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Some minimum broadcast graphs

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Abstract

In this paper, we give the size $B(n)$ of a minimum broadcast graph of order n for the values 20, 21 and 22 of n, improve the upper bound of $B(23)$, and by corollary of some other $B(n)$.

1. Introduction

Let G be a connected network of order n, in which any node knows a message at the start of the process.

Broadcasting is the problem of informing every other node of this message in a minimum time, starting from any node as originator. During the process, each call requires one unit of time, and at each unit of time, several calls may be executed only on a set of independent edges.

Given a vertex x as originator, we define the *broadcast time* of x to be the minimum number $b(x)$ of time units required to complete broadcasting from vertex x. Since at each unit of time, the number of informed vertices can at most double, an obvious lower bound for $b(x)$ is $\lceil \log_2 n \rceil$.

The *broadcast time* $b(G)$ of the graph is $\max\{b(x) \mid x \in G\}$. We say that G (of order n) is a *broadcast graph* if $b(G) = \lceil \log_2 n \rceil$.

The *broadcast function* $B(n)$ is the minimum number of edges in any broadcast graph of order n. A *minimum broadcast graph* is a broadcast graph of size $B(n)$. We also call it *mbg*.

2. Known results

In [2], Farley et al. studied the broadcast function $B(n)$; in particular they determined the values of $B(n)$ for $n \leqslant 15$ and proved that $B(2^k) = k2^{k-1}$. Mitchell and

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Hedetniemi [4] determined the value $B(17)$ and Bermond et al. [1] established the values of $B(n)$ for $n = 18, 19, 30$ and 31, the first value coming from [6]. They also give the value 35 as an upper bound for $B(23)$.

In an unpublished manuscript [6], Wang proved a lemma which is useful for giving a lower bound of $B(n)$ when $B(n-1)$ is known. We give a corollary of this result at the beginning of the next section. Other and better techniques are presented in [3].

3. Study of $B(n)$ for $20 \leqslant n \leqslant 23$

3.1. Preliminary results

We will use in our proofs the following corollary of the result of Wang.

Proposition 3.1. *For n not a power of 2, we have $B(n+1) \geqslant \min\{B(n)+1, 3(n+1)/2\}$.*

Proof. If a minimum broadcast graph on $n+1$ vertices has a minimum degree not less than 3, its size is at least the second value. Otherwise, it possesses a vertex of degree 1 or 2, and the contraction of an edge incident to this vertex gives a broadcast graph on n vertices with one less edge. □

Let us define, for integers t, Δ, δ, with $\Delta \geqslant \delta$, $\Delta \geqslant 2$, $\delta \geqslant 1$, the sequence of integers $n'(\Delta,t) = 0$ if $t < 0$, $n'(\Delta,0) = 1$, $n'(\Delta,t) = 1 + \sum_{i=1}^{\Delta-1} n'(\Delta,t-i)$ and the integers $n(\Delta,\delta,t) = 1 + \sum_{i=1}^{\delta} n'(\Delta,t-i)$ and $n(\Delta,t) = n(\Delta,\Delta,t)$. These values are easy to compute, for instance we give below the first values for $t \geqslant 0$:

$\Delta \backslash t$	0	1	2	3	4	5
2	1	2	3	4	5	6
3	1	2	4	7	12	20
4	1	2	4	8	15	28
5	1	2	4	8	16	31

In order to make explanations more precise, we call *T-degree* of a vertex its degree in a broadcast tree T. This value depends on the choice of the tree, but it is clear that it is at most its degree in the graph.

Intuitively, one can see (and we prove that fact in Proposition 3.3) that the value $n(\Delta,\delta,t)$ will give the order of maximum tree rooted at a vertex of degree δ, with maximum degree Δ and broadcasting in time t, the value $n'(\Delta,t-i)$ corresponding to the subtree rooted at the ith son. Note that the T-degree of a son in the subtree rooted at itself is no more than $\Delta - 1$, which explains why we need two distinct quantities

n and n'. Note that since the maximum degree in a tree broadcasting in time t is no more than t, the values of δ and Δ have no effect on those of $n'(\Delta, t)$ and $n(\Delta, \delta, t)$ that is to say, if the minimum degree is at least the time of broadcasting, $n(\Delta, \delta, t)$ gives the order of a maximum broadcast tree in time t.

