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Minimum Broadcast Time is NP-complete for 3-regular planar graphs and deadline 2

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Abstract

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The Minimum Broadcast Time (MBT) problem is whether a piece of information can be disseminated in a graph from a set of source nodes to all other nodes of the graph in at most k time steps. It is assumed that in one time step each node that contains the information can broadcast it to at most one of its neighbours. We show that MBT is NP-complete for 3-regular planar graphs and a constant deadline $k \geqslant 2$.

Keywords: Combinatorial optimization; information dissemination; algorithms; graph algorithms; NP-completeness

1. Introduction

We consider the problem of how fast a piece of information can be disseminated in a processor network from some source processors to all other processors. It is assumed that in one time step each processor that contains the information can broadcast it to at most one of its neighbours. A processor network is described by an undirected graph where nodes represent processors and edges represent the connections between processors. For many classes of processor networks the corresponding graphs have a small maximum degree. For a survey on broadcast problems in processor networks we refer to [3].

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Formally, let $G = (V, E)$ be a graph and $S(G) \subset V$ a set of *sources* in G. A *k-step broadcast scheme B for G* is a $(2k+1)$-tuple $(V_0, V_1, \ldots, V_k, E_1, E_2, \ldots, E_k)$ where $S(G) = V_0 \subset V_1 \subset \cdots \subset V_k \subset V$ and $E_1, E_2, \ldots, E_k \subset E$ such that for all $\{u, v\} \in E_i$ we have $u \in V_{i-1}$ and $v \in V_i$, and for each $u \in V_{i-1}$ there is at most one edge in E_i that contains u. The set V_k is the set of nodes which are *informed by B*. For two nodes u, v we say that *u directly informs v* if there is an edge $\{u, v\} \in E_i$ with $u \in V_{i-1}$ for an $i \in [k] := \{1, 2, \ldots, k\}$. A node *w gets information from* a node u if there is a sequence $u = w_1, w_2, \ldots, w_l = w$ of nodes such that w_i directly informs w_{i+1} for all $i \in [l-1]$. E_i is the set of edges *along which information is broadcasted in the ith step* of the broadcast scheme B, $i \in [k]$. Our problem can be formulated as follows [2]:

Minimum Broadcast Time (MBT)
Instance: A graph $G=(V, E)$ and a set $S \subset V$ of sources. A deadline $k \in \mathbb{N}$.
Question: Is there a k-step broadcast scheme B for G such that all nodes in V are informed by B?

It has been shown by Slater, Cockayne and Hedetniemi [6] that MBT is NP-complete even for a constant deadline $k \geqslant 4$. Recently, Jakoby, Reischuk and Schindelhauer [4] have investigated the complexity of MBT for graphs with constant maximum degree. They have shown that MBT is NP-complete for graphs with maximum degree 5 and a constant deadline $k \geqslant 3$. In this paper we show that MBT is NP-complete even for 3-regular planar graphs and a constant deadline $k \geqslant 2$. This result is optimal in the sense that MBT is polynomial time solvable for graphs with maximum degree 2 or for a deadline 1 (this is essentially the problem to find a maximum matching in bipartite graphs).

2. Preliminaries

In this section we introduce two graphs as gadgets that are needed in the proof of our theorem. There, the gadgets will occur as subgraphs of a larger graph. The node set of each gadget is partitioned into inner and outer nodes. The construction in the proof of the theorem will be such that all inner nodes of a gadget must get the information from sources inside the gadget. Only outer nodes can also get information from sources not in the gadget.

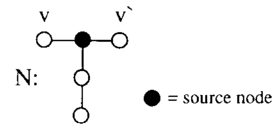

Fig. 1. Gadget N.

Definition 1. Let $G=(V, E)$ be a graph, $I(G) \subset V$ a set of *inner nodes* and $O(G)=V-I(G)$ a corresponding set of *outer nodes*. A broadcast scheme B for G is *$I(G)$-valid* if all nodes in $I(G)$ are informed by B. An $I(G)$-valid k-step broadcast scheme B for G is called *maximal* if no other $I(G)$-valid k-step broadcast scheme informs properly more nodes than B.

