N = (2^k - 1)2^{t+1-k}$ (case 1). The broadcast graph $G = (V, E)$, $|V| = n$, with $2^t < n < N$ shall be constructed later (case 2). The broadcast scheme in that graph shall demonstrate in the next section.

Case 1: For a given integer $t \geq 7$ and $k, 2 \leq k \leq \lfloor t/2 \rfloor - 1$, we construct a minimal broadcast graph $G = (V, E)$ with $|V| = N = (2^k - 1)2^{t+1-k}$.

The broadcast graph G is constructed of $2^k - 1$ binomial trees denoted $B_i, 1 \leq i \leq 2^k - 1$. Each $B_i, 1 \leq i \leq 2^k - 1$, is a B_{t+1-k} tree. Let $R = \{r_1, \dots, r_{2^k-1}\}$ be the set of the roots of the binomial trees $B_1, B_2, \dots, B_{2^k-1}$, respectively. For each $i, 1 \leq i \leq 2^k - 1$, r_i is of degree $t + 1 - k$.

Thus, $|V(G)| = N = (2^k - 1)2^{t+1-k}$.

It is easily observed that $\lceil \log N \rceil = t + 1$.

Denote by $V_1 = \{r_1, \dots, r_{k-1}\}$ the set of the roots of the trees B_1, \dots, B_{k-1} , respectively and by $V_2 = \{r_{k+1}, \dots, r_{2^k-1}\}$ the set of the roots of the trees $B_{k+1}, \dots, B_{2^k-1}$, respectively. Thus, $V_1 \cup \{r_k\} \cup V_2 = R$ with $|R| = 2^k - 1$.

Now we construct the set of edges $E(G)$ and then calculate its cardinality. Let $w \in B_1$ be the farthest leaf from the root r_1 . We connect the vertices of $R \cup \{w\}$ in a way that they form a hypercube of dimension k , denoted by Q^k . Let $Q^{k-1}, Q^{k-2}, \dots, Q^0, Q^{01}$, be the hypercube graphs that form Q^k such that $w \in Q^{01}$ (in fact, $w = Q^{01}$) and for each $0 \leq i \leq k - 1, r_{i+1} \in Q^i$. Let Q_1^{k-1} and Q_2^{k-1} be the two hypercube graphs of dimension $k - 1$ that form Q^k such that $Q_1^{k-1} = Q^{k-1}$ and $Q^{k-2}, \dots, Q^0, Q^{01}$ form Q_2^{k-1} . We connect each vertex $v, v \in V \setminus (R \cup \{w\})$, in which its root, $r, r \in Q_1^{k-1}$, to each of the vertices in $V_1 \cup \{r\}$. For the vertices $v, v \in V \setminus (R \cup \{w\})$, in which their root $r, r \in Q_2^{k-1}$, we do the following: if $r \in Q^i, 0 \leq i \leq k - 2$, we connect v to its root r , to each vertex in $V_1 \setminus \{r_{i+1}\}$ and to r_k .

Summary: The broadcast graph G constructed is a hypercube Q^k of dimension k , and 2^k vertices (the set $R \cup \{w\}$), where each of the vertices in R is a root of a binomial tree on 2^{t+1-k} vertices. Furthermore, each of the vertices of the binomial trees which are not on $R \cup \{w\}$ is adjacent to its root and to each of the vertices in $V_1 \cup \{r_k\}$, except to r_j , if that vertex belongs to Q^{j-1} , for $1 \leq j \leq k$.

Following is a calculation of the cardinality of $|E(G)|$.

First, the number of edges in the binomial trees is

$$|\bigcup_{i=1}^{2^k-1} E(B_i)| = \sum_{i=1}^{2^k-1} |E(B_i)| = (2^k - 1)(2^{t+1-k} - 1). \quad (6)$$

The number of edges in the hypercube induced on $R \cup \{w\}$ is

$$|E(Q^k)| = k2^{k-1}. \quad (7)$$

The number of edges that connect each non root vertex in G to its root is

$$(2^k - 1)[2^{t+1-k} - 1 - (t + 1 - k)] - 1. \quad (8)$$

The number of edges that connect the non root vertices in $Q_2^{k-1} \setminus \{w\}$ to r_k is

$$(2^{k-1} - 1)(2^{t+1-k} - 1) - 1. \quad (9)$$

The number of edges that connect each vertex of V_1 to all vertices of Q_1^{k-1} which are not roots (do not belong to R) is

$$(k-1)2^{k-1}(2^{t+1-k} - 1). \quad (10)$$

And finally, the number of edges that connect the vertices of V_1 to all the vertices in $Q_2^{k-1} \setminus \{w\}$ is

$$(k-2)(2^{k-1} - 1)[(2^{t+1-k} - 1) - 1]. \quad (11)$$

Thus, summing the values in (6) up to (11) and recalling that $N = (2^k - 1)2^{t+1-k}$ we obtain

$$|E(G)| = (k+1)N - \left(t + 2 - \frac{k}{2}\right)2^k + t + 2 - k. \quad (12)$$