Actually we can easily obtain the following results.

Proposition 3.2. (a) *For $t \leqslant \Delta - 1$, we have $n'(\Delta, t) = n(\Delta, t) = 2^t$.*

(b) *For $\Delta \leqslant t \leqslant 2\Delta - 1$, we have $n'(\Delta, t) = 2^t - (t - \Delta + 1)2^{t-\Delta}$ and $n(\Delta, t) = 2^t - (t - \Delta)2^{t-\Delta}$.*

Proof. (a) Since $t - \Delta < 0$ implies $n'(\Delta, t - \Delta) = 0$, we have $n(\Delta, t) = n'(\Delta, t)$. We obtain the value of $n'(\Delta, t)$ by induction on t. The result is obvious for $t = 0$ and if $t \leqslant \Delta - 1$, the induction hypothesis gives $n'(\Delta, t) = 1 + \sum_{i=1}^{\Delta-1} 2^{t-i} = 1 + (2^t - 1) = 2^t$.

(b) Since in this case $0 \leqslant t - \Delta \leqslant \Delta - 1$, implying by (a) $n'(\Delta, t - \Delta) = 2^{t-\Delta}$, the value of $n(\Delta, t) = n'(\Delta, t) + n'(\Delta, t - \Delta)$ comes immediately from that of $n'(\Delta, t)$.

We obtain this last value by induction hypothesis, (a) giving $n'(\Delta, \Delta) = 1 + \sum_{i=1}^{\Delta-1} 2^i = 1 + (2^\Delta - 2) = 2^\Delta - 1$ for the beginning of our induction. Now if t is between the two given values, the induction hypothesis together with (a) gives

$$\begin{aligned} n'(\Delta, t) &= 1 + \sum_{i=1}^{\Delta-1} 2^{t-i} - \sum_{j=0}^{t-\Delta-1} (j+1)2^j \\ &= 1 + (2^t - 2^{t-\Delta+1}) - ((t - \Delta - 1)2^{t-\Delta} + 1) \\ &= 2^t - (t - \Delta + 1)2^{t-\Delta}. \quad \square \end{aligned}$$

Proposition 3.3. *A broadcast tree rooted at a vertex of degree δ, with maximum degree Δ and broadcasting in time t can have at most $n(\Delta, \delta, t)$ vertices.*

Proof. The proof is by induction on t. The result is obvious for $t = 0$. Suppose it is true for $t - 1$ and let x be the root of the tree, $y_1, \ldots, y_\delta$ its neighbours labeled in the order of the broadcasting. The subtree rooted at y_i has by induction hypothesis at most $n(\Delta, \Delta - 1, t - i) = n'(\Delta, t - i)$ thus giving the result. $\square$

Note, as we already said, that if $t \leqslant \Delta$ we have for $n(\Delta, \delta, t)$ the value $2^t - 2^{t-\delta} + 1$, which is the upper bound for the number of nodes of a tree rooted at a vertex of degree δ, broadcasting in time t and with no restriction on the maximum degree.

The following result will be useful in the last part of Section 3.4.

Proposition 3.4. *Let k, l be nonnegative integers, and put $b(l, k) = \sum_{j \geqslant l} \binom{k}{j}$. Consider a broadcasting of a graph starting at any vertex. Then at most $b(l, k)$ new vertices at distance not less than l of the initializer, are informed at time unit k.*

Proof. By induction on k. Since by definition $b(l,0) = 0$ if $l > 0$ and 1 if $l = 0$ the result is true for $k = 0$. On the other hand if $l = 0$ we have $b(0,k) = 2^k$ and the result is well known. Now let k and l be positive: by induction hypothesis, at time $k-1$, at most $\sum_{i=0}^{k-1} b(l-1,i)$ vertices at distance $\geqslant l-1$ have been informed. The vertices at distance $\geqslant l$ must be informed by the previous, thus it remains to prove that this number equals $b(l,k)$. But if $l > k$ these two quantities are equal to 0, and otherwise

$$\sum_{i=0}^{k-1} b(l-1,i) = \sum_{i=0}^{k-1} \sum_{j \geqslant l-1} \binom{i}{j} = \sum_{i=l-1}^{k-1} \sum_{j=l-1}^{i} \binom{i}{j}$$

$$= \sum_{j \geqslant l-1} \sum_{i=j}^{k-1} \binom{i}{j} = \sum_{j \geqslant l-1} \binom{k}{j+1} = b(l,k). \qquad \square$$

3.2. *Minimum broadcast graph on 20 vertices*

Theorem 3.5. $B(20) = 26$.

