On two properties of the minimum broadcast time function

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Abstract

Broadcasting is the problem of dissemination of information in which one piece of information needs to be transmitted to a group of individuals connected by an interconnection network. A widely accepted communication model for this problem is the 1-port, constant model, in which a node of the network can transmit the message only to one neighbor at a time, and the transmission time is constant, regardless the length of the message. Finding an optimum strategy for broadcasting under this model, such that this process is accomplished in minimum time, has been proved to be NP-complete, for an arbitrary network.

If we model the interconnection network as an undirected graph, the minimum broadcast time function associates to each vertex an integer which represents the minimum time necessary to broadcast the information stored in that vertex to all other vertices. The values of the minimum broadcast time function are known for a very restricted class of graphs, mainly regular ones, and very little is known about this function in general.

In this paper we explore two new properties of this function. The first property establishes a connection between this function and the behavior of the optimal broadcast schemes. We prove an exact result for trees and we conjecture it for arbitrary graphs. The second property establishes a connection between this function and the density of the graph.

Keywords: broadcasting, minimum broadcast time, broadcast function, maximum diameter.

1. Introduction

Broadcasting is the problem of dissemination of information in an interconnection network in which, one member, called originator, knows a piece of information and has to transmit it to all other members of the network. The minimum broadcast time problem consists in finding a strategy, for a given graph G and an originator v, such that the broadcast time of v is minimized. This problem finds applications in parallel and distributed computing, Internet messaging, rumors and virus spreading, etc.

Different communication models have been proposed and analyzed for this problem. In this paper we consider the 1-*port*, *constant time* model. That is, once a member of the network knows the information, he can transmit it only to one neighbor at a time, and the transmitting time is constant.

We model the interconnected network as a simple undirected connected graph $G\,(V, E)$, in which the members of the network are the vertices of G, and the communication lines, the edges of G. Given such a graph G and an originator $v \in V$, the minimum broadcast time function, shortly the b-function, is defined as $b : V \to \mathrm{N}$, such that $b\,(v)$ represents be the minimum time necessary to inform all the vertices of G, starting from v. The broadcast time of G is defined as $b\,(G) = \max\{\, b\,(v) |\, v \in G\}$.

The minimum broadcast time problem has been proved to be NP complete [9], even for bounded degree graphs [2]. Good surveys on broadcasting can be found in [3] and [4]. The values of the minimum broadcast time function are known for a very restricted class of graphs, mainly regular ones, and very little is known about this function in general.

In this paper we investigate the connection between the values of the b-function and the behavior of the optimal broadcast schemes. We prove that this new property of the b-function is always true for arbitrary trees and we conjecture it for arbitrary graphs. Based on this property we managed to exhibit an iterative global algorithm for minimum broadcast time problem which performs very well in practice (see [5] for a detailed description of this algorithm and comparative results).

Due to the NP-completeness, some heuristic algorithms have been developed, in order to give an approximate solution for this problem (see [5] for further references). There are two main criteria to compare these heuristics: the complexity of the algorithm and the "optimality" of the solution. Since the first criterion is relatively easy to compute, the second one raises some questions, especially for random generated graphs, for which an optimal solution is generally not known a priory. There are three approaches to overcome this problem: to obtain theoretical results regarding the inapproximability of the problem, to run the algorithm on graphs with known broadcast time, mainly regular ones, or to do benchmarks on random graphs, generated using common generators [5, 6, 7]. Two main parameters of the random graph generators are the number of vertices and the density of the graph. In this paper we establish a connection between the minimum broadcast time function and these two parameters by upper bounding the range in which the b-function can take values.

The paper is organized as follows: Section 2 describes the first property, Section 3 describes the second property, and Section 4 draws the conclusions. The definitions will be given in the places where they are used. Throughout this paper we assume that the graphs are connected.

2. The *b*-function and optimal broadcast schemes

We denote by $G = (V, E)$ an undirected graph with V the set of vertices and E the set of edges. If we consider the broadcasting process originated in vertex v, at time-slot t, one already informed vertex, say u, will choose one of its uninformed neighbors, say w, and will transmit the information to w. Therefore, a broadcast scheme will assign to each vertex and time slot another vertex to inform. Therefore, we can associate to an optimal broadcast scheme originated in v a function $B_v : (V, \Gamma) \to V$, where $\Gamma = \{1, ..., b(G)\}$, such that $B_v(u, t) = w$ if and only if vertex u will transmit to vertex w at time t in that optimal broadcast scheme started from v.

