Improved Lower Bound on Broadcast Function Based on Graph Partition

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Abstract. Broadcasting is a one-to-all information spreading process in a communication network. The network is modeled as a graph. The broadcast time of a given vertex is the minimum time required to broadcast a message from that vertex to all vertices of the graph. The broadcast time of a graph is the maximum time required to broadcast from any vertex in the graph. A graph G on n vertices is a minimum broadcast graph if the broadcast time of G is the minimum possible time: $\lceil \log n \rceil$, and the number of edges in G is minimized. The broadcast function $B(n)$ denotes the number of edges in a minimum broadcast graph on n vertices. The exact value of $B(n)$ is only known for $n = 2^m$, $2^m - 2$, and some small values of $n < 64$. Finding $B(n)$ is very difficult due to the lack of tight lower bounds on $B(n)$. The existing lower bounds are based on the vertex degree of the originator vertex. However, most of the minimum broadcast graphs are not necessarily regular. In this paper, we present an improved general lower bound on $B(n)$ based on new observations about partitioning broadcast graphs.

1 Introduction

One-to-all information dissemination problem, named *broadcasting*, is one of the essential tasks in computer networks. The performance of broadcasting usually measures the overall efficiency of the network. To study the properties of broadcasting and the topology of a network, the network is defined as a simple connected graph $G = (V, E)$, where the vertex set V represents the nodes in the network, and the edge set E represents the communication links. A broadcast in a graph originates from one vertex, called the *originator*, and finishes when every vertex in the graph is informed.

In past decades, a long sequence of research papers study this topic under different models. These models differ at the number of originators, the number of destinations, the lifetime of each sender, the specific topologies of the graphs, the number of receivers at each time unit, the distance of each call, and other characteristics. In this paper, we focus on the classical model with the following assumptions.

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- the graph has only one originator;
- every call is a fundamental process of broadcasting and requires one time unit;
- every call is from exactly one informed vertex, the sender to one of its uninformed neighbors, the receiver.

With the assumptions above, we have the following formal definitions.

Definition 1. *A broadcast scheme is a sequence of parallel calls in a graph G originating from a vertex v. Each call, represented by a directed edge, defines a sender and a receiver. A broadcast scheme generates a broadcast tree, which is a spanning tree of the graph rooted at the originator.*

Definition 2. *Let G be a graph on n vertices and v be the broadcast originator in graph G, the broadcast time of vertex v, $b(G, v)$ defines the minimum number of time units required to broadcast from originator v in graph G. The broadcast time of graph G $b(G) = max\{b(G, v)|v \in V(G)\}$ defines the maximum of all broadcast times of any vertex in graph G.*

Since a sender can inform at most one receiver node during one time unit, the number of informed vertices at each time unit is at most doubled. Thus, $b(G) \geq \lceil \log_2 n \rceil = \lceil \log n \rceil$. For convenience, we will omit the base of all logarithms throughout this paper when the base is 2.

Definition 3. *A graph G on n vertices is a broadcast graph if $b(G) = \lceil \log n \rceil$. A broadcast graph with the minimum number of edges is a minimum broadcast graph. This minimum number of edges $B(n)$ is called the broadcast function.*

From the applications perspective minimum broadcast graphs are the cheapest graphs (with minimum number of edges) where broadcasting can be accomplished in minimum possible time.

The study of minimum broadcast graphs and broadcast function $B(n)$ has a long history. In [5], Farley et al. introduced minimum broadcast graphs, defined the broadcast function, determined the values of $B(n)$, for $n \leq 15$ and $n = 2^k$ and proved that hypercubes are minimum broadcast graphs. Khachatrian and Haroutunian [19] and independently Dinneen et al. [4] show that Knödel graphs, defined in [20], are minimum broadcast graphs on $n = 2^k - 2$ vertices. Park and Chwa prove that the recursive circulant graphs on 2^k vertices are minimum broadcast graphs [25]. The comparison of the three classes of minimum broadcast graphs mentioned above can be found in [6]. Besides these three classes, there is no other known infinite construction of minimum broadcast graphs. The values of $B(n)$ are also known for $n = 17$ [24], $n = 18, 19$ [3,30], $n = 20, 21, 22$ [23], $n = 26$ [26,31], $n = 27, 28, 29, 58, 61$ [26], $n = 30, 31$ [3], $n = 63$ [22], $n = 127$ [11] and $n = 1023, 4095$ [27].

