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Lower bounds for the size in four families of minimum broadcast graphs

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Abstract

In this paper, we give a lower bound for the size $B(n)$ of a minimum broadcast graph of order $n = 2^k - 4$, $2^k - 6$, $2^k - 5$ or $2^k - 3$ which is shown to be accurate in the cases when $k = 5$ and $k = 6$. This result provides, together with an upper bound obtained by a construction given in Bermond et al. (1992), an estimation of the value $B(n)$ for $n = 2^k - 4$.

1. Introduction

Let G be a connected net work or graph of order n, in which some node knows a message at the start of the process. The size of G will be denoted m in the whole article.

Broadcasting is the problem of informing every other node of this message in a minimum time, starting from any node as originator. During the process, each call requires one unit of time, and at each unit of time, several calls may be executed only on a set of independent edges. A broadcast originated by a vertex x determines a spanning tree rooted at x called a *broadcast tree* or *broadcast protocol* for x. We shall use for the description of a broadcast tree, a recursive enumeration, namely if $y_1, \dots, y_k$ are the sons of the root x taken in the order of information, we shall use the recursive function: Tree of $x = x$(Tree of y_1) (Tree of y_2) $\cdots$ (Tree of y_k). For instance, $a(b(c(d))(e))(f(g)(h))(i)$ describes the tree of Fig. 1, in which b is informed at time 1, c and f at time 2 and so on.

Given a vertex x as originator, we define the *broadcast time* of x as the minimum number $b(x)$ of time units required to complete broadcasting from vertex x. Since at each unit of time, the number of informed vertices can at most double, an obvious lower bound for $b(x)$ is $\lceil \log_2 n \rceil$.

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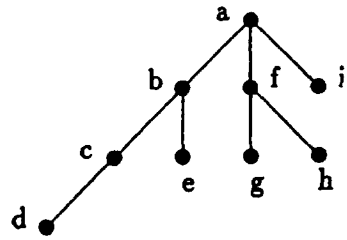

Fig. 1.

The *broadcast time* $b(G)$ of the graph is $\max\{b(x) \mid x \in G\}$. We say that G (of order n) is a *broadcast graph* if $b(G) = \lceil \log_2 n \rceil$.

The *broadcast function* $B(n)$ is the minimum number of edges in any broadcast graph of order n. A *minimum broadcast graph* is a broadcast graph of size $B(n)$. We also call it *mbg*. It is known [3] that $B(n) = (k-1)n/2$, when $n = 2^k - 2$, $k \geqslant 4$. One possible minimum broadcast graph of this order (actually the only one known at this time for every value of k) is a regular bipartite vertex-transitive graph.

In [1], Bermond et al. give a very efficient construction of broadcast graphs from known graphs of lower order, generalizing an idea coming from Harutounian and Khachatrian [4]. This construction provides upper bounds for $B(n)$ in some cases, in one of which we give a lower bound. In order to complete our result by giving an order of magnitude for $B(n)$ in this case, we now briefly describe this method.

They call *solid-1-cover* in a broadcast graph G of order n and size m, a set S of vertices which is a vertex cover for G, and has the following property: for every vertex x not in S, there exists a broadcast protocol originating at x such that at least one neighbour of x is idle at some time t between the epoch of its own information, and the end of the broadcast. That is to say, at time t, this neighbour does not communicate with any vertex in this broadcast protocol.

Suppose now that S is a solid-1-cover for G. Now they take two copies G' and G'' of the graph, and prove the following result.

Theorem. *The graph H obtained from G' and G'' by matching correspondent vertices of the solid-1-cover sets S' and S'' is a broadcast graph of order $2n$ and size $2m + |S|$.*

For instance, this construction, starting with $G = C_6$, for which any stable set of order 3 is a solid-1-cover, gives a broadcast graph H of order 12, actually a minimum broadcast graph in which the set $H \backslash (S' \cap S'')$ is also a solid-1-cover. Applying again the construction to this later graph provides a 4-regular broadcast graph of order 24. It is easy to see that this construction may be generalized, starting from the known minimum broadcast graph of order $2^{k-2} - 2$ to obtain a $(k-2)$-regular broadcast graph of order $2^k - 8$ [1]; in fact, this graph was previously discovered by Maheo and myself [5], but the theorem proves more easily that it *is* a broadcast graph.