Proof. It is known that $B(19) = 25$ [1], so Proposition 3.1 gives $B(20) \geqslant 26$. We are going to prove that the following graph G [see Fig. 1(a)] is a minimum broadcast graph on 20 vertices with size $B(20) = 26$.

In G we find four types of vertices:

- A vertex a of the first type has degree 2, is adjacent to another one of the same type, and to a vertex of degree 4.
- A vertex b of the second type has degree 4. Its neighbours are two vertices of the first type, one of the second type, and one of degree 3.
- A vertex c of the third type has degree 2, and two neighbours of degree 3.
- A vertex d of the fourth type has degree 3. Its neighbours are one vertex of the second type and two of the third type.

It is sufficient to describe a broadcast tree for each type. Remark that a broadcast tree for a vertex a whose first branch is ab gives a broadcast tree of root b with first

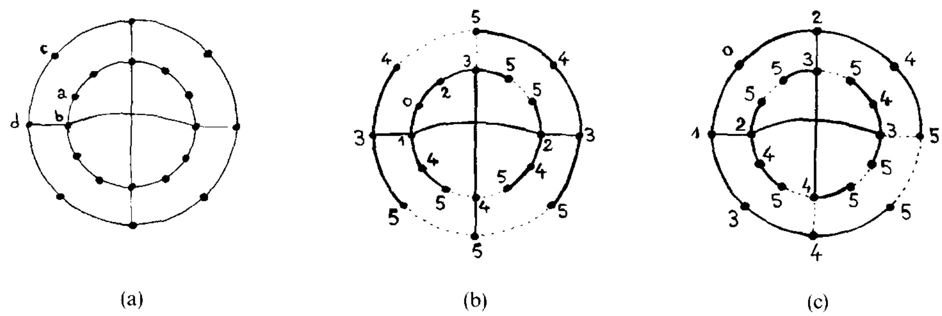


Fig. 1.

branch ba. An analogous remark allows us to give only a broadcast tree for a vertex c, whose first branch is cd.

Fig. 1(b) shows the first type of the broadcast tree and Fig. 1(c) the other one. □

Note that, by contracting either the edge ab or the edge cd, we obtain two nonisomorphic minimum broadcast graphs on 19 vertices. These two graphs are not isomorphic to the one known [1].

3.3. *Minimum broadcast graph on 21 vertices*

From Proposition 3.1, we know that $B(21) \geqslant 27$. The Fig. 2(a) describes a broadcast graph of size 28, thus $B(21) \leqslant 28$. Since there are only two types of vertices (the set of vertices of degree 2, and that of vertices of degree 4), it suffices to give a broadcast tree rooted at a vertex of degree 2 whose first branch is incident to a vertex of degree 4. This tree is shown in Fig. 2(b).

Theorem 3.6. $B(21) = 28$.

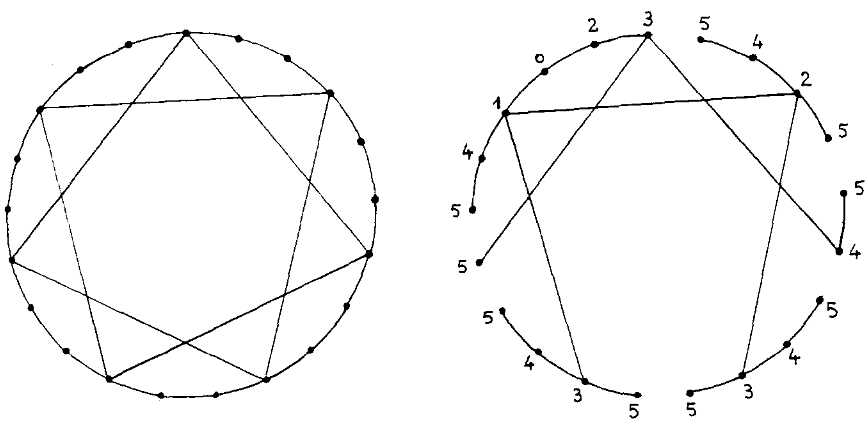
Proof. It remains to prove that no broadcast graph of order 21 can have 27 edges. Let us suppose that the contrary is true. It is clear that the minimum degree of such a graph G is 2. Since $n(3,2,5) = n'(3,5) = 20$, the maximum degree is at least 4.

