

The minimum broadcast time problem for several processor networks

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Abstract

Broadcasting is the information dissemination process in a communication network. A subset of processors $V_0 \subset V$ called originators knows an unique message which has to be transferred by calls between adjacent processors. Each call requires one time unit and each processor can participate in at most one call per time unit. The problem is to find a schedule such that the time needed to inform all processors is less than or equal to a deadline $k \in \mathbb{N}$. We present NP-completeness results for this problem restricted to several communication networks (bipartite planar graphs, grid graphs, complete grid graphs, split graphs and chordal graphs) with constant deadline $k = 2$ or one originator $V_0 = \{v\}$.

1. Introduction

Broadcasting is the information dissemination process in a processor network where all processors become informed of a message by calls over lines in the network. A communication network is modelled by an undirected graph $G = (V, E)$ consisting of a set V of vertices (processors) and a set E of edges (network connections). The process of information dissemination is described by the following constraints:

- (i) Each call involves two adjacent processors.
- (ii) Each call requires one time unit.
- (iii) Each processor can participate in at most one call per time unit.

More formally, the *broadcasting* can be defined as a sequence of sets $V_0 \subseteq \dots \subseteq V_k = V$ where each set V_i represents the processors informed after time unit i , $0 \leq i \leq k$. The vertices in V_0 are called *originators*. For each vertex $v \in V_i \setminus V_{i-1}$, there exists an adjacent vertex $\bar{v} \in V_{i-1}$ who has informed v . Moreover, for each pair

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$v, w \in V_i \setminus V_{i-1}$ with $v \neq w$ we have $\bar{v} \neq \bar{w}$. The times in which the processors are informed can be described by a mapping $t: V \rightarrow \{0, \dots, k\}$. Such a mapping is called a *broadcast scheme* or *schedule*.

The minimum value of k for a given network G and originator set V_0 is defined as the *broadcast time* of G with respect to V_0 . The minimum number of time units required to broadcast a message on a network with n processors and one originator is at least $\lceil \log(n) \rceil$. A *minimum broadcast network* with n processors is a network in which each processor can broadcast in $\lceil \log(n) \rceil$ time units. Constructing minimum broadcast networks has been studied extensively, see for example [1–4, 8, 13]. For a survey on the broadcast time problem and the related gossiping problem we refer to [9, 11].

Slater et al. [16] have described an algorithm to compute the broadcast time for an arbitrary tree network with $|V_0| = 1$. An algorithm for trees with a general number of originators is given in [6]. The broadcast time problem restricted to complete grid graphs and $|V_0| = 1$ is also solvable in polynomial time [5].

The general problem of determining the broadcast time for a given network has been shown algorithmic hard (NP-complete) by Garey and Johnson [7]. A proof of this result for an arbitrary graph with deadline $k = 4$ and unbounded degree is contained in [16]. Using a reduction from the three-dimensional matching problem, Jakoby et al. [10] proved the NP-completeness for graphs with constant maximum degree 5 and constant deadline $k \geq 3$. Using a complicated reduction, they showed that the problem remains NP-complete for planar graphs with maximum degree 3 and $|V_0| = 1$. The complexity of the problem for deadline $k = 2$ and an arbitrary network was unknown.

In this paper, we improve their results by simpler reductions from the planar satisfiability problem. We investigate on several processor networks as bipartite planar graphs, grid graphs, complete grid graphs, chordal graphs and split graphs. For grid graphs with maximum degree 3, complete grid graphs and split graphs, we show that broadcast time problem is NP-complete even for a deadline $k = 2$. For $|V_0| = 1$, the problem restricted to grid graphs and chordal graphs remains NP-complete. Independently, Middendorf [14] proved that the minimum broadcast time problem is NP-complete for 3-regular planar graphs with constant deadline $k \geq 2$.

2. Preliminaries

2.1. Planar 3-SAT

Our NP-completeness proofs are based on the planar 3-SAT problem which is NP-complete shown by Lichtenstein [12].

Input: A set of unnegated variables $X = \{x_1, \dots, x_n\}$ and negated variables $\bar{X} = \{\bar{x}_1, \dots, \bar{x}_n\}$, a collection of clauses C over $X \cup \bar{X}$ such that:

- (i) The graph $G = (X \cup C, E)$ with edge set $E = \{\{x, c\} \mid x \in C \vee \bar{x} \in C\}$ is planar.
- (ii) Each clause $c \in C$ contains two or three literals $y \in X \cup \bar{X}$.

Question: Does there exist a truth mapping for the variables such that each clause is satisfied?

We say x_i appears in c if $x_i \in c$ or $\bar{x}_i \in c$. In the first case x_i appears unnegated, and in the second case negated. In the reductions with constant deadline, we use a modification of the planar 3-SAT problem. In this problem, called planar 3,4-SAT we assume that each clause $c \in C$ contains exactly three literals and that each variable x_i appears in at most four clauses. The problem 3,4-SAT without the planarity constraint is NP-complete shown by Tovey [17].