By the difficulty of constructing minimum broadcast graphs, a long sequence of papers present different techniques to construct broadcast graphs in order to obtain upper bounds on $B(n)$. Furthermore, proving that a lower bound matches the upper bound is also extremely difficult, because most of the lower bound proofs are based on vertex degrees. However, minimum broadcast graphs except hypercubes and Knödel graphs on $2^k - 2$ vertices are not regular. Thus, in general upper bounds cannot match lower bounds.

Upper bounds on $B(n)$ are given by constructing sparse broadcast graphs. General Knödel graphs on even number of vertices give a good bound on $B(n)$ for even n. Compounding two broadcast graphs of smaller sizes is a powerful method to construct good broadcast graphs with larger size. In [1,12], compounding binomial trees, hypercubes, and Knödel graphs improves the upper bound on $B(n)$ for $2^{m-1}+1 \leq n \leq 2^m - 2^{\frac{m}{2}}$ if the value of n is separated by intervals $[2^m - 2^{m-1}+1, 2^m]$. The vertex addition method is also used in the constructions. In [13], authors add one vertex to Knödel graphs and improve the upper bound on $B(n)$ for even $2^m - 2^{\frac{m}{2}} + 1 \leq n \leq 2^m$. Ad hoc constructions sometimes also provides good upper bounds. This method usually constructs broadcast graphs by adding edges to a binomial tree [9,14]. The vertex deletion is studied in [3]. Several other constructions are presented in [3,8,9,14,28–30].

Lower bounds on $B(n)$ are also studied in the literature. In [8], Gargano and Vaccaro show $B(n) \geq \frac{n}{2}(\lfloor \log n \rfloor - \log(1 + 2^{\lceil \log n \rceil} - n))$, for any n. $B(n) \geq \frac{n}{2}(m-p-1)$ is proved in [21], where m is the length of the binary representation $a_{m-1}a_{m-2}...a_1a_0$ of n and p is the index of the leftmost 0 bit. Harutyunyan and Liestman study k-broadcasting (every sender can inform at most k neighbors in each time unit) and give a lower bound on k-broadcast graph in [15]. The latter bound is the best known general lower bound for our model of broadcasting (which corresponds to the case $k = 1$ in [15]). Below we summarize this lower bound for our model.

Theorem 1. *Let $n = 2^m - 2^k + 1 - d$, $1 \leq k \leq m-2$ and $0 \leq d \leq 2^k - 1$.*

$$B(n) \geq \frac{n}{2}(m-k)$$

Besides the general lower bounds, Labahn shows $B(n) \geq \frac{m^2(2^m-1)}{2(m+1)}$ for $n = 2^m - 1$ in [22] by considering the broadcast tree rooted at a vertex with the minimum degree. Saclé follows this method and gives tight lower bounds on $B(2^m-3)$, $B(2^m-4)$, $B(2^m-5)$ and $B(2^m-6)$ in [26]. Grigoryan and Harutyunyan show a better lower bound for $n = 2^m - 2^k + 1$.

Theorem 2. [10] $B(2^m - 2^k + 1) \geq \frac{2^m - 2^k + 1}{2}(m - k + \frac{m(2k-1)-(k^2+k-1)}{m(m-1)-(k-1)})$

Better lower bounds for $n = 24, 25$ are given in [2]. Note that $23 \leq n \leq 25$ are the only values of $n \leq 32$ for which $B(n)$ is not known. For more on broadcasting and gossiping in general see the following survey papers [7,16–18]. This paper is organized as follows. Section 2 introduces some important definitions and presents three useful observations. Section 3 gives a general lower bound on $B(n)$ by using the observations. Section 4 give conclusions and future work.

2 Definitions and Observations

Definition 4. *A binomial tree BT_m on 2^m vertices of order m consists of*

1. *a single vertex which is also the root, if $m = 1$;*
2. *two copies of binomial trees BT_{m-1} having the two roots connected by an edge, if $m > 1$.*

Definition 5. *Let BT_m and BT_k are two binomial trees of order m and k respectively, and $m > k$. u is the root of BT_m. $BT_m \backslash BT_k$ is a tree obtained by removing a complete binomial tree BT_k from u in BT_m except the root u.*

Figure 1 gives an example of a binomial tree and $BT_m \setminus BT_k$ for $m = 5$ and $k = 3$.

