# A UNIVERSAL PAIR OF GENUS-TWO CURVES

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Abstract. Let $\mathfrak{p}$ be any point in the moduli space of genus-two curves $\mathcal{M}_2$ and $K$ its field of moduli. We provide a universal pair of genus-two curves over its minimal field of definition $K[d]$ where $d^2$ is given in terms of $\mathfrak{p}$. Furthermore, $K$ is a field of definition if and only if $d^2$ is a complete square in $K$.

## 1. Introduction

In [5] Igusa described the moduli space $\mathcal{M}_2$ of genus-two curves over a field $k$. This is based on the invariants of the binary sextics, already known to classical invariant theorists. Most of the Igusa's paper focuses on the cases when char $k < 5$. For a much shorter approach in the case of characteristic zero which makes use of geometric invariant theory see [9].

In [11] Mestre introduced a method to construct the equation of a genus-two curve $\mathcal{C}$ from a point in the moduli space. It was already known to Clebsch and Bolza that the Weierstrass points of the genus-two curve were given as the intersection of a conic and a cubic. Mestre determined the equation of $\mathcal{C}$ by parametrizing the conic. However, this is only possible when the conic has a rational point. Replacing such parametrization in the equation of the cubic gives a degree six polynomial whose roots are the projections of the Weierstrass points of $\mathcal{C}$. More precisely, Mestre's method assumed that for any particular point $\mathfrak{p} \in \mathcal{M}_2(\mathbb{Q})$ the corresponding conic has a rational point. When the conic has no rational point, then there is no genus-two curve defined over $\mathbb{Q}$ (i.e., the field of moduli is not a field of definition) and the algorithm does not give an output. Moreover, the case of curves with automorphism group of order bigger than two was not considered. An equation for such cases was provided later independently by Cardona and the second author.

In this paper, for any moduli point $\mathfrak{p} \in \mathcal{M}_2(K)$, where $K$ is the field of definition of $\mathfrak{p}$, we construct a universal genus-two curve $\mathcal{C}$ defined over the minimal field of definition, which we prove is $K[d]$, where $d^2$ is given in terms of the Igusa invariants $I_2, I_4, I_6, I_{10}$ or in terms of the Siegel modular forms. To be precise, we will be constructing a universal pair of genus-two curves corresponding to the two solutions $\pm d$ of the aforementioned modular equation defining $d^2$.

The case when $d^2$ is a complete square in $K$ corresponds exactly to the genus-two curves for which the field of moduli is a field of definition. In contrast to Mestre's approach our equation is valid even when the field of moduli is not a field of definition, and it works for curves with extra automorphisms as well. Our main result is the following:

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**Theorem:** *For every point $\mathfrak{p} \in \mathcal{M}_2$ such that $\mathfrak{p} \in \mathcal{M}_2(k)$, for some number field $K$, there is pair of genus-two curves $\mathcal{C}^{\pm} : y^2 = \sum_{i=0}^{6} a_i^{\pm} x^i$, corresponding to $\mathfrak{p}$, such that $a_i^{\pm} \in K[d]$, $i = 0, \ldots, 6$ as given explicitly in Equation* (42)*. Moreover, $K[d]$ is the minimal field of definition of $\mathfrak{p}$.*

The paper is organized as follows. In Section 2 we give a brief summary of Siegel modular forms, classical invariants of binary sextics and the relations among them, and Igusa functions. While this material can be found in many places in the literature, there is plenty of confusion on the labeling of such invariants and relations among them. We also provide a short description of the Mestre's method and the results that we need in the following sections.

In Section 3 we give a description of the moduli space of curves via the Siegel modular forms. In Section 4 we determine the universal curve of genus two. We chose to express the coefficients of this curve in terms of the Clebsch invariants $A, B, C, D$. However, it is straight forward to convert our results to the global Igusa functions $j_1, j_2, j_3$ and the Siegel modular forms.

## 2. Preliminaries

2.1. **The Siegel modular three-fold.** The Siegel three-fold is a quasi-projective variety of dimension 3 obtained from the Siegel upper half-plane of degree two which by definition is the set of two-by-two symmetric matrices over $\mathbb{C}$ whose imaginary part is positive definite, i.e.,

(1)
$$\mathbb{H}_2 = \left\{ \underline{\tau} = \left( \begin{array}{cc} \tau_1 & z \\ z & \tau_2 \end{array} \right) \middle| \, \tau_1, \tau_2, z \in \mathbb{C}\,,\ \operatorname{Im}(\tau_1) \operatorname{Im}(\tau_2) > \operatorname{Im}(z)^2\,,\ \operatorname{Im}(\tau_2) > 0 \right\}\,,$$

quotiented out by the action of the modular transformations $\Gamma_2 := \operatorname{Sp}_4(\mathbb{Z})$, i.e.,

$$\mathcal{A}_2 = \mathbb{H}_2 / \Gamma_2\,. \tag{2}$$

Each $\underline{\tau} \in \mathbb{H}_2$ determines a principally polarized complex abelian surface

$$\mathbf{A}_{\underline{\tau}} = \mathbb{C}^2 / \langle \mathbb{Z}^2 \oplus \underline{\tau}\, \mathbb{Z}^2 \rangle$$

with period matrix $(\underline{\tau}, \mathbb{I}_2) \in \operatorname{Mat}(2, 4; \mathbb{C})$. Two abelian surfaces $\mathbf{A}_{\underline{\tau}}$ and $\mathbf{A}_{\underline{\tau}'}$ are isomorphic if and only if there is a symplectic matrix

$$M = \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \in \Gamma_2 \tag{3}$$

such that $\underline{\tau}' = M(\underline{\tau}) := (A\underline{\tau}+B)(C\underline{\tau}+D)^{-1}$. It follows that the Siegel three-fold $\mathcal{A}_2$ is also the set of isomorphism classes of principally polarized abelian surfaces. The sets of abelian surfaces that have the same endomorphism ring form sub-varieties of $\mathcal{A}_2$. The endomorphism ring of principally polarized abelian surface tensored with $\mathbb{Q}$ is either a quartic CM field, an indefinite quaternion algebra, a real quadratic field or in the generic case $\mathbb{Q}$. Irreducible components of the corresponding subsets in $\mathcal{A}_2$ have dimensions $0, 1, 2$ and are known as CM points, Shimura curves and Humbert surfaces, respectively.

The Humbert surface $H_\Delta$ with invariant $\Delta$ is the space of principally polarized abelian surfaces admitting a symmetric endomorphism with discriminant $\Delta$. It turns out that $\Delta$ is a positive integer $\equiv 0, 1 \mod 4$. In fact, $H_\Delta$ is the image inside $\mathcal{A}_2$ under the projection of the rational divisor associated to the equation

$$a\,\tau_1 + b\,z + c\,\tau_3 + d\,(z^2 - \tau_1\,\tau_2) + e = 0\,, \tag{4}$$

with integers $a, b, c, d, e$ satisfying $\Delta = b^2 - 4\,a\,c - 4\,d\,e$ and $\underline{\tau} = \left(\begin{smallmatrix} \tau_1 & z \\ z & \tau_2 \end{smallmatrix}\right) \in \mathbb{H}_2$. For example, inside of $\mathcal{A}_2$ sit the Humbert surfaces $H_1$ and $H_4$ that are defined as the images under the projection of the rational divisor associated to $z = 0$ and $\tau_1 = \tau_2$, respectively. In fact, the singular locus of $\mathcal{A}_2$ has $H_1$ and $H_4$ as its two connected components. As analytic spaces, the surfaces $H_1$ and $H_4$ are each isomorphic to the Hilbert modular surface

$$\Big( (\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})) \rtimes \mathbb{Z}_2 \Big) \backslash \Big( \mathbb{H} \times \mathbb{H} \Big) \,. \tag{5}$$

For a more detailed introduction to Siegel modular form, Humbert surfaces, and the Satake compactification of the Siegel modular threefold we refer to Freitag's book [3].

2.2. **Siegel modular forms.** In general, we can define the Eisenstein series $\psi_{2k}$ of degree $g$ and weight $2k$ (where we assume $2k > g + 1$ for convergence) by setting

$$\psi_{2k}(\underline{\tau}) = \sum_{(C,D)} \det\, (C \cdot \underline{\tau} + D)^{-2k} \,, \tag{6}$$

where the sum runs over non-associated bottom rows $(C, D)$ of elements in $\mathrm{Sp}_{2g}(\mathbb{Z})$ where non-associated means with respect to the multiplication by $\mathrm{GL}(g, \mathbb{Z})$. In the following, we will always assume $g = 2$ in the definition of $\psi_{2k}$. Using Igusa's definition [6, Sec. 8, p. 195] we define a cusp form of weight 10 by

$$\chi_{10}(\underline{\tau}) = -\frac{43867}{2^{12}\, 3^5\, 5^2\, 7 \cdot 53} \left( \psi_4(\underline{\tau})\, \psi_6(\underline{\tau}) - \psi_{10}(\underline{\tau}) \right) \,. \tag{7}$$

It is well known that the vanishing divisor of the cusp form $\chi_{10}$ is the Humbert surface $H_1$ because a period point $\underline{\tau}$ is equivalent to a point with $z = 0$ if and only if $\chi_{10}\big(\underline{\tau}\big) = 0$. Based on Igusa's definition [6, Sec. 8, p. 195] and the work in [10] we define a second cusp form $\chi_{12}$ of weight 12 by

$$\chi_{12}(\underline{\tau}) = \frac{131 \cdot 593}{2^{13}\, 3^7\, 5^3\, 7^2\, 337} \left( 3^2\, 7^2\, \psi_4^3(\underline{\tau}) + 2 \cdot 5^3\, \psi_6^2(\underline{\tau}) - 691\, \psi_{12}(\underline{\tau}) \right) \,. \tag{8}$$

Igusa proved [7, 8] that the ring of Siegel modular forms is generated by $\psi_4$, $\psi_6$, $\chi_{10}$, $\chi_{12}$ and by one more cusp form $\chi_{35}$ of odd weight 35 whose square is the following polynomial [7, p. 849] in the even generators

