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Minimum strictly fundamental cycle bases of planar graphs are hard to find

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ABSTRACT

In this paper, we consider the problem of finding a spanning tree in a graph that minimizes the sum over the lengths of the cycles induced by the chords of the tree. We show the $\mathcal{NP}$-completeness of this problem for planar graphs. The proof will be by reduction of a planar version of the Exact Cover By 3-Sets Problem. Finding such a minimum strictly fundamental cycle basis has various practical applications, e.g. in designing optimal periodic timetables in public transport.

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1. Introduction

For solving algorithmic problems on a graph, the characterization of its cyclic structure plays an important role. Often, cycle bases belong to the input of algorithms. In these cases, the running time of such an algorithm can depend on the size of the given cycle basis. In the Minimum Strictly Fundamental Cycle Basis Problem (Msfcb for short) one seeks for a spanning tree T that minimizes the sum over the lengths of the cycles induced by the edges which are not in T. Strictly fundamental cycle bases (SFCB) form a natural subclass of cycle bases, thus, we are interested in small strictly fundamental cycle bases.

In general, cycle bases have numerous applications, e.g. in the fields of periodic timetable optimization [14], coordination of traffic signals [17], electric engineering [2], and chemistry [8]. In many of these applications, the underlying graph is planar or almost planar. This motivated the interest in the Msfcb restricted to planar graphs.

The first polynomial time algorithm for finding a general minimum, not necessarily strictly fundamental cycle basis on an arbitrary undirected graph was given by Horton in [9]. Its running time of $\mathcal{O}(m^3n)$ was improved to $\mathcal{O}(m^2n + mn^2 \log n)$ in [13]. The currently best running time of $\mathcal{O}(m^2n/\log n)$ is due to Amaldi et al. [1]. More recently, better running times could be achieved for approximation algorithms [12].

Especially for the above mentioned periodic timetable optimization, integral cycle bases turned out to be of special importance, see e.g. [14,17]. The complexity of finding a minimum integral cycle basis is still open, but there are several results on subclasses of integral cycle bases. The $\mathcal{APX}$-hardness of finding a minimum weakly fundamental cycle basis was shown in [16]. A similar result for the Msfcb was obtained in [7]. On the class of integral cycle bases itself, there are heuristics for computing integral cycle bases of low weight [10]. Both, weakly fundamental and integral cycle bases are superclasses of strictly fundamental cycle bases.

Deo et al. established the $\mathcal{NP}$-completeness of the Msfcb for general graphs in [4]. In this paper, we prove the $\mathcal{NP}$-completeness for the Msfcb on planar graphs by a reduction of a planar version of Exact Cover By 3-Sets Problem. The complexity status of the Msfcb restricted to planar graphs was pointed out to be open in several publications, e.g. in [11,14,17].

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Note that the problem can be solved in polynomial time for the case of weakly fundamental cycle bases on planar graphs, see [14]. A preliminary version of the proof in this paper was presented in [15].

In Section 2 we give definitions and notations and define the problems P-MSFCB and P-X3C, the planar versions of MSFCB and EXACT COVER BY 3-SETS, respectively. In Section 3 we reduce P-X3C to P-MSFCB. We introduce several gadgets, illustrate their properties, and explain how the partial results on the individual gadgets interact to achieve the main result, the $\mathcal{NP}$-completeness of P-MSFCB. The interaction of the single statements is additionally summarized with a graphical overview at the end of this section.

2. Preliminaries

This section is dedicated to the elementary definitions on graph theory, where we additionally refer to standard textbooks like [5]. In the second part, we define the planar versions of MSFCB and X3C.

2.1. Basic notations

Throughout the paper, we consider only simple undirected graphs $G = (V, E)$ with finite node set $V(G) = V$ and finite edge set $E(G) = E$. The *degree* of a node $v \in V$ is denoted by $\deg(v)$. A *path* P of length ℓ in a graph is a sequence $P = (v_0, v_1, \dots, v_\ell)$ of pairwise disjoint nodes with $v_{i-1}v_i \in E$ for $1 \le i \le \ell$. The length of a shortest path between two nodes u and v in G is called the *distance* $\operatorname{dist}_G(u, v)$. A path from node u to node v is referred to as u–v-path.

A *circuit* C in G is a non-empty connected subgraph of G with $\deg(v) = 2$ for all $v \in V(C)$. We define $|C| := |V(C)| = |E(C)|$ as the *length* of a circuit C. For a spanning tree $T = (V, E(T))$ of G and an edge $e \in E \setminus E(T)$ define the *fundamental circuit* as the unique circuit $C_T(e)$ in $(V, E(T) \cup \{e\})$.

The *cycle space* $\mathcal{C}(G)$ of a graph $G = (V, E)$ is the vector subspace of $\mathrm{GF}(2)^E$ that is generated by the incidence vectors of the circuits in G. A *cycle basis* B of G is a set of $\nu = m - n + 1$ circuits whose incidence vectors form a basis of $\mathcal{C}(G)$. The *size* $\Phi(B)$ of a cycle basis B is defined as $\Phi(B) := \sum_{C \in B} |C|$. A cycle basis B of $G = (V, E)$ is designated *strictly fundamental* iff there is a spanning tree $T = (V, E(T))$ with $B = \{C_T(e) \mid e \in E \setminus E(T)\}$. The edge e is called the *private edge* of the circuit $C_T(e)$, because it is not contained in any other circuit in B. In this case, we denote T a *fundamental spanning tree* and write $\Phi(T)$ for the size of the induced cycle basis. The edges in $E \setminus E(T)$ are called *chords*.

