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# Broadcasting in DMA-bound bounded degree graphs

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*Abstract*

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Broadcasting is an information dissemination process in which a message is to be sent from a single originator to all members of a network by placing calls over the communication lines of the network. In [2], Bermond, Hell, Liestman and Peters studied the effect, on broadcasting capabilities, of placing an upper bound on the graph's degree. In this paper, we generalize their results by allowing calls to involve more than two participants. We give lower bounds and construct bounded degree graphs which allow rapid broadcasting. Our constructions use the notion of compounding graphs in de Bruijn digraphs. We also obtain asymptotic upper and lower bounds for broadcast time, as the maximum degree increases.

La diffusion correspond à un processus dans lequel un message doit être envoyé à tous les membres d'un réseau à partir d'un initiateur en utilisant les lignes de communication de ce réseau. Dans [2], Bermond, Hell, Liestman et Peters ont étudié les possibilités de diffusion des graphes de degré borné. Dans ce rapport, nous généralisons ces travaux en permettant des communications réunissant plus de deux participants. Nous donnons des bornes inférieures et construisons des réseaux de degré borné permettant une diffusion rapide. Nous utilisons la notion de composition de graphes dans les réseaux de de Bruijn. Nous obtenons aussi asymptotiquement des bornes inférieures et supérieures pour le temps de diffusion lorsque le degré augmente.

## 1. Introduction

A node in a typical network architecture consists of a processor, a memory, a fast bus, and several DMA (Direct Memory Access) channels. Each DMA channel connects the node to one of its neighbours. The memory and DMA channels are all connected to the fast bus, and the processor is connected to the memory. A processor communicates with a neighbour by writing the information in its memory. The information is then transmitted by the appropriate DMA channel via the bus to the neighbour's memory via its bus. This communication path between two processors will be called a *link*.

In a recent paper [25], Stout and Wager investigated communication in hypercubes under the assumption that communication time dominates local processing time to the extent that local processing costs can be ignored. More precisely, they considered problems in which extensive communication is needed because very long messages are being sent. They classified communication patterns in hypercubes into three types depending on where the communication bottleneck occurs. Typically, links are bidirectional, so, theoretically, a processor with $d$ neighbours can be involved in as many as $2d$ simultaneous communications. Stout and Wager [25] call this situation *link-bound*. In a *processor-bound* communication pattern, each processor can only communicate with one neighbour at any given time. In a *DMA-bound* system, there is an upper bound on the number of messages that can enter or leave any node simultaneously.

Stout and Wager [25] concentrated on link-bound hypercube systems. In this paper we shall investigate communication problems in DMA-bound systems, including, but not restricted to, hypercubes. As in [25], we assume that communication time dominates local processing time so that local processing costs can be ignored. We also assume that messages are short and therefore make the simplifying assumption that each call uses one unit of time.

DMA-bound systems can be described in terms of several parameters. The topology of the system can be described by an (undirected) graph or family of graphs. A node in a graph corresponds to a processor together with its memory and bus, while edges represent the channels connecting the buses. $n$ will be used to denote the number of processors (nodes) in the system, $d(v)$ will denote the degree of the node $v$ and $\Delta = \max\{d(v) \mid v \in V(G)\}$. $k$ is the number of distinct data streams that can be handled simultaneously by a bus in a node. Since the systems that we shall investigate are quite uniform, we assume that $k$ applies to all nodes unless otherwise specified.

Broadcasting refers to the process of message dissemination in a communication network whereby a message, originated by one node, is transmitted to all nodes of the network. Broadcasting is accomplished by placing a series of calls over the communication lines of the network. This is to be completed as quickly as possible subject to the constraints that each call involves only one informed node and $k$ or fewer of its neighbours, each call requires one unit of time, a vertex can participate in only one call per unit of time, and a vertex can only call its neighbours.

Given a connected graph $G$ and a message originator $u$, the *broadcast time of vertex* $u$, $b_k(u)$, is the minimum number of time units required to complete broadcasting from vertex $u$. It is easy to see that for any vertex $u$ in a connected graph $G$ with $n$ vertices, $b_k(u) \geq \lceil \log_{k+1} n \rceil$, since the number of informed vertices can at most be multiplied by $k+1$ during each time unit. The *broadcast time of a graph* $G$, $b_k(G)$, is defined to be the maximum broadcast time of any vertex $u$ in $G$, i.e., $b_k(G) = \max\{b_k(u) \mid u \in V(G)\}$. For the complete graph $K_n$ with $n \geq 2$ vertices, $b_k(K_n) = \lceil \log_{k+1} n \rceil$, yet $K_n$ is not minimal with respect to this property for any $n > 3$, while $n > k+1$. That is, we can remove edges from $K_n$ and still have a graph $G$ with $n$ vertices such that $b_k(G) = \lceil \log_{k+1} n \rceil$.

