

Algebra Outside

How algebra is used

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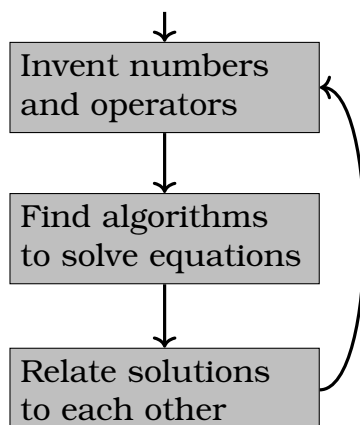


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Algebra today is taught chronologically by example. First groups, then rings, then fields, then modules, then bigger groups, then crazier rings, then fusing groups and rings with outside influences like topology, analysis, and geometry. Today there are few general-practice algebraist. There are instead geometric group theorist, algebraic geometers, commutative ring theorist, representation theorist, non-associative algebraist, computational algebraist (that's me), and even those topics are too general for any one theorist to master.

It bucks tradition (set out by Noether) to change this approach to algebra. Chances are high that the books that brought you here already laid out the subject in this order. But I speculate that there is a place in the sciences for a view of algebra as a whole. What is it that algebra does? Like many who work in math a likely response has a does of calculated propaganda to impress your parents, funding agency, or quite the passenger next to you. This text offers some clues about making that response a bit more honest in intention. I divide all of algebra into a 3 phase flow chart.



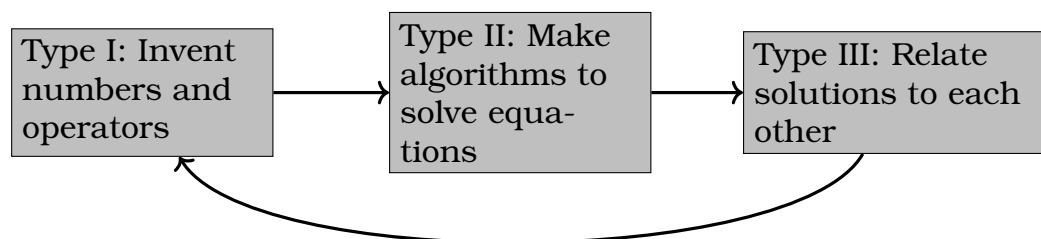
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What is algebra?

1

Algebra is the study of equations, for the most part equations involving variables. That is because applications have unknowns and if the shape of the equation can tell us anything about the options to solve it, we shall want to take advantage of this. Witness how we solve $ax^2 + bx + c = 0$ in one strategy despite it covering an infinite number of equations. Basic though it may seem that solution illustrates the general arc of an algebra investigation.



1.1 \mathbb{N} : the first Type I algebra

Childhood is mostly type I algebra. Children learn to count and give these counts the usual names

$1 := |$, $2 := ||$, $3 := |||$, $4 := ||||$, $5 := |||||$, ...

Nothing gets the name 0 (one theory is that the symbol reflects a dent in the sand left by removing the last pebble from a count.) Mathematicians and programmers see these *natural numbers* as produced by two rules, start with nothing, 0, or take a successor $S(k)$ to natural number k already created. In notation we separate the cases by | (read as “or”) and define \mathbb{N} by this pattern

$$\mathbb{N} = 0 | S(k) \quad (\text{Mathematics})$$

We say that 0 , $1 := S(0)$, $2 := S(S(0))$ and so on, are natural numbers. It is important to see this as creating data rather than explaining data we already have. To emphasize this point consider how this

translates into programs, they won't be packages of known data, they will be ways to generate data. Here and throughout we use coding displays in two of the three popular paradigms.

Remark 1.1. In programming, logic is recorded by matching a pattern of words and symbols called an *idiom*. Often an idiom attempts to mimic a natural language (commonly English) or a mix of natural language and mathematical notation. A *Programming Language (PL)* is a collection of idioms capable of expressing a system of logic. There are three major systems of logic, two based on functions, one based on relations. There are likewise three major PL families.

Procedural PLs that express a Lovelace-Turing-Curry *Combinatorial Logic*, a logic based on functions as sequences of primitive steps (combinators/assembly). E.g. Fortran (1957-), C (1972-), Python (1991-), Java (1995-).

Functional PLs that express a Gödel-Church λ -*calculus/recursive functions*, a logic based on functions as substitution rules. E.g. Lisp (1960-), ML (1973-), Haskell (1990-).

Relational PLs that express a Tarski-Codd *Relational Logic*, a logic based on relations rather than functions. E.g. SQL (1974-), Excel (1987-).

Most languages from 2000 and beyond use a mix of styles. We will concentrate on the first two families.