(9)

$$\begin{aligned}
\chi_{35}^2 = \frac{1}{2^{12}\, 3^9}\, \chi_{10} \Big( & 2^{24}\, 3^{15}\, \chi_{12}^5 - 2^{13}\, 3^9\, \psi_4^3\, \chi_{12}^4 - 2^{13}\, 3^9\, \psi_6^2\, \chi_{12}^4 + 3^3\, \psi_4^6\, \chi_{12}^3 \\
& - 2 \cdot 3^3\, \psi_4^3\, \psi_6^2\, \chi_{12}^3 - 2^{14}\, 3^8\, \psi_4^2\, \psi_6\, \chi_{10}\, \chi_{12}^3 - 2^{23}\, 3^{12}\, 5^2\, \psi_4\, \chi_{10}^2\, \chi_{12}^3 + 3^3\, \psi_6^4\, \chi_{12}^3 \\
& + 2^{11}\, 3^6\, 37\, \psi_4^4\, \chi_{10}^2\, \chi_{12}^2 + 2^{11}\, 3^6\, 5 \cdot 7\, \psi_4\, \psi_6^2\, \chi_{10}^2\, \chi_{12}^2 - 2^{23}\, 3^9\, 5^3\, \psi_6\, \chi_{10}^3\, \chi_{12}^2 \\
& - 3^2\, \psi_4^7\, \chi_{10}^2\, \chi_{12} + 2 \cdot 3^2\, \psi_4^4\, \psi_6^2\, \chi_{10}^2\, \chi_{12} + 2^{11}\, 3^5\, 5 \cdot 19\, \psi_4^3\, \psi_6\, \chi_{10}^3\, \chi_{12} \\
& + 2^{20}\, 3^8\, 5^3\, 11\, \psi_4^2\, \chi_{10}^4\, \chi_{12} - 3^2\, \psi_4\, \psi_6^4\, \chi_{10}^2\, \chi_{12} + 2^{11}\, 3^5\, 5^2\, \psi_6^3\, \chi_{10}^3\, \chi_{12} - 2\, \psi_4^6\, \psi_6\, \chi_{10}^3 \\
& - 2^{12}\, 3^4\, \psi_4^5\, \chi_{10}^4 + 2^2\, \psi_4^3\, \psi_6^3\, \chi_{10}^3 + 2^{12}\, 3^4\, 5^2\, \psi_4^2\, \psi_6^2\, \chi_{10}^4 + 2^{21}\, 3^7\, 5^4\, \psi_4\, \psi_6\, \chi_{10}^5 \\
& - 2\, \psi_6^5\, \chi_{10}^3 + 2^{32}\, 3^9\, 5^5\, \chi_{10}^6 \Big) \,.
\end{aligned}$$

Hence, the expression $Q := 2^{12}\, 3^9\, \chi_{35}^2 / \chi_{10}$ is a polynomial of degree 60 in the even generators. The following fact is well-known [4]:

**Proposition 1.** *The vanishing divisor of $Q$ is the Humbert surface $H_4$, i.e., a period point $\underline{\tau}$ is equivalent to a point with $\tau_1 = \tau_2$ if and only if $Q\big(\underline{\tau}\big) = 0$. Accordingly, the vanishing divisor of $\chi_{35}$ is the formal sum $H_1 + H_4$ of Humbert surfaces, that constitutes the singular locus of $\mathcal{A}_2$.*

In accordance with Igusa [6, Thm. 3] we also introduce the following ratios of Siegel modular forms

$$\mathbf{x}_1 = \frac{\psi_4\,\chi_{10}^2}{\chi_{12}^2}, \quad \mathbf{x}_2 = \frac{\psi_6\,\chi_{10}^3}{\chi_{12}^3}, \quad \mathbf{x}_3 = \frac{\chi_{10}^6}{\chi_{12}^5}\ , \tag{10}$$

as well as

$$\mathbf{y}_1 = \frac{\mathbf{x}_1^3}{\mathbf{x}_3} = \frac{\psi_4^3}{\chi_{12}}, \quad \mathbf{y}_2 = \frac{\mathbf{x}_2^2}{\mathbf{x}_3} = \frac{\psi_6^2}{\chi_{12}}, \quad \mathbf{y}_3 = \frac{\mathbf{x}_1^2\,\mathbf{x}_2}{\mathbf{x}_3} = \frac{\psi_4^2\,\psi_6\,\chi_{10}}{\chi_{12}}\ , \tag{11}$$

where we have suppressed the dependence of each Siegel modular form on $\underline{\tau}$. These ratios have the following asymptotic expansion as $z \to 0$ [6, pp. 180–182] in terms of ordinary Eisenstein series

$$\begin{aligned}
\mathbf{x}_1 &= E_4(\tau_1)\ E_4(\tau_2)\,(\pi z)^4 + O(z^5)\ ,\\
\mathbf{x}_2 &= E_6(\tau_1)\ E_6(\tau_2)\,(\pi z)^6 + O(z^7)\ ,\\
\mathbf{x}_3 &= \eta^{24}(\tau_1)\ \eta^{24}(\tau_2)\,(\pi z)^{12} + O(z^{13})\ ,
\end{aligned} \tag{12}$$

and

$$\begin{aligned}
\mathbf{y}_1 &= j(\tau_1)\,j(\tau_2) + O(z^2)\ ,\\
\mathbf{y}_2 &= \Big(1728 - j(\tau_1)\Big)\,\Big(1728 - j(\tau_2)\Big) + O(z^2)\ ,\\
\mathbf{y}_3 &= \frac{E_4^2(\tau_1)\,E_4^2(\tau_2)\,E_6(\tau_1)\,E_6(\tau_2)}{\eta^{24}(\tau_1)\,\eta^{24}(\tau_2)}\,(\pi z)^2 + O(z^3)\ ,
\end{aligned} \tag{13}$$

where we have set

$$\begin{aligned}
j(\tau_j) &= \frac{1728\,E_4^3(\tau_j)}{E_4^3(\tau_j) - E_6^2(\tau_j)} = \frac{E_4^3(\tau_j)}{\eta^{24}(\tau_j)}\ ,\\
1728 - j(\tau_j) &= \frac{1728\,E_6^2(\tau_j)}{E_4^3(\tau_j) - E_6^2(\tau_j)} = \frac{E_6^2(\tau_j)}{\eta^{24}(\tau_j)}\ .
\end{aligned} \tag{14}$$

2.3. **Sextics and Igusa invariants.** We write the equation defining a genus-two curve $\mathcal{C}$ by a degree-six polynomial or sextic in the form

$$\mathcal{C}:\ y^2 = f(x) = a_0\,\prod_{i=1}^{6}(x - \alpha_i) = \sum_{i=0}^{6} a_i\,x^{6-i}\ . \tag{15}$$

The roots $\{\alpha_i\}_{i=1}^6$ of the sextic are the six ramification points of the map $\mathcal{C} \to \mathbb{P}^1$. Their pre-images on $\mathcal{C}$ are the six Weierstrass points. The isomorphism class of $f$ consists of all equivalent sextics where two sextics are considered equivalent if there is a linear transformation in $\mathrm{GL}(2,\mathbb{C})$ which takes the set of roots to the roots of the other.

The ring of invariants of binary sextics is generated by the Igusa invariants $(I_2, I_4, I_6, I_{10})$ as defined in [9, Eq. (9)], which are the same invariants as the ones denoted by $(A', B', C', D')$ in [11, p.319] and also the same invariants as $(A, B, C, D)$ in [6, p.176]. For expressions of such invariants in terms of the coefficients $a_0, \ldots, a_6$ of the binary sextic, or $I_k \in \mathbb{Z}[a_0, \ldots, a_6]$ for $k \in \{2, 4, 6, 10\}$; see [9, Eq. (11)].

Thus, the invariants of a sextic define a point in a weighted projective space $[I_2 : I_4 : I_6 : I_{10}] \in \mathbb{WP}^3_{(2,4,6,10)}$. It was shown in [6] that points in the projective variety Proj $\mathbb{C}[I_2, I_4, I_6, I_{10}]$ which are not on $I_{10} = 0$ form the variety $\mathcal{U}_6$ of moduli of sextics. Equivalently, points in the weighted projective space $\{[I_2 : I_4 : I_6 : I_{10}] \in \mathbb{WP}^3_{(2,4,6,10)} : I_{10} \neq 0\}$ are in one-to-one correspondence with isomorphism classes of sextics.

Often the *Clebsch invariants* $(A, B, C, D)$ of a sextic are used as well. They are defined in terms of the transvectants of the binary sextics; see [2] for details. Clebsch invariants $(A, B, C, D)$ are related to the Igusa-Clebsch invariants by the equations

$$
\begin{aligned}
I_2 &= -\,120\,A\,, \\
I_4 &= -\,720\,A^2 + 6750\,B\,, \\
I_6 &= \phantom{-}\,8640\,A^3 - 108000\,A\,B + 202500\,C\,, \\
I_{10} &= -\,62208\,A^5 + 972000\,A^3\,B + 1620000\,A^2\,C \\
&\quad -\,3037500\,A\,B^2 - 6075000\,B\,C - 4556250\,D\,.
\end{aligned} \tag{16}
$$

Conversely, the invariants $(A, B, C, D)$ are polynomial expressions in the Igusa invariants $(I_2, I_4, I_6, I_{10})$ with rational coefficients:

$$
\begin{aligned}
A &= -\,\frac{1}{2^3 3\cdot 5}\,I_2\,, \\
B &= \phantom{-}\,\frac{1}{2^2 3^3 5^3}\left(I_2^2 + 20\,I_4\right)\,, \\
C &= -\,\frac{1}{2^5 3^5 5^6}\left(I_2^3 + 60\,I_2 I_4 - 600\,I_6\right)\,, \\
D &= -\,\frac{1}{2^8 3^9 5^{10}}\big(9\,I_2^5 + 700\,I_2^3 I_4 - 3600\,I_2^2 I_6 \\
&\qquad -12400\,I_2 I_4^2 + 48000\,I_4\,I_6 + 10800000\,I_{10}\big)\,.
\end{aligned} \tag{17}
$$

2.4. **Igusa functions.** The *Igusa functions* or absolute invariants (i.e., $\mathrm{GL}(2,\mathbb{C})$-invariants) are defined as

$$
j_1 = \frac{I_2^5}{I_{10}}, \quad j_2 = \frac{I_4 I_2^3}{I_{10}}, \quad j_3 = \frac{I_6 I_2^2}{I_{10}}
$$

Notice that other sets of absolute invariants have been used as well in the literature. In this paper we will use these invariants since they are defined everywhere on $\mathcal{M}_2$.

2.5. **Recovering the equation of the curve from invariants.** As above, let $\mathfrak{p} \in \mathcal{M}_2$ and $\mathcal{C}$ a genus-two curve corresponding to $\mathfrak{p}$ defined by a sextic polynomial $f$. Then, $\mathrm{Aut}(\mathfrak{p})$ is a finite group as described in [12]. The quotient space $\mathcal{C}/\mathrm{Aut}(\mathfrak{p})$ is a genus zero curve and therefore isomorphic to a conic. Since conics are in one to one correspondence with three-by-three symmetric matrices (up to equivalence), then let $M = [A_{ij}]$ be the symmetric matrix corresponding to this conic.

