ARTICLE IN PRESS

Discrete Applied Mathematics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Minimum multiple originator broadcast graphs[☆]

Arthur L. Liestman[a], Dana Richards[b,*]

[a] School of Computing Science, Simon Fraser University, Burnaby, BC V5A 1S6, Canada

[b] Department of Computer Science, George Mason University MSN 4A5, 4400 University Drive, Fairfax, VA 22030, USA

ARTICLE INFO

Article history:
Received 2 March 2015
Received in revised form 9 March 2016
Accepted 20 June 2016
Available online xxxx

Dedicated to Levon Khachatrian

Keywords:
Broadcast
Multiple originator
Minimum broadcast graph

ABSTRACT

Broadcasting from multiple originators is a variant of broadcasting in which any k vertices may be the originators of a message in a network of n vertices. A minimum broadcast graph has the fewest possible edges while still allowing minimum time broadcasting from any set of k originators. We provide a census of all known such graphs.

1. Introduction

Broadcasting is the process of message dissemination in a communication network in which a message, originated by one vertex, is transmitted to all vertices of the network by placing a series of calls over the communication lines of the network. This is to be completed as quickly as possible. Typically, it is assumed that each call involves only one informed vertex and one of its neighbors, each call requires one unit of time, a vertex can participate in only one call per unit of time, and a vertex can only call its neighbors. Here, we consider broadcasting from any set of k originators.

Given a connected graph $G = (V, E)$ and a subset of the vertices $V' \subseteq V$ the *broadcast time of the set* V', $b(V')$, is the minimum number of time units required to complete a broadcast from the vertices V'. Since the number of informed vertices can at most be doubled during each time unit, it is clear that for any set V' of k vertices in a connected graph G with n vertices, $b(V') \geq t(n, k)$, where $t(n, k) = \lceil \log_2 \frac{n}{k} \rceil$. The *k-originator broadcast time of a graph* G, denoted $b_k(G)$, is the maximum broadcast time of any such subset V' in G, with $|V'| = k$, i.e. $b_k(G) = \max\{b(V') | V' \subseteq V, |V'| = k\}$. We use the term *k-originator broadcast graph* to refer to any graph G on n vertices with $b_k(G) = t(n, k)$. The *k-originator broadcast function*, $B_k(n)$, is the minimum number of edges in any k-originator broadcast graph on n vertices. A *minimum k-originator broadcast graph* is a k-originator broadcast graph on n vertices having $B_k(n)$ edges.

Early work by Khachatrian appeared in [10,11]. For surveys of results on broadcasting and related problems, see Hedetniemi, Hedetniemi and Liestman [8], Fraigniaud and Lazard [5], Hromkovič, Klasing, Monien, and Peine [9] and Harutyunyan, Liestman, Peters and Richards [7]. More recent results are [2,1,6].

In this paper we will provide a census of multiple originator minimum broadcast graphs, for various values of k and n where they have been determined. The upper bounds are given by exhibiting such a graph and explaining how it works for

☆ This work was supported by the Natural Sciences and Engineering Research Council of Canada, grant OGP0001734.

* Corresponding author.
E-mail address: richards@cs.gmu.edu (D. Richards).

http://dx.doi.org/10.1016/j.dam.2016.06.015
0166-218X/

any subset of k vertices. The lower bounds can involve a detailed case analysis. To aid the reader these are grouped by the time necessary to broadcast. Table 1, a summary of our theorems, appears at the end of the paper.

2. Background

The concept of multiple originators has appeared in various restricted cases, such as the best way to position originators in a tree [4]. We however focus on the worst-case problem of unrestricted placement of originators. We have previously given asymptotic results on $B_k(n)$ [13]. In particular we have these upper and lower bounds, where $L(x)$ be the number of leading 1 bits in the binary representation of the integer x.

Theorem 2.1. *For every* $n \geq 1$, B_k *is* $\Omega\left(L(\lceil \frac{n}{k} \rceil - 1) \cdot (n-k)\right)$.

Theorem 2.2. *For every* $n > 1$, $B_k(n)$ *is* $O\left(L(\lceil \frac{n}{k} \rceil - 1) \cdot n \cdot k\right)$.

Since these bounds are not tight we directed our efforts in a very different direction. (They are tight for constant k but general interest is for variable numbers.) In this paper we consider the cases where we can prove $B_k(n)$ exactly. This will involve much detail and constrain us to small values of k and n.

A census of minimum broadcast graphs has previously been undertaken for $k = 1$. Farley, Hedetniemi, Mitchell and Proskurowski [3,14] presented such graphs for all $n \leq 15$. These have been extended over a series of publications as discussed in the survey papers. Some papers address a single value of n [12,15]. At this point $B_1(n)$ is known for these values of n: 1 to 22, 26 to 32, 58 to 64, and similar ranges below powers of two [7].

We will organize our search by considering examples which allow only two time units to broadcast, and then we will consider examples for three time units. When the broadcast must be completed in one time unit (when $k \geq \frac{n}{2}$) we have an exact value for $B_k(n)$ [13].

Theorem 2.3. $B_k(n) = \lceil \frac{n(n-k)}{2} \rceil$ *for* $k \geq \frac{n}{2}$.

We state some general results that will be used below.

Lemma 2.4. *If* $t(n,k) = t(n,k+1)$, *then* $B_k(n) \geq B_{k+1}(n)$.

Proof. This follows from the observation that if any set of k originators can broadcast in graph G in time t, then any set of $k+1$ originators can also broadcast in G in time t. □

Lemma 2.5. *If* $t(n,k) = t(n+1,k)$ *and a minimum* k *originator broadcast graph on* $n+1$ *vertices has a vertex of degree* 2, *then* $B_k(n) \leq B_k(n+1) - 1$.

Proof. Let G be such a graph, let a be such a vertex of degree 2 and let b and c be its neighbors. Let G' be the graph on n vertices formed by deleting a and its incident edges and adding the edge between b and c. G' has fewer edges than G. In any broadcasting scheme for k originators (not including a), if one neighbor, say b, calls a and a later calls c, replace the former call with a call from b to c. Otherwise, delete any calls involving a. The result is a valid k originator broadcasting scheme for G'. □

Similar results hold for higher degree vertices, such as:

Lemma 2.6. *If* $t(n,k) = t(n+1,k)$ *and a minimum* k *originator broadcast graph on* $n+1$ *vertices has a vertex of degree* 3, *then* $B_k(n) \leq B_k(n+1)$.

Proof. Let G be such a graph, let a be such a vertex of degree 3 and let b, c and d be its neighbors. Let G' be the graph on n vertices formed by deleting a and its incident edges and adding edges between b, c and d. G' has at most the same number of edges as G. In any broadcasting scheme for k originators (not including a), if b calls a and a later calls c and then (perhaps) d, replace the call from b to a with a call from b to c and replace the call from a to c with a call from c to d. Otherwise, delete any calls involving a. The result is a valid k originator broadcasting scheme for G'. □

3. Specific values for time 2

In this section we only discuss results for cases where $t(n,k) = 2$. The reader will not be reminded repeatedly that only two time units are allowed. This section is a case study for determining exact values of $B_k(n)$ for small values of n. Note that in two time units, each originator can inform at most three other vertices and it must have at least two non-originator neighbors to inform three. The theorems in this section are presented in an order that corresponds to filling in columns of Table 1 (at the end), which naturally leads to building upon previous proofs.

To prove lower bounds we need two lemmas. A vertex u is *isolated* if there are k vertices more than two steps away from u. This lemma follows since such a vertex cannot be informed if the originators are all too far away.

Please cite this article in press as: A.L. Liestman, D. Richards, Minimum multiple originator broadcast graphs, Discrete Applied Mathematics (2016), http://dx.doi.org/10.1016/j.dam.2016.06.015

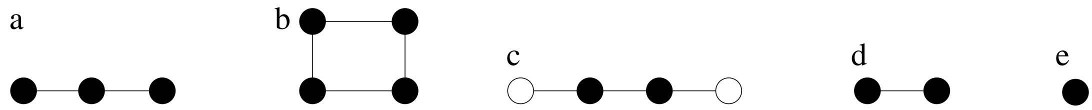

Fig. 1. Broadcast subgraphs for two time units.