2. Bounds for $B(2^k-4)$

Theorem. *The size $B(n)$ of a minimum broadcast graph on $n = 2^k - 4$ vertices, with $k \geqslant 4$ satisfies the inequalities*

$$\left\lceil \left(k-2+\frac{4}{2k+1}\right)\frac{n}{2} \right\rceil \leqslant B(n) \leqslant \left(k-2+\frac{1}{2}\right)\frac{n}{2}.$$

Proof. The construction already described in [1] provides a broadcast graph G_n of order $n = 2^k - 4$, thus giving the upper bound.

Let G be a minimum broadcast graph of order $2^k - 4$, $k \geqslant 4$. Since a complete broadcast tree (in time k) rooted at a vertex of degree $k-3$ has only $2^k - 7$ vertices, the minimum degree in G is at least $k-2$.

Let us first establish some properties according to the eventual degrees of the vertices of G, then translate these properties into inequalities, and finally conclude from these inequalities to the lower bound of the theorem. □

Property 1. *The complete broadcast tree with a root of degree $k-1$ and having $k-1$ sons of degree $k-2$ has $2^k - 5$ vertices. Therefore, in G no vertex of degree $\leqslant k-1$ may have all its neighbours of degree $k-2$.*

Property 2. *The complete broadcast tree with a root of degree $k-2$ and $k-3$ sons (from the second to the last) of degree $k-2$ has $2^k - 4$ vertices. Therefore, if a vertex of degree $k-2$ in G has all its neighbours but one of degree $k-2$, this last one has degree $\geqslant k$ and has another neighbour of degree $\geqslant k-1$ (the first son of the subtree rooted at it).*

Proof of Theorem (*Conclusion*). Let $V_{k-2}^{(1)}$ denote the number of vertices of this type, and in general $V_{k-2}^{(i)}$ for $i \geqslant 1$ the number of vertices of degree $k-2$ having exactly i neighbours of degree $\geqslant k-1$ (of degree $\geqslant k$ for $i = 1$). For the total number of vertices of degree p we use the notation V_p. We shall use this last notation all along this article.

Let E' denote the number of edges incident to two vertices of degree $k-2$, and E the number of edges incident to only one such vertex. We have obviously

$$E = \sum_{i=1}^{k-2} i V_{k-2}^{(i)} \quad \text{and} \quad 2E' = \sum_{i=1}^{k-2} (k-2-i) V_{k-2}^{(i)}.$$

Thus,

$$2(E + E') = \sum_{i=1}^{k-2} (k-2+i) V_{k-2}^{(i)} \geqslant kV_{k-2} - V_{k-2}^{(1)}. \tag{1}$$

On the other hand, from Property 1, the number of edges non-incident to a vertex of degree $k-2$ is at least $\frac{1}{2}V_{k-1}$. Therefore,

$$E + E' \leqslant m - \tfrac{1}{2}V_{k-1}. \tag{2}$$

Applying now Property 2, we obtain

$$V_{k-2}^{(1)} \leqslant \sum_{i \geqslant k} (i-1) V_i. \tag{3}$$

Combining (1)–(3) gives

$$kV_{k-2} + V_{k-1} - \sum_{i \geqslant k} (i-1) V_i \leqslant 2m. \tag{4}$$

We use now the basic equalities: $\sum_{i \geqslant k-2} V_i = n$, $\sum_{i \geqslant k-2} iV_i = 2m$. With these equations, (4) becomes

$$(2k-3) V_{k-2} + (k-1) V_{k-1} + n \leqslant 4m. \tag{5}$$

On the other hand, obviously,

$$2m \geqslant kn - V_{k-1} - 2V_{k-2} \tag{6}$$

At last, multiplying (5) by 2, (6) by $(2k-3)$ and adding

$$(4k+2)m \geqslant (2k^2 - 3k + 2)n + V_{k-1} \geqslant (2k^2 - 3k + 2)n$$

which gives the lower bound of the theorem. □

Note that this lower bound for $k=4$ is the value of 15 of $B(12)$. We are going to prove the same fact for $k=5$ and $k=6$.

