Review of Tutorials & Concepts in ECON7321

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T1.1

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  - *Independence:* $P(A \cap B) = P(A) \times P(B)$
  - *Mutually Exclusive*: $P(A \cup B) = P(A) + P(B)$
- **False**: If $P(A)$ or $P(B)$ is zero,

$$\begin{aligned} P(A \cup B) =& P(A) + P(B) + P(A \cap B) \\ =& P(A) + P(B) + P(A) \times P(B) \\ =& P(A) + P(B). \end{aligned}$$

T2.1

Let X be a random variable with

$$f_X(x) = \begin{cases} \frac{1}{2}e^{-x/2} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Find the distribution function.

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Let X be a random variable with

$$f_X(x) = \begin{cases} \frac{1}{2}e^{-x/2} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Find the distribution function.

$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt$$

$$= \begin{cases} \int_0^x \frac{1}{2}e^{-\frac{t}{2}}dt & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Now, $\int_0^x \frac{1}{2}e^{-\frac{t}{2}} = - e^{-\frac{t}{2}}\Big|_0^x = 1 - e^{-\frac{x}{2}}$ So,

$$F_X(x) = \begin{cases} 1 - e^{-\frac{x}{2}} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

T3.1

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- **TRUE**
  - $V(X) = E(X^2) - [E(X)]^2$
  - $V(X) \leq E(X^2)$

T3.2

Let X be a random variable with

$$f_X(x) = \begin{cases} 2x^{-3} & x \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the its expectation, mode and variance.

T3.2

Find $\mathbb{E}[X], \mathrm{Var}[X]$ and the mode of X

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x) dx = 2\int_1^{\infty} x^{-2} dx = -\, 2x^{-1}\Big|_1^{\infty} = 2$$

$$\mathbb{E}\left[X^2\right] = \int_{-\infty}^{\infty} x^2 f_X(x) dx = 2\int_1^{\infty} \frac{dx}{x} = 2\ln x\Big|_1^{\infty} = \infty$$

So, $\mathrm{Var}[X]$ is undefined.

$$\frac{df_X(x)}{dx} = \begin{cases} -6x^{-4} & x > 1 \\ 0 & \text{otherwise} \end{cases}$$

Since f_X is monotone decreasing in x, and x is bounded strictly, the mode does not exist

T4.1

Let X be a random variable with

$$f_X(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$f_{Y|X}(y|x) = \begin{cases} 1/x & 0 \leq y \leq x \\ 0 & \text{otherwise} \end{cases}$$

Find $E(Y)$.

T4.1

$$
\begin{aligned}
\mathbb{E}(Y) &= \mathbb{E}(\mathbb{E}(Y \mid X)) \quad \text{law of iterated expectations} \\
&= \mathbb{E}\left(\int_{-\infty}^{\infty} y f_{Y|X}(y \mid x) dy\right) \\
&= \mathbb{E}\left(\int_{0}^{x} \frac{y}{x} dy\right) \\
&= \frac{1}{2}\mathbb{E}(X) \\
&= \frac{1}{2}\int_{-\infty}^{\infty} x f_X(x) dx \\
&= \frac{1}{2}\int_{0}^{1} x dx \\
&= \frac{1}{4}
\end{aligned}
$$

T5.1

Let X be a random variable with moment generating function $m_X(t)$ and let $W = aX + b$ with known constants a and b. Find $m_W(t)$.

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Given $M_X(t)$ and $W = aX + b$, find $M_W(t)$

$$\begin{aligned} M_W(t) &= \mathbb{E}\left(e^{tW}\right) \\ &= \mathbb{E}\left(e^{t(aX+b)}\right) \\ &= \mathbb{E}\left(e^{bt}e^{(at)X}\right) \\ &= e^{bt}M_X(at) \end{aligned}$$

T6.1

Let X_1, X_2 be i.i.d. random variables each following a standard normal distribution. (i) Find the distribution of $Y = (X_1 + X_2)/2$ (ii) Find the distribution of $Z = X_1^2 + X_2^2$.

T6.1.i

Find the distribution of $Y = \frac{X_1+X_2}{2}$

$$\begin{aligned}
M_Y(t) &= \mathbb{E}\left(e^{tY}\right) \\
&= \mathbb{E}\left(e^{\frac{t}{2}(X_1+X_2)}\right) \\
&= \mathbb{E}\left(e^{\frac{t}{2}X_1}\right)\mathbb{E}\left(e^{\frac{t}{2}X_2}\right) \quad X_1, X_2 \text{ independent} \\
&= M_{X_1}\left(\frac{t}{2}\right)M_{X_2}\left(\frac{t}{2}\right) \\
&= \exp\left(\frac{1}{2}\left(\frac{t}{2}\right)^2\right)\exp\left(\frac{1}{2}\left(\frac{t}{2}\right)^2\right) \\
&= \exp\left(\frac{1}{2}\left(\frac{1}{2}\right)t^2\right)
\end{aligned}$$

which is the moment generating function of a $N\left(0, \frac{1}{2}\right)$ random variable. So, $Y \sim N\left(0, \frac{1}{2}\right)$.