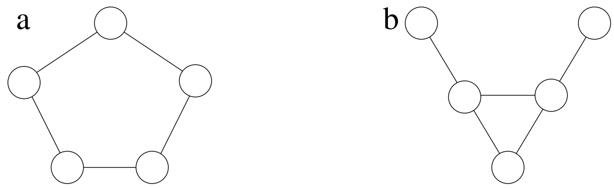

Fig. 2. Minimum two originator broadcast graphs on 5 vertices.

Lemma 3.1. *A k-originator broadcast graph, $t(n, k) = 2$, has no isolated vertices.*

Suppose the vertex set is partitioned into the sets S_1, S_2, and S_3, so that there is no edge connecting a vertex in S_1 to a vertex in S_3. In that case S_2 is a *separator*. Let $|S_1| = n_1$, $|S_2| = n_2$, and $|S_3| = n_3$. If the following lemma is violated we say the graph has a *bad separator*.

Lemma 3.2. *For a k-originator broadcast graph, $t(n, k) = 2$:*
If $k \le n_1$ then $n_3 \le n_2$.
If $k - n_2 \le n_1 < k$ then $n_3 \le 3(k - n_1) + (n_1 + n_2 - k)$.
If $n_1 < k - n_2$ then $n_3 \le 4(k - n_1 - n_2) + 3n_2$.

Proof. In the first case if all the originators are in S_1 only n_2 vertices can be reached in S_3 in two time steps. In the second case if all the vertices of S_1 are originators and the rest are in S_2 then those in S_2 can inform three vertices in S_3, while each of the remaining vertices in S_2 can be used to inform only one more vertex. In the third case, even when all of the vertices of S_1 and S_2 are originators there will be originators in S_3 and all those in S_2 and S_3 can inform three additional vertices. □

Each lower bound proof of $B_k(n) > e$ will be a contradiction of the assumption that e edges suffice; in particular we begin by assuming that the sum of the degrees is exactly $2e$. The proofs use degree sequences (ordered lists of the n degrees) and use the well-known property that there is an even number of vertices of odd degree. We will use an exponent shorthand for degree sequences, so that 222335 would be $2^3 3^2 5^1$. Some proofs can be abbreviated since, for a fixed k, a forbidden subgraph for n implies the subgraph is forbidden for $n + 1$, as will become clear below. Forbidden subgraphs that are paths or cycles will sometimes be denoted as in this example: $x - 2 - y - 2 - x$ denotes a cycle of length 4 beginning and ending at vertex x with two of the vertices being unnamed vertices of degree 2.

We say vertices x and y *meet* if they are both adjacent to the same vertex z.

We begin with an observation about some graphs of at most four vertices. In the proofs below, it will be convenient to describe broadcasting schemes for various sets of originators by referring to the subgraphs of Fig. 1.

Observation 3.3. *If the originator is any black vertex in one of the five graphs in Fig. 1, then the other vertices in its graph can be informed in two time units.*

Each upper bound proof of $B_k(n) \le e$ will proceed by exhibiting a graph. To show the graph works for all possible choices of k originators we will give a set of *schemes* and each is given as a partition of the graph into subgraphs of the types discussed in Observation 3.3. When there are symmetries in the graph the set of schemes can be small. If a careful enumeration shows that every choice of originators can map to black vertices in separate subgraphs we say the schemes are *exhaustive*.

We will give specific values for small n and k. Clearly $B_2(3) = 2$ and $B_2(4) = 3$.

Theorem 3.4. $B_2(5) = 5$.

Proof. [$B_2(5) \le 5$] We show that any two originators can broadcast in two steps in graph $G = C_5$ (the cycle of five vertices, graph (a) of Fig. 2). Observe that given any two originators in the cycle, we can delete two edges from the cycle leaving a path of three vertices and a path of two vertices, subgraphs *a* and *d* from Fig. 1, with one originator in each of the subgraphs.

Often there is more than one graph that works. To illustrate, graph (b) of Fig. 2 is an alternate solution. Depending on the two originators chosen, we can delete two edges from this graph either leaving subgraphs *a* and *d* with one originator in each of the subgraphs or leaving subgraphs *e* and *c* with one originator in *e* and one originator on one of the black vertices of *c*. Henceforth only one graph is given.

[$B_2(5) > 4$] There are two possible degree sequences: $1^2 2^3$ and $1^3 2^1 3^1$. In both cases a connected graph has a degree 1 vertex x connected to a degree 2 vertex. Hence x is isolated. □

Please cite this article in press as: A.L. Liestman, D. Richards, Minimum multiple originator broadcast graphs, Discrete Applied Mathematics (2016), http://dx.doi.org/10.1016/j.dam.2016.06.015

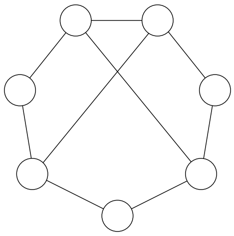

Fig. 3. Minimum two originator broadcast graph on 7 vertices.

Theorem 3.5. $B_2(6) = 6$.

Proof. $[B_2(6) \leq 6]$ Consider graph $G = C_6$. We can delete two edges from the cycle leaving two paths of three vertices each (two subgraphs a from Fig. 1) with one originator in each of the subgraphs.

$[B_2(6) > 5]$ There must be a degree 1 vertex x. Hence there is a bad separator, with $S_1 = \{x\}$. □

Theorem 3.6. $B_2(7) = 9$.

Proof. $[B_2(7) \leq 9]$ Consider graph G shown in Fig. 3. We regard this graph as consisting of an *outer* cycle on the seven vertices plus two chords. Given any two originators at least two edges apart on the outer cycle, we can delete two edges from the cycle and the two chords to leave subgraphs a and c of Fig. 1 with one originator in a and the other on one of the black vertices of c. Otherwise, the two originators are adjacent on the outer cycle. In this case, at least one of them must be of degree 3 and we can remove edges from G leaving subgraphs a and b with one originator in each subgraph or leaving subgraphs a and c with one originator in a and one originator on a black vertex of c.

$[B_2(7) > 8]$ Again, there can be no vertex of degree 1. The only possible degree sequence is $2^5 3^2$. Hence there must be two adjacent vertices of degree 2, u and v. There is a bad separator with $S_1 = \{u, v\}$. □

Theorem 3.7. $B_2(8) = 12$.

Proof. $[B_2(8) \leq 12]$ Any two originators can broadcast in time two in graph $G = Q_3$, the three dimensional hypercube. We can always remove four edges leaving two subgraphs b of Fig. 1 with one originator in each.

$[B_2(8) > 11]$ Again, there can be no vertex of degree 1. If there is a vertex x of degree 2 then there is a bad separator, with $S_1 = \{x\}$. Hence there is no possible degree sequence. □

Theorem 3.8. $B_3(7) = 7$.

Proof. $[B_3(7) \leq 7]$ Consider graph $G = C_7$. With any set of three originators, we can always remove three edges either leaving subgraphs a, a, and e of Fig. 1 with one originator in each subgraph or leaving subgraphs a, d, and d with one originator in each subgraph.

$[B_2(7) > 6]$ In any possible degree sequence there will be two degree 1 vertices u and v. If u and v meet at some vertex w then there is a bad separator, with $S_1 = \{u, v\}$. Suppose, now, that the degree 1 vertex u was adjacent to a degree 2 vertex x. In that case there is a bad separator, with $S_1 = \{u, v, x\}$. Hence each degree 1 vertex is paired with a vertex of degree 3 or more. It follows there is no possible degree sequence. □

Theorem 3.9. $B_3(8) = 9$.

Proof. $[B_3(8) \leq 9]$ Consider graph G shown in Fig. 4. If the three originators consist of any degree 2 vertex and both of its neighbors (one of degree 2 and the other of degree 3), then the two edges joining these originators and two other edges can be removed, leaving subgraphs a, c and e of Fig. 1 with the degree 3 originator on a black vertex of subgraph c. With any other set of three originators, we can always find a C_6 containing exactly two of the three originators leaving the other originator in a subgraph d of Fig. 1. It follows from the proof of Theorem 3.5.

$[B_3(8) > 8]$ Again, no two vertices of degree 1 can meet. Suppose there is degree 1 vertex x; to avoid being isolated x must be adjacent to a vertex of degree 5 or more. Hence there is no degree 1 vertex. The only possible degree sequence is 2^8. Such a graph must be single cycle which has a bad separator, with S_1 being three adjacent vertices. □

Theorem 3.10. $B_3(9) = 11$.

