ECON7350 - Tutorial 4 Solutions

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Error Correction Decomposition

- Lecture 4: 24/36
- Let $\alpha(L)$ be a polynomial in L of degree $r : \alpha(L) = \alpha_0 + \alpha_1 + ...\alpha_r L^r$.
- For every $\alpha(L)$, there exists a polynomial $\widetilde{\alpha}(L)$ of degree $r - 1$, such that:

$$\widetilde{\alpha}(L)(1 - L) = \alpha(L) - \alpha(1)L$$

- This property of polynomials is used to transform ARDL(p, l, s) into another useful representation called the error correction model (ECM) form.

Error Correction Decomposition

Let

$$a(L) = \widetilde{a}(L)(1 - L) + a(1)L,$$
$$b(L) = \widetilde{b}(L)(1 - L) + b(1)L,$$
$$c(L) = \widetilde{c}(L)(1 - L) + c(1)L,$$

and substitute these into

$$a(L)y_t = a_0 + b(L)x_t + c(L)w_t + \varepsilon_{y,t}$$

Define the difference operator $\Delta = 1 - L$ such that $\Delta y_t = y_t - y_{t-1}$ and

$$a(L)y_t = \widetilde{a}(L)\Delta y_t + a(1)y_{t-1}$$
$$b(L)x_t = \widetilde{b}(L)\Delta x_t + b(1)x_{t-1}$$
$$c(L)w_t = \widetilde{c}(L)\Delta w_t + c(1)w_{t-1}$$

ARDL Model

$$a(L)y_t = a_0 + b(L)x_t + c(L)w_t + \varepsilon_{y,t}$$

QUESTION 1: Derive the ECM representation of the following ARDL(1, 1, 2) model,

$$y_t = a_0 + a_1 y_{t-1} + b_0 x_t + b_1 x_{t-1} + c_0 w_t + c_1 w_{t-1} + c_2 w_{t-2} + \epsilon_{y,t}$$

Decomposing ARDL(1,1,2)

$$y_t = a_0 + a_1 y_{t-1} + b_0 x_t + b_1 x_{t-1} + c_0 w_t + c_1 w_{t-1} + c_2 w_{t-2} + \epsilon_{y,t} \tag{1}$$

We calculate the differences of the red terms w.r.t. previous period.

$$\begin{aligned} &= y_t = a_0 + \gamma - \gamma + y_{t-1} - y_{t-1} + a_1 y_{t-1} + b_0 x_t \\ &+ b_0 x_{t-1} - b_0 x_{t-1} + b_1 x_{t-1} + c_0 w_t + c_0 w_{t-1} - c_0 w_{t-1} \\ &+ c_1 w_{t-1} + c_2 w_{t-2} + c_2 w_{t-1} - c_2 w_{t-1} + \epsilon_{y,t} \end{aligned} \tag{2}$$

$$\begin{aligned} y_t - y_{t-1} &= a_0 + \gamma - \gamma - y_{t-1} + a_1 y_{t-1} + b_0 \Delta x_t + b_0 x_{t-1} \\ &+ b_1 x_{t-1} + c_0 \Delta w_t + c_0 w_{t-1} + c_1 w_{t-1} - c_2 \Delta w_{t-1} \\ &+ c_2 w_{t-1} + \epsilon_{y,t} \end{aligned} \tag{3}$$

$$\begin{aligned} \Delta y_t &= \gamma - y_{t-1} + a_1 y_{t-1} + (a_0 - \gamma) + b_0 \Delta x_t + b_0 x_{t-1} \\ &+ b_1 x_{t-1} + c_0 \Delta w_t + c_0 w_{t-1} + c_1 w_{t-1} - c_2 \Delta w_{t-1} \\ &+ c_2 w_{t-1} + \epsilon_{y,t} \end{aligned} \tag{4}$$

Decomposing ARDL(1,1,2)

$$
\begin{aligned}
\Delta y_t = \gamma & - y_{t-1} + a_1 y_{t-1} + (a_0 - \gamma) + b_0 \Delta x_t + b_0 x_{t-1} \\
& + b_1 x_{t-1} + c_0 \Delta w_t + c_0 w_{t-1} + c_1 w_{t-1} - c_2 \Delta w_{t-1} \\
& + c_2 w_{t-1} + \epsilon_{y,t} \qquad (5) \\
\Delta y_t = \gamma & - ((a_1 - 1) y_{t-1} + (a_0 - \gamma) + (b_0 + b_1) x_{t-1} + (c_0 + c_1 + c_2) w_{t-1}) \\
& + (a_0 - \gamma) + b_0 \Delta x_t + c_0 \Delta w_t - c_2 \Delta w_{t-1} + \epsilon_{y,t} \qquad (6)
\end{aligned}
$$

$$
\begin{aligned}
\Delta y_t = & \gamma + \alpha \left(y_{t-1} - \mu - \beta_x x_{t-1} - \beta_w w_{t-1} \right) + \\
& b_0 \Delta x_t + c_0 \Delta w_t - c_2 \Delta w_{t-1} + \epsilon_t \qquad (7)
\end{aligned}
$$

where,

$$
\alpha = -(1 - a_1), \quad \beta_x = \frac{b_0 + b_1}{1 - a_1}, \quad \beta_w = \frac{c_0 + c_1 + c_2}{1 - a_1}, \quad \mu = \frac{a_0 - \gamma}{1 - a_1}
$$

Question 2

2. Create a function in R to compute coefficients $\theta_0, \ldots, \theta_h$ in

$$\theta(L) = b(L)/a(L) = \theta_0 + \theta_1 L + \cdots + \theta_h L^h + \cdots$$

where $a(L) = a_0 + a_1 L + \cdots + a_p L^p$ and $(L) = b_0 + b_1 L + \cdots + b_q L^q$.

Need analytical solution for θ_k for all k.
(See Lecture 4 Slides 24-28/36)

2. Create a function in R to compute coefficients $\theta_0, \ldots, \theta_h$ in

$$\theta(L) = b(L)/a(L) = \theta_0 + \theta_1 L + \cdots + \theta_h L^h + \cdots$$

where $a(L) = a_0 + a_1 L + \cdots + a_p L^p$ and $(L) = b_0 + b_1 L + \cdots + b_q L^q$.

The analytical solution for θ_k for all k.

$$\theta_j = \begin{cases} b_0/a_0 & \text{if } j = 0 \\ b_j - \dfrac{\sum_{k=2}^{j} a_k \theta_{j-k+1}}{a_1} & \text{otherwise} \end{cases} \tag{8}$$

Question 3

3. Create a function in R to compute IRFs (to both one-off and permanent shocks) up to horizon h as well as the LRMs for the ARDL(p, l, s) :

$$a(L)y_t = a_0 + b(L)x_t + c(L)w_t + \epsilon_{y,t}.$$

Question 3

3. Create a function in R to compute IRFs (to both one-off and permanent shocks) up to horizon h as well as the LRMs for the ARDL(p, l, s) :

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- IRF: Lecture 4 Slides 15/36
  - One-off: θ_h
  - Permanent: $\theta_0 + ... + \theta_h$
- LRM: Lecture 4 Slides 16/36 ($h \to \infty$)
  - One-off: $\lim_{h\to\infty} \theta_h$
  - Permanent: $b(1)/a(1)$
- We need θ_h to calculate IRFs and LRMs. So, we will use Q2 to estimate θ_h. Therefore, we need the coefficients a and b.