The crux of an optimal broadcast scheme is the way of choosing the next neighbor to inform. If we denote by $N_v(u, t)$ the set of uninformed neighbors of vertex u in graph at time-slot t in a broadcast scheme originated in v, the following theorem establishes a connection between the minimum broadcast time function and optimal broadcast schemes, for arbitrary trees.

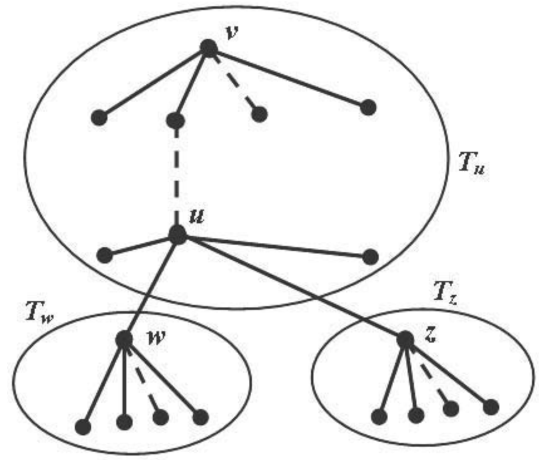

Figure 1. Tree T and the subtrees generated by u, w, and z.

Theorem 1 *Consider a tree T, an originator vertex v, and the b-function defined for each vertex of T. During any optimal broadcast scheme starting from v, at any time t, an informed vertex u will call vertex z, that has the minimum b-function among all uniformed neighbors of u. More formally, $B_v(u, t) = z$ implies $b(z) = \min\{b(N_v(u, t-1))\}$.*

Proof:

Let u be an informed vertex at time-slot $t - 1$. Assume, by contradiction, that vertex u will call at time-slot t one of its children, say w, with a greater broadcast time then z, which has the smallest broadcast time among all uninformed neighbors of u. Formally, we assume by contradiction that $B_v(u, t) = w$ and $b(w) > b(z) = \min\{b(N_v(u, t-1))\}$.

We denote by T_w the subtree induced by w, and by T_z the subtree induced by z, both in an optimal broadcast scheme having v as originator. We denote by $T_u = T - T_w - T_z$, the tree remaining after deleting T_w and T_z from T, rooted in u (see Figure 1). Also, we denote by:

- $b(w \to T_w)$ the minimum time needed for w to broadcast in subtree T_w
- $b(z \to T_z)$ the minimum time needed for z to broadcast in subtree T_z
- $b(u \to T_u)$ the minimum time needed for u to broadcast in subtree T_u

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There are 13 possible relationships between $b(w \to T_w)$, $b(z \to T_z)$, and $b(u \to T_u)$, with respect to "$<$" and "$=$". These 13 possibilities can be grouped in six cases:

a) $b(z \to T_z) = b(w \to T_w) = b(u \to T_u)$
$b(u \to T_u) < b(w \to T_w) = b(z \to T_z)$
In these cases $b(w) = b(z)$ (contradiction).

b) $b(u \to T_u) < b(z \to T_z) < b(w \to T_w)$
$b(z \to T_z) < b(u \to T_u) < b(w \to T_w)$
In these cases we have:
$b(w) \leqslant b(w \to T_w) + 1$, and
$b(z) \geqslant b(w \to T_w) + 2$
Adding we get $b(w) + 1 \leqslant b(z)$ (contradiction).

c) $b(u \to T_u) = b(z \to T_z) < b(w \to T_w)$
In this case we have:
$b(w) \leqslant b(w \to T_w) + 1$, and
$b(z) \geqslant b(w \to T_w) + 1$
Adding we get $b(w) \leqslant b(z)$ (contradiction).

d) $b(z \to T_z) < b(w \to T_w) < b(u \to T_u)$
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$b(w \to T_w) = b(z \to T_z) < b(u \to T_u)$
In these cases we have either:
$b(w) = b(z) = b(u \to T_u) + 1$, or
$b(w) = b(z) = b(u \to T_u) + 2$ (contradiction).

e) $b(z \to T_z) < b(w \to T_w) = b(u \to T_u)$
In this case:
$b(w) \leqslant b(w \to T_w) + 2$, and
$b(z) \geqslant b(w \to T_w) + 2$
Adding we get $b(w) \leqslant b(z)$ (contradiction).

f) $b(u \to T_u) < b(w \to T_w) < b(z \to T_z)$
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$b(w \to T_w) < b(u \to T_u) = b(z \to T_z)$
All these cases imply that:

$$b(w \to T_w) < b(z \to T_z) \tag{1}$$

Inequality (1) shows that that u cannot call w before z since the obtained broadcast scheme will not be optimal. More formally, $b(w \to T_w) < b(z \to T_z) \Rightarrow B_v(u,t) \neq w$. □

Note that the converse is not true. That is, $b(z) = \min\{b(N_v(z,t-1))\}$ does not imply $B_v(u,t) = z$, since there may be another vertex y such that $b(y) = \min\{b(N_v(z,t-1))\}$ and $B_v(u,t) = y$.