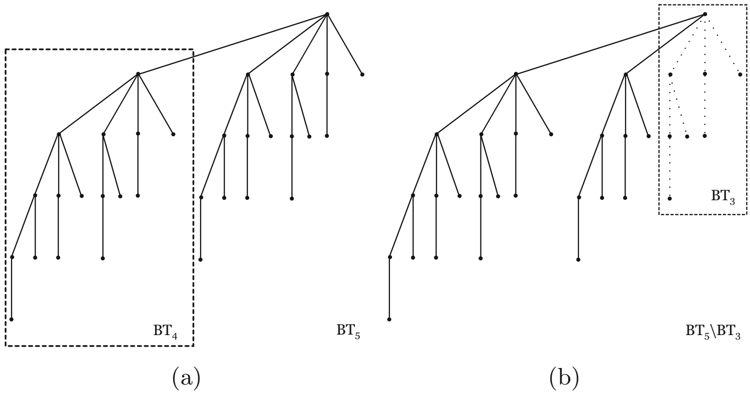

Fig. 1. (a) is an example of a binomial tree BT_5. (b) the solid edges and the associated vertices give an example of $BT_5 \setminus BT_3$.

Definition 6. *Let T be a broadcast tree of graph G originating from root u. Then $L_k(T)$, the first k broadcast level tree of T, consists of all the vertices of T which are informed in the first k time units following the broadcast scheme from originator vertex u in graph G.*

We know that any broadcast tree of graph G on n vertices is a subtree of a binomial tree $BT_{\lceil \log n \rceil}$. So, the first k broadcast level tree $L_k(T)$ is a subtree of a binomial tree BT_k. Figure 2 gives one example of a broadcast tree BT_4 and its first 3 broadcast level tree.

Let G be a minimum broadcast graph on $n = 2^m - 2^k + 1$ vertices, where $1 \le k \le m - 2$; u be a vertex of degree $m - k$ in G; T be the broadcast tree rooted at vertex u; and $L_k(T)$ be the first k broadcast level tree of T. If the neighbors of u are sorted in the decreasing order of their degrees and the i-th neighbor corresponds to the i-th branch, we have the following observations.

Observation 1. *$BT_m \setminus BT_k$ is a broadcast tree T of broadcast graph G on $n = 2^m - 2^k + 1$ vertices rooted at a vertex u of degree $m - k$.*

Proof. Graph G has $2^m - 2^k + 1$ vertices, so the broadcasting must be completed in m time units. It is clear that during this m time broadcasting from originator u in $BT_m \setminus BT_k$ there are no idle vertices (informed vertices but not transferring the message). Thus, branches of the root u are complete binomial trees BT_{m-1},

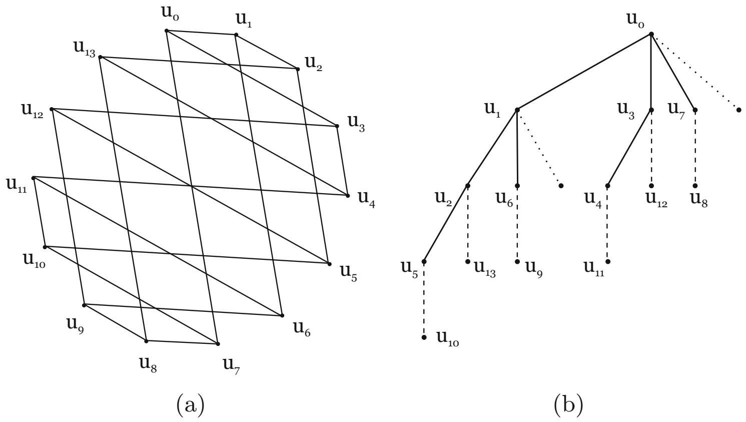

Fig. 2. (a) is a broadcast graph G on 14 vertices. (b) is a binomial tree BT_4 on 16 vertices. 14 vertices with labels among them together with the solid and the dashed edges give a broadcast tree T of G. And the solid edges form a first 3 broadcast level tree $L_3(T)$.

$BT_{m-2}, \cdots, BT_k$. There are in total $2^{m-1} + \cdots + 2^k + 1 = 2^m - 2^k + 1$ vertices, which is exactly the same as the number of vertices in G. Thus, the broadcast tree has to be $BT_m \setminus BT_k$.

Observation 2. *If $k \geq \frac{m}{2}$, the i-th branch of u has 2^{k-i} vertices in $L_k(T)$.*

Proof. If we ignore the first level (only one vertex: the root u), broadcast tree T becomes a forest of binomial trees BT_{m-1}, BT_{m-2}, $\cdots$, BT_k. So, the first k level broadcast tree $L_k(T)$ of T consists of the first $k-1$ level broadcast tree $L_{k-1}(BT_{m-1})$ of the first branch, $L_{k-2}(BT_{m-2})$ of the second branch, and $L_{k-i}(BT_{m-i})$ of the i-th branch in general. If $k \geq \frac{m}{2}$, then the last neighbor is informed at time unit $m-k \leq k$. Thus, $L_{2k-m}(BT_{m-k})$ is a complete binomial tree BT_{2k-m}. Then, each of $L_{k-i}(BT_{m-i})$ becomes a complete binomial tree BT_{k-i}. So, there are 2^{k-i} vertices on the i-th branch.

