THE SATAKE SEXTIC IN ELLIPTIC FIBRATIONS ON K3

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Abstract. We describe explicit formulas relevant to the F-theory/heterotic string duality that reconstruct from a specific Jacobian elliptic fibration on the Shioda-Inose surface covering a generic Kummer surface the corresponding genus-two curve using the level-two Satake coordinate functions. We derive explicitly the rational map on the moduli space of genus-two curves realizing the algebraic correspondence between a sextic curve and its corresponding Satake sextic. We will prove that it is not the original sextic defining the genus-two curve, but its corresponding Satake sextic which is manifest in the F-theory model, dual to the $\mathfrak{so}(32)$ heterotic string with an unbroken $\mathfrak{so}(28)\oplus\mathfrak{su}(2)$ gauge algebra.

1. Introduction

Constructing equations of algebraic curves from a given point in the moduli space or a given Jacobian has always been interesting to both mathematicians and physicists. The only case where such constructions can be made explicit is the case of genus two curves. There have been attempts by other authors before where equations of the genus two curve is written in terms of the thetanulls of the jacobian; see [26] and [28].

By a sextic curve we mean a projective curve of degree six. To each sextic curve one can associate another sextic curve, called the *Satake sextic*. The algebraic correspondence between these two sextics is quite complicated, and we give explicit formulas for its construction. In fact, starting with a plane curve, for example in Rosenhain normal form, the computation of the Igusa invariants provides an effective method for computing the corresponding Satake sextic. Conversely, starting with the roots of the Satake sextic we will derive explicit formulas for the reconstruction of the original sextic up to equivalence.

For a generic genus-two curve $\mathcal{C}$ the Jacobian variety $\operatorname{Jac}(\mathcal{C})$ is principally polarized abelian surface, and the minimal resolution of the quotient by the involution automorphism is a special K3 surface called the *Kummer surface* $\operatorname{Kum}(\operatorname{Jac}\mathcal{C})$. There is a closely related K3 surface, called the *Shioda-Inose surface* $\operatorname{SI}(\operatorname{Jac}\mathcal{C})$ which carries a *Nikulin involution*, i.e., an automorphism of order two preserving the holomorphic two-form, such that the quotient by this involution and blowing up the fixed points one recovers the Kummer surface. By using the Shioda-Inose surface $\operatorname{SI}(\operatorname{Jac}\mathcal{C})$ that covers the Kummer surface, one establishes a one-to-one correspondence between two different types of surfaces with the same Hodge-theoretic data, principally polarized abelian surfaces and algebraic K3 surfaces polarized by a special lattice, which is known as a *geometric two-isogeny*.

2010 *Mathematics Subject Classification.* 11F03, 14J28, 14J81.

In string theory, compactifications of the so-called type-IIB string in which the complex coupling varies over a base generically referred to as F-theory. The simplest such construction corresponds to a Jacobian elliptic fibration on a K3 surface. By taking this K3 surface to be the aforementioned Shioda-Inose surface $\mathrm{SI}(\mathrm{Jac}\,\mathcal{C})$ a phenomenon called *F-theory/heterotic string duality* is manifested as the aforementioned geometric two-isogeny. An important question is whether the original genus-two curve $\mathcal{C}$ is still manifest in this F-theoretic description of non-geometric heterotic string backgrounds. We will prove that it is not the original sextic defining the genus-two curve $\mathcal{C}$, but the corresponding Satake sextic which is manifest in the F-theoretic data. In fact, the ramification locus of the Satake sextic is the genus-two component of the fixed point set of the Nikulin involution.

This article is structured as follows: in Section 2 we give a brief review of principally polarized abelian surfaces, the thetanulls for genus two, and the Satake coordinate functions, as well as their relations to the Igusa invariants and Siegel modular forms of genus two. We then prove a Picard like result, which gives the Rosenhein roots of a genus-two curve in terms of the thetanulls and also in terms of the Satake coordinate functions. These explicit formulas are instrumental in our that the algebraic correspondence between the sextic and its corresponding Satake sextic defines a rational map on the moduli space of genus-two curves of degree 16. In Section 3 we describe the construction of the Kummer surface $\mathrm{Kum}(\mathrm{Jac}\,\mathcal{C})$ and Shioda-Inose surface $\mathrm{SI}(\mathrm{Jac}\,\mathcal{C})$, as well as the Jacobian elliptic fibrations on them which are relevant for the F-theory/heterotic string duality. We then prove that the positions of 7-branes with string charge $(1,0)$ in the F-theory model dual to the $\mathfrak{so}(32)$ heterotic string with an unbroken $\mathfrak{so}(28) \oplus \mathfrak{su}(2)$ gauge algebra and only one non-vanishing Wilson line form the ramification locus of the Satake sextic which is in algebraic correspondence with the genus-two curve $\mathcal{C}$.

2. The correspondence between a sextic and its Satake sextic

In section 2 we give a brief review of principally polarized abelian surfaces, the thetanulls for genus two, and the Satake coordinate functions, as well as their relations to the Igusa invariants and Siegel modular forms of genus two. We then prove a Picard like result, which gives the Rosenhein roots of genus two curve in terms of the thetanulls and also in terms of the Satake coordinate functions. We give the explicit formulas that prove that the algebraic correspondence between the sextic and its corresponding Satake sextic is of degree 16.

2.1. **Abelian surfaces.**

The *Siegel upper-half space* is the set of two-by-two symmetric matrices over $\mathbb{C}$ whose imaginary part is positive definite, i.e.,

$$\mathbb{H}_2 = \left\{ \tau = \left(\begin{array}{cc} \tau_1 & z \\ z & \tau_2 \end{array} \right) \middle| \, \tau_1, \tau_2, z \in \mathbb{C}\,,\ \mathrm{Im}(\tau_1)\,\mathrm{Im}(\tau_2) > \mathrm{Im}(z)^2\,,\ \mathrm{Im}(\tau_2) > 0 \right\} .$$

The Siegel three-fold is a quasi-projective variety of dimension three obtained from the Siegel upper half plane when quotienting out by the action of the modular transformations $\Gamma_2 := \mathrm{Sp}_4(\mathbb{Z})$, i.e.,

$$\mathcal{A}_2 = \mathbb{H}_2 / \Gamma_2 \ . \tag{1}$$

For each $\tau \in \mathbb{H}_2$ the columns of the matrix $[\mathbb{I}_2 | \tau]$ form a lattice Λ in $\mathbb{C}^2$ and determine a principally polarized complex abelian surface $\mathbf{A}_\tau = \mathbb{C}^2/\Lambda$. Two abelian surfaces $\mathbf{A}_\tau$ and $\mathbf{A}_{\tau'}$ are isomorphic if and only if there is a symplectic matrix

$M \in \Gamma_2$ such that $\tau' = M(\tau)$. It follows that the Siegel three-fold $\mathcal{A}_2$ is also the set of isomorphism classes of principally polarized abelian surfaces. The even Siegel modular forms of $\mathcal{A}_2$ are a polynomial ring in four free generators of degrees 4, 6, 10 and 12 that will be denoted by $\psi_4, \psi_6, \chi_{10}$ and χ_{12}, respectively. Igusa showed in [15] that for the full ring of modular forms, one needs an additional generator χ_{35} which is algebraically dependent on the others. We also define $\Gamma_2(2n) = \{M \in \Gamma_2 | \, M \equiv \mathbb{I} \mod 2n\}$ with corresponding Siegel modular threefold $\mathcal{A}_2(2)$ such that $\Gamma_2/\Gamma_2(2) \cong S_6$ where S_6 is the permutation group of order 720.

If $\mathcal{C}$ is an irreducible nonsingular projective curve with self-intersection $\mathcal{C} \cdot \mathcal{C} = 2$ then $\mathcal{C}$ is a smooth curve of genus two. We choose a symplectic homology basis for $\mathcal{C}$, say $\{A_1, A_2, B_1, B_2\}$, such that the intersection products $A_i \cdot A_j = B_i \cdot B_j = 0$ and $A_i \cdot B_j = \delta_{ij}$, where δ_{ij} is the Kronecker delta. We choose a basis $\{w_i\}$ for the space of holomorphic one-forms such that $\int_{A_i} w_j = \delta_{ij}$. The matrix

$$\tau = \left[\int_{B_i} w_j \right]$$

is the *period matrix* of $\mathcal{C}$ and $\text{Jac}\,(\mathcal{C}) = \mathbf{A}_\tau$ is the Jacobian of $\mathcal{C}$. Moreover, the map $\jmath_{\mathcal{C}} : \mathcal{C} \to \text{Jac}(\mathcal{C})$ is an embedding of the variety of moduli of curves of genus two $\mathcal{M}_2$ into the space of principally polarized abelian surfaces, i.e.,

$$\mathcal{M}_2 \hookrightarrow \mathcal{A}_2 \; ,$$

where the hermitian form associated to the divisor class $[\mathcal{C}]$ is a principal polarization ρ on $\text{Jac}(\mathcal{C})$. Moreover, a curve $\mathcal{C}$ of genus-two is called generic if the Néron-Severi lattice is generated by $[\mathcal{C}]$, i.e., $\text{NS}(\text{Jac}\,\mathcal{C}) = \mathbb{Z}[\mathcal{C}]$. Since we have $\rho^2 = 2$, the transcendental lattice is $T(\text{Jac}\,\mathcal{C}) = H \oplus H \oplus \langle -2 \rangle$ in this case.[1] Conversely, one can always regain $\mathcal{C}$ from the pair $(\text{Jac}\,\mathcal{C}, \rho)$ where ρ is a principal polarization.

The Humbert surface H_Δ with invariant Δ is the space of principally polarized abelian surfaces admitting a symmetric endomorphism with discriminant Δ. It turns out that Δ is always a positive integer $\equiv 0, 1 \mod 4$. In fact, H_Δ is the image associated to the equation

$$a\,\tau_1 + b\,z + c\,\tau_3 + d\,(z^2 - \tau_1\,\tau_2) + e = 0 \; , \tag{2}$$

with integers a, b, c, d, e satisfying $\Delta = b^2 - 4\,a\,c - 4\,d\,e$ and $\tau = \left(\begin{smallmatrix} \tau_1 & z \\ z & \tau_2 \end{smallmatrix}\right) \in \mathbb{H}_2$. Therefore, inside of $\mathcal{A}_2$ sit the Humbert surfaces H_1 and H_4 that are defined as the images of the rational divisors associated to $z = 0$ and $\tau_1 = \tau_2$, respectively. In fact, H_1 and H_4 form the two connected components of the singular locus of $\mathcal{A}_2$, and the formal sum $H_1 + H_4$ of Humbert surfaces is the vanishing divisor of χ_{35}.

Furthermore, Torelli's theorem states that the map sending a curve $\mathcal{C}$ to its Jacobian variety $\text{Jac}(C)$ induces a birational map from the moduli space $\mathcal{M}_2$ of genus-two curves to the complement of the Humbert surface H_1 in $\mathcal{A}_2$. This locus is expressed in terms of modular forms as $\mathcal{A}_2 \backslash \operatorname{supp}(\chi_{10})_0$. That is, a period point τ is equivalent to a point with $z = 0$, i.e., $\tau \in H_1$, if and only if $\chi_{10}(\tau) = 0$, if and only if the principally polarized abelian surface $\mathbf{A}_\tau$ is a product of two elliptic curves $\mathbf{A}_\tau = \mathcal{E}_{\tau_1} \times \mathcal{E}_{\tau_2}$. In turn, the transcendental lattice is to $T(\mathbf{A}_\tau) = H \oplus H$.