Theorem 2.1. *Planar 3,4-SAT is NP-complete.*

Proof. We give a reduction from planar 3-SAT. First, for each variable x which appears in $n > 3$ clauses, we define a cycle $(y_1 \vee \bar{y}_2) \wedge (y_2 \vee \bar{y}_3) \wedge \cdots \wedge (y_n \vee \bar{y}_1)$ and replace the i th occurrence of x with y_i , $1 \leq i \leq n$. After this step, each variable appears in at most three clauses and each clause contains at most three literals.

In the second step, we replace each clause $(x \vee y)$ containing two literals in the following way. We introduce a new variable z such that the clause becomes $(x \vee y \vee \bar{z})$ and force that z must be true. To achieve this, we add the following clauses:

- (i) $(\bar{a}_i \vee b_i \vee d_i), (\bar{a}_i \vee \bar{b}_i \vee c_i), (\bar{b}_i \vee \bar{c}_i \vee d_i)$ for each $i = 1, 2, 3$.
- (ii) $(z \vee a_i \vee d_i)$ for each $i = 1, 2, 3$.
- (iii) $(\bar{d}_1 \vee \bar{d}_2 \vee \bar{d}_3)$.

Assume that z is false. Then $(a_i \vee d_i)$ must be true for $i = 1, 2, 3$. Using (iii), at least one variable d_j must be false and the corresponding variable a_j must be true. But (i), the clauses $(b_j), (\bar{b}_j \vee c_j)$ and $(\bar{b}_j \vee \bar{c}_j)$ are true. This gives a contradiction. Therefore, z is forced to be true.

It is possible to insert these clauses so that the corresponding graph $G = (X \cup C, E)$ remains planar (see also Fig. 1). In total, we obtain a planar 3-SAT instance where each variable appears in at most four clauses. \square

For reductions with $|V_0| = 1$, we use another modification of planar 3-SAT. In this problem, called planar separable 3-SAT, we assume that the graph $G = (X \cup C, E \cup E')$ with edge set $E = \{\{x, c\} \mid x \in c \vee \bar{x} \in c\}$ and $E' = \{\{x_i, x_{i+1}\} \mid 1 \leq i \leq n-1\} \cup \{x_n, x_1\}$ is planar. Moreover, each variable x_i appears in at most three clauses and each unnegated or negated variable at least once. Furthermore, the following separability constraint holds. For each variable x_i , the edges $\{x_{i-1}, x_i\}$ and $\{x_i, x_{i+1}\}$ separate the edges $\{x_i, c\} \in E$ such that all edges representing unnegated variables are incident to one side and all edges representing negated variables are incident to the other side. An illustration of this separability constraint is given in Fig. 2.

Theorem 2.2. *Planar separable 3-SAT is NP-complete.*

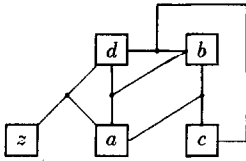


Fig. 1. Clauses for planar 3,4-SAT.

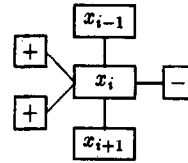


Fig. 2. The separability constraint.

Proof. See Lichtenstein [12]. \square

2.2. Embedding of a planar graph

For the NP-completeness results of grid graphs, we compute for a planar graph G a rectilinear planar layout. This layout maps vertices of G to horizontal line segments and edges to vertical line segments with all endpoints at positive integer coordinates. Two horizontal line segments are connected by a vertical one if and only if the corresponding vertices are adjacent in G . An example for this transformation is given in Fig. 3.

For a planar graph with n vertices a rectilinear planar layout can be computed in $O(n)$ time. The height of the layout is at most n and the width is at most $2n - 4$. For details regarding the algorithm used to obtain a rectilinear planar layout, we refer to [15].

For our reduction we use a modified layout which can be obtained directly from the planar rectilinear layout. Let d be a positive even integer.

(i) Vertices are mapped to disjoint axes parallel rectangles of height d with integer endpoints. If ℓ is the length of the horizontal line, then the width of the rectangle is $d \cdot \ell + d/2$.

(ii) Edges are mapped to disjoint axes parallel rectangles of width $d/2$ with integer endpoints. If ℓ is the length of the original line, then the height of the rectangle is $d \cdot (2 \cdot \ell - 1)$.

(iii) Two horizontal rectangles are connected by a vertical rectangle if and only if the corresponding vertices are adjacent. In this case the vertical rectangle touches the upper horizontal rectangle at the lower side and the lower horizontal rectangle at the upper side.