A graph is *planar* if it can be drawn in the plane such that its edges do only intersect at their end nodes. A *plane graph* is a graph that is embedded into the plane, where the embedding of a planar graph is not unique, in general. A plane graph divides the plane into *faces*. For a planar graph, a plane embedding can be computed in linear time [3]. Thus, we define the problems as usual with planar graphs, while we consider plane graphs in the reduction. For stringent definitions of these terms we refer again to [5].

2.2. The problems

The MSFCB restricted to planar graphs reads as follows.

P-MSFCB	
Instance:	Planar graph $G_S = (V_S, E_S)$, positive integer k.
Question:	Does G_S have an SFCB B with $\Phi(B) \le k$, resp. does G_S have a spanning tree T with $\sum_{e \in E \setminus E(T)} C_T(e) \le k$?

Lemma 1. *If a fundamental spanning tree is given then the size of the corresponding SFCB can be computed in quadratic time. Hence, especially the* P-MSFCB *is in* $\mathcal{NP}$.

Proof. Given a fundamental spanning tree, the length of a fundamental circuit C can be determined by performing a depth first search on the tree edges from one end node of the private edge of C to the other one. This can be done in time $\mathcal{O}(n)$. Since there are $\nu \in \mathcal{O}(m)$ chords, it is possible in time $\mathcal{O}(nm)$ to decide whether the size of a given strictly fundamental cycle basis is at most k. □

We will reduce the planar version of the EXACT COVER BY 3-SETS Problem to P-MSFCB. The $\mathcal{NP}$-completeness of the former problem has been established in [6].

P-X3C	
Instance:	Integers n, m, set $X = \{1, \dots, n\}$, subset $\mathcal{S} \subset \mathcal{P}(X)$ with $|\mathcal{S}| = m$ and $|S| = 3$ for all $S \in \mathcal{S}$, where the graph $G_X = (V_X, E_X)$ with $V_X = X \cup \mathcal{S}$ and $E_X = \bigcup_{S \in \mathcal{S}} \{\{S, x\} \mid x \in S\}$ is planar.
Question:	Is there a subset $\mathcal{S}' \subseteq \mathcal{S}$ such that for all $x \in X$ there is exactly one $S \in \mathcal{S}'$ with $x \in S$?

We remark that the problem can only have a solution if $3 \le n \le 3m$ and if n is a multiple of 3.

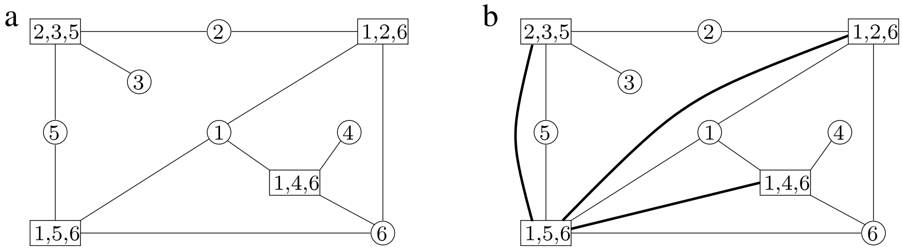


Fig. 1. (a) Embedded instance of the P-X3C Problem. (b) The instance after the insertion of $m-1$ spinal edges of the spinal tree.

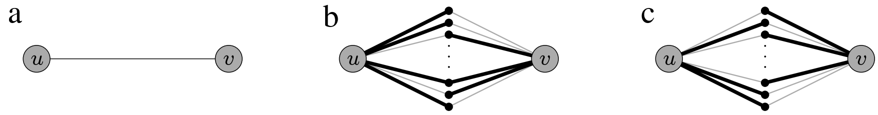


Fig. 2. (a) An edge uv, (b) a regular and (c) an irregular thick bunch. Tree edges are drawn as thick lines.

3. $\mathcal{NP}$-completeness of Msfcb on planar graphs

Given an instance of P-X3C with a plane graph G_X, we reduce this graph to a plane graph G_S and compute a number k such that there is a fundamental spanning tree T on G_S with $\Phi(T) \leq k$ iff there is a subset $\mathscr{S}' \subseteq \mathscr{S}$ as posed in the definition of P-X3C. The nodes of G_X in $\mathscr{S}$ are called *set nodes*, while the nodes in X are referred to as *element nodes*.

The first step of the reduction is to construct a kind of a *spinal tree*, which connects the set nodes to each other. Therefore, connect the m set nodes in $\mathscr{S}$ with a set E_{SP} of $m-1$ *spinal edges* in accordance with the following rules.

R 1. There is a unique path consisting only of spinal edges between each pair of set nodes.
R 2. Between each pair of element nodes around a set node of the graph G_S, at most one spinal edge can be connected with this set node.

For the sake of accuracy, we have to formulate a third rule after the definition of the *gear*. Note that this construction can always be realized while preserving planarity because the graph G_X is bipartite and simple and thus, the boundary of each face of G_X contains at least two set nodes. Fig. 1 illustrates this construction with an example with $X = \{1, 2, \ldots, 6\}$ and $\mathscr{S} = \{\{2, 3, 5\}, \{1, 2, 6\}, \{1, 4, 6\}, \{1, 5, 6\}\}$. The element nodes are depicted as numbered circles, while the rectangles represent the set nodes.