On the other hand, it is known that the vanishing divisor of $Q = 2^{12}\,3^9\,\chi_{35}^2/\chi_{10}$ is the Humbert surface H_4 [11], that is, a period point τ is equivalent to a point with $\tau_1 = \tau_2$, i.e., $\tau \in H_4$, if and only if $Q(\tau) = 0$. In turn, the transcendental lattice degenerates to $T(\mathbf{A}_\tau) = H \oplus \langle 2 \rangle \oplus \langle -2 \rangle$. Bolza [4] described the possible

[1] H is the standard hyperbolic lattice with the quadratic form $q = x_1 x_2$.

automorphism groups of genus-two curves defined by sextics. In particular, he proved that a sextic curve $Y^2 = F(X)$ defining the genus-two curve $\mathcal{C}$ with $\mathbf{A}_\tau = \mathrm{Jac}(C)$ has an extra involution, which can be represented as $(X, Y) \mapsto (-X, Y)$, if and only if $Q = 0$.

2.2. **Thetanulls for genus two.** For any $z \in \mathbb{C}^2$ and $\tau \in \mathbb{H}_w$ *Riemann's theta function* is defined as

$$\theta(z, \tau) = \sum_{u \in \mathbb{Z}^w} e^{\pi i (u^t \tau u + 2 u^t z)}$$

where u and z are two-dimensional column vectors and the products involved in the formula are matrix products. The fact that the imaginary part of τ is positive makes the series absolutely convergent over any compact sets. Therefore, the function is analytic. The theta function is holomorphic on $\mathbb{C}^2 \times \mathbb{H}_2$ and satisfies

$$\theta(z + u, \tau) = \theta(z, \tau), \quad \theta(z + u\tau, \tau) = e^{-\pi i (u^t \tau u + 2 z^t u)} \cdot \theta(z, \tau),$$

where $u \in \mathbb{Z}^2$; see [23] for details. Any point $e \in \mathrm{Jac}\,(\mathcal{C})$ can be written uniquely as $e = (b, a) \begin{pmatrix} \mathbb{I}_2 \\ \tau \end{pmatrix}$, where $a, b \in \mathbb{R}^2$. We shall use the notation $[e] = \begin{bmatrix} a \\ b \end{bmatrix}$ for the characteristic of e. For any $a, b \in \mathbb{Q}^2$, the theta function with rational characteristics is defined as

$$\theta \begin{bmatrix} a \\ b \end{bmatrix} (z, \tau) = \sum_{u \in \mathbb{Z}^2} e^{\pi i ((u+a)^t \tau (u+a) + 2(u+a)^t (z+b))}.$$

When the entries of column vectors a and b are from the set $\{0, \frac{1}{2}\}$, then the characteristics $\begin{bmatrix} a \\ b \end{bmatrix}$ are called the *half-integer characteristics*. The corresponding theta functions with rational characteristics are called *theta characteristics*. A scalar obtained by evaluating a theta characteristic at $z = 0$ is called a *theta constant*. Points of order n on $\mathrm{Jac}\,(\mathcal{C})$ are called the $\frac{1}{n}$-*periods*. Any half-integer characteristic is given by

$$\mathfrak{m} = \frac{1}{2} m = \frac{1}{2} \begin{pmatrix} m_1 & m_2 \\ m_1' & m_2' \end{pmatrix}$$

where $m_i, m_i' \in \mathbb{Z}$. For $\gamma = \begin{bmatrix} \gamma' \\ \gamma'' \end{bmatrix} \in \frac{1}{2}\mathbb{Z}^4 / \mathbb{Z}^4$ we define $e_*(\gamma) = (-1)^{4 (\gamma')^t \gamma''}$. Then,

$$\theta[\gamma](-z, \tau) = e_*(\gamma) \theta[\gamma](z, \tau).$$

We say that γ is an **even** (resp. **odd**) characteristic if $e_*(\gamma) = 1$ (resp. $e_*(\gamma) = -1$).

For any genus-two curve we have six odd theta characteristics and ten even theta characteristics; see [26] for details. The following are the sixteen theta characteristics, where the first ten are even and the last six are odd. We denote the even theta constants by

$$\theta_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \theta_2 = \begin{bmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \theta_3 = \begin{bmatrix} 0 & 0 \\ \frac{1}{2} & 0 \end{bmatrix}, \theta_4 = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \ \theta_5 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix},$$

$$\theta_6 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \theta_7 = \begin{bmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}, \theta_8 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix}, \theta_9 = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}, \theta_{10} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix},$$

where we write

$$\theta_i(z) \quad \text{instead of} \quad \theta \begin{bmatrix} a^{(i)} \\ b^{(i)} \end{bmatrix} (z, \tau) \quad \text{where } i = 1, \dots, 10, \tag{3}$$

and $\theta_i = \theta_i(0)$. Similarly, the odd theta functions correspond to the following characteristics

$$\begin{bmatrix} 0 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix}, \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix}.$$

Thetanulls are modular forms of $\mathcal{A}_2(2)$, and the even theta fourth powers define a compactification of $\mathcal{A}_2(2)$ as $\text{Proj}[\theta_1^4 : \cdots : \theta_{10}^4]$, called the *Satake compactfication*. $\theta_1, \dots, \theta_4$ are called *fundamental thetanulls*; see [26] for details. They are determined via the Göpel systems. We have the following Frobenius identities relating the remaining theta constants to the fundamental thetanulls

$$\begin{array}{rclrcl} \theta_5^2\theta_6^2 & = & \theta_1^2\theta_4^2 - \theta_2^2\theta_3^2\,, & \theta_5^4 + \theta_6^4 & = & \theta_1^4 - \theta_2^4 - \theta_3^4 + \theta_4^4\,, \\ \theta_7^2\theta_9^2 & = & \theta_1^2\theta_3^2 - \theta_2^2\theta_4^2\,, & \theta_7^4 + \theta_9^4 & = & \theta_1^4 - \theta_2^4 + \theta_3^4 - \theta_4^4\,, \\ \theta_8^2\theta_{10}^2 & = & \theta_1^2\theta_2^2 - \theta_3^2\theta_4^2\,, & \theta_8^4 + \theta_{10}^4 & = & \theta_1^4 + \theta_2^4 - \theta_3^4 - \theta_4^4\,, \end{array} \tag{4}$$

as well as the mixed relations

$$\theta_5^2\theta_9^2 \;=\; \theta_3^2\theta_8^2 - \theta_4^2\theta_{10}^2\,, \qquad \theta_5^2\theta_7^2 \;=\; \theta_1^2\theta_8^2 - \theta_2^2\theta_{10}^2. \tag{5}$$

Let a genus-two curve $\mathcal{C}$ be given by

$$Y^2 = F(X) = X(X-1)(X-\lambda_1)(X-\lambda_2)(X-\lambda_3). \tag{6}$$

The ordered tuple $(\lambda_1, \lambda_2, \lambda_3)$ where the λ_i are all distinct and different from $0, 1, \infty$ determines a point in $\mathcal{M}_2(2)$, the moduli space of genus-two curves together with a level-two structure, and, in turn, a level-two structure on the corresponding Jacobian variety, i.e., a point in $\mathcal{A}_2(2)$. As functions on $\mathcal{M}_2(2)$, the Rosenhain invariants generate its coordinate ring $\mathbb{C}(\lambda_1, \lambda_2, \lambda_3)$ and hence generate the function field of $\mathcal{A}_2(2)$, that is the three-dimensional moduli space of principally polarized abelian surfaces with level-two structure.

The three λ-parameters in the Rosenhain normal (6) can be expressed as ratios of even theta constants by Picard's lemma. There are 720 choices for such expressions since the forgetful map $\mathcal{M}_2(2) \to \mathcal{M}_2$ is a Galois covering of degree $720 = |S_6|$ where S_6 acts on the roots of F by permutations. Any of the 720 choices may be used, we picked the one from [27]:

Lemma 1. *If $\mathcal{C}$ is a genus-two curve with period matrix τ and $\chi_{10}(\tau) \neq 0$, then $\mathcal{C}$ is equivalent to the curve (6) with Rosenhain parameters $\lambda_1, \lambda_2, \lambda_3$ given by*

$$\lambda_1 = \frac{\theta_1^2\theta_3^2}{\theta_2^2\theta_4^2}\,, \quad \lambda_2 = \frac{\theta_3^2\theta_8^2}{\theta_4^2\theta_{10}^2}\,, \quad \lambda_3 = \frac{\theta_1^2\theta_8^2}{\theta_2^2\theta_{10}^2}\,. \tag{7}$$

Conversely, given three distinct complex numbers $(\lambda_1, \lambda_2, \lambda_3)$ different from $0, 1, \infty$ there is complex abelian surface $\mathbf{A}_\tau$ with period matrix $[\mathbb{I}_2|\tau]$ such that $\mathbf{A}_\tau = \text{Jac}(\mathcal{C})$ where $\mathcal{C}$ is the genus-two curve with period matrix τ.

2.3. **Igusa functions and Siegel modular forms.** Let $I_2, \dots, I_{10}$ denote Igusa invariants of the binary sextic $Y^2 = F(X)$ as defined in [29, Eq. 9] and explicitly given by Equations (44) for the curve (6) in Rosenhain normal form. The Igusa functions or absolute invariants are defined as

$$(j_1, j_2, j_3) = \left(\frac{I_2^5}{I_{10}}, \frac{I_4 I_2^3}{I_{10}}, \frac{I_6 I_2^2}{I_{10}} \right)$$

Two genus-two curves $\mathcal{C}$ and $\mathcal{C}'$ are isomorphic if and only if

$$(j_1, j_2, j_3) = (j_1', j_2', j_3') \ .$$

Moreover, j_1, j_2, j_3 are given as rational functions of fourth powers of the fundamental theta functions $\theta_1, \dots, \theta_4$.

The even Siegel modular forms of $\mathcal{A}_2$ are a polynomial ring in four free generators of degrees 4, 6, 10 and 12 denoted by $\psi_4, \psi_6, \chi_{10}$ and χ_{12}, respectively. Igusa [15, p. 848] proved that the relation between the Igusa invariants and the even Siegel modular forms are as follows:

$$\begin{aligned} I_2(F) &= -2^3 \cdot 3 \, \frac{\chi_{12}(\tau)}{\chi_{10}(\tau)} \ , \\ I_4(F) &= \quad 2^2 \, \psi_4(\tau) \ , \\ I_6(F) &= -\frac{2^3}{3} \, \psi_6(\tau) - 2^5 \, \frac{\psi_4(\tau) \, \chi_{12}(\tau)}{\chi_{10}(\tau)} \ , \\ I_{10}(F) &= -2^{14} \, \chi_{10}(\tau) \ . \end{aligned} \tag{8}$$

Notice that the Igusa invariant I_{10} is the discriminant of the sextic $Y^2 = F(X)$, i.e., $\Delta_F = I_{10}(F)$. Conversely, for $r \neq 0$ the point $[I_2 : I_4 : I_6 : I_{10}]$ in weighted projective space equals

$$\Big[2^3 \, 3 \, (3r\chi_{12}) \ : \ 2^2 3^2 \, \psi_4 \, (r\chi_{10})^2 \ : \ 2^3 \, 3^2 \, \Big(4\psi_4 \, (3r\chi_{12}) + \psi_6 \, (r\chi_{10}) \Big) \, (r\chi_{10})^2 \ : 2^2 \, (r\chi_{10})^6 \Big].$$

