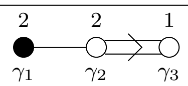
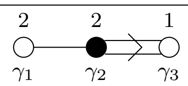
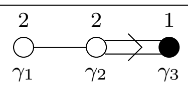
Special Vogan diagrams

In this project we want to present a combinatorial property of Vogan diagrams which may be studied both directly and by deep learning techniques. Let $\mathfrak{g}$ be a real semisimple Lie algebra of rank ℓ with Vogan diagram having indices of painted nodes P and set of positive roots Δ^+. Let $\{\gamma_1, \ldots, \gamma_\ell\}$ be the set of simple roots and $\{\varphi_1, \ldots, \varphi_\ell\}$ be the set of fundamental dominant weights. We assign to each positive root α a coefficient ε_α which is -1 if α is compact and 1 otherwise. Define $\eta = \sum_{\alpha\in\Delta^+} \varepsilon_\alpha \alpha$ and consider the vector

$$\varphi_P := \eta - 2 \sum_{\alpha\in\mathrm{span}\{\gamma_i | i\in P^c\}\cap\Delta^+} \alpha,$$

which may be expressed in the basis of fundamental dominant weights as $\sum_{i\in P} a_i\varphi_i$, with $a_i \in \mathbb{Z}$. We say that a Vogan diagram is *special* if the a_i's are all negative, zero or positive. In particular, the diagram is said to be *symplectic general type*, *symplectic Calabi-Yau* and *symplectic Fano* respectively. Thus, the problem is the following: given a Vogan diagram, determining whether it is special or not. This issue turns out to be purely combinatorial in terms of root data. In particular, for connected Vogan diagrams, it may be studied directly through to the algorithm in `specialVoganDiagrams.sage`. The speciality of Vogan diagrams is relevant in the almost-Kähler geometry of adjoint orbits of semisimple Lie groups, as it allows to determine when an orbit admits a canonical metric. For more details, see [1].

Example 1. Consider the Vogan diagrams of classical type B. The special Vogan diagrams are listed in the following table.

Vogan diagram	φ_P	Type	$\mathfrak{g}$
	φ_1	sGT	$\mathfrak{so}(2,5)$
	φ_2	sGT	$\mathfrak{so}(4,3)$
	φ_3	sF	$\mathfrak{so}(6,1)$

The acronyms sGT, sCY, sF stands for symplectic general type, symplectic Calabi-Yau and symplectic Fano respectively, while in the last column we report the real form of $\mathfrak{sl}(7,\mathbb{C})$ corresponding to the Vogan diagram on the same line.

On the other hand, the problem may be studied also with deep learning techniques, especially to make predictions on higher ranks connected Vogan diagrams. We use the algorithm in `specialOrbits.sage` to prepare data, in particular we run the algoritm for each Vogan diagram up to rank 11 and we pair each diagram with a label (0 if it is special, 1 otherwise). Then we build a simple fully connected deep neural network and we train it with the prepared data. After many temptatives, one may see that the neural network gives quite precise prediction for non-special Vogan diagrams up to rank 11, while it is less accurate for special Vogan diagrams. Moreover, it seems that it is not able to extrapolate predictions for higher rank Vogan diagrams. We tried also with a convolutional neural network, but the results are quite the same as for the fully connected neural network.

References

[1] A. Della Vedova and A. Gatti, *Almost Kaehler geometry of adjoint orbits of semisimple Lie groups*, arXiv:1811.06958v2 [math.DG].