Proof. $[B_3(9) \leq 11]$ Consider graph G shown in Fig. 5. Scheme structures are shown in Fig. 6. These schemes are exhaustive.

$[B_3(9) > 10]$ This is proven in an identical manner to the proof of $B_4(9) > 10$ below. □

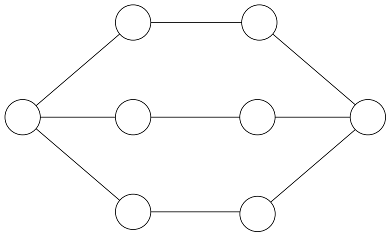

Fig. 4. Minimum three originator broadcast graph on 8 vertices.

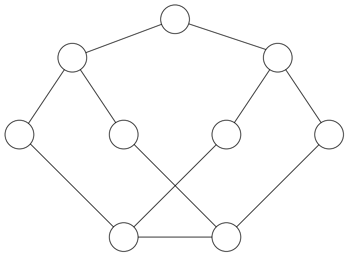

Fig. 5. Minimum three originator broadcast graph on 9 vertices.

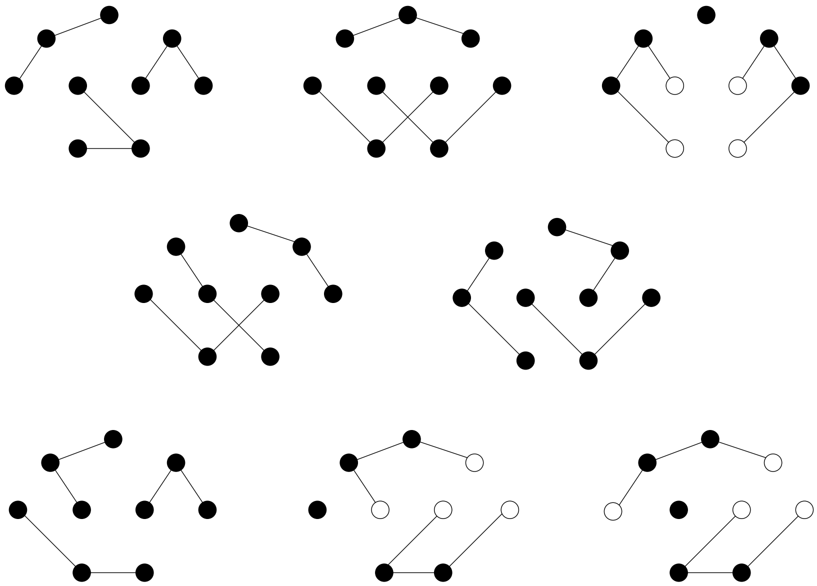

Fig. 6. Broadcast schemes for 3 originators on 9 vertices.

Theorem 3.11. $B_3(10) = 15$.

Proof. $[B_3(10) \leq 15]$ Consider graph G shown in Fig. 7. The exhaustive scheme structures are shown in Fig. 8.

$[B_3(10) > 14]$ As above, no two vertices of degree 1 can meet, and a degree 1 vertex must be adjacent to a vertex of degree 7 or more. Any degree 2 vertex x gives a bad separator with $S_1 = \{x\}$. Hence every possible degree sequence has a sum at least 30. □

Theorem 3.12. $B_3(11) = 18$.

Proof. $[B_3(11) \leq 18]$ Consider graph G shown in Fig. 9 with the exhaustive scheme structures shown in Fig. 10.

$[B_3(11) > 17]$ Again, no two vertices of degree 1 can meet, a degree 1 vertex must be adjacent to a vertex of degree 8 or more, and there can be no vertex of degree 2. Hence the only possible degree sequence is $3^{10}4^1$. Consider the four degree 3 neighbors of the vertex of degree 4. If two of these four degree 3 neighbors are connected then these two with the degree 4 vertex can inform only seven other vertices. Hence there is a set of three degree 3 vertices forming a cycle of length 4 with the degree 4 vertex. These require three time units. □

Please cite this article in press as: A.L. Liestman, D. Richards, Minimum multiple originator broadcast graphs, Discrete Applied Mathematics (2016), http://dx.doi.org/10.1016/j.dam.2016.06.015

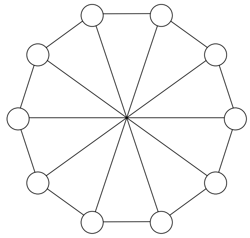

Fig. 7. Minimum three originator broadcast graph on 10 vertices.

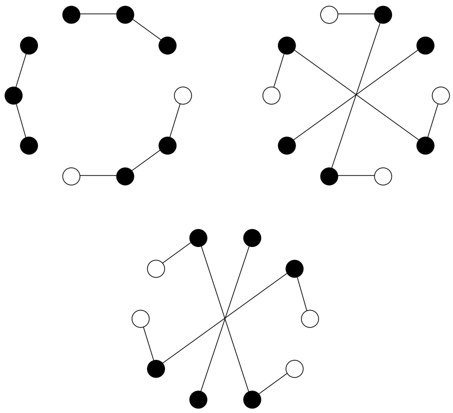

Fig. 8. Broadcast schemes for 3 originators on 10 vertices.

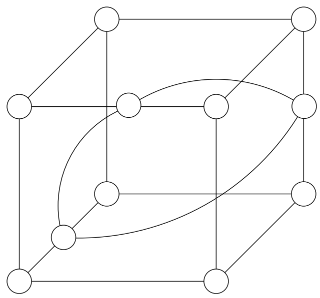

Fig. 9. Minimum three originator broadcast graph on 11 vertices.

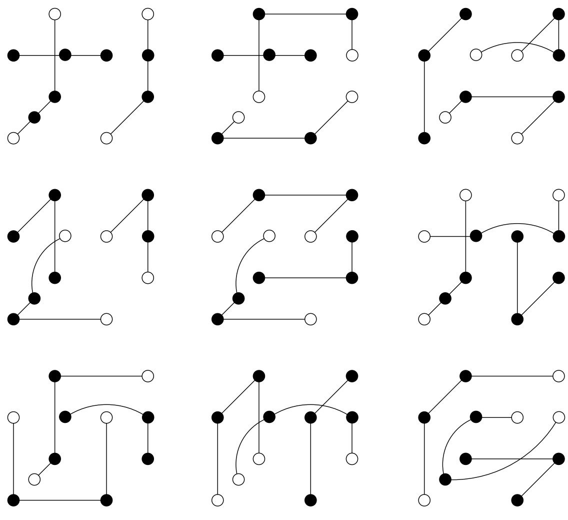

Fig. 10. Broadcast schemes for 3 originators on 11 vertices.

Please cite this article in press as: A.L. Liestman, D. Richards, Minimum multiple originator broadcast graphs, Discrete Applied Mathematics (2016), http://dx.doi.org/10.1016/j.dam.2016.06.015

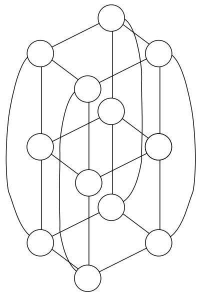

Fig. 11. Minimum three originator broadcast graph on 12 vertices.

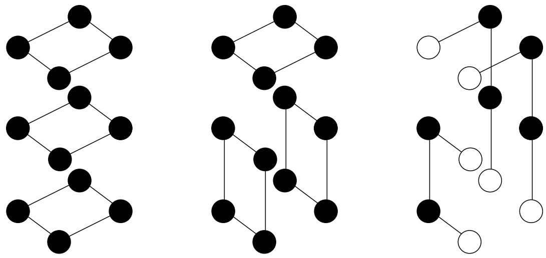

Fig. 12. Broadcast schemes for 3 originators on 12 vertices.

Theorem 3.13. $B_3(12) = 24$.

Proof. $[B_3(12) \leq 24]$ Consider graph G shown in Fig. 11 with the exhaustive scheme structures shown in Fig. 12. Note the symmetries here use the 3 cycles.