T6.1.ii

Find the distribution of $Z = X_1^2 + X_2^2$

$$
\begin{aligned}
M_{X_1^2}(t) &= \mathbb{E}\left(e^{X_1^2 t}\right) \\
&= \int_{-\infty}^{\infty} e^{tx_1^2} f_{X_1}(x_1)\, dx_1 \\
&= \int_{-\infty}^{\infty} e^{tx_1^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x_1^2}\, dx_1 \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(x_1\sqrt{\frac{1}{2}-t}\right)^2}\, dx_1
\end{aligned}
$$

T6.1.ii

Find the distribution of $Z = X_1^2 + X_2^2$

$$M_{X_1^2}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(x_1\sqrt{\frac{1}{2}-t}\right)^2} dx_1$$

Let $z = x_1\sqrt{\frac{1}{2} - t}$. Then $dz = dx_1\sqrt{\frac{1}{2} - t}$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(x_1\sqrt{\frac{1}{2}-t}\right)^2} dx_1 = \frac{1}{\sqrt{2\pi}\sqrt{\frac{1}{2} - t}} \int_{-\infty}^{\infty} e^{-z^2} dz$$

$$= \left(\frac{\frac{1}{2}}{\frac{1}{2} - t}\right)^{\frac{1}{2}}$$

which is the moment generating function of a $\Gamma\left(\frac{1}{2}, \frac{1}{2}\right)$ random variable. So, $X_1^2, X_2^2 \sim \chi_1^2$. And, since $\chi_n^2 + \chi_m^2 \sim \chi_{n+m}^2$, $Z \sim \chi_2^2$

T7.1

Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with expectation μ and variance σ^2 and consider the estimators

$$\overline{X}_2 = \frac{X_1 + X_2}{2}, \overline{X}_3 = \frac{X_1 + X_2 + X_3}{3}, \overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

of μ. (a) Show that all three estimators are unbiased (b) Show that $\overline{X}_n$ is the most efficient estimator (c) Find the variance of $\overline{X}_n$.

T7.1.a

(a) Show the estimators are all unbiased.

$$\mathbb{E}\left(\bar{X}_2\right) - \mu = \mathbb{E}\left(\frac{X_1 + X_2}{2}\right) - \mu = \frac{\mu + \mu}{2} - \mu = 0$$

$$\mathbb{E}\left(\bar{X}_3\right) - \mu = \mathbb{E}\left(\frac{X_1 + X_2 + X_3}{3}\right) - \mu = \frac{\mu + \mu + \mu}{3} - \mu = 0$$

$$\mathbb{E}\left(\bar{X}_n\right) - \mu = \mathbb{E}\left(\frac{1}{n}\sum^{n} X_i\right) - \mu = \frac{n\mu}{n} - \mu = 0$$

T7.1.b-c

(b) Show that $\bar{X}_n$ is the most efficient estimator of μ

$$\mathrm{Var}\left(\bar{X}_2\right) = \mathrm{Var}\left(\frac{X_1 + X_2}{2}\right) = \frac{\sigma^2 + \sigma^2}{4} = \frac{\sigma^2}{2}$$

$$\mathrm{Var}\left(\bar{X}_3\right) = \mathrm{Var}\left(\frac{X_1 + X_2 + X_3}{3}\right) = \frac{\sigma^2 + \sigma^2 + \sigma^2}{9} = \frac{\sigma^2}{3}$$

$$\mathrm{Var}\left(\bar{X}_n\right) = \mathrm{Var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$$

So, if $n > 3, \bar{X}_n$ is the most efficient estimator for μ as $\mathrm{Var}\left(\bar{X}_n\right) < \mathrm{Var}\left(\bar{X}_3\right) < \mathrm{Var}\left(\bar{X}_2\right)$

(c) The variance of $\bar{X}_n$ is $\frac{\sigma^2}{n}$.

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- **True/False**: An unbiased estimator is always consistent
- Unbiasedness: $E[\hat{\theta}] = \theta$
- Consistency: $\lim_{n\to\infty} \Pr(|\hat{\theta} - \theta| > \epsilon) = 0$
- **FALSE**
  - *Counter Example*: Average of the first two observations in a sample. **Unbiased but not consistent.**

T8.3

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- FALSE

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- FALSE
- Only true if the estimator is *unbiased*.

T9.1

- **True/False**: Let $X_1, ..., X_4$ be a random sample from a *Bernoulli*(p) distribution and consider the estimators $\hat{p}_1 = \overline{X}$ and $\hat{p}_2 = \overline{X}/2 + 1/4$. The MSE of $\hat{p}_1$ is always smaller than the MSE of $\hat{p}_2$.

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- $\text{MSE}(\hat{\theta}) = \mathbb{E}\left[(\hat{\theta} - \theta)^2\right]$

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- $\text{MSE}(\hat{\theta}) = \mathbb{E}\left[(\hat{\theta} - \theta)^2\right]$
- **FALSE:** The mean squared error (MSE) of the estimator $\hat{p}_1 = \overline{X}$ is not always smaller than the MSE of $\hat{p}_2 = \overline{X}/2 + 1/4$ for a random sample from a Bernoulli distribution.

T9.2

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- **FALSE**

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- Type II: $P(\textit{Fail to Reject } H_0 | H_1)$

T10.4

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- Type I: $P(\text{Reject } H_0 | H_0)$
- Type II: $P(\text{Fail to Reject } H_0 | H_1)$
- **FALSE:** Trade-off between I and II.