The same property as in Theorem 1 can be similarly proven for a path P_n or for the product graph $P_2 \times P_n$. We have explored a large variety of small irregular graphs with known broadcast scheme without finding any counterexample to this property. Also, based on this property, we have designed an iterative global heuristic in order to find the minimum broadcast time function, which works very well in practice [5]. All these results convinced us to make the following conjecture.

Conjecture *Given an undirected connected graph G, an arbitrary originator v, and a label attached to each vertex corresponding to its minimum broadcast time, during an optimal broadcast process starting from v, an informed vertex will always call one uniformed neighbor with the smallest label.*

3. The *b*-function and the graph density

Some heuristics, developed to approximately solve the minimum broadcast time problem, have been tested on random graphs. Since most of the random graph generators have as main parameters the number of vertices and the density of the graph, we considered opportune to study the relationship between the b-function and these two parameters. More precisely, considering vertex v having the smallest broadcast time $b_{\min}$, and vertex w having the greatest broadcast time $b_{\max}$, we give an upper bound for the difference $b_{\max} - b_{\min}$, in terms of graph density ρ and the number of vertices n.

Definition 1 *For an undirected graph $G = (V_G, E_G)$ on n vertices and m edges, the graph density ρ is defined by:*

$$\rho = \frac{|E_G|}{|E_{K_n}|} = \frac{2m}{n(n-1)} \tag{2}$$

where $|E_{K_n}|$ represents the number of edges in the complete graph on n vertices K_n.

Although the result of the following lemma seems to be somehow trivial, it turns out to be helpful to our study.

Lemma 1 *If there is an edge between vertices v and w in graph, then $|b(v) - b(w)| \leqslant 1$.*

Proof: Less formally, Lemma 1 claims that the minimum broadcast time of two adjacent vertices cannot differ by more than one.

Without loss of generality, assume that $b(v) > b(w)$ and $b(v) - b(w) > 1$, which implies $B_v(v,1) \neq w$. Considering a new broadcast scheme B' in which v is calling w in the first time-slot, $B'(v,1) = w$. After this call, we can follow the broadcast scheme having w as originator. We obtain for v a broadcast time $b'(v) = b(w) + 1 < b(v)$. This contradicts the definition of $b(v)$ as being the minimum broadcast time for v. □

This lemma has a straightforward corollary, which bounds $|b(v) - b(w)|$ for two arbitrary vertices v and w in graph, in terms of graph's diameter.

Corollary 1 *Given a graph $G(V, E)$ with diameter D, $|b(v) - b(w)| \leqslant D$, for any $v, w \in V$.*

Using this corollary we can now bound the range in which the minimum broadcast function can take values.

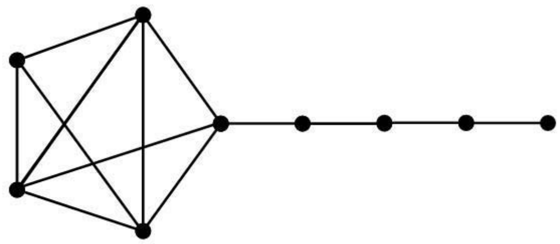

Figure 2. Diameter critical graph on 9 vertices with diameter 6.

Theorem 2 *For any connected graph $G(V, E)$ on n vertices, having density ρ, the following inequality holds:*

$$|b(v) - b(w)| \leqslant \left\lfloor \frac{2n+1-\sqrt{4(\rho n(n-1)-2n)+17}}{2} \right\rfloor,$$

for any $v, w \in V$.

