Homework 2

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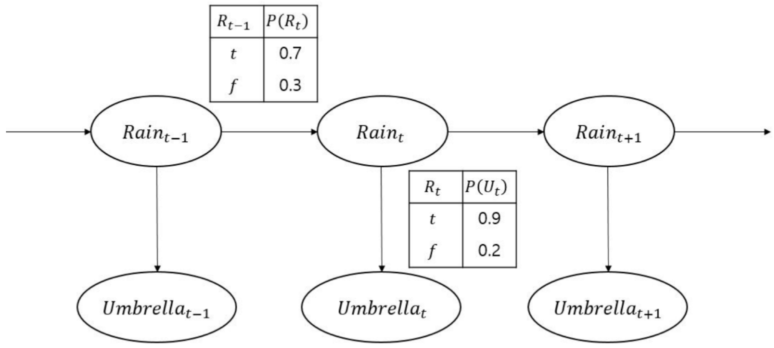
1)



a) Probability of rain on current day → filtering (predict rainy knowing $u_{1:t}$)

$$P(R_t \mid u_{1:t}) = \alpha P(u_t \mid R_t) \sum_{R_{t-1}} P(R_t \mid R_{t-1}) P(R_{t-1} \mid u_{1:t-1})$$

if there's fixed point $\Rightarrow$ $P(R_t \mid u_{1:t-1}) = P(R_t \mid u_t) = \langle m, 1-m \rangle$

$$\langle m, 1-m \rangle = \alpha\left(\langle 0{,}9, 0{,}2 \rangle \left(\langle 0{,}7; 0{,}3 \rangle m + \langle 0{,}3; 0{,}7 \rangle (1-m) \right) \right)$$

$$// \quad = \alpha\left(\langle 0{,}9; 0{,}2 \rangle \left(\langle 0{,}4m; -0{,}4m \rangle + \langle 0{,}3; 0{,}7 \rangle \right)\right)$$

where $\alpha = \dfrac{1}{0{,}9(0{,}3+0{,}4m) + 0{,}2(0{,}7-0{,}4m)}$

$$// \quad = \alpha\left(\langle 0{,}9; 0{,}2 \rangle . \langle 0{,}3+0{,}4m; 0{,}7-0{,}4m \rangle \right)$$

using wolfram:

$m = 0{,}89$

b) Forecasting, Bayes

from $t-1 \to t$ model

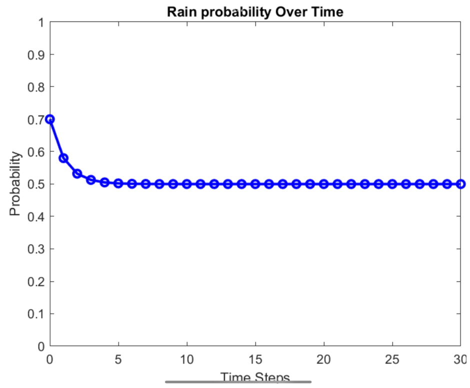
$$p(r_{2+k} \mid u_1, u_2) = \sum_{r_{1+k}} P(R_{2+k} \mid R_{1+k}) P(R_{1+k} \mid u_1, u_2)$$

$$P(r_{2+k} \mid u_1, u_2) = 0{,}7\, P(R_{1+k} \mid u_1, u_2) + 0{,}3\,(1 - P(R_{1+k} \mid u_1, u_2))$$

$P = P(r_{2+k} \mid u_1, u_2) = 0{,}3 + 0{,}4\, P(R_{1+k} \mid u_1, u_2) = 0{,}3 + 0{,}4P$ at convergence.