(iv) If ℓ is the distance between two vertical lines, the distance between the left sides of the corresponding rectangles is $\ell \cdot d$. If ℓ is the distance between two horizontal lines, the distance between the upper sides of the corresponding rectangles is $2 \cdot \ell \cdot d$.

We obtain the modified layout, which we call channel layout, by stretching the horizontal and vertical lines to rectangles with height d and width $d/2$, respectively. We call d the stretching factor. Clearly, this transformation can be done in polynomial time. For our example in Fig. 3 we get the channel layout illustrated in Fig. 4.

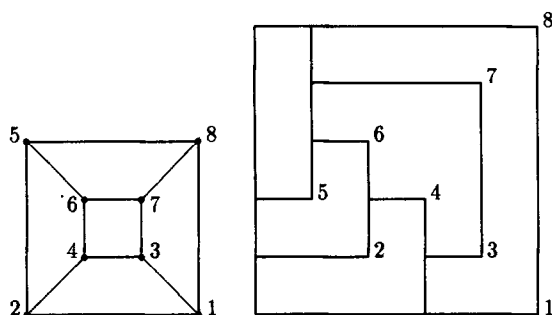


Fig. 3. A planar graph and its rectilinear planar layout.

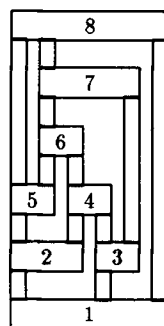


Fig. 4. The channel layout.

In the following, we give a transformation for a planar graph with maximum degree $\Delta(G) \leq 3$. Using this transformation, we can draw a planar graph as a grid graph G' where each edge is replaced by a path of the same length ℓ , with $\ell = O(|V|^2)$.

Lemma 2.3. *Let $G = (V, E)$ be a planar graph with $\Delta(G) \leq 3$. Then, there is a positive integer $\ell = O(|V|^2)$ such that the graph $G' = (V \cup V', E')$ with $V' = \{a_{e,j} \mid e \in E, 1 \leq j \leq \ell - 1\}$ and*

$$E' = \bigcup_{e=\{u,v\} \in E} \{\{u, a_{e,1}\}, \{a_{e,1}, a_{e,2}\}, \dots, \{a_{e,\ell-1}, v\}\}$$

is a grid graph.

Proof. We assume that G contains no isolated vertices. Clearly, isolated vertices can be added after the transformation. In the first step we compute a rectilinear planar layout for G where w.l.o.g. each line segment has even length. Let ℓ_{\max} be the maximum length of the line segments.

In the second step, we generate a channel layout with stretching factor $d = 4 \cdot \ell_{\max}$. For each vertex $v \in V$, we place into the corresponding horizontal rectangle a horizontal path P_v of length $d \cdot \ell$ (where ℓ is the length of the line segment) and with vertices of distance 1. For each edge $e = \{u, v\} \in E$, we insert into the corresponding vertical rectangle a vertical path P_e which connects the horizontal paths P_u and P_v . We place the vertical path at the left side of the vertical rectangle and the horizontal path in the middle of the horizontal rectangle. In dependence on the degree $\delta(v)$ of the vertices, we obtain different components and define an unique point $p(v)$ on P_v as follows (see also Fig. 5).

- (1) If $\delta(v) = 1$, then $p(v)$ is the rightmost point on P_v .
- (2) If $\delta(v) = 2$, then $p(v)$ is the middle point on P_v .
- (3) If $\delta(v) = 3$, then $p(v)$ is the point on P_v adjacent to the middle vertical path.

Since each horizontal path has even length, a middle point always exists. For each edge $e = \{u, v\}$ in G , we obtain a path between $p(u)$ and $p(v)$ consisting of at most two

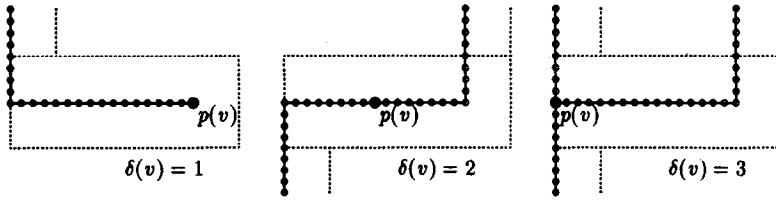
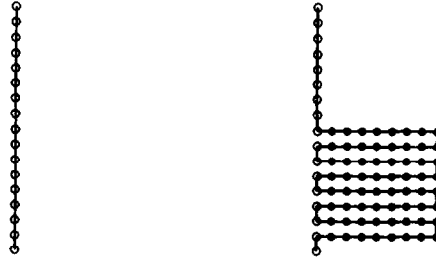
Fig. 5. The grid graphs in dependence on the degrees of vertices $v \in V$.