The *broadcast function*, $B_k(n)$, is the minimum number of edges in any graph on $n$ vertices such that each vertex in the graph can broadcast in minimum time, that is, in time $\lceil \log_{k+1} n \rceil$. A *minimum broadcast graph* (mbg) is a graph $G$ on $n$ vertices having $B_k(n)$ edges and $b_k(G) = \lceil \log_{k+1} n \rceil$. From an application perspective, minimum broadcast graphs represent the cheapest possible communication networks (having the fewest communication lines) in which broadcasting can be accomplished, from any vertex, as fast as theoretically possible.

## 2. Previous results

If one wants to concentrate on the hypercube problem, a lot of results have been found by Fraigniaud [11], Johnsson and Ho [16,18], Saad and Schultz [24] and Stout and Wager [25]; but apart from the hypercube, most of the previous work in this area has been on the processor-bound problem, that is $k=1$. For a survey of results on broadcasting and related problems, see Hedetniemi, Hedetniemi and Liestman [13] and Fraigniaud and Lazard [27]. Johnson and Garey [17] showed that the problem of determining $b_1(v)$ for a vertex $v$ in an arbitrary graph $G$ is NP-complete. In [10] Farley, Hedetniemi, Mitchell and Proskurowski studied $B_1(n)$. In particular, they determined the values of $B_1(n)$ for $n \leq 15$ and noted that $B_1(2^p) = p2^{p-1}$ (the $p$-cube is an mbg on $n = 2^p$ vertices). Mitchell and Hedetniemi [23] determined the value for $B_1(17)$, Wang [26] found the value of $B_1(18)$, and Bermond, Hell, Liestman and Peters [3] found the values of $B_1(19)$, $B_1(30)$ and $B_1(31)$. On the directed broadcasting problem, that is broadcasting in digraphs, Liestman and Peters [21] studied $\vec{B}(n)$, the minimum number of arcs in a broadcast digraph on $n$ vertices.

Since these studies suggest that mbgs are extremely difficult to find, several authors have devised methods to construct graphs with small numbers of edges which allow minimum time broadcasting from each vertex. In [9] Farley designed several techniques for constructing broadcast graphs with $n$ vertices and approximately $\frac{1}{2}\log_2 n$ edges. Chau and Liestman [7] presented constructions based on Farley's techniques which yield somewhat sparser graphs for most values of $n$. In [12] Grigni and Peleg showed that $B_1(n) \in \Theta(L(n)n)$ where $L(n)$ denotes the exact number of consecutive leading 1's in the binary representation of $n-1$. More generally, they showed that $B_k(n) \in \Theta(kL_k(n)n)$ where $L_k(n)$ denotes the exact number of consecutive leading $k$'s in the $(k+1)$-ary representation of $n-1$. Asymptotically, Grigni and Peleg's construction (which establishes their upper bound) produces the best results for most values of $n$.

In Section 3, we shall give some results for minimum broadcasting graphs in DMA-bound systems and especially some values of $B_2(n)$, $B_3(n)$ and $B_4(n)$ for small values of $n$.

So far, the emphasis in this research has been on obtaining graphs in which each vertex can broadcast in minimum time. If these graphs are to be used in the design of actual networks, other considerations may override the need for minimum time broadcasting. In particular, the constructions of Farley, and of Chau and Liestman result in graphs with $n$ vertices and average degree $O(\log_2 n)$ while the construction of Peleg yields graphs with some vertices of degree $O(L_k(n)+\log_{k+1}\log n)$. It may be more realistic to use a graph with fixed maximum degree (see [1, 15]) in which every vertex can broadcast "quickly". We shall use the term ***bounded degree broadcast graph*** (bdbg) to describe a graph $G$ on $n$ vertices with maximum degree $\Delta$ such that $b_k(G)$ is "close to" $b_k(n,\Delta)=\min\{b_k(H) \mid H$ has $n$ vertices and maximum degree $\Delta\}$.

Liestman and Peters [20] first investigated broadcasting in bounded degree graphs. Then, Bermond and Peyrat [4] considered broadcasting in de Bruijn and Kautz graphs. More recently, Bermond, Hell, Liestman and Peters [2], and Capocelli, Gargano and Vaccaro [6] presented general lower bounds on the time required to broadcast in bounded degree graphs and reported the best of the known upper bounds on the time required to broadcast in bounded degree graphs. Recently, Heydemann, Opatrny and Sotteau [14] improved the results of Bermond and Peyrat for broadcasting time in de Bruijn and Kautz graphs.

In this paper, we generalize the results obtained by Bermond, Hell, Liestman and Peters [2] to DMA-bound broadcast graphs, where each vertex can call simultaneously several of its neighbours. In Section 3, we give some basic results on minimum broadcast graphs and some examples of those graphs for small values of $n$. In Section 4, we present general lower bounds on the time required to broadcast in DMA-bound bounded degree graphs. In Section 5, we give the definitions of the de Bruijn digraph and of compound graphs and summarize some of the work of Bermond, Hell, Liestman and Peters [2] which will be useful in the last section. Finally, in Section 6, we show how the use of compound graphs gives upper bounds on the time required to broadcast in DMA-bound bounded degree graphs.