In the next paragraphs, we introduce several gadgets and describe their properties, in particular, we state what form a spanning tree on these gadgets can have. We start with *thick bunches* and *long bunches* which both will replace edges. The gadgets ensure that a spanning tree induces a small SFCB if the tree path between the end nodes of the replaced edge runs through the replacing gadget. Another gadget is the *switcher* which models whether an element $x \in X$ is an element in a 3-set $S \in \mathscr{S}$ or not. Three of the switchers and six of the long bunches are assembled together to constitute a *gear* which represents a 3-set from $\mathscr{S}$. A gear gets three special *spinal nodes* on which the spinal tree can be attached.

Moreover, the integer k for the P-Msfcb instance is computed. Directly after the description in each paragraph, we compute the part of the parameter k on the assumption that the tree is regular on the component. Our notion of *regularity* of a spanning tree is declared later and must not be confused with the property of a graph in which all nodes have the same degree. The corresponding part of k, where the tree is irregular, is computed below in several extra paragraphs, after the computation of the integer k itself.

Thick bunches. Each of the $3m$ edges in E_X and each of the $m-1$ spinal edges is substituted by a *thick bunch*. A thick bunch arises out of an edge uv by replacing it with m^7 paths of length 2. The m^7 new nodes are denoted *center nodes*. There are essentially two possibilities of a spanning tree restricted to a thick bunch. When one center node has degree two in the spanning tree, the thick bunch is called *regular*. Otherwise, if all center nodes have degree one, the thick bunch is termed *irregular*. In what follows, we generalize regularity of several components to spanning subgraphs. See Fig. 2 for a regular and an irregular thick bunch.

If all thick bunches are regular, the chords contained in the thick bunches induce circuits with total length of

$$\Phi^r_{\text{tb}} = (\underbrace{4m-1}_{\substack{\text{number of}\\ \text{thick bunches}}}) \cdot (\underbrace{m^7-1}_{\substack{\text{circuits per}\\ \text{thick bunch}}}) \cdot \underbrace{4}_{\substack{\text{length of}\\ \text{one circuit}}} . \quad (1)$$

Long bunches. *Long bunches* are parts of gears, which are described later. Gears will replace the set nodes in $\mathscr{S}$, and each gear will contain six long bunches. Hence, the graph G_S will contain $6m$ long bunches. A long bunch is constructed by subdividing an edge uv with two additional nodes and by replacing each of the three originated edges with a *slice* of m^5 paths with lengths $6m^2$. If all edges in a path are in the tree, the path is called a *complete path*. Otherwise, i.e. one edge of the path is

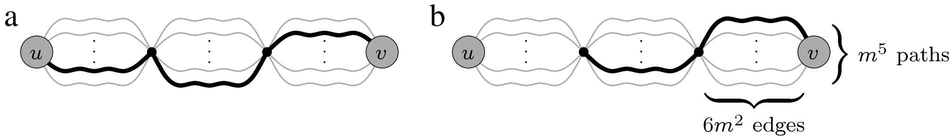


Fig. 3. (a) A regular resp. (b) an irregular long bunch. Every edge of a fat path is contained in the tree while exactly one edge of a gray path is missing in the tree.

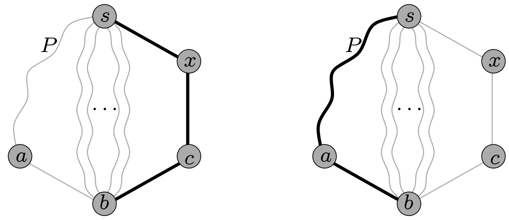


Fig. 4. Active (left) and inactive switcher (right).

missing in the tree, it is referred to as *incomplete path*. The terms *regular* and *irregular* are similarly used as in the context of thick bunches. See also Fig. 3.

Again, we assume that all long bunches are regular for the moment. Then, the chords contained in the long bunches induce circuits of length

$$\Phi_{\text{lb}}^{r} = \underbrace{6m}_{\substack{\text{number of}\\\text{long bunches}}} \cdot \underbrace{3}_{\substack{\text{slices per}\\\text{long bunch}}} \cdot \underbrace{(m^5-1)}_{\substack{\text{circuits}\\\text{per slice}}} \cdot \underbrace{2\cdot 6m^2}_{\substack{\text{length of}\\\text{one circuit}}}. \tag{2}$$

Switchers. These components help to control whether the gears which model the 3-sets belong to $\mathscr{S}'$ or $\mathscr{S} \setminus \mathscr{S}'$. See Fig. 4 for an illustration.

The a–s-path P has a length of $5m$ edges. In the interior of a switcher are $130m^3$ *interior b–s-paths*, each of length $16m^2$. For the interior paths and the path P, the terms *complete* and *incomplete path* are used equivalently to the case of long bunches. Contrary to thick or long bunches, there are basically two types of regular switchers. A switcher is *regular* if exactly one of the conditions below is fulfilled.

- The edges sx, cx and bc are contained in the corresponding tree. In this case, the switcher is referred to as *active*.
- The spanning tree contains the complete path P and the edge ab. These regular switchers are called *inactive*.

All other switchers are *irregular*, i.e. one of the $130m^3$ interior b–s-paths is a complete path or the unique b–s-path in the tree uses edges outside of the switcher. Note that an active switcher is allowed to contain the complete path P or the edge ab. Similarly, an inactive switcher can contain one or two of the edges sx, cx or bc. Anyway, we will see that all these cases are not possible if the switcher is assembled in a gear whose long bunches are all regular.

