Improved Upper and Lower Bounds for k-Broadcasting

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We continue the investigation of k-broadcasting, a variant of broadcasting in which an informed vertex can call up to k of its neighbors in each time unit. A focus of the investigation into broadcasting is the function $B_k(n)$, which is the minimum number of edges in any n vertex graph such that each vertex can originate a k-broadcast that completes in minimum time. We give several methods to construct graphs which allow minimum-time k-broadcasting from each vertex. These constructions give improvements to the best current upper bounds on $B_k(n)$. We also give an improvement to the best existing lower bound on $B_k(n)$. In addition, a few new exact values of $B_k(n)$ are determined.

Keywords: broadcast; k-broadcast; minimum broadcast graphs

1. INTRODUCTION AND DEFINITIONS

Broadcasting is the process of message dissemination in a communication network in which a message, originated by one vertex, is transmitted to all vertices of the network by placing a series of calls over the communication lines of the network. This is to be completed as quickly as possible. Typically, it is assumed that each call involves only one informed vertex and one of its neighbors, each call requires one unit of time, a vertex can participate in only one call per unit of time, and a vertex can only call its neighbors. Here, we consider k-broadcasting in which each call involves a caller who sends the message to k (or fewer) of its neighbors in one time unit.

Given a connected graph G and a message originator, vertex u, the *k-broadcast time of vertex u*, $b_k(u)$, is the minimum number of time units required to complete a k-broadcast from the vertex u. For any vertex u in a connected graph G with n vertices, $b_k(u) \geq \lceil \log_{k+1} n \rceil$, since the number of informed vertices can, at most, be multiplied by $k + 1$ during each time unit. The *k-broadcast time of a graph G*, denoted $b_k(G)$, is the maximum k-broadcast time of any vertex u in G, that is, $b_k(G) = \max\{b_k(u) | u \in V(G)\}$. For the complete graph K_n with $n \geq 2$ vertices, $b_k(K_n) = \lceil \log_{k+1} n \rceil$, the smallest k-broadcast time for any graph on n vertices. However, K_n is not minimal with respect to this property for any $n \geq 4$, when $n \geq k + 2$, that is, we can remove edges from K_n and still have a graph G with n vertices such that $b_k(G) = \lceil \log_{k+1} n \rceil$. We use the term *$k$-broadcast graph* to refer to any graph G on n vertices with $b_k(G) = \lceil \log_{k+1} n \rceil$. The *$k$-broadcast function*, $B_k(n)$, is the minimum number of edges in any k-broadcast graph on n vertices. A *minimum k-broadcast graph* is a k-broadcast graph on n vertices having $B_k(n)$ edges.

Most of the previous work in this area has been for k=1. For a survey of results on broadcasting and related problems, see Hedetniemi et al. [6]. Grigni and Peleg [5] showed that $B_k(n) \in \Theta(kL_k(n)n)$ where $L_k(n)$ denotes the exact number of consecutive leading k's in the $(k+1)$-ary representation of $n-1$. Asymptotically, Grigni and Peleg's construction produces the best-known upper bounds on $B_k(n)$ for most values of n. Lazard [9] studied minimum k-broadcast graphs and, in particular, gave some values of $B_2(n)$, $B_3(n)$ and $B_4(n)$ for small values of n. More generally, Lazard showed that for $n \leq k + 1$, $B_k(n) = \frac{1}{2}n(n-1)$. He also showed that $B_k(k+2) = k+1$ and that $B_k([k+1]^m) = \frac{1}{2}km(k+1)^m$ for $m \geq 1$. König and Lazard [8] generalized some results on 1-broadcasting, found minimum k-broadcast graphs for all n in the range $k+3 \leq n \leq 2k+3$, and gave a more precise statement of the Grigni and Peleg lower bound on $B_k(n)$. In addition, they showed that $B_k(n) \leq \frac{nk}{2}\lceil \log_{k+1} n \rceil$ for $n \geq k$.

In this paper, we continue the search for minimum k-broadcast graphs and for improved bounds on $B_k(n)$. In Section 2, we present a minimum 3-broadcast graph on

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11 vertices. In Section 3, we give an improvement to the best existing lower bound. In Section 4, we remark on the connection of this problem to the (Δ, D) problem and use (Δ, D) graphs to obtain some exact values of $B_k(n)$ for small k and n and some upper bounds on $B_k(n)$ for larger n. In Section 5, we give improvements to Grigni and Peleg's upper bounds on $B_k(n)$. For some n and k, these also improve on the upper bound of König and Lazard. Finally, in Section 6, we show that the well-known compounding method of constructing 1-broadcast graphs can be easily generalized to construct k-broadcast graphs. (This has recently been shown independently by Lee and Ventura [10].)

2. AN AD HOC CONSTRUCTION

We begin by presenting a new minimum 3-broadcast graph on 11 vertices.

Theorem 1. $B_3(11) = 18$.

Proof. Let G be a 3-broadcast graph on 11 vertices. Every vertex of G must have degree at least 3 since a vertex of degree 2 cannot 3-broadcast to 11 vertices in 2 time units. If G has a vertex of degree greater than 5 (and no vertices of degree less than 3), then it must have at least 18 edges. If G has a vertex of degree 3, then it must have at least one neighbor of degree ≥ 4. If G has a vertex of degree 5, it must also have another vertex of degree at least 4 and, thus, at least 18 edges. If G has no vertices of degree greater than 4, then it must have at least 3 vertices of degree 4 and at least 18 edges. Hence, $B_3(11) \geq 18$.