Fig. 6. A stretching of a vertical path.

horizontal subpaths of P_u and P_v and of the vertical path P_e . With exception of the endpoints, these paths are pairwise disjoint. The total length of a path between points $p(u)$ and $p(v)$ is $(a_1 + 2 \cdot a_2 + a_3) \cdot d$ with $a_1, a_2, a_3 \leq \ell_{\max}$ and $a_2 \geq 2$ (note that each line segment has even length). The maximum possible length is $4 \cdot \ell_{\max} \cdot d = 16 \cdot \ell_{\max}^2$.

In the third step, we replace each vertical path P_e , for $e = \{u, v\} \in E$, by a path P'_e such that the path between x_u and x_v has the length $16 \cdot \ell_{\max}^2$. To stretch a path with length $(a_1 + 2 \cdot a_2 + a_3) \cdot d$, we insert

$$[(\ell_{\max} - a_1) + 2 \cdot (\ell_{\max} - a_2) + \ell_{\max} - a_3] \cdot d < 16 \cdot \ell_{\max}^2$$

new points. An original vertical path with white coloured points and a stretched path with new black coloured points are given in Fig. 6. Since the length of a vertical path P_e is at least $4 \cdot d = 16 \cdot \ell_{\max}$ and since the width of a vertical rectangle is $2 \cdot \ell_{\max}$, the construction is always possible. The grid graph constructed in this way has the desired properties; where each edge is replaced by a path of length $16 \cdot \ell_{\max}^2 = O(|V|^2)$. \square

3. Planar bipartite graphs

In this section we analyse the complexity of the broadcast time problem for bipartite planar graphs.

Theorem 3.1. *The broadcast time problem is NP-complete for bipartite planar graphs, deadline $k = 2$ and maximum degree $\Delta(G) \leq 3$.*

Proof. By reduction from planar 3,4-SAT. Given an instance with planar graph $(X \cup C, E)$ we construct the reduction graph as follows. Each vertex $x \in X$ is replaced by a variable component and each vertex $c \in C$ is replaced by a clause component. We describe two variants of the variable component. Both are working here, but later on we use the small one in case of complete grids and the big one in case of split and chordal graphs. The components are depicted in Figs. 7–9.

The construction of the reduction graph is complete by identifying corresponding b and t or f vertices. For each edge $\{x, c\}$ in $(X \cup C, E)$ we identify a t -vertex of the variable component of x with a b -vertex of the clause component of c if $x \in c$. Otherwise ($\bar{x} \in c$) we identify an f -vertex of the variable component of x with a b -vertex of the clause component of c . This identification is made with respect to the cyclic ordering of the neighbourhood of each vertex in $(X \cup C, E)$ such that the resulting reduction graph is planar. It is easy to see that the reduction graph is bipartite too. We define the set V_0 of originators to be the set of all x and all y -vertices.

Suppose there is a truth mapping $b: X \rightarrow \{t, f\}$ satisfying all clauses of C . We define $a(x) = t$ if $b(x) = f$ and $a(x) = f$ if $b(x) = t$ for all $x \in X$. Then there is a broadcasting scheme of our reduction graph with deadline $k = 2$. In the first step the vertices $x \in X$ inform their neighbours $b(x)$. In the second step the vertices $c \in C$ are informed by one of its neighbours informed in the first step, i.e. if $x \in c$ and $b(x) = t$ then the common neighbour t of x and c informs c , otherwise ($\bar{x} \in c$ and $b(x) = f$) the common neighbour f informs c . Also in the second step the p -vertices are informed by their neighboured

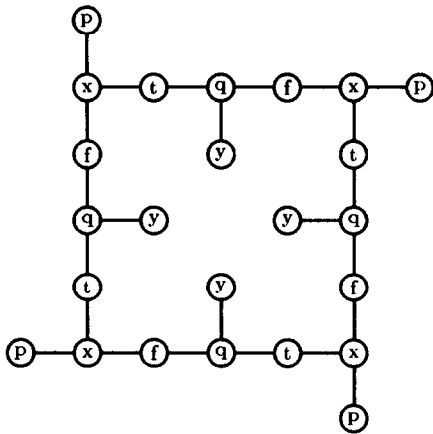


Fig. 7. Small variable component.

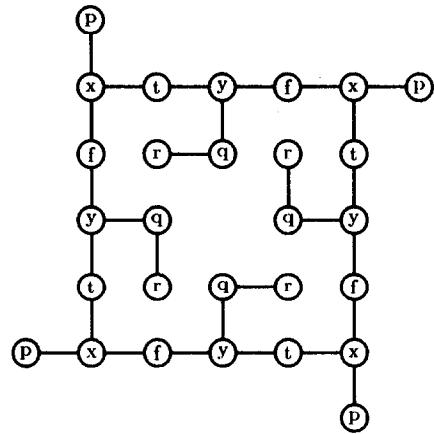


Fig. 8. Big variable component.