Let $\mathbf{X} = [X_1 : X_2 : X_3] \in \mathbb{P}^2$ and

$$
\mathcal{Q} : \qquad \mathbf{X}^t \cdot M \cdot \mathbf{X} = \sum_{i,j=1}^{3} A_{ij}\,X_i X_j = 0
$$

be the conic corresponding to $M$. Clebsch [2] determined the entries of this matrix $M$ as follows

$$
\begin{aligned}
A_{11} &= 2\,C + \frac{1}{3}\,A\,B\,, \\
A_{22} &= A_{13} = \,D\,, \\
A_{33} &= \frac{1}{2}\,B\,D + \frac{2}{9}\,C\,(B^2 + A\,C)\,, \\
A_{23} &= \frac{1}{3}\,B\,(B^2 + A\,C) + \frac{1}{3}\,C\,(2\,C + \frac{1}{3}\,A\,B)\,, \\
A_{12} &= \frac{2}{3}\,(B^2 + A\,C)\,.
\end{aligned}
\tag{18}
$$

One can obtain from the sextic $f$ three binary quadrics $\mathsf{y}_i(x)$ with $i = 1, 2, 3$ by an operation called 'Überschiebung' [11, p. 317]. The quadrics $\mathsf{y}_i$ for $i = 1, 2, 3$ have the property that their coefficients are polynomial expressions in the coefficients of $f$ with rational coefficients. Moreover, under the operation $f(x) \mapsto \tilde{f}(x) = f(-x)$ the quadrics change according to $\mathsf{y}_i(x) \mapsto \tilde{\mathsf{y}}_i(x) = \mathsf{y}_i(-x)$ for $i = 1, 2, 3$. Hence, they are not invariants of the sextic $f$. The coefficients $A_{ij}$ in Equations (18) satisfy $A_{ij} = (\mathsf{y}_i\mathsf{y}_j)_2$.[1] Therefore, the coefficients $A_{ij}$ are invariant under the operation $f(x) \mapsto \tilde{f}(x) = f(-x)$, and the locus $D = 0$ is equivalent to

$$
D = 0 \quad \Leftrightarrow \quad (\mathsf{y}_1\mathsf{y}_3)_2 = (\mathsf{y}_2\mathsf{y}_2)_2 = 0\,. \tag{19}
$$

We define $R$ to be $1/2$ times the determinant of the three binary quadrics $\mathsf{y}_i$ for $i = 1, 2, 3$ with respect to the basis $x^2, x, 1$. If one extends the operation of Überschiebung by product rule [11, p. 317], then $R$ can be re-written as

$$
R = -(\mathsf{y}_1\mathsf{y}_2)_1\,(\mathsf{y}_2\mathsf{y}_3)_1\,(\mathsf{y}_3\mathsf{y}_1)_1, \tag{20}
$$

or, equivalently, as

$$
\begin{aligned}
R \;=\; -\frac{1}{2}\Big(&\mathsf{y}_{1,yy}\,\mathsf{y}_{2,xy}\,\mathsf{y}_{3,xx} - \mathsf{y}_{1,yy}\,\mathsf{y}_{2,xx}\,\mathsf{y}_{3,xy} - \mathsf{y}_{1,xy}\,\mathsf{y}_{2,yy}\,\mathsf{y}_{3,xx} \\
&+ \mathsf{y}_{1,xy}\,\mathsf{y}_{2,xx}\,\mathsf{y}_{3,yy} + \mathsf{y}_{1,xx}\,\mathsf{y}_{2,yy}\,\mathsf{y}_{3,xy} - \mathsf{y}_{1,xx}\,\mathsf{y}_{2,xy}\,\mathsf{y}_{3,yy}\Big)\,.
\end{aligned}
\tag{21}
$$

It is then obvious that under the operation $f(x) \mapsto \tilde{f}(x) = f(-x)$ the determinant $R$ changes its sign, i.e., $R(f) \mapsto R(\tilde{f}) = -R(f)$. A straightforward calculation shows that

$$
R^2 = \frac{1}{2}\begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{12} & A_{22} & A_{23} \\ A_{13} & A_{23} & A_{33} \end{vmatrix}\,, \tag{22}
$$

where $A_{ij}$ are the Clebsch invariants in Equations (18). Like the coefficients $A_{ij}$, $R^2$ is invariant under the operation $f(x) \mapsto \tilde{f}(x) = f(-x)$ and must be a polynomial in $(I_2, I_4, I_6, I_{10})$. Bolza [1] described the possible automorphism groups of genus-two curves defined by sextics. In particular, he provided effective criteria for the cases when the automorphism group of the sextic curve in Equation (15) is nontrivial. For a detailed discussion of the automorphism groups of genus-two curve defined over any field $k$ and the corresponding loci in $\mathcal{M}_2$ see [12].

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[1] For two binary forms $f, g$ of degree $m$ and $n$, respectively, we denote the Überschiebung of order $k$ by $(fg)_k = (-1)^k(gf)_k$. For $\tilde{f}(x) = f(-x)$ and $\tilde{g}(x) = g(-x)$ and $m = n = k$, we have $(fg)_m = (-1)^m(\tilde{f}\tilde{g})_m$.

We have the following lemma summarizing our discussion:

**Lemma 2.** *We have the following statements:*

(1) *$R^2$ is a order 30 invariant of binary sextics expressed as a polynomial in $(I_2, I_4, I_6, I_{10})$ as in* [12, Eq. (17)] *given by plugging Equations (17) and (18) into Equation (22).*

(2) *The locus of curves $\mathfrak{p} \in \mathcal{M}_2$ such that $V_4 \hookrightarrow Aut(\mathfrak{p})$ is a two-dimensional irreducible rational subvariety of $\mathcal{M}_2$ given by the equation $R^2 = 0$ and a birational parametrization given by the $u, v$-invariants as in* [12, Thm. 1].

Notice that the Igusa invariants $I_2, I_4, I_6, I_{10}$ are denoted by $J_2, \dots, J_{10}$ in [12, Eq. (17)]. From now on we will denote $I_{30} := R^2$. The relation between all aforementioned invariants and Siegel modular forms, in particular the relation between $\chi_{35}$ and $R^2$, will be given in the next section.

2.5.1. *The cubic curve.* In the previous chapter we introduced the invariant $R^2$ for any binary sextic given by $f$. To the corresponding symmetric matrix $M$ with $R^2 = \det M/2$ and the coefficients $A_{ij} = (\mathsf{y}_i\mathsf{y}_j)_2$ of order zero invariant under the operation $f(x) \mapsto \tilde{f}(x) = f(-x)$, we associated a conic $\mathcal{Q}$. Similarly, there is also a cubic curve given by the equation

$$\mathcal{T}: \qquad \sum_{1\leq i,j,k\leq 3} a_{ijk}\, X_i X_j X_k = 0\,,$$

where the coefficients $a_{ijk}$ are of order zero and invariant under $f(x) \mapsto \tilde{f}(x) = f(-x)$. In terms of Überschiebung the coefficients are given by

$$a_{ijk} = (f\mathsf{y}_i)_2\,(f\mathsf{y}_j)_2\,(f\mathsf{y}_k)_2\;. \tag{23}$$

The coefficients $a_{ijk}$ are given explicitly as follows:

$$\begin{aligned}
36\,a_{111} &= 8(A^2C - 6BC + 9D),\\
36\,a_{112} &= 4(2B^3 + 4ABC + 12C^2 + 3AD),\\
36\,a_{113} &= 36\,a_{122} = 4(AB^3 + 4/3\,A^2BC + 4B^2C + 6AC^2 + 3BD),\\
36\,a_{123} &= 2(2B^4 + 4AB^2C + 4/3\,A^2C^2 + 4BC^2 + 3ABD + 12CD),\\
36\,a_{133} &= 2\,\big(AB^4 + 4/3\,A^2B^2C + 16/3\,B^3C\\
&\quad +26/3\,ABC^2 + 8C^3 + 3B^2D + 2ACD\big)\,,\\
36\,a_{222} &= 4(3B^4 + 6AB^2C + 8/3\,A^2C^2 + 2BC^2 - 3CD),\\
36\,a_{223} &= 2(-2/3\,B^3C - 4/3\,ABC^2 - 4C^3 + 9B^2D + 8ACD),\\
36\,a_{233} &= 2(B^5 + 2AB^3C + 8/9\,A^2BC^2 + 2/3\,B^2C^2 - BCD + 9D^2),\\
36\,a_{333} &= -2B^4C - 4AB^2C^2 - 16/9\,A^2C^3 - 4/3\,BC^3\\
&\quad + 9B^3D + 12ABCD + 20C^2D.
\end{aligned} \tag{24}$$

The intersection of the conic $\mathcal{Q}$ with the cubic $\mathcal{T}$ consists of six points which are the zeroes of a polynomial $f(t)$ of degree 6 in the parameter $t$. The roots of this polynomial are the images of the Weierstrass points under the hyperelliptic projection. Hence, the affine equation of the genus-two curve corresponding to $\mathfrak{p}$ is given by $y^2 = f(t)$.

Mestre's method is briefly described as follows: if the conic $\mathcal{Q}$ has a rational point over $\mathbb{Q}$, then this leads to a parametrization of $\mathcal{Q}$, say $(h_1(t), h_2(t), h_3(t))$. Substitute $X_1, X_2, X_3$ by $h_1(t), h_2(t), h_3(t)$ in the cubic $\mathcal{T}$ and we get the degree 6 polynomial $f(t)$. However, if the conic has no rational point or $R^2 = \det M = 0$ the method obviously fails. In section 4 we determine such intersection $\mathcal{T} \cap \mathcal{Q}$ over a quadratic extension which is always possible. Therefore, our approach works in all cases and provides an equation for the genus-two curve over its minimal field of definition (cf. Section 4).

## 3. The moduli space of genus-two curves

Suppose that $\mathcal{C}$ is an irreducible projective non-singular curve. If the self-intersection is $\mathcal{C} \cdot \mathcal{C} = 2$ then $\mathcal{C}$ is a curve of genus two. For every curve $\mathcal{C}$ of genus two there exists a unique pair $(\mathrm{Jac}(\mathcal{C}), j_{\mathcal{C}})$ where $\mathrm{Jac}(\mathcal{C})$ is an abelian surface, called the Jacobian variety of the curve $\mathcal{C}$, and $j_{\mathcal{C}} : \mathcal{C} \to \mathrm{Jac}(\mathcal{C})$ is an embedding. One can always regain $\mathcal{C}$ from the pair $(\mathrm{Jac}(\mathcal{C}), \mathcal{P})$ where $\mathcal{P} = [\mathcal{C}]$ is the class of $\mathcal{C}$ in the Néron-Severi group $\mathrm{NS}(\mathrm{Jac}(\mathcal{C}))$. Thus, if $\mathcal{C}$ is a genus-two curve, then $\mathrm{Jac}(\mathcal{C})$ is a principally polarized abelian surface with principal polarization $\mathcal{P} = [\mathcal{C}]$, and the map sending a curve $\mathcal{C}$ to its Jacobian variety $\mathrm{Jac}(\mathcal{C})$ is injective. In this way, the variety of moduli of curves of genus two is also the moduli space of their Jacobian varieties with canonical polarization. Furthermore, Torelli's theorem states that the map sending a curve $\mathcal{C}$ to its Jacobian variety $\mathrm{Jac}(\mathcal{C})$ induces a birational map from the moduli space $\mathcal{M}_2$ of genus-two curves to the complement of the Humbert surface $H_1$ in $\mathcal{A}_2$, i.e., $\mathcal{A}_2 - \mathrm{supp}(\chi_{10})_0$.