A minimum 3-broadcast graph on 11 vertices with 18 edges is given in Figure 1.

This graph has diameter 2, three vertices of degree 4, and eight vertices of degree 3. A 3-broadcast scheme for any originator of degree 3 is obvious. A 3-broadcast scheme for a vertex of degree 4 is presented in

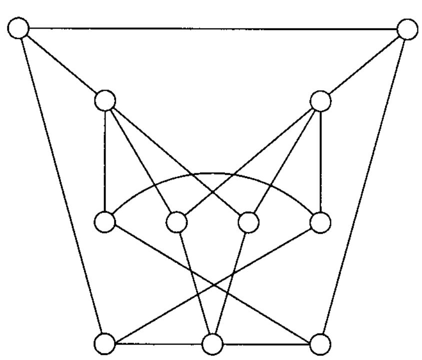

FIG. 1. 3-broadcast graph on 11 vertices.

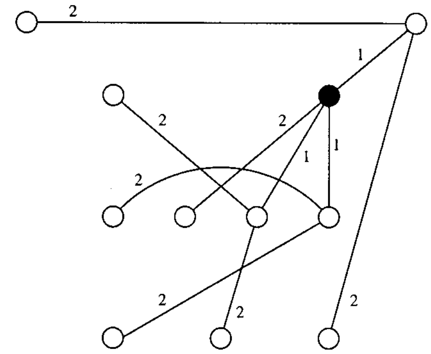

FIG. 2. Broadcast scheme for a vertex of degree 4 in the 3-broadcast graph on 11 vertices.

Figure 2. □

Previously, the best upper bound on $B_3(11)$ was 25 [10]. Although not terribly important on its own, this improved value can be used to improve other upper bounds by the compounding method introduced in Section 6.

3. IMPROVED LOWER BOUND

We know that $B_k([k+1]^m) = \frac{1}{2}km(k+1)^m$. For values of n other than powers of $k+1$, we wish to improve upon the existing lower bound on $B_k(n)$ [8]. Consider the $(k+1)$-ary representation of the integer $n-1$: $n-1 = (\gamma_{m-1}\gamma_{m-2}...\gamma_0)_{k+1}$, where $0 \leq \gamma_i \leq k$ for $i = 0, ..., m-1$ and $\gamma_{m-1} \neq 0$. Let $p = max\{i | \gamma_i \neq k\}$, that is, let p be the index of the leftmost digit which is not equal to k. We will have $n-1 = k(k+1)^{m-1} + ... + k(k+1)^{p+1} + \gamma_p(k+1)^p + \gamma_{p-1}(k+1)^{p-1} + ... + \gamma_0$.

König and Lazard [8] (see also [5]) showed that $B_k(n) \geq \frac{nk}{2}(m-p-1)$. In this section, we improve upon this lower bound. For a given n and k, we define $\beta = \gamma_p$ if $p = 0$ or if $\gamma_0 = \gamma_1 = ... = \gamma_{p-1} = 0$. Otherwise, $\beta = \gamma_p + 1$. Thus, $\beta 00...0 \geq \gamma_p\gamma_{p-1}...\gamma_0$.

Theorem 2. *When n is not a power of $k+1$, $B_k(n) \geq \frac{nk}{2}(m-p-1) + \frac{n}{2}\beta$, where m, p and β are as described above.*

Proof. Consider any originator v in a k-broadcast graph G on n vertices. To inform $k(k+1)^{m-1}$ or more vertices in m time units, vertex v must call k neighbors at time 1. In general, to inform at least $\sum_{i=j}^{m-1} k(k+1)^i$ vertices in m time units, vertex v must call k "new" neighbors at each time unit i for $i = 1, ..., m-j$. By our choice of p, to k-broadcast to n vertices in m time units, v must call k distinct neighbors at each time unit i for $i = 1, ..., m-p-1$, and at time $m-p$, it must call at least γ_p additional neighbors. If $\gamma_i > 0$ for any i,

$0 \le i \le p-1$, then the originator v must inform at least one more neighbor at time $m-p$ or later. Thus, v must call at least $k(m-p-1)+\beta$ distinct neighbors and we obtain the result. □

This new lower bound improves on the old lower bound whenever $\beta > 0$. In particular, this occurs for all n and k except when $n = (100...0)_{k+1}$ where the exact value is known and when $n = (kk...k00...01)_{k+1}$, where the number of 0's is p with $0 \le p \le m-2$. Thus, it is improved for all but m values in each range $(k+1)^{m-1} < n \le (k+1)^m$.

We compared our lower bounds to the upper bounds reported by Lee and Ventura [10] and discovered that our lower bound matches their upper bound for $k = 2$ and $n = 24$, yielding the new exact value of $B_2(24) = 48$. Our lower bounds are close to their upper bounds for some other values of n and k, showing, for example, that $45 \le B_2(23) \le 48$ and $323 \le B_2(81) \le 324$.