## 3. Minimum broadcast graphs

Let $k$ be the maximum number of neighbours a vertex can inform in a single call in one unit of time. We have the following results:

(1) For $n \le k+1$, $B_k(n)=\frac{1}{2}n(n-1)$, and a minimum broadcast graph is $K_n$, the complete graph with $n$ vertices.

(2) $B_k(k+2)=k+1$ and a minimum broadcast graph with $k+2$ vertices is the star with $k+1$ edges around a central vertex.

(3) $B_k([k+1]^p)=\frac{1}{2}kp(k+1)^p$ for $p \ge 1$ and a minimum broadcast graph with $(k+1)^p$ vertices is $(K_{k+1})^p=K_{k+1}\square\cdots\square K_{k+1}$, the cartesian product of complete graphs[1].

[1] The *cartesian product* (*also called cartesian sum*) of two simple graphs $G$ and $H$ is the simple graph $G\square H$ with vertex set $V(G)\times V(H)$, in which $(u,v)$ is adjacent to $(u',v')$ if and only if either $u=u'$ and $vv'\in E(H)$ or $v=v'$ and $uu'\in E(G)$. Let $(K_n)^p=K_n\square\cdots\square K_n$, then $(K_2)^p$ is the usual $p$-hypercube family (see [5]).

**Proofs.** (1) and (2) The proofs are immediate.

(3) Let $G$ be a minimum broadcast graph on $(k+1)^p$ vertices. The broadcast must be completed in $p$ units of time. As $(k+1)^p$ vertices must be informed, a vertex must never become idle during the broadcast. Therefore, each vertex must be of degree at least $kp$. So $G$ must have at least $\frac{1}{2}kp(k+1)^p$ edges: $B([k+1]^p) \geq \frac{1}{2}kp(k+1)^p$.

If we consider the graph $(K_{k+1})^p$, we can see that, by sending along one "dimension" at a time, the broadcast can be completed in $p$ units of time. As $(K_{k+1})^p$ has $\frac{1}{2}kp(k+1)^p$ edges, it follows that $(K_{k+1})^p$ is a minimum broadcast graph on $(k+1)^p$ vertices and that $B([k+1]^p) = \frac{1}{2}kp(k+1)^p$. (If we take $k=1$, we have the usual $p$-hypercube family.) (Proof also given in [12].) $\square$

Apart from those results, we have been able to calculate some values of $B_2(n)$, $B_3(n)$ and $B_4(n)$ for small values of $n$. To obtain those values of $B_i(n)$, we first exhibit a graph $G$ on $n$ vertices and $l$ edges that broadcasts in the minimum time, which shows that $B_k(n) \leq l$. We then show that a minimum broadcast graph with $l-1$ edges is impossible by studying, for example, a broadcast tree (a subgraph of $G$ showing the calls made to complete broadcast), which proves $B_k(n) \geq l$, hence the results given in Table 1. We also give some examples of minimum broadcast graphs for those values of $n$ in Figs. 1, 2 and 3.

## 4. Lower bounds

If we want to use graphs with good broadcasting capabilities, we rapidly see that we have to use graphs with a large degree. Unfortunately, practical graphs have a limited degree, bounded by the technology. Transputers, for example, currently have a degree of 4. We therefore need to calculate the broadcasting capabilities of bounded degree graphs. This is the purpose of this section which gives lower bounds on broadcasting time in bounded degree graphs.

Let $G$ be a graph on $n$ vertices with a maximum degree $\Delta$. $b_k(G)$ is the broadcast time of $G$. $b_k(n, \Delta) = \min\{b_k(G) \mid G$ has $n$ vertices and is of maximum degree $\Delta\}$.

We wish to prove a lower bound on $b_k(n, \Delta)$. It will be more convenient to first consider the quantity $a_{t,k}^{\Delta}$ which denotes the maximum number of vertices that can be informed in time $t$ in any graph of maximum degree $\Delta$ (for clarity, we shall omit the subscript $k$). An upper bound on $a_t^{\Delta}$ will clearly translate to a lower bound on $b_k(n, \Delta)$.

Table 1. Some values of $B_k(n)$

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $B_2(n)$ | 0 | 1 | 3 | 3 | 5 | 7 | 10 | 12 | 18 | 12 | 13 | 15 |
| $B_3(n)$ | 0 | 1 | 3 | 6 | 4 | 7 | 9 | 11 | | | | |
| $B_4(n)$ | 0 | 1 | 3 | 6 | 10 | 5 | 9 | 11 | | | | |