As a general guide, procedural PLs have preferred the $\sin(x)$ style of functions whereas functional PLs have largely embraced the $\sin x$ convention.

While dialects of languages differ, below are two pseudo-code samples repeating the definition $\mathbb{N} := 0|S(k)$ in two of the dominant forms of programming.

```
data Nat = Z | S k
```

(Functional Program)

```
class Nat
  case Zero ext Nat
  case Next (k:Nat) ext Nat
sealed
```

(Procedural Program)

Writing $n \in \mathbb{N}$, $n : \mathbb{N}$, $0 : \text{Nat}$, and `uint n` (represented an “unsigned integer”) are all ways that we communicate that we have some data n that was created as a natural number. This means when it comes time to use n we can assume one of two things $n = 0$ or $n = S(k)$.

Example 1.2. A function to decide if a natural number is positive.

$$p(n) = \begin{cases} \top & n = 0 \\ \perp & n = S(k) \end{cases}$$

Here are some equivalents as programs.

```
isPositive n =
  match n with
    Z => true
    (S k) => false
```

```
def isPos(n) =
  match n with
    Zero => true
    Next(k) => false
```

For example, mathematicians might write

$$p(n) = \begin{cases} \top & n = 0 \\ \perp & n = S(k) \end{cases} \quad e(n) = \begin{cases} \top & n = 0 \\ \perp & n = S(0) \\ e(k) & n = S(S(k)) \end{cases}$$

The first function return \top (which we read “true”) if, and only if, $n = 0$; otherwise it returns \perp (which reads as “false”). So p is a test of positivity. Meanwhile e uses an extra case so that $e(0)$ is true, $e(1)$ is false and $e(2+k) = e(k)$, where $2+k$ is short hand for $S(S(k))$. Extrapolating the pattern, we end with true if, and only if, n is

even. We could render these as code in roughly the same way but with idioms of a particular programming language.

```
isPositive : Nat ->
  Boolean
isPositive 0 = true
isPositive (S k) = false

isEven: Nat->Boolean
isEven n = match n with
           0 => true
           S 0 => false
           S (S k) =>
             isEven k
```

This creation of numbers inspires many others. Imagine a string of characters in an alphabet `Char:=['a','b',...,'z']`. We start with either nothing (an empty string) or we add a character to a string we already have. There is a little more going on here, we could add a character before or after the string we have.

```
data String = Empty
            | Prepend( head:Char, tail:String)
            | Append( head:String, tail:Char)
```

Writing `head:Char` or `tail:String` indicates that `head` must come from the alphabet we chose and `tail` must be some already produced string, possibly empty. Some readers might relate to a different dialect of programming such as the following

```
class String
  case Empty extends String
  case Prepend( head:Char, tail:String) extends String
sealed // No further cases
```

The head here carries around what we put in the list and the tail is what comes next in the list. Observe the similarities:

$$2 := S(S(0)) \quad (\mathbb{N})$$

$$\text{"me"} := \text{Prepend}('m', \text{Prepend}('e', \text{Empty})) \quad (\text{String})$$

The left-hand sides are merely notation for what the data really is on the right. Both the successor and the `Prepend` are operators that generate new values. So part of algebra is to generate new data; so, it is no wonder that it closely connections to computation.

Exercise

1. Mimic the String data type to make a list of integers (that is switch from the alphabet to integers).
2. Mimic the String data type to make a list of fixed by unknown data of type A , call it `List [A]`.¹

Historically `Empty` for lists is called `Nil` and `Prepend` is called `Cons`.

Children also learn to add, whatever that means conceptually, it obeys another simple pattern.

$$m + n := \begin{cases} n & m = 0 \\ S(n + k) & m = S(k) \end{cases}$$

So does $2 + 4 = 6$? We can test this out.

$$\begin{aligned} || + |||| &= | (| + ||||) \\ &= | (| (_ + ||||)) \\ &= | (| ||||) \\ &= | \#\# \end{aligned}$$

A natural number is either 0, or a successor $S(k)$ to another natural number k . Often this is expressed as a definition where $|$ stands for separating cases, for example:

$$\mathbb{N} := 0 \mid S(k)$$

So 0 is “zero”, and 1 is just a symbol representing $S(0)$, $2 := S(S(0))$ and so on. Replace S ’s with tally marks (not to be confused with ‘|’ used as an “or” above) we recover childhood counting:

$$0 := _, 1 := |, 2 := ||, 3 := |||, 4 := ||||, 5 := \#\#, \dots$$

The point is, the successors are not so much a function moving around the numbers we have, it actually is a producer of numbers.