4. Δ-BROADCAST GRAPHS FROM (Δ, D) GRAPHS

A problem of continuing interest in graph theory is the construction of graphs with as many vertices as possible for a given degree and diameter. A (Δ, D) *graph* is a graph with maximum degree Δ and diameter at most D. The number of vertices of a graph with degree $\Delta \ge 3$ of diameter D is bounded above by $1 + \Delta + \Delta(\Delta-1) + ... + \Delta(\Delta-1)^{D-1} = \frac{\Delta(\Delta-1)^D-2}{\Delta-2}$. This value is called the *Moore bound* and a graph which achieves this bound is called a *Moore graph*. It is known that for $D \ge 2$ and $\Delta \ge 3$ Moore graphs exist only for $D = 2$ and $\Delta = 3, 7$, and (perhaps) 57, (see [3] for a proof). The Petersen graph is a $(3, 2)$ Moore graph with 10 vertices. Hoffman and Singleton [7] obtained a $(7, 2)$ Moore graph with 50 vertices (see, e.g., [1], [7]).

Theorem 3. *$B_\Delta(\Delta^2+1) = \frac{\Delta(\Delta^2+1)}{2}$ if and only if there exists a $(\Delta, 2)$ Moore graph.*

Proof. Let $B_\Delta(\Delta^2+1) = \frac{\Delta(\Delta^2+1)}{2}$. Thus, there is a graph G on Δ^2+1 vertices with $\frac{\Delta(\Delta^2+1)}{2}$ edges which can Δ-broadcast in 2 time units. It follows that the diameter of G is at most 2. If G contains a vertex u of degree $\le \Delta - 1$, then if u is the originator, at time 1, u can inform at most $\Delta-1$ vertices. At time 2, at most $(\Delta-1)\Delta$ additional vertices can be informed, giving a total of at most $1 + (\Delta-1) + (\Delta-1)\Delta = \Delta^2$ informed vertices. Thus, every vertex of G has degree at least Δ. Since G has $\frac{\Delta(\Delta^2+1)}{2}$ edges, all vertices must have degree Δ and G is a $(\Delta, 2)$ Moore graph.

If G is a $(\Delta, 2)$ Moore graph, then G has $\frac{\Delta(\Delta^2+1)}{2}$ edges and diameter 2 and is Δ-regular. Let u be any vertex of such a graph G. It follows from the constraints on the diameter and degree that there exists a spanning tree T of G rooted at u such that all neighbors of u are adjacent to u in T and each of these neighbors are adjacent to $\Delta-1$ distinct vertices from the remaining $\Delta^2-\Delta$ vertices.

We can Δ-broadcast in G as follows: At time 1, u calls all Δ of its neighbors. At time 2, these Δ neighbors each call their $\Delta-1$ uninformed neighbors, completing the Δ-broadcast. □

The above theorem gives $B_3(10) = 15$ and $B_7(50) = 175$. In addition, if there exists a $(57, 2)$ Moore graph, then $B_{57}(3250) = 92,625$.

Other (Δ, D) graphs that are not Moore graphs are Δ-broadcast graphs, giving upper bounds on $B_\Delta(n)$.

Lemma 4. *Any (Δ, D) graph on n vertices, with $(\Delta+1)^{D-1}+1 \le n \le \frac{\Delta(\Delta-1)^D-2}{\Delta-2}$, is a Δ-broadcast graph and $B_\Delta(n) \le \frac{\Delta n}{2}$.*

Proof. Let G be a (Δ, D) graph on n vertices and consider a vertex u of G. At time 1, u sends the message to its Δ neighbors. At time $i > 1$, any vertex informed at time $i-1$ sends the message to its $\Delta-1$ remaining neighbors. Since the diameter of G is D, Δ-broadcast will complete at time D. Since $\lceil \log_{\Delta+1} n \rceil = D$, G is a Δ-broadcast graph. □

The size of the current largest known (Δ, D) graphs are available online [4]. Table 1 shows the upper bounds on $B_\Delta(n)$ that can be obtained from the current largest known (Δ, D) graphs with at most 200 vertices. Note that the included bounds for $B_3(10)$ and $B_7(50)$ are exact values. All the values in the table are improvements to known upper bounds for these cases except that $B_3(10) = 15$ was noted by Lee and Ventura [10]. Further improvements to the (Δ, D) graphs may result in new bounds for $B_\Delta(n)$.

5. CONSTRUCTIONS USING (k + 1)-NOMIAL TREES

We describe two constructions of k-broadcast graphs based on $(k+1)$-nomial trees. Our first construction is

TABLE 1. Upper bounds on $B_\Delta(n)$.

Δ	n	Bound	Δ	n	Bound
3	10	15	8	57	228
3	20	30	9	74	333
4	15	30	10	91	455
4	41	82	11	98	539
5	24	60	12	133	798
5	72	180	13	162	1053
6	32	96	14	183	1281
6	110	330	15	186	1395
7	50	175	16	198	1584
7	148	518			

essentially the same as that of Peleg [11] which was reported in Grigni and Peleg [5]. These papers gave asymptotic bounds on $B_k(n)$ and the authors did not carefully count the number of edges in their upper-bound construction. We will describe the construction and broadcast schemes in more detail than in [5] and give an exact value for the number of edges in these graphs. We then give an improved construction which yields a better upper bound.

The *b-nomial tree* T_b^m of dimension m has b^m vertices and can be constructed recursively. The tree T_b^0 is a single vertex. For $m \geq 1$, the tree T_b^m is obtained from b copies of T_b^{m-1} by connecting the roots of $b-1$ copies of T_b^{m-1} to the root u of the remaining copy of T_b^{m-1}. This vertex u is the root of T_b^m.