Observation 3. *If a vertex w is in $L_k(T)$, then the corresponding vertex in broadcast tree T has degree greater than $m-k$.*

Proof. Observation 1 ensures that on i-th branch, BT_{m-i} is a complete binomial tree. So, $L_{k-i}(BT_{m-i})$ is indeed a complete binomial tree BT_{k-i} of order $k-i$, and it can be obtained by replacing every vertex in BT_{k-i} by a binomial tree BT_{m-k}. Thus, if a vertex w in $L_{k-i}(BT_{m-i})$ (which is BT_{m-k}) has degree a, then vertex w has degree $a+m-k$ in BT_{m-i} (also in broadcast tree T). Every leaf in any tree has the minimum degree 1. Therefore, any leaf in $L_k(T)$ gives the minimum degree $m-k+1 > m-k$ in broadcast tree T. Figure 3 shows an example of broadcast tree T when $k \geq \frac{m}{2}$.

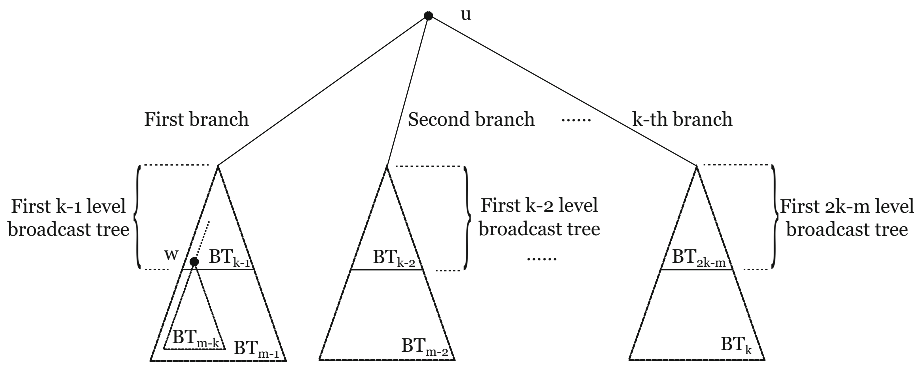

Fig. 3. An example of a broadcast tree rooted at vertex u of degree $m-k$. The triangle at the i-th branch is a binomial tree BT_{m-i}. The smaller triangle is the first $k-i$ broadcast level tree $L_{k-i}(BT_{m-i})$. And leaf w is an example of a vertex in $L_k(T)$. w has degree 1 in $L_k(T)$ and degree $m-k$ in $T-L_k(T)$. So, the degree is $m-k+1$ in total.

3 New Lower Bound

In this section, we first give a lower bound on $B(n)$ when $n = 2^m - 2^k + 1 - d$, where $\frac{m}{2} \leq k \leq m-2$ and $d = 0$. Then, we generalize the lower bound for any $0 \leq d \leq 2^k - 1$. That is we give a lower bound on $B(n)$ for all $2^{m-1} + 1 \leq n \leq 2^m - 2^{\frac{m}{2}+1} + 1$.

Theorem 3. *If* $n = 2^m - 2^k + 1$ *and* $\frac{m}{2} \leq k \leq m-2$,

$$B(n) \geq \frac{n}{2}(m-k+\frac{1}{2}-\frac{1}{4m-4k+2}) + \frac{2^{k+1}-2^{2k-m+1}-m+k}{2m-2k+1}$$

Proof. Observation 1 shows that the minimum degree of any vertex in G is $m-k$. So, we partition the vertices of G into V_{m-k}, the vertices of degree $m-k$; and V_{other}, other vertices. We also partition the edges into E_{m-k}, the edges connecting two vertices in V_{m-k}; E_{inter}, the edges connecting one vertex in V_{m-k} and one vertex in V_{other}; and E_{other}, the edges connecting two vertices in V_{other}. Let v_{m-k}, v_{other}, e_{m-k}, e_{inter}, and e_{other} be the cardinality of each of the respective sets. It is easy to see $n = v_{m-k} + v_{other}$ and $e = e_{m-k} + e_{inter} + e_{other}$.

