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Minimum broadcast digraphs*

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Abstract

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Broadcasting is an information dissemination process in which a message is to be sent from a single originator to all members of a network by placing calls over the communication lines of the network. Numerous previous papers have investigated ways to construct sparse graphs (networks) in which this process can be completed in minimum time from any originator. In this paper, we consider the broadcasting problem in directed graphs. We show that several of the upper and lower bounds which have been produced for the undirected problem have analogs in the directed case. We describe several techniques to construct sparse digraphs on n vertices in which broadcasting can be completed in minimum time from any originator. For several values of n, these techniques produce the sparsest possible digraphs of this type (called minimum broadcast digraphs). For other values of n, these techniques produce the sparsest known digraphs of this type.

Keywords. Broadcasting, directed graphs, networks.

1. Definitions

Broadcasting refers to the process of message dissemination in a communication network whereby a message, originated by one member, is transmitted to all members of the network. Broadcasting is accomplished by placing a series of calls over the communication lines of the network. This is to be completed as quickly as possible subject to the constraints that each call involves only two vertices, each call requires one unit of time, a vertex can participate in only one call per unit of time, and a vertex can only call a vertex to which it is adjacent.

Given a strongly connected digraph G and a message originator, vertex u, we define the *directed broadcast time of vertex u*, $\vec{b}(u)$, to be the minimum number of

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time units required to complete broadcasting from vertex u. It is easy to see that $\vec{b}(u) \geq \lceil \log_2 n \rceil$ for any vertex u in a strongly connected digraph G with n vertices since the number of informed vertices can at most double during each time unit.

We define the *broadcast time of a digraph G*, $\vec{b}(G)$, to be the maximum broadcast time of any vertex u in G, i.e., $\vec{b}(G) = \max\{\vec{b}(u) \mid u \in V(G)\}$. For the complete symmetric digraph K_n^* with $n \geq 2$ vertices, $\vec{b}(K_n^*) = \lceil \log_2 n \rceil$, yet K_n^* is not minimal with respect to this property for any $n > 2$. That is, we can remove edges from K_n^* and still have a digraph G with n vertices such that $\vec{b}(G) = \lceil \log_2 n \rceil$. We use the term *broadcast digraph* to refer to any digraph G on n vertices with $\vec{b}(G) = \lceil \log_2 n \rceil$.

We define the *directed broadcast function*, $\vec{B}(n)$, to be the minimum number of edges in any broadcast digraph on n vertices. A *minimum broadcast digraph* (mbd) is a broadcast digraph on n vertices having $\vec{B}(n)$ edges. From an applications perspective, minimum broadcast digraphs represent the cheapest possible communication networks (having the fewest directed communication lines) in which broadcasting can be accomplished, from any vertex, as fast as theoretically possible.

Analogous definitions have previously been given for (undirected) graphs [6]. Given a connected graph G and a message originator, vertex u, we use $b(u)$ to denote the *broadcast time of vertex u*, $b(G)$ to denote the *broadcast time of a graph G*, and $B(n)$ to denote the *broadcast function*, that is, the number of edges in a *minimum broadcast graph* (mbg).

2. Previous results

Most of the previous work on broadcasting has been on the undirected problem and only a few values of $B(n)$ are known. Farley, Hedetniemi, Mitchell and Proskurowski [4] determined the values of $B(n)$ for $n \leq 15$ and noted that $B(2^k) = k2^{k-1}$ (the k-cube is an mbg on $n = 2^k$ vertices). Mitchell and Hedetniemi [7] determined the value for $B(17)$. Wang [8] gave the value of $B(18)$ and the values of $B(n)$ for $n = 19$, 30, and 31 were presented by Bermond, Hell, Liestman and Peters [1].

Since minimum broadcast graphs are very difficult to find, several authors have devised methods to construct broadcast graphs with small numbers of edges. Farley designed several techniques for constructing such graphs with n vertices and approximately $(n/2)\log_2 n$ edges [3]. Chau and Liestman presented constructions based on Farley's techniques which yield somewhat improved graphs, that is, graphs with fewer edges, for most values of n [2]. Grigni and Peleg [5] showed that $B(n) \in \Theta(L(n-1)n)$ for $n \geq 1$ where $L(k)$ denotes the exact number of consecutive leading 1's in the binary representation of k. Asymptotically, Grigni and Peleg's construction (which establishes their upper bound) produces the best results for most values of n. For a survey of results on undirected broadcasting and related problems, see Hedetniemi, Hedetniemi and Liestman [6].

Comparatively little has been done on the directed broadcasting problem. Grigni and Peleg [5] showed that $\vec{B}(n) \in \Theta(L(n-1)n)$ for $n \geq 1$. While these bounds are good asymptotically, they do not provide much information for small values. In fact, the problems of constructing minimum broadcast digraphs and determining exact values for $\vec{B}(n)$ have not been addressed. Our goals in this paper are to find better bounds for "practical" values of n and to determine exact values for $\vec{B}(n)$ where possible. We adapt the techniques of Bermond, Hell, Liestman and Peters [1] for use in directed graphs.

3. General bounds

The following simple lemma, noted by Grigni and Peleg [5], will serve as a useful starting point for our search for better bounds.

Lemma 1. $B(n) \leq \vec{B}(n) \leq 2B(n)$.

The lower bound follows from the observation that the underlying graph of any minimum broadcast digraph on n vertices must be a broadcast graph on n vertices. Note that $B(n) \geq n-1$ and $\vec{B}(n) \geq n$ follow trivially from the fact that every vertex must be able to send a message. The upper bound follows from the observation that replacing each edge of a minimum broadcast graph on n vertices with a symmetric pair of directed edges yields a broadcast digraph on n vertices. Unfortunately, as noted above, the value of $B(n)$ is only known for $n = 2^k$ and for a few small values of n. For other values, bounds on $\vec{B}(n)$ can be obtained from bounds on $B(n)$ as noted in the following corollary.

Corollary 2. *If* $f(n) \leq B(n) \leq g(n)$, *then* $f(n) \leq \vec{B}(n) \leq 2g(n)$.

The techniques of Farley [3] and of Chau and Liestman [2] for constructing broadcast graphs can be modified to construct broadcast digraphs and give upper bounds on $\vec{B}(n)$. These techniques involve combining broadcast graphs on fewer than n vertices with some new edges to form a broadcast graph on n vertices.