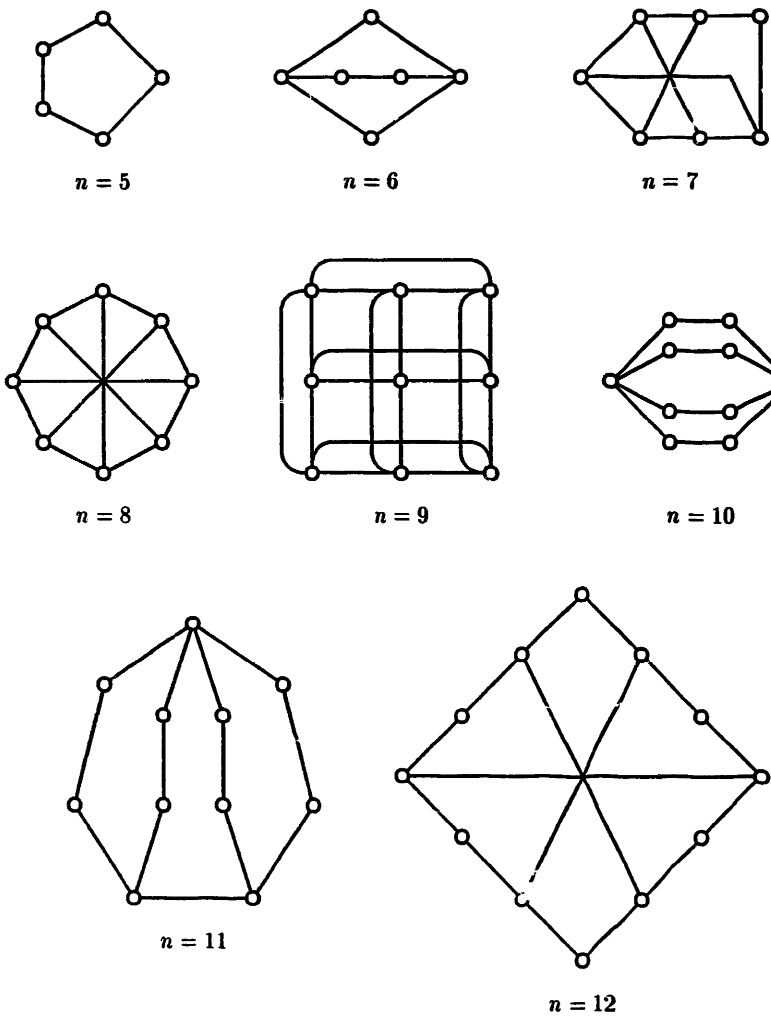



Fig. 1. Some minimum broadcast graphs for $k = 2$.

Let

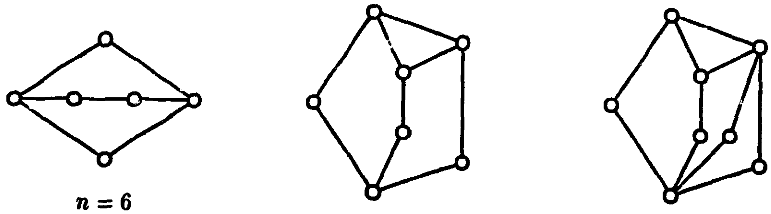
$$\begin{cases} q = (\Delta - 1) \operatorname{div} k, \\ p = (\Delta - 1) \bmod k. \end{cases}$$

Hence,

$$\Delta - 1 = qk + p.$$

A vertex which is informed, calls its neighbours: in the best case it can call $q$ groups of $k$ neighbours and then one group of $p$ neighbours, then it becomes idle. Thus we obtain the following inequality:

$$a^{\Delta}_{t+q+2} \leq (k+1)a^{\Delta}_{t+q+1} - (k-p)(a^{\Delta}_{t+1} - a^{\Delta}_{t}) - k a^{\Delta}_{t}.$$



$n=6$ $n=7$ $n=8$

Fig. 2. Some minimum broadcast graphs for $k=3$.

Therefore an upper bound on $a_t^{\Delta}$ is the solution to the following recurrence:

$$\begin{cases} b_t^{\Delta}=(k+1)^t & \text{for } 0\le t\le q, \\ b_{t+q+2}^{\Delta}=(k+1)b_{t+q+1}^{\Delta}-(k-p)(b_{t+1}^{\Delta}-b_t^{\Delta})-kb_t^{\Delta} & \text{for } t\ge 0. \end{cases} \tag{1}$$

To solve (1), consider its characteristic equation (see also [22]):

$$X^{q+2}-(k+1)X^{q+1}+(k-p)X+p=0. \tag{2}$$

We state below some properties concerning the roots of this equation; proofs which use different tools of calculus and complex analysis are omitted; they are given in full detail in the report [19].

**Theorem 4.1.** (2) *has only one real solution greater than* 1.

**Theorem 4.2.** *The other complex solutions of* (2) *have an absolute value smaller than* 1.

Let $r$ be the only real solutions of (2) greater than 1. With Theorem 4.2, we can say that: $b_t^{\Delta}\approx c_0\cdot r^t$, for some $c_0>0$, for $t\to\infty$. If $t$ is the broadcast time, we have: $t\approx(1/\log_{k+1} r)\log_{k+1} n$ for $n$ large enough.