¹Alternatives include `List a` and `List<A>`. Search for *generics* in your programming language to learn more.

, find distances of length $\sqrt{2}$, learn about imaginary numbers and π . The shift to π signals the shift to type II because π is such a fiddly number that it makes sense to leave it alone moving it round solely by rules of algebra until the end when we might replace with 3.14 or a better approximation. For practical purposes π is the first quantity used as a variable.

High school algebra is nearly all type II, and mostly following al Khwarizmi's *al Jabr*—the method of balancing parts, which today is called Algebra. Move over Euclid, al Khwarizmi's method made people money, settled land disputes, calculated grain totals and arguably did more to make math a necessity for average people than any other source. His name has passed down to modern life as the word *algorithm*. At this stage students are told that polynomials of degree d have d complex roots. Perhaps because some non-algebraist call this fact “the fundamental theorem of algebra” (it's not) this seems a good place to leave algebra for calculus or leave math for good. Most people never reach a type 3 problem.

You are one of the few who have heard of type 3 algebra. You have seen the solving linear equations gives you back often infinitely many answers and you don't have time nor the interest to write down infinitely many answers. You know that this can be done with a basis. The next example is even more greedy. Why write down d roots to a d degree polynomial?! Instead, you want to write down one solution and sprinkle in some pattern to explain how you might find all the others. Think back to $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ solving $ax^2 + bx + c = 0$ (with $a \neq 0$). That ‘ \pm ’ trick saved you thinking of two roots. Ever solved $x^6 - 1 = 0$ by using $x = e^{(2\pi i/6)t}$ thinking of rotating around by 60 degrees? These relationships turned out to have a name (groups) and were the first type 3 algebra.

This is where the heavy weights got involved: Lagrange made a formula for quartic polynomials from a primitive concept of groups but he got distracted by a fluke concerning symmetric polynomials and missed out on the eventual full solution by Galois. Gauss showed that construction by ruler and compass produces only numbers that are iterated applications of square-roots by observing what in today's language would say that the group of relations is nilpotent. Abel found a specific obstacle with $x^5 - x + 1 =$

0 by showing that if its relations were to be solvable they would create contradictions with facts from calculus, the first proof that not all polynomials are solvable by radicals. Galois characterizes all such polynomials by describing what their type 3 algebra would look like. Both men died tragically and without their work being understood until after their young deaths.

Here I should settle a confusion in the subject. Abel and Galois did not show that the quintic has no solutions! That would fly in the face of Gauss' "fundamental theorem of algebra" (a 5th degree polynomial has 5 complex roots). What Abel and Galois proved is that roots of these polynomials were not of the form hinted at by the solutions for 1,2,3 and 4th degree polynomials. Take for example al Khwarizmi's quadratic formula gives roots of the form $\frac{-b}{2a} \pm \frac{1}{2a} \sqrt{b^2 - 4ac}$, where a, b, c are numbers we already created. Cardano's cubic formula goes a step further. Given numbers a, \dots, g already created, then solutions to the cubic can be made in the following form.

$$a + b\sqrt{c} + d\sqrt[3]{e + f\sqrt{g}}.$$

Lagrange's quartic formula gives solutions now involving rational linear combinations of fourth roots. When a number is a linear combination of n -th roots of iteratively created numbers then we say that number is *radical*. That the quintic is not solvable by radicals just means that to factor quintics and higher students need to go back to a type 1 problem: invent new constructions of numbers. In fact, history had already done this for degrees 2,3. For example, if $b^2 - 4ac < 0$ then the solutions may be written as

$$x = \sqrt{\frac{c}{a}} \cos \theta \pm \sqrt{\frac{c}{a}} \sin \theta \quad \theta = \cos^{-1} \frac{-b}{2\sqrt{ac}}$$

The cubic equations can likewise be solved with trigonometry. So Abel and Galois point the way to revisiting how we build numbers and finding new methods. That is, the outcome of type 3 algebra was a need to build new type 1 algebra.

This feedback look has been hard at work for more than a century. These new type 1 problems lead to rings and modules. That lead to algorithms and representation theory to solve type

2 problems with those numbers. Those solutions generated new relations, more groups (type 3), but also new type 1 algebras (Clifford, Lie, Hecke, and Hopf to name a few). The iteration continues.

As a final remark, we should settle one last potential confusion. The fundamental theorem of algebra already told us the shape of roots, i.e. complex numbers. Why not stop here? Answer this yourself: what are the roots of $x^2 = 2$? If you thought $x = \pm\sqrt{2}$ and not $x \approx \pm 1.41$ then you appreciate the difference. By prescribing $x = \pm\sqrt{2}$ we both solve nothing (its just notation) and everything (this solution truly works). This gets a philosophical position that numbers are made up to mean what they can do. Real numbers are made up to mean that finite patterns that stay close together (Cauchy sequences) can in fact be considered as numbers (they converge to themselves, which means nothing but lets there be real numbers).