$[B_3(12) > 23]$ Again, no two vertices of degree 1 can meet, a degree 1 vertex must be adjacent to a vertex of degree 9 or more, and there can be no vertex of degree 2. Further if there is a degree 3 vertex x then there would be a bad separator with $S_1 = \{x\}$. □

Theorem 3.14. $B_4(9) = 11$.

Proof. $[B_4(9) \leq 11]$ We know that $B_4(9) \leq 11$ by combining Lemma 2.4 and Theorem 3.10.

$[B_4(9) > 10]$ Again, no two vertices of degree 1 can meet, and any degree 1 vertex must be adjacent to a vertex of degree 5 or more. Therefore the only possible degree sequence is $2^7 3^2$. There must be a vertex x which is adjacent to just two degree 2 vertices, and x must be isolated. □

Theorem 3.15. $B_4(10) = 12$.

Proof. $[B_4(10) \leq 12]$ Consider graph G shown in Fig. 13 with the exhaustive scheme structures shown in Fig. 14.

$[B_4(10) > 11]$ As in the previous proof, there is no degree 1 vertex. The only possible degree sequence is $2^8 3^2$. There must be a vertex x which is adjacent to two degree 2 vertices, and x must be isolated. □

Theorem 3.16. $B_4(11) = 16$.

Proof. $[B_4(11) \leq 16]$ Consider graph G shown in Fig. 16 with the exhaustive scheme structures shown in Fig. 17.

$[B_4(11) > 15]$ There are many cases in this proof so we state clearly which subgraphs are forbidden (with an indication of the reason).

Fact 1: There are no two vertices of degree 1. (Let S_1 be those two vertices.)

Fact 2: There is no 2-2, that is two adjacent degree 2 vertices. (Let S_1 be those two vertices.)

Fact 3: There is no 3-2-3. (Degree 2 isolated)

Fact 4: There is no x-2-3-3-2-x. (Let S_1 be those four vertices other than x.)

Fact 5: There is no degree 3 vertex connected to three degree 2 vertices. (Let S_1 be those.)

Fact 6: There is no x-2-y-2-x. (Let S_1 be the degree 2 vertices.)

Fact 7: There is no x-2-3-2-y-2-x. (Let S_1 be those vertices except x and y.)

Fact 8: There is no x-2-3-x. (Let S_1 be the degree 3 neighborhood.)

We proceed by examining each possible degree sequence, eliminating each one. As we progress we will omit appeals to arguments that are repetitive, especially appeals to Facts 2, 3, and 6.

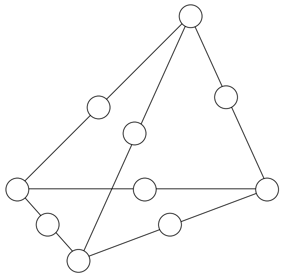

Fig. 13. Minimum four originator broadcast graph on 10 vertices.

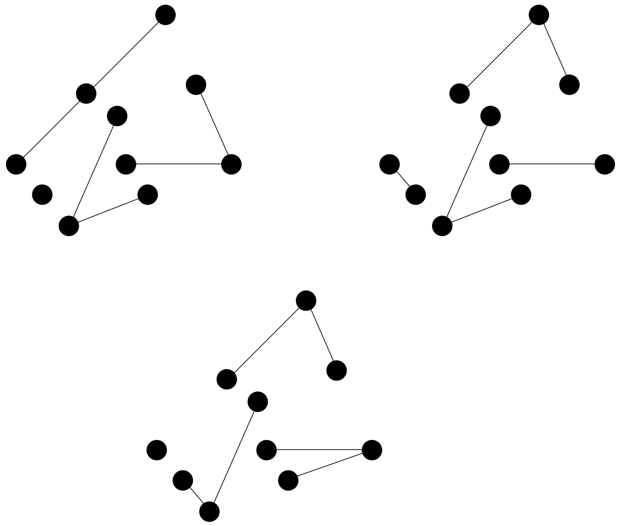

Fig. 14. Broadcast schemes for 4 originators on 10 vertices.

- $1^1 2^a \cdots$: By Fact 1 this is the only case with a degree 1 vertex. A degree 1 vertex w, to avoid being isolated, must be adjacent to a vertex z of degree 7 or more; hence $a \geq 5$. If $a = 5$ the only possible degree sequence is $1^1 2^5 3^4 7^1$. By Facts 2 and 3 all degree 2 vertices are adjacent to the degree 7 vertex. Fact 8 implies two of the degree 2 vertices must meet at a degree 3 vertex, violating Fact 6. If $a = 6$ the degree sequence is $1^1 2^6 3^2 4^1 7^1$ or $1^1 2^6 3^3 8^1$. In all cases Fact 6 will be violated unless three degree 2 vertices are adjacent to z, but then Fact 2 will be violated. If $a > 6$ there are fewer cases, all similar.
- $2^3 3^8$: Fact 2 or 3 must be violated.
- $2^4 3^6 4^1$: By Facts 2 and 3, each degree 2 vertex is connected to the degree 4 vertex and a unique degree 3 vertex. By Fact 4 those four degree 3 vertices must be adjacent only to the two remaining degree 3 vertices, which is impossible.
- $2^5 3^4 4^2$: Let the two degree 4 vertices be x and y; each degree 2 vertex is connected to x or y by Fact 3. If a degree 2 vertex is connected to both x and y then, by Facts 6 and 7, the other degree 2 vertices are each connected to a unique degree 3 vertex. Fact 4 forces each degree 4 vertex to have three degree 2 neighbors. If the degree 4 vertices are connected then S_1 is a bad separator, where S_1 is x and its degree 2 neighbors. Otherwise it contains degree 2 vertices, u and v, connected to degree 3 vertices w and z, such that w is connected to z and z is connected to x (or y). There is bad separator with $S_1 = \{u, v, w, z\}$.

  The remaining case is each degree 2 vertex is connected to a degree 4, x or y, and a degree 3 vertex. If x is connected to four degree 2 vertices they would, by Fact 6, need to be connected to unique degree 3 vertices and so Fact 4 must be violated. Hence, wlog, x is connected to three degree 2 vertices, u_1, u_2, u_3 and y is connected to two degree 2 vertices, v_1 and v_2. By Fact 6 each w_i has a distinct degree 3 neighbor w_i; wlog w_1 is not connected to v_1 or v_2. By Fact 8 u_1 is isolated from $\{w_2, w_3, v_1, v_2\}$.
- $2^5 3^5 5^1$: By Facts 2, 3, and 6, each degree 2 vertex is adjacent to the degree 5 vertex and a unique degree 3 vertex. A violation of Fact 4 follows.
- $2^6 3^2 4^3$: If there is no 2-3-2 Fact 6 must be violated. If there is exactly one 2-3-2, case avoiding the more common Facts will lead to a violation of Fact 7. Now we can assume there are two 2-3-2's, and they are disjoint by Facts 3 and 5. Facts 2, 3, 6, and 7 imply that at least those edges in Fig. 15 are forced. By Fact 8 the degree 3 vertices are connected to the bottom vertex. The graph is unique and it fails if the originators are 3-2-4-2 down the left side of the figure.

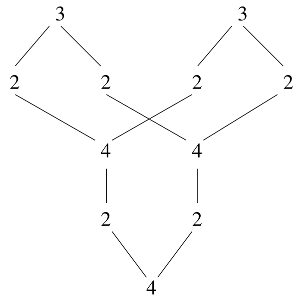

Fig. 15. Forced edges.

- $2^6 3^3 4^1 5^1$: Assume there is 4-2-5. In that case each remaining degree 2 vertex is connected to a degree 3 vertex and either the degree 4 or degree 5 vertex. There is a degree 3 vertex connected to two degree 2 vertices and this vertex will either violate Fact 6 or 7. So we can assume there is no 4-2-5. It follows there are three disjoint 2-3-2's and one degree 3 vertex is connected to the degree 4 vertex; this will violate Fact 6 (or 8).
- $2^6 3^4 6^1$: Each degree 2 vertex is connected to the degree 6 vertex and a degree 3 vertex, which must violate Fact 6.
- $2^a \cdots$: If $a > 6$ then are there not enough vertices to prevent a violation of Fact 2 or 3.

This completes the case analysis. □

Theorem 3.17. $B_4(12) = 18$.

Proof. $[B_4(12) \le 18]$ Consider graph G shown in Fig. 18 with the exhaustive scheme structures shown in Fig. 19.