Furthermore, Igusa showed in [15] that for the full ring of modular forms, one needs an additional generator χ_{35} which is algebraically dependent on the others. In fact, its square can be written as follows:

$$\begin{aligned} \chi_{35}^2 = \frac{1}{2^{12} \, 3^9} \, \chi_{10} \Big(& 2^{24} \, 3^{15} \, \chi_{12}^5 - 2^{13} \, 3^9 \, \psi_4^3 \, \chi_{12}^4 - 2^{13} \, 3^9 \, \psi_6^2 \, \chi_{12}^4 + 3^3 \, \psi_4^6 \, \chi_{12}^3 \\ & - 2 \cdot 3^3 \, \psi_4^3 \, \psi_6^2 \, \chi_{12}^3 - 2^{14} \, 3^8 \, \psi_4^2 \, \psi_6 \, \chi_{10} \, \chi_{12}^3 - 2^{23} \, 3^{12} \, 5^2 \, \psi_4 \, \chi_{10}^2 \, \chi_{12}^3 + 3^3 \, \psi_6^4 \, \chi_{12}^3 \\ & + 2^{11} \, 3^6 \, 37 \, \psi_4^4 \, \chi_{10}^2 \, \chi_{12}^2 + 2^{11} \, 3^6 \, 5 \cdot 7 \, \psi_4 \, \psi_6^2 \, \chi_{10}^2 \, \chi_{12}^2 - 2^{23} \, 3^9 \, 5^3 \, \psi_6 \, \chi_{10}^3 \, \chi_{12}^2 \\ & - 3^2 \, \psi_4^7 \, \chi_{10}^2 \, \chi_{12} + 2 \cdot 3^2 \, \psi_4^4 \, \psi_6^2 \, \chi_{10}^2 \, \chi_{12} + 2^{11} \, 3^5 \, 5 \cdot 19 \, \psi_4^3 \, \psi_6 \, \chi_{10}^3 \, \chi_{12} \\ & + 2^{20} \, 3^8 \, 5^3 \, 11 \, \psi_4^2 \, \chi_{10}^4 \, \chi_{12} - 3^2 \, \psi_4 \, \psi_6^4 \, \chi_{10}^2 \, \chi_{12} + 2^{11} \, 3^5 \, 5^2 \, \psi_6^3 \, \chi_{10}^3 \, \chi_{12} - 2 \, \psi_4^6 \, \psi_6 \, \chi_{10}^3 \\ & - 2^{12} \, 3^4 \, \psi_4^5 \, \chi_{10}^4 + 2^2 \, \psi_4^3 \, \psi_6^3 \, \chi_{10}^3 + 2^{12} \, 3^4 \, 5^2 \, \psi_4^2 \, \psi_6^2 \, \chi_{10}^4 + 2^{21} \, 3^7 \, 5^4 \, \psi_4 \, \psi_6 \, \chi_{10}^5 \\ & - 2 \, \psi_6^5 \, \chi_{10}^3 + 2^{32} \, 3^9 \, 5^5 \, \chi_{10}^6 \Big) \ . \end{aligned} \tag{9}$$

Hence, $Q = 2^{12} \, 3^9 \, \chi_{35}^2 / \chi_{10}$ is a polynomial of degree 60 in the even generators.

2.4. **The Satake coordinate functions.** For a symplectic matrix $T \in \mathrm{Sp}_4(\mathbb{Z})$, there is an induced action on the characteristics of the theta constants $\mathfrak{m} \mapsto T \cdot \mathfrak{m}$ such that the characteristic $T \cdot \mathfrak{m}$ has the same parity as $\mathfrak{m}$ and $T \cdot \mathfrak{m} = \mathfrak{m}$ if $T \equiv \mathbb{I}(2)$. The latter property implies that $\Gamma_2/\Gamma_2(2) \cong \mathrm{Sp}_4(\mathbb{F}_2)$ acts on the characteristics. It turns out that this action is transitive on the six odd characteristics and gives an isomorphism between the permutation group S_6 and $\mathrm{Sp}_4(\mathbb{F}_2)$ [12].

On any function $f : \mathbb{H}_2 \to \mathbb{C}$, a right action of $T = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Sp}_4(\mathbb{R})$ is given by setting $f \circ [T](\tau) := \det\,(c\tau + d)^{-2} f(T \cdot \tau)$. It then follows that $\theta^4_{\mathfrak{m}} \circ [T^{-1}] = \pm\theta^4_{T\cdot\mathfrak{m}}$ for all $T \in \mathrm{Sp}_4(\mathbb{Z})$ with $\Gamma_2(2)$ acting trivially. Thus, S_6 acts on the vector space $\mathrm{Mat}_2\,(\Gamma(2))$ spanned by the ten even theta fourth powers. The vector space $\mathrm{Mat}_2\,(\Gamma(2))$ is five-dimensional vector space. We will use the set of theta functions

$$\left\{\theta_1^4, \theta_2^4, \theta_3^4, \theta_4^4, \theta_5^4\right\}$$

as a basis for the space $\mathrm{Mat}_2\,(\Gamma_2(2))$. In fact, the other even theta fourth powers are represented in terms of this basis as

$$\begin{aligned} \theta_6^4 &= \theta_1^4 - \theta_2^4 - \theta_3^4 + \theta_4^4 - \theta_5^4, \\ \theta_7^4 &= \theta_3^4 - \theta_4^4 + \theta_5^4, \\ \theta_8^4 &= \theta_2^4 - \theta_4^4 + \theta_5^4, \\ \theta_9^4 &= \theta_1^4 - \theta_2^4 - \theta_5^4, \\ \theta_{10}^4 &= \theta_1^4 - \theta_3^4 - \theta_5^4. \end{aligned} \tag{10}$$

If we set $u_k = \sum_{\mathfrak{m}} \theta_{\mathfrak{m}}^{4k}$ it can be checked using the Frobenius identities (4) that $u_2^2 = 4u_4$; see [12]. Therefore, Equations (10) realize the Satake compactification of $\mathcal{A}_2(2)$ as the quartic threefold $u_2^2 = 4u_4$ in $\mathrm{Proj}[\theta_1^4 : \cdots : \theta_5^4]$.

The following functions are linear combinations of the fourth-powers of even generators and are called *level-two Satake coordinate functions*

$$\begin{aligned} x_1 &= -\theta_1^4 + 2\,\theta_2^4 + 2\,\theta_3^4 - \theta_4^4 + 3\,\theta_5^4, \\ x_2 &= -\theta_1^4 + 2\,\theta_2^4 - \theta_3^4 - \theta_4^4, \\ x_3 &= -\theta_1^4 - \theta_2^4 - \theta_3^4 + 2\,\theta_4^4, \\ x_4 &= 2\,\theta_1^4 - \theta_2^4 - \theta_3^4 - \theta_4^4, \\ x_5 &= -\theta_1^4 - \theta_2^4 + 2\,\theta_3^4 - \theta_4^4, \\ x_6 &= 2\,\theta_1^4 - \theta_2^4 - \theta_3^4 + 2\,\theta_4^4 - 3\,\theta_5^4. \end{aligned} \tag{11}$$

It is obvious that $\sum_i x_i = 0$. A direct computation also shows that the permutation group S_6 acts on $(x_1, \dots, x_6)$ by permutation [12].

We have the following lemma:

Lemma 2. *For $\tau \in \mathcal{A}_2$ with $\chi_{10}(\tau) \neq 0$, the level-two Satake coordinate functions $x_1, \dots x_6$ determine a curve in Rosenhain normal from (6) with $\Delta_F = -2^{14}\,\chi_{10}(\tau)$ and Igusa invariants given by (8) where the Rosenhain roots $(\lambda_1, \lambda_2, \lambda_3)$ are given by relations*

$$\lambda_1 = \frac{1}{2} + \frac{\theta_1^4\theta_3^4 - \theta_7^4\theta_9^4}{2\,\theta_2^4\theta_4^4}, \quad \lambda_2 = \frac{1}{2} + \frac{\theta_3^4\theta_8^4 - \theta_5^4\theta_9^4}{2\,\theta_4^4\theta_{10}^4}, \quad \lambda_3 = \frac{1}{2} + \frac{\theta_1^4\theta_8^4 - \theta_5^4\theta_7^4}{2\,\theta_2^4\theta_{10}^4}, \tag{12}$$

and

$$\begin{array}{rclrcl} \theta_1^4 &=& -\frac{1}{3}\left(x_2 + x_3 + x_5\right), & \theta_2^4 &=& -\frac{1}{3}\left(x_3 + x_4 + x_5\right), \\ \theta_3^4 &=& -\frac{1}{3}\left(x_2 + x_3 + x_4\right), & \theta_4^4 &=& -\frac{1}{3}\left(x_2 + x_4 + x_5\right), \\ \theta_5^4 &=& \frac{1}{3}\left(x_1 + x_3 + x_4\right), & \theta_6^4 &=& -\frac{1}{3}\left(x_1 + x_2 + x_5\right), \\ \theta_7^4 &=& \frac{1}{3}\left(x_1 + x_4 + x_5\right), & \theta_8^4 &=& \frac{1}{3}\left(x_1 + x_2 + x_4\right), \\ \theta_9^4 &=& -\frac{1}{3}\left(x_1 + x_2 + x_3\right), & \theta_{10}^4 &=& -\frac{1}{3}\left(x_1 + x_3 + x_5\right). \end{array} \tag{13}$$

Proof. The proof follows when using the Frobenius identities (4) to re-write the Rosenhain roots in terms of fourth powers of theta functions and solving Equations (11) for $\theta_1^4, \dots, \theta_{10}^4$. □

Define the j-th power sums s_j are defined by

$$s_j = \sum_{i=1}^{6} x_i^j \ .$$

Apart from the obvious identity $s_1 = 0$, the relation $u_2^2 = 4\,u_4$ implies the relation $s_2^2 = 4\,s_4$. Therefore, Equations (11) define an embedding of the Satake compactification of $\mathcal{A}_2(2)$, into $\mathbb{P}^5 \ni [x_1 : x_2 : x_3 : x_4 : x_5 : x_6]$. The image in $\mathbb{P}^5$, known as the *Igusa quartic*, is the intersection of the hyperplane $s_1 = 0$ and the quartic hypersurface $s_2^2 = 4\,s_4$.

We have the following lemma describing the descent to the Igusa invariants:

Lemma 3. *We have the following relations between s_2, s_3, s_5, s_6 and the Igusa invariants*

$$\begin{aligned} s_2 &= 3\,I_4, \\ s_3 &= \frac{3}{2} I_2 I_4 - \frac{9}{2} I_6, \\ s_5 &= \frac{15}{8} I_2 I_4^2 - \frac{45}{8} I_4 I_6 + 1215 I_{10}, \\ s_6 &= \frac{27}{16} I_4^3 + \frac{3}{8} I_2^2 I_4^2 - \frac{9}{4} I_2 I_4 I_6 + \frac{27}{8} I_6^2 + \frac{729}{4} I_2 I_{10} \ , \end{aligned} \tag{14}$$

and conversely

$$\begin{aligned} I_2 &= \frac{5}{3} \frac{3\,s_2^3 + 8 s_3^2 - 48 s_6}{5 s_2 s_3 - 12 s_5}, \\ I_4 &= \frac{1}{3} s_2, \\ I_6 &= \frac{1}{27} \frac{15\,s_2^4 + 10 s_2 s_3^2 - 240 s_2 s_6 + 72 s_3 s_5}{5\,s_2 s_3 - 12 s_5}, \\ I_{10} &= -\frac{1}{2916} s_2 s_3 + \frac{1}{1215} s_5. \end{aligned} \tag{15}$$

Proof. Using the definition of the Igusa invariants we prove the lemma by explicit computation. □

2.5. **The Satake sextic.** We combine the level-two Satake functions in another plane sextic curve, called *Satake sextic*, given by

$$f(x) = \prod_{i=1}^{6} (x - x_i)$$

The coefficients of the Satake sextic are polynomials in $\mathbb{Z}\left[\frac{1}{2}, \frac{1}{3}, s_2, s_3, s_4, s_6\right]$. In fact, we obtain

$$f(x) = x^6 + \sum_{i=1}^{6} \frac{(-1)^i}{i!} \mathcal{B}_i(Z)\, x^{6-i}$$

where $Z = \{s_1, -s_2, 2!\, s_3, -3!\, s_4, 4!\, s_5, -5!\, s_6\}$ and $\mathcal{B}_i(Z)$ is the complete Bell polynomial of order i in the variables contained in Z. The following proposition follows:

Proposition 4. *Level-two Satake coordinate functions $x_1, \dots, x_6$ are the roots of the Satake polynomial $f \in \mathbb{Z}\left[\frac{1}{2}, \frac{1}{3}, I_2, I_4, I_6, I_{10}\right][x]$ given by*

$$\begin{aligned} f(x) = & x^6 - \frac{1}{2} s_2 x^4 - \frac{1}{3} s_3 x^3 + \frac{1}{16} s_2^2 x^2 \\ & + \left(\frac{1}{6} s_2 s_3 - \frac{1}{5} s_5\right) x + \left(\frac{1}{96} s_2^3 + \frac{1}{18} s_3^2 - \frac{1}{6} s_6\right), \end{aligned} \tag{16}$$

which defines a genus-two curve $\mathcal{S}$ by $y^2 = f(x)$ iff the discriminant does not vanish, i.e.,

$$\Delta_f = I_{10}(f) = 2^{52} 3^{21}\, Q \neq 0\ .$$

Proof. The proof follows from explicit computation of the Bell polynomials and using the relations $s_1 = 0$ and $s_2^2 = 4\, s_4$. One then checks that the discriminant of Equation (16) after using relations (14) and (8) coincides up to factor with $Q = 2^{12}\, 3^9\, \chi_{35}^2 / \chi_{10}$ where χ_{35}^2 was given in Equation (9). □

The proposition and Equations (14) prove that the Igusa invariants

$$\Big[I_2(F) : \, I_4(F) : \, I_6(F) : \, I_{10}(F) \Big] \tag{17}$$

of a sextic curve $Y^2 = F(X)$ defining a genus-two curve $\mathcal{C}$ determine the Satake sextic polynomial (16) in Proposition 4. The roots of the Satake sextic polynomial in turn determine by means of Equation (12) and Equation (13) the Rosenhaim roots of an equivalent genus-two curve corresponding to a curve in Rosenhain normal form (6), with the original Igusa invariants (17). Therefore, we get a map

$$\begin{aligned} \Phi : \mathcal{M}_2 & \to \mathcal{M}_2 \\ \mathcal{C} & \mapsto \mathcal{S} \end{aligned} \tag{18}$$

by mapping the genus-two curve $\mathcal{C}$ with period matrix τ defined by the sextic $Y^2 = F(X)$ with F given in Equation (6) to the Satake sextic $y^2 = f(x)$ with f given in Equation (16). Note that this map is defined if and only if $\Delta_F = \chi_{10}(\tau) \neq 0$ and $\Delta_f = Q(\tau) \neq 0$ because only then define the sextics two genus-two curves in the domain and range with period matrixes τ and τ', respectively. Though not by explicit formulas it was proved in [10] that this map is a rational map of degree 16. The following lemma provides the explicit formulas:

Proposition 5. *Let (j_1, j_2, j_3) and (j_1', j_2', j_3') be the absolute invariants of $\mathcal{C}$ and $\mathcal{S}$, respectively. The map $\Phi : \mathcal{M}_2 \backslash \operatorname{supp}(\chi_{35})_0 \to \mathcal{M}_2$ in Equation (18) is given by*

(19)
$$j_1' = \frac{64}{729}\, \frac{g^{(1)}(j_1, j_2, j_3)}{h(j_1, j_2, j_3)}, \quad j_2' = \frac{4}{729}\, \frac{g^{(2)}(j_1, j_2, j_3)}{h(j_1, j_2, j_3)}, \quad j_3' = \frac{1}{729}\, \frac{g^{(3)}(j_1, j_2, j_3)}{h(j_1, j_2, j_3)},$$

where the rational functions $g^{(n)}$ for $n = 1, 2, 3$ and h are given in Equations (46) in the appendix.

Proof. Let $\mathcal{C}$ be a genus-two curve defined by the sextic $Y^2 = F(X)$ with F given in Equation (6). Then the corresponding moduli point in $\mathcal{M}_2$ is determined by the Igusa functions (j_1, j_2, j_3). Indeed, we have

$$\begin{aligned} s_2 = 3 j_1 j_2, \quad s_3 = \frac{3}{2}\, j_1^2\, (j_2 - 3 j_3), \quad s_5 = \frac{15}{8}\, j_1^3\, \big(j_2^2 - 3\, j_2\ , j_3 + 648\, j_1\big)\,, \\ s_6 = \frac{3}{16}\, j_1^3\, \left(2\, j_1\, j_2^2 - 12\, j_1\, j_2\, j_3 + 18\, j_1\, j_3^2 + 9 j_2^3 + 972\, j_1^2\right). \end{aligned}$$

Let $y^2 = f(x)$ with f given in Equation (16) define a second genus-two curve $\mathcal{S}$. Then, the Igusa functions (j_1', j_2', j_3') of the genus-two curve $\mathcal{S}$ are given by where the rational functions $g^{(n)}$ for $n = 1, 2, 3$ and h are given in Equations (46). In particular, we have

$$Q(\tau) = 2^{-63}\, j_1^{15}\, h(j_1, j_2, j_3)\ .$$

It is clear from these expressions that $[k(j_1, j_2, j_3)/k(j_1', j_2', j_3')] = 16$ and therefore $\deg \Phi = 16$. If we denote the period matrix of $\mathcal{S}$ by τ' such that $j_n' = j_n(\tau')$ for $n = 1, 2, 3$ we find that

$$Q' = Q(\tau') = 2^{210} 3^{132}\, Q(\tau)\, M(\tau)^2\ ,$$

where M is a polynomial of degree 90 in the even generators. $\square$

3. Jacobian elliptic Kummer and Shioda-Inose surfaces

In this section we describe the construction of the Kummer surface $\mathrm{Kum}(\mathrm{Jac}\,\mathcal{C})$ and the Shioda-Inose surface $\mathrm{SI}(\mathrm{Jac}\,\mathcal{C})$ together with the Jacobian elliptic fibrations on these surfaces which are relevant to the F-theory/heterotic string duality.

3.1. **Jacobian elliptic fibrations.**

A surface is called Jacobian elliptic fibration if it is a (relatively) minimal elliptic surface $\pi : \mathcal{X} \to \mathbb{P}^1$ over $\mathbb{P}^1$ with a distinguished section S_0. The complete list of possible singular fibers has been given by Kodaira [17]. It encompasses two infinite families $(I_n, I_n^*, n \geq 0)$ and six exceptional cases $(II, III, IV, II^*, III^*, IV^*)$. To each Jacobian elliptic fibration $\pi : \mathcal{X} \to \mathbb{P}^1$ there is an associated Weierstrass model $\bar{\pi} : \bar{\mathcal{X}} \to \mathbb{P}^1$ with a corresponding distinguished section $\bar{S}_0$ obtained by contracting all components of fibers not meeting S_0. The fibers of $\bar{\mathcal{X}}$ are all irreducible whose singularities are all rational double points, and $\mathcal{X}$ is the minimal desingularization. If we choose $t \in \mathbb{C}$ as a local affine coordinate on $\mathbb{P}^1$, we can present $\bar{\mathcal{X}}$ in the Weierstrass normal form

$$Y^2 = 4\, X^3 - g_2(t)\, X - g_3(t)\ , \tag{20}$$

where g_2 and g_3 are polynomials in t of degree four and six, respectively, because $\mathcal{X}$ is a K3 surface. Type of singular fibers can then be read off from the orders of vanishing of the functions g_2, g_3 and the discriminant $\Delta = g_2^3 - 27\, g_3^2$ at the various singular base values. Note that the vanishing degrees of g_2 and g_3 are always less or equal three and five, respectively, as otherwise the singularity of $\bar{X}$ is not a rational double point.

For a family of Jacobian elliptic surfaces $\pi : \mathcal{X} \to \mathbb{P}^1$, the two classes in Néron-Severi lattice $\mathrm{NS}(\mathcal{X})$ associated with the elliptic fiber and section span a sub-lattice $\mathcal{H}$ isometric to the standard hyperbolic lattice H with the quadratic form $q = x_1 x_2$, and we have the following decomposition as a direct orthogonal sum

$$\mathrm{NS}(\mathcal{X}) = \mathcal{H} \oplus \mathcal{W}\ .$$

The orthogonal complement $T(\mathcal{X}) = \mathrm{NS}(\mathcal{X})^{\perp} \in H^2(\mathcal{X}, \mathbb{Z}) \cap H^{1,1}(\mathcal{X})$ is called the transcendental lattice and carries the induced Hodge structure.

3.2. **The Kummer surface.**

For the Jacobian variety $\mathrm{Jac}(\mathcal{C})$ of a genus-two curve $\mathcal{C}$, let $\imath$ be the involution automorphism on $\mathrm{Jac}(\mathcal{C})$ given by $\imath : a \mapsto -a$. The quotient, $\mathrm{Jac}(\mathcal{C})/\{\mathbb{I}, \imath\}$, is a singular surface with sixteen ordinary double points. Its minimum resolution, $\mathrm{Kum}(\mathrm{Jac}\,\mathcal{C})$, is a special K3 surface called the *Kummer surface* associated to $\mathrm{Jac}\,\mathcal{C}$.

On the Kummer surface $\mathrm{Kum}(\mathrm{Jac}\,\mathcal{C})$, there are two sets of sixteen (-2)-curves, called nodes and tropes, which are either the exceptional divisors corresponding to blowup of the 16 two-torsion points of the Jacobian $\mathrm{Jac}(\mathcal{C})$ or they arise from the embedding of $\mathcal{C}$ into $\mathrm{Jac}(\mathcal{C})$ as symmetric theta divisors. These two sets of smooth rational curves have a rich symmetry, the so-called 16_6-configuration, where each node intersects exactly six tropes and vice versa [13].

Using curves and symmetries in the 16_6-configuration one can define various elliptic fibrationson $\mathrm{Kum}(\mathrm{Jac}\,\mathcal{C})$, since all irreducible components of a reducible fiber in an elliptic fibration are (-2)-curves [17]. In fact, for the Kummer surface of a generic curve of genus two all inequivalent elliptic fibrations where determined explicitly by Kumar in [19]. In particular, Kumar computed elliptic parameters and Weierstrass equations for all twenty five different fibrations that appear, and analyzed the reducible fibers and Mordell-Weil lattices.

3.2.1. *A first elliptic fibration on* $\mathrm{Kum}(\mathrm{Jac}\,\mathcal{C})$. Given a genus-two curve $\mathcal{C}$ defined by a sextic $y^2 = f(x)$, the Jacobian variety $\mathrm{Jac}(\mathcal{C})$ is birational to the symmetric product of two copies of $\mathcal{C}$, i.e., $(\mathcal{C} \times \mathcal{C})/\{\mathbb{I}, \pi\}$, where we have set $\pi(x_1) = x_2$ and $\pi(y_1) = y_2$. The function field is the sub-field of $\mathbb{C}[x_1, x_2, y_1, y_2]$ such that $y_i^2 = f(x_i)$ for $i = 1, 2$ which is fixed under π.