For example, to construct a broadcast graph on n vertices, Farley's two-way split method is as follows. First, construct broadcast graphs G_1 and G_2 on n_1 and n_2 vertices, respectively, such that $n_1 + n_2 = n$, $n_1 \geq n_2$, and $\lceil \log_2 n_1 \rceil = \lceil \log_2 n_2 \rceil = \lceil \log_2 n \rceil - 1$. Then, add n_1 edges between G_1 and G_2 in such a way that each vertex in G_1 is adjacent to a vertex in G_2 and vice versa. This process yields graphs with fewer than $\frac{1}{2}n\lceil \log_2 n \rceil$ edges. To mimic this construction for directed graphs, first construct broadcast digraphs D_1 and D_2 on n_1 and n_2 vertices, respectively. Then, add n_2 directed edges so that there is an edge from each vertex in D_2 to some vertex in D_1. Also, add n_2 vertex-disjoint directed edges from n_2 distinct vertices of D_1 to the n_2 vertices of D_2. This gives the following lemma.

Lemma 3. *$\vec{B}(n) \le \vec{B}(n_1) + \vec{B}(n_2) + 2n_2$ where $n_1 + n_2 = n \ge 4$, $n_1 \ge n_2$, and $\lceil \log_2 n_1 \rceil = \lceil \log_2 n_2 \rceil = \lceil \log_2 n \rceil - 1$.*

To construct a broadcast graph on n vertices where $2^{\lfloor \log_2 n \rfloor} < n \le 3 * 2^{\lceil \log_2 n \rceil - 2}$, Farley's three-way split method is as follows. First, construct broadcast graphs G_1, G_2 and G_3 on n_1, n_2 and n_3 vertices, respectively, such that $n_1 + n_2 + n_3 = n$, $n_1 \ge n_2 \ge n_3$, and $\lceil \log_2 n_i \rceil = \lceil \log_2 n \rceil - 2$ for each i. If n is even, connect each vertex of each component (graph) to a member of a different component using $\frac{1}{2}n$ edges in total. If n is odd, do as above for $n-1$ vertices and then connect the remaining vertex to a vertex of another component to which no other member of its own component is already connected. This uses $\lceil \frac{1}{2}n \rceil$ edges in total and yields graphs with fewer than $\frac{1}{2}n \lceil \log_2 n \rceil - \frac{1}{2}n$ edges. To mimic this construction for directed graphs, first construct broadcast digraphs D_1, D_2, and D_3 on n_1, n_2 and n_3 vertices, respectively. Then, add n directed edges so that every vertex has an edge to a vertex in another component and an edge from a vertex in another component. This gives the following lemma.

Lemma 4. *$\vec{B}(n) \le \vec{B}(n_1) + \vec{B}(n_2) + \vec{B}(n_3) + n$ for $n \ge 5$ in the range $2^{\lfloor \log_2 n \rfloor} < n \le 3 * 2^{\lceil \log_2 n \rceil - 2}$ where $n_1 + n_2 + n_3 = n$ and $\lceil \log_2 n_i \rceil = \lceil \log_2 n \rceil - 2$ for each i.*

Additional bounds may be obtained by modifying the constructions of Chau and Liestman [2] to yield directed graphs.

The following lemma is analogous to a result for undirected graphs which appears in [1]. It can be used to argue either lower or upper bounds. If an upper bound for $\vec{B}(n)$ is known, the lemma supplies an upper bound on $\vec{B}(n-1)$. On the other hand, given a lower bound for $\vec{B}(n-1)$, the result supplies a lower bound on $\vec{B}(n)$. The proof is omitted since it is similar to the proof in [1].

Lemma 5. *If a broadcast digraph on n vertices, $2^{k-1} + 1 < n \le 2^k$, with e edges has a vertex of indegree α and outdegree β, then $\vec{B}(n-1) \le e + \alpha\beta + \beta^2 - \alpha - 2\beta$.*

Lower bounds on $\vec{B}(n)$ can be obtained by considering the outdegrees required of the vertices. One such bound, obtained by considering the outdegree of the originator of the broadcast, is the following result of Grigni and Peleg [5].

Lemma 6. *$\vec{B}(n) \ge n(\lceil \log_2 n \rceil - \lfloor \log_2 (2^{\lceil \log_2 n \rceil} - (n-1)) \rfloor)$.*

The above lemma establishes a lower bound on the minimum outdegree of each vertex. An improved lower bound on $\vec{B}(n)$ can be argued for some values of n. In particular, if we know that each vertex must have outdegree $\ge \Delta$, we can consider the maximum number of vertices which can be informed when every vertex has outdegree $= \Delta$. If n vertices cannot be informed in such a graph, we know that some vertex must have outdegree $> \Delta$, giving the following improvement in the lower bound. (A more detailed examination could yield even better lower bounds.)

Lemma 7. *If $n > d^{\Delta}_{\lceil \log_2 n \rceil}$, where d^{Δ}_t is an upper bound on the number of vertices that can be informed in time t in any digraph of maximum outdegree Δ, and $\Delta = \lceil \log_2 n \rceil - \lfloor \log_2 (2^{\lceil \log_2 n \rceil} - (n-1)) \rfloor$, then $\vec{B}(n) > \Delta n$.*

Proof. Let a^{Δ}_t denote the maximum number of vertices that can be informed in time t in any digraph of maximum outdegree Δ. In any broadcasting scheme which achieves this maximum a^{Δ}_t we may assume that a vertex does not remain idle if it has already been informed and it still has uninformed out-neighbors. Therefore, if there is one informed vertex at time 0, then all the vertices that were informed by time t have informed all of their out-neighbors and must be idle after time $t+\Delta$. Hence, $a^{\Delta}_{t+\Delta+1} \le a^{\Delta}_{t+\Delta} + (a^{\Delta}_{t+\Delta} - a^{\Delta}_t) = 2a^{\Delta}_{t+\Delta} - a^{\Delta}_t$. It is also clear that $a^{\Delta}_t \le 2^t$ because at each time a vertex can inform at most one other vertex. Thus an upper bound on a^{Δ}_t is the solution to the recurrence $d^{\Delta}_{t+\Delta+1} = 2d^{\Delta}_{t+\Delta} - d^{\Delta}_t$ for $t \ge \Delta$, and $d^{\Delta}_t = 2^t$ for $t = 0, 1, \ldots, \Delta$. This upper bound on a^{Δ}_t can be translated into a lower bound on $\vec{B}(n)$. In particular, we know from Lemma 6 that no vertex can have degree less than $\Delta = \lceil \log_2 n \rceil - \lfloor \log_2 (2^{\lceil \log_2 n \rceil} - (n-1)) \rfloor$. So, if $n > a^{\Delta}_{\lceil \log_2 n \rceil}$, then n vertices cannot be informed by time $\lceil \log_2 n \rceil$ in any digraph of maximum outdegree Δ and $\vec{B}(n) > \Delta n$. Since $a^{\Delta}_t \le d^{\Delta}_t$, the result follows. □