Consider the problem of k-broadcasting from the root of T_{k+1}^m, a $(k+1)$-nomial tree on $(k+1)^m$ vertices. The root of this tree, denoted v^m, has km children $v_1^{m-1}, v_2^{m-1}, ..., v_k^{m-1}$, $v_1^{m-2}, v_2^{m-2} ...$, $v_k^{m-2}, ...,$ $v_1^0, v_2^0, ..., v_k^0$. Each vertex v_j^i, where $0 \leq i \leq m-1$, $1 \leq j \leq k$, is the root of a $(k+1)$-nomial tree T_j^i on $(k+1)^i$ vertices. To k-broadcast from v^m, the root calls vertices v_1^{m-i}, v_2^{m-i},...,v_k^{m-i} at time i for each $1 \leq i \leq m$. Each vertex which learns the message at time i calls k uninformed neighbors in order of decreasing subtree size at each time unit $i+1, i+2, ..., m$.

The k-broadcast schemes for the graphs constructed in this section are based on this k-broadcast scheme which we will refer to as the *$(k+1)$-nomial tree scheme*. In particular, beginning with an n vertex subgraph of $T_{k+1}^{\lceil \log_{k+1} n \rceil}$, we add edges so that each vertex can (essentially) play the role of $v^{\lceil \log_{k+1} n \rceil}$ in the scheme above. Since $B_k(n)$ is known exactly for $n \leq k+2$ and for $n = (k+1)^m$, we will not consider these cases in our constructions. Note that although the graphs we construct are not trees this underlying structure is of particular importance and we will often refer to the position of vertices in this underlying tree such as being the root of a particular subtree.

5.1. *First Construction*

To construct a k-broadcast graph G_n^k on n vertices, let $n = (\alpha_{m-1}\alpha_{m-2}...\alpha_0)_{k+1}$ be the $(k+1)$-ary representation of the integer n, where $0 \leq \alpha_i \leq k$ for $i = 0, ..., m-1$ and $m = \lceil \log_{k+1} n \rceil$. If $n \neq (kk...k)_{k+1}$, then let $p = max\{i | \alpha_i \neq k\}$.

Beginning with T_{k+1}^m, we delete the root v^m and its incident edges. The resulting graph has $(k+1)^m - 1$ vertices. We describe the rest of the construction in three cases:

Case 1: If $n = (k+1)^m - 1$, that is, if $n = (kk...k)_{k+1}$, we complete the construction of G_n^k by adding an edge from each vertex u to every child of the (deleted) root, that is, to $v_1^{m-1}, v_2^{m-1}, ..., v_k^{m-1}$, $v_1^{m-2}, v_2^{m-2} ...$, $v_k^{m-2}, ...,$ $v_1^0, v_2^0, ..., v_k^0$ unless such an edge is already in the graph. Thus, in the constructed graph G_n^k, each vertex u is connected to all of the v_j^i. As an example, the tree structure of G_{26}^2 is shown in Figure 3.

In addition to the edges shown, each vertex is connected to each of the vertices labeled v_j^i. The graph G_{26}^2 has a total of 143 edges.

Consider k-broadcasting from any originator v_j^i with $0 \leq i \leq m-1$ and $1 \leq j \leq k$ in G_n^k. In the $(k+1)$-nomial tree scheme S described above, v_j^i receives the message at time $m-i$ and then initiates a k-broadcast within the subtree rooted at v_j^i at time $m-i+1$. In G_n^k, the originator v_j^i plays the role of v^m during time units $1, 2, ..., m-i-1$, that is, it calls $v_1^{m-1}, v_2^{m-1}, ..., v_k^{m-1}$ at time 1, $v_1^{m-2}, v_2^{m-2} ..., v_k^{m-2}$ at time 2 ,..., and $v_1^{i+1}, v_2^{i+1}, ..., v_k^{i+1}$ at time $m-i-1$. Each of these roots initiates a k-broadcast in their subtrees exactly as in S. At time $m-i$, v_j^i calls $v_1^i, v_2^i, ..., v_{j-1}^i, v_{j+1}^i, ..., v_k^i$ and one of its children that it would call at time $m-i+1$ in S. This child, v', thus learns the message one time unit earlier than in S. In subsequent time units, v_j^i calls as in S except that it can always call one child "early" in each time unit. Except for v_j^i and these "early" children such as v', the other vertices continue exactly as in S. Vertex v' calls $v_1^{i-1}, v_2^{i-1}, ..., v_k^{i-1}$ at time $m-i+1$ and then continues as in S. Thus, the vertices $v_1^{i-1}, v_2^{i-1}, ..., v_k^{i-1}$ are informed at the same time as in S. Similarly, each "early" child of v_j^i can call the next

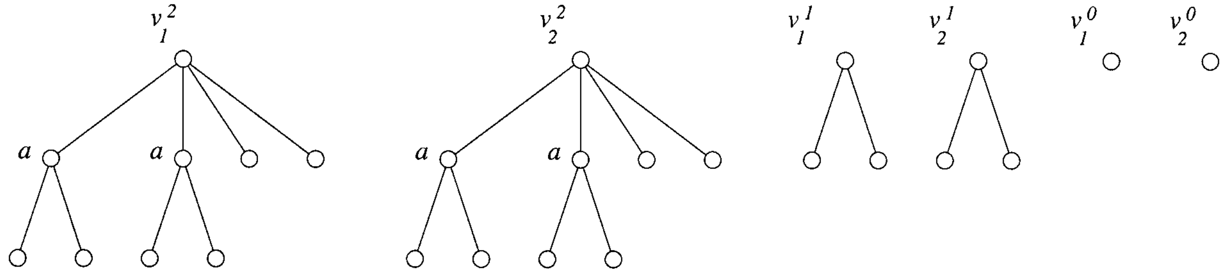

FIG. 3. Tree structure of G_{26}^2.

set of vertices $v_1^l, v_2^l, ..., v_k^l$ at time $m - l$ and, otherwise, the scheme is as in S.