The estimation of the combined circuit length induced by chords in regular switchers is caught up on in the next paragraph, because the circuits contain also edges outside of the regular switcher.

Gears. A *gear* G^S substitutes a set node $S \in \mathscr{S}$. One gear consists of three switchers and six long bunches which are arranged as shown in Fig. 5. A gear is called *regular* if all six long bunches and all three switchers are regular.

The *X-nodes* x_1, x_2 and x_3 in the gear G^S are connected via thick bunches to the three element nodes which are in the 3-set S. The *spinal nodes* s_1, s_2 and s_3 can be used to connect the gears to each other with thick bunches which replace the spinal edges of the spinal tree introduced in the first step of the reduction. Lemma 2 will guarantee that there are spinal nodes for each spinal edge. Clearly, not all spinal nodes will be used therefore, because there are $3m$ spinal nodes but only $m-1$ spinal edges which connect them. Fig. 6 illustrates how all components are assembled with a complete example. With this figure in mind, we are able to formulate the third rule for the construction of the spinal tree.

R 3. If two switchers S_1 and S_2 of two different gears G_1 and G_2 are connected to the same element node, then G_1 and G_2 must not be connected with a spinal edge between S_1 and S_2.

Lemma 2. *Let G_R be an embedding of the graph G_X as in the description of* P-X3C *where the set nodes in $\mathscr{S}$ are replaced by gears. G_R shall not contain the spinal tree, yet. Then, for an arbitrary gear G^S, the boundary of every adjacent face contains at least one of the gear's spinal nodes.*

Proof. When looking at a single gear G^S, this is due to the fact that X-nodes and spinal nodes of one gear G^S alternate in the boundary of the unbounded face of G^S. Therefore, take a look at Fig. 5, which shows a single gear. □

It follows that the described reduction is actually possible while preserving planarity.

Proposition 3. *The constructed graph is planar. Moreover, the reduction can be done in polynomial time.* □

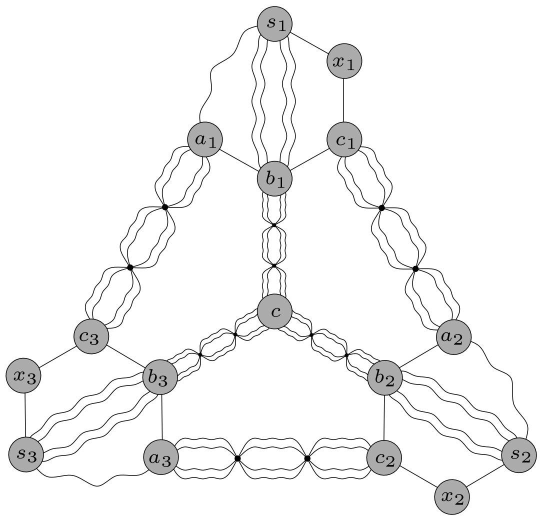


Fig. 5. Arrangement of the three switchers and the six long bunches in a gear.

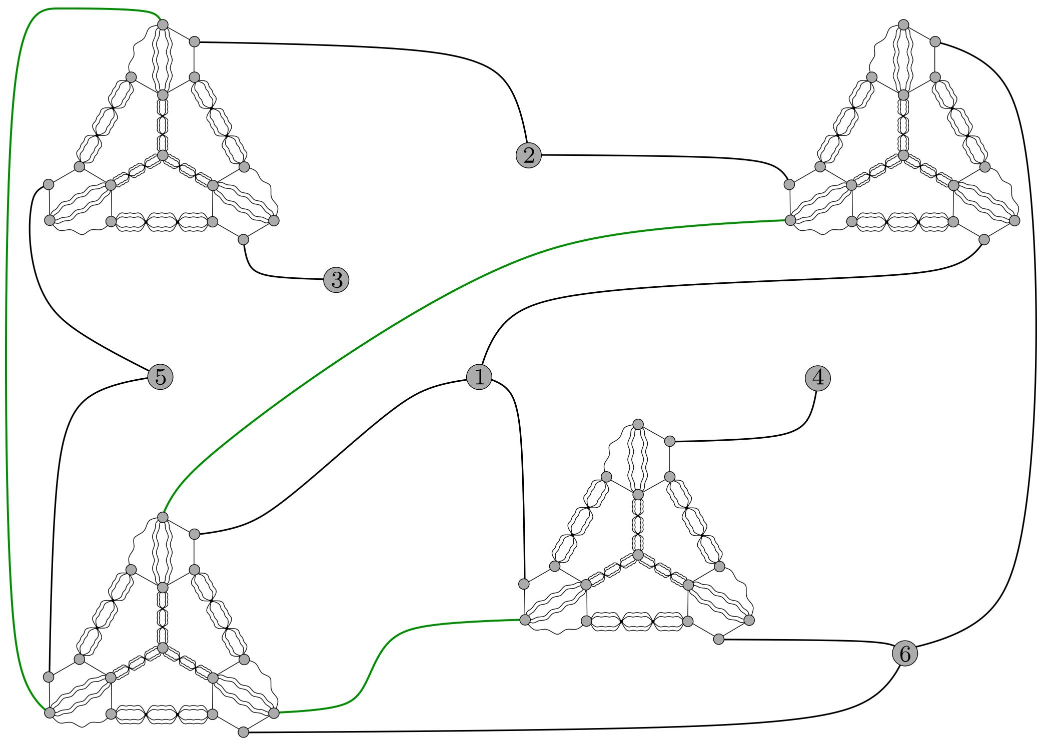


Fig. 6. Complete example for the instance $X = \{1, 2, 3, 4, 5, 6\}$, $\mathscr{S} = \{\{2, 3, 5\}, \{1, 2, 6\}, \{1, 4, 6\}, \{1, 5, 6\}\}$, the same as in Fig. 1. Thick bunches which replaced the edges in E_X are drawn as thick black lines. The fat green lines represent thick bunches which replaced the spinal edges. (For interpretation of the references to colour in this article, the reader is referred to the web version of this article.)