Proposition. $B(28) = 48$.

This result was independently found by Chen Xiebin [2].

Proof. The theorem gives for $k = 5$, $B(28) \geqslant 48$. We give the construction of a broadcast graph, having 48 edges, therefore a minimum broadcast graph on 28 vertices. Let us take a cycle on 22 vertices labelled by Z_{22} together with six vertices labelled by letters from a to f and adding the edges joining:

— a to c, e, 0, 6, 16
— b to 1, 7, 13, 19
— c to 2, 8, 12, 18
— d to 3, 9, 15, 21
— e to 4, 10, 14, 20
— f to b, d, 5, 11, 17.

There are two automorphisms of this graph, namely,

- $\forall i \in Z_{22}$, $i \to 11 - i$, $a \leftrightarrow f$, $b \leftrightarrow e$, $c \leftrightarrow d$,
- $\forall i \in Z_{22}$, $i \to 22 - i$, $b \leftrightarrow d$, $c \leftrightarrow e$.

Therefore, it suffices to give a broadcast tree for the vertices 0 to 5, in which the first son (which is the first informed neighbour) is the neighbour labelled by a letter, since such a broadcast tree may be used as well for this first son.

One broadcast tree is described below with the help of the already described notation.

- $0(a(c(12(13(14))(11))(18(19))(8))(e(10(9))(4))(16(17))(6))(1(b(f(5))(7))(2(3)))$ $(21(d(15))(20)$.

The other broadcast trees are given in [6]. □

Proposition. $B(60) = 130$.

Proof. The lower bound of the theorem for $k = 6$ gives $B(60) \geqslant \lceil 120 + 120/13 \rceil = 130$. Let us now consider the following graph G: take a cycle C of order 50 with vertices indexed by $(i, 0)$, $i \in Z_{50}$, and a set S of 10 additional vertices indexed by $(i, 1)$, $i \in Z_{10}$. Add edges from $(i, 1)$ to $(i + 10k, 0)$, $0 \leqslant k \leqslant 4$ between the two sets, chords from $(2i, 0)$ to $(2i + 7, 0)$ in the cycle C, and edges from $(i, 1)$ to $(i + 5, 1)$ in S. The vertices of C are now of degree 4 and those of S are of degree 6, so G is of size 130. According to the automorphisms of G all vertices of C are equivalent and so are all vertices of S. It suffices to give a broadcast tree for a vertex in C with first edge joining it to its neighbour in S in order to prove that G is a broadcast graph, hence a minimum broadcast graph. This is shown in Fig. 2. □

3. Lower bound for $B(2^k - 6)$

Theorem. *The size $B(n)$ of a minimum broadcast graph on $n = 2^k - 6$ vertices, with $k \geqslant 4$ satisfies the inequality*

$$\left\lceil \left(k - 2 + \frac{1}{k}\right)\frac{n}{2} \right\rceil \leqslant B(n).$$

Proof. Since the complete broadcast tree in time k rooted at a vertex of degree $k - 3$ has $2^k - 7$ vertices, the minimum degree in a broadcast tree with $2^k - 6$ vertices is at least $k - 2$.

On the other hand, the complete broadcast tree in time k with root and every son of the root of degree $k - 2$ also has $2^k - 7$ vertices. Therefore, in a broadcast tree G with $n = 2^k - 6$ vertices every vertex of degree $k - 2$ must have at least one neighbour of degree $\geqslant k - 1$. So, if V_{k-2} denotes the number of vertices of degree $k - 2$ in G, this property implies the inequality on the size: $2m \geqslant 2V_{k-2} + (k - 3)V_{k-2} = (k - 1)V_{k-2}$. Since we have obviously $2m \geqslant (k - 1)n - V_{k-2}$, we obtain by combining these two inequalities $2m\,k \geqslant (k - 1)^2 n$ which yields the lower bound of the theorem. □

Note that this lower bound is for $k = 4$ the value 12 of $B(10)$. We prove that it is also the case for $k = 5$ and $k = 6$.