$[B_4(12) > 17]$ As with the previous proof it will be necessary to begin by stating which subgraphs are forbidden. Actually Facts 1 through 8 still hold and we will use the same numbering. However Facts 3, 5 and 7 can be strengthened and we add a Fact 9.

Fact 3′: There is no 3-2-3 or 3-2-4. (Degree 2 isolated.)

Fact 5′;: There is no 2-3-2. (Let S_1 be those three vertices.)

Fact 7′: There is no x-2-y-2-z-2-x. (Let S_1 be x and the degree 2 vertices.)

Fact 9: If the degree sequence is $2^a 3^b 4^c$ then $a \le \binom{c}{2}$.

Fact 9 follows since, by Fact 3′, each degree 2 vertex x is connected to two degree 4 vertices, and, by Fact 6, x must not use the same two degree 4 vertices used by another degree 2 vertex. We proceed by examining each possible degree sequence, eliminating each one.

- $1^1 2^a \cdots$: By Fact 1 this is the only case with a degree 1 vertex. A degree 1 vertex w, to avoid being isolated, must be adjacent to a vertex z of degree 8 or more, and so $a \ge 5$. So there must be two degree 2 vertices adjacent to z. If two degree 2 vertices, x and y, meet at z then these vertices $\{w, x, y, z\}$ cannot inform the rest.
- $2^2 3^{10}$: Fact 3′ must be violated.
- $2^3 3^8 4^1$: Fact 3′ must be violated.
- $2^4 3^6 4^2$: Fact 9 is violated.
- $2^4 3^7 5^1$: By Facts 3′ and 6, each degree 2 is adjacent to the degree 5 and to a distinct degree 3 vertex. At least two of these degree 3 vertices, connected to the degree 2 vertices, must be adjacent, violating Fact 4.
- $2^5 3^4 4^3$: Fact 9 is violated.
- $2^5 3^5 4^1 5^1$: By Facts 2 and 3′, each degree 2 vertex is adjacent to the degree 5 vertex. By Fact 6 only one degree 2 vertex can now be adjacent to the degree 4 vertex. Among the at least four degree 3 vertices adjacent to degree 2 vertices there must be two that are adjacent. This violates Fact 4.
- $2^5 3^6 6^1$: This is very similar to the previous case, again violating Fact 4.
- $2^6 3^2 4^4$: By Fact 2 and 3′ and the proof of Fact 9 each degree 2 vertex is connected to a distinct pair of degree 4 vertices. Now, by symmetry, it can be seen that there is only one way to form a graph for this degree sequence: the two degree 3 vertices are adjacent and connected to the degree 4 vertices. A degree 2 vertex connected to the same vertices as a degree 3 vertex is isolated.
- $2^6 3^3 4^2 5^1$: By Facts 2 and 3′ we know that a degree 2 vertex is connected to both degree 4 vertices and the remaining five degree 2 vertices are connected to the degree 5 vertex. By Fact 6, three of the remaining degree 2 vertices are connected to distinct degree 3 vertices and the remaining two degree 2 vertices are connected to distinct degree 4 vertices. Now it follows that two of the degree 3 vertices are adjacent, violating Fact 4.
- $2^6 3^4 4^1 6^1$: By Facts 2 and 3′ each degree 2 vertex must be adjacent to either the degree 4 or the degree 6 vertex but, by Fact 6, only one can be adjacent to both. By the pigeon-hole principle, one degree 3 vertex is adjacent to two degree 2 vertices, violating Fact 5′.
- $2^6 3^4 5^2$: This is very similar to the previous case.
- $2^6 3^5 7^1$: All degree 2 vertices are connected to the degree 7 vertex, so one degree 3 vertex must violate Fact 5′.

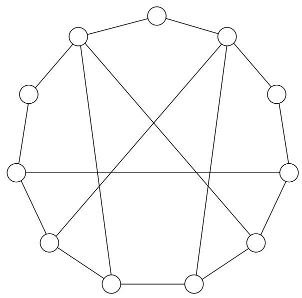

Fig. 16. Minimum four originator broadcast graph on 11 vertices.

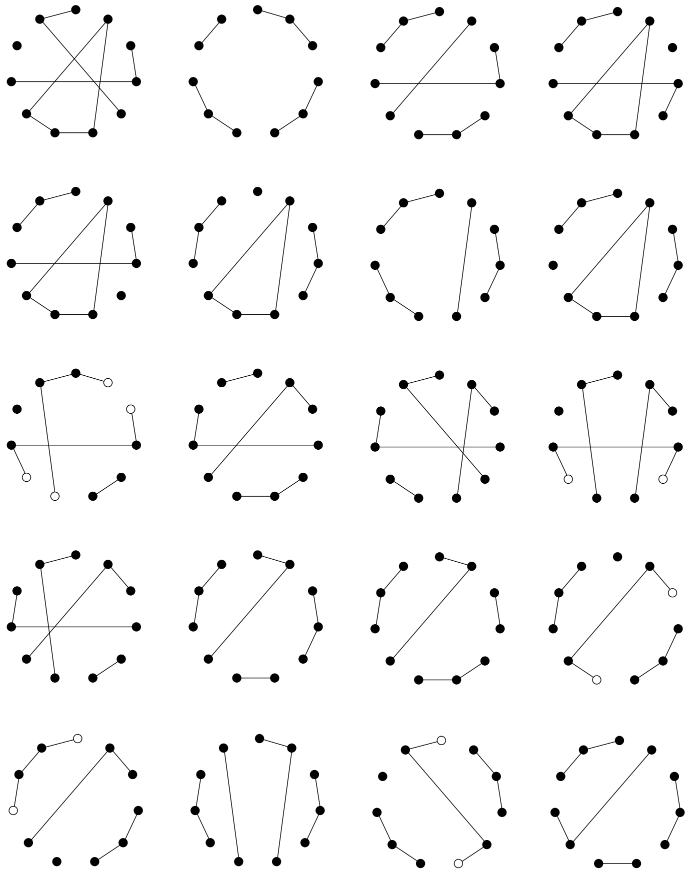

Fig. 17. Broadcast schemes for 4 originators on 11 vertices.

- $2^7 3^1 4^3 5^1$: Let y be the degree 3 vertex and z be the degree 5 vertex. First, y and z meet at unique degree 2 vertex x: by Fact 6 there cannot be more than one and, analogously to Fact 8, if no degree 2 was adjacent to y then two of the seven degree 2 vertices would have to violate Fact 6. Now y and z cannot be adjacent, otherwise there is a bad separator, with $S_1 = \{x, y, z\}$. By Fact 5′, y must be adjacent to two degree 4 vertices, u and v. However there is a bad separator, with $S_1 = \{x, y, u, v\}$.
- $2^7 3^2 4^1 5^2$: Suppose that five of the seven degree 2 vertices are adjacent to degree 3 vertices. By Facts 2 and 3′ these five vertices are also adjacent to the two degree 5 vertices. Now two of the these five degree 2 vertices must violate Fact 6 with a degree 3 and a degree 5 vertex. Hence there must be three degree 2 vertices – x, y, and z – adjacent to only the degree 4 vertex, w and the two degree 5 vertices. Note, by Fact 6, that w must be adjacent to at least two of these three vertices. There is a bad separator, with $S_1 = \{w, x, y, z\}$.
- $2^7 3^2 4^2 6^1$: This is very similar to the previous case.
- $2^7 3^3 4^1 7^1$: This is very similar to the case $2^6 3^4 4^1 6^1$.
- $2^7 3^3 5^1 6^1$: This is very similar to the previous case.
- $2^7 3^4 8^1$: This is very similar to the case $2^6 3^5 7^1$.

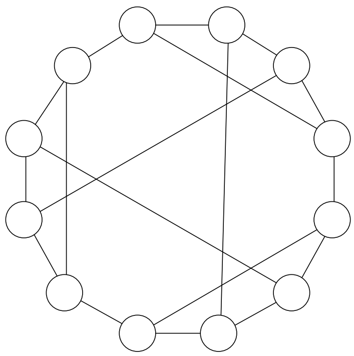

Fig. 18. Minimum four originator broadcast graph on 12 vertices.