Radical numbers like $\sqrt{2}$ are another form of invention, a number that when squared is 2. It is freeing to think this way because we can make such numbers precisely with ease. For example $\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$ squares to $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ which gives precise calculations. But for some situations precision wont be important and we can use the approximation $1.41^2 \approx 2$. Algebra therefore sees the matrix and the decimal examples of $\sqrt{2}$ as interchangeable. To algebra, $\sqrt{2}$ is a property, not so much a number. Because of this we can imagine many more creative algorithms to work with these numbers.

For the purposes of solving it is important to have both but exact and approximate solutions. To expose this difference look no further than finding the eigenvalues of a matrix.

$$\begin{bmatrix} -9 & 7 & -12 & 6 \\ -3 & 5 & -4 & 6 \\ 7 & -5 & 8 & -2 \\ 2 & -2 & 4 & -4 \end{bmatrix}$$

With 64 bit floating points in Julia, my computer printed out

```
julia> eigvals([-9 7 -12 6; -3 5 -4 6; 7 -5 8 -2; 2
               -2 4 -4])
4-element Vector{ComplexF64}:
```

```

-0.0005295477729425071 - 0.0005295190134323146im
-0.0005295477729425071 + 0.0005295190134323146im
 0.0005295477729438736 - 0.0005295765047025466im
 0.0005295477729438736 + 0.0005295765047025466im

```

But actually this matrix has a single eigenvector 0 and is not diagonalizable but instead it is conjugate to a nilpotent matrix:

$$\begin{bmatrix} -9 & 7 & -12 & 6 \\ -3 & 5 & -4 & 6 \\ 7 & -5 & 8 & -2 \\ 2 & -2 & 4 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 & 2 \\ -1 & 2 & -1 & 2 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 & 2 \\ -1 & 2 & -1 & 2 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

Solutions to equations are not stable under approximations.

1.2 Everything is variable

You probably already know every color, shape, and pattern of equation you will ever need. Compare these two equations

$$x^2 + y^2 \equiv 0 \pmod{541} \qquad \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

Can you stop yourself from seeing them as related? These equations concern entirely different things. On the left 0 can equal 541. On the right 0 is functions on the xy -plane. Yet, the similarities as equations shine through. Why? Is it because 0, +, and powers of 2 are general concepts, “abstractions”. This doesn’t connect them to polarizing art movements nor render the concept inapplicable. Abstract here, and everywhere, means to study by limited attributes. That’s how we do all math and science. So when we abstract the equation on the left we forget about mod 541 and the precise meaning of these numbers. On the right we forget about functions and the notions of derivatives. We are left with just 0, +, and squares and where they sit. We abstract both to a common equation

$$x^2 + y^2 = 0.$$

In fact, even the equality was an abstraction which could vary from context to context. With such flexibility a small number

of symbols and their grammar are enough to capture the huge variety of equations we encounter in real life.

The power of algebra is that every symbol in an equation is a variable, especially the equals sign.

1.3 Inventing numbers

Now since every symbol in an equation is a variable we have new powers to solve equations. Look to the humble

$$x^2 + 1 = 0.$$

By our own view, right now this is nothing but variables, so it means nothing to solve this. But, drop this into a context such as decimal numbers \mathbb{R} and the understanding is to replace $1 := 1.0$, $0 := 0.0$, $+$ is substituted for addition of decimals, and square is by multiplying decimals. Equality now means two equal decimals, or in practice two decimal numbers that are close enough to be considered equal. The only remaining unknown is x , but as everyone knows, $0 \leq x$ or $x < 0$ so in both cases $-1 < 0 \leq x^2$.

The power of variable everything is that we are not stuck with the real numbers. Let us replace everything with complex numbers \mathbb{C} . Substitute $0 := 0+0i$, $1 = 1+0i$, $(a+bi)+(c+di) := (a+b)+(c+d)i$, and $(a+bi)^2 = (a^2 - b^2) + 2abi$. Now we find $\pm i$ are the solutions. Solutions do exist! Since they exist we can return to a problem and ask if the solutions we found will do the job. Maybe not, or maybe we should revisit our models and see these solutions as predicting the presence of previously unknown realities.