Case 1. If there is no vertex of degree $m-k$ in graph G, then the minimum degree is $m-k+1$, we have

$$e \geq \frac{n}{2}(m-k+1) \tag{1}$$

Case 2. If there is a vertex of degree $m-k$ in graph G, we consider the broadcast tree T originating from such a vertex u. In order to inform all vertices in graph

G within m time units, every vertex except originator u cannot be idle during the minimum time broadcasting in G. So, the vertices informed by u (also the neighbors of u in broadcast tree T) must have degree m, $m-1$, ..., $k+1$. In other words the broadcast tree of originator u must be $BT_m \setminus BT_k$.

Since $k \geq \frac{m}{2}$, then the last neighbor of u has degree $k+1 > m-k$. Thus, there is no vertex of degree $m-k$ having a neighbor of degree $m-k$. Furthermore, if an edge is attached to a vertex of degree $m-k$, then it must be attached to a vertex of degree at least $m-k+1$.

$$e_{m-k} = 0$$

$$e_{inter} = (m-k)v_{m-k}$$

Again we consider the broadcast tree T and estimate e_{other}. By Observation 3, every vertex in the first k broadcast level tree $L_k(T)$ except the root u has degree greater than $m-k$. Thus, every edge except the ones on the first level in $L_k(T)$ has both of its endpoints of degree greater than $m-k$. And by Observation 2, $L_k(T)$ becomes a forest of $L_{k-1}(BT_{m-1})$, $\cdots$, $L_{2k-m}(BT_{m-k})$ by ignoring the root and its incident edges. Then, e_{other} can be estimated by counting the number of the edges in the forest. Therefore,

$$e_{other} \geq 2^{k+1} - 2^{2k-m+1} - (m-k)$$

Combining e_{m-k}, e_{inter}, and e_{other},

$$\begin{aligned}
e &= e_{m-k} + e_{inter} + e_{other} \\
e &\geq (m-k)v_{m-k} + 2^{k+1} - 2^{2k-m+1} - (m-k) \\
v_{m-k} &\leq \frac{e - 2^{k+1} - 2^{2k-m+1} - (m-k)}{m-k} \\
n - v_{m-k} &\geq n - \frac{e - 2^{k+1} + 2^{2k-m+1} + (m-k)}{m-k} \\
v_m + \cdots + v_{m-k+1} &\geq n - \frac{e - 2^{k+1} + 2^{2k-m+1} + (m-k)}{m-k} \qquad (2)
\end{aligned}$$

We have the following trivial inequalities.

$$\begin{aligned}
2e &\geq (m-k)v_{m-k} + \cdots + mv_m \\
2e &\geq (m-k)n + v_{m-k+1} + 2v_{m-k+2} + \cdots + kv_m \\
2e &\geq (m-k)n + v_{m-k+1} + v_{m-k+2} + \cdots + v_m \qquad (3)
\end{aligned}$$

By substituting inequality (2) we get

$$\begin{aligned}
2e &\geq (m-k)n + n - \frac{e - 2^{k+1} + 2^{2k-m+1} + (m-k)}{m-k} \\
e &\geq \frac{n}{2}(m - k + \frac{1}{2} - \frac{1}{4m-4k+2}) \\
&\quad + \frac{2^{k+1} + 2^{2k-m+1} + (m-k)}{2m-2k+1} \qquad (4)
\end{aligned}$$

Now we combine inequality (1) and inequality (4) given by the two different cases. Let RHS_1 and RHS_2 be the right hand side of the two inequalities respectively.

$$\begin{aligned}
RHS_1 - RHS_2 &= \frac{m-k+1}{2(2m-2k+1)}n - \frac{2^{k+1} - 2^{2k-m+1} - m + k}{2m-2k+1} \\
&= \frac{1}{2(2m-2k+1)}((m-k+1)(2^m - 2^k + 1) \\
&\quad - (2^{k+2} - 2^{2k-m+2} - 2m + 2k)) \\
&\geq \frac{1}{2(2m-2k+1)}(3(2^{k+2} - 2^k + 1) \\
&\quad - (2^{k+2} - 2^{2k-m+2} - 2m + 2k)) \\
&= \frac{1}{2(2m-2k+1)}((2^{k+3} + 2^{k+1} + 3) \\
&\quad - (2^{k+2} - 2^{2k-m+2} - 2m + 2k)) \\
&> 0
\end{aligned}$$

Thus, inequality (4) is the worst case and gives the lower bound, which completes the proof. □

By simple comparison, we can see that Theorem 3 does not give a better bound than Theorem 2. But Theorem 3 can be generalized to other n.