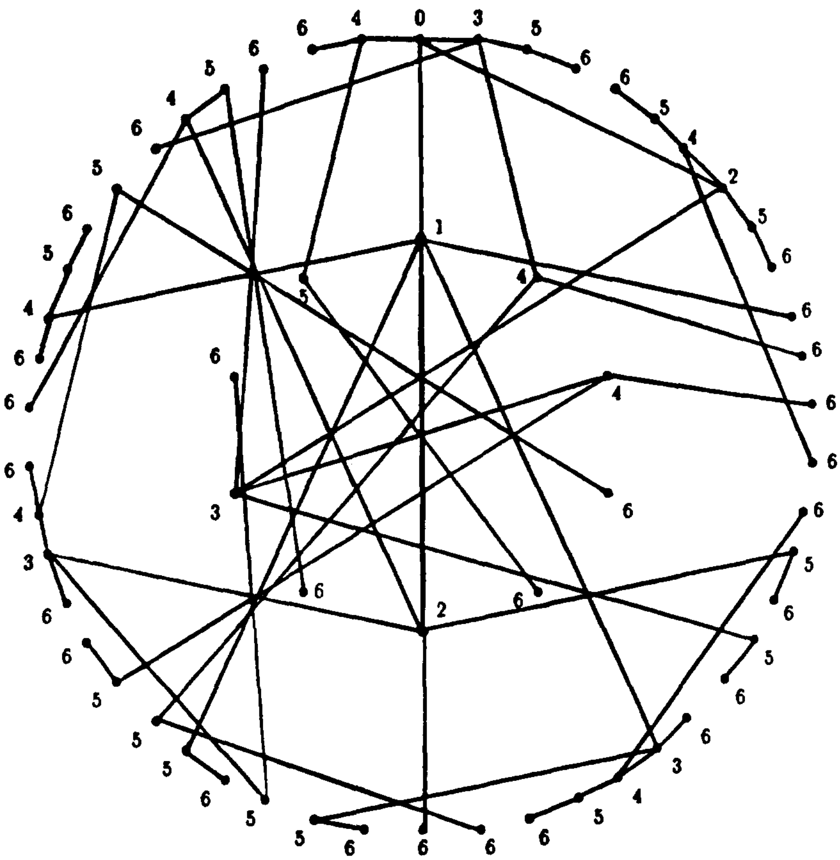

Fig. 2. Broadcast tree of G.

Proposition. $B(26) = 42$.

(This result was also independently found by Chen [2].)

Proof. The theorem gives, for $k = 5$, $B(26) \geqslant 42$ and the following graph G is a broadcast graph on 26 vertices and of size 42: take a cycle C_{20} with vertices labeled by indices $i \in Z_{20}$, and six other vertices labeled by letters a to f, and add edges joining:

— a to 0, 5, 10, 15
— b to e, 6, 11, 16
— c to f, 1, 12, 17
— d to 2, 7, 13, 18
— e to 3, 8, 19
— f to 4, 9, 14.

We give a broadcast tree for the couple of initiators $(0, a)$:

- $0(a(10(9(8(7))(f))(11(12)))(15(14(13))(16))(5(6)))(1(2(3(4))(d))(c(17)))$
  $(19(e(b))(18))$.

Due to the symmetry of the graph, we only need give a broadcast tree for half the vertices. We describe them in [6]. □

Proposition. $B(58) = 121$.

Proof. The theorem gives for $k = 6$: $B(58) \geqslant 121$ and the following graph G is a broadcast graph on 58 vertices and of size 121: take a cycle C_{48} with vertices labeled by indices $i \in Z_{48}$, with chords $(2p, 2p + 7)$, and ten other vertices labeled by letters a to j, and add edges joining:

— a to 0, 10, 19, 29, 38
— b to 1, 11, 20, 30, 39
— c to h, 2, 12, 21, 40
— d to 3, 13, 22, 31, 41
— e to 4, 14, 23, 32, 42
— f to 5, 15, 24, 33, 43
— g to 6, 16, 25, 34, 44
— h to 7, 26, 35, 45
— i to 8, 17, 27, 36, 46
— j to 9, 18, 28, 37, 47

This graph has an internal automorphism $p \to 47 - p$, $a \leftrightarrow j$ and so on, thus we only need give broadcast trees for half the vertices. We give the first of these trees, the others may be found in [6].