Now, consider k-broadcasting from vertex u of G_n^k which is not a child of the (deleted) root. Let x denote the parent of u in the tree T_{k+1}^m and let t be the time that u is informed from x in the $(k+1)$-nomial tree scheme S. As above, in G_n^k, the originator u plays the role of v^m during time units $1, 2, ..., t$ and then initiates a k-broadcast to the subtree rooted at u at time $t+1$. The vertices $v_1^{m-1}, v_2^{m-1}, ..., v_k^{m-1}$, $v_1^{m-2}, v_2^{m-2} ...$, $v_k^{m-2}, ...$, $v_1^0, v_2^0, ..., v_k^0$ and the subtrees rooted at these vertices behave as in S. Thus, x is informed at some time before t. At time t, x need not inform u but can instead inform another child u' one time unit early. This vertex u' can call $v_1^{m-t-1}, v_2^{m-t-1}, ..., v_k^{m-t-1}$ at time $t+1$ and then continue as in S. At each subsequent time, x can similarly inform a child one time unit early. This child can call the next set of vertices $v_1^l, v_2^l, ..., v_k^l$ before proceeding as in S. Each set $v_1^l, v_2^l, ..., v_k^l$ and the subtrees rooted at these vertices proceed as in S.

Case 2: If $n = (kk...k\alpha_p 00...0)_{k+1}$, where $0 \le \alpha_p \le k-1$, we must remove more vertices from T_{k+1}^m than just the root v^m. In this case, we leave only the vertices in the subtrees rooted at $v_1^{m-1}, v_2^{m-1}, ..., v_k^{m-1}$, $v_1^{m-2}, v_2^{m-2} ...$, $v_k^{m-2}, ..., v_1^{p+1}, v_2^{p+1}, ..., v_k^{p+1}$, and $v_1^p, v_2^p, ..., v_{\alpha_p}^p$. As before, we connect every vertex to each of the children of the (deleted) root. This gives the graph G_n^k. As an example, the tree structure of G_{25}^2 is shown in Figure 4.

In addition to the edges shown, each vertex is connected to each of the vertices labeled v_j^i. The graph G_{25}^2 has a total of 118 edges.

To k-broadcast from any originator in G_n^k, we proceed exactly as in the previous case.

Case 3: Finally, let $n = (kk...k\alpha_p\alpha_{p-1}...\alpha_0)_{k+1}$, where $0 \le \alpha_p \le k-1$ and $\sum_{i=0}^{p-1} \alpha_i > 0$. In this case, from T_{k+1}^m, leave only the subtrees rooted at $v_1^{m-1}, v_2^{m-1}, ..., v_k^{m-1}$, $v_1^{m-2}, v_2^{m-2} ...,$ $v_k^{m-2}, ...,$ $v_1^{p+1}, v_2^{p+1}, ..., v_k^{p+1}$, $v_1^p, v_2^p, ..., v_{\alpha_p}^p$ and $v_{\alpha_p+1}^p$. We remove some vertices from the subtree rooted at $v_{\alpha_p+1}^p$. Remove the vertices one at a time, each time removing a leaf furthest from $v_{\alpha_p+1}^p$ until only $\sum_{i=0}^{p-1} \alpha_i(k+1)^i$ vertices remain in the subtree. The resulting structure has n vertices. To this, we add edges as before connecting each vertex to each of the roots v_j^i. The resulting graph is G_n^k. As an example, the tree structure of G_{23}^2 is shown in Figure 5.

In addition to the edges shown, each vertex is connected to each of the vertices labeled v_j^i. The graph G_{23}^2 has a total of 90 edges.

To k-broadcast from any originator in G_n^k, we proceed exactly as above.

The graph G_n^k constructed here is a k-broadcast graph and yields an upper bound on $B_k(n)$. Note that this construction is essentially the same as that described by Peleg [11], but the bound is stated more precisely here.

Theorem 5. *When n is not a power of $k+1$,*

$$B_k(n) \le \begin{cases} n(\tau+1) - (\frac{\tau^2+3\tau}{2}) - k^2(\frac{m^2-m}{2}) \\ \quad \textit{in case 1} \\ n(\tau+1) - (\frac{\tau^2+3\tau}{2}) - k^2(\frac{m^2-m-p^2-p}{2}) - kp\alpha_p \\ \quad \textit{in case 2} \\ n(\tau+1) - (\frac{\tau^2+3\tau}{2}) - k^2(\frac{m^2-m-p^2-p}{2}) - kp(\alpha_p+1) \\ \quad \textit{in case 3a} \\ n(\tau+1) - (\frac{\tau^2+3\tau}{2}) - k^2(\frac{m^2-m-p^2-p}{2}) \\ \quad -kp\alpha_p - \sum_{i=0}^{p-1} \alpha_i(k+1)^i + 1 \\ \quad \textit{in case 3b.} \end{cases}$$

In all cases, $m = \lceil \log_{k+1} n \rceil$.