4. Construction of broadcast digraphs

Several techniques have proven valuable in attempting to construct sparse broadcast digraphs for small n. Since these techniques result in highly symmetric digraphs, only a few broadcast schemes need to be exhibited to prove that a digraph is a broadcast digraph. We have, in fact, been able to show that many of our broadcast digraphs are minimum broadcast digraphs. Some of these techniques can be used to find broadcast digraphs for larger n.

In the following sections, several figures are used to depict broadcast digraphs and broadcasting schemes. A pair of symmetric directed edges between a pair of vertices is represented in the figures as a single edge with arrowheads at both ends. A broadcast scheme is depicted as a tree which consists of the edges used in the scheme for an originator isomorphic to the vertex indicated by "+". Other vertices are labelled with the times at which they receive the message under the scheme.

4.1. Simple cases

Theorem 8. $\vec{B}(3) = 3$.

Proof. The directed cycle on 3 vertices is a broadcast digraph with 3 edges. With two time units allowed for broadcasting, each originator can simply send the message to its only out-neighbor at time 1 who can relay it to the third vertex at time 2. Since every vertex must have an out-neighbor, at least 3 edges are needed in any broadcast digraph on 3 vertices. □

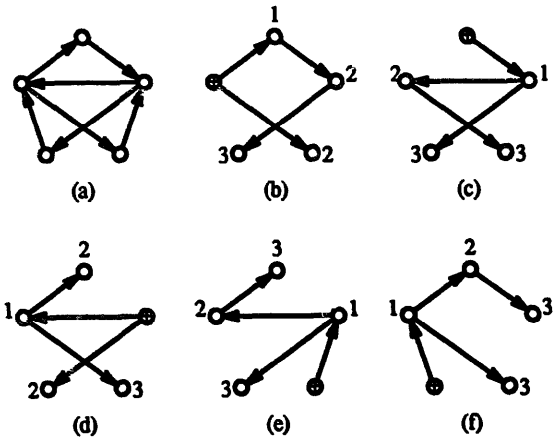


Fig. 1. (a) Broadcast digraph on 5 vertices and (b)–(f) broadcast schemes.

Theorem 9. $\vec{B}(5)=7$.

Proof. The digraph in Fig. 1(a) is a broadcast digraph with 7 edges. Broadcast schemes for the 5 vertices are shown in Fig. 1(b)–(f).

Lemma 7 shows that at least 6 edges are needed in any broadcast digraph on 5 vertices, so suppose that G is a broadcast digraph on 5 vertices with 6 edges. G must have 4 vertices with outdegree 1 and 1 vertex with outdegree 2. If a vertex of outdegree 1 is to inform 5 vertices in 3 time units, it must call the vertex of outdegree 2 during time unit 1. Since all of the edges from vertices of outdegree 1 must connect to the vertex of outdegree 2, some vertex of outdegree 1 has indegree 0 and cannot receive a message broadcast from any other originator. □

4.2. *Symmetric minimum broadcast digraphs*

As noted in the proof of Lemma 1, a broadcast digraph on n vertices can be obtained by replacing each edge of a minimum broadcast graph on n vertices with a symmetric pair of directed edges. We use the term *symmetric minimum broadcast digraph* (smbd) to denote such a digraph. For some values of n we can show that an smbd is a minimum broadcast digraph on n vertices.

Theorem 10. $\vec{B}(2^k)=k2^k$, *for* $k\geq 1$.

Proof. It has been shown (see [4]) that $B(2^k)=k2^{k-1}$ and that the k-cube is an mbg on $n=2^k$ vertices. Replacing each edge of the k-cube with a symmetric pair of directed edges yields a broadcast digraph on n vertices with $k2^k$ directed edges. Lemma 6 provides a matching lower bound on $\vec{B}(2^k)$. □

In addition to the digraph described in the above proof, any broadcast digraph on $n=2^k$ vertices constructed by the method of Lemma 3 also has $k2^k$ edges.

Theorem 11. $\vec{B}(6)=12$.

Proof. In [4] it is shown that $B(6)=6$ and that the cycle of length 6 is an mbg on 6 vertices. Replacing each edge of the cycle with a symmetric pair of directed edges yields a broadcast digraph on n vertices with 12 directed edges, matching the lower bound from Lemma 6. □

It should be noted that any broadcast digraph on 6 vertices constructed by the method of Lemma 3 also has 12 edges.

4.3. Cycles with chords

One type of construction which yields several broadcast digraphs can be described as a directed cycle with appropriately chosen chords. Let $\vec{C}_n$ be the directed cycle on n vertices $0,1,\dots,n-1$ with directed edges $(i,i+1)$, $0\le i\le n-1$. (Note that addition is assumed to be performed modulo n throughout this section.) The figures in this section are drawn with vertex labels increasing clockwise with 0 at the top.

We can construct a 2-regular digraph by choosing α and adding edges $(i,i+\alpha)$ for all i, $0\le i\le n-1$. (Note: An *r-regular* digraph is a digraph in which each vertex has indegree r and outdegree r.) With this technique, we can construct minimum broadcast digraphs to determine the values of $\vec{B}(7)$ and $\vec{B}(10)$.

Theorem 12. $\vec{B}(7)=14$.

Proof. The digraph in Fig. 2(a) is a 2-regular broadcast digraph with 14 edges constructed by choosing $\alpha=3$. All of the vertices of the digraph are isomorphic. A broadcast scheme for a vertex of the digraph is shown in Fig. 2(b). (Another 2-regular broadcast digraph with 14 edges can be constructed by choosing $\alpha=5$.)

Lemma 6 shows that at least 14 edges are needed in any broadcast digraph on 7 vertices. □

Theorem 13. $\vec{B}(10)=20$.