Why stop here? Quaternions have $\hat{i}^2 = \hat{j}^2 = \hat{k}^2 = -1$; so, at least 6 solutions to $x^2 + 1 = 0$. Even more. Try (2×2) -matrices, I bet you can find infinitely many matrices M where $M^2 = -I_2$. This is the method of algebra: dream up new numbers that might be used to solve equations. Alter their properties, e.g. drop the order of real numbers and you can get complex solutions. Drop commutative multiplication of complex numbers and you might get infinite solutions. Learn enough by this process and we can begin to predict if solutions are to be expected and fathom algorithms to find them

when they do exist. When the solutions become infinite we find ways to parameterize them with smaller data such as a basis.

Don't forget equality is variable too! Suppose we wanted to solve $x^{541} + x + 1 = 0$ using only integers. Replace equality with \equiv modulo 2 and ask for a whole number x that solves

$$x^{541} + x + 1 \equiv 0 \pmod{2}.$$

All integers in this equality become either 0 or 1, but neither will solve this equation. By varying equality we confirm there are no solutions.

This is why so much of algebra today is concerned with making new numbers and searching out which equalities (congruences) can be used on new numbers. It is the front line of algebra, the first question that must be concurred. And it part of why so much research on algebra doesn't even feature solving equations because the goal is to have numbers ready for whatever equations show up.

The first method of algebra is to invent numbers: their constants, their operations, their rules. We call these “algebras”.

1.4 Computation

Algebra is the computational wing of mathematics.

Look to calculus. We obviously work with limits and derivatives to recover Rolle's theorem and the significance of $f'(x) = 0$. But when we want to find such x we inevitably are left to solve equations, like $x^2 - x - 3 = 0$, which we do by algebra.

Switch to topology. Roll up a stretchy sheet of plastic along the following Van Kampen patterns viewed in Figure 1.1 to make a torus \mathbb{T}^2 and a Klein bottle \mathbb{K} . These are different, right? Why? Of course we use topology to prove the interesting bit, like the Seifert-van Kampen theorem: the loops $\pi_1(M^2)$ on a surfaces M^2 can each be described as a sequence of blue loops b and red loops r . Loops that contract to nothing can be read along the boundary.

Hence,

$$\pi_1(\mathbb{T}^2) = \text{words over } \{b, r\} \text{ rewritten by } br = rb;$$

$$\pi_1(\mathbb{K}^2) = \text{words over } \{b, r\} \text{ rewritten by } r = brb.$$

We use algebra to compute that those numbers are not the same so neither are the two spaces.

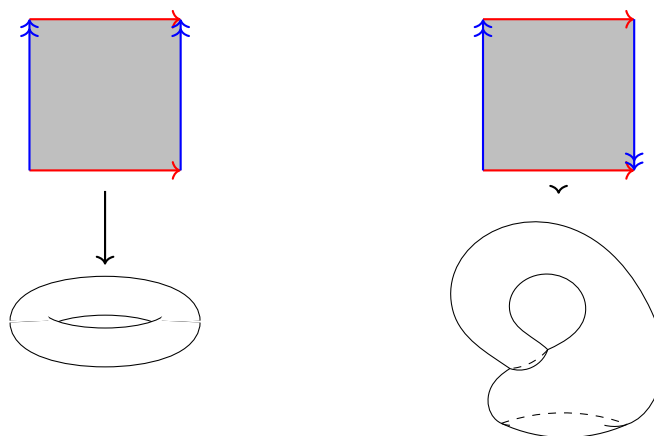


Figure 1.1: A torus and Klein bottle, with thanks to <https://tex.stackexchange.com/q/77606/86>.

Other fields? How many times in applied math, number theory, or combinatorics do we end up solving something even if just a linear equation, in order to get a final answer? You don't have to be convinced but the evidence is there.

Yet most of us do not want to be computers. So what does an algebraist do when the questions come down to computing? We put our energy into crafting algorithms to solve these problems. In fact the word “algorithm” comes from al Khwarizmi whose influential book *al Jabr* (the method of parts) gave use the first detailed explanations of algebra. Half of that book is a list of story problems solved by various algorithms in algebra. If you are concerned that this now trends to computer science, well know that computer science cares more about the data structures than about any algorithms. Just like counting get faster with an abacus, algebraic algorithms get better with data structures. So these two

worlds interact but care deeply about different parts of the problem.

How to operate

2

One day + means to add natural numbers, the next day polynomials, later matrices. You can even add colors “Yellow=Blue+Green”. When you program you learn to add strings

```
In [1] print "Algebra " + " is " + " computation"
```

```
Out [1] Algebra is computation
```

The + is in fact a variable stand in for what we call a *binary operator* or *bi-valent operator*. It takes in a pair, according to the grammar $\square + \square$. The valence (the number of inputs) and the grammar of an operator comprise its *signature*.