Case 1 applies when $n = (kk...k)_{k+1}$. In case 1, $\tau = km$.

In the other cases, p is the position of the leftmost digit which is not k in the $(k+1)$-ary representation of the integer n and α_p is that digit.

Case 2 applies when $n = (kk...k\alpha_p 00...0)_{k+1}$ with $0 \le \alpha_p \le k-1$; here, $\tau = (m-p-1)k + \alpha_p$.

The remaining n are subdivided into Case 3a and Case 3b. In both these cases, $n = (kk...k\alpha_p\alpha_{p-1}...\alpha_0)_{k+1}$, with $0 \le \alpha_p \le k-1$ and $\sum_{i=0}^{p-1} \alpha_i > 0$, and $\tau = (m-p-1)k + \alpha_p + 1$. Case 3a applies when $\sum_{i=0}^{p-1} \alpha_i(k+1)^i \ge pk+1$ and case 3b applies when $\sum_{i=0}^{p-1} \alpha_i(k+1)^i < pk+1$.

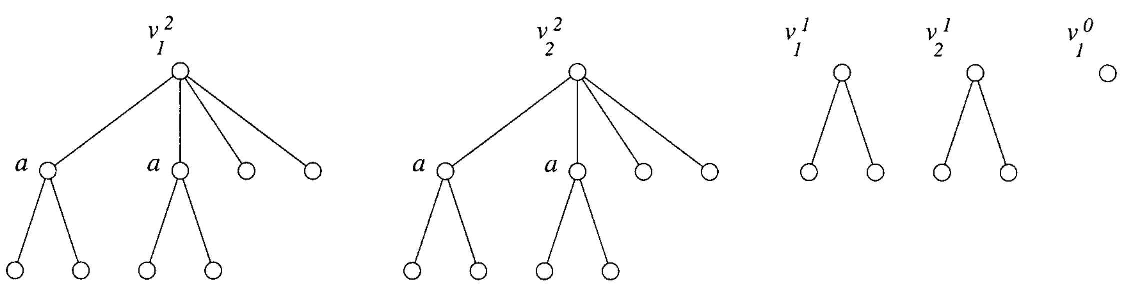

FIG. 4. Tree structure of G_{25}^2.

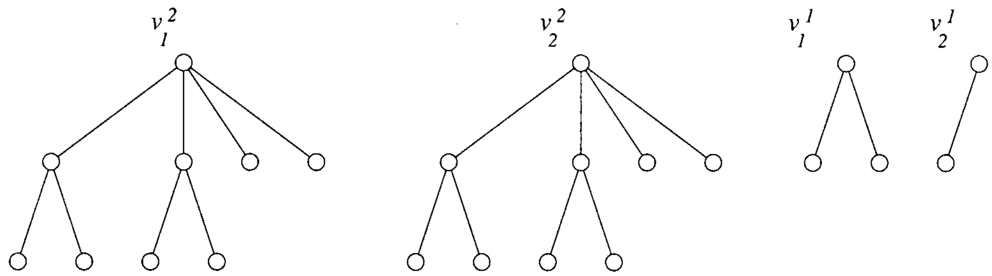

FIG. 5. Tree structure of G_{23}^2.

Proof. The construction of G_n^k was described for three cases. In each case, vertices (and edges) were removed from T_{k+1}^m, leaving n vertices in a forest of τ trees for some $\tau > 0$. Thus, $n - \tau$ edges of T_{k+1}^m remain in G_n^k. Subsequently, edges were added connecting each vertex to the remaining children of the deleted root v^m. The number of added edges is at most $n\tau$, giving an upper bound of $n - \tau + n\tau$ for $B_k(n)$. (This is the bound reported by Grigni and Peleg [5].) The $n\tau$ is a generous bound and, in particular, causes double counting of some existing edges. To count exactly the number of edges in G_n^k, we specify τ and are careful not to overcount the number of edges added.

When $n = (kk...k)_{k+1}$, $\tau = km$. In this case, $n\tau$ includes τ loops (one from each vertex v_j^i to itself), the $\binom{\tau}{2}$ edges among the v_j^i are counted twice, and for each v_j^i, the ik edges from v_j^i to its children are counted twice. Thus, the total number of edges is $n - \tau + n\tau - \tau - \binom{\tau}{2} - k\sum_{i=0}^{m-1} ik = n(\tau + 1) - \frac{\tau^2+3\tau}{2} - k^2(\frac{m^2-m}{2})$.

When $n = (kk...k\alpha_p 00...0)_{k+1}$ with $0 \le \alpha_p \le k-1$, $\tau = (m-p-1)k+\alpha_p$. Here, $n\tau$ includes τ loops (one from each vertex v_j^i to itself), the $\binom{\tau}{2}$ edges among the v_j^i are counted twice, and for each v_j^i, the ik edges from v_j^i to its children are counted twice. In this case, the number of edges is $n - \tau + n\tau - \tau - \binom{\tau}{2} - k\sum_{i=p+1}^{m-1} ik - \alpha_p pk = n(\tau + 1) - \frac{\tau^2+3\tau}{2} - k^2(\frac{m^2-m-p^2-p}{2}) - kp\alpha_p$.