Let us denote by V_i the set of vertices of degree i in the graph G, and n_i the cardinality of this set. We have the relations: $\sum_{i \geqslant 2} n_i = 21$ and $\sum_{i \geqslant 2} i n_i = 54$ hence

$$n_2 = 9 + \sum_{i \geqslant 4} (i-3) n_i \tag{1}$$

and

$$n_3 = 12 - \sum_{i \geqslant 4} (i-2) n_i. \tag{2}$$



(a) (b)

Fig. 2.

We will first establish some restrictions on the degrees of a vertex and its neighbours, then translate them into equations, which will induce a few possibilities for G, and finally conclude by contradiction. Many easy details will be left to the reader, others may be found in [5]. Inspection of broadcast trees rooted at various vertices give the following properties.

• If there is a vertex x whose neighbours are all of degree 2, its degree is at least 5. In particular, a vertex of degree 2 has at most one neighbour of the same degree. Let V'_2 (resp. V''_2) be the set of vertices in V_2 having none (resp. one) neighbour in V_2.

• Let x be a vertex in V''_2, and denote by y its neighbour not in V''_2. Then y has at least degree 4 and has at least one neighbour of degree $\geqslant 3$. Moreover if y is of degree 4, it has at most two neighbours in V_2.

• Let us put a weight on the edges between V_2 and the other vertices: $w(e) = 1$ if e is incident to V'_2 and $w(e) = 2$ if e is incident to V''_2. Now for every $x \notin V_2$ we put $q(x) = \sum_{e \ni x} w(e)$. We have the equality $\sum_{x \notin V_2} q(x) = 2n_2$. By the previous remarks, we obtain $q(x) \leqslant 2$ for any $x \in V_3$, $q(x) \leqslant 4$ for any $x \in V_4$ and $q(x) \leqslant 2(i-1)$ for any $x \in V_i$, $i \geqslant 5$. Thus

$$2n_2 \leqslant 2n_3 + 4n_4 + \sum_{i \geqslant 5} 2(i-1)n_i \tag{3}$$

and from (1) and (2) we obtain

$$n_4 + \sum_{i \geqslant 5} (i-4)n_i \leqslant 3. \tag{4}$$

Thus $n_i = 0$ for $i \geqslant 8$.

• Let x be a vertex in V'_2 whose neighbours y, z are in V_3. Then y has x as only neighbour in V_2, so $q(y) = 1$. But there are at least $\lambda = n_2 - 3n_4 - \sum_{i \geqslant 5} in_i$ vertices in V_2 of the same type as x (and at least as many vertices of the type of y) and we can rewrite inequality (3) as follows: $2n_2 \leqslant 2n_3 + 4n_4 + \sum_{i \geqslant 5} 2(i-1)n_i - \lambda$ which is equivalent to $3n_2 \leqslant 2n_3 + 7n_4 + \sum_{i=5}^{7}(3i-2)n_i$ giving with (1) and (2): $3 \leqslant \sum_{i=5}^{7}(11-2i)n_i$ which is at most equal to n_5.

The only possibility between this last inequality and (4) is $n_5 = 3$, $n_4 = n_6 = n_7 = 0$, $n_2 = 15$, $n_3 = 3$. But this is the case only if $q(x) = 2$ for every $x \in V_3$ and $q(x) = 8$ for every $x \in V_5$ which implies that the vertices of degree 5 have four neighbours in V''_2. If we look again at the broadcast tree of a vertex $x \in V''_2$, the vertex y is in V_5 with three other neighbours in V''_2 and thus must have its last neighbour u in V_5. But in the tree u has a neighbour v not in V_2 distinct from y and this is a contradiction. □

3.4. Minimum broadcast graph on 22 vertices

Theorem 3.7. $B(22) = 31$.