Proof:

By Corollary 1, we have transformed the problem of upper bounding $b_{\max} - b_{\min}$ in the problem of upper bounding the diameter of the graph in terms of graph density and number of vertices. This problem has been considered in [8] and later in [1], in a slightly different context. Since we are looking for a closed formula for the maximum diameter of a graph function of the number of vertices and the graph density, we will use Theorem 3.1 from [8], which gives an upper bound for the number of edges of a graph $m_{\max}$, given the diameter D, and the number of vertices n:

$$m_{\max} \leqslant D + \frac{1}{2}(n - D - 1)(n - D + 4) \qquad (3)$$

In order to apply this result, we have to introduce the *diameter critical* graphs: a graph G is called diameter critical if the addition of any edge decreases the diameter (Figure 2).

Ore proved in [8] that the upper bound from (3) is attained for diameter critical graphs. Therefore, given a graph with n vertices and m edges, the maximum possible diameter, $D_{\max}$, satisfies the following inequalities:

$$m \leqslant D_{\max} + \frac{1}{2}(n - D_{\max} - 1)(n - D_{\max} + 4) \qquad (4)$$

$$m > D_{\max} + 1 + \frac{1}{2}(n - D_{\max} - 2)(n - D_{\max} + 3) \qquad (5)$$

Inequality (5) is obtained applying Ore's theorem for the case in which the diameter would be $D_{\max} - 1$. Solving the last two inequalities for $D_{\max}$ we get:

$$D_{\max} \in \left(\left\lceil \frac{2n-1-S}{2} \right\rceil, \left\lfloor \frac{2n+1-S}{2} \right\rfloor \right] \cup \left[\left\lceil \frac{2n-1+S}{2} \right\rceil, \left\lfloor \frac{2n+1+S}{2} \right\rfloor \right) \qquad (6)$$

where $S = \sqrt{8(m-n)+17}$.

Since $\left\lceil \frac{2n-1+\sqrt{8(m-n)+17}}{2} \right\rceil > n - 1$ for any $m > n - 1$, the maximum diameter must be in the interval:

$$D_{\max} \in \left(\left\lceil \frac{2n-1-S}{2} \right\rceil, \left\lfloor \frac{2n+1-S}{2} \right\rfloor \right] \qquad (7)$$

We observe that the limits of the interval from (7) satisfy:

$$\left\lceil \frac{2n-1-S}{2} \right\rceil = \left\lfloor \frac{2n+1-S}{2} \right\rfloor \qquad (8)$$

excepting the case when $\sqrt{8(m-n)+17}$ is an odd integer. In this case:

$$\left\lceil \frac{2n-1-S}{2} \right\rceil = \left\lfloor \frac{2n+1-S}{2} \right\rfloor - 1 \qquad (9)$$

Since the left side of the interval from (7) is opened, we obtain the following formula for the maximum diameter of a graph:

$$D_{\max} = \left\lfloor \frac{2n+1-\sqrt{8(m-n)+17}}{2} \right\rfloor \qquad (10)$$

If we substitute $m = \rho n(n-1)/2$ in (10) we obtain the claimed bound. □

Observation

The upper bound from Theorem 2 is tight by an additive factor of 2. This can be proved by finding a graph for which $b_{\max} - b_{\min} = D - 2$. We can draw such a graph by joining the extremity of a path on n_1 vertices with the center of a star on n_2 vertices (see Figure 3). It can be seen that if n_1 is smaller than the number of neighbors of v then $b(v) - b(w) = (n_1 + n_2 - 1) - n_2 = n_1 - 1 = D - 2$.

4. Conclusions

In this paper we explore two new properties of the minimum broadcast time function. The first property establishes a connection between the values of this function and the behavior of the optimal broadcast schemes. We prove an exact result for arbitrary trees and we conjecture it for arbitrary graphs. Based on this property we managed to exhibit in [5] an iterative heuristic, which, to our knowledge, is the first iterative approach to an NP-complete problem. This opens the possibility to apply the same technique to other NP-complete problems, in which, starting from an instance and an approximate solution, to be able to find a better solution. The second property of the minimum broadcast function described in this paper establishes a connection between the values of this function and the density of the graph. We also prove that the obtained upper bound is tight by an additive factor of two.

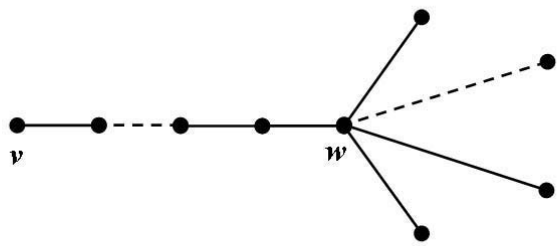

Figure 3. A path of length n_1 joined by an edge with a star on n_2 vertices

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