Observation 2. In any 2-step broadcast scheme for a graph at most three nodes can be informed from a source node s; two of them forming a path of length 2 with s and the third one forms a path of length 1 with s.

Our first gadget N is very simple (see Fig. 1). The set of outer nodes is $O(N)=\{v, v'\}$.

Fact 3. *Each $I(N)$-valid maximal 2-step broadcast scheme for N informs exactly one of the outer nodes v and v'.*

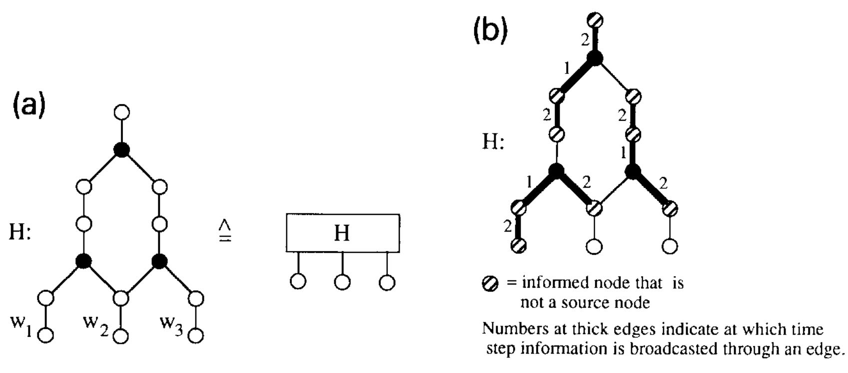

Fig. 2. (a) Graph H. (b) Example of an $I(H)$-valid maximal broadcast scheme for H.

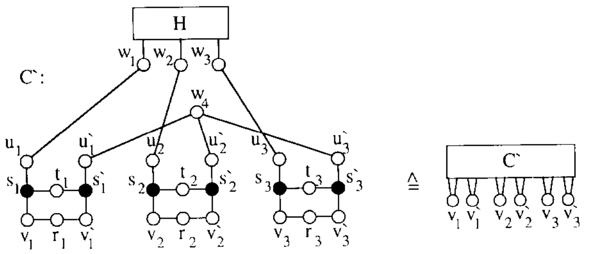

Fig. 3. Graph C'.

Now, we consider the graph H of Fig. 2(a) with set of outer nodes $O(H)=\{w_1, w_2, w_3\}$. The following fact is easy to derive. It is illustrated by the example in Fig. 2(b).

Fact 4. *Any $I(H)$-valid maximal 2-step broadcast scheme for H informs exactly one outer node. For each outer node $w_i \in O(H)$ there exists an $I(H)$-valid maximal 2-step broadcast scheme for H that informs w_i.*

The graph H is used to define a graph C' which forms the basis of our second gadget (see Fig. 3). The set of outer nodes is $O(C')=\{v_1, v_2, v_3, v_1', v_2', v_3'\}$.

Fact 5. (1) *Each $I(C')$-valid maximal s-step broadcast scheme for C' informs exactly three outer nodes.*

(2) *For an $I(C')$-valid maximal 2-step broadcast scheme B for C' we have:*

(i) *For every $i \in [3]$ exactly one of the nodes v_i, v_i' is informed.*

(ii) *B informs exactly one of the nodes v_1, v_2, v_3 and exactly two of the nodes v_1', v_2', v_3'.*

Proof. We have $|S(C')|=9$ and $|I(C')-S(C')|=24$. With Observation 2 follows that at most 3 outer nodes are informed by an $I(C')$-valid 2-step broadcast scheme for C'. Hence, (1) holds. See Fig. 4 for an example.