The following observation eliminates further cases in the estimation of the circuit lengths induced by chords in regular switchers.

Lemma 4. *In a regular gear, the switchers have the same orientation, i.e. all three switchers are either active or all three are inactive. With Figs. 4 and 5 in mind, if all three switchers are active then edge ab and one edge of P is missing in the tree. On the other hand, when the switchers are inactive edge bc and at least one of cx and sx are chords.*

Proof. If there was an active *and* an inactive switcher in a regular gear, T would contain a circuit as all long bunches are regular. This is a contradiction since T is a tree. If all three switchers are active then both, a tree edge *ab* or a complete path P would induce a circuit. If all three switchers are inactive, *bc* has to be a chord to avoid a circuit. Finally, if *cx and sx* are in the tree, they would induce a circuit similarly to the path P in the case of three active switchers. □

As we will see later in Corollary 10, actually all three edges have to be chords. If all three switchers of a regular gear are (in)active, the gear itself is also denoted *(in)active*.

Lemma 5. *If all thick bunches, all long bunches, and all switchers are regular then each element node x is connected to at most one active gear. This implies that at most n switchers can be active.*

Proof. Otherwise, there would be an element node $x \in X$ which is connected to two active switchers. The corresponding gears are also connected to each other via the replaced spinal edges, and hence, T would contain a circuit. □

On the other hand, it is possible that less than n switchers are active. An element in X could also be connected to an inactive switcher if the edges cx or sx are in the tree. Anyway, Lemma 9 will ensure that the path P avoids too many inactive switchers.

Lemma 6. *Given that all thick bunches, all long bunches, and all switchers are regular as well as exactly n switchers are active, then there is an exact cover for the* P-X3C *iff there is a spanning tree T' of G_S.*

Proof. Let $\mathscr{S}'$ be an exact cover. We construct a subgraph T' as follows. Set all switchers in gears corresponding to $\mathscr{S}'$ as active. With a look at Fig. 4, this is done by putting the edges sx, cx and bc of each switcher into T'. Then each $x \in X$ is connected to exactly one gear via an active switcher. Since the gears are connected via the thick bunches which replaced the spinal edges, the constructed subgraph is a spanning tree.

Now assume that there is no solution of the P-X3C instance. Then, for every choice $\mathscr{S}' \subseteq \mathscr{S}$ there is an $x \in X$ with

$$|\{S \in \mathscr{S}' | x \in S\}| = 0 \quad \text{or} \tag{3}$$

$$|\{S \in \mathscr{S}' | x \in S\}| \geq 2. \tag{4}$$

We identify the possibilities to set the switchers of the gears with the choice $\mathscr{S}'$, i.e. $S \in \mathscr{S}'$ if and only if the switchers of the corresponding gear are active. Then x is not connected to the rest of the graph (3) or it induces a circuit since it is connected to at least two gears (4), which are also connected with spinal edges. Thus, the constructed subgraph is not a spanning tree. □

Now, we are able to determine the size of the circuits induced by chords in active switchers. By Lemma 5, there are at most n active switchers if all thick and long bunches are regular. Fig. 7 shows the setting for one active switcher in a regular gear. We describe the circuits which are closed by chords in the lower switcher. The colour of the terms below the braces corresponds to the part of the gear in Fig. 7. The chords induce circuits with a total length of

$$\Phi^r_{\text{as}} = \underbrace{n}_{\substack{\text{number of}\\ \text{active switchers}}} \cdot \Big[\underbrace{130m^3}_{\substack{\text{number of}\\ \text{int. paths}}} \cdot \underbrace{(16m^2 + 3)}_{\substack{\text{int. path lengths}\\ \text{and } e_2,\, e_3,\, e_4}} \tag{5}$$

$$+ \overbrace{(\underbrace{2}_{e_1, e_5} + \underbrace{3 \cdot 3 \cdot 6m^2}_{\text{3 long bunches}})}^{e_1} \tag{6}$$

$$+ \overbrace{(\underbrace{5m}_{P} + \underbrace{3 \cdot 3 \cdot 6m^2}_{\text{3 long bunches}} + \underbrace{4}_{\substack{e_2, e_3,\\ e_4, e_5}})}^{e_P} \Big]. \tag{7}$$

The first summand in the squared brackets in Line (5) is the total size of the circuits closed by the $130m^3$ interior paths and the tree edges e_2, e_3 and e_4. The chord e_1 in the lower switcher induces a circuit with three long bunches and the edge e_5, see Line (6). There is one chord e_P in the path P which closes a circuit with the three long bunches, e_5 in the upper switcher, and the edges e_2 to e_4. This is expressed in Line (7).