One can then ask what the Igusa-Clebsch invariants of a genus-two curve $\mathcal{C}$ defined by a sextic curve $f$ are in terms of $\underline{\tau}$ such that $(\underline{\tau}, \mathbb{I}_2) \in \mathrm{Mat}(2,4;\mathbb{C})$ is the period matrix of the principally polarized abelian surface $\mathbf{A}_{\underline{\tau}} = \mathrm{Jac}(\mathcal{C})$. Based on the asymptotic behavior in Equations (12) and (13), Igusa [7, p. 848] proved that the relations are as follows:

$$
\begin{aligned}
I_2 &= -2^3 \cdot 3 \, \frac{\chi_{12}(\underline{\tau})}{\chi_{10}(\underline{\tau})} \,, \\
I_4 &= \phantom{-} 2^2 \, \psi_4(\underline{\tau}) \,, \\
I_6 &= -\frac{2^3}{3} \, \psi_6(\underline{\tau}) - 2^5 \, \frac{\psi_4(\underline{\tau}) \, \chi_{12}(\underline{\tau})}{\chi_{10}(\underline{\tau})} \,, \\
I_{10} &= -2^{14} \, \chi_{10}(\underline{\tau}) \,.
\end{aligned} \tag{25}
$$

Thus, we find that the point $[I_2 : I_4 : I_6 : I_{10}]$ in weighted projective space equals

$$
\Big[2^3 \, 3 \, (3r\chi_{12}) \,:\, 2^2 3^2 \, \psi_4 \, (r\chi_{10})^2 \,:\, 2^3 \, 3^2 \Big(4\psi_4 \, (3r\chi_{12}) + \psi_6 \, (r\chi_{10})\Big) \, (r\chi_{10})^2 : 2^2 \, (r\chi_{10})^6\Big] \tag{26}
$$

with $r \neq 0$. Substituting (25) into Equations (16), (18) it also follows that

$$
\begin{aligned}
I_{30} = R^2 &= -2^6 \, 3^{-27} \, 5^{-20} \, \frac{Q\Big(\psi_4(\underline{\tau}), \psi_6(\underline{\tau}), \chi_{10}(\underline{\tau}), \chi_{12}(\underline{\tau})\Big)}{\chi_{10}(\underline{\tau})^3} \\
&= -\left(2^9 \, 3^{-9} \, 5^{-10} \, \frac{\chi_{35}(\underline{\tau})}{\chi_{10}(\underline{\tau})^2}\right)^2 \,,
\end{aligned} \tag{27}
$$

where $Q$ and $R^2$ where defined in Equation (9) and (22), respectively. In terms of Igusa invariants we have

$$\chi_{35}^2 = -\frac{3^{18}\,5^{20}}{2^{74}}\,I_{10}^4\,I_{30}. \tag{28}$$

For $I_2 \neq 0$ we use the variables $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ from Equations (10) to write

(29)
$$\Big[I_2 : I_4 : I_6 : I_{10}\Big] = \left[1 : \frac{1}{2^4\,3^2}\,\mathbf{x}_1 : \frac{1}{2^6\,3^4}\,\mathbf{x}_2 + \frac{1}{2^4\,3^3}\,\mathbf{x}_1 : \frac{1}{2\cdot 3^5}\,\mathbf{x}_3\right] \in \mathbb{WP}^3_{(2,4,6,10)}\,.$$

We summarize the above discussion as follows:

**Remark 3.**

(1) *The modulus point $\underline{\tau}$ is equivalent to a point with $\tau_1 = \tau_2$ or $[\underline{\tau}] \in H_4 \subset \mathcal{A}_2$ such that the corresponding sextic curve has an extra automorphism if and only if $Q(\underline{\tau}) = 0$.*

(2) *The modulus point $\underline{\tau}$ is equivalent to a point with $z = 0$ or $[\underline{\tau}] \in H_1 \subset \mathcal{A}_2$ such that the principally polarized abelian surface is a product of two elliptic curves $\mathbf{A}_{\underline{\tau}} = E_{\tau_1} \times E_{\tau_2}$ if and only if $\chi_{10}(\underline{\tau}) = 0$. The elliptic modular parameters are then determined by Equations (29) and (12).*

Since the invariants $I_4, I_6, I_{10}$ vanish simultaneously at sextics with triple roots all such abelian surfaces are mapped to $[1:0:0:0] \in \mathbb{WP}^3_{(2,4,6,10)}$ with uniformizing affine coordinates $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ around it. Blowing up this point gives a variety that parameterizes genus-two curves with $I_2 \neq 0$ and their degenerations. In the blow-up space we have to introduce additional coordinates that are obtained as ratios of $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ and have weight zero. Those are precisely the coordinates $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3$ already introduced in Equation (11). It turns out that the coordinate ring of the blown-up space is $\mathbb{C}[\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3]$.

If a Jacobian variety corresponds to a product of elliptic curves then $\underline{\tau}$ is equivalent to a point with $z = 0$, i.e., $\underline{\tau}$ is located on the Humbert surface $H_1$. We then have $\chi_{10}(\underline{\tau}) = 0, \chi_{12}(\underline{\tau}) \neq 0$ and $[I_2 : I_4 : I_6 : I_{10}] = [1:0:0:0]$. Equations (12) and (13) imply $\mathbf{x}_1 = \mathbf{x}_2 = \mathbf{x}_3 = \mathbf{y}_3 = 0$ and $\mathbf{y}_1 = j(\tau_1)\,j(\tau_2)$ and $\mathbf{y}_2 = (1728 - j(\tau_1))\,(1728 - j(\tau_2))$.

## 4. A universal genus-two curve from the moduli space

We start with a point $p \in \mathcal{M}_2$. This point can be given by $\mathfrak{p} = [j_1, j_2, j_3, 1]$. The goal of this section is to explicitly determine the equation of a genus-two curve over its minimal field of definition corresponding to this point $\mathfrak{p}$.

Associated to the symmetric matrix $M$ whose determinant is given by $R^2$ in Equation (22) is the plane conic $\mathcal{Q}$ in the variables $[X_1 : X_2 : X_3] \in \mathbb{P}^2$ given by

$$A_{11}\,X_1^2 + A_{22}\,X_2^2 + A_{33}\,X_3^2 + 2A_{12}\,X_1X_2 + 2A_{13}\,X_1X_3 + 2A_{23}\,X_2X_3 = 0\,. \tag{30}$$

Over the field extension $\mathbb{Q}[d]$ where $d$ is given by

$$d^2 = -2\,A_{2,2}\,R^2 \tag{31}$$

we can re-express $A_{1,1}$ in terms of $d$ and the other coefficients as

$$A_{1,1} = -\frac{d^2}{(A_{2,2}A_{3,3} - A_{2,3}^2)\,A_{2,2}} + \frac{A_{1,2}^2A_{3,3} - 2A_{1,2}A_{1,3}A_{2,3} + A_{1,3}^2A_{2,2}}{A_{2,2}A_{3,3} - A_{2,3}^2}\,. \tag{32}$$

We can then find a rational point of the conic $\mathcal{Q}$ over $\mathbb{Q}[d,A,B,C,D]$. In fact, a rational point $[X_1^{(0)} : X_2^{(0)} : X_3^{(0)}]$ on the conic $\mathcal{Q}$ is given by

$$
\begin{aligned}
X_1^{(0)} &= A_{2,2}(A_{2,2}A_{3,3} - A_{2,3}^2)\;, \\
X_2^{(0)} &= \mp A_{2,3}d - A_{2,2}\big(A_{1,2}A_{3,3} - A_{1,3}A_{2,3}\big)\;, \\
X_3^{(0)} &= A_{2,2}\;\left(\pm d + A_{1,2}A_{2,3} - A_{1,3}A_{2,2}\right)\;.
\end{aligned}
\tag{33}
$$

We substitute

$$
X_3^{(0)}X_2 = X_2^{(0)}X_3 + t\left(X_3^{(0)}X_1 - X_1^{(0)}X_3\right)
\tag{34}
$$

into Equation (30). We already know one of the roots of this quadratic, since it must be satisfied if $[X_1 : X_2 : X_3] = [X_1^{(0)} : X_2^{(0)} : X_3^{(0)}]$. The second root is given by

$$
\begin{aligned}
X_1 =& A_{2,2}^3(A_{2,2}A_{3,3} - A_{2,3}^2)^2\, t^2 + 2A_{1,2}A_{2,2}^2(A_{2,2}A_{3,3} - A_{2,3}^2)\, t \\
&+ A_{2,2}(A_{2,2}A_{3,3} - A_{2,3}^2)\left(A_{1,2}^2A_{2,2}A_{3,3} - 2A_{1,2}A_{1,3}A_{2,2}A_{2,3}\right. \\
&\quad\left. + A_{1,3}^2A_{2,2}^2 \pm 2(A_{1,2}A_{2,3} - A_{1,3}A_{2,2})\, d + d^2\right), \\
X_2 =& -A_{2,2}^2(A_{2,2}A_{3,3} - A_{2,3}^2)(A_{1,2}A_{2,2}A_{3,3} - A_{1,3}A_{2,2}A_{2,3} \pm A_{2,3}d)\, t^2 \\
&+ 2A_{2,2}(A_{2,2}A_{3,3} - A_{2,3}^2) \\
&\quad \times(-A_{1,2}^2A_{2,2}A_{3,3} + A_{1,2}A_{1,3}A_{2,2}A_{2,3} \mp A_{1,3}A_{2,2}d + d^2)\, t \\
&+ (A_{1,2}A_{2,2}A_{3,3} - A_{1,3}A_{2,2}A_{2,3} \pm A_{2,3}d) \\
&\quad \times(-A_{1,2}^2A_{2,2}A_{3,3} + 2A_{1,2}A_{1,3}A_{2,2}A_{2,3} - A_{1,3}^2A_{2,2}^2 + d^2), \\
X_3 =& A_{2,2}^3(A_{2,2}A_{3,3} - A_{2,3}^2)(A_{1,2}A_{2,3} - A_{1,3}A_{2,2} \pm d)\, t^2 \\
&+ 2A_{1,2}A_{2,2}^2(A_{2,2}A_{3,3} - A_{2,3}^2)(A_{1,2}A_{2,3} - A_{1,3}A_{2,2} \pm d)\, t \\
&- A_{2,2}(A_{1,2}A_{2,3} - A_{1,3}A_{2,2} \pm d) \\
&\quad \times(-A_{1,2}^2A_{2,2}A_{3,3} + 2A_{1,2}A_{1,3}A_{2,2}A_{2,3} - A_{1,3}^2A_{2,2}^2 + d^2).
\end{aligned}
\tag{35}
$$

Using $A_{1,3} = A_{2,2}$ and $d^2 = -2\,A_{2,2}\,R^2$ the point $[X_1 : X_2 : X_3]$ is easily shown to be equivalent to

$$
\begin{aligned}
X_1 &= A_{2,2}^2(A_{2,2}A_{3,3} - A_{2,3}^2)^2\, t^2 + 2A_{1,2}A_{2,2}(A_{2,2}A_{3,3} - A_{2,3}^2)\, t \\
&- A_{1,1}A_{2,2}^2A_{3,3} + A_{1,1}A_{2,2}A_{2,3}^2 + 2A_{1,2}^2A_{2,2}A_{3,3} - 4A_{1,2}A_{2,2}^2A_{2,3} \\
&\pm 2\,(A_{1,2}A_{2,3} - A_{2,2})\, d, \\
X_2 &= -A_{2,2}(A_{1,2}A_{2,2}A_{3,3} - A_{2,2}^2A_{2,3} \pm A_{2,3}d)\, t^2 \\
&- 2A_{2,2}(A_{1,1}A_{2,2}A_{3,3} - A_{1,1}A_{2,3}^2 + A_{1,2}A_{2,2}A_{2,3} - A_{2,2}^3 \pm A_{2,2}d)\, t \\
&- A_{1,1}(A_{1,2}A_{2,2}A_{3,3} - A_{2,2}^2A_{2,3} \pm A_{2,3}d), \\
X_3 &= A_{2,2}^2(A_{1,2}A_{2,3} - A_{2,2}^2 \pm d)\, t^2 + 2A_{1,2}A_{2,2}(A_{1,2}A_{2,3} - A_{2,2}^2 \pm d)\, t \\
&+ A_{1,1}A_{2,2}(A_{1,2}A_{2,3} - A_{2,2} \pm d).
\end{aligned}
\tag{36}
$$

Equations (36) give for any $t \in \mathbb{Q}$ a rational parametrization of the conic $\mathcal{Q}$ over $\mathbb{Q}[d,A,B,C,D]$.