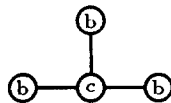


Fig. 9. Clause component.

x -vertices. The remaining vertices ($q, a(x)$ and in case of the big variable component r) receive information originating in y -vertices. This takes two steps.

On the other hand, suppose there is a broadcast scheme of our reduction graph with deadline $k = 2$. Observe that in this case all x -vertices in a variable component inform in the first step their neighbours t , or all of them inform their neighbours f . Otherwise it is impossible to inform the remaining vertices of the component in a second step. We define a truth mapping $b: X \rightarrow \{t, f\}$ such that the x -vertices of the variable component of x inform in the first step their neighbours $b(x)$. We consider a clause $c \in C$. The corresponding neighbours of the vertex c in our reduction graph are t or f -vertices of variable components. Hence c is informed in the second step by a vertex $b(x)$ informed in the first step and the variable x or \bar{x} satisfies the clause c . \square

A graph $G = (V, E)$ is called a *split graph* if the vertex set V can be partitioned into a clique C of G and an independent set I of G , i.e. $V = C \cup I$, $C \cap I = \emptyset$ and $\binom{C}{2} \subseteq E \subseteq \binom{V}{2} \setminus \binom{I}{2}$. A *chord* of a cycle in a graph G is an edge of G connecting two nonconsecutive vertices of the cycle. G is a *chordal graph* if each cycle of length at least four admits a chord. Clearly, split graphs are chordal graphs.

Corollary 3.2. *The broadcast time problem with deadline $k = 2$ is NP-complete for split graphs.*

Proof. If we use the big variable components in the above proof, then all the c, r, x and y -vertices belong to the first colour class and all the t, f, p and q -vertices belong to the second colour class of the bipartite reduction graph. We add $4|X| - |C|$ dummy d -vertices connected with all t, f -vertices and obtain a split graph by completing the second colour class. Clearly, added edges cannot be used in a broadcast scheme with deadline $k = 2$ to inform c -vertices. \square

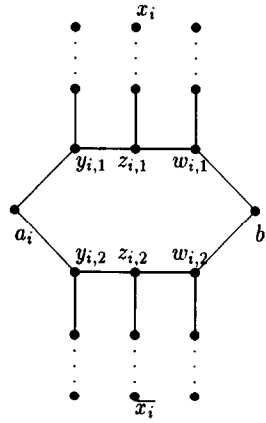
In the following we consider the problem with the constraint $|V_0| = 1$. We show that the broadcast problem remains NP-complete for a planar graph.

Theorem 3.3. *The broadcast time problem is NP-complete even if the graph G is planar with maximum degree $\Delta(G) \leq 3$ and $|V_0| = 1$.*

Proof. By reduction from planar separable 3-SAT. Given an instance with planar graph $G = (X \cup C, E \cup E')$, we construct a planar graph G^* as follows.

First, we define a graph G_i given in Fig. 10, for each variable $x_i \in X$. The vertices $y_{i,1}, y_{i,2}, z_{i,1}, z_{i,2}, w_{i,1}$ and $w_{i,2}$ are adjacent to disjoint paths. The number of vertices in the pending paths is $5 \cdot (|X| - i + 1) - 3$ for $y_{i,1}$ and $y_{i,2}$ and $5 \cdot (|X| - i) - 1$ for the other. The last vertices in the pending paths for $z_{i,1}$ and $z_{i,2}$ are the variable x_i and the negated variable \bar{x}_i , respectively.

For each edge $\{x_i, x_{i+1}\}$ in G with $1 \leq i \leq n - 1$, we insert an edge $\{b_i, a_{i+1}\}$ which connects the graphs G_i and G_{i+1} . If x_i appears as unnegated variable in the clause c ,

Fig. 10. The graph G_i .

we add an edge $\{x_i, c\}$, and if x_i appears negated, we add $\{\bar{x}_i, c\}$. The set $V_0 = \{a_1\}$ and the deadline k is $5 \cdot |X| - 1$. Since G is planar and separable, the constructed graph G^* is planar, too.

For a feasible broadcast of G^* within k time steps, we get

(i) $a_i \in V_{5(i-1)}$ and $b_i \in V_{5i-1}$ and

(ii) either $y_{i,1} \in V_{5(i-1)+1}$, $z_{i,1} \in V_{5(i-1)+2}$, $w_{i,1} \in V_{5(i-1)+3}$, $x_i \in V_{5|X|-3}$ or $y_{i,2} \in V_{5(i-1)+1}$, $z_{i,2} \in V_{5(i-1)+2}$, $w_{i,2} \in V_{5(i-1)+3}$, $\bar{x}_i \in V_{5|X|-3}$.