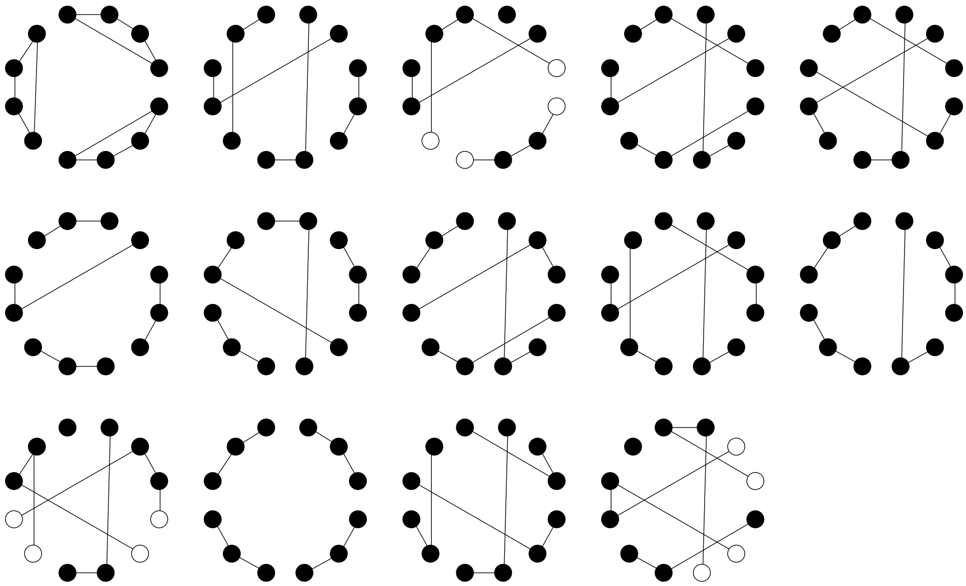

Fig. 19. Broadcast schemes for 4 originators on 12 vertices.

- $2^7 4^5$: By Fact 2 each degree 2 vertex is connected to two degree 4 vertices. (Note the graph is essentially a complete graph, K_5, with seven edges subdivided by a degree 2 vertex.) There are two cases. First, three of the degree 4 vertices – u, v, and w – are in a 3-cycle. Let x and y be degree 2 vertices connecting u and v to another vertex. The set $\{u, v, x, y\}$ cannot broadcast. Second, there is a 4-cycle with just one degree 2 vertex w, but w is isolated.
- $2^8 \cdots$: By Fact 2 each of the eight degree 2 vertices connect to two of the remaining four vertices. Since $8 > \binom{4}{2}$ Fact 6 is violated. Similarly, for any degree sequence $2^a \cdots$, with $a > 8$.

This completes the case analysis. □

Theorem 3.18. $B_5(11) = 14$.

Proof. [$B_5(11) \leq 14$] Consider graph G shown in Fig. 20 with the exhaustive scheme structures shown in Fig. 21.

[$B_5(11) > 13$] As in the previous proofs, no two vertices of degree 1 can meet, and any degree 1 vertex must be adjacent to a vertex of degree 6 or more. For any degree 2 vertex x, if the sum of the degrees of the neighbors of x is 5 or less then x will be isolated. If there is a degree 1 vertex the only sequences are $1^1 2^8 3^1 6^1$ and $1^1 2^9 7^1$; however all the degree 2 vertices need to be adjacent to the high degree vertex. If there is a degree 1 vertex the only sequence is $1^1 2^8 3^1 6^1$; however all degree 2 vertices must be attached to the degree 6 vertex. There are five possible degree sequences. Consider $2^7 3^4$. At least one degree 2 vertex will be adjacent to another degree 2 vertex and so will be isolated. Consider $2^8 3^2 4^1$. To avoid isolation each of the eight degree 2 vertices must be adjacent to the single degree 4 vertex, which is not possible. Similar reasoning applies to the remaining cases: $2^9 3^1 5^1$, $2^9 4^2$, and $2^{10} 6^1$. □

Theorem 3.19. $B_5(12) = 17$.

Please cite this article in press as: A.L. Liestman, D. Richards, Minimum multiple originator broadcast graphs, Discrete Applied Mathematics (2016), http://dx.doi.org/10.1016/j.dam.2016.06.015

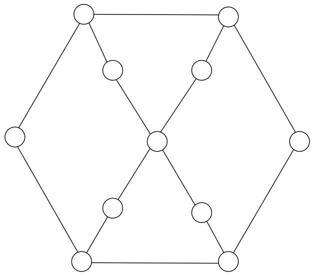

Fig. 20. Minimum five originator broadcast graph on 11 vertices.

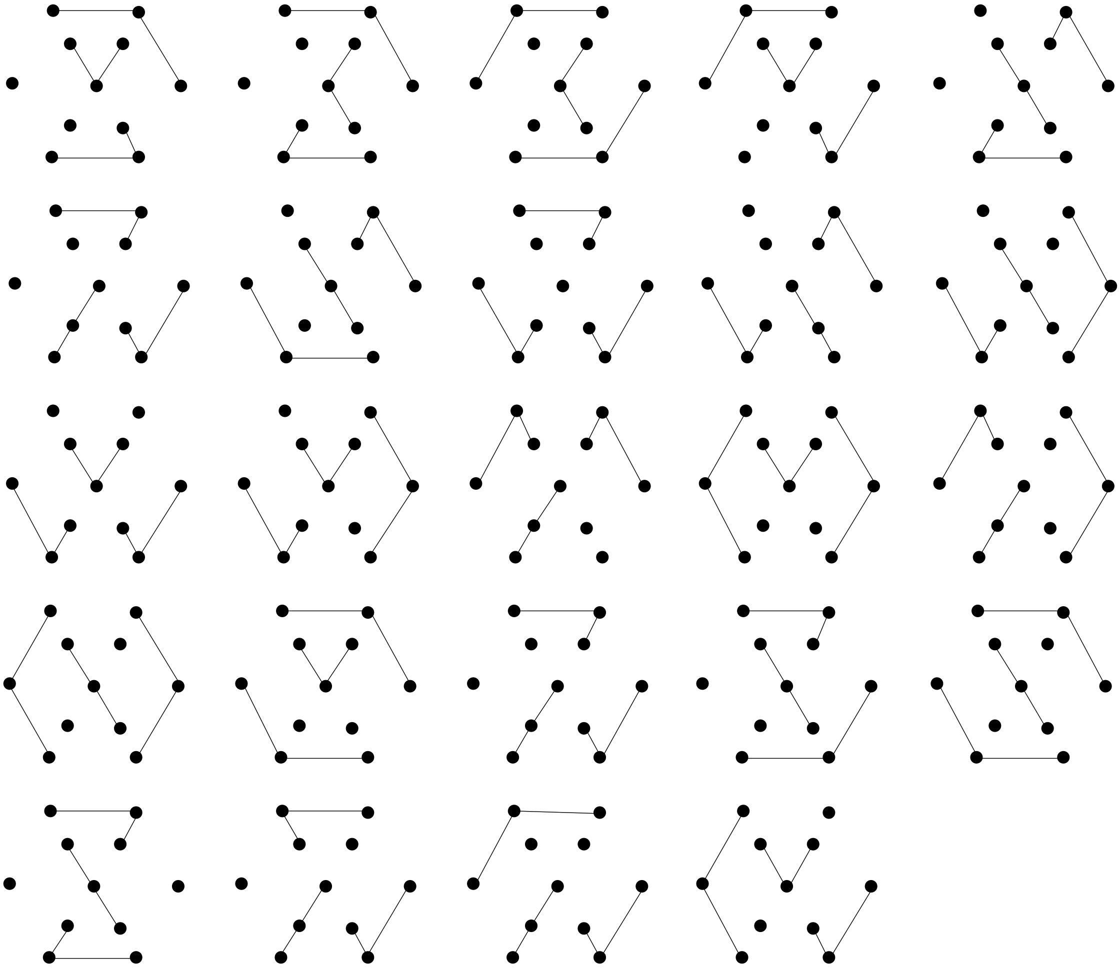

Fig. 21. Broadcast schemes for 5 originators on 11 vertices.

Proof. [$B_5(12) \leq 17$] Consider graph G shown in Fig. 22 with the exhaustive scheme structures shown in Fig. 23. Note that in many of these schemes only four components appear and that is because only four of the five originators need to be invoked (which reduces the case analysis).