Similarly, associated to the coefficients $(a_{ijk})$ in Equation (24) is a plane cubic curve $\mathcal{T}$ in the variables $[X_1 : X_2 : X_3] \in \mathbb{P}^3$ given by

$$\begin{aligned} & a_{111}\, X_1^3 + a_{222}\, X_2^3 + a_{333}\, X_3^3 + 6\, a_{123}\, X_1X_2X_3 \\ & + 3\, a_{112}\, X_1^2X_2 + 3\, a_{113}\, X_1^2X_3 + 3\, a_{122}\, X_1X_2^2 + 3\, a_{223}\, X_2^2X_3 \\ & + 3\, a_{133}\, X_1X_3^2 + 3\, a_{233}\, X_2X_3^2 = 0\ . \end{aligned} \tag{37}$$

Plugging the rational parametrization of the conic $\mathcal{Q}$ from Equations (36) into the cubic $\mathcal{T}$ in Equation (37), one obtains the ramificattion locus of sextic curve. The ramification locus is equivalent to

$$0 = \sum_{i=0}^{6} \underbrace{18^{-\lfloor \frac{i+1}{2} \rfloor}\, \kappa_i\, \left( \delta_i\, (54D)^{\lfloor \frac{i+1}{2} \rfloor} \pm 54 \cdot 3^{\lfloor \frac{(i-3)^2}{2} \rfloor - 3 \lfloor \frac{(i-3)^2}{5} \rfloor}\, \epsilon_i\, (54D)^{\lfloor \frac{i}{2} \rfloor}\, d \right)}_{=:\, a^{\pm}_{6-i}} t^i\ , \tag{38}$$

where $\delta_i, \epsilon_i$ are irreducible polynomials in $\mathbb{Z}[A, B, C, D]$ and $\kappa_i = 1, 12, 15B$, 360, 15, 12, 1 for $i = 0, \ldots, 6$ such that $a^{\pm}_{6-i} \in \mathbb{Q}[d, A, B, C, D]$.

$(A, B, C, D)$ are given as polynomial in terms of the invariants $(I_2, I_4, I_6, I_{10})$ in Equations (17). Thus, we can express all coefficients of the sextic as polynomials in $\mathbb{Q}[d, I_2, I_4, I_6, I_{10}]$, and we have

$$d^2 = \frac{I_{30}}{2^7\, 3^9\, 5^{10}}\, (9I_2^5 + 700I_2^3I_4 - 3600I_2^2I_6 - 12400I_2I_4^2 + 48000I_4I_6 + 10800000I_{10}). \tag{39}$$

Notice that $d^2$ has two significant factors: one is $I_{30}$ which correspond exactly to the locus of the curves with extra involutions, and the other one is the Clebsch invariant $D$. Next we have our main result:

**Theorem 4.** *For every point $\mathfrak{p} \in \mathcal{M}_2$ such that $\mathfrak{p} \in \mathcal{M}_2(k)$, for some number field $K$, there is a pair of genus-two curves $\mathcal{C}^{\pm}$ given by*

$$\mathcal{C}^{\pm} : \quad y^2 = \sum_{i=0}^{6} a^{\pm}_{6-i}\, x^i\ ,$$

*corresponding to $\mathfrak{p}$, such that $a_i^{\pm} \in K(d)$, $i = 0, \ldots, 6$ as given explicitly in Equation (42). Moreover, $K(d)$ is the minimal field of definition of $\mathfrak{p}$.*

*Proof.* From the above discussion we know that there is a genus-two curve $\mathcal{C} : y^2 = f(t)$ corresponding to $\mathfrak{p}$, where $f(t)$ is given in Equation (38). The coefficients $a_i^{\pm} \in \mathbb{Q}[d, I_2, I_4, I_6, I_{10}]$ as displayed in Equation (42). The field of moduli $K$ of the point $\mathfrak{p}$ is $K = \mathbb{Q}(j_1, j_2, j_3)$. Therefore, $a_i^{\pm} \in K\,[d]$, for $i = 0, \ldots, 6$. This completes the proof.

□

Computing expressions for $a_i^{\pm} \in K[d] = \mathbb{Q}(j_1, j_2, j_3)[d]$ is straightforward, but we choose not to display them here.

**Corollary 5.** *Let $j_1, j_2, j_3$ be transcendentals. There exists a pair of genus-two curves $\mathcal{C}^{\pm}$ defined over $\mathbb{Q}(j_1, j_2, j_3)[d]$ such that*

$$j_1(\mathcal{C}^{\pm}) = j_1, \quad j_2(\mathcal{C}^{\pm}) = j_2, \quad j_3(\mathcal{C}^{\pm}) = j_3,$$

*where $d^2$ is given in terms of $(j_1, j_2, j_3)$ in Equation (41).*

The result in Theorem 4 answers completely the question; for what curves the field of moduli is a field of definition. The answer is exactly for curves such that $d$ is a complete square in the field of moduli $K$. Obviously, when the curve has extra involutions, then $I_{30} = 0$ and therefore the field of moduli is a field of definition.

**Corollary 6.** *Let $D$ be the Clebsch invariant and $\mathfrak{p} \in \mathcal{M}_2$. If $D(\mathfrak{p}) = 0$ then there is a genus-two curve $\mathcal{C}$ defined over the field of moduli.*

**Remark 7.** *The locus $D = 0$ is given in terms of Siegel modular forms by*

$$\begin{aligned} 0 = &250\, \chi_{10}^5 \psi_4 \psi_6 + 675\, \chi_{10}^4 \chi_{12} \psi_4^2 + 86400000\, \chi_{10}^6 - 2700\, \chi_{10}^3 \chi_{12}^2 \psi_6 \\ &- 13500\, \chi_{10}^2 \chi_{12}^3 \psi_4 + 34992\, \chi_{12}^5 , \end{aligned}$$

*and in terms of the absolute invariants $(j_1, j_2, j_3)$ by*

$$0 = 9\, j_1^2 + 700\, j_2 j_1 - 3600\, j_3 j_1 - 12400\, j_2^2 + 48000\, j_2 j_3 + 10800000\, j_1 \; .$$

*Therefore, the locus $D = 0$ is parametrized by $j_1, j_2 \in \mathbb{C}$ and $j_3 = h(j_1, j_2)$ with*

$$h(j_1, j_2) = 3000 + \frac{j_1}{400} + \frac{41\, j_2}{180} - \frac{j_2}{9} \cdot \frac{11\, j_2 - 1080000}{3\, j_1 - 40 j_2} \; .$$

Notice that $d = 0$ is equivalent to $D \cdot \chi_{35} = 0$. Hence, we have the following observation:

**Corollary 8.** *Let $\mathfrak{p} \in \mathcal{M}_2$. If*

$$\begin{aligned} \chi_{35}\, \big( &250\, \chi_{10}^5 \psi_4 \psi_6 + 675\, \chi_{10}^4 \chi_{12} \psi_4^2 + 86400000\, \chi_{10}^6 - 2700\, \chi_{10}^3 \chi_{12}^2 \psi_6 \\ &-13500\, \chi_{10}^2 \chi_{12}^3 \psi_4 + 34992\, \chi_{12}^5 \big) = 0, \end{aligned}$$

*then the field of moduli of $\mathfrak{p}$ is also a field of definition.*

The above theorem gives two twists of the curve $\mathcal{C}$ which have moduli point $\mathfrak{p}$. Whether such equation can be modified so that it can be minimal is unknown.

## 5. Appendix

In Equations (42) we will display the polynomials $\delta_i, \epsilon_i$ for $i = 0, \dots, 6$ that determine the pair of genus-two curves $\mathcal{C}^{\pm}$ given by

$$y^2 = \sum_{i=0}^{6} a_{6-i}^{\pm}\, x^i = \sum_{i=0}^{6} 18^{-\lfloor \frac{i+1}{2} \rfloor}\, \kappa_i \left( \delta_i\, (54D)^{\lfloor \frac{i+1}{2} \rfloor} \pm 54 \cdot 3^{\lfloor \frac{(i-3)^2}{2} \rfloor - 3 \lfloor \frac{(i-3)^2}{5} \rfloor}\, \epsilon_i\, (54D)^{\lfloor \frac{i}{2} \rfloor}\, d \right) x^i \tag{40}$$

where $(A, B, C, D)$ are the Clebsch invariants and $\kappa_i = 1, 12, 15B, 360, 15, 12, 1$ for $i = 0, \dots, 6$. The Clebsch invariants are given as polynomial in terms of $(I_2, I_4, I_6, I_{10})$ in Equations (17). The square $d^2$ is given in terms of $(j_1, j_2, j_3)$ by

$$\begin{aligned}
d^2 = \frac{1}{2^{22} 3^{36} 5^{30}\, j_1^9} \Big( & j_2^4 j_1^3 - 12\, j_2^3 j_3 j_1^3 + 54\, j_2^2 j_3^2 j_1^3 - 108\, j_2 j_3^3 j_1^3 + 81\, j_3^4 j_1^3 + 78\, j_2^5 j_1^2 - 1332\, j_2^4 j_3 j_1^2 + 8910\, j_2^3 j_3^2 j_1^2 - 29376\, j_2^2 j_3^3 j_1^2 + 47952\, j_2 j_3^4 j_1^2 - 31104\, j_3^5 j_1^2 \\
& - 159\, j_2^6 j_1 + 1728\, j_2^5 j_3 j_1 - 6048\, j_2^4 j_3^2 j_1 + 6912\, j_2^3 j_3^3 j_1 + 80\, j_2^7 - 384\, j_2^6 j_3 - 972\, j_2^2 j_1^4 + 5832\, j_2 j_3 j_1^4 - 8748\, j_3^2 j_1^4 - 77436\, j_2^3 j_1^3 + 870912\, j_2^2 j_3 j_1^3 \\
& - 3090960\, j_2 j_3^2 j_1^3 + 3499200\, j_3^3 j_1^3 + 592272\, j_2^4 j_1^2 - 4743360\, j_2^3 j_3 j_1^2 + 9331200\, j_2^2 j_3^2 j_1^2 - 41472\, j_2^5 j_1 + 236196\, j_1^5 + 19245600\, j_2 j_1^4 - 104976000\, j_3 j_1^4 \\
& - 507384000\, j_2^2 j_1^3 + 2099520000\, j_2 j_3 j_1^3 + 125971200000\, j_1^4 \Big) \Big( 9\, j_1^2 + 700\, j_2 j_1 - 3600\, j_3 j_1 - 12400\, j_2^2 + 48000\, j_2 j_3 + 10800000\, j_1 \Big).
\end{aligned} \tag{41}$$

