PLEASE DON'T CONTRADICT ME

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1. Introduction

We all have argued something is false by considering the consequences of assuming it is true. The idea occurs throughout cultures and eras and so it seems sound. Mathematics has numerous examples including the following claims.

- $\sqrt{2}$ is irrational.
- There are infinitely many primes.
- The decimal numbers outnumber the integers.

In each of these we argue by assuming the opposite, for example that $\sqrt{2} = a/b$, and we work towards a difficulty we cannot accept. Many authors and speakers of today describe such proofs as *indirect*. Some even say they are *proofs by contradiction*. The very bold will invoke latin titles *reducto ad absurdum*, a nod to the rich history.

The latin titles not withstanding, western philosophy actually spent most of the last 2 millennia concerned with a poetic schemes for reasoning described by Socrates and known as *syllogisms*. There are 256 syllogisms, of which 24 are called valid. Even the form of syllogisms is different from what today's authors mean by indirect proof, or proof by contradiction. Those are much more recent concepts due in large part to the polymath David Hilbert, who published a widely read treatise on the foundations of math in 1927. Contemporaries to Hilbert, namely Brouwer, Heyting, and Kolmogorov (BHK) are far less known, but their work was almost exclusively on foundations which permitted them to notice and craft a far more nuanced ontology of arguments.

Mathematics today is such a broad field with connections to so many areas that it benefits enormously from the informal and expedient treatment given by Hilbert's system. Yet, a growing number of fields including Computer Science and Linguistics, as well as the mathematical disciplines of Universal Algebra, Lattices, Topology, & Category Theory; are today turning towards the lessons of his contemporaries. Arguably the most influential difference is the clarity that the *BHK interpretation* brings to the study of negatives.

2. What is a negative?

In symbolic form a negative is a sentence beginning in negation, often written $\neg P$. In our examples language can serve to hide this negation so it is a good practice to begin teasing out what is hidden.

- It is *not true* that $\sqrt{2}$ is rational.
- It is false that the set of primes is finite.
- There is no bijection $\mathbb{R} \to \mathbb{Z}$

To make this even more stark we can use some symbolic logic.

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- $P \equiv \sqrt{2} = a/b$. **Claim** $\neg P$.
- $Q \equiv (\exists n \in \mathbb{N})(|\mathsf{Primes}| = n)$. **Proposition** $\neg Q$
- $R \equiv (\exists f : \mathbb{R} \to \mathbb{Z})(g : \mathbb{Z} \to \mathbb{R})(f \circ g = 1) \wedge (g \circ f = 1)$. **Theorem** $\neg R$.

Choices. Already you may notice a bit of a predicament. We seem to have options. In the case of primes we appealed to a definition of finite, for instance

"a set is finite if it has bijection to a set $\{1, \dots, n\}$ for some natural number $n \in \mathbb{N} = \{0, 1, \dots\}$"

We might have wondered instead if we could use a direct definition of infinite instead, perhaps Cantor's definition:

"a set with a bijection to a proper subset".

If we did so then "There are infinitely many primes" would cease to be a negative.

Of course if we switch definitions then it is on us to also demonstrate that those definitions agree. There in hides our missing negative. In our setting, if your approach to proving that there are infinitely many primes will somewhere use a phrase like

"Let $p_1, \dots, p_n$ be the assumed list of all primes..."

then plainly you are using the first of the two possible definitions and your proof is in fact proving a negative. We shall get to ways to prove negatives in a moment but now is a good time to acknowledge our problems with the concept.

3. Psychology of negatives

"You cannot prove a negative!" This is a premise often used in arguments both mindless and seemingly profound. It is picked up at an early age rooting its philosophy deep into a our early intuition. If it is true we would do well to steer clear of arguments that prove negatives and attach different rationalizations to our arguments, such as "proof by contradiction, modus tolens" and such. But is this phrase even true?

To begin with the idiom itself is a negative. It is equivalent to stating:

$$P \equiv \text{“You can prove a negative”}$$
$$\neg P \equiv \text{“You cannot prove a negative.”}$$

So the sentence, if true, would need to be a proof of a negative, which the sentence argues cannot be done. So we ought to ignore this sentence entirely for logical purposes.

However it is worth acknowledging what this phrase has done to readers of proofs. As psychologist Stephen Law observes in *Psychology Today*, Sept. 15, 2011, this phrase often adequately summarizes the nature of *doubt*, not *truth*. As scientists of reason we should like to remove doubt and so arguments seen as "proving a negative" do indeed deserve close scrutiny.

Negatives arguments read like false arguments. One reason for so much doubt in proving negatives is that the arguments lie adjacent to many known logical fallacies (invalid arguments).

For instance, here is a flawed conclusion

> **Arugment from ignorance** "I've never seen aliens; so, they do not exist."

Here is a

These are not proofs of anything but they do rather well at expressing a degree of belief and doubt. If we lower the doubt below a reader's threshold the result seems indistinguishable from proof. But it is not, in the end, a proof.

Mathematicians do not escape these issues in their proofs, but they have instead established conventions on the permitted use of these sort of fallacies. For example, how many proofs can be found that argue with phrases like these:

> *It is enough to consider the special case ...*
> *...the argument follows similarly for other cases.*
> *Clearly...*
> *...mutatits mutandis.*

3.1. **Linguistic negatives.** Neither *Argument from Ignorance* nor *Burden of Proof* fallacies are unique to negatives. The fundamental struggle is that when proving a negative we begin precisely doubting what we claim. So the very language of our argument entertains the vary vocabulary that often leads us to speak forcefully but in error.

