Markov Chains, Review, (Joe Wilck)

Stochastic Process:

Suppose we observe some characteristic of a system at discrete points in time. Let X_t be the value of the system characteristic at time *t*. In most situations, X_t is not known with certainty before time *t* and may be viewed as a random variable. A **discrete-time stochastic process** is simply a description of the relation between the random variables X_0, X_1, X_2, etc. A **continuous –time stochastic process** is simply the stochastic process in which the state of the system can be viewed at any time, not just at discrete instants in time. For example, the number of people in a supermarket *t* minutes after the store opens for business may be viewed as a continuous-time stochastic process.

Markov Chain:

One special type of discrete-time is called a Markov Chain. **Definition:** A discrete-time stochastic process is a **Markov chain** if, for $t = 0,1,2\ldots$ and all states $P(\mathbf{X}_{t+1} = i_{t+1}|\mathbf{X}_t = i_t, \mathbf{X}_{t-1}=i_{t-1},\ldots,\mathbf{X}_1=i_1, \mathbf{X}_0=i_0) = P(\mathbf{X}_{t+1}=i_{t+1}|\mathbf{X}_t = i_t)$. Essentially this says that the probability distribution of the state at time t+1 depends on the state at time $t(i_t)$ and does not depend on the states the chain passed through on the way to i_t at time *t*.

In our study of Markov chains, we make further assumption that for all states *i* and *j* and all *t*, $P(X_{t+1} = j|X_t = i)$ is independent of *t*. This assumption allows us to write $P(X_{t+1} = j|X_t = i) = p_{ij}$ where p_{ij} is the probability that given the system is in state *i* at time *t*, it will be in a state *j* at time *t*+1. If the system moves from state *i* during one period to state *j* during the next period, we that a **transition** from *i* to *j* has occurred.

The p_{ij}'s are often referred to as the **transition probabilities** for the Markov chain. This equation implies that the probability law relating the next period's state to the current state does not change over time. It is often called the **Stationary Assumption** and any Markov chain that satisfies it is called a **Stationary Markov chain**. We also must define q_i to be the probability that the chain is in state *i* at the time 0; in other words, $P(\mathbf{X}_0=i) = q_i$.

We call the vector $\mathbf{q}= [q_1, q_2, \ldots q_s]$ the **initial probability distribution** for the Markov chain. In most applications, the transition probabilities are displayed as an *s* x *s* **transition probability matrix** *P*. The transition probability matrix *P* may be written as:

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1s} \\ p_{21} & p_{11} & \cdots & p_{11} \\ \vdots & \vdots & \ddots & \vdots \\ p_{s1} & p_{s2} & \cdots & p_{ss} \end{bmatrix}$$

For each *i*: $\sum_{j=1}^{j=s} p_{ij} = 1$. We also know that each entry in the *P* matrix must be nonnegative. Hence, all entries in the transition probability matrix are nonnegative, and the entries in each row must sum to 1.

Definitions

Path: able to go from i to j by a sequence of transitions (a positive probability of occurring - doesn't have to be a direct link)

Reachable: a state j is reachable from state i if there is a path leading from i to j

Communicate: can go from i to j and j to i

Closed Set: a set of states S in a MC is a closed state if no state outside of S is reachable from any state in S

Absorbing State: a state i is an absorbing state if $p_{ii} = 1$

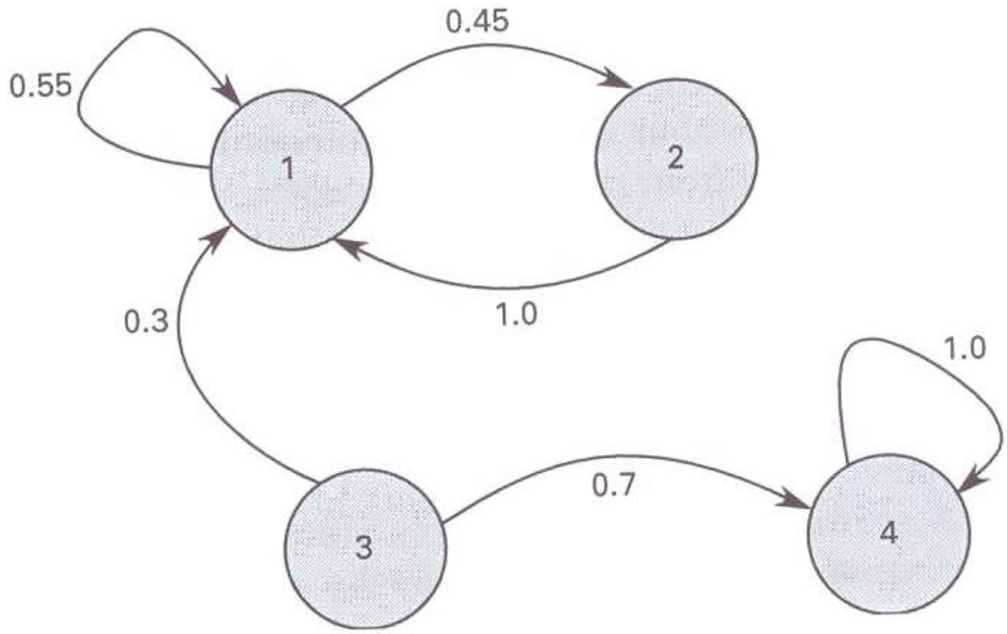
Transient State: a state i is transient if there exists a state j that is reachable from i, but the state i is not reachable from state j