Proof. The graph of Fig. 3(a) is a minimum broadcast graph of size 31, thus giving $B(22) \leqslant 31$. Since there is only one symmetry in this graph, we have to give a broadcast

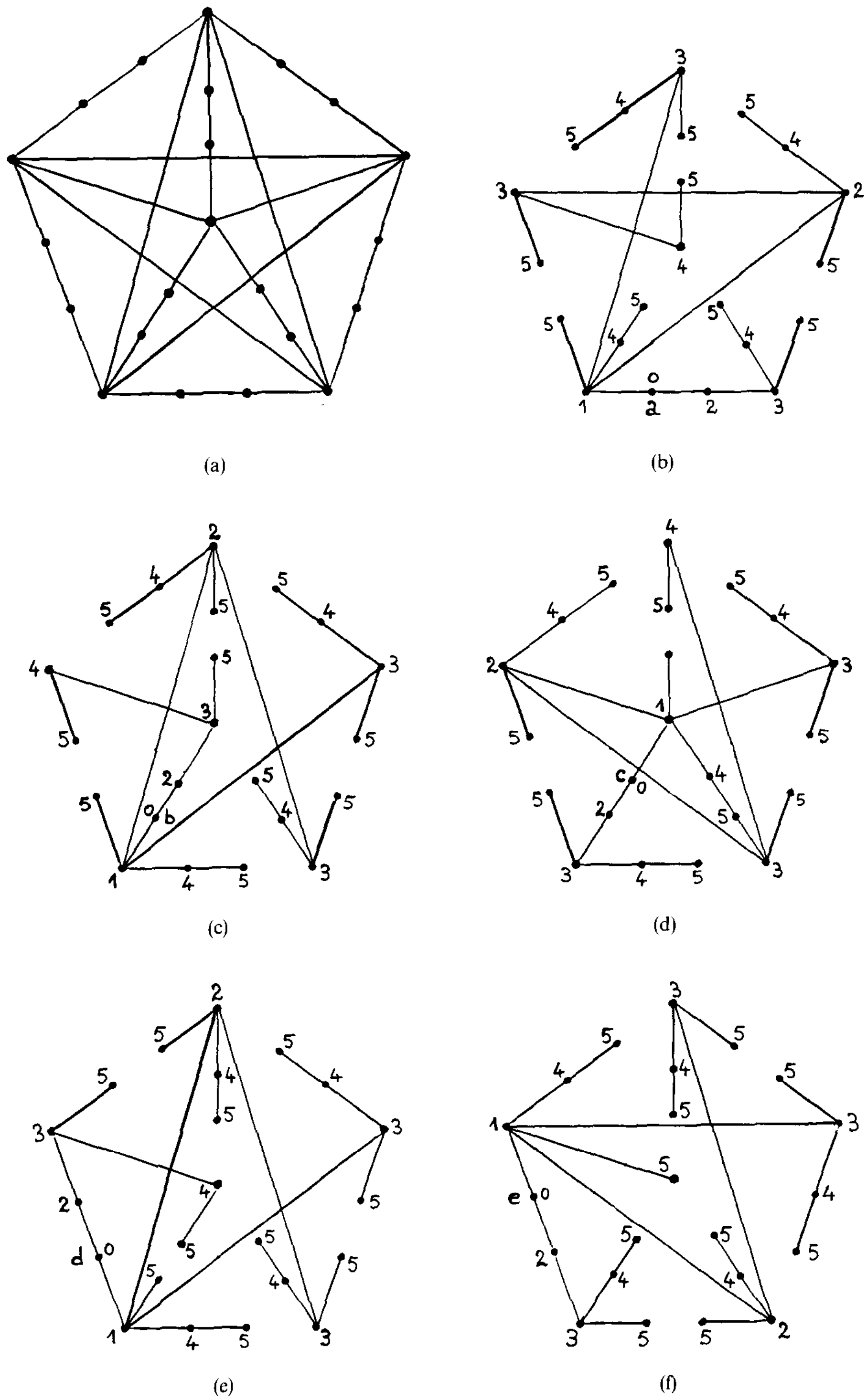


Fig. 3.