Consider an $I(C')$-valid maximal 2-step broadcast scheme B for C'. It is easy to see that no inner node in $I(H)$ of the subgraph H is informed from one of the sources s_1, s_2, s_3, s_1', s_2', s_3'. Fact 4 implies that two of the nodes w_1, w_2, w_3 must be informed from the sources s_1, s_2, s_3. Assume w_i and w_j, $1 \leqslant i \leqslant j \leqslant 3$ are informed

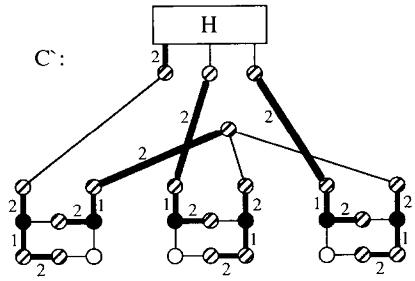

Fig. 4. Example of an $I(C')$-valid maximal broadcast scheme for C'.

from the sources s_1, s_2, s_3. Obviously, w_i (w_j) is informed from s_i (respectively s_j). Consequently the inner node r_i (r_j) is informed from s_i' (respectively s_j') via the node v_i' (respectively v_j'). Also u_i' (u_j') is informed from s_i' (respectively s_j'). This implies that t_i (t_j) is informed from s_i (respectively s_j). Then v_i and v_j cannot be informed.

Now, the only node that can inform w_4 is s_h', $i \neq h \neq j$. By a similar reasoning we get that r_h is informed from s_h via v_h and t_h is informed from s_h'. Moreover, v_h' cannot be informed. Altogether, (ii) follows. □

To reduce the degree of the outer nodes v_1', v_2', v_3' of C' we add some nodes to C'. The resulting graph C is our second gadget. See Fig. 5 for this. The set of outer nodes is $O(C)=\{v_1, v_2, v_3, \bar{v}_1, \bar{v}_2, \bar{v}_3\}$. We have the following fact:

Fact 6. *Fact 5 holds for the graph C with v_i' replaced with $\bar{v}_i$.*

3. The theorem

Theorem 7. MBT *is* NP-*complete for planar* 3-*regular graphs with a constant deadline* $k \geqslant 2$.

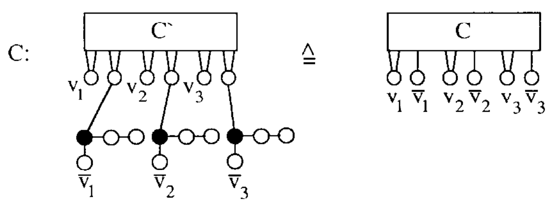

Fig. 5. Gadget C.

Proof. Obviously, MBT is in NP. The proof of its completeness proceeds in three steps. In Step 1 we prove the completeness only for graphs that are not necessarily planar and have maximum degree 3. In Step 2 it is proved that we can additionally assume planarity. Finally, in Step 3 we show that the graphs used in Step 2 can be made 3-regular.

Step 1. We reduce the *exactly-one-in-three* 3SAT problem to our problem. Let a set $C=\{C_1, C_2,\dots,C_m\}$ of clauses each of size 3 over a set $\Sigma=\{v_1, v_2,\dots,v_n\}$ of variables be an instance of *exactly-one-in-three* 3SAT. Without loss of generality we can assume that no clause contains a negated literal (This restriction is known to be NP-complete [2].) Recall that the *exactly-one-in-three* 3SAT problem asks whether there is a truth assignment of the variables in Σ such that each clause in C contains exactly one true literal. We now construct a graph $G=(V, E)$ with maximum degree 3 and a set $S(G)$ of sources in G. Our construction is such that for each variable $v_j\in\Sigma$ the graph G contains a subgraph G_{v_j} isomorphic to the gadget N. Furthermore, for each clause $C_p\in C$ the graph G contains a subgraph G_{C_p} isomorphic to the gadget C. The way in which all these subgraphs of G intersect reflects the structure of the given instance C.

For each variable $v_i\in\Sigma$ we define the graph G_{v_i} to be isomorphic to the gadget N and to have set of outer nodes $O(G_{v_i})=\{v_i^1, v_i^{k_i+1}\}$ where k_i is the number of clauses in C that contain v_i.