$0{,}6P = 0{,}3$

$P = 0{,}5 \longrightarrow$



2)

$$P(\mathbf{X}_{t+1}|\mathbf{e}_{1:t+1}) = \alpha P(\mathbf{e}_{t+1}|\mathbf{X}_{t+1}) \sum_{\mathbf{x}_t} P(\mathbf{X}_{t+1}|\mathbf{x}_t) P(\mathbf{x}_t|\mathbf{e}_{1:t})$$

$$\boxed{\mathbf{f}_{1:t+1} = \alpha \mathbf{O}_{t+1}\mathbf{T}^T\mathbf{f}_{1:t}}$$

$L_{1:t} = P(e_{1:t}) = \sum_{x_t} \ell_{1:t}(x_t)$ in matrix form. Since $\ell_{1:t+1} = \text{FORWARD}(\ell_{1:t}, e_{t+1})$ Pg.486

for ℓ as a column vector, $\ell_i = P(X_t = i, e_{1:t})$ for each step

$$\ell_{1:t+1} = \alpha O_{t+1} T^t \ell_{1:t} \quad \text{where} \quad T_{ij} = P(X_t = j \mid X_{t-1} = i)$$

$$O_{ii} = P(e_t \mid X_t = i) \text{ else } O_{ij\neq i} = 0$$

3) From equation 14.18:

$$P(x_1|z_1) = \alpha P(z_1|x_1)P(x_1) = \alpha\, e^{-\frac{1}{2}\left(\frac{(z_1-x_1)^2}{\sigma_z^2}\right)}\, e^{-\frac{1}{2}\left(\frac{(x_1-\mu_0)^2}{\sigma_0^2+\sigma_x^2}\right)}$$

expand

$$= \alpha \exp\left[-\frac{1}{2}\left[\frac{(\sigma_0^2+\sigma_x^2)(z_1^2+x_1^2-2z_1x_1)+\sigma_z^2(x_1^2+\mu_0^2-2x_1\mu_0)}{\sigma_z^2(\sigma_0^2+\sigma_x^2)}\right]\right]$$

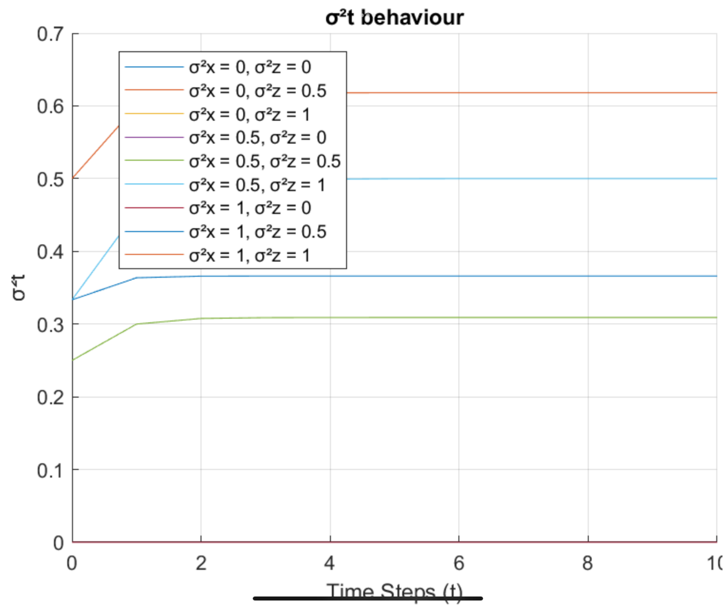
$$= \alpha \exp\left[-\frac{1}{2}\cdot\frac{(\sigma_0^2+\sigma_x^2+\sigma_z^2)x_1^2 - 2\left((\sigma_0^2+\sigma_x^2)z_1+\sigma_z^2\mu_0\right)x_1 + (\sigma_0^2+\sigma_x^2)z_1^2+\sigma_z^2\mu_0^2}{\sigma_z^2(\sigma_0^2+\sigma_x^2)}\right]$$

taking out the constants

$$= \alpha \exp\left[-\frac{1}{2}\cdot\frac{\left(x_1 - \frac{(\sigma_0^2+\sigma_x^2)z_1+\sigma_z^2\mu_0}{\sigma_0^2+\sigma_x^2+\sigma_z^2}\right)^2}{\sigma_z^2(\sigma_0^2+\sigma_x^2)/(\sigma_0^2+\sigma_x^2+\sigma_z^2)}\right]$$