Case 2: We construct now a broadcast graph $G' = (V', E')$, $|V'| = n$, where $2^t < n < (2^k - 1)2^{t+1-k}$. We start by constructing a broadcast graph, $G = (V, E)$, with $|V| = N = (2^k - 1)2^{t+1-k}$ as described in Case 1. Then, we obtain G' from G by deleting vertices and edges from G , in a way described below.

Define $d = N - n$, $x = \lfloor \frac{d}{2^{t+1-k}} \rfloor$, $y = d - x2^{t+1-k}$ and

$$p = \begin{cases} \lfloor \log_2(x+1) \rfloor & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Note that $0 \leq x < 2^{k-1}$, $0 \leq y < 2^{t+1-k}$ and $1 \leq p < k$.

In order to construct G' we delete vertices from G as needed according to the value of d . Since $d = 2^{t+1-k}x + y$, the deletion process is done as follows:

1. If $x = 0$, $d = y$, we delete y vertices from some binomial tree in a way that we start deleting from the leaves and each vertex is deleted after all its descendants in the binomial tree are already deleted.
2. If $x > 0$, $d = 2^{t+1-k}x + y$, we delete $2^p - 1$ complete binomial trees and additional $2^{t+1-k}[x - (2^p - 1)] + y$ non root vertices and then add $2^p - 1$ edges. This is done in the following way:
 - (a) Delete all the vertices that are in the binomial trees in which their roots form Q^0, Q^1, \dots, Q^{p-1} . Here, we delete $2^p - 1$ binomial trees, where p of these trees are rooted by vertices from V_1 . Note that the hypercubes Q^0, Q^1, \dots, Q^{p-1} are deleted from Q_2^{k-1} .
 - (b) Delete $2^{t+1-k}(x - (2^p - 1)) + y$ non root vertices from the trees in which their roots are in $Q_2^{k-1} \setminus (\cup_{i=0}^{p-1} Q^i)$. Note that since $p = \lfloor \log_2(x+1) \rfloor$, the number of vertices that we delete here is less than $2^{t+1-k} \cdot 2^p$.
 - (c) For each vertex $b \in Q_1^{k-1} \cap R$, in which we have deleted its neighbor in Q_2^{k-1} , we connect b to some vertex that remained in Q^{k-2} . Those edges that we add here replace the edges that connected b to some other root in Q_2^{k-1} that we have deleted in (a). This addition of edges is crucial in order to keep each vertex in the hypercube Q_1^{k-1} matched to another vertex in Q_2^{k-1} .

After the deletion process is ended we obtain in G' the following sets: R' is the set of the binomial trees roots. Then, $R' = V'_1 \cup \{r_k\} \cup V'_2$, $|R'| = 2^k - 1 - (2^p - 1) = 2^k - 2^p$, where $V'_1 = \{r_1 \dots r_{k-1-p}\}$, $|V'_1| = k - 1 - p$, $V'_2 = R' \setminus (V'_1 \cup \{r_k\})$ and $|V'_2| = 2^k - 1 - k - (2^p - 1 - p) = 2^k - 2^p + p - k$.

2.1.1. Example: minimal broadcast graph construction

Now we calculate the number of edges that are deleted from G in order to obtain the graph G' (see Fig. 1).

First, we count the edges that are adjacent to each non root vertex in the $2^p - 1$ complete binomial trees that were deleted from G . The degree of each vertex v in $V \setminus R$ is $k + j + 1$, where j is the distance of v to the farthest leaf in its subtree. Indeed, j edges connect v to its direct siblings, k edges connect v to vertices in R and one edge connects v to its direct ancestor. Since we delete a vertex after all its siblings are already deleted, the number of edges deleted each time we delete a vertex in $V \setminus R$ is $k + 1$. Therefore, the number of such edges that are deleted is

$$(k+1)(2^{t+1-k} - 1)(2^p - 1). \quad (13)$$

Since the degree of each vertex in Q^k is k , the number of edges that we delete from Q^k is $k(2^p - 1)$. By adding the $2^p - 1$ edges, we actually omit from Q^k , as described,

$$(k-1)(2^p - 1) \quad (14)$$

edges.