Unlike these notes, addition should only be used when it is grammatically correct e.g. *infix* $2 + 3$ rather than *prefix* $+, 2, 3$ or *postfix* $2, 3, +$. Think of this like any other language where there could be a dialect that evolves the operator’s grammar and lexicon. Famously HP calculators were postfix for some time to match engineering requirements. It turns out humans will adapt to technology easier than technology adapting to humans. A program to add two lists could get away with the following linguistic drift:

```
In [2] cat [3,1,4] [1,5,9]
       [3,1,4] + [1,5,9]
```

```
Out [2] [3,1,4,1,5,9]
       [4,6,13]
```

Using `cat` reminded us to concatenate and avoided confusion with the later + concept. It was the better choice. Challenge yourself to see both as addition and you will find addition everywhere.

Since we are evolving, we may as well permit multiplication as a binary operator symbol, changing the signature to $\square \cdot \square$, i.e. $2 \cdot 4$; or $\square \square$, e.g. xy . Avoid $\square \times \square$, we need that symbol elsewhere. These days composition $\square \circ \square$ is written as multiplication; so, you can use that symbol however you like. Addition is held to high standards in algebra (that it will evolve into linear algebra). So

when you are considering a binary operation with few if any good properties, use a multiplication inspired notation instead.

Valence 1, *unary*, operators include the negative sign $-$ to create -2 . The transpose of a matrix is a unary operator. Programming languages add several others such as $++i$, $--i$ which are said to *increment* or *decrement* the counter i (change it by ± 1).

We can combine binary and unary operators to make useful (and useless) ternary operators. One of my favorite (but generally useless) ternary products multiplies (2×3) -matrices, that is right, rectangles not squares. The product goes like this:

$$[A, B, C] = AB^{\dagger}C$$

where B^{\dagger} is the transpose. A more serious product comes up in symmetric matrices where we need what is called the *Jordan Triple product*

$$\{A, B, C\} = \frac{1}{6}(ABC + CBA)$$

This is part of an whole family of Jordan products including

$$A \bullet B = \frac{1}{2}(AB + BA)$$

$$\langle A_1, \dots, A_{\ell} \rangle = \frac{1}{\ell!}(A_1 \cdots A_{\ell} + A_{\ell} \cdots A_1).$$

Notice in all these case if $A_i = A_i^{\dagger}$ then $\langle A_1, \dots, A_{\ell} \rangle = \langle A_1, \dots, A_{\ell} \rangle^{\dagger}$. So these are products that explain how symmetric matrices behave, and if you know about quantum mechanics that means these are the products you can use on observable quantum events.¹

Stranger ternary products showup in places where we wish we had easier binary products to explain things. For example in geometry at small dimensions (dimension 2) we do not always have coordinates to explain what makes some set of points into a line. So Marshall Hall decided why not make a line ℓ be described by a ternary operator

$$- \otimes - \oplus -$$

¹Pascual Jordan was a physicist inventing math for quantum mechanics, not be be confused with Camile Jordan of Jordan-Holder and Jordan Normal form fame.

This is just notation but put in a formula like $y = m \otimes x \oplus b$ and you start to imagine the result as an algebraic line. With this ternary product he was able to associated every projective plane to coordinates in some ternary ring. Once you have such a ternary ring you can go to work to see if it might actually decompose into two binary operations of multiplication and addition, e.g. by locating a “one” and a “zero” where $x = 1 \otimes x \oplus 0$. Then you reverse the process and define $m \cdot x := m \otimes x \oplus 0$ and $x + b := 1 \otimes x \oplus b$ to get a more familiar ring-like structure.

Programs also exploit a ternary (valence 3) operator:

```
if (...) then (...) else (...)
```

The words, while helpful, are unimportant. Some programming languages replace them with symbols emphasizing their “operator-ness”

$_? _ : _$

Here for example is division with remainder of positive integers

```
div(m,n) = (m >= n) ? (div(m-n,n) + (1,0)) : (0,m)
```

Programs go further making operators with huge valence. If you find that interesting look into *variadic* operators to see how far this idea goes.

A number of subtle problems are mounting. For example, we probably want a Boolean (true/false) to go in the the left-most spot of an if-then-else-, and we cannot compose just any two functions and get expected results. We scale a vector on one side by not the other. Grammar must be more than just $\square \cdot \square$.