[$B_5(12) > 16$] There are several facts that quickly reduce the cases. A degree 2 vertex is isolated if the sum of the degree of its neighbors is not at least 7. A degree 3 connected to only degree 2 vertices is isolated. Two degree 2 vertices cannot meet at x and y and the sum of those two degrees is less than 12. Other forbidden subgraphs, because of bad separators, are: triangles 4-3-2, 4-4-2 and 5-3-2; x-2-3-3-2-x; x-2-2-x; x-2-2-y-2-x; x-2-2-y-2-2-x.

- $1^a \ldots$: As in the previous proofs, no two vertices of degree 1 can meet, and any degree 1 vertex must be adjacent to a vertex of degree 7 or more. If $a = 2$ then the sequence is $1^2 2^8 7^2$ and there are two degree 2 vertices connected to the same degree 7 vertex; these cannot both be reached. When $a = 1$ let z be the vertex adjacent to the degree 1 vertex. If x and y are two degree 2 vertices adjacent to z then x and y do not meet at another vertex or connect to each other; further

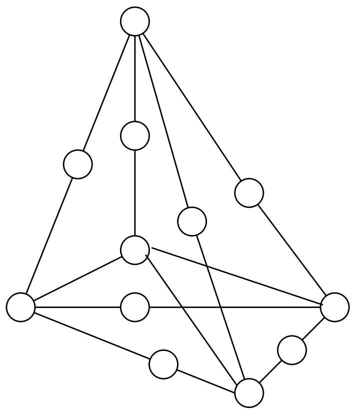

Fig. 22. Minimum five originator broadcast graph on 12 vertices.

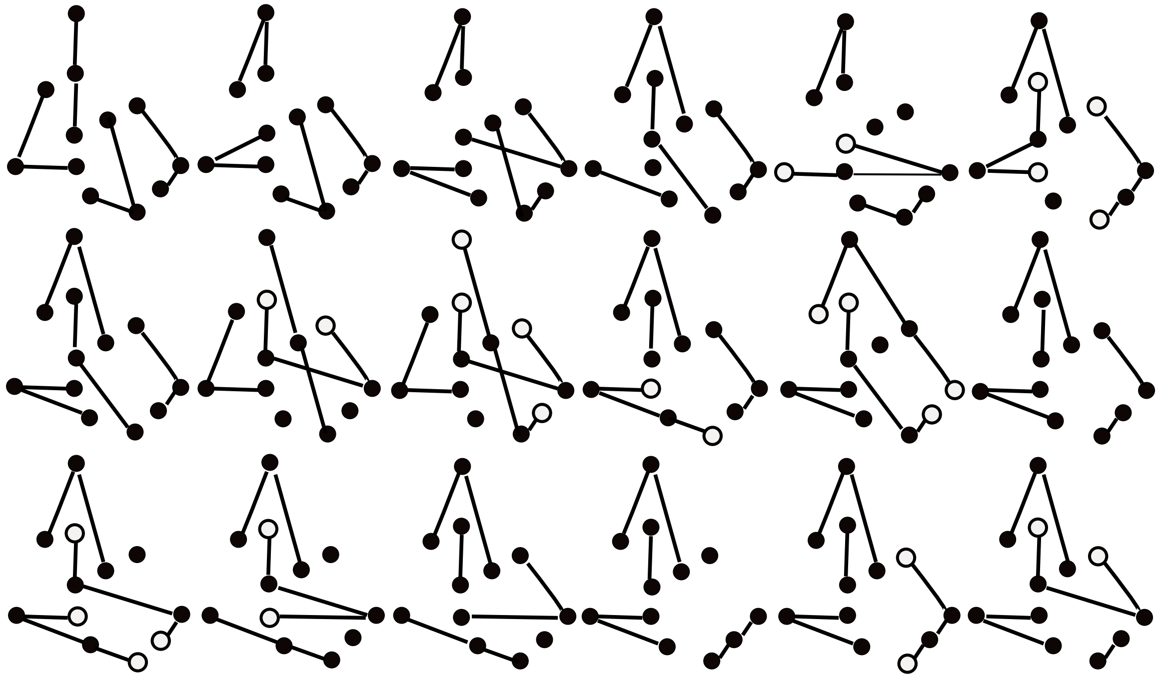

Fig. 23. Broadcast schemes for five originators on 12 vertices.

x cannot connect to another degree 2 vertex. There are twelve possible degree sequences, from $1^1 2^6 3^4$ to $1^1 2^{10} 11^1$. The above facts, with simple case analyses, exclude each and every one of these sequences.

- $2^5 3^6 4^1$: All the degree 2 vertices need to attach to the degree 4 vertex. Similarly for $2^6 3^5 5^1$, $2^7 3^4 6^1$, $2^8 3^3 7^1$, $2^9 3^2 8^1$, $2^{10} 3^1 9^1$, and $2^{11} 10^1$.
- $2^6 3^4 4^2$: There can be at most one 4-2-4. If there are two degree 3 vertices not connected to other degree 3 vertices the facts preclude any graph. If there is just one degree 3 vertex not connected to other degree 3 vertices a 4-3-2 triangle exists. So there are two vertex-disjoint 3-3 pairs and x-2-3-3-2-x is forced.
- $2^7 3^2 4^3$: Let S_2 be the degree 4 vertices. If it has four 4-2-4's it has a bad separator with S_2. Otherwise two degree 2 vertices meet at a degree 3 and degree 4 vertices.
- $2^7 3^3 4^1 5^1$: Avoiding the other facts there must x-2-3-3-2-x.
- $2^8 3^2 4^1 6^1$: Each degree 3 vertex connects to only degree 2 vertices. Similarly for $2^8 3^1 4^2 5^1$ and $2^8 3^2 5^2$.
- $2^8 8 4^4$: Avoiding the other facts there must be a x-2-2-x. Similarly for $2^9 3^1 4^1 7^1$
- $2^9 3^1 5^1 6^1$: Avoiding the other facts there must be a x-2-2-y-2-2-x.
- $2^9 4^2 6^1$: Avoiding the other facts there must be a x-2-2-y-2-x. Similarly for $2^9 4^1 5^2$.
- $2^{10} 5^1 7^1$: Avoiding the other facts there must be a x-2-2-y-2-2-x. Similarly for $2^{10} 6^2$. □

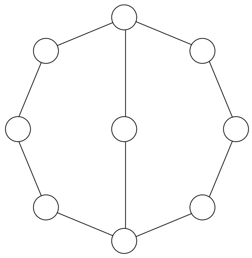

Fig. 24. Minimum two originator broadcast graph on 9 vertices.

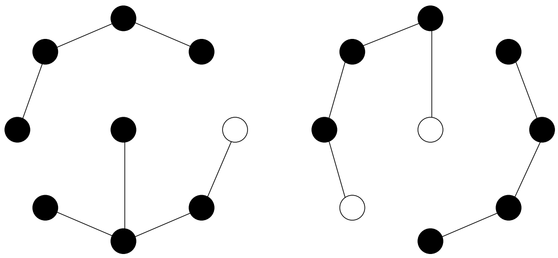

Fig. 25. Broadcast schemes for 2 originators on 9 vertices.

4. Specific values for time 3

In three time units, each originator can inform at most seven other vertices, and to do so, it must have at least three non-originator neighbors. The lower bounds here follow the same notation and conventions as in the previous section. In particular, a vertex x is *isolated* if there are at least k vertices that cannot be reached in three steps from x. The upper bounds are similar to those of the previous section. As in the previous section, these theorems fill in a missing column of Table 1.

Theorem 4.1. $B_2(9) = 10$.

Proof. $[B_2(9) \leq 10]$ Consider graph G shown in Fig. 24 with the exhaustive scheme structures shown in Fig. 25.

$[B_3(9) > 9]$ A degree 1 vertex must be adjacent to a vertex with degree at least 7 to avoid being isolated. If two degree 1 vertices, u and v, meet at a vertex, then u and v cannot be successful as originators. Since any degree sequence with a degree 1 vertex must have a sum at least 22, it follows that there are no degree 1 vertices. Hence the degree sequence must be 2^9 corresponding to C_9. Two adjacent originators in a C_9 cannot succeed. □

Theorem 4.2. $B_2(10) = 12$.