What we now need to do is to estimate $r$, which is the only real solution greater than 1 of the equation (2): $X^{q+2}-(k+1)X^{q+1}+(k-p)X+p=0$.

**Theorem 4.3.** *Let* $\alpha=(k+1)-k/(k+1)^{q+1}$, *then* $r<\alpha$.

$n=7$ $n=8$

Fig. 3. Some minimum broadcast graphs for $k=4$.

*Let* $0<\varepsilon<1$, *let* $\beta=(k+1)-k/(k+1-\varepsilon)^q$, *then, for* $q>-\log(\varepsilon/k)/\log(k+1-\varepsilon)$, *we have* $\beta<r$.

**Theorem 4.4.** *For all* $\Delta$ *and for* $n$ *large enough, we have*

$$b_k(n,\Delta)\geq\left(1+\frac{k\cdot\log_{k+1}\mathrm{e}}{(k+1)^{q+2}}\right)\log_{k+1}n+\mathrm{O}(1).$$

**Proof.** Let $\Delta$ be given and $\alpha=(k+1)-k/(k+1)^{q+1}$. If $b_t^\Delta\geq n$ and $n$ is large enough, then $t$ will be large enough so that, for a $c_1>0$, we have

$$\begin{aligned}&b_t^\Delta\leq c_1\cdot\alpha^t\\&\quad\Leftrightarrow c_1\cdot\alpha^t\geq n\\&\quad\Leftrightarrow t\geq\frac{1}{\log_{k+1}\alpha}\log_{k+1}n+\mathrm{O}(1).\end{aligned}$$

We also have

$$\log_{k+1}\alpha=1+\log_{k+1}\mathrm{e}\cdot\ln\left(1-\frac{k}{(k+1)^{q+2}}\right)$$

and

$$(\forall u\mid 0<u<1)(\forall v\mid v>0)\,\frac{1}{1+v\ln(1-u)}>1+uv,$$

thus we obtain

$$t\geq\left(1+\frac{k\cdot\log_{k+1}\mathrm{e}}{(k+1)^{q+2}}\right)\log_{k+1}n+\mathrm{O}(1).\qquad\square$$

**Theorem 4.5.** *For any* $c>\log_{k+1}\mathrm{e}$ *and for* $n$ *and* $\Delta$ *large enough, we have*

$$b_k(n,\Delta)\leq\left(1+\frac{kc}{(k+1)^{q+1}}\right)\log_{k+1}n.$$

**Proof.** As in the previous demonstration we have

$$b_k(n,\Delta)\leq\frac{1}{\log_{k+1}\beta}\log_{k+1}n+\mathrm{O}(1)$$

and

$$b_k(n,\Delta)<\left(1+\frac{k}{(k+1)^{q+1}}\log_{k+1}\mathrm{e}+\mathrm{o}\left(\frac{1}{(k+1)^{q+1}}\right)\right)\log_{k+1}n,$$

thus we obtain

$$(\forall c>\log_{k+1}\mathrm{e})\quad b_k(n,\Delta)<\left(1+\frac{kc}{(k+1)^{q+1}}\right)\log_{k+1}n.\qquad\square$$

This way we have an approximation of $b_k(n,\Delta)$, but if one wants to obtain the exact factor in front of the $\log_{k+1}n$, one just has to calculate $1/\log_{k+1}r$, where $r$ is the solution of (2). Table 2 gives the exact solution of (2) for $\Delta$ and $k$ smaller than 10.

Table 2. Some values of $1/\log_{k+1} r$

| $\Delta$ | $k$ | $1/\log_{k+1} r$ equals | $\Delta$ | $k$ | $1/\log_{k+1} r$ equals |
|---|---|---|---|---|---|
| 3 | 1 | 1.440420090 | 7 | 5 | 1.087740159 |
| 3 | 2 | 1.584962501 | 7 | 6 | 1.086033133 |
| 4 | 1 | 1.137466951 | 8 | 1 | 1.005842216 |
| 4 | 2 | 1.246477436 | 8 | 2 | 1.016265198 |
| 4 | 3 | 1.261859507 | 8 | 3 | 1.028883529 |
| 5 | 1 | 1.056214652 | 8 | 4 | 1.047843132 |
| 5 | 2 | 1.093089412 | 8 | 5 | 1.065728862 |
| 5 | 3 | 1.160308872 | 8 | 6 | 1.070094827 |
| 5 | 4 | 1.160964047 | 8 | 7 | 1.068621562 |
| 6 | 1 | 1.025404040 | 9 | 1 | 1.002873979 |
| 6 | 2 | 1.055654946 | 9 | 2 | 1.007846727 |
| 6 | 3 | 1.091401382 | 9 | 3 | 1.018492951 |
| 6 | 4 | 1.114850625 | 9 | 4 | 1.022176363 |
| 6 | 5 | 1.113282753 | 9 | 5 | 1.046439458 |
| 7 | 1 | 1.012034454 | 9 | 6 | 1.055732406 |
| 7 | 2 | 1.025341228 | 9 | 7 | 1.057852106 |
| 7 | 3 | 1.040210375 | 9 | 8 | 1.056641667 |
| 7 | 4 | 1.078141287 | | | |

What we can see from those results is that graphs are not too hampered in their broadcasting capabilities by having a bounded degree, except maybe for the first few values. If $k=1$, we have the same results Bermond, Hell, Liestman and Peters [2] found.