Lets consider an operator

Another operator takes a pair A and B of sets or more general types of data and creates ordered pairs or disjoint unions.

$$\begin{array}{c}
 \frac{A, B : Type}{A \times B : Type} \quad (F_{\times}) \\
 \frac{a : A \quad b : B}{(a, b) : A \times B} \quad (I_{\times}) \\
 \frac{x : A \times B}{\pi_A(x) : A} \quad \frac{x : A \times B}{\pi_B(x) : B} \quad (E_{\times}) \\
 \frac{a : A \quad b : B}{\pi_A(a, b) := a} \quad \frac{a : A \quad b : B}{\pi_B(a, b) := b} \quad (C_{\times})
 \end{array}$$

In code you may encounter this as `Pair[A, B]` or as `(A, B)`.

```

Either A B = Left (a:A) | Right (b:B)

apply (f:A->C, g:B->C, x:Either[A,B]) : C =
  match x with
    Left a => f(a)
    Right b => g(b)

```

A special case of the disjoint union operator is to extend a type by one term:

$$A^? := A \sqcup \{*\}.$$

It is said that $x : A^?$ is “maybe an A ”, or “optionally A ”. In code you will find these written out this way.

```

Maybe A = Just (a:A) | None

```

```

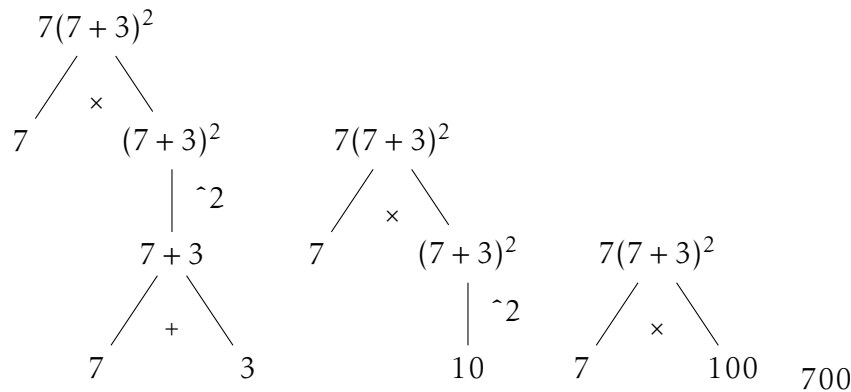
class Option[A]
  case Some(a:A) extends Option[A]
  case None extends Options[All]
sealed // You cannot add to these cases

```

2.1 Grammar school

“... small number of symbols and their grammar are enough to capture the huge variety of equations...”

Ask yourself how you know what to do when asked to calculate $7(7+3)^2$. Parenthesis, Exponents, Multiplication, Division, Addition, Subtraction (PEMDAS) correct? Whether obvious or not, this is parsing grammar. We can make this visual with a *parse tree*.



We can read the tree like step-by-step instructions. Start at the leaves and join them by whatever operation is displayed on adjacent branches. We start at the bottom with 7, 3, and join them as $7+3$ (computing 10), then the next step is to square (now 100), then multiply by 7, we reach 700.

You may have been taught induction through stories of falling dominos. Good. But what if induction was more like climbing, and the domino illustration was bottling up the experience of climbing stairs? Surely its more fun to climb trees and mountains. Climbing can go up (induct) or down (recurse), but of course I drew the tree upside-down; so, the metaphor must be rotated. My goal is to turn all of your views on induction on their head. There are many inductions, one for each free algebra, but that's the forest and we should look first at some more trees.

Complex inductions can be specified by grammar.

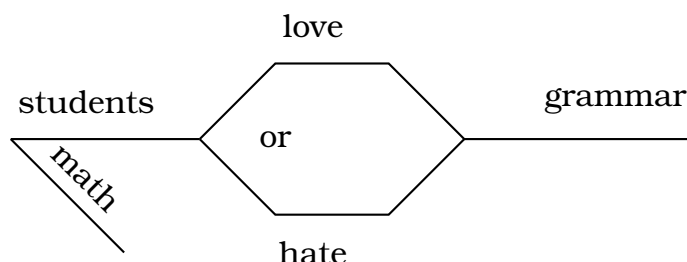
2.2 Grammar

Climbing could find many routes, even go in cycles, but evaluating a formula was a precise algorithm without ambiguity. The reason was that we had a tree. Trees have unique paths between any two

vertices. So if we start at the bottom we have a unique direction to reach the top. Recursion on trees is so special it has a special name: *traversing* the tree.

Parsing grammars in natural language is not always that uniform. English grammar may have cycles.

“Math students love or hate grammar.”