For each clause $C_p=\{v_h, v_i, v_j\}$, where $h<i<j$, $p\in[m]$, define the graph G_{C_p} to be isomorphic to the gadget C with set of outer nodes

$$O(G_{C_p})=\{v_h^{t_h}, v_h^{t_h+1}, v_i^{t_i}, v_i^{t_i+1}, v_j^{t_j}, v_j^{t_j+1}\},$$

where t_h (t_i, t_j) denotes the number of clauses in the subset $\{C_1, C_2,\dots,C_p\}$ of C containing v_h (v_i, v_j) (see Fig. 6(a)).

Set $\mathscr{G}=\{G_{v_1}, G_{v_2},\dots,G_{v_n}, G_{C_2},\dots,G_{C_m}\}$. We assume that the sets of inner nodes of each two different graphs in $\mathscr{G}$ are disjoint.

Now define the graph $G=\{V, E\}$ such that V is the union of the node sets of the graphs in $\mathscr{G}$ and E is the union of the edges sets of the graphs in $\mathscr{G}$. The set of sources $S(G)$ of G is the union of the sources of the subgraphs in $\mathscr{G}$. Similarly, the set of outer nodes $O(G)$ is the union of the outer nodes of the subgraphs in $\mathscr{G}$, i.e. $O(G)=\{v_i^1, v_i^2,\dots,v_i^{k_i+1}\mid i\in[n]\}$.

The construction of G is such that the subgraphs of G of $\mathscr{G}$ share common (outer) nodes in the following way: For $i\in[n]$ let $C_{i_1}, C_{i_2},\dots,C_{i_{k_i}}$ be the clauses that contain v_i, $i_1<i_2<\cdots<i_{k_i}$. The, as illustrated by Fig. 6(b):

(1) – G_{v_i} and $G_{C_{i_1}}$ are the only subgraphs in $\mathscr{G}$ containing the node v_i^1.
– G_{v_i} and $G_{C_{i_{k_i}}}$ are the only subgraphs in $\mathscr{G}$ containing the node $v_i^{k_i+1}$.
– $G_{C_{i_j}}$ and $G_{C_{i_{j+1}}}$ are the only subgraphs in $\mathscr{G}$ that contain the node $v_i^{i_j+1}$.

Note, that G has maximum degree 3. We show the following:

Claim. *There is a 2-step broadcast scheme for G that informs all nodes iff there exists a truth assign-*

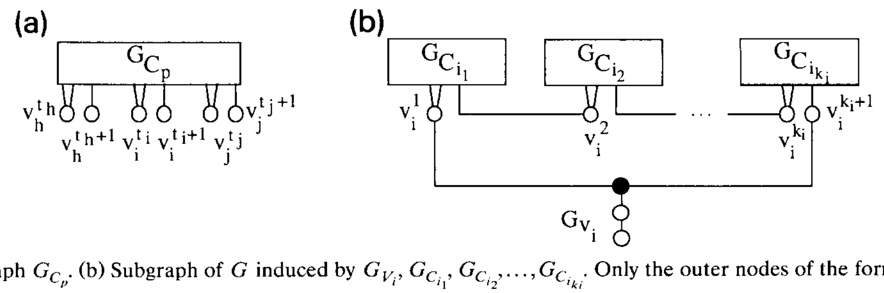

Fig. 6. (a) Graph G_{C_p}. (b) Subgraph of G induced by G_{V_i}, $G_{C_{i_1}}$, $G_{C_{i_2}}$,...,$G_{C_{i_{k_i}}}$. Only the outer nodes of the form v_i^j are shown.

ment of the variables in Σ such that each clause in $\mathscr{C}$ contains exactly one true literal.

Proof. Let B be a 2-step broadcast scheme for G that informs all nodes. From our construction the following is easily derived:

(2) No source node of a subgraph in $\mathscr{G}$ informs an inner node of another subgraph in $\mathscr{G}$.