Recurrent State: if a state is not transient, it is called a recurrent state

Periodic: a state i is periodic with period k > 1 if k is the smallest number such that all paths leading from state i back to state i have a length that is a multiple of k.

Ergodic: all states in a chain are recurrent, aperiodic, and communicate with each other.

1: For the following transition diagram of a discrete Markov chain, identify whether the states are transient, absorbing, or recurrent.



A. All states are recurrent
B. State 1 is recurrent, states 2 and 3 are transient, and state 4 is absorbing.
C. State 4 is absorbing, and states 1, 2, and 3 are transient.
D. States 1 and 2 are recurrent, state 3 is transient, and state 4 is absorbing.

Solution:
States 1 and 2 are recurrent (i.e., the state communicates with itself; once it leaves itself it will eventually return to itself). State 3 is transient. State 4 is absorbing.
Thus, the answer is D.

2: A national sports team uses an automated answering service for taking ticket orders. The service is always in one of two modes: answering the phone, or off the phone. Calls come in at the rate of 10 per hour. The service can handle 15 calls per hour. Due to a high demand for tickets, the service runs 24 hours a day. The probability that the service will be answering call three weeks from now is most nearly.

A. 0.30 B. 0.40 C. 0.50 D. 0.60

Solution:
This is a continuous Markov chain. Need to find steady-state probabilities.
State 1: "off the phone" State 2: "on the phone"

$$P = \begin{pmatrix} -10 & 10 \\ 15 & -15 \end{pmatrix}$$

Steady-State:

$$\begin{pmatrix} \pi_1 & \pi_2 \end{pmatrix} \begin{pmatrix} -10 & 10 \\ 15 & -15 \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix}$$

1) $-10\pi_1 + 15\pi_2 = 0$
2) $10\pi_1 - 15\pi_2 = 0$
3) $\pi_1 + \pi_2 = 1$

3 Equations, 2 Unknowns, Must use last equation
Using equation 1: $-10\pi_1 + 15\pi_2 = 0 \rightarrow \pi_1 = 1.5\pi_2$

Using equation 3: $1.5\pi_2 + \pi_2 = 1 \rightarrow \pi_2 = 0.4$
Thus, $\pi_1 = 0.6$

Thus, the answer is B.

3: Consider two stocks. Stock 1 always sells for either \$20 or \$40. If stock 1 is selling for \$20 today, there is a 70% chance that it will sell for \$20 tomorrow. If it is selling for \$40 today, there is an 85% chance it will sell for \$40 tomorrow. Stock 2 always sells for either \$20 or \$35. If stock 2 is selling for \$20 today, there is a 90% chance that it will sell for \$20 tomorrow. If it is selling for \$35 today there is an 80% chance it will sell for \$35 tomorrow. What is the expected value of stock 2?

(A) \$20 (B) \$25 (C) \$28 (D) \$46

Solution:
Take steady-state of Stock 2. Then multiply steady state probabilities times the values to obtain expected value.
Let state 1 represent \$20 and let state 2 represent \$35

$$P = \begin{pmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{pmatrix}$$

Steady-State:
$\pi P = \pi$
1) $0.9\pi_1 + 0.2\pi_2 = \pi_1$
2) $0.1\pi_1 + 0.8\pi_2 = \pi_2$
3) $\pi_1 + \pi_2 = 1$

3 Equations, 2 Unknowns, Must use last equation
Using equation 1: $0.9\pi_1 + 0.2\pi_2 = \pi_1 \rightarrow \pi_1 = 2\pi_2$
Using equation 3: $2\pi_2 + \pi_2 = 1 \rightarrow \pi_2 = 1/3$
Thus, $\pi_1 = 2/3$
Thus, expected value is:
(2/3)(\$20) + (1/3)(\$35) = \$25

Thus, the answer is B.