Theorem 4.

$$B(n) \geq \frac{n}{2}(m - k + \frac{1}{2} + \frac{\alpha - 1}{4m - 4k - 2\alpha + 2}) + \frac{2^{k+1} - 2^{2k-m+1} - m + k - d}{2m - 2k + 1}$$

where

$$\alpha = \lfloor \frac{-W_{-1}(-2^{-d-2^{2k-m+1}+2k-m+1}\ln(2))}{\ln(2)} \rfloor - d - 2^{2k-m+1}$$

and $W_{-1}(x)$ is the lower branch of Lambert-W function.

Proof. Observation 1 is not true for general n, but in this case the minimum degree is always $m-k$. Assume a vertex r has degree $m-k-1$ in a broadcast graph G on $n = 2^m - 2^k + 1 - d$ vertices, where $0 \leq d \leq 2^k - 1$. Minimum time broadcasting from originator r informs at most 2^{m-1} vertices on the first branch, 2^{m-2} vertices on the second branch, $\cdots$, and 2^{k+1} vertices on the last branch. Together with the originator r, there are $2^m - 2^{k+1} + 1$ vertices in total, which is $2^m - 2^{k+1} + 1 > 2^m - 2^{k+1} \geq 2^m - 2^k + 1 - d$. Thus, the minimum degree has to be $m-k$. Then, we have the two cases similar to Theorem 3.

Case 1. If the minimum degree is greater than $m-k$, then

$$e \geq \frac{n}{2}(m-k+1) \tag{5}$$

Case 2. If the minimum degree is $m - k$, we again have e_{m-k}, e_{inter}, and e_{other} indicating the cardinality of the different edge set as in the proof of Theorem 3. However, the value of e_{m-k}, e_{inter}, and e_{other} are different after removing d vertices.

Let u be a vertex of degree $m - k$. Assume α neighbors of u become degree $m - k$ after removing d vertices.

$$\begin{aligned} e_{m-k} + e_{inter} &\geq \frac{1}{2}\alpha v_{m-k} + (m - k - \alpha)v_{m-k} \\ &= \frac{1}{2}(2m - 2k - \alpha)v_{m-k} \end{aligned}$$

$e_{m-k} + e_{inter}$ is minimized when α is maximized, which is the worst case for the lower bound. Consider the broadcast tree T rooted at vertex u. The neighbors $s_1, s_2, \cdots, s_{m-k}$ of u have degree $m, m-1, \cdots, k+1$ respectively. And s_i is the root of a binomial tree BT_{m-i}. To maximize α, we remove vertices and make neighbors of u of degree $m - k$ from s_{m-k} to s_1, because the last neighbor s_{m-k} has the smallest degree. So, $2k - m + 1$ neighbors of s_{m-k} are removed. $2^{2k-m+1} - 1$ vertices are removed from the last branch. And the binomial tree BT_k attached to s_{m-k} becomes $BT_k \setminus BT_{2k-m+1}$.

In general, to make s_i of degree $m - k$, $2^{k-i+1} - 1$ vertices are removed from the i-th branch. Thus, if α neighbors of u are of degree $m-k$, we need to remove $2^{2k-m+1} - 1 + 2^{2k-m+2} - 1 + \cdots + 2^{2k-m+\alpha} - 1 = 2^{2k-m+\alpha+1} - 2^{2k-m+1} - \alpha$ vertices from broadcast tree T. Since the number of removed vertices cannot exceed d, we have the following inequality:

$$\begin{aligned} d &\geq 2^{2k-m+\alpha+1} - 2^{2k-m+1} - \alpha \\ 2^{2k-m+1}2^{\alpha} &\leq d + \alpha + 2^{2k-m+1} \\ 2^{\alpha} &\leq 2^{2k-m+1}\alpha + 2^{-(2k-m+1)}d + 1 \end{aligned}$$

Let $\alpha = -x - d - 2^{2k-m+1}$

$$\begin{aligned} 2^{-x-d-2^{2k-m+1}} &\leq -x2^{-(2k-m+1)} \\ -2^{-2^{2k-m+1}-d+2k-m+1} &\geq x2^{x} \\ -2^{-2^{2k-m+1}-d+2k-m+1}\ln(2) &\geq x\ln(2)e^{x\ln(2)} \end{aligned} \tag{6}$$