Hence, B restricted to a subgraph G' of G in $\mathscr{G}$ is an $I(G')$-valid 2-step broadcast scheme for G'. Now, observe that $|O(G)| = 3m + n$. By Fact 6 an $I(G_{C_i})$-valid 2-step broadcast scheme for a graph G_{C_i}, $i \in [m]$ informs at most 3 outer nodes. Further, by Fact 3 an $I(G_{v_i})$-valid-2-step broadcast scheme for a graph G_{v_i}, $i \in [n]$ informs at most 1 outer node. These observations and (2) imply (3) and (4):

(3) In B no outer node of G gets information from two different nodes.

(4) B restricted to a subgraph G' in $\mathscr{G}$ is an $I(G')$-valid maximal 2-step broadcast scheme for G'.

From (1), (3), and (4) we derive that for each variable v_i, $i \in [n]$, where $C_{i_1}, C_{i_2}, \ldots, C_{i_{k_i}}$, $i_1 < i_2 < \cdots < i_{k_i}$ are the clauses that contain v_i, we have:

(5) (i) If v_i^1 gets information from the source in G_{v_i}, then v_i^{j+1} gets information from a source in $G_{C_{i_j}}$ for all $j \in [k_i]$;
(ii) If $v_i^{k_i+1}$ gets information from the source in G_{v_i}, then v_i^j gets information from a source in $G_{C_{i_j}}$ for all $j \in [k_i]$.

Therefore, we get a truth assignment of Σ if we set v_i true iff (5)(ii) holds. The 2nd part of Fact 6, (3), and (4) imply that there is exactly one true variable in each clause.

The other direction of the proof is easy using the ideas above. □

Step 2. In this step we give only the idea. Details are left to the reader. We reduce the planar version of the *exactly-one-in-three* 3SAT problem (see [1]), i.e., we can assume for the given instance $\mathscr{C}$ that the graph $G_C = (\{C \cup \Sigma\}, E_C)$ with edge set $E_C = \{(C_i, v_j) \mid v_j \in C_i \text{ or } \bar{v}_j \in C_i, i \in [m], j \in [n]\}$ is planar. It is not difficult to show that we can again assume that no clause contains a negated literal. The construction then proceeds similar to Step 1. There are two main differences. One is that we use a planar version of our gadget C. The other difference is that we possible have to renumber the nodes of the form $v_i^{t_i}$ in our graph G to make sure that G is planar. To this we consider a planar embedding of G_C. For the definition of the subgraph G_{C_p} that we define for each clause $C_p = \{v_h, v_i, v_j\}$, $h < i < j$, $p \in [m]$, we assume that v_j, v_i, v_h are the neighbours of the node C_p in the embedding of G_C in this clockwise order. For the set of outer nodes

$$O(G_{C_p}) = \{v_h^{t_h}, v_h^{t_h+1}, v_i^{t_i}, v_i^{t_i+1}, v_j^{t_j}, v_j^{t_j+1}\}$$

we assume that C_p is the t_hth (t_ith, t_jth) of the neighbours of the node v_h (v_i, v_j) with respect to a clockwise ordering in the embedding of G_C (see Fig. 6(a)). It is straightforward to see that the so obtained graph G is planar.

To find a planar version of C we have to find a planar version of the subgraph C' of C. Look at the embedding of C' in Fig. 3. There are two pairs of crossing edges. These are $(\{u_1', w_4\}, \{u_2, w_2\})$ and $(\{u_3', w_4\}, \{u_3, w_3\})$. These four edges share the property that all 2-step broadcast schemes for C' broadcast information along the edge only in step 2 and only in the same direction. This follows easily since each edge has one node with minimum distance 2 and one with minimum distance 1 from the sources. We simulate such crossing edges with a planar gadget K that will replace the crossing edges. Consider the pair of edges $(\{u_1', w_4\}, \{u_2, w_2\})$. There, u_1', u_2 are the nodes with minimum distance 1 and w_2, w_4 are the nodes with minimum distance 2 from the sources in C'. We replace the subgraph of C' induced by the nodes u_1', u_2, w_2, w_4 by the gadget K as shown in Fig. 7. Our gadget must be able to simulate the following four cases:

Case 1: No information is broadcasted along the edges $\{u_1', w_4\}$ and $\{u_2, w_2\}$.