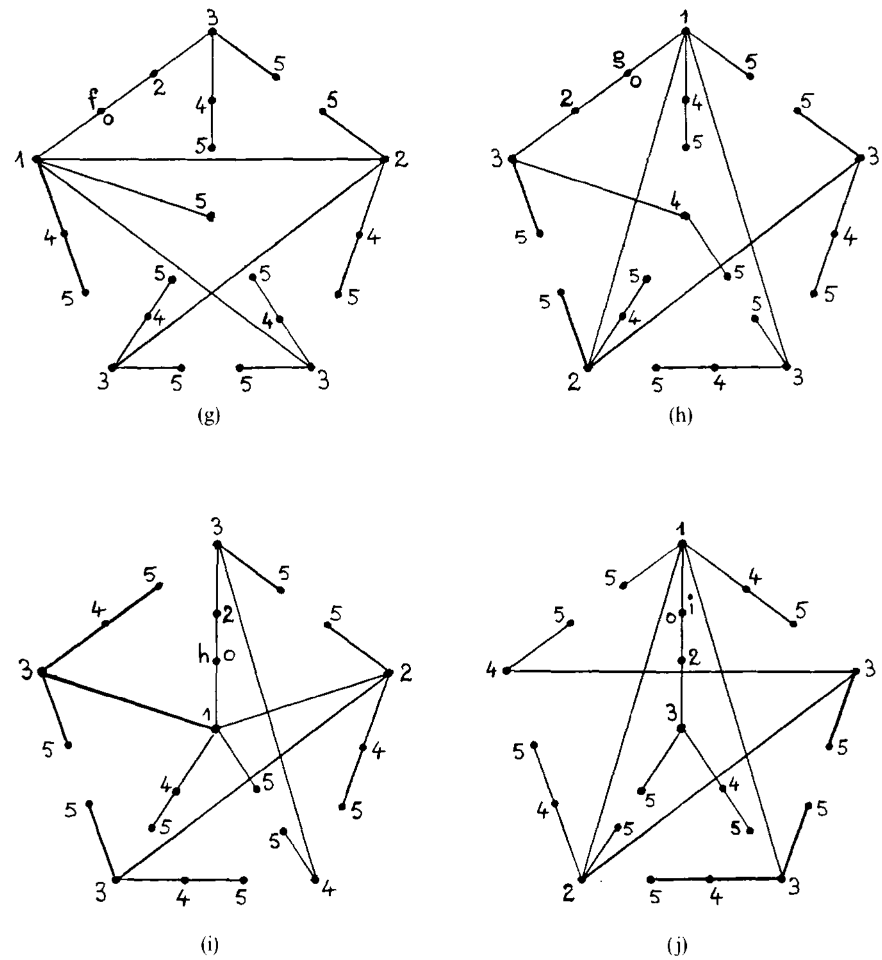


Fig. 3 (continued)

tree for every distinct type of node of degree 2, each vertex of degree 5 being implied as first neighbour in at least one of these trees, and thus allowed to use this tree as its own broadcast tree. This is shown in Figs. 3(b)–3(j).

From Proposition 3.1, we have $B(22) \geqslant B(21) + 1 = 29$. It remains to be proved that no graph of order 22 and size $\leqslant 30$ can broadcast in time 5. Such a graph must have 2 as minimum degree, and its maximum degree is at least 4. We can use again the conditions studied in the case $n = 21$, but they must be applied more restrictively to our purpose.

- No vertex has all its neighbours of degree 2.

We call *saturate* a vertex having all but one neighbour of degree 2. As above, V_2'' is simply the set of saturate vertices of degree 2. Denote by S_i, $i \geqslant 3$ the set of saturate vertices of degree i and by s_i the cardinality of this set.

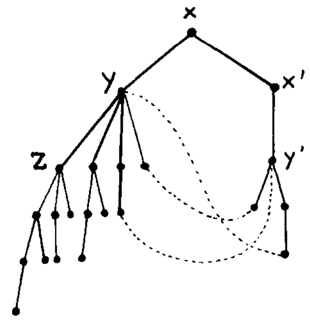


Fig. 4.

• Let x be a vertex in V_2'' and y its neighbour not in V_2''. Then y must be of T-degree 5, have a neighbour z of T-degree 4. Moreover both y and z have one own neighbour of T-degree 3, so they are not saturate. The neighbour $x' \in V_2''$ of x has the same properties by symmetry, but z may be a common neighbour to y and y', in which case the degree k of z is at least 5 and z has not more than $k - 3$ neighbours in V_2.

A more inquisitive inspection of the tree shows that there may exist at most 4 couples of adjacent vertices of V_2'' between y and y', including the couple x, x' (see the doted lines in Fig. 4). But it is possible only if y and y' are of degrees at least 6, otherwise the number of such couples do not exceed 3.