Now let us turn our focus to inactive switchers. As we will see in Lemma 9, there are at most $3m - n$ inactive switchers. Contrary to the case of active switchers, we are now only able to give an upper bound on the circuit lengths. Therefore, we introduce the term $\tilde{\Phi}^r_{\text{is}}$ as the upper bound, while the real size is denoted by Φ^r_{is}. Now, consider the switcher in the upper left corner of Fig. 8. The total size of circuits induced by chords in inactive switchers is

$$\Phi^r_{\text{is}} \leq \tilde{\Phi}^r_{\text{is}} = \underbrace{(3m - n)}_{\substack{\text{number of}\\ \text{inact. switchers}}} \cdot \Big[\underbrace{130m^3}_{\substack{\text{number of}\\ \text{int. paths}}} \cdot \underbrace{(16m^2 + 5m + 1)}_{\substack{\text{int. path lengths,}\\ P_1 \text{ and } e_1}} + \overbrace{(\underbrace{2}_{e_2, e_5} + \underbrace{3 \cdot 3 \cdot 6m^2}_{\text{3 long bunches}})}^{e_2}$$

$$+ \underbrace{2}_{e_3, e_4} \cdot \Big(\underbrace{4}_{\substack{\text{via } x\\ \text{to } x_1}} + \underbrace{(5 + 2 \cdot 3 \cdot 6m^2)}_{\substack{\text{crossing the first}\\ \text{active gear } G_1}} \tag{8}$$

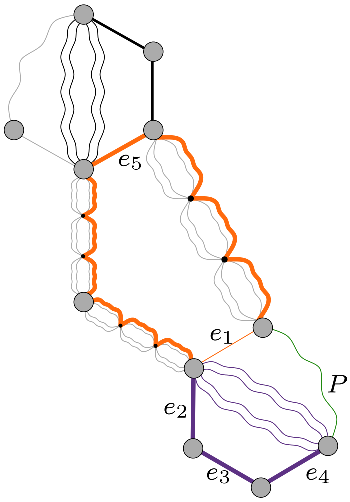


Fig. 7. Circuits closed by chords in the lower active switcher. By Lemma 4, the upper switcher is also active since the gear is regular. Tree edges which are *not* in incomplete paths are depicted as thick lines, chords and incomplete paths as thin lines. The colors correspond to the terms in the equation below, Lines (5) to (7).

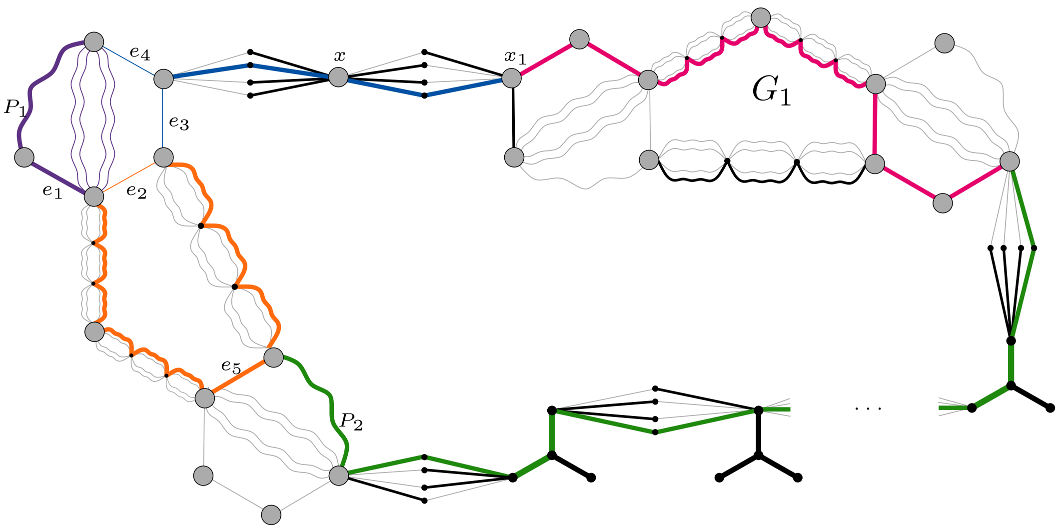


Fig. 8. Circuits induced by chords in an inactive switcher.

$$+\overbrace{(n-1)(\underbrace{2\cdot 3\cdot 6m^2+6}_{\substack{\text{crossing of an}\\ \text{active gear}}}+\underbrace{2}_{\substack{\text{thick bunch to}\\ \text{the next gear}}})}^{\text{crossing at most } n-1 \text{ further active gears}} \tag{9}$$

$$+\overbrace{(3m-n)(\underbrace{2\cdot 3\cdot 6m^2}_{\substack{\text{2 long}\\ \text{bunches}}}+\underbrace{2\cdot(5m+1)}_{\substack{2\times\text{Path } P\\ \text{and edge } ab}}+\underbrace{2}_{\substack{\text{thick bunch to}\\ \text{the next gear}}})}^{\text{crossing at most } 3m-n \text{ inactive gears}}\Big)\Big]. \tag{10}$$

Similarly to the switcher in Fig. 7, the switcher in the upper left corner of Fig. 8 contains $130m^3$ interior paths. But here, they induce circuits of lengths $16m^2+5m+1$. The chord e_2 closes a circuit of the same length as e_1 in Fig. 7. Lines (8)–(10) are dedicated to estimate the circuit lengths induced by e_3 and e_4. This is done simultaneously for both edges. At first, the circuits pass node x and meet the X-node x_1, indicated by blue edges. Then, they cross the active gear G_1, highlighted in magenta. Afterwards, the circuits have to cross at most $n-1$ further active and at most $3m-n$ inactive gears via green edges. Both, active and inactive gears are indicated in Fig. 8 by very fat drawn $K_{3,1}$'s. Note that the last inactive gear is the gear which contains the edges e_3 and e_4.