The Kummer surface $\mathrm{Kum}(\mathrm{Jac}\,\mathcal{C})$ is birational to the quotient $\mathrm{Jac}(\mathcal{C})/\{\mathbb{I}, \imath\}$ with $\imath(x_i) = x_i$ and $\imath(y_i) = -y_i$ for $i = 1, 2$. Its function field is the sub-field of $\mathbb{C}[x_1, x_2, y_1, y_2]$ with $y_i^2 = f(x_i)$ for $i = 1, 2$ which is fixed under both π and $\imath$. Thus, the function field of $\mathrm{Kum}(\mathrm{Jac}\,\mathcal{C})$ is generated by $Y = y_1 y_2$, $t = x_1 x_2$, and $X = x_1 + x_2$. We have the following lemma:

Lemma 6. *The function field of the Kummer surface* $\mathrm{Kum}(\mathrm{Jac}\,\mathcal{C})$ *for the genus-two curve* $\mathcal{C}$ *given by the sextic (6) is generated by* η, ξ, t *subject to the relation*

$$\mathcal{K}(Y, X, t): \; Y^2 = t\,\big(1 - X + t\big)\,\big(\lambda_1^2 - \lambda_1\,X + t\big)\,\big(\lambda_2^2 - \lambda_2\,X + t\big)\,\big(\lambda_3^2 - \lambda_3\,X + t\big). \tag{21}$$

Equation (21) defines a Jacobian elliptic fibration $\bar{\pi} : \bar{\mathcal{X}} \to \mathbb{P}^1$ with a distinguished section $\bar{S}_0$ on $\mathcal{X} = \mathrm{Kum}(\mathrm{Jac}\,\mathcal{C})$ by choosing t as the elliptic parameter and the point at infinity in each fiber for the section. In fact, this fibration is well-known and labeled fibration '1' in [19]. The following lemma is immediate and follows by comparison with the explicit results in [19].

Lemma 7. *Equation (21) is a Jacobian elliptic fibration* $\bar{\pi} : \bar{\mathcal{X}} \to \mathbb{P}^1$ *with 6 singular fibers of type* I_2*, two singular fibers of type* I_0^**, and the Mordell-Weil group of sections* $\mathrm{MW}(\bar{\pi}) = (\mathbb{Z}/2)^2 \oplus \langle 1 \rangle$.

The sextic curve $\mathcal{C}$ is recovered directly from the Jacobian elliptic fibration by the following corollary:

Corollary 8. *The sextic associated with the genus-two curve* $\mathcal{C}$ *is recovered from the Jacobian elliptic fibration (21) on* $\mathrm{Kum}(\mathrm{Jac}\,\mathcal{C})$ *by letting* $X \to \infty$ *while keeping* $t/X = x$, $Y^2/X^5 = y^2$ *fixed, i.e.,*

$$\lim_{\epsilon \to 0} \epsilon^{10}\,\mathcal{K}\left(Y = \frac{y}{\epsilon^5}, X = \frac{1}{\epsilon^2}, t = \frac{x}{\epsilon^2}\right) = \mathcal{C}\,. \tag{22}$$

Proof. The proof follows by explicit computation. □

3.2.2. *A second elliptic fibration on* $\operatorname{Kum}(\operatorname{Jac}\mathcal{C})$. There is another elliptic fibration $\mathcal{X} \to \mathbb{P}^1$ on $\operatorname{Kum}(\operatorname{Jac}\mathcal{C})$ which is more relevant for us, labeled fibration '23' in [19]. Kumar proved the following [18]:

Proposition 9. *A Jacobian elliptic fibration* $\bar{\pi} : \bar{\mathcal{X}} \to \mathbb{P}^1$ *with 6 singular fibers of type* I_2*, one fiber of type* I_5^**, and one fiber of type* I_1*, and a Mordell-Weil group of sections* $\mathrm{MW}(\bar{\pi}) = \mathbb{Z}/2$ *is given by the Weierstrass equation*

$$
\begin{aligned}
Y^2 = X^3 &- 2\left(t^3 - \frac{I_4}{12}t + \frac{I_2I_4 - 3I_6}{108}\right) X^2 \\
&+ \left(\left(t^3 - \frac{I_4}{12}t + \frac{I_2I_4 - 3I_6}{108}\right)^2 + I_{10}\left(t - \frac{I_2}{24}\right)\right) X \,.
\end{aligned} \tag{23}
$$

An immediate corollary is the following:

Corollary 10. *The positions of the* I_2 *fibers in the elliptic fibration (23) on the Kummer surface* $\operatorname{Kum}(\operatorname{Jac}\mathcal{C})$ *are given by the roots of the polynomial*

$$
\left(t^3 - \frac{I_4}{12}t + \frac{I_2I_4 - 3I_6}{108}\right)^2 + I_{10}\left(t - \frac{I_2}{24}\right) = 0 \,, \tag{24}
$$

or equivalently by

$$
\left(t^3 - \frac{\psi_4}{3}t + \frac{2\,\psi_6}{27}\right)^2 - 2^{14}\,(\chi_{10}\,t + \chi_{12}) = 0 \,. \tag{25}
$$

Equivalently, the loci of I_2 *fibers form the ramification locus of the Satake sextic (16) if we set* $t = -x/3$*.*

3.3. **The Shioda-Inose surface.** A K3 surface $\mathcal{Y}$ has a Shioda-Inose structure if it admits an involution fixing the holomorphic two-form, such that the quotient is Kummer surface $\operatorname{Kum}(\mathbf{A})$ of a principally polarized abelian surface $\mathbf{A}$ and the rational quotient map $p : \mathcal{Y} \dashrightarrow \operatorname{Kum}(\mathbf{A})$ of degree two induces a Hodge isometry[2] between the transcendental lattices $T(\mathcal{Y})(2)$[3] and $T(\operatorname{Kum}\mathbf{A})$. Morrison proved that a K3 surface $\mathcal{Y}$ admits a Shioda-Inose structure if and only if there exists a Hodge isometry between the following transcendental lattices $T(\mathcal{Y}) \cong T(\mathbf{A})$.

The Shioda-Inose $\mathcal{Y}$ of the Kummer surface $\operatorname{Kum}(\operatorname{Jac}\mathcal{C})$ for a generic genus-two curve $\mathcal{C}$ is a K3 surface of Picard-rank 17 and has a transcendental lattice isomorphic to $H \oplus H \oplus \langle -2 \rangle$ by Morrison's criterion. It was shown in [22] that for fixed $\mathcal{C}$ this K3 surface $\mathcal{Y}$ is in fact unique. In the following, we always let $\mathcal{Y} := \mathrm{SI}(\operatorname{Jac}\mathcal{C})$ be this K3 surface with Shioda-Inose structure.

Clingher and Doran proved in [6] that as the genus-two curve $\mathcal{C}$ varies the K3 surface $\mathcal{Y}$ admits a birational model isomorphic to a quartic surface with canonical $H \oplus E_8(-1) \oplus E_7(-1)$[4] lattice polarization[5] that fits into the following four-parameter family in $\mathbb{P}^3$ [6, Eq. (3)] given in terms of the variables $[\mathbf{W} : \mathbf{X} : \mathbf{Y} : \mathbf{Z}] \in$

[2] A Hodge isometry between two transcendental lattices is an isometry preserving the Hodge structure.

[3] The notation $T(\mathcal{Y})(2)$ indicates that the bilinear pairing on the transcendental lattice $T(\mathcal{Y})$ is multiplied by 2.

[4] Here, $E_8(-1)$ and $E_7(-1)$ are the negative definite lattice associated with the exceptional root systems of E_8 and E_7, respectively.

[5] A lattice polarization of a K3 surface $\mathcal{Y}$ is a primitive embedding of a lattice $L' \hookrightarrow L = H_2(\mathcal{Y}, \mathbb{Z})$ such that the image of L' lies in the Néron-Severi group $\mathrm{NS}(\mathcal{Y}) = L \cap H^{1,1}(\mathcal{Y})$ and contains a pseudo-ample class.

$\mathbb{P}^3$ by the equation

$$\mathbf{Y}^2\mathbf{Z}\mathbf{W} - 4\,\mathbf{X}^3\mathbf{Z} + 3\,\alpha\,\mathbf{X}\mathbf{Z}\mathbf{W}^2 + \beta\,\mathbf{Z}\mathbf{W}^3 + \gamma\,\mathbf{X}\mathbf{Z}^2\mathbf{W} - \frac{1}{2}(\delta\,\mathbf{Z}^2\mathbf{W}^2 + \mathbf{W}^4) = 0. \tag{26}$$

They also found the parameters $(\alpha, \beta, \gamma, \delta)$ in terms of the standard even Siegel modular forms $\psi_4, \psi_6, \chi_{10}, \chi_{12}$ (cf. [14]) given by

$$(\alpha, \beta, \gamma, \delta) = \big(\psi_4, \psi_6, 2^{12}3^5\,\chi_{10}, 2^{12}3^6\,\chi_{12}\big)\ , \tag{27}$$

or, equivalently, in terms of the Igusa-Clebsch invariants using Equations (8) by

$$(\alpha, \beta, \gamma, \delta) = \left(\frac{1}{4}I_4, \frac{1}{8}I_2\,I_4 - \frac{3}{8}I_6, -\frac{243}{4}I_{10}, \frac{243}{32}I_2\,I_{10}\right)\ , \tag{28}$$

where I_n for $n = 2, 4, 6, 10$ are the Igusa invariants of the sexic curve (6) defining the genus-two curve $\mathcal{C}$ if $I_{10} \neq 0$. The Shioda-Inose surface $\mathcal{Y} = \mathrm{SI}(\mathrm{Jac}\,\mathcal{C})$ of a generic genus-two curve $\mathcal{C}$ admits two Jacobian elliptic fibrations realizing the two inequivalent ways of embedding of H into the lattice $H \oplus E_8(-1) \oplus E_7(-1)$. These two elliptic fibrations were described in [7, 18]. A similar picture was developed in earlier work for the case of a $H \oplus E_8(-1) \oplus E_8(-1)$ lattice polarization in [5, 30] that generalized a special two-parameter family of K3 surfaces introduced by Inose in [16].

From the point of view of K3 geometry, if the periods are preserved by a reflection of δ with $\delta^2 = -2$, then δ must belong to the Néron-Severi lattice of the K3 surface. That is, the lattice $H \oplus E_8(-1) \oplus E_7(-1)$ must be enlarged by adjoining δ. It is not hard to show (using methods of [25], for example), that there are only two ways this enlargement can happen (if we have adjoined a single element only): either the lattice is extended to $H \oplus E_8(-1) \oplus E_8(-1)$ or it is extended to $H \oplus E_8(-1) \oplus E_7(-1) \oplus \langle -2 \rangle$.

3.3.1. *The alternate fibration.* The first Jacobian elliptic fibration on (26), called the *alternate fibration*, has two disjoint sections and a singular fiber of Kodaira-type I_{10}^*. For convenience, let us introduce the parameters (a, b, c, d, e) given by

$$a = -\frac{I_4}{12},\ b = \frac{I_2\,I_4 - 3\,I_6}{108},\ c = -1,\ d = \frac{I_2}{24},\ e = \frac{I_{10}}{4}. \tag{29}$$

The alternate fibration is obtained by setting

$$\mathbf{X} = \frac{t\,x^3}{2^{29}\,3^5}\ , \quad \mathbf{Y} = \frac{\sqrt{6}\,i\,x^2y}{2^{29}\,3^5}\ , \quad \mathbf{W} = -\frac{x^3}{2^{28}\,3^6}\ , \quad \mathbf{Z} = \frac{x^2}{2^{28}\,3^9}\ , \tag{30}$$

in Equation (26), and given in Weierstrass form by

$$y^2 = x^3 + (t^3 + at + b)\,x^2 + e\,(ct + d)\,x\ . \tag{31}$$

The fibration (31) has special fibers of Kodaira-types I_{10}^* and I_2, and six fibers of Kodaira-type I_1, and a second section $(x, y) = (0, 0)$. This proves the following:

Proposition 11. *For a generic genus-two curve $\mathcal{C}$ there is a Jacobian elliptic fibration $\bar{\pi}_{alt} : \bar{\mathcal{Y}} \to \mathbb{P}^1$ on $\mathcal{Y} = \mathrm{SI}(\mathrm{Jac}\,\mathcal{C})$ given by Equation (31) with 6 singular fibers of type I_1, one fiber of type I_{10}^*, and one fiber of type I_2, and a Mordell-Weil group of sections $\mathrm{MW}(\bar{\pi}) = \mathbb{Z}/2$ with elliptic parameter $t = t_{alt}$.*

The discriminant of Equation (31) is given by

$$\Delta = 16\,e^2\,(ct + d)^2\,\Big((t^3 + at + b)^2 - 4\,e\,(ct + d)\Big)\ . \tag{32}$$

The following corollary is crucial:

Corollary 12. *The positions of the I_1 fibers in the Jacobian elliptic fibration (23) are given by the roots of the polynomial*

$$f(t) = (t^3 + at + b)^2 - 4\,e\,(ct + d) = 0\,. \tag{33}$$

Equivalently, the loci of I_1 fibers form the ramification locus of the Satake sextic (16) if we set $t = -x/3$.