4: In a planned continuity-assurance system for a 10,000-circuit control facility for life-support devices in an intensive-care ward, each circuit sends a binary signal once per second. For a typical circuit, the probability of a yes signal is 0.92175 if it follows a previous yes signal, or 0.71344 otherwise.

The expected number of yes signals received per second is most nearly:
(A) 6000 (B) 7000 (C) 8000 (D) 9000

Solution:
Need to take the steady-state, and multiply by 10,000 to get expected number.
Let state 1 be yes, and state 2 be otherwise.

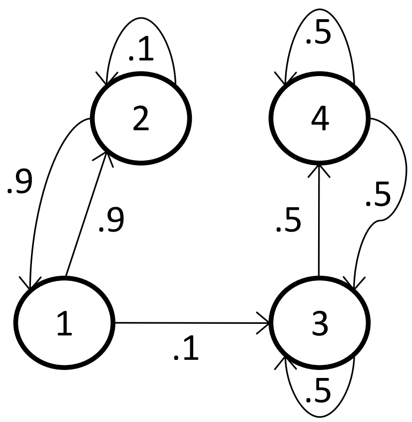
$$P = \begin{pmatrix} 0.92175 & 1-0.92175 \\ 0.71344 & 1-0.71344 \end{pmatrix} = \begin{pmatrix} 0.92175 & 0.07825 \\ 0.71344 & 0.28656 \end{pmatrix}$$

Steady-State:
$\pi P = \pi$
1) $0.92175\pi_1 + 0.71344\pi_2 = \pi_1$
2) $0.07825\pi_1 + 0.28656\pi_2 = \pi_2$
3) $\pi_1 + \pi_2 = 1$

3 Equations, 2 Unknowns, Must use last equation
Using equation 1: $0.92175\pi_1 + 0.71344\pi_2 = \pi_1 \rightarrow \pi_1 = 9.11744408946\pi_2$
Using equation 3: $9.11744408946\pi_2 + \pi_2 = 1 \rightarrow \pi_2 = 0.098839192108$
Thus, $\pi_1 = 0.901160807892$
Thus, the expected value is:
(0.901160807892)(10,000) = 9,011.6

Thus, the answer is D.

5: Consider the following Markov Chain $\{X_n, n = 0, 1, 2, \ldots\}$ with states {1,2,3,4}.



A. Provide P, the one-step transition probability matrix.

$$P = \begin{bmatrix} 0 & .9 & .1 & 0 \\ .9 & .1 & 0 & 0 \\ 0 & 0 & .5 & .5 \\ 0 & 0 & .5 & .5 \end{bmatrix}$$

B. Which state is (or states are) recurrent? **3 and 4**
C. Which state is (or states are) absorbing? **None**
D. Which state is (or states are) transient? **1 and 2**
E. Is the Markov Chain ergodic? **No**

6: Consider the following Markov Chain $\{X_n, n = 0, 1, 2, \ldots\}$ with states {1, 2} and a one-step transition probability matrix, along with mean first passage time formulas.

$$P = \begin{bmatrix} 0.75 & 0.25 \\ 0.20 & 0.80 \end{bmatrix}$$ Where: $\pi_1 = 4/9$, $\pi_2 = 5/9$ $$m_{ii} = \frac{1}{\pi_i}$$ $$m_{ij} = 1 + \sum_{k \neq j} p_{ik} m_{kj}$$

A. Give the mean first passage time from state 2 to state 1 (m_{21}). (Give the formula with substitution for full credit.)

$$m_{ij} = 1 + \sum_{k \neq j} p_{ik} m_{kj} \quad m_{21} = 1 + p_{22} m_{21} \quad m_{21} = 1 + .8 m_{21} \quad m_{21} = 1/.2 = 5$$

B. Is the Markov Chain ergodic? **Yes**

7: Consider the following Markov Chain $\{X_n, n = 0, 1, 2, \ldots\}$ with states {1, 2, 3} and a one-step transition probability matrix, compute the following.

$$P = \begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.2 & 0.5 & 0.3 \end{bmatrix}$$