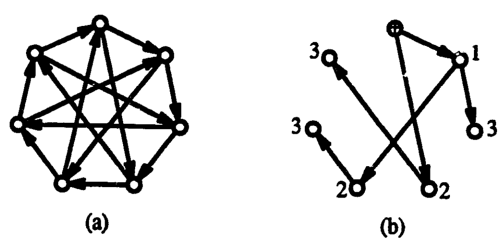


Fig. 2. (a) Broadcast digraph on 7 vertices and (b) broadcast scheme.

Proof. A 2-regular broadcast digraph on 10 vertices with 20 edges can be constructed by choosing $\alpha = 4$. The broadcast scheme is omitted.

Lemma 6 shows that at least 20 edges are needed in any broadcast digraph on 10 vertices. $\square$

This technique can be generalized by adding chords of several lengths from all vertices of the directed cycle. For example, we can construct a 3-regular digraph by choosing integers α and β and adding chords $(i, i+\alpha)$ and $(i, i-\beta)$ to the directed cycle $\vec{C}_n$ for all i, $0 \le i \le n-1$.

Theorem 14. $\vec{B}(14) = 42$.

Proof. The digraph in Fig. 3(a) is a 3-regular broadcast digraph with 42 edges constructed by choosing $\alpha = 5$ and $\beta = 3$.

All of the vertices of the digraph are isomorphic. A broadcast scheme for a vertex of the digraph is shown in Fig. 3(b).

Lemma 6 shows that at least 42 edges are needed in any broadcast digraph on 14 vertices. $\square$

Theorem 15. $\vec{B}(15) = 45$.

Proof. A 3-regular broadcast digraph on 15 vertices with 45 edges can be constructed by choosing $\alpha = 7$ and $\beta = 3$. The broadcast scheme is omitted.

Lemma 6 shows that at least 45 edges are needed in any broadcast digraph on 15 vertices. $\square$

This technique can also be generalized by adding chords of different lengths from various subsets of the vertices. This is illustrated by the following examples.

Theorem 16. $\vec{B}(12) = 24$.

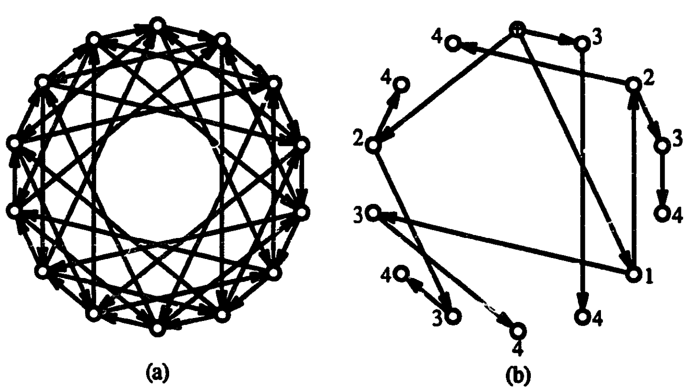


Fig. 3. (a) Broadcast digraph on 14 vertices and (b) broadcast scheme.

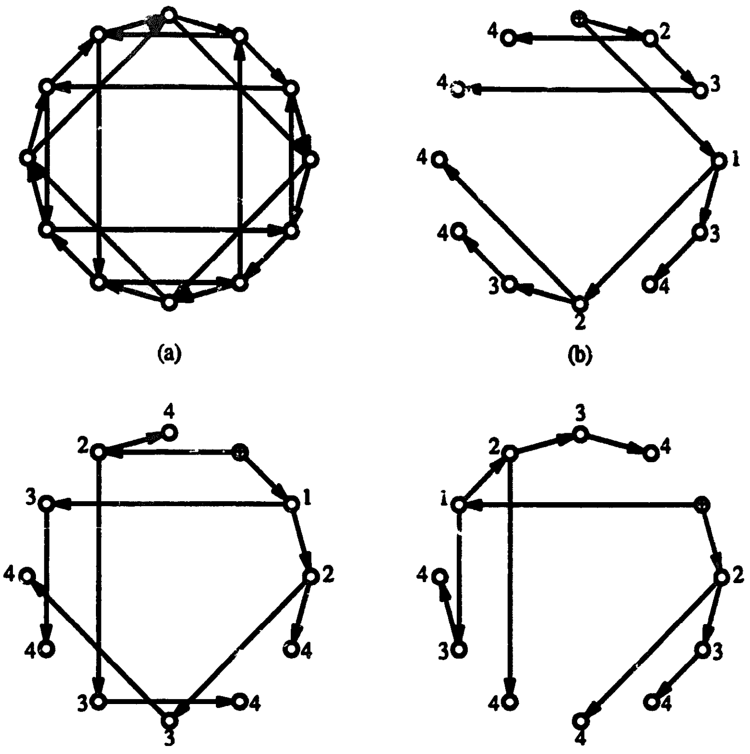


Fig. 4. (a) Broadcast digraph on 12 vertices and (b)-(d) broadcast schemes.

Proof. The digraph in Fig. 4(a) is a 2-regular broadcast digraph with 24 edges. The digraph was constructed by adding edges $(i, i+3)$ where $i \equiv 0 \bmod 3$, $(i, i+10)$ where $i \equiv 1 \bmod 3$, and $(i, i+8)$ where $i \equiv 2 \bmod 3$ to $\vec{C}_{12}$.

The vertices of the digraph can be partitioned into 3 isomorphism classes. Broadcast schemes for originators from these classes are shown in Fig. 4(b)-(d). (Another 2-regular broadcast digraph can be constructed by adding edges $(i, i+3)$ where $i \equiv 0 \bmod 3$, $(i, i+7)$ where $i \equiv 1 \bmod 3$, and $(i, i+8)$ where $i \equiv 2 \bmod 3$ to $\vec{C}_{12}$.)

Lemma 6 shows that at least 24 edges are needed in any broadcast digraph on 12 vertices. □

Theorem 17. $\vec{B}(18) = 36$.

Proof. A 2-regular broadcast digraph on 18 vertices with 36 edges can be constructed by adding edges $(i, i+4)$ for even i and $(i, i-4)$ for odd i to the directed cycle $\vec{C}_{18}$. The broadcast scheme is omitted.

Lemma 6 shows that at least 36 edges are needed in any broadcast digraph on 18 vertices. □

Theorem 18. $45 \le \vec{B}(22) \le 55$.

