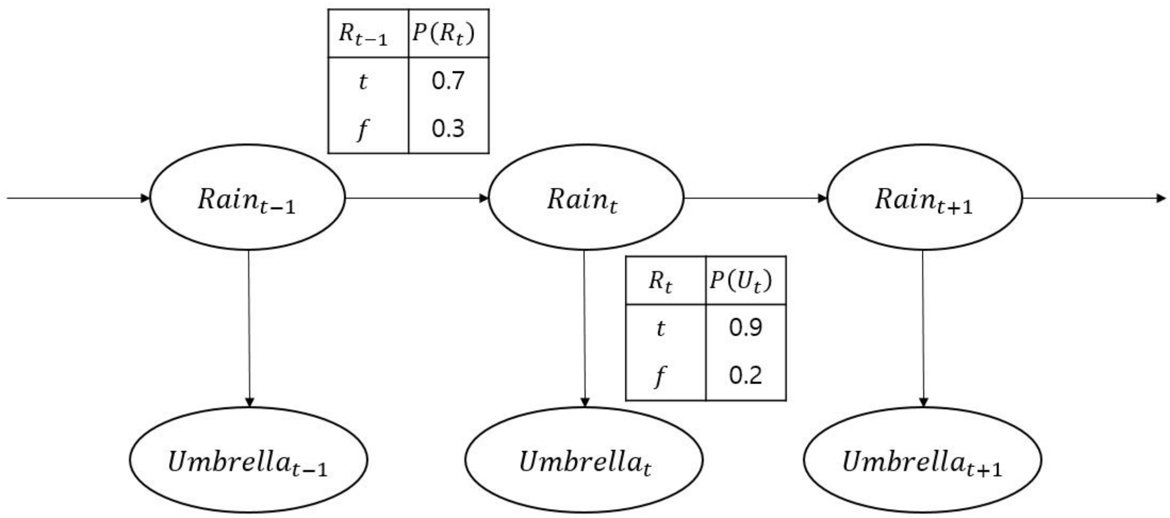
Homework 2 (due: 10/11/2023)

1. In this exercise, we examine what happens to the probabilities in the umbrella world in the limit of long time sequences.



   (a) Suppose we observe an unending sequence of days on which the umbrella appears. Show that, as the days go by, the probability of rain on the current day increases monotonically toward a fixed point. Calculate this fixed point.

   (b) Now consider *Forecasting* further and further into the future, given just the first two umbrella observations. First, compute the probability $P(r_{2+k}|u_1, u_2)$ for $k = 1 \ldots 20$ and plot the results. You should see that the probability converges towards a fixed point. Prove that the exact value of this fixed point is 0.5.

$$L_{1:t} = P(\mathbf{e}_{1:t}) = \sum_{\mathbf{x}_t} l_{1:t}(\mathbf{x}_t) \tag{14.7}$$

$$\mathbf{f}_{1:t+1} = \alpha \mathbf{O}_{t+1} \mathbf{T}^T \mathbf{f}_{1:t} \tag{14.12}$$

2. Equation (14.12) describes the filtering process for the matrix formulation of HMMs. Give a similar equation for the calculation of likelihoods, which was described generically in Equation (14.7).

$$P(x_1 \mid z_1) = \alpha \exp\left(-\frac{1}{2} \frac{\left(x_1 - \frac{(\sigma_0^2 + \sigma_x^2) z_1 + \sigma_z^2 \mu_0}{\sigma_0^2 + \sigma_x^2 + \sigma_z^2} \right)^2}{(\sigma_0^2 + \sigma_x^2)\sigma_z^2 / (\sigma_0^2 + \sigma_x^2 + \sigma_z^2)} \right) \tag{14.19}$$

3. Complete the missing step in the derivation of Equation (14.19), the first update step for the one-dimensional Kalman filter.

$$\mu_{t+1} = \frac{(\sigma_t^2 + \sigma_x^2) z_{t+1} + \sigma_z^2 \mu_t}{\sigma_t^2 + \sigma_x^2 + \sigma_z^2} \quad \text{and} \quad \sigma_{t+1}^2 = \frac{(\sigma_t^2 + \sigma_x^2)\sigma_z^2}{\sigma_t^2 + \sigma_x^2 + \sigma_z^2} \tag{14.20}$$

4. Let us examine the behavior of the variance update in Equation (14.20).
   (a) Plot the value of σ_t^2 as a function of t, given various values for σ_x^2 and σ_z^2.
   (b) Show that the update has a fixed point σ^2 such that $\sigma_t^2 \to \sigma^2$ as $t \to \infty$, and calculate the value of σ^2.
   (c) Give a qualitative explanation for what happens as $\sigma_x^2 \to 0$ and $\sigma_z^2 \to 0$.