Note that if $x \neq 0$, the tree B_1 rooted in r_1 ($r_1 = Q^0$) is deleted from G . Since $w \in B_1$, w is deleted from G . The calculation in (14) includes the k edges that connect w to Q^k .

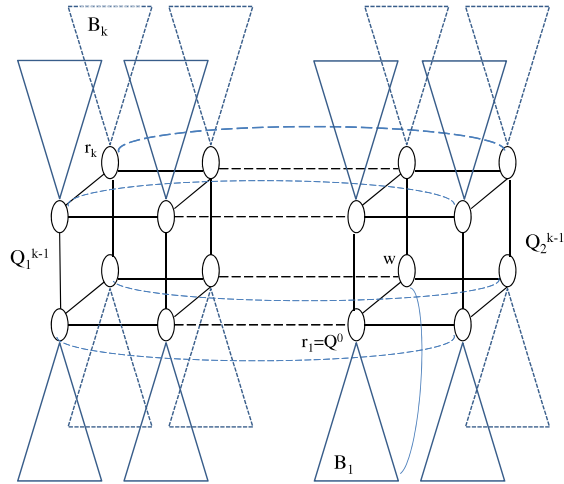


Fig. 1. This figure demonstrates the broadcast graph construction for $k = 4$. The graph is constructed of $2^k - 1$ binomial trees of dimension $t + 1 - k$. The set of binomial trees roots is R . The vertex w is a leaf in B_1 . The vertices in $R \cup \{w\}$ form a hypercube Q^k of dimension k . The two hypercubes of dimension $k - 1$ that form Q^k are Q_1^{k-1} and Q_2^{k-1} . Each vertex v , $v \in V \setminus (R \cup \{w\})$, in which its root, r , $r \in Q_1^{k-1}$, is connected to $k - 1$ roots in Q_2^{k-1} (the set V_1) and to $\{r\}$. Each vertex in $V \setminus (R \cup \{w\})$, in which its root r , $r \in Q_2^{k-1}$ and $r \in Q^i$, $0 \leq i \leq k - 2$, is connected to its root r , to each vertex in $V_1 \setminus \{r_{i+1}\}$ and to r_k .

Now, we count the number of edges that connected the p roots that were deleted from V_1 to all the non root vertices that remained in G' . This number is

$$[n - (2^k - 2^p)]p. \quad (15)$$

Finally, we count $k + 1$ edges for each of the $2^{t+1-k}(x - (2^p - 1)) + y$ non root vertices that we delete from Q_2^{k-1} , which is:

$$(k + 1)[2^{t+1-k}(x - (2^p - 1)) + y]. \quad (16)$$

Summing (13)–(16), the total number of edges that we delete from G in order to construct G' is

$$np + (k + 1)d - p2^k + (p - 2)2^p + 2. \quad (17)$$

Now, by subtracting (17) from (12), recalling that $d = N - n$, we obtain that the number of edges in G' :

$$|E(G')| = (k + 1 - p)n - \left(t - \frac{k}{2} + p + 2\right)2^k + t - k - (p - 2)2^p. \quad (18)$$

This completes the proof of the construction of a broadcast graph for $2^t < n \leq N$.

Observation. One can easily observe that if $n = N$ and thus, $x = p = 0$, we obtain $E(G') = (k + 1)n - \left(t - \frac{k}{2} + 2\right)2^k + t - k + 2$ as in (12).

Remark. For odd n we can have $k \leq \lceil \frac{t}{2} \rceil - 1$.

2.2. Broadcasting scheme

Let u be an originator. We demonstrate a broadcast scheme in the constructed graphs of cases 1 and 2.

Case 1: $|V| = n = (2^k - 1)2^{t+1-k}$.

Case 1.1: Let $u \in R \cup \{w\}$.

The broadcast scheme in that case is as follows: Since the vertices of $R \cup \{w\}$ form a hypercube of 2^k vertices, at most k time units are needed to complete broadcasting in $R \cup \{w\}$ (see Lemma 2.2).

Case 1.2: $u \in Q_1^{k-1} \setminus R$.