3.2. **Detecting negatives.**

4. Arguing negatives to a machines

Since human language has such a p

The actual name is *proving a negative*.

Many books introducing scholars to reasoning do not even mention this type of proof by name. Some authors even make the effort to credit historical studies given methods of proofs their latin titles *modus ponens* and *tolens*, and *reducto ad absurdum*. Have we lost track of the methods of proofIf it seems did we come so far and loose track of the method and name for a proof we use so often?

Knowing the difference makes certain that our arguments on based on firm reasoning that ends in the desired conclusion. Without it we may succeed in an argument that lowers our doubt below what we tolerate but leave open the door that others may still suspect our claims.

5. Examples.

Proposition 1. $\sqrt{2}$ *is* irrational.

First we identify the negativity. Here *ir*-rational is a linguistic trick to mean *not* rational. So we actually mean the following.

Proposition 2 (Formal Form)**.** $\neg(\sqrt{2} \in \mathbb{Q})$

Second we identify the context, in this case that means to explain some possible meaning of rational.

Definition 3. *A* rational number *is a solution to an equation $a = bx$ for integers a, b with $b \neq 0$. We write $x \in \mathbb{Q}$.*

So we we break up the proof to see what is happening.

Lemma 4. *If there are integers a, b for which $b\sqrt{2} = a$ then $b = 0$.*

This lemma is the heart of the proof and can be proved directly but we leave that proof to an appendix so as to stay focussed on negativity.

Proposition 5 (Formal Form)**.** $(\sqrt{2} \notin \mathbb{Q}) \equiv (\sqrt{2} \in \mathbb{Q} \Rightarrow \bot)$

Proof. Assume $\sqrt{2} \in \mathbb{Q}$. That means that there are integer a, b such that $a = b\sqrt{2}$ and $b \neq 0$. Next by Lemma 4 we also know that if $b\sqrt{2} = a$ then $b = 0$. Finally $b \neq 0$ means $(b = 0) \Rightarrow \bot$, and $b = 0$; so, we deduce $\bot$. Dismissing the assumed hypothesis we have shown $(\sqrt{2} \in \mathbb{Q}) \Rightarrow \bot$. That is, $\sqrt{2} \notin \mathbb{Q}$. □

Proposition 6. $(|\mathbb{N}| \neq |\mathbb{R}|) \equiv [(|\mathbb{N}| = |\mathbb{R}|) \Rightarrow \bot]$.

Lemma 7. *If $f : \mathbb{N} \to \mathbb{R}$ is a function then there is an $x^f \in \mathbb{R}$ such that the k-th decimal place of x^f is*

$$x^f_k = \begin{cases} f(k)_k + 1 & f(k) > 0 \\ 9 & f(k) = 0 \end{cases}$$

This proof is again the heart of the proposition but it is not a negative, though in this case case it does have an embedded negative in premise "x is not in the image of f". However the full sentence reads and so this is not a negative itself.

Proposition 8. $\neg(\exists f : \mathbb{N} \to \mathbb{R})(\forall x \in \mathbb{R})(\exists k \in \mathbb{Z})(f(k) = x)$

Proof. Assume for there is an $f : \mathbb{N} \to \mathbb{R}$ where for every $x \in \mathbb{R}$ has some $k \in \mathbb{Z}$ such that $f(k) = x$. Using the x^f from Lemma 7 then there is some $k \in \mathbb{Z}$ such that $f(k) = x^f$. Therefore the k-th decimal places of both agree: $f(k)_k = x^f_k$. Therefore $f(k)_k = f(k)_k + 1$ or $0 = f(k)_k = 9$. In the first case $0 = 1$ and in the second $0 = 9$. Since $0 \neq 1$ and $0 \neq 9$ in either case we deduce $\bot$. Dismissing the assumption, if there is a surjection $f : \mathbb{N} \twoheadrightarrow \mathbb{R}$ then $\bot$ follows. So $\nexists f : \mathbb{N} \twoheadrightarrow \mathbb{R}$. □

Appendix A. Proofs for the curious

Proof of Lemma 4. To prove the implication, assume the hypothesis, in this case that for integers a, b $b\sqrt{2} = a$. If $b < 0$ then replace with $(-a, -b)$. So $S = \{b \in \mathbb{N} \mid (\exists a \in \mathbb{Z})(b\sqrt{2} = a)\}$ is non-empty. Let $b \in S$ be the least element of b, and let a be such that $b\sqrt{2} = a$. Square both sides to find $2b^2 = a^2$. So $2|aa$ so by Euclid's Lemma $2|a$. Thus $4|a^2 = 2b^2$ and so $2|b^2$ which by Euclid's lemma proves $2|b$. Therefore $(b/2)\sqrt{2} = (a/2)$ so $b/2 \in S$. Since b is minimum $b/2 = b$ Therefore $b = 0$. Since $b \neq 0 \equiv (b = 0 \Rightarrow \bot)$ and $b = 0$, $\bot$ follows. Therefore $(b\sqrt{2} = a) \Rightarrow \bot$. So $\sqrt{2}$ is irrational. □