Consider the second case of (ii) with $y_{i,2} \in V_{5(i-1)+1}$ and $y_{i,1} \in V_{5(i-1)+2}$. Since $y_{i,1} \in V_{5(i-1)+2}$, $y_{i,1}$ must inform in time step $5(i-1) + 3$ the first vertex in his pending path (otherwise the last vertex there gets the information too late). Therefore, if $y_{i,1} \in V_{5(i-1)+2}$ then $z_{i,1} \in V_{5(i-1)+4}$. Again, using the number of pending vertices of $w_{i,1}$, $w_{i,1}$ must lie in $V_{5(i-1)+5}$. Since the first vertex in the pending path of $z_{i,1}$ lies in $V_{5(i-1)+6}$, x_i is informed in the last step $k = 5|X| - 1$. Therefore, using the number of vertices in the pending paths, only the following two cases are possible:

(i) $y_{i,2} \in V_{5(i-1)+2}$, $z_{i,2} \in V_{5(i-1)+4}$, $w_{i,2} \in V_{5(i-1)+5}$ and \bar{x}_i is informed in the last step k .

(ii) $y_{i,1} \in V_{5(i-1)+2}$, $z_{i,1} \in V_{5(i-1)+4}$, $w_{i,1} \in V_{5(i-1)+5}$ and x_i is informed in the last step k .

Therefore, the instance of planar separable 3-SAT is satisfiable if and only if there is a broadcast for the graph G^* and $V_0 = \{a_1\}$ within $k = 5|X| - 1$ time steps. \square

Using a similar transformation, we can prove the following result.

Theorem 3.4. *The broadcast time problem is NP-complete even if the graph G is bipartite and planar with maximum degree $\Delta(G) \leq 3$ and $|V_0| = 1$.*

4. Grid graphs

In this section we extend our results to grid graphs.

Theorem 4.1. *The broadcast time problem is NP-complete even for a grid graph with deadline $k = 2$ and maximum degree $\Delta(G) \leq 3$.*

Proof. By reduction from planar 3,4-SAT. Given an instance, a planar bipartite graph $G = (X \cup C, E)$, we construct a channel layout with stretching factor $d = 10$. We place into the horizontal and vertical rectangles a set of points (at integer positions) and a set of edges such that we obtain a grid graph. In the following, we give the designs for the connection between a variable and a clause, for the variable setting and for the clause function.

Variable setting. In each horizontal rectangle corresponding to a variable we place a component as illustrated in Fig. 11. This component contains the same gadget as for the bipartite planar graph. The vertices coloured black belong to V_0 and the other vertices to $V \setminus V_0$. For each occurrence of a variable in a clause we construct a path connected with a t -vertex for an unnegated variable and connected with an f -vertex for a negated variable. The height of each variable component is 10. We observe (see Fig. 11) that the vertices at the upper and lower side belong to V_0 . For synchronisation, the distance between two vertical paths connecting a variable with a clause is a multiple of 10.

Connections. For a vertical connection between a variable, we use a path of length $10 \cdot (2\ell - 1)$. A subpath of length 10 is given in Fig. 12. The idea of the signal flow can be described as follows. Assume that the left-most vertex in V_0 informs the next two

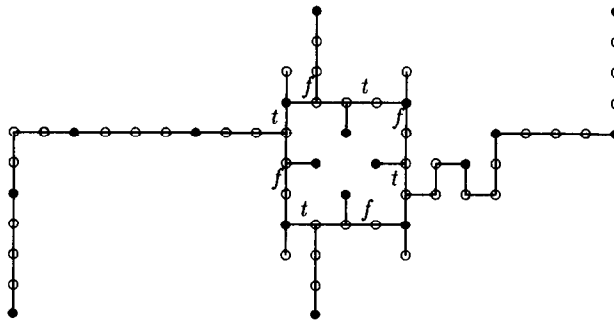


Fig. 11. The variable setting.

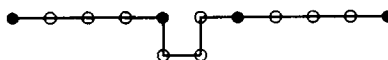


Fig. 12. A path of length 10.

right vertices (step 1 and 2). Then, the second vertex in V_0 can inform also the next two vertices. If only the right neighbour is informed by the left-most vertex in V_0 , the second vertex in V_0 can only inform his right neighbour. In other words, a positive and a negative signal can be transferred through the path and a negative signal cannot be converted into a positive.

Clause component. A clause component of height 10 is given in Fig. 13. It contains three paths corresponding to the three literals of the clause connected at a square. This square is informed in two time units, only if at least one positive signal arrives in a connecting path. Clearly, this simulates the function of clause.