4)

a)



b) show $\sigma_t^2 \to \sigma^2$ as $t \to \infty \wedge \sigma > 0$

then $\sigma_{t-1} : \sigma_t$ from 14.20

$$\sigma^2 = \frac{(\sigma^2 + \sigma_x^2)\sigma_z^2}{\sigma^2 + \sigma_x^2 + \sigma_z^2} \quad , \text{ for } \sigma^2 = y$$

$$y^2 + y\sigma_x^2 + \cancel{y\sigma_z^2} = \cancel{y\sigma_z^2} + \sigma_x^2 \sigma_z^2$$

$$y^2 + \sigma_x^2 y - \sigma_x^2 \sigma_z^2 = 0$$

$$y = \frac{-\sigma_x^2 + \sqrt{\sigma_x^4 + 4\sigma_x^2\sigma_z^2}}{2}$$

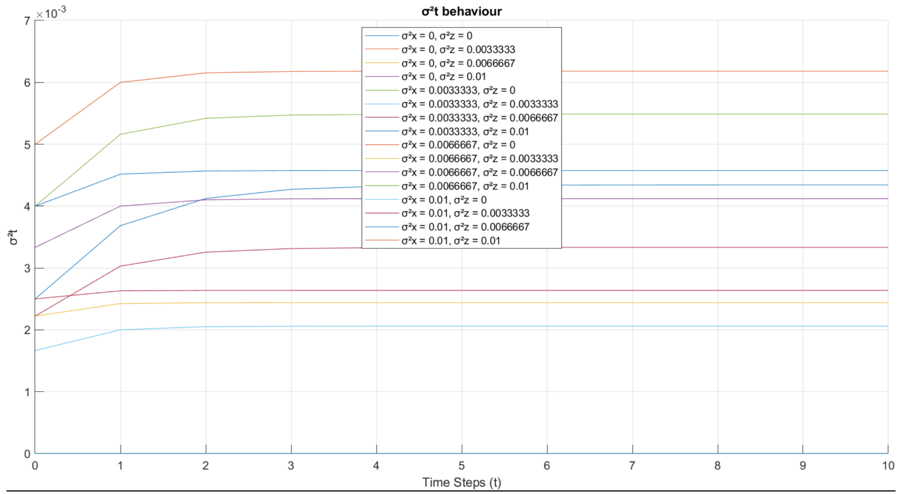
Proving convergence: Since it is monotonically increasing:
stablish upper bound for $\sigma_x^2 \wedge \sigma_z^2$.

$$\sigma_t^2 \leq \frac{(\sigma_t^2 + P)\cancel{P}}{2\cancel{P}} = \frac{\sigma_t^2 + P}{2}$$

then we say σ_t^2 is always increasing and bounded, then there must exist a limit of convergence.

c) $\sigma_x^2 \wedge \sigma_z^2 \to 0$

We see that $\sigma^2 \to 0$ since the next action is deterministic, less variance, that until the variables are completely providing information. In other words, the delta function describe the posterior when there's an exact observation.



5)

a)

Filtering Result (Posterior Probabilities):

0.1200	0.0318	0.0053	0.0012	0.0005	0.0001	0.0000	0.0000	0.0000	0.0000
0.1000	0.0450	0.0057	0.0017	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000
0.0600	0.0132	0.0133	0.0022	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000

b)

```
Emission Probabilities (B matrix):
State 1 (Weather Sunny):
Observation  Activity
  1:         Walking: 0.3000
  2:         Shopping: 0.2000
  3:         Cleaning: 0.3500
  4:         Movies: 0.1500

State 2 (Weather Cloudy):
Observation  Activity
  1:         Walking: 0.5000
  2:         Shopping: 0.2000
  3:         Cleaning: 0.1000
  4:         Movies: 0.2000

State 3 (Weather Rainy):
Observation  Activity
  1:         Walking: 0.1500
  2:         Shopping: 0.2500
  3:         Cleaning: 0.1500
  4:         Movies: 0.4500
```

c)

```
Likelihood of the observation sequence: 1264802.733456
```

d)