## 5. Broadcasting in compound graphs

We shall use the notion of compound graph that was defined in [2]. A quick way of describing it would be to say that the compound graph $G[B]$ is a graph where all the vertices of $B$ have been replaced by a copy of $G$ and where the edges of $B$ are redistributed on the vertices of $G$.

To determine the broadcast time of $G[B]$ we introduce a new parameter $\bar{b}_k(G)$, the smallest possible "average" time needed to transmit a message originated in a copy of $G$ to its outneighbour copies. Formally, let $G$ be a graph to which $d$ outgoing arcs $o_0, \ldots, o_{d-1}$ have been attached. For a vertex $u$ of $G$ and a particular broadcasting scheme for $u$ in $G$, let $t^i_{k,u}$ denote the time at which message originated at $u$ at time 0 will be sent on the arc $o_i$. The value $\bar{t}_{k,u} = 1/d(t^0_{k,u} + \cdots + t^{d-1}_{k,u})$ is the average time for a message originated at $u$ to leave $G$ along the arcs $o_i$ under the given broadcasting scheme. If we let $\bar{b}_k(G,u)$ be the minimum of $\bar{t}_{k,u}$ over all possible broadcasting schemes for originator $u$ in $G$, then $\bar{b}_k(G) = \max\{\bar{b}_k(G,u) \mid u \in V(G)\}$.

**Theorem 5.1.** $b_k(G[B(d,D)]) \le (D+1)\bar{b}_k(G) + b_k(G)$.

**Proof.** In [2], the authors give a demonstration of this theorem for the case $k=1$. Their proof can be very easily generalized for any $k\geq 1$. Therefore we refer the reader to the original article [2]. □

## 6. Upper bounds

In this section, we give examples of the construction of specific bounded degree graphs and determine the broadcast time for these graphs. We have especially studied the broadcasting capabilities of compounding the infinite family of cartesian products of complete graphs, $(K_{k+1})^p$, in de Bruijn digraphs (see [8]), for large values of $\Delta$.

If we have $\Delta=3$, $k=2$, let $B=B(2,D)$ and $G=K_2$. The resulting compound graph $H=K_2[B(2,D)]$ is a 3-regular graph on $n=2^{D+1}$ vertices and has broadcast time $b_2(H)\leq(D+1)\frac{3}{2}+1=1.5\log_2 n+1=2.377\log_3 n+1$. However if we choose the 3-star as $G$ and compound this graph with $B=B(3,D)$, we get $H'=G[B(3,D)]$, a graph on $4\cdot 3^D$ vertices. Since $\bar{b}_2(G)=\frac{7}{3}$, we get $b_2(H')\leq\frac{7}{3}D+\frac{7}{3}+2=\frac{7}{3}\log_3 n-\frac{7}{3}\log_3 4+\frac{13}{3}=2.333\log_3 n+1.389$. This is the best value we obtained.

Similar calculations have been done using various graphs for small values of $\Delta$. The best values that we have obtained are shown in Table 3. The table shows the degree of the graph constructed ($\Delta$), the graph $G$, the average time ($\bar{b}_k(G)$) needed to transmit a message originated in a copy of $G$ to all of the outneighbour copies, the indegree (outdegree) of the de Bruijn digraph used ($d$), the upper bound obtained by this graph and the best lower bound known. $T_6$ is used to denote a special tree on six vertices also shown in [2]. $C_5$, $C_7$ and $C_8$ denote the cycles on five, seven and eight vertices, respectively. 40-sbg represents a sparse broadcast graph on 40 vertices (see [3]). 3-cube indicates the 3-dimensional cube. This table shows the best values that we have been able to obtain with this technique using known graphs, but, clearly, most of these values are not optimal.

We can also obtain some "asymptotic results" by using the only known infinite family of minimum broadcast graphs, the cartesian product of complete graphs,