In total, we derive that the instance of planar 3,4-SAT is satisfiable if and only if there is a broadcast for G within two time steps. \square

Theorem 4.2. *The broadcast time problem remains NP-complete for a grid graph with $|V_0| = 1$ (and maximum degree $\Delta(G) \leq 4$).*

Proof. By reduction from planar separable 3-SAT. First, we modify the graph G^* constructed in Theorem 3.3. For the vertices $y_{i,1}, y_{i,2}, z_{i,1}, z_{i,2}, w_{i,1}, w_{i,2}$ we use pending paths with additional vertices. The number of vertices for $y_{i,1}$ and $y_{i,2}$ is $5 \cdot (|X| - i) + 4$ and the number for the other vertices is $5 \cdot (|X| - i) + 2$. Next, we connect b_n with a new vertex a_{n+1} and get a planar graph G with maximum degree $\Delta(G) \leq 3$. As described in Lemma 2.3, we compute for G a grid graph G' where each edge in G is replaced by a path of the same length ℓ , with $\ell = O(|V|^2)$. Finally, we add the following vertices:

- (i) A path of length three connected with $p(a_{n+1})$.
- (ii) A path of length two connected with the leaf of the path pending at $p(y_{i,j})$, for each $i \in \{1, \dots, |X|\}$, $j \in \{1, 2\}$.
- (iii) One vertex connected with the leaf of the path pending at $p(w_{i,j})$, for each $i \in \{1, \dots, |X|\}$, $j \in \{1, 2\}$.
- (iv) One vertex connected with the clause point $p(c)$, for each clause $c \in C$.

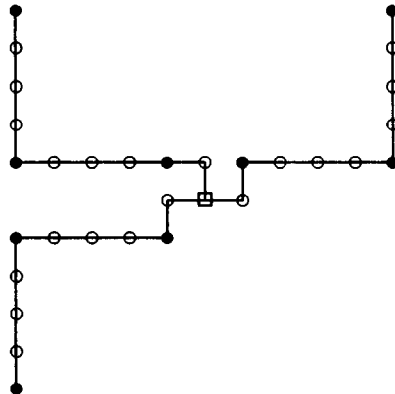


Fig. 13. The clause function.

Clearly, the so constructed graph G'' is a grid graph and can be generated in polynomial time. Using the same calculation as in Theorem 3.3, the instance of planar separable 3-SAT is satisfiable if and only if there is broadcast for G'' and $V_0 = \{p(a_1)\}$ within $k = 5 \cdot \ell \cdot |X| + 3$ time units. \square

5. Complete grid graphs

In this section we analyse the complexity of the broadcast time problem for complete grid graphs. We will show that our problem remains NP-complete when restricted to this class and with deadline $k = 2$.

In what follows we describe the main idea. We use the reduction graph for grid graphs and stretch all paths outside any variable component such that there is an embedding in a complete grid with each such path of length 8 is embedded with at most one bend. We fix such an embedding. All the vertices of the embedded graph form the *channels*. The complete host grid is chosen such that the vertices on its boundary have distance at least 3 from the channels. The vertices of distance 3 or more from the channels form *islands*. In particular, all the vertices of degree 2 or 3 of the complete grid belong to one island. The vertices between the islands and the channel form strips of breadth two, the *banks*. We add the islands to V_0 . The vertices in the islands inform the vertices in the neighboured bank. Vertices in a channel are informed by other vertices in the channel as before. This does not work correctly around bends and components. Therefore we have to transform the islands in these areas.

First we consider the bend. The transformed island around a bend is shown in Fig. 14. Thick points indicate vertices in V_0 , thick lines indicate the boundary of the channel and dotted lines indicate the boundary of the islands. The channel vertices in

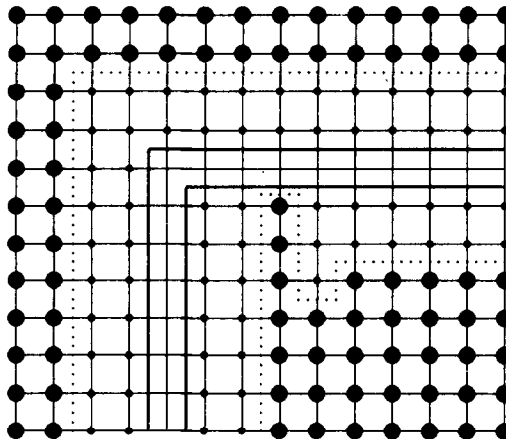


Fig. 14. The bend.

V_0 are not indicated, the islands have equal shape around the bend independent from what happens in the channel.

It is easy to see that the vertices of the islands are able to inform the banks in two steps. Now we consider a broadcasting scheme with deadline $k = 2$. That vertex in an island which has a neighbour in the channel is called the *point*. The point is the only vertex in an island able to inform vertices in the channel. In this case, for each vertex in the channel, receiving information originating in the point, there is a vertex in the bank receiving information originating in the channel. Therefore we may assume that vertices in the island inform vertices in the bank only.