- 0(7(h(26(33(32(31))(f))(25(24))(27))(c(40(41))(21))(35(34))(45))(8(15(16(23))(14)) (i(36))(9))(6(5(4))(g)))(a(19(12(13(d))(11))(18(17))(20))(38(37(30))(39))(29(22))(10)) (1(42(43(44))(e))(2(3))(b))(47(j(28))(46)). □

4. Some new upper bounds for $B(2n)$

The previous results, together with the construction given in [1] provides the following corollary.

Corollary. $B(56) \leqslant 111$, $B(120) \leqslant 290$ *and* $B(52) \leqslant 99$.

Proof. Let us consider the broadcast graph of order 28 given in Section 2. The vertices labeled $2k$ together with the vertices a, b, d, f of this graph, form a solid-1-cover since it covers every edge and in every given protocol of broadcasting, the first informed neighbour of degree 3 is idle at time unit 5. Therefore, one obtains $B(56) \leqslant 111$.

In the same way the vertices $(2k, 0)$ and $(2k + 1, 1)$ cover all the edges in the mbg of order 60 of the same section, and the first informed neighbour of degree 4 in both protocols is idle at time unit 6, therefore giving a solid-1-cover of order 30, and the upper bound of the corollary.

Finally, it is easy to verify that in the mbg of order 26 in Section 3, the set $\{0, 1, 3, 5, 7, 8, 10, 12, 13, 15, 17, 19, b, d, f\}$ is a solid-1-cover. So we obtain the third upper bound of the corollary. □

The bounds on $B(56)$ and $B(120)$ are of peculiar interest, since these orders of graphs are of the form $2^k - 8$, a family for which is known a general construction of regular broadcast graph, construction recalled in [1]. This result shows that the regular model is not the best possible, contrarily to the cases 2^k and $2^k - 2$ in which the regular models are also mbg.

5. Lower bounds for $B(2^k - 3)$

Theorem. *The size $B(n)$ of a minimum broadcast graph on $n = 2^k - 3$ vertices, with $k \geqslant 4$ satisfies the inequality*

$$B(n) \geqslant \left\lceil \left(k - 2 + \frac{3k-5}{k^2 - k - 1}\right)\frac{n}{2} \right\rceil.$$

Proof. The minimum value for the degree of a vertex is $k - 2$ (see Section 2) and the complete broadcast tree with a root of degree $k - 2$ is of order $2^k - 3$. Therefore, a vertex of degree $k - 2$ must have a neighbour of degree at least k and another of degree at least $k - 1$. On the other hand, a root of degree at least k with all its sons of maximum degree $k - 2$ should give a broadcast tree of order no more than $2^k - 4$.

We may write for the number of vertices of degree $k - 2$, taking account of their neighbours of degree $\geqslant k$, the inequality

$$V_{k-2} \leqslant \sum_{i \geqslant k} (i-1) V_i = (2m - n) - (k-2) V_{k-1} - (k-3) V_{k-2}$$

and if we also take into account their neighbours of degree $k - 1$, the other one:

$$2V_{k-2} \leqslant \sum_{i \geqslant k-1} (i-1) V_i = (2m - n) - (k-3) V_{k-2}.$$

We can rewrite these inequalities in the following way:

$$n + (k-1) V_{k-2} \leqslant 2m, \qquad n + (k-2)(V_{k-2} + V_{k-1}) \leqslant 2m$$

and use the trivial inequality: $kn - V_{k-1} - 2V_{k-2} \leqslant 2m$ in order to obtain by combining them (with respective coefficients $k - 2$, $k - 1$ and $(k-1)(k-2)$):

$$n(k^3 - 3k^2 + 4k - 3) \leqslant 2m(k^2 - k - 1)$$

which gives the lower bound of the theorem. □

Note that for $k = 4$ this bound is the value 18 of $B(13)$. We prove that it is also the case for $k = 5$ and $k = 6$.