The irreducible polynomials $\delta_i, \epsilon_i$ in $\mathbb{Z}[A, B, C, D]$ for $i = 0, \dots, 6$ are given by

$$\begin{aligned}
\delta_6 = {} & -2048\, A^5 B^3 C^5 - 9216\, A^4 B^5 C^4 - 16128\, A^3 B^7 C^3 - 13824\, A^2 B^9 C^2 - 5832\, A B^{11} C - 972\, B^{13} - 9216\, A^4 B^2 C^6 - 36096\, A^3 B^4 C^5 \\
& - 50112\, A^2 B^6 C^4 - 29808\, A B^8 C^3 - 6480\, B^{10} C^2 - 6912\, A^4 B C^5 D - 35712\, A^3 B^3 C^4 D - 13824\, C^7 A^3 B - 55728\, A^2 B^5 C^3 D - 48384\, C^6 A^2 B^3 \\
& - 34992\, A B^7 C^2 D - 49248\, C^5 A B^5 - 7776\, B^9 C D - 15552\, C^4 B^7 + 25920\, A^3 B^2 C^3 D^2 - 10368\, A^3 C^6 D + 54432\, A^2 B^4 C^2 D^2 - 90720\, A^2 B^2 C^5 D \\
& - 6912\, C^8 A^2 + 37908\, A B^6 C D^2 - 114048\, A B^4 C^4 D - 25920\, C^7 A B^2 + 8748\, B^8 D^2 - 38880\, B^6 C^3 D - 15552\, C^6 B^4 + 136080\, A^2 B C^4 D^2 \\
& + 208008\, A B^3 C^3 D^2 - 108864\, A B C^6 D + 79704\, B^5 C^2 D^2 - 77760\, B^3 C^5 D - 5184\, C^8 B + 7776\, A^2 C^3 D^3 + 46656\, A B^2 C^2 D^3 + 139968\, A C^5 D^2 \\
& + 34992\, B^4 C D^3 + 116640\, B^2 C^4 D^2 - 62208\, C^7 D - 52488\, A B C D^4 - 19683\, B^3 D^4 + 23328\, B C^3 D^3 - 139968\, C^2 D^4, \\
\epsilon_6 = {} & -128\, A^3 B^2 C^3 - 288\, A^2 B^4 C^2 - 216\, A B^6 C - 54\, B^8 - 384\, C^4 A^2 B - 576\, C^3 A B^3 - 216\, B^5 C^2 - 48\, A^2 C^3 D + 108\, A B^2 C^2 D \\
& - 288\, C^5 A + 108\, B^4 C D - 216\, B^2 C^4 + 324\, A B C D^2 + 243\, B^3 D^2 + 288\, B C^3 D + 864\, C^2 D^2, \\
\delta_5 = {} & 1024\, A^5 B^4 C^4 + 3072\, A^4 B^6 C^3 + 3456\, A^3 B^8 C^2 + 1728\, A^2 B^{10} C + 324\, A B^{12} + 8064\, A^4 B^3 C^5 + 20352\, A^3 B^5 C^4 + 18648\, A^2 B^7 C^3 \\
& + 7236\, A B^9 C^2 + 972\, C B^{11} - 6912\, A^4 B^2 C^4 D - 25056\, A^3 B^4 C^3 D + 20736\, C^6 A^3 B^2 - 32832\, A^2 B^6 C^2 D + 38880\, C^5 A^2 B^4 - 18630\, A B^8 C D \\
& + 23544\, C^4 A B^6 - 3888\, B^{10} D + 4536\, C^3 B^8 - 5184\, A^3 B^3 C^2 D^2 - 17712\, A^3 B C^5 D - 7776\, A^2 B^5 C D^2 - 56376\, A^2 B^3 C^4 D + 21600\, C^7 A^2 B
\end{aligned} \tag{42}$$

$$
\begin{aligned}
& - 2916\, AB^7D^2 - 53784\, AB^5C^3D + 24624\, C^6AB^3 - 16092\, B^7C^2D + 6480\, C^5B^5 - 3888\, A^3C^4D^2 - 46656\, A^2B^2C^3D^2 - 11664\, A^2C^6D \\
& - 51030\, AB^4C^2D^2 - 40824\, AB^2C^5D + 7776\, C^8A - 13608\, B^6CD^2 - 23328\, B^4C^4D + 2592\, C^7B^2 + 48600\, A^2BC^2D^3 + 83835\, AB^3CD^3 \\
& - 78732\, ABC^4D^2 + 34992\, B^5D^3 - 49572\, B^3C^3D^2 - 11664\, BC^6D + 6561\, AB^2D^4 + 96228\, AC^3D^3 + 97686\, B^2C^2D^3 - 52488\, C^5D^2 \\
& + 41553\, BCD^4 - 78732\, D^5, \\
\epsilon_5 &= 128\, A^3B^4C^3 + 288\, A^2B^6C^2 + 216\, AB^8C + 54\, B^{10} + 576\, A^2B^3C^4 + 864\, AB^5C^3 + 324\, B^7C^2 + 1296\, A^2B^2C^3D + 2052\, AB^4C^2D \\
& + 864\, C^5AB^2 + 810\, B^6CD + 648\, C^4B^4 - 3456\, A^2BC^2D^2 - 5508\, AB^3CD^2 + 3240\, ABC^4D - 2187\, B^5D^2 + 2592\, B^3C^3D + 432\, C^6B \\
& - 4860\, AC^3D^2 - 4050\, B^2C^2D^2 + 1944\, C^5D - 243\, BCD^3 + 8748\, D^4, \\
\delta_4 &= 2048\, A^6B^4C^5 + 7168\, A^5B^6C^4 + 9984\, A^4B^8C^3 + 6912\, A^3B^{10}C^2 + 2376\, A^2B^{12}C + 324\, AB^{14} + 15360\, A^5B^3C^6 + 40704\, A^4B^5C^5 \\
& + 38208\, A^3B^7C^4 + 13392\, A^2B^9C^3 - 648\, B^{13}C - 20736\, A^5B^2C^5D - 93312\, A^4B^4C^4D + 41472\, A^4C^7B^2 - 167184\, A^3B^6C^3D \\
& + 66816\, A^3C^6B^4 - 149040\, A^2B^8C^2D + 16416\, A^2C^5B^6 - 66096\, AB^{10}CD - 18144\, AB^8C^4 - 11664\, B^{12}D - 7776\, B^{10}C^3 - 5184\, A^4B^3C^3D^2 \\
& - 51840\, A^4BC^6D - 12960\, A^3B^5C^2D^2 - 194400\, A^3B^3C^5D + 48384\, A^3BC^8 - 10692\, A^2B^7CD^2 - 272160\, A^2B^5C^4D + 8640\, A^2B^3C^7 \\
& - 2916\, AB^9D^2 - 168480\, AB^7C^3D - 62208\, AB^5C^6 - 38880\, B^9C^2D - 31104\, B^7C^5 - 10368\, A^4C^5D^2 - 81648\, A^3B^2C^4D^2 - 31104\, A^3C^7D \\
& - 141912\, A^2B^4C^3D^2 - 93312\, A^2B^2C^6D + 20736\, C^9A^2 - 88128\, AB^6C^2D^2 - 93312\, AB^4C^5D - 57024\, C^8AB^2 - 17496\, B^8CD^2 - 31104\, B^6C^4D \\
& - 51840\, C^7B^4 + 132192\, A^3BC^3D^3 + 369360\, A^2B^3C^2D^3 - 139968\, A^2BC^5D^2 + 344088\, AB^5CD^3 - 221616\, AB^3C^4D^2 + 104976\, B^7D^3 \\
& - 81648\, B^5C^3D^2 - 31104\, C^9B + 256608\, A^2C^4D^3 + 6561\, AB^4D^4 + 501552\, AB^2C^3D^3 - 139968\, AC^6D^2 + 268272\, B^4C^2D^3 - 139968\, B^2C^5D^2 \\
& + 52488\, ABC^2D^4 + 91854\, B^3CD^4 - 69984\, BC^4D^3 - 209952\, ACD^5 - 236196\, B^2D^5, \\
\epsilon_4 &= -128\, A^4B^3C^3 - 288\, A^3B^5C^2 - 216\, A^2B^7C - 54\, AB^9 + 576\, A^2B^4C^3 + 864\, AB^6C^2 + 324\, B^8C - 1584\, A^3BC^3D - 3924\, A^2B^3C^2D \\
& + 864\, A^2BC^5 - 3348\, AB^5CD + 2376\, AB^3C^4 - 972\, B^7D + 1296\, B^5C^3 + 324\, A^2B^2CD^2 - 2160\, A^2C^4D + 243\, AB^4D^2 - 2808\, AB^2C^3D \\
& + 864\, C^6A - 1080\, B^4C^2D + 1296\, C^5B^2 + 972\, ABC^2D^2 + 486\, B^3CD^2 + 1296\, BC^4D + 3888\, ACD^3 + 4374\, B^2D^3, \\
\delta_3 &= -512\, A^6B^2C^6 - 3456\, A^5B^4C^5 - 8864\, A^4B^6C^4 - 11464\, A^3B^8C^3 - 8028\, A^2B^{10}C^2 - 2916\, AB^{12}C - 432\, B^{14} + 384\, A^5B^3C^4D \\
& - 1536\, A^5BC^7 + 864\, A^4B^5C^3D - 11904\, A^4B^3C^6 + 648\, A^3B^7C^2D - 28320\, A^3B^5C^5 + 162\, A^2B^9CD - 29976\, A^2B^7C^4 - 14832\, AB^9C^3 \\
& - 2808\, B^{11}C^2 - 192\, A^5C^6D + 3024\, A^4B^2C^5D - 1152\, A^4C^8 + 6120\, A^3B^4C^4D - 16416\, A^3B^2C^7 + 3768\, A^2B^6C^3D - 35856\, A^2B^4C^6 \\
& + 720\, AB^8C^2D - 27936\, AB^6C^5 - 7344\, B^8C^4 + 2592\, A^4BC^4D^2 + 15120\, A^3B^3C^3D^2 + 10224\, A^3BC^6D + 25434\, A^2B^5C^2D^2 + 12312\, A^2B^3C^5D \\
& - 12960\, A^2BC^8 + 16848\, AB^7CD^2 + 3024\, AB^5C^4D - 22464\, AB^3C^7 + 3888\, B^9D^2 - 360\, B^7C^3D - 9504\, B^5C^6 - 972\, A^3B^2C^2D^3
\end{aligned}
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\begin{aligned}
& + 6048\, A^3C^5D^2 - 729\, A^2B^4CD^3 + 44388\, A^2B^2C^4D^2 + 7776\, A^2C^7D + 58968\, AB^4C^3D^2 - 2592\, AB^2C^6D - 5184\, AC^9 + 21600\, B^6C^2D^2 \\
& - 5184\, B^4C^5D - 5184\, B^2C^8 - 14580\, A^2BC^3D^3 - 15552\, AB^3C^2D^3 + 34992\, ABC^5D^2 - 3888\, B^5CD^3 + 29160\, B^3C^4D^2 - 7776\, BC^7D \\
& - 3888\, A^2C^2D^4 - 16767\, AB^2CD^4 - 29160\, AC^4D^3 - 8748\, B^4D^4 - 20412\, B^2C^3D^3 + 11664\, C^6D^2 - 24786\, BC^2D^4 + 17496\, CD^5, \\
\epsilon_3 &= -512\, A^5B^3C^4 - 1536\, A^4B^5C^3 - 1728\, A^3B^7C^2 - 864\, A^2B^9C - 162\, AB^{11} - 2304\, A^4C^5B^2 - 4416\, A^3C^4B^4 - 2160\, A^2C^3B^6 + 324\, AC^2B^8 \\
& + 324\, CB^{10} - 1728\, A^4BC^4D - 8064\, A^3B^3C^3D - 3456\, A^3BC^6 - 12636\, A^2B^5C^2D - 1728\, A^2B^3C^5 - 8262\, AB^7CD + 3240\, AB^5C^4 - 1944\, B^9D \\
& + 1944\, B^7C^3 + 1296\, A^3B^2C^2D^2 - 2592\, A^3C^5D + 1944\, A^2B^4CD^2 - 14904\, A^2B^2C^4D - 1728\, C^7A^2 + 729\, AB^6D^2 - 18792\, AB^4C^3D \\
& + 3888\, C^6AB^2 - 6804\, B^6C^2D + 3888\, C^5B^4 + 12636\, A^2BC^3D^2 + 17982\, AB^3C^2D^2 - 9720\, ABC^5D + 6318\, B^5CD^2 - 7776\, B^3C^4D + 2592\, BC^7 \\
& + 3888\, A^2C^2D^3 + 15309\, AB^2CD^3 + 17496\, AC^4D^2 + 8748\, B^4D^3 + 14580\, B^2C^3D^2 - 3888\, C^6D + 16038\, BC^2D^3 - 17496\, CD^4, \\
\delta_2 &= 6144\, A^7B^4C^5 + 23552\, A^6B^6C^4 + 36096\, A^5B^8C^3 + 27648\, A^4B^{10}C^2 + 10584\, A^3B^{12}C + 1620\, A^2B^{14} + 39936\, A^6C^6B^3 + 80640\, A^5B^5C^5 \\
& - 17856\, A^4B^7C^4 - 168432\, A^3B^9C^3 - 168048\, A^2B^{11}C^2 - 68688\, AB^{13}C - 10368\, B^{15} - 34560\, A^6B^2C^5D - 114048\, A^5B^4C^4D + 96768\, A^5C^7B^2 \\
& - 140400\, A^4B^6C^3D - 76032\, A^4B^4C^6 - 76464\, A^3B^8C^2D - 733536\, A^3B^6C^5 - 15552\, A^2B^{10}CD - 981504\, A^2B^8C^4 - 515808\, AB^{10}C^3 \\
& - 97200\, B^{12}C^2 - 36288\, A^5B^3C^3D^2 - 93312\, A^5BC^6D - 80352\, A^4B^5C^2D^2 + 75168\, A^4B^3C^5D + 103680\, A^4C^8B - 59292\, A^3B^7CD^2 \\
& + 803520\, A^3B^5C^4D - 675648\, A^3B^3C^7 - 14580\, A^2B^9D^2 + 1143072\, A^2B^7C^3D - 1829952\, A^2B^5C^6 + 632448\, AB^9C^2D - 1425600\, AB^7C^5 \\
& + 124416\, B^{11}CD - 357696\, B^9C^4 - 20736\, A^5C^5D^2 - 423792\, A^4B^2C^4D^2 - 62208\, A^4C^7D - 740664\, A^3B^4C^3D^2 + 1041984\, A^3B^2C^6D \\
& + 41472\, A^3C^9 - 434808\, A^2B^6C^2D^2 + 2799360\, A^2B^4C^5D - 1104192\, A^2B^2C^8 - 81648\, AB^8CD^2 + 2317248\, AB^6C^4D - 1710720\, AB^4C^7 \\
& + 622080\, B^8C^3D - 642816\, B^6C^6 + 256608\, A^4BC^3D^3 + 412128\, A^3B^3C^2D^3 - 855360\, A^3BC^5D^2 + 163296\, A^2B^5CD^3 - 746496\, A^2B^3C^4D^2 \\
& + 1399680\, A^2BC^7D + 128304\, AB^5C^3D^2 + 2426112\, AB^3C^6D - 746496\, ABC^9 + 159408\, B^7C^2D^2 + 1026432\, B^5C^5D - 559872\, B^3C^8 \\
& + 52488\, A^3B^2CD^4 + 419904\, A^3C^4D^3 + 32805\, A^2B^4D^4 - 1539648\, A^2B^2C^3D^3 - 699840\, A^2C^6D^2 - 2939328\, AB^4C^2D^3 + 489888\, AB^2C^5D^2 \\
& + 559872\, AC^8D - 1119744\, B^6CD^3 + 629856\, B^4C^4D^2 + 559872\, B^2C^7D - 186624\, C^{10} + 1364688\, A^2BC^2D^4 + 1758348\, AB^3CD^4 \\
& - 4199040\, ABC^4D^3 + 629856\, B^5D^4 - 3359232\, B^3C^3D^3 + 839808\, BC^6D^2 - 419904\, A^2CD^5 + 2519424\, AC^3D^4 + 1810836\, B^2C^2D^4 \\
& - 1679616\, C^5D^3 + 2519424\, BCD^5 - 1889568\, D^6, \\
\epsilon_2 &= 128\, A^5B^3C^3 + 288\, A^4B^5C^2 + 216\, A^3B^7C + 54\, A^2B^9 + 1152\, A^4C^4B^2 + 2112\, A^3C^3B^4 + 1224\, A^2C^2B^6 + 216\, ACB^8 - 3024\, A^4BC^3D \\
& - 5868\, A^3B^3C^2D + 2592\, A^3C^5B - 3564\, A^2B^5CD + 1944\, A^2C^4B^3 - 648\, AB^7D - 864\, AC^3B^5 - 648\, B^7C^2 - 324\, A^3B^2CD^2 - 4320\, A^3C^4D \\
& - 243\, A^2B^4D^2 + 11664\, A^2B^2C^3D + 1728\, A^2C^6 + 24624\, AB^4C^2D - 2592\, AC^5B^2 + 9936\, B^6CD - 2592\, B^4C^4 - 13608\, A^2BC^2D^2
\end{aligned}
$$