The right hand side of inequality (6) has the form $z \cdot e^z$. It can be solved by Lambert-W function $W(z \cdot e^z) = z$. However, $W(z)$ is a multivalued relation. $W(z)$ increases when $z \geq -\frac{1}{e}$ and $W(z) \geq -1$; while it decreases when $-\frac{1}{e} \leq z < 0$ and $W(z) \leq -1$. Let $W_0(z)$ and $W_{-1}(z)$ define the two single-valued function for the two different branches of $W(z)$ respectively. We need to estimate the value of $x\ln(2)$ to decide which single-valued function is used. We know that $\alpha \geq 0$, $0 \leq d \leq 2^k - 1$, and $\frac{m}{2} \leq k \leq m - 2$.

$$
\begin{aligned}
-x - d - 2^{2k-m+1} &\geq 0 \\
-x - 2^{2k-m+1} &\geq 0 \\
-x &\geq 2 \\
x\ln(2) &< -1
\end{aligned}
$$

Thus, $W_{-1}(z)$ is used.

$$W_{-1}(-2^{-2^{2k-m+1}-d+2k-m+1}\ln(2)) \leq x\ln(2)$$

Solve α by substitution.

$$\alpha \leq -\frac{W_{-1}(-2^{-2^{2k-m+1}-d+2k-m+1}\ln(2))}{\ln(2)} - d - 2^{2k-m+1}$$

Since α is an integer,

$$\alpha = \lfloor -\frac{W_{-1}(-2^{-2^{2k-m+1}-d+2k-m+1}\ln(2))}{\ln(2)} \rfloor - d - 2^{2k-m+1}$$

e_{other} is analyzed as in the proof of Theorem 3 by counting the number of vertices in the first k broadcast level tree $L_k(T)$. If all the removed d vertices are in $L_k(T)$, then we have a trivial bound as follows.

$$e_{other} \geq 2^{k+1} - 2^{2k-m+1} - (m-k) - d$$

Therefore, we have the following inequality

$$e \geq \frac{1}{2}(2m - 2k - \alpha)v_{m-k} + 2^{k+1} - 2^{2k-m+1} - (m-k) - d$$

After reformatting,

$$v_m + \cdots + v_{m-k+1} \geq n - \frac{2e - 2^{k+1} + 2^{2k-m+1} + (m-k) + d}{2m - 2k - \alpha}$$

Then, by substituting the inequality to $2e \geq (m-k)v_{m-k} + \cdots + mv_m$ and by the similar technique given in the proof of Theorem 3,

$$e \geq \frac{n}{2}(m-k+1)\frac{2m-2k-\alpha}{2m-2k-\alpha+1} + \frac{2^{k+1} - 2^{2k-m+1} - m + k - d}{2m-2k+1} \tag{7}$$

Again by the similar comparison, we can see that this bound is worse than bound 5 given in the first case. Thus, inequality (7) is the general lower bound on broadcast function, which completes the proof. □

4 Conclusion

Simple comparison shows that Theorem 4 always gives a better bound than Theorem 1 when $d \geq 0$. In the future, we can further improve this bound, because inequality (3) unifies the coefficients of $v_{m-k+1}, \cdots v_m$ to be 1. This step over shrinks the lower bound. So, the gap between the current bound and the optimal bound may not be small.

Another work can be done in the future is a further generalization. Since Theorem 4 restricts n in the range $2^{m-1}+1 \leq n \leq 2^m - 2^{\frac{m}{2}+1} + 1$, we can further explore the lower bound for the other side of interval $[2^{m-1}+1, 2^m]$.