In our equations we will note μ the number of couples of adjacent vertices of degree 2 in G. So the cardinality of V_2'' is 2μ.

• Every vertex of V_2 has a neighbour of degree at least 4.

Moreover we have the following new restrictions.

• Suppose $x \in S_3$ is adjacent to a vertex $y \in V_3$. Then y is not saturate and has a neighbour of T-degree 4 not saturate.

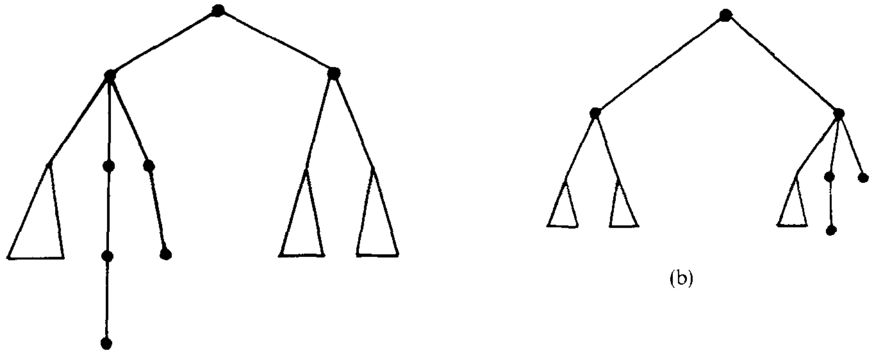
• Suppose $x, y \in V_2$ are adjacent to a same vertex of V_4. Then they cannot have another common neighbour in V_3 (which actually would be in S_3).

• Let $x \in S_4$ and suppose one of its neighbours $y \in V_2'$ has for second neighbour a vertex z of degree 3. There may be two possibilities for the broadcast tree of y [see Figs. 5(a) and (b)], with 22 nodes in either case.

So, in the first case z is nonsaturate with a nonsaturate neighbour of T-degree 4, or if it is not the case, x is adjacent to a nonsaturate vertex of T-degree 4. In particular, if all the vertices of degree $\geqslant 4$ are saturate, then every vertex of V_2 adjacent to a vertex of V_4 has no neighbour in V_3.

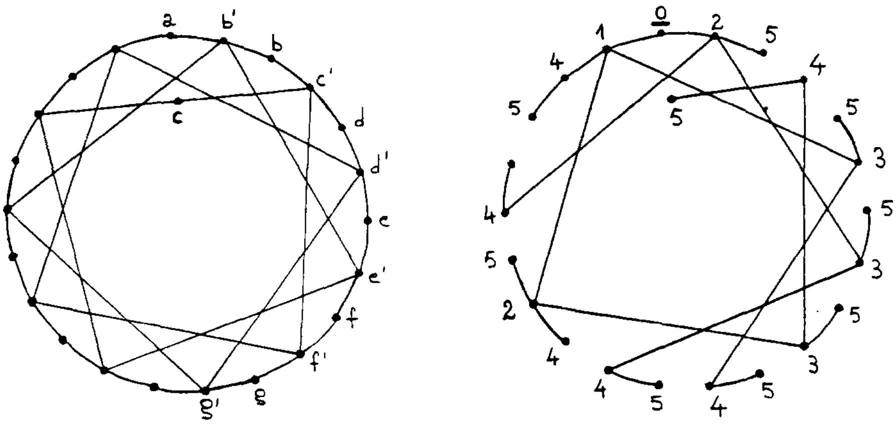
• No vertex in S_4 can have its fourth neighbour in S_3.

From these considerations we can establish relations between the parameters n_i of the graph. These relations eliminate but few cases which must be studied with the help of more specifical techniques. The proof is roughly analogous to that of Theorem 3.6, but more complicated. It is omitted in this paper, but details may be found in [5]. □



(a) (b)

Fig. 5.



(a) (b)

Fig. 6.

3.5. Broadcast graph on 23 vertices

Theorem 3.8. $B(23) \leqslant 34$.

Proof. Figs. 6(a) and (b) present a broadcast graph of size 34 with 11 vertices of degree 4 and 12 of degree 2, together with a broadcast tree in 5 time units for one type of nodes of degree 2. The remaining broadcast trees may be found in [5].

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