Now it is easy to see that the clause component shown in Fig. 15 with the islands around works as in the prior cases.

Our variable component is based on the small variable component for bipartite planar graphs. First we add four further vertices to be informed by y -vertices in the second step and a z -vertex in V_0 . This enlarged component is shown in Fig. 16 and works as before.

The shape of the islands around the variable component depends on sequence of negated and unnegated appearances. Some examples are given in Figs. 17–20.

If one is not convinced that the variable components and clause components are connectable as before, then consider the following construction. For each component, either variable component or clause component, we create a sufficiently large box, say 99×99 , in the host grid. Then we put the component in the box at central position. For each channel leaving the component we append pieces of ordinary channel and up to three bends, such that the channels leave the box at a midpoint of a side of the box. Clearly, the remaining part of the box is filled with vertices in V_0 . Outside these boxes the channels use the grid on rows and columns divisible by 100 only.

Theorem 5.1. *The broadcast time problem with deadline $k = 2$ is NP-complete for complete grid graphs.*

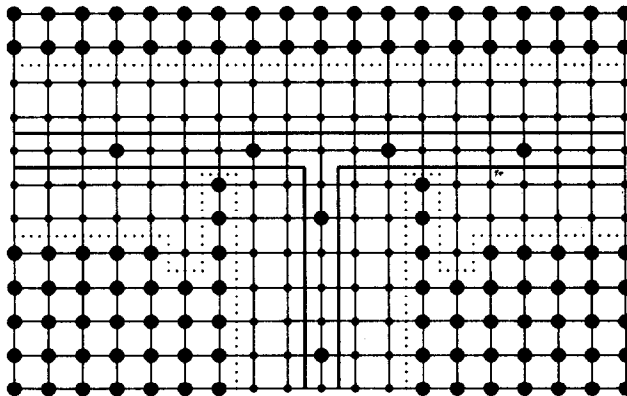


Fig. 15. The clause component.

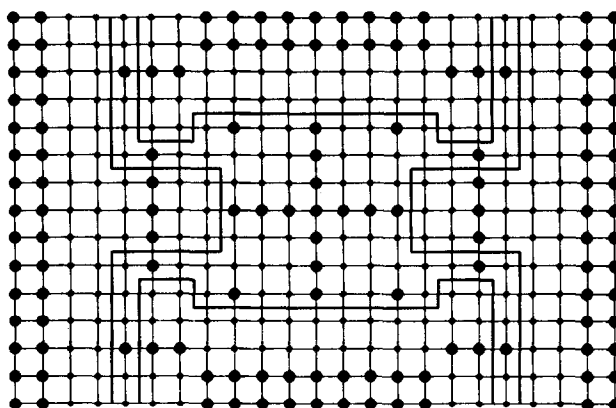


Fig. 18. The variable component for a variable appearing twice unnegated and twice negated in alternate order.

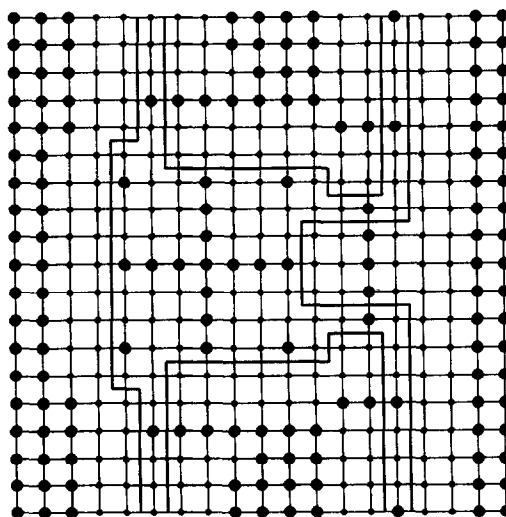


Fig. 19. The variable component for a variable appearing twice unnegated and twice negated in consecutive order.

MINIMUM BROADCAST TIME problem restricted to chordal graphs with $|V_0| = 1$.

In case of bipartite planar graphs and in case of grid graphs the NP-completeness of MBT for $k = 2$ implies the NP-completeness for any deadline $k \geq 2$. This is not easy to see for complete grid graphs and split graphs. But we conjecture that MBT remains NP-complete for constant deadline $k \geq 2$ on these classes too. Another open problem in this field is MTP for split graphs with $|V_0| = 1$.

Theorem 6.1. *The broadcast time problem with one originator is NP-complete for chordal graphs.*

Table 1

Communication network	Constant deadline $k = 2$	Constant number of originators $ V_0 = 1$
Planar bipartite graphs	NP-complete	NP-complete
Grid graphs	NP-complete	NP-complete
Complete grid graphs	NP-complete	Polynomial
Trees	Polynomial	Polynomial
Chordal graphs	NP-complete	NP-complete
Split graphs	NP-complete	Open

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