5. (MATLAB) You have a friend whose name is Tiffany. You live far from her but talk to her on a daily basis about what she did that day. Tiffany's activities are limited to *four* kinds: Walking in the park, Shopping, Cleaning her house, and Watching movies. Her activities are completely dependent on the weather where Tiffany lives, which has *three* states: Sunny, Cloudy, and Rainy. You have no definite information about the weather where Tiffany lives. You want to guess what the weather must have been like based on what Tiffany told you each day.

   You know the trends of weather changes from the weather center as follows:

Today	The next day		
	Sunny	Cloudy	Rainy
Sunny	0.5	0.4	0.1
Cloudy	0.3	0.3	0.4
Rainy	0.2	0.4	0.4

   You also know the initial state distribution as:

Initial Probability	Sunny	Cloudy	Rainy
	0.4	0.2	0.4

   In addition, you know the relation between the weather and Tiffany's activity as follows:

Weather	Activity			
	Walking	Shopping	Cleaning	Movies
Sunny	0.30	0.20	0.35	0.15
Cloudy	0.50	0.20	0.10	0.20
Rainy	0.15	0.25	0.15	0.45

   The following is a sequence of Tiffany's activities for the last ten days:

   [Walking, Walking, Movies, Shopping, Cleaning, Cleaning, Shopping, Movies, Shopping, Walking].

   (a) Given the observation sequence, implement a filtering algorithm to compute $P(X_k|e_{1:k})$ for each $1 \leq k \leq 10$. Use the skeleton code `hmm_main.m` and implement the forward recursion. Report posterior probabilities for each k.
   (b) Implement a smoothing algorithm and compute $P(X_k|e_{1:10})$ for all $0 \leq k \leq 10$. Use the skeleton code `hmm_main.m` and implement the forward-backward algorithm. Report posterior probabilities for each k.
   (c) Visualize the results of (a) and (b) using a bar graph. (You can use the function `hmm_visualize_bar.m`.)
   (d) Compute the likelihood of the observation sequence.
   (e) Find the most likely sequence of weather based on the observation sequence by implementing the Viterbi algorithm. Report the weather sequence.

6. (MATLAB) The file KF_data_a.mat contains states x and noisy observation z, respectively. The model is of a particle moving in a plane under random forces and damping. The 4-dimension state x of the particle at a given time step is $x = [x^1, \dot{x}^1, x^2, \dot{x}^2]^T$, where x^1 and x^2 represent the particle's coordinates in the plane, and $\dot{x}^1$ and $\dot{x}^2$ are its velocities in the two dimensions. The state in each dimension evolves as $x_{t+1}^i = x_t^i + \dot{x}_t^i$, and

the velocity in each dimension evolves as $\dot{x}^i_{t+1} = 0.98\dot{x}^i_t + w^i_t$, where $w^i_t \sim \mathcal{N}(0, 1)$ for $i = 1, 2$. Lastly, the observations of z are noisy measurements of the particle's position. In detail, we have

$$F = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0.98 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0.98 \end{bmatrix} \quad H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \Sigma_z = \begin{bmatrix} 2500 & 0 \\ 0 & 2500 \end{bmatrix}$$

The system model is following:

$$x_{t+1} = Fx_t + w_t$$
$$z_t = Hx_t + v_t$$
$$v_t \sim N(0, \Sigma_z)$$

The initial state x_0 is distributed as $\mathcal{N}(0, I)$. Now we want to estimate the particle's state from the noisy measurements $\{z_t\}$.

(a) Plot the evolution of the particle's true position (x^1_t, x^2_t) using the ground-truth data x in KF_data_a.mat. All other plots should be plotted on top of this one.

(b) Plot the noisy observations z_t of the particle's positions from KF_data_a.mat.

(c) Implement the Kalman filter, find $\hat{x}_t$ for each t using the model above and the observations z_t. Plot the resulting estimates of the particle's positions ($\hat{x}^1_t, \hat{x}^2_t$) for all t.