Proposition. $B(29) = 52$.

Proof. The theorem gives, for $k = 5$, $B(29) \geqslant 52$. Let us then describe a broadcast graph of order 29 and size 52.

Take a cycle C on 27 vertices, adding the chords joining each vertex of index $3k$ to the vertex of index $3(k+1)$ (these chords form a cycle C' on 9 vertices). Then identify, on the one hand, the two vertices following the vertex 0, and on the other hand, the two preceding it on the cycle C. We obtain a cyclic $\tilde{C}$ on 25 vertices with chords, which we label by $\boldsymbol{Z}/25\boldsymbol{Z}$, the vertex of old label 0 keeping this label. Now add a set of four additional vertices a, b, c, d, successively joining them to the 16 vertices of $\tilde{C} \setminus \tilde{C}'$, and complete by the two edges (ac) and (bd). For instance, the neighbours of the vertex a in this graph are $(c, 1, 7, 13, 19)$ and those of d are $(b, 6, 12, 18, 24)$. The homomorphism $(i \leftrightarrow 25 - i,\ a \leftrightarrow d,\ b \leftrightarrow c)$ is an involution of the graph. We need only broadcast protocols for the vertices with indices between 0 and 12, provided that a and b be the first sons in at least two of these protocols. We give here one such protocol, the others may be found in [6]:

- 0(5(11(17(16(15))(18))(10(9))(12))(6(7(8)))(4(3))(20(21(22(23))(b))(14(13))(19))(1(a(c))(2))(24(d)). □

Proposition. $B(61) = 136$.

Proof. The lower bound of the theorem is, for $k = 6$, exactly 136. We need a minimum broadcast graph of this size.

Take a cycle C on 55 vertices, with the chords joining each vertex labeled $5k + 1$ to the vertex $5k + 19$, and each vertex $5k$ to the vertices $5k \pm 8$, with the exception that $+10$ must be joined to -2 instead of $+2$, and symmetrically -10 to $+2$ instead of -2. Now identify $+1$ with $+2$ on one side, and -1 with -2 on the other side, call this new cycle (on 53 vertices) $\tilde{C}$ and relabel it by $Z/53Z$ in such a way that the vertex having label 0 in C keeps this label in $\tilde{C}$. Call $\tilde{S}$ the set of vertices which had labels $5k$ in C.

Add a set of 8 additional vertices labeled from a to h, and successively join them to the vertices of $\tilde{C} \setminus (\tilde{S} \cup \{\pm 1\})$. For instance, a is joined to the vertices (2, 12, 22, 32, 42) of $\tilde{C}$. Complete the graph with the following edges incident to $\tilde{S}$: $(0, \pm 24)$ $(4, e)$ $(9, a)$ $(14, f)$ $(19, b)$ $(34, g)$ $(39, c)$ $(44, h)$ $(49, d)$. We obtain a graph of required order and size. Since it possesses a unique involution $(i \mapsto -i,\ a \leftrightarrow h,\ b \leftrightarrow g,\ c \leftrightarrow f,\ d \leftrightarrow e)$ we must give broadcast trees having as root or first son the vertices of the set $\{[0..26]\} \cup \{a, b, c, d\}$. One of these trees is given below, the others may be found in [6]:

- 1(0(24(32(a(12(13))(42))(31(30))(33))(23(22(21))(5))(16(15))(25))(7(e(47(39))(4))(6(14))(8))(52(51(50))(9))(29(28))(46))(44(36(35(34(g))(c))(37(38))(d))(h(11(10))(41))(43(b))(45))(18(19(27(26))(20))(f(48))(17))(2(3(40))(49)). □

6. Lower bounds for $B(2^k-5)$

Theorem. *The size $B(n)$ of a minimum broadcast graph on $n = 2^k - 5$ vertices, with $k \geqslant 4$ satisfies the inequality*

$$\left\lceil \left(k-2+\frac{2}{2k-1}\right)\frac{n}{2}\right\rceil \leqslant B(n).$$