Proof. A broadcast digraph on 22 vertices with 55 edges can be constructed by adding edges $(i, i+4)$ and $(i, i-4)$ for even i and edges $(i, i-6)$ for odd i to the directed cycle $\vec{C}_{22}$. The details of the broadcast schemes for this digraph are omitted.

Lemma 7 shows that at least 45 edges are needed in any broadcast digraph on 22 vertices. □

Theorem 19. $49 \leq \vec{B}(24) \leq 66$.

Proof. The digraph in Fig. 5 is a broadcast digraph with 66 edges. The digraph was constructed by adding edges $(i, i+4)$ and $(i, i-4)$ for even i, edges $(i, i-6)$ for odd i, and edges $(i, i+12)$ for $i \equiv 1 \bmod 4$ to the directed cycle $\vec{C}_{24}$. The details of the broadcast schemes for this digraph are omitted.

Lemma 7 shows that at least 49 edges are needed in any broadcast digraph on 24 vertices. □

Theorem 20. $\vec{B}(26) = 78$.

Proof. A 3-regular broadcast digraph on 26 vertices with 78 edges can be constructed by adding edges $(i, i+6)$ and $(i, i-6)$ for even i and edges $(i, i+4)$ and $(i, i-4)$ for odd i to the directed cycle $\vec{C}_{26}$. The details of the broadcast schemes for this digraph are omitted.

Lemma 6 shows that at least 78 edges are needed in any broadcast digraph on 26 vertices. □

4.4. Interconnection of cycles

Another type of construction which yields several new sparse broadcast digraphs can be described as the interconnection of cycles. In particular, when n can be

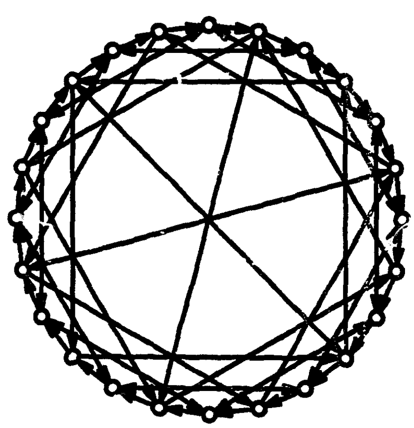


Fig. 5. Broadcast digraph on 24 vertices.

factored as $n=ab$, we can construct a digraph on n vertices by interconnecting b directed cycles of length a. Start by labelling the n vertices with the pairs (i,j) where $0 \le i \le a-1$ and $0 \le j \le b-1$ and add the edges $((i,j),((i+1) \bmod a, j))$ for all i and j. This creates b cycles of length a where (i,j) is the label of the vertex in position i of cycle j.

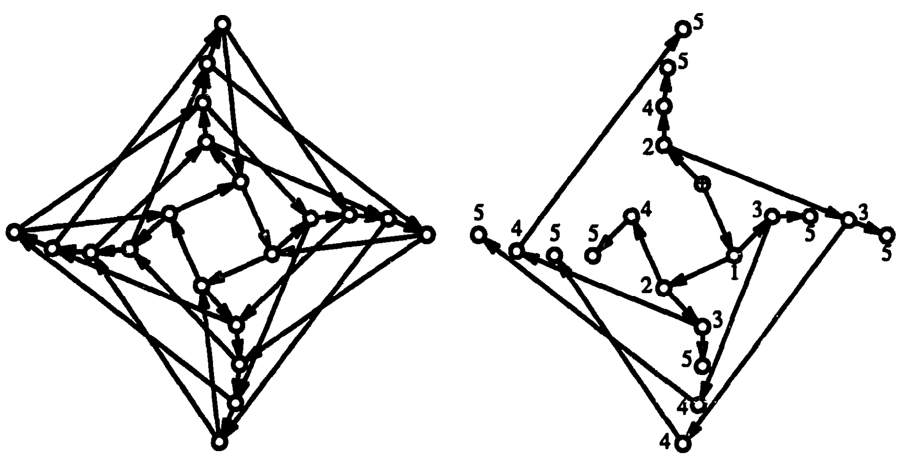
The interconnections among the cycles can be described by one or more pairs of mappings $f:\{0,1,\dots,a-1\} \to \{0,1,\dots,a-1\}$ and $g:\{0,1,\dots,a-1\} \to \{0,1,\dots,b-1\}$ so that vertex (i,j) is adjacent to vertex $((i+f(i)) \bmod a, (j+g(i)) \bmod b)$. Notice that there are at most a isomorphism classes of vertices corresponding to the a positions on the cycles.

The 20-vertex digraph in Fig. 6(a) is an interconnection of 4 cycles of length 5. If the positions of each cycle are labelled clockwise starting with the innermost vertex in position 0, and the cycles are also labelled clockwise, then the interconnection functions are $f(0)=0$, $f(1)=2$, $f(2)=4$, $f(3)=1$, $f(4)=3$, and $g(j)=1$ for $0 \le j \le 4$. Interconnection functions among cycles can also be described more succinctly by a *template* with a columns corresponding to the positions on cycles and 2 rows which list the values of the interconnection functions f and g. For example, the template for the digraph in Fig. 6(a) is $\begin{smallmatrix}02413\\11111\end{smallmatrix}$.

Theorem 21. $\vec{B}(20)=40$.

Proof. The digraph in Fig. 6(a) is a 2-regular broadcast digraph with 40 edges which was constructed as described above. Figure 6(b) describes a broadcast scheme for vertex 0 of cycle 0. The other vertices in position 0 of their cycles are isomorphic and can use the same broadcast scheme. Broadcast schemes for the other vertices are omitted.

Lemma 6 shows that at least 40 edges are needed in any broadcast digraph on 20 vertices. $\square$



(a) (b)

Fig. 6. (a) Broadcast digraph on 20 vertices and (b) a broadcast scheme.

Theorem 22. $51 \leq \vec{B}(25) \leq 75$.

Proof. A 3-regular broadcast digraph on 25 vertices with 75 edges can be constructed as an interconnection of 5 directed cycles of 5 vertices using the two templates $\genfrac{}{}{0pt}{}{10000}{12342}$ and $\genfrac{}{}{0pt}{}{22220}{43213}$. The details of the broadcast schemes for this digraph are omitted.