Case 2: Information is broadcasted only along $\{u_1', w_4\}$ from u_1' to w_4.

Case 3: Information is broadcasted only along $\{u_2, w_2\}$ from u_2 to w_2.

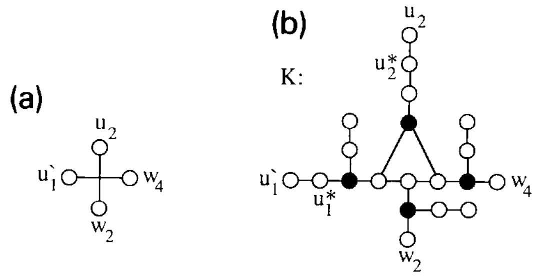

Fig. 7. (a) Crossing edges. (b) Gadget K.

Case 4: Information is broadcasted along both edges $\{u'_1, w_4\}$ and $\{u_2, w_2\}$ from u'_1 to w_4 and from u_2 to w_2.

Define the set of outer nodes of K to be $O(K)=\{u'_1, u^*_1, u_2, u^*_2, w_2, w_4\}$. We can show the following:

(6) Exactly four $I(K)$-valid maximal 2-step broadcast schemes for K exists:
(i) One that informs all nodes except u'_1, w_4, u_2, w_2 (see Fig. 8(a)).
(ii) One that informs all nodes except u'_1, u^*_1, u_2, w_2 (see Fig. 8(b)).
(iii) One that informs all nodes except u'_1, w_4, u_2, u^*_2 (see Fig. 8(c)).
(iv) One that informs all nodes except u'_1, u^*_1, u_2, u^*_2 (see Fig. 8(d)).

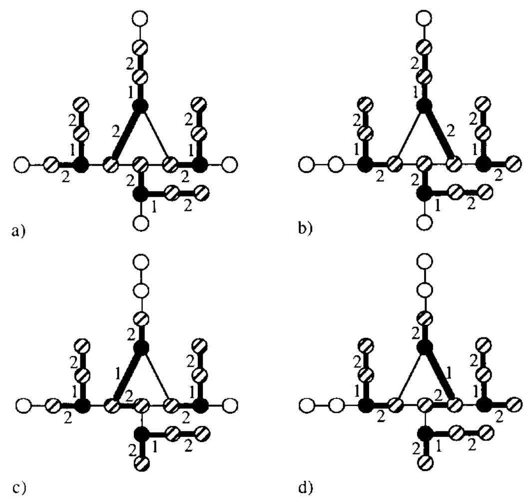

Fig. 8. $I(K)$-valid maximal broadcast schemes for K.

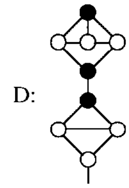

Fig. 9. Gadget D.

Thus, (i), (ii), (iii), and (iv) show that the corresponding cases (1), (2), (3), and (4) can be simulated by K. To finish our construction we also replace the subgraph induced by the nodes u'_3, u_3, w_3, w_4 by a graph isomorphic to K. The so created graph is denoted by C^*. Let C^* have the same outer nodes as C', i.e. $O(C^*)=\{v_1, v_2, v_3, v'_1, v'_2, v'_3\}$. It is not difficult to show with the help of (6) that Fact 5 holds for C^*.

Step 3. It remains to show that we can further assume the graphs to be 3-regular. For this we add the gadget D (see Fig. 9) to each node v with degree 2 in the graph G from Step 2 of this proof.

This gadget D has the property that in any 2-step broadcast scheme the node v cannot be informed from a node in D. Further, there is a 2-step broadcast scheme for D such that all nodes of D are informed. □

Note added in revision

K. Jansen and H. Müller have shown independently that MBT is NP-complete for planar bipartite graphs and constant deadline $k \geqslant 2$ [5].

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