Proof. The minimum value for the degree of any vertex is obviously $(k-2)$. A vertex of degree $(k-2)$ must have one neighbour of degree at least $(k-1)$. The complete broadcast tree with a root of degree $(k-2)$, the first son of degree $(k-1)$, and the second of degree $(k-2)$ is of order exactly 2^k-5. Therefore, if a vertex of degree $(k-2)$ has all its neighbours of the same degree, except one of degree $(k-1)$, this one has at most $(k-2)$ neighbours of degree $(k-2)$. Let V' design the number of vertices of degree $(k-2)$ which have the previous neighbourhood, V'' the number of vertices of degree $(k-2)$ having all neighbours of degree $(k-2)$ except one of degree at least k, and V''' the number of the remaining vertices of degree $(k-2)$.

We may write the following relations:

$$V' + V'' + V''' = V_{k-2},$$
$$V' \leqslant (k-2)V_{k-1}, \quad V'' \leqslant \sum_{j\geqslant k} jV_j, \quad V' + V'' + 2V''' \leqslant \sum_{j\geqslant k-1} jV_j,$$

and so,

$$2V_{k-2} \leqslant (2k-3)V_{k-1} + 2\sum_{j\geqslant k} jV_j = 4m - V_{k-1} - (2k-4)V_{k-2}$$

or

$$(2k-2)V_{k-2} + V_{k-1} \leqslant 4m.$$

On the other hand, we have the trivial inequalities:

$$kn - 2V_{k-2} - V_{k-1} \leqslant 2m, \qquad (k-1)n - V_{k-2} \leqslant 2m.$$

Combining the three last inequalities give

$$(2k^2 - 5k + 4)n \leqslant (4k-2)m$$

and we obtain the bound of the theorem. □

Note that for $k = 4$ this bound is the value 13 of $B(11)$. We prove the same fact for $k = 5$ and $k = 6$.

Proposition. $B(27) = 44$.

(This result was also independently found by Chen [2]).

Proof. The theorem gives, for $k = 5$, $B(27) \geqslant 44$. It remains to construct a minimum broadcast graph of order 27 and size 44.

Take a cycle on 20 vertices labeled on $Z/20Z$ and a set of 7 additional vertices labeled from a to g. Join

— a to c, f, 0, 8
— b to e, 1, 7, 14
— c to 3, 11, 15
— d to g, 2, 9, 16
— e to 4, 10, 18
— f to 5, 13, 17
— g to 6, 12, 19.

Unfortunately, there is no symmetry at all and we need 20 broadcast trees which are given in [6], the first being

- $0(a(c(11(10(9))(12))(15(16))(3))(f(5(6))(13))(8(7)))(19(18(e(4))(17))(g(d)))$ $(1(b(14))(2))$. □

Proposition. $B(59) = 124$.

Proof. The theorem gives, for $k = 6$, $B(59) \geqslant 124$. Let us give a construction for a minimum broadcast graph of order 59 and size 124.

Take a cycle on 48 vertices labeled by $\boldsymbol{Z}/48\boldsymbol{Z}$, with chords $(2i, 2i + 9)$. Replace the two edges (0, 1) and (0, 47) by a single edge (1, 47). Take apart the vertex 0 and add 11 additional vertices labeled from a to k to form with it a set of 12 special vertices S. Join in S the jth element to the $(j + 6)$th, and join the jth element of S to the vertices $(j + 12i)$ of the cycle (recall that 0 is also joined to the vertex 9 of the cycle). This graph is now of order 59 and size 124 but has no symmetry at all. We must give a broadcast tree for every vertex remaining in the cycle. We give the first one below, the others may be found in [6]:

- $1(a(g(19(20(21(22))(h))(10(9))(18))(31(30(f))(32))(7(6))(43))(25(24(33(i))(0))$ $(16(15))(26))(37(28(27))(36))(13(12))))(47(k(e(17(8))(29))(35(34))(23))(46(45(44))$ $(j))(38(39))(2(3(4(5))(c))(b(14))(11))(40(41(42))(d))$. □

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