Lemma 7 shows that at least 51 edges are needed in any broadcast digraph on 25 vertices. □

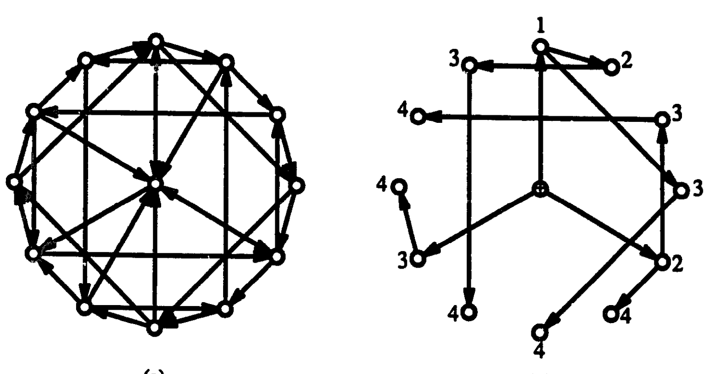
This method has also been used to construct additional minimum broadcast digraphs for $n=6$ using the template $\genfrac{}{}{0pt}{}{000}{111}$ with 2 directed cycles on 3 vertices, for $n=10$ using the template $\genfrac{}{}{0pt}{}{44444}{11111}$ with 2 directed cycles on 5 vertices, and for $n=15$ using the templates $\genfrac{}{}{0pt}{}{30000}{11111}$ and $\genfrac{}{}{0pt}{}{13341}{22222}$ with 3 directed cycles on 5 vertices.

4.5. Vertex addition

Many of the broadcast schemes above leave vertices idle during the last time unit. These vertices could, potentially, be used to inform additional vertices. It is sometimes possible to add new vertices and edges such that the existing broadcast schemes can be extended to include the new vertices, and the new vertices can originate their own minimum time broadcasts. In some cases, the numbers of edges added are small enough to give good broadcast digraphs.

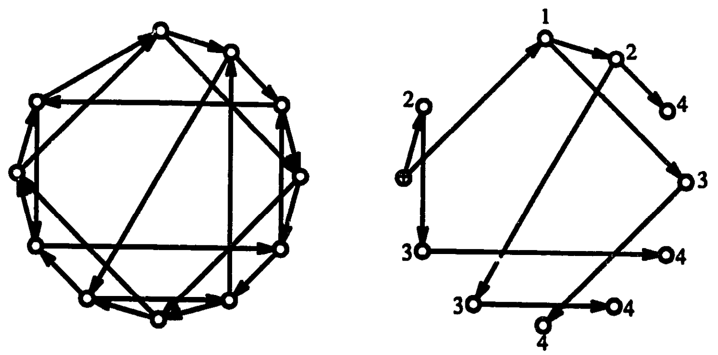
Theorem 23. $29 \leq \vec{B}(13) \leq 33$.

Proof. The digraph in Fig. 7(a) is a broadcast digraph which was constructed by adding 1 vertex and 9 edges to the digraph of Fig. 4(a). Considering the broadcast schemes in Fig. 4(b)–(d), we observe that regardless of the originator, at least 1 of the vertices 0, 1, 4, 6, 7, and 10 is idle at time 4. (Note, for example, that the scheme



(a) (b)

Fig. 7. (a) Broadcast digraph on 13 vertices and (b) a broadcast scheme.



(a) (b)

Fig. 8. (a) Broadcast digraph on 11 vertices and (b) a broadcast scheme.

shown for originator 0 indicates that vertices 0 and 3 are not involved in calls at time 4. If the originator was vertex 3 (which is isomorphic to vertex 0), vertices 3 and 6 would be idle at time 4.) Thus, if we add edges from vertices 0, 1, 4, 6, 7, and 10 to the new vertex, we can easily extend the broadcast schemes of Fig. 4(b)-(d) to inform the new vertex by time 4. The broadcast scheme given in Fig. 7(b) completes the proof of the upper bound. To prove the lower bound, notice that all vertices must have outdegree at least 2 and that a vertex with outdegree 2 can only broadcast to 12 other vertices in 4 time units if its first call is to a vertex with outdegree at least 3. By examining the broadcast tree for a vertex with outdegree 2, it is easy to argue that there must be at least 3 vertices which outdegree at least 3. □

4.6. Vertex deletion

For some values of n, our best result has come from applying the construction contained in the proof of Lemma 5 (or variations of it) to digraphs constructed by other techniques.

Theorem 24. $\vec{B}(11)=22$.

Proof. $\vec{B}(11)\leq 26$ follows from $\vec{B}(12)=24$ by using Lemma 5 to replace any of the vertices in the digraph G of Fig. 4(a). With a more detailed examination of the possible broadcast schemes for this digraph, an 11-vertex broadcast digraph G' with only 22 edges (see Fig. 8(a)) can be constructed. As in the proof of Lemma 5, a vertex, say vertex 11, is removed along with its incident edges. Rather than adding the 6 edges called for in the proof, we add only the edges (10, 0) and (1, 7) to form G'.

We must now verify that each vertex in G' can broadcast in 4 time units. Vertex 11, which was removed, is of the form 2 mod 3. Examining the broadcast schemes of Fig. 4(b)-(d), we note that if the originator is any vertex $u\equiv 0 \bmod 3$, all vertices of the form 2 mod 3 are informed at time 4 with the exception of vertex $u+2$.

Since the vertices which are informed at time 4 do not participate in any other calls, they may be deleted without affecting the broadcast. Thus, if the originator is any vertex $u \equiv 0 \bmod 3$ where $u \neq 9$, then the broadcast schemes for G also work for G'. A broadcast scheme G' with originator 9 is shown in Fig. 8(b). Similarly, we can show that the schemes for G also work for some originators of the form $u \equiv 1 \bmod 3$ or $u \equiv 2 \bmod 3$ in G'. Additional schemes need only be produced for originators 1, 2, 5, and 10 when vertex 11 is removed. The details of these schemes are omitted.

Lemma 6 shows that at least 22 edges are needed in any broadcast digraph on 11 vertices. □

Theorem 25. $24 \leq \vec{B}(17) \leq 34$.

Proof. $\vec{B}(17) \leq 38$ follows from $\vec{B}(18) = 36$ by using Lemma 5 to delete any of the vertices in a broadcast digraph on 18 vertices as described in the proof of Theorem 17. With a more detailed examination of the possible broadcast schemes for this digraph, a 17-vertex broadcast digraph G' with only 34 edges (see Fig. 9) can be constructed. As in the proof of Lemma 5, a vertex, say vertex 17, is removed along with its incident edges. Rather than adding the 6 edges called for in the proof, we add only the edges $(16, 0)$ and $(3, 13)$ to form G'.