A. Is the Markov Chain ergodic? **Yes**

B. Probability that $X_5 = 1$ given that $X_4 = 1$ and $X_3 = 2$. (Give the formula with substitution for full credit.)

$$P(X_5 = 1 | X_4 = 1, X_3 = 2) = P(X_5 = 1 | X_4 = 1) = 0.2$$

C. Probability that $X_5 = 1$ and $X_4 = 1$ given that $X_3 = 2$. (Give the formula with substitution for full credit.)

$$P(X_5 = 1, X_4 = 1 | X_3 = 2) = P(X_5 = 1 | X_4 = 1) \times P(X_4 = 1 | X_3 = 2) = 0.2 \times 0.5 = 0.1$$

D. Probability that $X_6 = 3$ given that $X_4 = 1$. (Give the formula with substitution for full credit.)

$$P(X_6 = 3 | X_4 = 1) = P(X_6 = 3 | X_5 = 1) \cdot P(X_5 = 1 | X_4 = 1) + P(X_6 = 3 | X_5 = 2) \cdot P(X_5 = 2 | X_4 = 1) +$$
$$P(X_6 = 3 | X_5 = 3) \cdot P(X_5 = 3 | X_4 = 1) = (.5)(.2) + (.5)(.3) + (.3)(.5) = .4$$

E. The limiting probabilities π_1, π_2, and π_3 exist. Write all of the equations (necessary and unnecessary) needed to determine the limiting probabilities. Do NOT solve for the limiting probabilities.

$$\pi_1 = .2\pi_1 + .5\pi_2 + .2\pi_3 \qquad \pi_3 = .5\pi_1 + .5\pi_2 + .3\pi_3$$
$$\pi_2 = .3\pi_1 + 0\pi_2 + .5\pi_3 \qquad \pi_1 + \pi_2 + \pi_3 = 1$$

8: Consider two stocks. Stock 1 always sells for \$10 or \$20. If stock 1 is selling for \$10 today, there is a 0.80 chance that it will sell for \$10 tomorrow. If it is selling for \$20 today, there is a 0.20 chance that it will sell for \$20 tomorrow. Stock 2 always sells for \$10 or \$25. If stock 2 sells today for \$10, there is a 0.90 chance that it will sell tomorrow for \$10. If it sells today for \$25, then there is a 0.10 chance that it will sell tomorrow for \$25. On the average, which stock will sell for a higher price? *Clearly define Markov Chain(s) and State(s).*

Stock 1:
Let state 1 represent \$10 and let state 2 represent \$20

$$P = \begin{pmatrix} 0.8 & 0.2 \\ 0.8 & 0.2 \end{pmatrix}$$

Steady-State:
$\pi_1 = 0.8$, $\pi_2 = 0.2$
Thus, expected value is:
(.8)(\$10) + (.2)(\$20) = \$12

Stock 2:
Let state 1 represent \$10 and let state 2 represent \$25

$$P = \begin{pmatrix} 0.9 & 0.1 \\ 0.9 & 0.1 \end{pmatrix}$$

Steady-State:
$\pi_1 = 0.9$, $\pi_2 = 0.1$
Thus, expected value is:
(.9)(\$10) + (.1)(\$25) = \$11.50

Thus, Stock 1 will sell for a higher price.

9: Consider the following Markov Chain $\{X_n, n = 0, 1, 2, \ldots\}$ with states $\{1,2,3,4,5\}$.

$$P = \begin{bmatrix} .4 & .6 & 0 & 0 & 0 \\ .5 & .5 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & .5 & .4 & .1 \\ 0 & 0 & 0 & .3 & .7 \end{bmatrix}$$

A. Is the Markov Chain ergodic? **No**

B. Which state is (or states are) recurrent? **1, 2**

C. Which state is (or states are) absorbing? **3**

D. Which state is (or states are) transient? **4, 5**

10: Consider the following Markov Chains:

A. $P = \begin{bmatrix} 0.1 & 0.4 & 0.5 \\ 0.2 & 0 & 0.8 \\ 0.2 & 0.4 & 0.4 \end{bmatrix}$ A. Is the Markov Chain ergodic? **Yes**

B. $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ B. Is the Markov Chain ergodic? **No**

C. $P = \begin{bmatrix} .5 & .5 & 0 & 0 \\ .4 & .6 & 0 & 0 \\ 0 & 0 & .2 & .8 \\ 0 & 0 & .6 & .4 \end{bmatrix}$ C. Is the Markov Chain ergodic? **No**

11: Consider the following Markov Chain $\{X_n, n = 0, 1, 2, \ldots\}$ with states $\{1, 2, 3\}$ and a one-step transition probability matrix, along with mean first passage time formulas.