At time unit $t = 1$, u transmits to its root, which needs another $k - 1$ time units to accomplish broadcasting to all members of Q_1^{k-1} . At time unit i , $2 \leq i \leq k$, u transmits to r_{k-i+1} that needs another $k - i$ time units to accomplish broadcasting in Q^{k-i} . Broadcasting in Q^{k-i} completes after time unit k and therefore broadcasting in Q_2^{k-1} completes at time unit k (see Lemma 2.2). Therefore, broadcasting in Q^k completes within k time units.

Case 1.3: $u \in Q_2^{k-1} \setminus (R \cup \{w\})$.

At the first time unit u transmits the message to r_k , which needs another $k - 1$ time units to accomplish broadcasting to all members of Q_1^{k-1} . Suppose $u \in Q_1^j$, $0 \leq j \leq k - 2$. Then, at time unit i , $2 \leq i \leq k$, $i \neq j$, u transmits the message to r_{k-i+1} that needs another $k - i$ time units to accomplish broadcasting in Q^{k-i} and thus, broadcasting in Q^{k-i} completes after time unit k . At time unit j , u transmits the message to its root that needs another j time units to accomplish broadcasting in Q^j . Therefore, broadcasting in Q_2^{k-1} completes at time unit k and broadcasting in Q^k complete within k time units (see Lemma 2.2).

Now, in all three cases, after the first k time units, each root in R needs at most additional $t + 1 - k$ time units to complete broadcasting in its binomial tree (see Lemma 2.1). Thus, broadcasting in G completes within at most $k + t + 1 - k = t + 1$ time units, which is $b(u) \leq t + 1$, $\forall u \in V(G)$.

Case 2: $2^t < n < (2^k - 1)2^{t+1-k}$.

In this section we recall the definitions of d , x and p defined in case 2 in the previous section: $d = N - n$, $x = \lfloor d/2^{t+1-k} \rfloor$, $p = \lfloor \log_2(x + 1) \rfloor$, where $0 \leq x < 2^{k-1}$ and $0 \leq p < k - 1$.

Case 2.1: $u \in R'$.

At the first time unit u transmits the message to the other half of Q^k . Meaning, if $u \in Q_2^{k-1}$ then u transmits the message to its neighbor in Q_1^{k-1} , or, $u \in Q_1^{k-1}$, and it transmits the message to its neighbor in Q_2^{k-1} . That is possible, since each vertex in Q_1^{k-1} is connected to one of the vertices in Q_2^{k-1} . Thus, after the first time unit $k - 1$ more time units are needed to accomplish broadcasting in Q_1^{k-1} and Q_2^{k-1} . Therefore, broadcasting in R' is completing within at most k time units.

Case 2.2: $u \in Q_1^{k-1} \setminus R'$.

At time unit $t = 1$, u transmits to its root, that needs another $k - 1$ time units to accomplish broadcasting to all members of Q_1^{k-1} . At time unit i , $2 \leq i \leq k - p$, u transmits to r_{k-i+1} that needs another $k - i$ time units to accomplish broadcasting in Q^{k-i} . Broadcasting in Q^{k-i} completes after time unit k and therefore broadcasting in Q_2^{k-1} completes at time unit k (see Lemma 2.2). Thus, broadcasting in Q^k completes within k time units.

Case 2.3: $u \in Q_2^{k-1} \setminus (R' \cup \{w\})$.

At the first time unit u transmits the message to r_k , which needs another $k - 1$ time units to accomplish broadcasting to all members of Q_1^{k-1} . Furthermore, $u \in Q_1^j$, $p \leq j \leq k - 2$. Then, at time unit i , $2 \leq i \leq k - p$, $i \neq j$, u transmits the message to r_{k-i+1} that needs another $k - i$ time units to accomplish broadcasting in Q^{k-i} . Thus, broadcasting in Q^{k-i} completes after time unit k . At time unit j , u transmits the message to its root that needs another j time units to accomplish broadcasting in Q^j . Therefore, broadcasting in Q_2^{k-1} completes at time unit k and broadcasting in Q^k complete within k time units (see Lemma 2.2).

Now, in all three cases, each root in R' needs at most $t + 1 - k$ additional time units to complete broadcasting in its binomial tree (see Lemma 2.1). Thus, broadcasting in G' completes within at most $k + t + 1 - k = t + 1$ time units

Hence, $b(u) \leq t + 1$, $\forall u \in V(G')$.

This completes the proof of Theorem 1.2, in both cases.

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Further reading

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