As in the proof of Theorem 24, we observe that the broadcast schemes for G also work for G' for most originators. In particular, additional schemes need only be produced for originators 1, 2, 3, 7, 15 and 16. The details of these schemes are omitted.

Lemma 1 shows that at least 22 edges are needed in any broadcast digraph on 17 vertices. The lower bound can be improved to 24 by noticing that any broadcast originating from a vertex with outdegree 1 requires a vertex with outdegree at least 4, a vertex with outdegree at least 3, and 2 vertices with outdegree at least 2. □

Theorem 26. $38 \leq \vec{B}(19) \leq 39$.

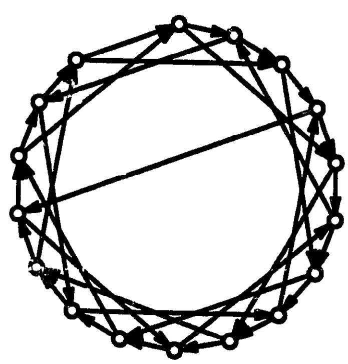


Fig. 9. Broadcast digraph on 17 vertices.

Proof. $\vec{B}(19) \leq 42$ follows from $\vec{B}(20) = 40$ by using Lemma 5 to replace any of the vertices in the digraph G of Fig. 6. With a more detailed examination of the possible broadcast schemes for this digraph, a 19-vertex broadcast digraph G' with only 39 edges can be constructed. As in the proof of Lemma 5, a vertex, say vertex $(4, 3)$ (vertex 4 of cycle 3), is removed along with its incident edges. Rather than adding the 6 edges called for in the proof, we only add edges $((3, 2), (3, 3))$, $((3, 3), (0, 3))$, and $((3, 3), (2, 0))$.

As in the proof of Theorem 24, we observe that the broadcast schemes for G also work for G' for most originators. The details of the additional required schemes are omitted.

Lemma 6 shows that at least 38 edges are needed in any broadcast digraph on 19 vertices. □

Theorem 27. $43 \leq \vec{B}(21) \leq 53$.

Proof. $\vec{B}(21) \leq 57$ follows from $\vec{B}(22) \leq 55$ by using Lemma 5 to replace any of the odd numbered vertices in the broadcast digraph on 22 vertices described in the proof of Theorem 18. With a more detailed examination of the possible broadcast schemes for this digraph, a 21-vertex broadcast digraph G' with only 53 edges can be constructed. As in the proof of Lemma 5, a vertex, say vertex 21, is removed along with its incident edges. Rather than adding the 6 edges called for in the proof, we add only the edges $(20, 0)$ and $(5, 15)$ to form G'.

As in the proof of Theorem 24, we observe that the broadcast schemes for G also work for G' for most originators. The details of the additional required schemes are omitted.

Lemma 7 shows that at least 43 edges are needed in any broadcast digraph on 21 vertices. □

Theorem 28. $47 \leq \vec{B}(23) \leq 64$.

Proof. $\vec{B}(23) \leq 68$ follows from $\vec{B}(24) \leq 66$ by using Lemma 5 to replace any of the vertices of the form 3 mod 4 in the digraph G of Fig. 5. With a more detailed examination of the possible broadcast schemes for this digraph, a 23-vertex broadcast digraph G' with only 64 edges can be constructed. As in the proof of Lemma 5, a vertex, say vertex 23, is removed along with its incident edges. Rather than adding the 6 edges called for in the proof, we add only the edges $(22, 0)$ and $(5, 15)$ to form G'.

As in the proof of Theorem 24, we observe that the broadcast schemes for G also work for G' for most originators. The details of the additional required schemes are omitted.

Lemma 7 shows that at least 47 edges are needed in any broadcast digraph on 23 vertices. □

5. Additional constructions

The methods of Lemmas 3 and 4 can be employed to establish new upper bounds on $\vec{B}(n)$ for several values of n. These methods use broadcast digraphs for small n to build broadcast digraphs for larger values of n. Thus, the discovery of additional values of $\vec{B}(n)$, or the improvement of the upper bound on $\vec{B}(n)$ for small values of n, is likely to lead to improvements in the upper bounds on $\vec{B}(n)$ for larger values of n.

Theorem 29. $\vec{B}(30)=120$.

Proof. A broadcast digraph on 30 vertices can be constructed by joining a pair of minimum broadcast digraphs on 15 vertices with 30 directed edges as described in the proof of Lemma 3. The resulting graph has 120 edges. (This upper bound also follows from Lemma 1.) Lemma 6 shows that at least 120 edges are needed in any broadcast digraph on 30 vertices. □

Theorem 30. $\vec{B}(9)=16$.

Proof. To show that $\vec{B}(9)\geq 12$, note that any vertex with outdegree 1 must be adjacent to a vertex which can inform the remaining 7 vertices in 3 time units. So, a vertex with outdegree 1 must be adjacent to a vertex with outdegree at least 3 which must, in turn, be adjacent to a vertex with outdegree at least 2. The situation is shown in Fig. 10(a) in which vertex s has outdegree 1, t has outdegree at least 3, and u has outdegree at least 2. Furthermore, all of the vertices in Fig. 10(a) must be distinct. If w has outdegree 1, then y must have outdegree at least 3. If w has outdegree at least 2, then y could have outdegree 1. Similarly, either x has outdegree at least 2 or z has outdegree at least 3. This shows that $\vec{B}(9)\geq 14$. It also shows that there is at least one vertex with outdegree at least 3 and at least 3 more vertices with outdegree at least 2.

Suppose that $\vec{B}(9)=15$. It follows from the argument above that there are only 3 possible ways to choose the outdegrees of the vertices; there can be 1 vertex with outdegree 4 and 3 vertices with outdegree 2, 1 vertex with outdegree 3 and 4 with outdegree 2, or 2 vertices with outdegree 3 and 2 with outdegree 2.

First consider the case in which there is 1 vertex with outdegree 4 and 3 with outdegree 2. Then w and x both have outdegree 2, and y, v, z, and r all have outdegree 1 and must be adjacent to t, which must be the vertex with outdegree 4. The situation is shown in Fig. 10(b). The only edges that have not been fixed are 1 edge leaving w, 1 edge leaving x, and 1 edge leaving t. Furthermore, 1 of these edges must enter s. If w is adjacent to s, then broadcasting from w is impossible. Similarly, x cannot be adjacent to s, so t must be the vertex adjacent to s.