In mathematics two paths to the same place are said to be a relation, they are related. So systems that have no cycles, i.e. trees, are free of relations, or simply *free*. What is free in this case is the grammar we used for arithmetic formulas. That is not to be confused with saying that we couldn't somehow rewrite a formula to mean the same thing. For example $7(7+3)(7+3)$ is a different formula with the same result. But if we diagrammed that formula as parse tree it would be quite different, and it would not have a cycle.

That we got a tree in math formulas means we have a rather boring grammar, what Chomsky's *Syntactic Structure* calls *context-free* grammars.² Don't be too disheartened. Virtually every programming language has a context-free grammar and programs can communicate a lot of hefty ideas.

To specify a context-free grammar we specify the atoms of our language. For natural numbers we can use digits $0, 1, \dots, 9$. Then we write down patterns of how to combine digits. The separate cases are designated by the auxiliary symbol '|' (which reads as "or"). Each rule is given a name called a *token* (or *tag*) and denoted $\langle \text{Name} \rangle$. Since the $\text{Walrus} :=$ is our assignment of variables,

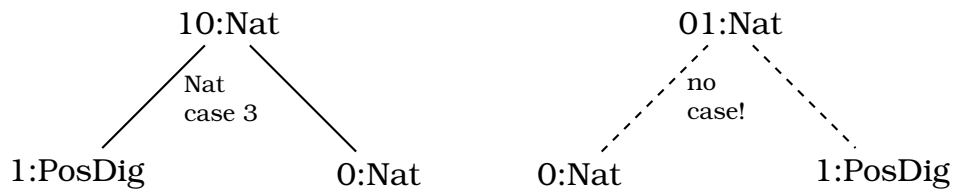
²If an algebraist starts a talk with a story that "...It was thought all natural languages were context-free until some obscure dialect in the alps or Africa was found...", then tune out until they return to equations. Linguists never had such illusions. Even English is not context-free, read James Higginbotham.

we use the “astonished Walrus” $::=$ as assignment of token rules. Listing 2.2.1 shows the grammar for natural numbers as digits.

Listing 2.2.1 The grammar for natural numbers.

```
<PosDig> ::= 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9
<Nat>    ::= 0 | <PosDig> | <PosDig><Nat>
```

The natural number grammar is said to *accept* 0 as a natural number, written “0:Nat”. It also accepts 541:Nat and 10:Nat but it rejects “01:Nat” as that case does not exist as a production rule for any token in this grammar.



Induction is to apply production rules, so it can only ever deliver a sentence accepted by the grammar. The strings accepted by our grammar are known as the *language* for that grammar. They are the source of one of the most important algebras out there, the free algebras.

Deciding to accept or reject is recursion, and so it may start with some string using the alphabet but with the wrong grammar and reach a point of failure, or it might get stuck trying to decide. Context-free grammars fortunately never get stuck, they do not even need a full strength computer to run (a push-down automata will do, like the chip you might need to run a microwave). Yet Moore’s law makes actual computers abundant so lets skip the microwave in favor of a general purpose program to parse.

Listing 2.2.2 Parsing

```

posDigitize( token:Char):Option[PosDig] =
  match token with
    '1' => Some 1:PosDig;   '2' => Some 2:PosDig;
    '3' => Some 3:PosDig;   '4' => Some 4:PosDig;
    '5' => Some 5:PosDig;   '6' => Some 6:PosDig;
    '7' => Some 7:PosDig;   '8' => Some 8:PosDig;
    '9' => Some 9:PosDig;
    _   => None

digitize( stream:String ):Option[Digit] =
  match stream with
    nil => None
    head::tail =>
      match head with
        '0' => match tail with
          nil => Some <0,0>:Nat
          _   => None
        _   => match posDigitize(head) with
          Some (x:PosDig) =>
            match digitize(tail) with
              Some (y:Nat) =>
                <(10^(y.digits)*x+y),1+y.digits>:Nat
              None => None
          None => Some (<x,1>:Nat)

```

Remark 2.1. The symbols $::=$, $\langle \text{Token} \rangle$ and $|$ are Backus-Naur Form (BNF) notation, which is popular in computer science. It is actually subject to its own grammar, admittedly basic and fixed in length. But you can be forgiven for wondering if this is all circular reasoning. When this occurs, mathematicians like to attach the word *meta*, which means literally “self-referential”. So BNF would be called a *meta-language*. Sometimes self-referential can be turned into paradoxes (Russell’s paradox, Gödel’s Incompleteness, Turing’s Halting problem). So you may worry. But I suppose if you didn’t believe in language, why would you be reading?

All these ideas appear under different notation in 600 BC by Panini's rules of Sanskrit grammar. Good ideas get rediscovered; great ideas get reappropriated.

Context-free grammars parse into trees leading complex inductions that are still follow a unique path.

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