The integer k. After the construction of the graph G_S, we go on with the other part of the P-MSFCB instance, namely the integer k. Its value is the sum

$$k := \Phi^r_{tb} + \Phi^r_{lb} + \Phi^r_{as} + \tilde{\Phi}^r_{is} \tag{11}$$

$$= 232m^8 - 4m^7 + 6240m^6 + 1950m^5 + (1038 - 650n)m^4 \tag{12}$$

$$+ (126 + 44n)m^3 + (72 - 66n)m^2 + (20n^2 + 5n - 4)m \tag{13}$$

$$+ (4 + 2n - 8n^2). \tag{14}$$

In this polynomial, only the coefficients in Line (12) will be of interest. To yield a strictly fundamental cycle basis with a size of at most k, it is necessary that all components – thick bunches, long bunches, and switchers – are regular. Furthermore, exactly $3m - n$ switchers may be inactive. At next, we will show that for a sufficiently large m, a strictly fundamental cycle basis has a size greater than k if the inducing spanning tree is irregular on any component or if more than $3m - n$ switchers are inactive. For instances where m is not large enough the P-MSFCB can be attacked by a classical brute-force approach.

Irregular thick and long bunches. Assume at first that p_{tb} thick bunches are irregular. Since G_X was bipartite, each chord in an irregular thick bunch induces a circuit with length of at least 5. Thus, the chords of all thick bunches together close circuits with total length

$$\Phi^i_{tb} \geq (4m - 1 - p_{tb}) \cdot (m^7 - 1) \cdot 4 + p_{tb} \cdot m^7 \cdot 5. \tag{15}$$

Now let p_{lb} long bunches be irregular. In an irregular long bunch, two of the three slices contain a complete path like in all three slices in a regular long bunch. This is regarded by the first summand inside of the squared brackets in Line (17). In the remaining slice, all m^5 chords induce circuits with lengths of at least three times the path length of a path in a slice, i.e. $3 \cdot 6m^2$. This is considered by the second summand in Line (17). Thus, the circuit lengths of chords in long bunches sum up to

$$\Phi^i_{lb} \geq (6m - p_{lb}) \cdot 3 \cdot (m^5 - 1) \cdot 2 \cdot 6m^2 \tag{16}$$

$$+ p_{lb} \cdot \left[2 \cdot (m^5 - 1) \cdot 2 \cdot 6m^2 + m^5 \cdot 3 \cdot 6m^2\right]. \tag{17}$$

Thick and long bunches are now viewed simultaneously. Then, after adding Φ^i_{tb} and Φ^i_{lb}, the cycle basis has the size

$$\Phi > \Phi^i_{tb} + \Phi^i_{lb} \geq 232m^8 + (p_{tb} + 6p_{lb} - 4)m^7 + o(m^7). \tag{18}$$

Looking now at the coefficient of m^7 and compare it with its counterpart in Line (12), it can only be at most -4 if $p_{tb} = p_{lb} = 0$. Lemma 7 follows.

Lemma 7. *An SFCB can only have a value of at most k if all thick and all long bunches are regular.* □

Irregular switchers. Remember that a switcher is irregular, if one of the interior b–s-paths is completely in the tree or if the unique b–s-path in the tree uses edges outside of this switcher. Thus, let p_{ip} switchers contain a complete interior path and let p_{np} be the number of further irregular switchers without such a path. The remaining $3m - p_{ip} - p_{np}$ switchers can be regarded as active since active switchers cause smaller induced circuits than inactive switchers. Although this is in general not possible by Lemma 5, it is sufficient for the estimation and makes it less involved. In a switcher with one complete interior path P_i in the tree, the remaining $130m^3 - 1$ paths close circuits with P_i, their total length is

$$\Phi^i_{ip} = p_{ip} \cdot (130m^3 - 1) \cdot 2 \cdot 16m^2. \tag{19}$$

Now, take a look at Fig. 5 and assume that in the upper switcher the unique b_1–s_1-path in the tree contains edges outside of the switcher. Then, the $130m^3$ circuits induced by the interior paths contain at least one long bunch. This long bunch can be one of those starting at the a_1, b_1 or c_1. Otherwise, assume that the edges b_1c_1 and c_1x_1 are in these circuits. Due to the second and third rule for the construction of the spinal tree, there must be such a long bunch in another gear. Together, these circuits have a total length of

$$\Phi^i_{np} \geq p_{np} \cdot 130m^3 \cdot (16m^2 + 3 \cdot 6m^2). \tag{20}$$

For the remaining regular and active switchers, the interior paths close circuits with total length of exactly

$$\Phi^r_s = (3m - p_{ip} - p_{np}) \cdot 130m^3 \cdot (16m^2 + 3). \tag{21}$$

With the assumption that all thick and long bunches are regular, we obtain a cycle basis of length

$$\Phi > \Phi^r_{tb} + \Phi^r_{lb} + \Phi^i_{ip} + \Phi^i_{np} + \Phi^r_s \tag{22}$$

$$\geq 232m^8 - 4m^7 + 6240m^6 + (2080p_{ip} + 2340p_{np})m^5 + o(m^5). \tag{23}$$

Analogously to the last paragraph, look at the coefficient of m^5 and compare it with its counterpart in Line (12). This coefficient can only be 1950 or less if $p_{ip} = p_{np} = 0$. Thus, we can strengthen Lemma 7 to the following Lemma.