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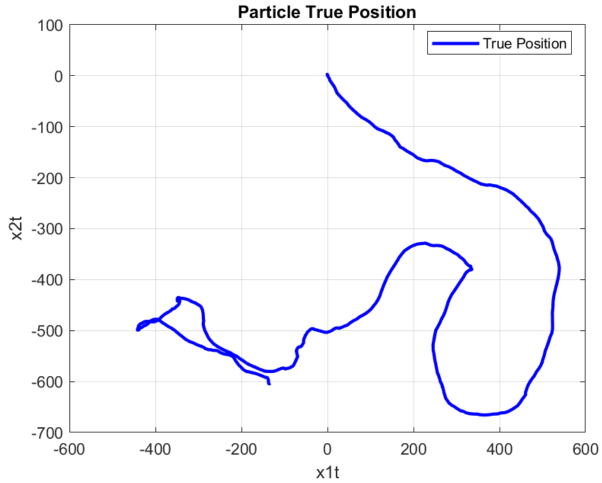
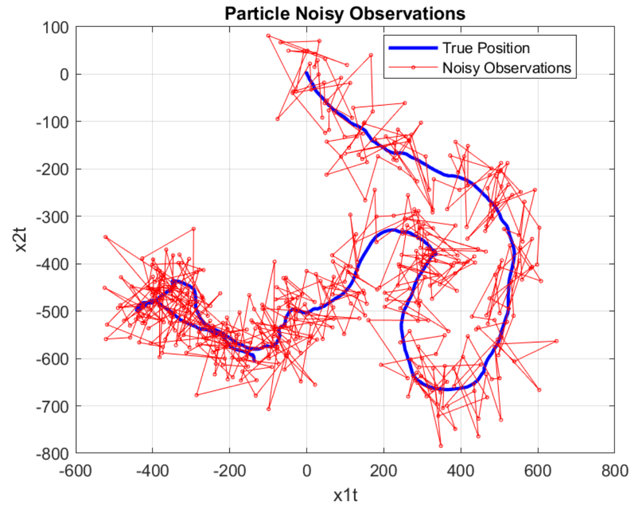
Most likely sequence of weather based on observation seq:
    {'Sunny'}    {'Cloudy'}    {'Rainy'}    {'Cloudy'}    {'Sunny'}    {'Sunny'}    {'Cloudy'}    {'Rainy'}    {'Rainy'}
```

6)

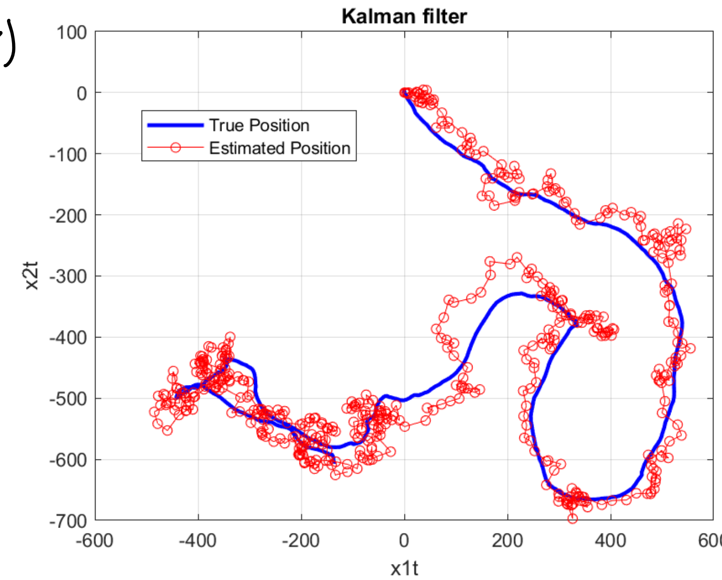
a)


Particle True Position
True Position
x1t
x2t
100
0
-100
-200
-300
-400
-500
-600
-700
-600
-400
-200
0
200
400
600

b)


Particle Noisy Observations
True Position
Noisy Observations
x1t
x2t
100
0
-100
-200
-300
-400
-500
-600
-700
-800
-600
-400
-200
0
200
400
600
800

c)


Kalman filter
True Position
Estimated Position
x1t
x2t
100
0
-100
-200
-300
-400
-500
-600
-700
-600
-400
-200
0
200
400
600