$P = \begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.3 & 0.2 & 0.5 \\ 0.2 & 0.5 & 0.3 \end{bmatrix}$ Where: $\pi_1 = 31/132$, $\pi_2 = 23/66$, $\pi_3 = 5/12$ $\quad m_{ii} = \dfrac{1}{\pi_i}$ $\quad m_{ij} = 1 + \sum_{k \neq j} p_{ik} m_{kj}$

Give the mean first passage time from state 2 to state 1 (m_{21}). *(Give the formula with substitution for full credit.)*

$$m_{ij} = 1 + \sum_{k \neq j} p_{ik} m_{kj} \rightarrow m_{21} = 1 + p_{22} m_{21} + p_{23} m_{31} = 1 + 0.2 m_{21} + 0.5 m_{31}$$

12: Consider the following Markov Chain $\{X_n, n = 0, 1, 2, \ldots\}$ with states $\{1, 2, 3\}$ and a one-step transition probability matrix, compute the following.

$$P = \begin{bmatrix} 0.9 & 0.1 & 0 \\ 0.2 & 0.3 & 0.5 \\ 0.2 & 0.7 & 0.1 \end{bmatrix}$$

A. Probability that $X_5 = 2$ given that $X_4 = 1$ and $X_3 = 3$. *(Give the formula with substitution for full credit.)*

$$P(X_5 = 2 \mid X_4 = 1, X_3 = 3) = P(X_5 = 2 \mid X_4 = 1) = 0.1$$

B. Probability that $X_5 = 2$ and $X_4 = 1$ given that $X_3 = 3$. *(Give the formula with substitution for full credit.)*

$$P(X_5 = 2, X_4 = 1 \mid X_3 = 3) = P(X_5 = 2 \mid X_4 = 1) \times P(X_4 = 1 \mid X_3 = 3) = 0.1 \times 0.2 = 0.02$$

C. Probability that $X_6 = 2$ given that $X_4 = 1$. *(Give the formula with substitution for full credit.)*

$$P(X_6=2\mid X_4=1)=P(X_6=2\mid X_5=1)\cdot P(X_5=1\mid X_4=1)+P(X_6=2\mid X_5=2)\cdot P(X_5=2\mid X_4=1)+$$
$$P(X_6=2\mid X_5=3)\cdot P(X_5=3\mid X_4=1)=(.1)(.9)+(.3)(.1)+(.7)(0)=.12$$

OR, first row times the second column added together = $(.9)(.1)+(.1)(.3)+(0)(0.7)=.12$

D. The limiting probabilities π_1, π_2, and π_3 exist. Write all of the equations (necessary and unnecessary) needed to determine the limiting probabilities. *Do NOT solve for the limiting probabilities.*

$$\pi_1=.9\pi_1+.2\pi_2+.2\pi_3 \qquad \pi_3=0\pi_1+.5\pi_2+.1\pi_3$$
$$\pi_2=.1\pi_1+.3\pi_2+.7\pi_3 \qquad \pi_1+\pi_2+\pi_3=1$$

13: Payoff Insurance Company charges a customer according to his or her accident history. A customer who has had no accident during the last two years is charged a \$100 annual premium. Any customer who has had an accident during each of the last two years is charged a \$400 annual premium. A customer who has had an accident during only one of the last two years is charged a \$300 annual premium. A customer who has had an accident during the last year has a 10% chance of having an accident during the current year. If a customer has not had an accident during the last year, then there is only a 3% chance he or she will have an accident during the current year. During a given year, what is the average premium paid by a Payoff customer? (Hint: Consider a Markov Chain with four states. You will need to calculate the limiting probabilities.)

States:
1. Accident during both previous two years.
2. Accident last year, but no accident two years ago.
3. No accident last year, but an accident 2 years ago.
4. No accidents during both previous two years.

$$P=\begin{bmatrix} 0.1 & 0 & 0.9 & 0 \\ 0.1 & 0 & 0.9 & 0 \\ 0 & 0.03 & 0 & 0.97 \\ 0 & 0.03 & 0 & 0.97 \end{bmatrix}$$

$$\pi_1=\frac{1}{310}\approx 0.0032$$
$$\pi_2=\frac{9}{310}\approx 0.0290$$
$$\pi_3=\frac{9}{310}\approx 0.0290$$
$$\pi_4=\frac{291}{310}\approx 0.9387$$

Expected Premium:
$\$100(291/310)+\$300[(9/310)+(9/310)]+\$400(1/310)=\$112.5806$