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\begin{aligned}
& - 20412\, AB^3CD^2 + 34992\, ABC^4D - 7776\, B^5D^2 + 27216\, B^3C^3D - 2592\, BC^6 + 7776\, A^2CD^3 + 2916\, AB^2D^3 - 23328\, AC^3D^2 - 20412\, B^2C^2D^2 \\
& + 15552\, C^5D - 29160\, BCD^3 + 34992\, D^4, \\
\delta_1 = {} & -1024\, A^7B^6C^4 - 3072\, A^6B^8C^3 - 3456\, A^5B^{10}C^2 - 1728\, A^4B^{12}C - 324\, A^3B^{14} + 3072\, A^7C^6B^3 + 31104\, A^6B^5C^5 + 120384\, A^5B^7C^4 \\
& + 209496\, A^4B^9C^3 + 181764\, A^3B^{11}C^2 + 77436\, A^2B^{13}C + 12960\, B^{15}A + 41472\, A^6B^4C^4D + 82944\, A^6C^7B^2 + 130464\, A^5B^6C^3D \\
& + 705024\, A^5C^6B^4 + 153360\, A^4B^8C^2D + 2112480\, A^4C^5B^6 + 79866\, A^3B^{10}CD + 2975832\, A^3B^8C^4 + 15552\, A^2B^{12}D + 2151144\, A^2B^{10}C^3 \\
& + 772416\, AB^{12}C^2 + 108864\, B^{14}C + 38016\, A^6BC^6D + 5184\, A^5B^5C^2D^2 + 352080\, A^5B^3C^5D + 228096\, A^5C^8B + 7776\, A^4B^7CD^2 \\
& + 456840\, A^4B^5C^4D + 2178144\, A^4C^7B^3 + 2916\, A^3B^9D^2 - 246456\, A^3B^7C^3D + 5903280\, A^3B^5C^6 - 771876\, A^2B^9C^2D + 6724944\, A^2B^7C^5 \\
& - 474336\, AB^{11}CD + 3446064\, AB^9C^4 - 93312\, B^{13}D + 657072\, B^{11}C^3 - 260496\, A^5B^2C^4D^2 + 82944\, A^5C^7D - 972000\, A^4B^4C^3D^2 \\
& + 382320\, A^4B^2C^6D + 165888\, A^4C^9 - 1404054\, A^3B^6C^2D^2 - 1235736\, A^3B^4C^5D + 2744928\, A^3C^8B^2 - 896184\, A^2B^8CD^2 - 3742848\, A^2B^6C^4D \\
& + 6757344\, A^2B^4C^7 - 209952\, AB^{10}D^2 - 2978856\, AB^8C^3D + 5632416\, AB^6C^6 - 751680\, B^{10}C^2D + 1531872\, B^8C^5 - 229392\, A^4B^3C^2D^3 \\
& - 1492992\, A^4BC^5D^2 - 359397\, A^3B^5CD^3 - 5794092\, A^3B^3C^4D^2 - 497664\, A^3BC^7D - 139968\, A^2B^7D^3 - 7986924\, A^2B^5C^3D^2 \\
& - 4210704\, A^2B^3C^6D + 1866240\, A^2BC^9 - 4537296\, AB^7C^2D^2 - 5668704\, AB^5C^5D + 3779136\, AB^3C^8 - 886464\, B^9CD^2 - 2072304\, B^7C^4D \\
& + 1726272\, B^5C^7 - 139968\, A^4C^4D^3 - 6561\, A^3B^4D^4 - 708588\, A^3B^2C^3D^3 - 1819584\, A^3C^6D^2 + 1010394\, A^2B^4C^2D^3 - 8275608\, A^2B^2C^5D^2 \\
& - 419904\, A^2C^8D + 2239488\, AB^6CD^3 - 9710280\, AB^4C^4D^2 - 2729376\, AB^2C^7D + 653184\, C^{10}A + 839808\, B^8D^3 - 3289248\, B^6C^3D^2 \\
& - 1959552\, B^4C^6D + 839808\, C^9B^2 + 1399680\, A^3BC^2D^4 + 2464749\, A^2B^3CD^4 + 1994544\, A^2BC^4D^3 + 1102248\, AB^5D^4 + 7269588\, AB^3C^3D^3 \\
& - 4304016\, ABC^6D^2 + 4199040\, B^5C^2D^3 - 3464208\, B^3C^5D^2 - 419904\, BC^8D + 314928\, A^2B^2D^5 + 3569184\, A^2C^3D^4 + 5616216\, AB^2C^2D^4 \\
& + 3044304\, AC^5D^3 + 1784592\, B^4CD^4 + 5196312\, B^2C^4D^3 - 1889568\, C^7D^2 - 472392\, ABCD^5 - 1889568\, B^3D^5 + 1495908\, BC^3D^4 \\
& - 1889568\, AD^6 - 2834352\, C^2D^5, \\
\epsilon_1 = {} & -1024\, A^6B^4C^4 - 3200\, A^5B^6C^3 - 3744\, A^4B^8C^2 - 1944\, A^3B^{10}C - 378\, A^2B^{12} + 1536\, A^5C^5B^3 + 22848\, A^4C^4B^5 + 58944\, A^3B^7C^3 \\
& + 61668\, B^9C^2A^2 + 29160\, AB^{11}C + 5184\, B^{13} + 10368\, A^5B^2C^4D + 40176\, A^4B^4C^3D + 20736\, A^4C^6B^2 + 56484\, A^3B^6C^2D + 124704\, A^3C^5B^4 \\
& + 34506\, A^2B^8CD + 211464\, A^2C^4B^6 + 7776\, AB^{10}D + 141264\, AB^8C^3 + 33048\, B^{10}C^2 + 9504\, A^4B^3C^2D^2 + 57024\, A^4BC^5D + 14580\, A^3B^5CD^2 \\
& + 220968\, A^3B^3C^4D + 38016\, A^3C^7B + 5589\, A^2B^7D^2 + 291600\, A^2B^5C^3D + 183600\, A^2C^6B^3 + 160056\, AB^7C^2D + 215136\, AC^5B^5 + 31104\, B^9CD \\
& + 73872\, B^7C^4 + 5184\, A^4C^4D^2 - 32076\, A^3B^2C^3D^2 + 62208\, A^3C^6D - 199746\, A^2B^4C^2D^2 + 274104\, A^2B^2C^5D + 20736\, C^8A^2 - 223236\, AB^6CD^2 \\
& + 291600\, AB^4C^4D + 98496\, C^7AB^2 - 69984\, B^8D^2 + 91368\, B^6C^3D + 64800\, C^6B^4 - 108864\, A^3BC^2D^3 - 190755\, A^2B^3CD^3 - 244944\, A^2BC^4D^2
\end{aligned}
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\begin{aligned}
&- 81648\, AB^5D^3 - 612360\, AB^3C^3D^2 + 139968\, ABC^6D - 312012\, B^5C^2D^2 + 93312\, B^3C^5D + 15552\, C^8B - 17496\, A^2B^2D^4 - 256608\, A^2C^3D^3 \\
&- 352836\, AB^2C^2D^3 - 221616\, AC^5D^2 - 93312\, B^4CD^3 - 332424\, B^2C^4D^2 + 69984\, C^7D + 157464\, ABCD^4 \\
&+ 209952\, B^3D^4 - 8748\, BC^3D^3 + 209952\, AD^5 + 314928\, C^2D^4, \\
\delta_0 &= 10240\, A^8B^6C^5 + 39936\, A^7B^8C^4 + 62208\, A^6B^{10}C^3 + 48384\, A^5B^{12}C^2 + 18792\, A^4B^{14}C + 2916\, A^3B^{16} + 98304\, A^8B^3C^7 + 580608\, A^7B^5C^6 \\
&+ 1208064\, A^6B^7C^5 + 1082688\, A^5B^9C^4 + 267984\, A^4B^{11}C^3 - 220320\, A^3B^{13}C^2 - 162648\, A^2B^{15}C - 31104\, AB^{17} + 172800\, A^7B^4C^5D \\
&+ 442368\, A^7C^8B^2 + 620928\, A^6B^6C^4D + 1110528\, A^6C^7B^4 + 895536\, A^5B^8C^3D - 2439936\, A^5C^6B^6 + 649296\, A^4B^{10}C^2D - 11284704\, A^4C^5B^8 \\
&+ 237168\, A^3B^{12}CD - 15591456\, A^3C^4B^{10} + 34992\, A^2B^{14}D - 10362816\, A^2C^3B^{12} - 3386448\, AB^{14}C^2 - 435456\, B^{16}C + 331776\, A^7BC^7D \\
&- 67392\, A^6B^5C^3D^2 + 4648320\, A^6B^3C^6D + 663552\, A^6C^9B - 147744\, A^5B^7C^2D^2 + 14486688\, A^5B^5C^5D - 3103488\, A^5C^8B^3 \\
&- 107892\, A^4B^9CD^2 + 21505824\, A^4B^7C^4D - 25468992\, A^4C^7B^5 - 26244\, A^3B^{11}D^2 + 18424800\, A^3B^9C^3D - 52265088\, A^3C^6B^7 \\
&+ 9572256\, A^2B^{11}C^2D - 47708352\, A^2C^5B^9 + 2846016\, AB^{13}CD - 20497536\, AC^4B^{11} + 373248\, B^{15}D - 3382560\, B^{13}C^3 - 2519424\, A^6B^2C^5D^2 \\
&+ 497664\, A^6C^8D - 7235568\, A^5B^4C^4D^2 + 12845952\, A^5B^2C^7D + 331776\, A^5C^{10} - 5445144\, A^4B^6C^3D^2 + 31290624\, A^4B^4C^6D \\
&- 10886400\, A^4C^9B^2 + 1279152\, A^3B^8C^2D^2 + 31648320\, A^3B^6C^5D - 50258880\, A^3C^8B^4 + 2886840\, A^2B^{10}CD^2 + 17184960\, A^2B^8C^4D \\
&- 74442240\, A^2C^7B^6 + 839808\, AB^{12}D^2 + 5520960\, AB^{10}C^3D - 45567360\, AC^6B^8 + 886464\, B^{12}C^2D - 9984384\, B^{10}C^5 - 925344\, A^5B^3C^3D^3 \\
&- 6718464\, A^5BC^6D^2 - 1924560\, A^4B^5C^2D^3 + 23001408\, A^4B^3C^5D^2 + 15676416\, A^4BC^8D - 1347192\, A^3B^7CD^3 + 116301744\, A^3B^5C^4D^2 \\
&+ 9859968\, A^3B^3C^7D - 10948608\, A^3C^{10}B - 314928\, A^2B^9D^3 + 148785984\, A^2B^7C^3D^2 - 20435328\, A^2B^5C^6D - 40217472\, A^2C^9B^3 \\
&+ 76667472\, AB^9C^2D^2 - 20808576\, AB^7C^5D - 42177024\, AC^8B^5 + 13996800\, B^{11}CD^2 - 4945536\, B^9C^4D - 13561344\, B^7C^7 - 1119744\, A^5C^5D^3 \\
&+ 104976\, A^4B^4CD^4 - 36181728\, A^4B^2C^4D^3 - 3359232\, A^4C^7D^2 + 59049\, A^3B^6D^4 - 96799536\, A^3B^4C^3D^3 + 145006848\, A^3B^2C^6D^2 \\
&+ 8584704\, A^3C^9D - 104941008\, A^2B^6C^2D^3 + 380642976\, A^2B^4C^5D^2 - 40310784\, A^2B^2C^8D - 3732480\, C^{11}A^2 - 52488000\, AB^8CD^3 \\
&+ 311148864\, AB^6C^4D^2 - 67184640\, AB^4C^7D - 12877056\, C^{10}AB^2 - 10077696\, B^{10}D^3 + 81671328\, B^8C^3D^2 - 23887872\, B^6C^6D \\
&- 7838208\, B^4C^9 + 11757312\, A^4BC^3D^4 + 11354904\, A^3B^3C^2D^4 - 115893504\, A^3BC^5D^3 - 6337926\, A^2B^5CD^4 - 257191200\, A^2B^3C^4D^3 \\
&+ 193155840\, A^2BC^7D^2 - 5668704\, AB^7D^4 - 203443488\, AB^5C^3D^3 + 353139264\, AB^3C^6D^2 - 47029248\, ABC^9D - 55007424\, B^7C^2D^3 \\
&+ 151585344\, B^5C^5D^2 - 33592320\, B^3C^8D - 1119744\, C^{11}B + 1259712\, A^3B^2CD^5 + 15116544\, A^3C^4D^4 + 708588\, A^2B^4D^5 \\
&- 151795296\, A^2B^2C^3D^4 - 116733312\, A^2C^6D^3 - 251233812\, AB^4C^2D^4 - 159983424\, AB^2C^5D^3 + 85660416\, AC^8D^2 - 89439552\, B^6CD^4 \\
&- 71803584\, B^4C^4D^3 + 95738112\, B^2C^7D^2 - 13436928\, C^{10}D + 117153216\, A^2BC^2D^5 + 178564176\, AB^3CD^5 - 319336992\, ABC^4D^4
\end{aligned}
$$