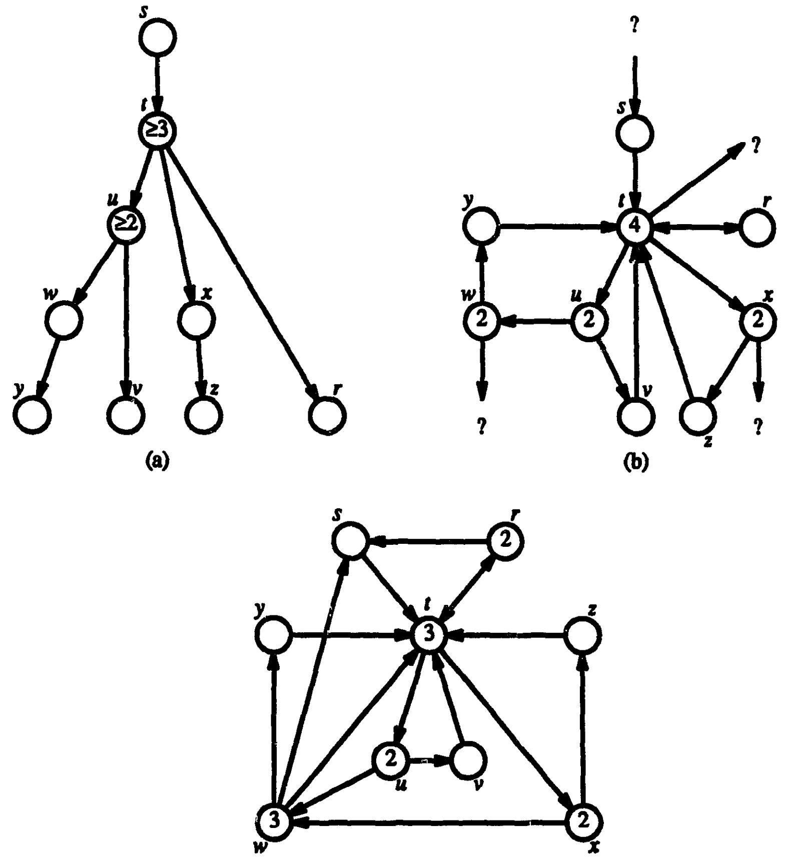


Fig. 10. Broadcast digraph on 9 vertices.

Consider a broadcast originating from v. v informs t at time 1 and t then has the choice of informing either u or x at time 2. If t informs u at time 2, then u calls w at time 3, and t is forced to call x at time 3 and s at time 4. It is now impossible to inform all three of y, z and r because only w and x remain available to make calls at time 4. Suppose that t informs x at time 2 instead of u. There are now 2 time units remaining and 6 vertices that have not been informed, so t and x must each call a vertex with outdegree 2 at time 3. The only possibility is that t calls u and x calls w. Since t must call s at time 4, w must be adjacent to r, but this makes broadcasting from w impossible. This completes the proof of the case in which there is 1 vertex with outdegree 4 and 3 with outdegree 2.

If there is 1 vertex with outdegree 3 and 4 with outdegree 2, then the vertex with outdegree 3 is t, and 3 of the vertices with outdegree 2 must be u, w, and x. There are 4 possible choices for the remaining vertex with outdegree 2. If y, v, or z has outdegree 2, then r must be adjacent to t and broadcasting from r is impossible. If r has outdegree 2, then y, v, and z are all adjacent to t. There are now only 4 edges that have not been fixed. The only possible configuration is that r is adjacent to s,

Table 1. Upper and lower bounds on $\vec{B}(n)$ (* indicates optimality)

n	Lower	Upper	n	Lower	Upper	n	Lower	Upper	n	Lower	Upper
1	0	0*	9	16	16*	17	24	34	25	51	75
2	2	2*	10	20	20*	18	36	36*	26	78	78*
3	3	3*	11	22	22*	19	38	39	27	81	93
4	8	8*	12	24	24*	20	40	40*	28	84	104
5	7	7*	13	29	33	21	43	53	29	88	115
6	12	12*	14	42	42*	22	45	55	30	120	120*
7	14	14*	15	45	45*	23	47	64	31	124	130
8	24	24*	16	64	64*	24	49	66	32	160	160*

and the other 3 unfixed edges (from w, x, and r) go to vertices with outdegree at least 2. However, this makes broadcasting from v impossible.

The only remaining possibility is 2 vertices with outdegree 3 and 2 vertices with outdegree 2. There are 5 possible locations for the second vertex with outdegree 3 (u, w, y, x, z) and each choice fixes the locations for the vertices with outdegree 2 and also fixes some of the edges leaving vertices with outdegree 1. A naive enumeration by computer of the digraphs satisfying these constraints (approximately 20,000 digraphs) was used to establish that none of them is a broadcast digraph. Therefore $\vec{B}(9) \geq 16$.

The upper bound is established by the digraph of Fig. 10(c) which was derived from the digraph of Fig. 10(b). The broadcast schemes can be summarized as follows. If the originator is s, v, w, y, z, or r, then the originator calls t during the first time unit. If the originator is u, then u calls v first and then informs w while v is informing t. If the originator is x, then x calls w first and then informs z while w is informing t. In most cases, t makes its first call to u and its second call to x. The exceptions are broadcasts originated by x and z when t calls r after calling u and broadcasts originated by u and v when t first calls x and then r. The remaining details of the broadcast schemes are now easy to deduce. □

6. Known bounds for small n

The current best known bounds on $\vec{B}(n)$ for $1 \leq n \leq 32$ are shown in Table 1. Note that the (nontrivial) upper bounds which were not explicitly discussed above are derived using Lemmas 1 and 3. In particular, the upper bounds given for $n = 28$ and 31 are from Lemma 1 while the upper bounds for $n = 27$ and 29 are from Lemma 3. The smallest n for which Lemma 4 gives the best upper bound is $n = 33$. The (nontrivial) lower bounds which were not explicitly discussed above are derived using Lemmas 6 and 7. The lower bounds given for $n = 27$, 28 and 31 are from Lemma 6 while the lower bound for $n = 29$ is from Lemma 7.

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