Given the discussion at the end of previous section, the following corollaries are immediate:

Corollary 13. *For the Jacobian elliptic fibration (31) two I_1 fibers coalesce and form a fiber of type I_2 if and only if the discriminant of the Satake sextic (33) vanishes, i.e.,*

$$\Delta_f = 2^{52}\,3^{-9} Q = 2^{64}\,\chi_{35}^2/\chi_{10} = q(a,b,c,d,e) = 0\,,$$

with

$$\begin{aligned} q(a,b,c,d,e) = 2^{12}\,e^3 \Big(& 16\,a^7c^2d - 16\,a^6bc^3 + 16\,a^5c^4e + 16\,a^6d^3 \\ & +216\,a^4b^2c^2d + 888\,a^4c^2d^2e - 216\,a^3b^3c^3 - 3420\,a^3bc^3de + 2700\,a^2b^2c^4e \\ & +4125\,a^2c^4de^2 - 5625\,abc^5e^2 + 3125\,c^6e^3 + 216\,a^3b^2d^3 + 864\,a^3d^4e \\ & -2592\,a^2bcd^3e + 729\,ab^4c^2d - 5670\,ab^2c^2d^2e + 16200\,ac^2d^3e^2 - 729\,b^5c^3 \\ & +6075\,b^3c^3de - 13500\,bc^3d^2e^2 + 729\,b^4d^3 - 5832\,b^2d^4e + 11664\,d^5e^2 \Big). \end{aligned} \tag{34}$$

Equivalently, $\Delta_f = 0$ iff the lattice polarization $H \oplus E_8(-1) \oplus E_7(-1)$ of the family (31) extends to $H \oplus E_8(-1) \oplus E_7(-1) \oplus \langle -2 \rangle$.

Remark 14. *We remark that each of the I_1-fiber will coalesce with the I_2-fiber to form a fiber of type III if and only if*

$$e^3\,(a\,c^2\,d - b\,c^3 + d^3) = -\frac{2^{36}}{3^3}\left(2\,\psi_6\,\chi_{10}^3 + 9\,\psi_4\,\chi_{10}^2\,\chi_{12} - 27\,\chi_{12}^3\right) = 0$$

However, it is easy to show that this does not change the lattice polarization of the family (31).

If we use a normalization consistent with F-theory [20] and set

$$\mathbf{X} = \frac{t\,x^3}{2^9\,3^5}\,, \quad \mathbf{Y} = \frac{x^2\,y}{2^{15/2}\,3^{9/2}}\,, \quad \mathbf{W} = \frac{x^3}{2^{10}\,3^6}\,, \quad \mathbf{Z} = \frac{x^2}{2^{16}\,3^9}\,, \tag{35}$$

and obtain from Equation (26) the Jacobian elliptic fibration

$$y^2 = x^3 + \left(t^3 - \frac{\psi_4}{48}\,t - \frac{\psi_6}{864}\right) x^2 - \Big(4\,\chi_{10}\,t - \chi_{12}\Big)\,x\,, \tag{36}$$

which is equivalent to Equation (36) for $\chi_{10}(\tau) \neq 0$. Equation (36) remains well-defined for $\chi_{10}(\tau) = 0$ when the principally polarized abelian surface $\mathbf{A}_\tau$ degenerates to a product of two elliptic curves $\mathbf{A}_\tau = \mathcal{E}_{\tau_1} \times \mathcal{E}_{\tau_2}$. It follows:

Corollary 15. *For the Jacobian elliptic fibration (36) the two fibers of type I_2 and I_{10}^* coalesce and form a fiber of type I_{12}^* if and only if the discriminant of the sextic (6) vanishes, i.e.,*

$$\Delta_F = I_{10} = -2^{14}\,\chi_{10} = 4\,e = 0\,.$$

Equivalently, $\Delta_F = 0$ iff the lattice polarization $H \oplus E_8(-1) \oplus E_7(-1)$ of the family (36) extends to $H \oplus E_8(-1) \oplus E_8(-1)$.

Using fibration (31) and fibration (23), the rational quotient map

$$p : \mathcal{Y} = \mathrm{SI}(\mathrm{Jac}\,\mathcal{C}) \dashrightarrow \mathcal{X} = \mathrm{Kum}(\mathrm{Jac}\,\mathcal{C})$$

can be realized as a fiberwise two-isogeny between elliptic surfaces, also known as a *Van Geemen-Sarti involution*. Together with the dual isogeny one obtains a chain of rational maps $\mathcal{Y} \dashrightarrow \mathcal{X} \dashrightarrow \mathcal{Y}$ called a *Kummer sandwich* in [30]. In fact, the translation of the elliptic fiber $\mathcal{E} = \mathcal{Y}_t$ in Equation (31) by a two-torsion point $\bar{S}_1 : (x, y) = (0, 0)$ yields the two-isogeneous fiber $\mathcal{E}' = \mathcal{E}/\{S_0, S_1\}$ given by

$$Y^2 = X^3 - 2(t^3 + at + b)\, X^2 + \Big((t^3 + at + b)^2 - 4\, e\, (ct + d)\Big)\, X, \tag{37}$$

which is precisely the fibration (23). The fibrewise isogeny $\mathcal{E} \to \mathcal{E}' = \mathcal{X}_t$ is given by

$$(x, y) \mapsto (X, Y) = \left(\frac{y^2}{x^2}, \frac{y\,(e\,(ct + d) - x^2)}{x^2}\right), \tag{38}$$

and the dual isogeny $\mathcal{E}' \to \mathcal{E}$ is given by

$$(X, Y) \mapsto (x, y) = \left(\frac{Y^2}{4X^2}, \frac{Y\,\Big((t^3 + at + b)^2 - 4e(ct + d) - X^2\Big)}{X^2}\right)\,. \tag{39}$$

The resulting *Nikulin involution* φ on the K3 surface $\mathcal{Y}$, i.e., the automorphism of order two preserving the holomorphic two-form, in this case the two-form $dt \wedge dx/y$, is given by mapping (x, y) to

$$\left(-\frac{\Big(x^2 - e\big(ct + d\big)\Big)^2}{4\, y^2}, \frac{\Big(x^2 - e\big(ct + d\big)\Big)\,\Big(y^4 - f(t)\, x^4\Big)}{8\, x^2 y^3}\right)$$

in each generic fiber $\mathcal{Y}_t$. A fiber of type I_1 is a rational curve with a node, whereas a fiber of type I_2 looks like two copies of $\mathbb{P}^1$ intersecting in two distinct points. The involution φ is free on the generic fibers and has exactly 8 fixed points, namely the nodal points on the I_1 fibers and the intersecting points on the I_2 fiber. We have the following corollary:

Corollary 16. *The positions of the I_1 fibers in the Jacobian elliptic fibration (23) are contained in the fixed point set of the Nikulin involution φ.*

3.3.2. *The standard fibration.* The second elliptic fibration, called the *standard fibration*, is the Jacobian elliptic fibration with two distinct special fibers of Kodaira-types II^* and III^*, respectively. By setting

$$\mathbf{X} = -\frac{2^7\, \chi_{10}^3\, t\, x}{3^5}\;, \quad \mathbf{Y} = \frac{2^7\, \sqrt{6}\, i\, \chi_{10}^3\, y}{3^5}\;, \quad \mathbf{W} = \frac{2^8\, \chi_{10}^3\, t^3}{3^6}\;, \quad \mathbf{Z} = \frac{\chi_{10}^2\, t^2}{2^4\, 3^9}\;, \tag{40}$$

in Equation (26) we obtain

$$y^2 = x^3 + t^3\, (a\, t + c)\, x + t^5\, (e\, t^2 + b\, t + d)\;. \tag{41}$$

The fibration (41) was investigated in [18, Theorem 11]. The birational transformation between the standard and the alternate fibration is given by

$$\Big(t, x, y\Big)_{\text{std}} = \Big(\frac{x}{e}, \frac{tx^2}{e^2}, -\frac{x^2 y}{e^3}\Big)_{\text{alt}}\;, \tag{42}$$

and combining it with Equation (3.3.1) recovers the Nikulin involution for the standard elliptic fibration given by (cf. [18, Theorem 11])

$$\Big(t, x, y\Big) \mapsto \left(\frac{cx + dt^2}{et^3}, \frac{x(cx + dt^2)^2}{e^2t^8}, -\frac{y(cx + dt^2)^3}{e^3t^{12}}\right). \tag{43}$$

It then follows:

Proposition 17. *For a generic genus-two curve $\mathcal{C}$ there is a Jacobian elliptic fibration $\bar{\pi}_{std} : \bar{\mathcal{Y}} \to \mathbb{P}^1$ on $\mathcal{Y} = \mathrm{SI}(\mathrm{Jac}\,\mathcal{C})$ given by Equation (41) with 5 singular fibers of type I_1, one fiber of type II^*, one fiber of type III^*, and a trivial Mordell-Weil group with elliptic parameter $t = t_{std}$.*

For the standard fibration, there are statements analogous to Corollary 13 or Corollary 15 when for the fibration (41) two I_1 fibers are coalescing to form a fiber of type II or a fiber from type III^* goes to type II^*, respectively [20].

3.4. **Relation to string theory.** In string theory a nontrivial connection appears as the eight-dimensional manifestation of a phenomenon called *F-theory/heterotic string duality*. This correspondence leads to a geometric picture that links together moduli spaces for two seemingly distinct types of geometrical objects: Jacobian elliptic fibrations on K3 surfaces and flat bundles over elliptic curves [9].

In string theory compactifications of the so-called type-IIB string in which the complex coupling varies over a base are generically referred to as F-theory. The simplest such construction corresponds to K3 surfaces that are elliptically fibered over $\mathbb{P}^1$ with a section, in physics equivalent to type-IIB string theory compactified on $\mathbb{P}^1$ and hence eight-dimensional in the presence of 7-branes [2]. In this way, a Jacobian elliptic K3 surface with elliptic fibers $\mathcal{E}_\tau = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau)$ defines an F-theory vacuum in eight dimensions where the complex-valued scalar field τ of the type-IIB string theory is now allowed to be multi-valued. The Kodaira-table of singular fibers gives a precise dictionary between the characteristics of the Jacobian elliptic fibrations and the content of the 7-branes present in the physical theory.

To make contact with the F-theory/heterotic string duality one considers Jacobian elliptic fibrations on a special K3 surface, namely the Shioda-Inose surface $\mathrm{SI}(\mathrm{Jac}\,\mathcal{C})$ of the principally polarized abelian surface $\mathrm{Jac}(\mathcal{C})$ that was defined in Section where $\mathcal{C}$ is a generic genus-two curve. The K3 surface $\mathcal{Y} = \mathrm{SI}(\mathrm{Jac}\,\mathcal{C})$ carries a Nikulin involution φ such that the quotient $\mathcal{Y}/\{\mathbb{I}, \varphi\}$ is birational to the Kummer surface $\mathrm{Kum}(\mathrm{Jac}\,\mathcal{C})$ and we have a Hodge-isometry between the transcendental lattices $T(\mathcal{Y}) \cong T(\mathrm{Jac}\,\mathcal{C})$. In this way, a one-to-one correspondence between two different types of surfaces with the same Hodge-theoretic data is established: the principally polarized abelian surfaces $\mathrm{Jac}\,\mathcal{C}$ and the algebraic K3 surfaces $\mathcal{Y}$ polarized by the rank-17 lattice $H \oplus E_8(-1) \oplus E_7(-1)$.