$$
\begin{aligned}
& + 68024448\, B^5 D^5 - 268791048\, B^3 C^3 D^4 + 42830208\, B C^6 D^3 - 15116544\, A^2 C D^6 + 11337408\, A B^2 D^6 + 249422976\, A C^3 D^5 \\
& + 229582512\, B^2 C^2 D^5 - 120932352\, C^5 D^4 + 158723712\, B C D^6 - 136048896\, D^7, \\
\epsilon_0 & = 128\, A^6 B^5 C^3 + 288\, A^5 B^7 C^2 + 216\, A^4 B^9 C + 54\, A^3 B^{11} + 6144\, A^6 B^2 C^5 + 22272\, A^5 B^4 C^4 + 28992\, A^4 B^6 C^3 + 16272\, A^3 B^8 C^2 + 3348\, A^2 B^{10} C \\
& + 48\, A^5 B^3 C^3 D + 18432\, A^5 C^6 B + 468\, A^4 B^5 C^2 D + 6048\, A^4 C^5 B^3 + 756\, A^3 B^7 C D - 108936\, A^3 C^4 B^5 + 324\, A^2 B^9 D - 178992\, A^2 C^3 B^7 \\
& - 103896\, A C^2 B^9 - 20736\, C B^{11} + 2304\, A^5 C^5 D - 324\, A^4 B^4 C D^2 + 73872\, A^4 B^2 C^4 D + 13824\, A^4 C^7 - 243\, A^3 B^6 D^2 + 237096\, A^3 B^4 C^3 D \\
& - 151200\, A^3 C^6 B^2 + 292392\, A^2 B^6 C^2 D - 460080\, A^2 C^5 B^4 + 157680\, A B^8 C D - 400896\, A C^4 B^6 + 31104\, B^{10} D - 110160\, C^3 B^8 - 51840\, A^4 B C^3 D^2 \\
& - 88020\, A^3 B^3 C^2 D^2 + 235008\, A^3 B C^5 D - 36450\, A^2 B^5 C D^2 + 648000\, A^2 B^3 C^4 D - 228096\, A^2 C^7 B + 589680\, A B^5 C^3 D - 432864\, A C^6 B^3 \\
& + 174096\, B^7 C^2 D - 191808\, B^5 C^5 - 72576\, A^3 C^4 D^2 - 1458\, A^2 B^4 D^3 + 326592\, A^2 B^2 C^3 D^2 + 212544\, A^2 C^6 D + 615276\, A B^4 C^2 D^2 \\
& + 427680\, A B^2 C^5 D - 93312\, A C^8 + 238464\, B^6 C D^2 + 225504\, B^4 C^4 D - 108864\, B^2 C^7 - 489888\, A^2 B C^2 D^3 - 833976\, A B^3 C D^3 + 828144\, A B C^4 D^2 \\
& - 326592\, B^5 D^3 + 659016\, B^3 C^3 D^2 - 15552\, B C^6 D + 93312\, A^2 C D^4 - 979776\, A C^3 D^3 - 1032264\, B^2 C^2 D^3 + 373248\, C^5 D^2 - 559872\, B C D^4 \\
& + 839808\, D^5.
\end{aligned}
$$

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