References

1. Averbuch, A., Shabtai, R.H., Roditty, Y.: Efficient construction of broadcast graphs. Discrete Appl. Math. **171**, 9–14 (2014)
2. Barsky, G., Grigoryan, H., Harutyunyan, H.A.: Tight lower bounds on broadcast function for n = 24 and 25. Discrete Appl. Math. **175**, 109–114 (2014)
3. Bermond, J.-C., Hell, P., Liestman, A.L., Peters, J.G.: Sparse broadcast graphs. Discrete Appl. Math. **36**, 97–130 (1992)
4. Dinneen, M.J., Fellows, M.R., Faber, V.: Algebraic constructions of efficient broadcast networks. In: Mattson, H.F., Mora, T., Rao, T.R.N. (eds.) AAECC 1991. LNCS, vol. 539, pp. 152–158. Springer, Heidelberg (1991). https://doi.org/10.1007/3-540-54522-0_104
5. Farley, A.M., Hedetniemi, S., Mitchell, S., Proskurowski, A.: Minimum broadcast graphs. Discrete Math. **25**, 189–193 (1979)
6. Fertin, G., Raspaud, A.: A survey on Knödel graphs. Discrete Appl. Math. **137**, 173–195 (2004)
7. Fraigniaud, P., Lazard, E.: Methods and problems of communication in usual networks. Discrete Appl. Math. **53**, 79–133 (1994)
8. Gargano, L., Vaccaro, U.: On the construction of minimal broadcast networks. Networks **19**, 673–689 (1989)
9. Grigni, M., Peleg, D.: Tight bounds on mimimum broadcast networks. SIAM J. Discrete Math. **4**, 207–222 (1991)
10. Grigoryan, H., Harutyunyan, H.A.: New lower bounds on broadcast function. In: Gu, Q., Hell, P., Yang, B. (eds.) AAIM 2014. LNCS, vol. 8546, pp. 174–184. Springer, Cham (2014). https://doi.org/10.1007/978-3-319-07956-1_16
11. Harutyunyan, H.A.: An efficient vertex addition method for broadcast networks. Internet Math. **5**(3), 197–211 (2009)
12. Harutyunyan, H.A., Li, Z.: A new construction of broadcast graphs. In: Govindarajan, S., Maheshwari, A. (eds.) CALDAM 2016. LNCS, vol. 9602, pp. 201–211. Springer, Cham (2016). https://doi.org/10.1007/978-3-319-29221-2_17
13. Harutyunyan, H.A., Li, Z.: Broadcast graphs using new dimensional broadcast schemes for Knödel graphs. In: Gaur, D., Narayanaswamy, N.S. (eds.) CALDAM 2017. LNCS, vol. 10156, pp. 193–204. Springer, Cham (2017). https://doi.org/10.1007/978-3-319-53007-9_18
14. Harutyunyan, H.A., Liestman, A.L.: More broadcast graphs. Discrete Appl. Math. **98**, 81–102 (1999)
15. Harutyunyan, H.A., Liestman, A.L.: Improved upper and lower bounds for k-broadcasting. Networks **37**, 94–101 (2001)
16. Harutyunyan, H.A., Liestman, A.L., Peters, J.G., Richards, D.: Broadcasting and gossiping. In: Handbook of Graph Theorey, pp. 1477–1494. Chapman and Hall, New York (2013)

17. Hedetniemi, S.M., Hedetniemi, S.T., Liestman, A.L.: A survey of gossiping and broadcasting in communication networks. Networks **18**, 319–349 (1988)
18. Hromkovič, J., Klasing, R., Monien, B., Peine, R.: Dissemination of information in interconnection networks (broadcasting & gossiping). In: Du, D.Z., Hsu, D.F. (eds.) Combinatorial Network Theory. Applied Optimization, vol. 1, pp. 125–212. Springer, Boston (1996). https://doi.org/10.1007/978-1-4757-2491-2_5
19. Khachatrian, L.H., Harutounian, H.S.: Construction of new classes of minimal broadcast networks. In: Conference on Coding Theory, Dilijan, Armenia, pp. 69–77 (1990)
20. Knödel, W.: New gossips and telephones. Discrete Math. **13**, 95 (1975)
21. König, J.-C., Lazard, E.: Minimum k-broadcast graphs. Discrete Appl. Math. **53**, 199–209 (1994)
22. Labahn, R.: A minimum broadcast graph on 63 vertices. Discrete Appl. Math. **53**, 247–250 (1994)
23. Maheo, M., Saclé, J.-F.: Some minimum broadcast graphs. Discrete Appl. Math. **53**, 275–285 (1994)
24. Mitchell, S., Hedetniemi, S.: A census of minimum broadcast graphs. J. Combin. Inform. Syst. Sci **5**, 141–151 (1980)
25. Park, J.-H., Chwa, K.-Y.: Recursive circulant: a new topology for multicomputer networks. In: International Symposium on Parallel Architectures, Algorithms and Networks, ISPAN 1994, pp. 73–80. IEEE (1994)
26. Saclé, J.-F.: Lower bounds for the size in four families of minimum broadcast graphs. Discrete Math. **150**, 359–369 (1996)
27. Shao, B.: On K-broadcasting in graphs. Ph.D. thesis, Concordia University (2006)
28. Ventura, J.A., Weng, X.: A new method for constructing minimal broadcast networks. Networks **23**, 481–497 (1993)
29. Weng, M.X., Ventura, J.A.: A doubling procedure for constructing minimal broadcast networks. Telecommun. Syst. **3**, 259–293 (1994)
30. Xiao, J., Wang, X.: A research on minimum broadcast graphs. Chin. J. Comput. **11**, 99–105 (1988)
31. Zhou, J.-G., Zhang, K.-M.: A minimum broadcast graph on 26 vertices. Appl. Math. Lett. **14**, 1023–1026 (2001)