When $n = (kk...k\alpha_p\alpha_{p-1}...\alpha_0)_{k+1}$, where $0 \le \alpha_p \le k-1$ and $\sum_{i=0}^{p-1}\alpha_i > 0$, $\tau = (m-p-1)k + \alpha_p + 1$. This case has one tree more than in the previous case and the formula produced for that case is correct except for double counting the edges between $v_{\alpha_p+1}^p$ and its children. To conclude this case, we need only count the number of children of $v_{\alpha_p+1}^p$ remaining in G_n^k. Let $R = \sum_{i=0}^{p-1}\alpha_i(k+1)^i$. R is the number of vertices in the subtree rooted at $v_{\alpha_p+1}^p$ which remain in G_n^k.

If $R \ge pk + 1$, then $v_{\alpha_p+1}^p$ has all of its pk children remaining in G_n^k and we have overcounted by pk. Otherwise, only $R - 1$ of these children remain and the overcounting is by $R - 1$. Thus, if $\sum_{i=0}^{p-1}\alpha_i(k+1)^i \ge pk + 1$, then the number of edges in G_n^k is $n(\tau+1) - \frac{\tau^2+3\tau}{2} - k^2(\frac{m^2-m-p^2-p}{2}) - kp(\alpha_p + 1)$, and if $\sum_{i=0}^{p-1}\alpha_i(k+1)^i \le pk$, then the number of edges in G_n^k is $n(\tau+1) - \frac{\tau^2+3\tau}{2} - -k^2(\frac{m^2-m-p^2-p}{2}) - kp\alpha_p - \sum_{i=0}^{p-1}\alpha_i(k+1)^i + 1$. □

5.2. *Improved Construction*

The construction of Section 5.1 includes some edges which are not used during the k-broadcast scheme described above. In particular, calls are never made on some of the edges added to T_{k+1}^m. By deleting these edges, we can obtain a smaller upper bound on $B_k(n)$.

Theorem 6. *The upper bound of Theorem 5 can be decreased by*

$$(k+1)^m - (1 + km + k^2(\tfrac{m^2-m}{2})) \quad \textit{in case 1}$$

$$(\alpha_p + 1)(k+1)^{m-p-1} - 1 - k(m-p-1) - k^2(\tfrac{m^2-3m+p^2+3p-2mp+2}{2}) - \alpha_p(k(m-p-1)+1) \quad \textit{in case 2}$$

$$(\alpha_p + 2)(k+1)^{m-p-1} - 1 - k(m-p-1) - k^2(\tfrac{m^2-3m+p^2+3p-2mp+2}{2}) - (\alpha_p+1)(k(m-p-1)+1) \quad \textit{in cases 3a and 3b.}$$

Proof. To improve on the upper bound in Theorem 5, we remove edges of G_n^k that connect a root v_j^i to a vertex u (not one of the $v_{j'}^{i'}$) that is the root of some induced subtree (of T_{k+1}^m) of height greater than i. These edges are not used in the k-broadcast scheme described above.

Let T^s be an induced subtree of height s rooted at one of the children $v_{j'}^s$ of the root v^m. The number of vertices of T^s that are roots of induced subtrees of height h, $i+1 \le h \le s-1$, is $k(k+1)^{s-h-1}$. Thus, we can remove

$\sum_{h=i+1}^{s-1} k(k+1)^{s-h-1} = (k+1)^{s-i-1} - 1$ edges between T^s and v_j^i from G_n^k.

We can remove edges from v_j^i to such vertices in each tree T^s rooted at $v_{j'}^s$, where $s > i$. Thus, in total, $D(i) = k\sum_{s=i+1}^{m-1}((k+1)^{s-i-1} - 1) = (k+1)^{m-i-1} - km + ki + k - 1$ edges incident on v_j^i can be deleted.

In Case 1, that is when $n = (kk...k)_{k+1}$, there are k vertices of the form v_j^i for each i, $0 \le i \le m-2$. (Note that no edges are deleted incident on any of the vertices v_j^{m-1}.) Summing over all i, we get the total number of deleted edges: $k\sum_{i=0}^{m-2} D(i) = (k+1)^m - (1 + km + k^2(\frac{m^2-m}{2}))$. Note that in the example of G_{26}^2 represented in Figure 3 the connections from v_1^0 and v_2^0 to the vertices labeled a are deleted, giving an improvement of eight edges in this case.

In Case 2, that is, when $n = (kk...k\alpha_p 00...0)_{k+1}$, where $0 \le \alpha_p \le k-1$, there are α_p vertices of the form v_j^p and k vertices of the form v_j^i for each i with $p+1 \le i \le m-2$. Again, we get the total number of deleted edges: $k\sum_{i=p+1}^{m-2} D(i) + \alpha_p D(p) = (\alpha_p+1)(k+1)^{m-p-1} - 1 - k(m-p-1) - k^2(\frac{m^2-3m+p^2+3p-2mp+2}{2}) - \alpha_p(k(m-p-1)+1)$. Note that in the example of G_{25}^2 represented in Figure 4 the connections from v_1^0 to the vertices labeled a are deleted, giving an improvement of four edges in this case.