Lemma 8. *An SFCB on G_S can only have a size of at most k if all thick bunches, all long bunches, and all switchers are regular.* □

Note that Lemma 8 does only hold for sufficiently large m, since several coefficients of the lower terms in Eqs. (18) and (23) are actually smaller than the corresponding coefficients in Lines (12)–(14).

From Lemma 5 it is known that at most n switchers can be active if all thick and long bunches, as well as all switchers are regular. We now go on to show that n active switchers are necessary to obtain an SFCB of size k or less. Therefore, assume that $3m - n + p_{\text{is}}$ are inactive and $n - p_{\text{is}}$ are active. Bearing the regular thick and long bunches in mind, one obtains

$$\begin{aligned}\Phi &> \Phi^r_{\text{tb}} + \Phi^r_{\text{lb}} + (3m - n + p_{\text{is}}) \cdot 130m^3 \cdot (16m^2 + 5m + 1) + (n - p_{\text{is}}) \cdot 130m^3 \cdot (16m^2 + 3) \\ &\geq 232m^8 - 4m^7 + 6240m^6 + 1950m^5 \qquad (24) \\ &\quad + (390 - 650n + 650p_{\text{is}})m^4 + o(m^4). \qquad (25)\end{aligned}$$

Again, we compare the coefficients of m^4 in Lines (25) and (12). Obviously, it holds $(390 - 650n + 650p_{\text{is}}) \leq (1038 - 650n)$ iff $p_{\text{is}} = 0$. Hence, additionally to the regularity of all bunches and switchers, an SFCB of size k or less can only be achieved if at most $3m - n$ switchers are inactive. Together with Lemma 5, Lemma 9 follows.

Lemma 9. *An SFCB on G_S can only attain a size of at most k if – in addition to the regularity of all bunches and switchers – at most $3m - n$ switchers are inactive. Together with Lemma 5 it follows that exactly n switchers are active.* □

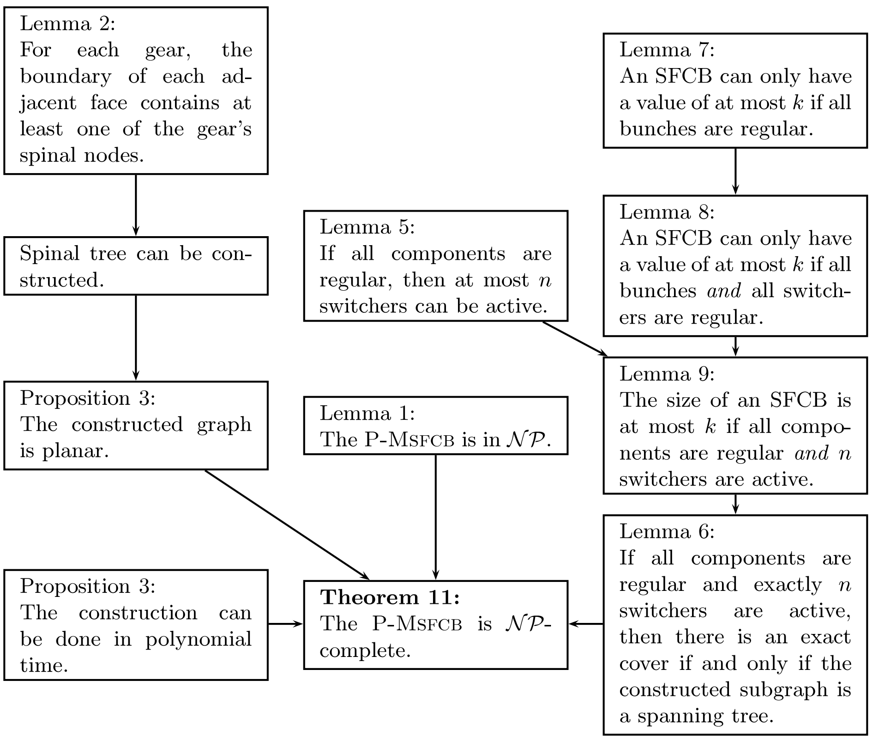
Furthermore, since the parameter $p_{\text{is}} = 0$, Corollary 10 follows.

Corollary 10. *Lemma 9 ensures that an element $x \in X$ cannot be connected to an inactive switcher by using edge sx or cx, see Fig. 4.* □

We introduced several graph components, derived an integer k, and verified that all components have to be regular if an SFCB on the reduced graph shall not exceed k (Lemma 8). In Lemma 9, it is stated that additionally exactly n switchers have to be active. Putting all these regular components together results in a spanning subgraph. Lemma 6 points out that this subgraph is a spanning tree if and only if there is an exact cover on the P-X3C instance. Proposition 3 reveals the preservation of planarity and states that the construction can be done in polynomial time. Finally, Lemma 1 ensures the membership of the P-MSFCB in $\mathcal{NP}$. All these statements together lead to the following theorem.

Theorem 11. *The* MINIMUM STRICTLY FUNDAMENTAL CYCLE BASIS *Problem is $\mathcal{NP}$-complete on planar graphs.* □

The diagram below summarizes how Theorem 11 is derived from the partial results in the last section.



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