Proof. $[B_2(10) \leq 12]$ Consider the graph G shown in Fig. 26 with the exhaustive scheme structures shown in Fig. 27.

$[B_3(10) > 11]$ Again, there are no degree 1 vertices. Hence the degree sequence must be $2^8 3^2$. Such a graph must have a degree 2 vertex x adjacent to two degree 2 vertices, y and z. However, x and y cannot be successful as originators. □

Theorem 4.3. $B_2(12) = 15$.

Proof. $[B_2(12) \leq 15]$ To see that $B_2(12) \leq 15$, we show that any two originators can broadcast in time three in graph G shown in Fig. 28. Scheme structures are shown in Fig. 29. In these structures, any two black vertices, one from each connected component, may broadcast in 3 time units. The schemes (or their reflections and rotations) cover all possible sets of two originators.

$[B_2(12) > 14]$ Again, there are no degree 1 vertices, and no adjacent degree 2 vertices. There are only five possible degree sequences: $2^8 3^4$, $2^9 3^2 4^1$, $2^{10} 3^1 5^1$, $2^{10} 4^2$, and $2^{11} 6^1$. In all five cases there must be two adjacent degree 2 vertices, a contradiction. □

Theorem 4.4. $B_2(11) = 14$.

Proof. $[B_2(11) \leq 14]$ $B_2(11) \leq 14$ follows from Lemma 2.5 and the fact that $B_2(12) = 15$.

$[B_2(11) > 13]$ Again, there are no degree 1 vertices. Suppose the graph has degree 2 vertex x adjacent to a degree 2 vertex y; x and y cannot be successful as originators. There are only five possible degree sequences: $2^7 3^4$, $2^8 3^2 4^1$, $2^9 3^1 5^1$, $2^9 4^2$, and $2^{10} 6^1$. In all five cases there must a two adjacent degree 2 vertices. □

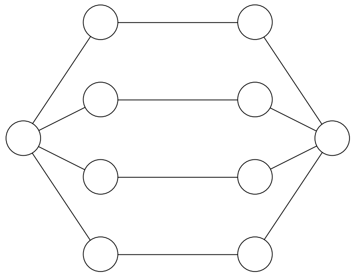

Fig. 26. Minimum two originator broadcast graph on 10 vertices.

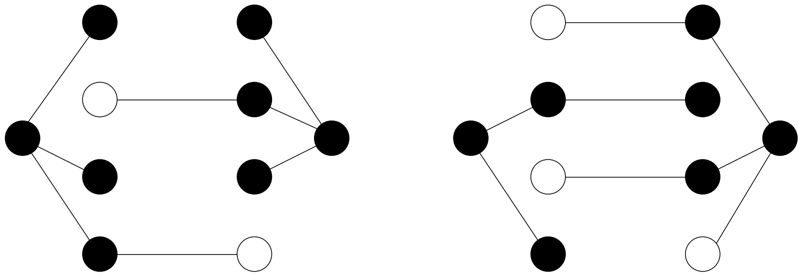

Fig. 27. Broadcast schemes for 2 originators on 10 vertices.

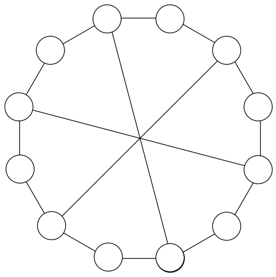

Fig. 28. Minimum two originator broadcast graph on 12 vertices.

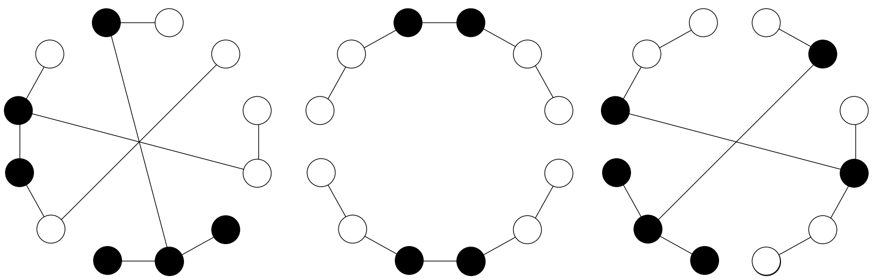

Fig. 29. Broadcast schemes for 2 originators on 12 vertices.

5. Conclusions

Except for $k = 1$ there are no prior results. All other minimum broadcast graph are in this paper. Table 1 summarizes all known values of $B_k(n)$ for $n \leq 13$. The values for $k = 1$ are from the surveys.

Prior research for $k = 1$ leads us to expect larger cases will be difficult. However by emphasizing the parallel methods used in our various proofs we believe that these methods can be extend somewhat further than we have gone, but as the cases increase computer-aided search may be needed to find an exhaustive set of schemes.

Again for $k = 1$ prior research has succeeded in find minimum broadcast graph for families of larger ns, in particular for groups of values below powers of 2. There is hope similar results can be attained for larger ks.

Table 1
Known values of $B_k(n)$.

$n:k$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	1														
3	2	2													
4	4	3	2												
5	5	5	5	3											
6	6	6	9	6	3										
7	8	9	7	11	7	4									
8	12	12	9	16	12	8	4								
9	10	10	11	11	18	14	9	5							
10	12	12	15	12	25	20	15	10	5						
11	13	14	18	16	14	28	22	17	11	6					
12	15	15	24	18	17	36	30	24	18	12	6				
13	18						39	33	26	20	13	7			

References

[1] A. Averbuch, R. Hollander Shabtai, Y. Roditty, Efficient construction of broadcast graphs, Discrete Appl. Math. 171 (2014) 9–14.
[2] M.J. Dinneen, J.A. Ventura, M.C. Wilson, G. Zakeri, Compound constructions of broadcast networks, Discrete Appl. Math. 171 (1999) 205–232.
[3] A. Farley, S. Hedetniemi, S. Mitchell, A. Proskurowski, Minimum broadcast graphs, Discrete Math. 25 (1979) 189–193.
[4] A. Farley, A. Proskurowski, Broadcasting in trees with multiple originators, SIAM J. Algebr. Discrete Methods 2 (1981) 381–386.
[5] P. Fraigniaud, E. Lazard, Methods and problems of communication in usual networks, Discrete Appl. Math. 53 (1994) 79–133.
[6] H. Grigoryan, H.A. Harutyunyan, New lower bounds on broadcast function, Lecture Notes in Comput. Sci. 8546 (2014) 174–184.
[7] H.A. Harutyunyan, A.L. Liestman, J.G. Peters, D.S. Richards, Broadcasting and gossiping, in: Handbook of Graph Theory, Chapman and Hall, 2013, pp. 1477–1494.
[8] S.T. Hedetniemi, S.M. Hedetniemi, A.L. Liestman, A survey of broadcasting and gossiping in communication networks, Networks 18 (1988) 319–349.
[9] J. Hromkovič, R. Klasing, B. Monien, R. Peine, Dissemination of information in interconnection networks (broadcasting & gossiping), in: Combinatorial Network Theory, Science Press & AMS, 1994, pp. 125–212.
[10] L.H. Khachatrian, H.A. Harutyunyan, Construction of new classes of minimal broadcast networks, in: Proc. International Colloquium on Coding Theory, Armenia, 1990, pp. 69–77.
[11] L.H. Khachatrian, H.A. Harutyunyan, On optimal broadcast graphs, in: Proc. International Colloquium on Coding Theory, Armenia, 1991, pp. 65–72.
[12] R. Labahn, A minimal broadcast graph on 63 vertices, Discrete Appl. Math. 53 (1994) 247–250.
[13] A.L. Liestman, Dana Richards, L. Stacho, Broadcasting from multiple originators, Discrete Appl. Math. 157 (2009) 2886–2891.
[14] S. Mitchell, S. Hedetniemi, A census of minimum broadcast graphs, J. Comb. Inf. Syst. Sci. 5 (1980) 141–151.
[15] J.-G. Zhou, K.-M. Zhang, A minimal broadcast graph on 26 vertices, Appl. Math. Lett. 14 (2001) 1023–1026.

Please cite this article in press as: A.L. Liestman, D. Richards, Minimum multiple originator broadcast graphs, Discrete Applied Mathematics (2016), http://dx.doi.org/10.1016/j.dam.2016.06.015