In Case 3, that is, when $n = (kk...k\alpha_p\alpha_{p-1}...\alpha_0)_{k+1}$, where $0 \le \alpha_p \le k-1$ and $\sum_{i=0}^{p-1}\alpha_i > 0$, there are $\alpha_p + 1$ vertices of the form v_j^p and k vertices of the form v_j^i for each i with $p+1 \le i \le m-2$. Again, we get the total number of deleted edges: $k\sum_{i=p+1}^{m-2} D(i) + (\alpha_p + 1)D(p) = (\alpha_p+2)(k+1)^{m-p-1} - 1 - k(m-p-1) - k^2(\frac{m^2-3m+p^2+3p-2mp+2}{2}) - (\alpha_p+1)(k(m-p-1)+1)$. The example given in Figure 5 is too small to show any improvements. □

The new upper bound improves on the bound of Grigni and Peleg [5] for almost all n and k and on the bound of König and Lazard [8] when $p \ge \lfloor \frac{m}{2} \rfloor$. Thus, the new upper bound is the smallest current upper bound for many n and k. These new bounds do not, however, improve on the upper bounds reported by Lee and Ventura for $2 \le k \le 4$ and $n \le 128$. However, we note that the Grigni and Peleg upper bound (as stated more carefully in Theorem 5) is, in fact, better than are the values reported by Lee and Ventura for $k = 3$ in the range $41 \le n \le 48$ and for $k = 4$ in the range $56 \le n \le 100$.

6. CONSTRUCTION BY COMPOUNDING

One can construct k-broadcast graphs by forming the compound of two k-broadcast graphs. This method has proven effective for 1-broadcast graphs (see, e.g., [2]). In particular, the compound of graphs on n_1 and n_2 vertices is a graph on $n_1 n_2$ vertices.

To describe the compounding of two graphs, we will use the terminology of Bermond et al. [2]. For a k-broadcast scheme in a graph G on n vertices, we say that vertex u is *idle* at time $t \le \lceil \log_{k+1} n \rceil$ if and only if u is aware of the message at time unit t and u does not send the message to any of its neighbors during time unit t.

A subset C of vertices of a given k-broadcast graph G is a *solid-1 cover* if and only if C is a vertex cover of G, and for each $u \notin C$, there exists a k-broadcast scheme for u such that at least one neighbor of u is idle at some time during the k-broadcast.

If $G = (V(G), E(G))$ and $H = (V(H), E(H))$ are graphs, then the *compound of G into H relative to a set $S \subseteq V(G)$*, denoted $G_S[H]$, is obtained by replacing each vertex x of H by a graph G_x isomorphic to G and adding a matching between the two sets S_x and S_y if x and y are adjacent in H. This matching between S_x and S_y connects each vertex in S_x with its copy in S_y. For any vertex $u \in S$, we use H_u to denote the graph isomorphic to H which interconnects the copies of $u \in V(G)$.

The following theorem is a simple generalization of Theorem 2 of Bermond et al. [2] for any k and we omit the proof.

Theorem 7. *For $k \ge 1$, let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be k-broadcast graphs on n_1 and n_2 vertices, respectively, and let C be a solid 1-cover of G. If $\lceil \log_{k+1} n_1 n_2 \rceil = \lceil \log_{k+1} n_1 \rceil + \lceil \log_{k+1} n_2 \rceil$, then $G_C[H]$ is a k-broadcast graph and $B_k(n_1 n_2) \le n_2|E(G)| + |C| \cdot |E(H)|$.*

A similar result was recently noted independently by Lee and Ventura [10]. Their paper included several specific upper bounds obtained using compounding.

7. SUMMARY

When $k > 1$ and n is large, the exact value of $B_k(n)$ is known only for $n = (k+1)^m$. In particular, Lazard [9] showed that $B_k([k+1]^m) = \frac{km(k+1)^m}{2}$. For small n and k, some specific values of $B_k(n)$ are known (see [8];[9];[10]). Various upper bounds on $B_k(n)$ have been shown by constructing k-broadcast graphs. König and Lazard [8] showed that $B_k(n) \le \frac{nk}{2}\lceil \log_{k+1} n \rceil$. Grigni and Peleg [5] showed that $B_k(n) \in \Theta(kL_k(n)n)$, where $L_k(n)$ denotes the exact number of consecutive leading k's in the $(k+1)$-ary representation of $n-1$.

In Section 2, we showed that $B_3(11) = 18$ by constructing a minimum 3-broadcast graph on 11 vertices. In Section 3, we improved the existing lower bounds [5] for almost all n and k. In particular, by combining this new lower bound with an upper bound from Lee and Ventura [10], we obtained $B_2(24) = 48$. In Section

4, we showed that (Δ, D) graphs can be used to produce upper bounds on $B_k(n)$. In addition, by using (Δ, D) graphs, we were able to show that $B_3(10) = 15$ and that $B_3(50) = 175$. In Section 5, we gave a more precise statement of Grigni and Peleg's upper bound [5] and then improved this bound by constructing a k-broadcast graph on n vertices with fewer edges. For some small n and k, the more precisely stated version of Grigni and Peleg's bound is better than are the upper bounds reported by Lee and Ventura [10]. For most large n and k, our improved bound is currently the best upper bound. Finally, in Section 6, we showed that the method of compounding can be used to construct graphs which provide upper bounds on $B_k(n)$.

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