To see the connection to the heterotic string theory, let us first consider the limit as the Jacobian variety degenerates to a product of two elliptic curves $\mathcal{E}_1 \times \mathcal{E}_2$ and the involved K3 surfaces have Picard-rank 18, obtained by letting $\chi_{10} \to 0$. This limit describes a well-understood case of the F-theory/heterotic string duality in the absence of any additional data given by so-called Wilson lines. In fact, the moduli space of Jacobian elliptic K3 surfaces with $H \oplus E_8(-1) \oplus E_8(-1)$ lattice polarization is identified with the moduli space of the heterotic string vacua with gauge algebra $\mathfrak{e}_8 \oplus \mathfrak{e}_8$ and $\mathfrak{so}(32)$, respectively, compactified on a two-torus T^2 (cf. [1]). If any flat connection on T^2 is assumed to be trivial, the only two moduli of such a string theory, i.e, the Kähler metric and the B-field of T^2, identify the torus with the elliptic curves $\mathcal{E}_1$ and $\mathcal{E}_2$, respectively. Notice that the existence of two inequivalent elliptic fibrations, the standard and the alternate fibration, is essential and corresponds to the two possible gauge groups of the heterotic string.

The first author together with David Morrison studied in [20] the *non-geometric* heterotic string compactified on T^2 that produces an eight-dimensional effective theory corresponding to the Jacobian elliptic K3 surfaces with Picard-rank 17 when $\chi_{10} \neq 0$. The corresponding heterotic models were called non-geometric because the Kähler and complex structures on T^2, and the Wilson line values, are not distinguished but instead are mingled together. The fibration in Equation (41) then describes a model dual to the $\mathfrak{e}_8 \oplus \mathfrak{e}_8$ heterotic string, with an unbroken gauge algebra of $\mathfrak{e}_8 \oplus \mathfrak{e}_7$ ensuring that only a single Wilson line expectation value is nonzero and all remaining Wilson lines values associated to the $E_8(-1) \oplus E_7(-1)$ sublattice be trivial. Similarly, the fibration in Equation (41) gives the analogous story for the $\mathfrak{so}(32)$ heterotic string: the fibration in Equation (31) describes a model dual to the $\mathfrak{so}(32)$ heterotic string, with an unbroken gauge algebra of $\mathfrak{so}(28) \oplus \mathfrak{su}(2)$. By a result of Vinberg [31] and its interpretation in string theory given in [20], the function field of the Narain moduli space of these heterotic theories turns out to be generated by the ring of Siegel modular forms with $g = 2$ *of even weight.* This is the physical manifestation of why the fibrations (41) and (31) only depend on the polynomial ring in the four free generators of degrees 4, 6, 10 and 12 given by the even Siegel modular forms.[6]

Generic non-geometric compactification constructed from the lattice-polarized K3 surfaces in Equation (26) will have two types of five-branes, corresponding to the situations discussed in Corollary 13 and Corollary 15. From the heterotic side, these five-brane solitons are easy to see: when $\Delta_f = \chi_{35}^2/\chi_{10} = 0$, we have an additional gauge symmetry enhancement by a factor of $\mathfrak{su}(2)$, and the parameters of the theory will include a Coulomb branch on which the Weyl group $W_{\mathfrak{su}(2)} = \mathbb{Z}_2$ acts. Therefore, there is a five-brane solution in which the field has a $\mathbb{Z}_2$ ambiguity encircling the location in the moduli space of enhanced gauge symmetry. The other five-brane solution is similar: when $\Delta_F = \chi_{10} = 0$, the gauge group enhances to $\mathfrak{so}(32)$ gauge symmetry, and a similar $\mathbb{Z}_2$ acts on the moduli space, leading to a solution with a $\mathbb{Z}_2$ ambiguity. These two brane solutions are the analogue of the simplest brane (a single D7-brane) in F-theory.

More general degenerations of the multi-parameter family of K3 surfaces in Equation (26) are then obtained from degenerations of the underlying genus-two curves. As we have seen, the parameters in Equation (26) are Siegel modular forms of even degree or, equivalently, the Igusa-Clebsch invariants of a binary sextic. Namikawa and Ueno gave a geometrical classification of all (degenerate) fibers in pencils of curves of genus two in [24]. For each such pencil allowed by their classification one can now apply the heterotic/F-theory duality map to express the heterotic background in terms of F-theory compactifications. Each resulting F-theory compactification will be a family of Jacobian elliptic K3 surfaces. Notice that any such degenerating pencil of genus-two curves is not the description of a heterotic model itself, but rather a computational tool for providing an interesting class of degenerations and their associated five-branes. Moreover, the F-theory background dual

[6] In contrast, Igusa showed in [15] that for the full ring of modular forms, one needs an additional generator χ_{35} which is algebraically dependent on the others.

to a given five-brane defect on the heterotic side can be highly singular. In some cases the singularities can be resolved by performing a finite number of blow-ups in the base. For some cases the resulting smooth geometry was constructed in [8].

Conversely, the combination of Proposition 4 and Corollary 12 give a computational recipe for how a degenerating pencil of genus-two curves is obtained from an F-theory background dual to the $\mathfrak{so}(32)$ string with only one non-vanishing Wilson line. In comparison, the work of the authors in [21] always allows for the construction of an explicit pencil of sextic curves given any family of Igusa invariants over a quadratic extension of the full ring of modular forms. However, this construction does not use the F-theoretic data of the $\mathfrak{so}(32)$ string background, i.e., the Jacobian elliptic fibration, and requires lifting of the family to a covering space of the moduli space. In contrast, Corollary 12 shows that the Satake sextic is inherently manifest in the Jacobian elliptic fibration (31).

We rephrase Corollary 12 according to the discussion in this section as follows:

Corollary 18. *The positions of the 7-branes with string charge $(1,0)$ in the F-theory model, dual to the $\mathfrak{so}(32)$ heterotic string with an unbroken gauge algebra of $\mathfrak{so}(28)\oplus\mathfrak{su}(2)$ and only a single non-vanishing Wilson line expectation value and no additional gauge-extension, are given by the loci of I_1 fibers in the Jacobian elliptic fibration (31) on the Shioda-Inose surface $\mathrm{SI}(\mathrm{Jac}\,\mathcal{C})$ of a generic genus-two curve $\mathcal{C}$ and form the ramification locus of the Satake sextic (24) given in terms of the Igusa invariants the sextic defining $\mathcal{C}$.*

Remark 19. *The section $(x,y)=(0,0)$ defines an element of order 2 in the Mordell-Weil group of the Jacobian elliptic fibration (31). It follows as in [1,3] that the actual gauge group of this heterortic model is $(\mathrm{Spin}(28)\times SU(2))/\mathbb{Z}_2$.*

In turn, the roots of the Satake sextic then determine a sextic curve (6) with full level-two structure by using Equation (13) and Equation (12).

4. Appendix

The Igusa-Clebsch invariants for the curve (6) in Rosenhain normal form are given by the following expressions:

$$
\begin{aligned}
I_2 &= 40\,\lambda_1\lambda_2\lambda_3 - 16\,(1+\lambda_1+\lambda_2+\lambda_3)(\lambda_1\lambda_2\lambda_3+\lambda_2\lambda_1+\lambda_3\lambda_1+\lambda_2\lambda_3) + 6\,(\lambda_2\lambda_1+\lambda_3\lambda_1+\lambda_2\lambda_3+\lambda_1+\lambda_2+\lambda_3)^2\,,\\
I_4 &= -12\,(\lambda_1+\lambda_2+\lambda_3)^3\,\lambda_1\lambda_2\lambda_3 + 4\,(\lambda_1+\lambda_2+\lambda_3)^2(\lambda_2\lambda_1+\lambda_3\lambda_1+\lambda_2\lambda_3)^2 - 4\,(\lambda_1+\lambda_2+\lambda_3)^2(\lambda_2\lambda_1+\lambda_3\lambda_1+\lambda_2\lambda_3)\,\lambda_1\lambda_2\lambda_3\\
&\quad + 4\,(\lambda_1+\lambda_2+\lambda_3)^2\,\lambda_1^2\lambda_2^2\lambda_3^2 + 12\,(\lambda_1+\lambda_2+\lambda_3)^2\,\lambda_1\lambda_2\lambda_3 - 4\,(\lambda_1+\lambda_2+\lambda_3)(\lambda_2\lambda_1+\lambda_3\lambda_1+\lambda_2\lambda_3)^2\\
&\quad + 44\,(\lambda_1+\lambda_2+\lambda_3)(\lambda_2\lambda_1+\lambda_3\lambda_1+\lambda_2\lambda_3)\,\lambda_1\lambda_2\lambda_3 - 12\,(\lambda_2\lambda_1+\lambda_3\lambda_1+\lambda_2\lambda_3)^3 + 12\,(\lambda_2\lambda_1+\lambda_3\lambda_1+\lambda_2\lambda_3)^2\,\lambda_1\lambda_2\lambda_3\\
&\quad - 12\,(\lambda_2\lambda_1+\lambda_3\lambda_1+\lambda_2\lambda_3)\,\lambda_1^2\lambda_2^2\lambda_3^2 - 12\,(\lambda_1+\lambda_2+\lambda_3)\,\lambda_1\lambda_2\lambda_3 + 4\,(\lambda_2\lambda_1+\lambda_3\lambda_1+\lambda_2\lambda_3)^2 - 72\,\lambda_1^2\lambda_2^2\lambda_3^2,\\
I_6 &= -24\,(\lambda_1+\lambda_2+\lambda_3)^3\,\lambda_1\lambda_2\lambda_3 + 10\,(\lambda_1+\lambda_2+\lambda_3)^2\,\lambda_1^2\lambda_2^2\lambda_3^2 + 32\,(\lambda_2\lambda_1+\lambda_3\lambda_1+\lambda_2\lambda_3)^2\,\lambda_1\lambda_2\lambda_3 + 150\,(\lambda_2\lambda_1+\lambda_3\lambda_1+\lambda_2\lambda_3)\,\lambda_1^2\lambda_2^2\lambda_3^2\\
&\quad + 8\,(\lambda_1+\lambda_2+\lambda_3)^2(\lambda_2\lambda_1+\lambda_3\lambda_1+\lambda_2\lambda_3)^2\,\lambda_1^2\lambda_2^2\lambda_3^2 + 118\,(\lambda_1+\lambda_2+\lambda_3)^3(\lambda_2\lambda_1+\lambda_3\lambda_1+\lambda_2\lambda_3)\,\lambda_1\lambda_2\lambda_3\\
&\quad - 194\,(\lambda_1+\lambda_2+\lambda_3)^2(\lambda_2\lambda_1+\lambda_3\lambda_1+\lambda_2\lambda_3)\,\lambda_1^2\lambda_2^2\lambda_3^2 + 118\,(\lambda_1+\lambda_2+\lambda_3)(\lambda_2\lambda_1+\lambda_3\lambda_1+\lambda_2\lambda_3)^3\,\lambda_1\lambda_2\lambda_3\\
&\quad - 66\,(\lambda_1+\lambda_2+\lambda_3)(\lambda_2\lambda_1+\lambda_3\lambda_1+\lambda_2\lambda_3)^2\,\lambda_1^2\lambda_2^2\lambda_3^2 + 76\,(\lambda_1+\lambda_2+\lambda_3)(\lambda_2\lambda_1+\lambda_3\lambda_1+\lambda_2\lambda_3)\,\lambda_1^3\lambda_2^3\lambda_3^3\\
&\quad - 194\,(\lambda_1+\lambda_2+\lambda_3)(\lambda_2\lambda_1+\lambda_3\lambda_1+\lambda_2\lambda_3)^2\,\lambda_1\lambda_2\lambda_3 + 412\,(\lambda_1+\lambda_2+\lambda_3)(\lambda_2\lambda_1+\lambda_3\lambda_1+\lambda_2\lambda_3)\,\lambda_1^2\lambda_2^2\lambda_3^2\\
&\quad + 20\,(\lambda_1+\lambda_2+\lambda_3)^4(\lambda_2\lambda_1+\lambda_1\lambda_3+\lambda_2\lambda_3)\,\lambda_1\lambda_2\lambda_3 - 36\,(\lambda_1+\lambda_2+\lambda_3)^3(\lambda_2\lambda_1+\lambda_1\lambda_3+\lambda_2\lambda_3)^2\,\lambda_1\lambda_2\lambda_3\\
&\quad + 20\,(\lambda_1+\lambda_2+\lambda_3)^3(\lambda_2\lambda_1+\lambda_1\lambda_3+\lambda_2\lambda_3)\,\lambda_1^2\lambda_2^2\lambda_3^2 - 8\,(\lambda_1+\lambda_2+\lambda_3)^2(\lambda_2\lambda_1+\lambda_1\lambda_3+\lambda_2\lambda_3)^3\,\lambda_1\lambda_2\lambda_3\\
&\quad + 8\,(\lambda_1+\lambda_2+\lambda_3)^2(\lambda_2\lambda_1+\lambda_1\lambda_3+\lambda_2\lambda_3)^2 - 252\,\lambda_1^3\lambda_2^3\lambda_3^3 - 36\,\lambda_1^4\lambda_2^4\lambda_3^4 - 24\,(\lambda_2\lambda_1+\lambda_1\lambda_3+\lambda_2\lambda_3)^5 + 48\,(\lambda_2\lambda_1+\lambda_1\lambda_3+\lambda_2\lambda_3)^4\\
&\quad - 24\,(\lambda_2\lambda_1+\lambda_1\lambda_3+\lambda_2\lambda_3)^3 + 8\,(\lambda_1+\lambda_2+\lambda_3)^4(\lambda_2\lambda_1+\lambda_1\lambda_3+\lambda_2\lambda_3)^2 - 8\,(\lambda_1+\lambda_2+\lambda_3)^3(\lambda_2\lambda_1+\lambda_1\lambda_3+\lambda_2\lambda_3)^3\\
&\quad + 8\,(\lambda_1+\lambda_2+\lambda_3)^2(\lambda_2\lambda_1+\lambda_1\lambda_3+\lambda_2\lambda_3)^4 - 8\,(\lambda_1+\lambda_2+\lambda_3)^3(\lambda_2\lambda_1+\lambda_1\lambda_3+\lambda_2\lambda_3)^2 - 36\,(\lambda_1+\lambda_2+\lambda_3)^2(\lambda_2\lambda_1+\lambda_1\lambda_3+\lambda_2\lambda_3)^3\\
&\quad + 20\,(\lambda_1+\lambda_2+\lambda_3)(\lambda_2\lambda_1+\lambda_1\lambda_3+\lambda_2\lambda_3)^4 + 20\,(\lambda_1+\lambda_2+\lambda_3)(\lambda_2\lambda_1+\lambda_1\lambda_3+\lambda_2\lambda_3)^3 - 36\,\lambda_1^2\lambda_2^2\lambda_3^2 - 24\,(\lambda_1+\lambda_2+\lambda_3)^5\,\lambda_1\lambda_2\lambda_3\\
&\quad + 48\,(\lambda_1+\lambda_2+\lambda_3)^4\,\lambda_1^2\lambda_2^2\lambda_3^2 - 24\,(\lambda_1+\lambda_2+\lambda_3)^3\,\lambda_1^3\lambda_2^3\lambda_3^3 + 24\,(\lambda_1+\lambda_2+\lambda_3)^4\,\lambda_1\lambda_2\lambda_3 - 136\,(\lambda_1+\lambda_2+\lambda_3)^3\,\lambda_1^2\lambda_2^2\lambda_3^2\\
&\quad + 32\,(\lambda_1+\lambda_2+\lambda_3)^2\,\lambda_1^3\lambda_2^3\lambda_3^3 + 24\,(\lambda_2\lambda_1+\lambda_1\lambda_3+\lambda_2\lambda_3)^4\,\lambda_1\lambda_2\lambda_3 - 24\,(\lambda_2\lambda_1+\lambda_1\lambda_3+\lambda_2\lambda_3)^3\,\lambda_1^2\lambda_2^2\lambda_3^2 + 150\,(\lambda_1+\lambda_2+\lambda_3)\,\lambda_1^3\lambda_2^3\lambda_3^3\\
&\quad - 136\,(\lambda_2\lambda_1+\lambda_1\lambda_3+\lambda_2\lambda_3)^3\,\lambda_1\lambda_2\lambda_3 + 10\,(\lambda_2\lambda_1+\lambda_1\lambda_3+\lambda_2\lambda_3)^2\,\lambda_1^2\lambda_2^2\lambda_3^2 - 42\,(\lambda_2\lambda_1+\lambda_1\lambda_3+\lambda_2\lambda_3)\,\lambda_1^3\lambda_2^3\lambda_3^3\\
&\quad - 42\,(\lambda_1+\lambda_2+\lambda_3)\,\lambda_1^2\lambda_2^2\lambda_3^2 + 76\,(\lambda_1+\lambda_2+\lambda_3)(\lambda_2\lambda_1+\lambda_1\lambda_3+\lambda_2\lambda_3)\,\lambda_1\lambda_2\lambda_3 - 66\,(\lambda_1+\lambda_2+\lambda_3)^2(\lambda_2\lambda_1+\lambda_1\lambda_3+\lambda_2\lambda_3)\,\lambda_1\lambda_2\lambda_3,\\
I_{10} &= \lambda_1^2\lambda_2^2\lambda_3^2\,(\lambda_3-1)^2(\lambda_2-1)^2(-\lambda_3+\lambda_2)^2(\lambda_1-1)^2(-\lambda_3+\lambda_1)^2(-\lambda_2+\lambda_1)^2\,.
\end{aligned}
\tag{44}
$$

The components of the rational map $\Phi : \mathcal{M}_2 \backslash \operatorname{supp}(\chi_{35})_0 \to \mathcal{M}_2$ are given by

$$j_1' = \frac{64}{729} \, \frac{g^{(1)}(j_1, j_2, j_3)}{h(j_1, j_2, j_3)}, \quad j_2' = \frac{4}{729} \, \frac{g^{(2)}(j_1, j_2, j_3)}{h(j_1, j_2, j_3)}, \quad j_3' = \frac{1}{729} \, \frac{g^{(3)}(j_1, j_2, j_3)}{h(j_1, j_2, j_3)} \tag{45}$$

with

$$\begin{aligned}
h(j_1, j_2, j_3) = {} & j_1^5 \Big(j_2^4 j_1^3 - 12\, j_1^3 j_2^3 j_3 + 54\, j_1^3 j_2^2 j_3^2 - 108\, j_1^3 j_2 j_3^3 + 81\, j_1^3 j_3^4 + 78\, j_2^5 j_1^2 - 1332\, j_1^2 j_2^4 j_3 + 8910\, j_1^2 j_2^3 j_3^2 - 29376\, j_1^2 j_2^2 j_3^3 + 47952\, j_1^2 j_2 j_3^4 \\
& - 31104\, j_1^2 j_3^5 - 159\, j_1 j_2^6 + 1728\, j_1 j_2^5 j_3 - 6048\, j_1 j_2^4 j_3^2 + 6912\, j_1 j_2^3 j_3^3 + 80\, j_2^7 - 384\, j_2^6 j_3 - 972\, j_1^4 j_2^2 + 5832\, j_1^4 j_2 j_3 - 8748\, j_1^4 j_3^2 - 77436\, j_1^3 j_2^3 \\
& + 870912\, j_1^3 j_2^2 j_3 - 3090960\, j_1^3 j_2 j_3^2 + 3499200\, j_1^3 j_3^3 + 592272\, j_2^4 j_1^2 - 4743360\, j_1^2 j_2^3 j_3 + 9331200\, j_1^2 j_2^2 j_3^2 - 41472\, j_1 j_2^5 + 236196\, j_1^5 \\
& + 19245600\, j_2 j_1^4 - 104976000\, j_1^4 j_3 - 507384000\, j_2^2 j_1^3 + 2099520000\, j_1^3 j_2 j_3 + 125971200000\, j_1^4 \Big), \\
g^{(1)}(j_1, j_2, j_3) = {} & \Big(- j_2^2 j_1 + 6\, j_2 j_3 j_1 - 9\, j_3^2 j_1 + j_2^3 + 540\, j_1^2 \Big)^5, \\
g^{(2)}(j_1, j_2, j_3) = {} & \Big(j_2^4 j_1^2 - 12\, j_1^2 j_2^3 j_3 + 54\, j_1^2 j_2^2 j_3^2 - 108\, j_1^2 j_2 j_3^3 + 81\, j_1^2 j_3^4 - 2\, j_1 j_2^5 + 12\, j_1 j_2^4 j_3 - 18\, j_1 j_2^3 j_3^2 + j_2^6 - 756\, j_2^2 j_1^3 + 4536\, j_1^3 j_2 j_3 - 6804\, j_1^3 j_3^2 \\
& + 5130\, j_1^2 j_2^3 - 17496\, j_1^2 j_2^2 j_3 + 131220\, j_1^4 - 2332800\, j_2 j_1^3 \Big) \Big(- j_2^2 j_1 + 6\, j_2 j_3 j_1 - 9\, j_3^2 j_1 + j_2^3 + 540\, j_1^2 \Big)^3, \\
g^{(3)}(j_1, j_2, j_3) = {} & \Big(- j_1^3 j_2^6 + 18\, j_1^3 j_2^5 j_3 - 135\, j_1^3 j_2^4 j_3^2 + 540\, j_1^3 j_2^3 j_3^3 - 1215\, j_1^3 j_2^2 j_3^4 + 1458\, j_1^3 j_2 j_3^5 - 729\, j_1^3 j_3^6 + 3\, j_1^2 j_2^7 - 36\, j_1^2 j_2^6 j_3 + 162\, j_1^2 j_2^5 j_3^2 - 324\, j_1^2 j_2^4 j_3^3 \\
& + 243\, j_1^2 j_2^3 j_3^4 - 3\, j_1 j_2^8 + 18\, j_1 j_2^7 j_3 - 27\, j_1 j_2^6 j_3^2 + j_2^9 + 1350\, j_1^4 j_2^4 - 16200\, j_1^4 j_2^3 j_3 + 72900\, j_1^4 j_2^2 j_3^2 - 145800\, j_1^4 j_2 j_3^3 + 109350\, j_1^4 j_3^4 - 6345\, j_1^3 j_2^5 \\
& + 52650\, j_1^3 j_2^4 j_3 - 144585\, j_1^3 j_2^3 j_3^2 + 131220\, j_1^3 j_2^2 j_3^3 + 4995\, j_1^2 j_2^6 - 14580\, j_1^2 j_2^5 j_3 - 599724\, j_1^5 j_2^2 + 3598344\, j_1^5 j_2 j_3 - 5397516\, j_1^5 j_3^2 + 4175226\, j_1^4 j_2^3 \\
& - 15390648\, j_1^4 j_2^2 j_3 + 4898880\, j_1^4 j_2 j_3^2 - 1961496\, j_1^3 j_2^4 + 87392520\, j_1^6 - 881798400\, j_1^5 j_2 - 1259712000\, j_1^5 j_3 \Big) \\
& \times \Big(- j_1 j_2^2 + 6\, j_1 j_2 j_3 - 9\, j_1 j_3^2 + j_2^3 + 540\, j_1^2 \Big)^2.
\end{aligned} \tag{46}$$

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