Table 3. Best bounds

| $\Delta$ | $k$ | $G$ | $\bar{b}_k(G)$ | $d$ | Upper bound | Lower bound |
|---|---|---|---|---|---|---|
| 3 | 1 | $T_6$ | 15/4 | 4 | $1.875000\log_2 n$ | $1.440420\log_2 n$ |
| 3 | 2 | 3-star | 7/3 | 3 | $2.333333\log_3 n$ | $1.584963\log_3 n$ |
| 4 | 1 | $C_8$ | 34/8 | 8 | $1.416667\log_2 n$ | $1.137467\log_2 n$ |
| 4 | 2 | $C_7$ | 20/7 | 7 | $1.613072\log_3 n$ | $1.246477\log_3 n$ |
| 4 | 3 | $C_5$ | 11/5 | 5 | $1.894977\log_4 n$ | $1.261860\log_4 n$ |
| 5 | 1 | 40-sbg | 266/40 | 40 | $1.249547\log_2 n$ | $1.056215\log_2 n$ |
| 5 | 2 | 3-cube | 23/8 | 8 | $1.518922\log_3 n$ | $1.093089\log_3 n$ |
| 5 | 3 | 3-cube | 21/8 | 8 | $1.750000\log_4 n$ | $1.160309\log_4 n$ |
| 5 | 4 | 3-cube | 20/8 | 8 | $1.934940\log_5 n$ | $1.160964\log_5 n$ |

$(K_{k+1})^p$. This graph on $n=(k+1)^p$ vertices is regular of degree $kp$ and all vertices complete their calls $p+1$ time units after the originator receives the message. We shall consider in some detail the case $k=2$ and give results for the general case. For more details, see also [19].

### 6.1. $k=2$

A graph of degree $\Delta$ can be constructed by replacing each vertex of a de Bruijn digraph $B(d,D)$ by a graph $(K_3)^p$.

#### 6.1.1. $\Delta$ even

We compound the graph $G=(K_3)^{(\Delta-2m)/2}$ in the de Bruijn digraph $B(m3^{(\Delta-2m)/2},D)$ for any $1\le m\le\lfloor(\Delta-2)/2\rfloor$, and distribute the arcs from the de Bruijn digraph so that each vertex of each copy of $G$ is given $m$ inarcs and $m$ outarcs. When the broadcast is completed in $G$, each vertex of $G$ can inform its $m$ neighbours, two by two, at times $((\Delta-2m)/2+1)$, $((\Delta-2m)/2+2),\dots,((\Delta-2m)/2+\lfloor m/2\rfloor)$ and eventually one more neighbour at time $((\Delta-2m)/2+\lceil m/2\rceil)$ if $m$ is odd.

Therefore we have the following results:

(1) $m\equiv 0 \pmod 2$, then

$$\bar{b}_2(G)=\frac{2\sum_{i=1}^{m/2}((\Delta-2m)/2+i)(3^{(\Delta-2m)/2})}{m(3^{(\Delta-2m)/2})}$$

$$=\tfrac{1}{2}\left(\Delta-\frac{3m}{2}+1\right).$$

Applying Theorem 5.1, we obtain

$$b_2(G[B(d,D)])\le\tfrac{1}{2}\left(\Delta-\frac{3m}{2}+1\right)\log_{m(3^{(\Delta-2m)/2})}n+\mathrm{O}(1)$$

$$\le\left(1+\frac{c_m}{\Delta-d_m}\right)\log_3 n+\mathrm{O}(1)$$

where $c_m$ and $d_m$ are constants depending on $m$, $c_m=m/2+1-2\log_3 m$. The smallest constant $c_m$ that can be obtained is $c_4\approx 0.4763$.

(2) $m\equiv 1 \pmod 2$, then

$$\bar{b}_2(G)=\frac{\Delta-2m}{2}+\frac{1}{m}\left(\frac{m+1}{2}\right)^2.$$

Applying Theorem 5.1, we obtain

$$b_2(G[B(d,D)])\le\left(1+\frac{c'_m}{\Delta-d'_m}\right)\log_3 n+\mathrm{O}(1)$$

where $c'_m=(m+1)^2/(2m)-2\log_3 m$. The minimum, for $m\equiv 1 \pmod 2$ is obtained for $m=3$, with $c'_3\approx 0.6667$.

Therefore, if $\Delta$ is even, we must take $m=4$ and we have:

$$b_2(G[B(d,D)]) \le \left(1+\frac{3-2\log_3 4}{\Delta-8+2\log_3 4}\right)\log_3 n + O(1) \approx \left(1+\frac{0.4763}{\Delta}\right)\log_3 n.$$

*6.1.2. $\Delta$ odd*

We compound the graph $G=(K_3)^{(\Delta-2m-1)/2}$ in the de Bruijn digraph $B((2m+1)/2)3^{(\Delta-2m-1)/2-1/2}, D)$ for any $1\le m\le \lfloor(\Delta-3)/2\rfloor$. There are $(3^{(\Delta-2m-1)/2}-1)/2$ vertices which have $m+1$ outarcs, and $(3^{(\Delta-2m-1)/2}+1)/2$ vertices which have $m$ outarcs.

For $\Delta$ large enough, we have

$$\bar{b}_2(G)\approx \tfrac{1}{2}\left(\Delta-\frac{3m^2+m-1}{2m+1}\right).$$

Considering again $m\equiv 0 \pmod 2$ and $m\equiv 1 \pmod 2$, it appears that the minimum is obtained for $m=3$ and we have:

$$b_2(G[B(d,D)]) \approx \left(1+\frac{0.5765}{\Delta}\right)\log_3 n.$$

*6.2. General case*

If we don't worry about the very best constructions available, here is an easy construction, showing a broadcast time of $(1+O(k/\Delta))\log_{k+1} n$.

Let $\Delta\equiv x \pmod k$. If $x=0$, we shall use $x=k$ and if $x=1$, we shall use $x=k+1$.

If $x$ is even, we leave $x/2$ inarcs and $x/2$ outarcs on each vertex of $G$, and if $x$ is odd, we leave $(x-1)/2$ inarcs and $(x+1)/2$ outarcs on half of the vertices and the opposite on the other half. In all cases, we have:

$$b_k(G[B(d,D)]) \equiv \left(1+O\left(\frac{k}{\Delta}\right)\right)\log_{k+1} n.$$

We present in Table 4 the best upper bounds obtained by compounding in the de Bruijn graphs. The two first columns specify the case. The third column indicates how many arcs we leave on each vertex of $G$. Let $i$ be the number of inarcs and $o$ the number of outarcs. If $i$ differs from $o$, it means that the values given are for half the vertices and the opposite on the other half. We therefore use in all cases $G=(K_{k+1})^{(\Delta-i-o)/k}$. Finally, the last column shows the value of $b_k(G[B(d,D)])$.

## 7. Conclusion

In this paper, we have shown that the results obtained by Bermond, Hell, Liestman and Peters in [2] can be generalized, but there is still a lot of work to do:

There is no systematic construction of minimum broadcast graphs. It would be

*Table 4. Upper bounds*

| Case | Number of arcs we leave on each vertex of $G$ | $b_k(G[B(d,D)])$ |
|---|---|---|
| $k$ odd: | | |
| $\Delta \equiv x \pmod k$ | $k=3$ 5 in, 5 out | $\left(1+\frac{0.7171}{\Delta}\right)\log_4 n$ |
| $x$ odd | $k \geq 5$ $\frac{x-1+2k}{2}$ in & out | $\left(1+\frac{k(1+x/(2k+x))-k\log_{k+1}(2k+x)/2}{\Delta}\right)\log_{k+1} n$ |
| $\Delta \equiv x \pmod k$ | $\frac{x}{2}$ in, $\frac{x}{2}$ out | $\left(1+\frac{k(1-\log_{k+1}(x/2))}{\Delta}\right)\log_{k+1} n$ |
| $x$ even | $\frac{x}{2}+k$ in, $\frac{x}{2}+k$ out | $\left(1+\frac{k(1+x/(x+2k)-\log_{k+1}((x/2)+k)}{\Delta}\right)\log_{k+1} n$ |
| | $\frac{k+x-1}{2}$ in, $\frac{k+x+1}{2}$ out | $\left(1+\frac{k(1-\log_{k+1}(k+x)/2)}{\Delta}\right)\log_{k+1} n$ |
| $k$ even: | | |
| $\Delta$ even | $\frac{x+k}{2}$ in, $\frac{x+k}{2}$ out | $\left(1+\frac{k(1-\log_{k+1}(k+x)/2)}{\Delta}\right)\log_{k+1} n$ |
| $\Delta \equiv x \pmod k$ | $\frac{x}{2}+k$ in, $\frac{x}{2}+k$ out | $\left(1+\frac{k(1+x/(x+2k)-\log_{k+1}(k+(x/2)))}{\Delta}\right)\log_{k+1} n$ |
| | $\frac{x+3k}{2}$ in, $\frac{x+3k}{2}$ out | $\left(1+\frac{k(1+(k+x)/(3k+x)-\log_{k+1}(3k+x)/2)}{\Delta}\right)\log_{k+1} n$ |
| $\Delta$ odd | $\frac{x-1+k}{2}$ in, $\frac{x+1+k}{2}$ out | $\left(1+\frac{k(1-\log_{k+1}(k+x)/2)}{\Delta}\right)\log_{k+1} n$ |
| $\Delta \equiv x \pmod k$ | $\frac{x-1}{2}+k$ in, $\frac{x+1}{2}+k$ out | $\left(1+\frac{k(1+x/(2k+x)-\log_{k+1}(2k+x)/2))}{\Delta}\right)\log_{k+1} n$ |

interesting to find families of mbgs on $n=f(k)$ vertices like the *star* for $n=k+2$. There is still a reasonably large gap between the best known upper and lower bounds, and this gap cannot be eliminated even by substituting arbitrarily large mbgs (which remain to be discovered) into de Bruijn digraphs. Also, asymptotically, our lower bound decreases like $(1+\mathrm{O}(k/(k+1)^{\Delta/k}))\log_{k+1} n$ whereas our upper bound decreases like $(1+\mathrm{O}(k/\Delta))\log_{k+1} n$. On the other hand, one may want to improve the usefulness of this study by using different hypotheses, especially on the communication laws: for example, if $m$ is the length of a message, we could say that each call uses $f(m,k)$ time instead of simply one unit of time.

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