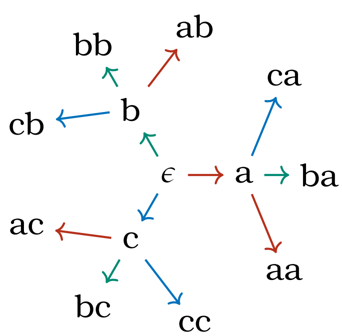
Talking Tensors

A short coarse on tensors

2024-03-22

James B. Wilson



Department of
Mathematics
Colorado State
University

James.Wilson@ColoState.Edu

James.Wilson@ColoState.Edu

Contents

The distributive property 1

The distributive law will be the star of this show. What does this require?

$$u * (v + \acute{v}) = u * v + u * \acute{v} \qquad (u + \acute{u}) * v = u * v + \acute{u} * v. \tag{1.1}$$

We definitely need additions, but we should not jump to assume that u, v, and w are of the same type. Just look at matrix multiplication (we use $\mathbb{R}^{a\times b}$ to denote $(a \times b)$-matrices of real numbers)

$$* : \mathbb{R}^{a\times b} \times \mathbb{R}^{b\times c} \to \mathbb{R}^{a\times c} \qquad (u * v)_{ij} = \sum_k u_{ik} v_{kj}.$$

So we except this is a study of *heterogeneous* algebra, so we wont be captivated by homomorphism but rather what we will call *hetero*morphisms. So we could think of three types of data U, V and W each with a + each combined by a function $* : U \times V \to W$ that satisfies the distributive law. Because this is the start it will get its own notation, we write $\rightarrowtail$, that is

$$* : U \times V \rightarrowtail W$$

denotes a distributive function on additive structures U, V, and W. As we go along we will prefer to use U, V and W in just this way so that we can get up to speed on examples as quickly as possible.

1.1 Properties of addition

It is tempting now to start assuming that U, V, and W are something family—vector spaces, modules, or at least abelian groups. However this would rob the distributive law of its power and leave us to think addition and its common attributes are the reason tensors work. But the distributive law is already claiming a strong interaction of two operations so maybe it should be explored on its own a little while longer to appreciate what it already says about the individual operations. More examples will demonstrate the value of a general point of view.

We will use a number of spaces

$$\begin{aligned} \mathbb{R}^d &:= \{u : [d] \to \mathbb{R}\}, \\ R^{m\times n} &:= \{M : [m]\times[n] \to \mathbb{R}\} \\ R^{\ell\times m\times n} &:= \{\Gamma : [\ell]\times[m]\times[n] \to \mathbb{R}\} \\ &\vdots \end{aligned}$$

Define the following operations.

$$\mathbb{R}^m \oplus \mathbb{R}^n := \mathbb{R}^{m+n}$$

$$\begin{bmatrix} \mathbb{R}^{a\times n} \\ \mathbb{R}^{b\times n} \end{bmatrix} := \mathbb{R}^{(a+b)\times n} \qquad \begin{bmatrix} \mathbb{R}^{m\times c} & \mathbb{R}^{m\times d} \end{bmatrix} := \mathbb{R}^{m\times(c+d)}$$

$$\vdots$$

Now we add a multiplication.

$$\mathbb{R}^m \otimes \mathbb{R}^n := \mathbb{R}^{m\times n}$$

Example 1.1. The distributive law with vector space operators.

$$\begin{aligned} (\mathbb{R}^a \oplus \mathbb{R}^b) \otimes \mathbb{R}^n &= \begin{bmatrix} \mathbb{R}^a \otimes \mathbb{R}^n \\ \mathbb{R}^b \otimes \mathbb{R}^n \end{bmatrix} \\ \mathbb{R}^m \oplus (\mathbb{R}^c \oplus \mathbb{R}^d) &= \begin{bmatrix} \mathbb{R}^m \otimes \mathbb{R}^c & \mathbb{R}^m \otimes \mathbb{R}^d \end{bmatrix} \end{aligned}$$

Lets combine the left and right distributive laws.

$$\begin{tikzcd}[column sep=tiny] & (u+\acute{u})*(v+\acute{v}) \arrow[dl, equal] \arrow[dr, equal] & \\ (u+\acute{u})*v+(u+\acute{u})*\acute{v} \arrow[d, equal] & & u*(v+\acute{v})+\acute{u}*(v+\acute{v}) \arrow[d, equal] \\ (u*v+\acute{u}*v)+(u*\acute{v}+\acute{u}*\acute{v}) & & (u*v+u*\acute{v})+(\acute{u}*v+\acute{u}*\acute{v}) \end{tikzcd}$$

Thus the values $a, b, c, d, \ldots$ in the image of a distributive product $* : U \times V \rightarrowtail W$ must satisfy the following identity.

$$(a+b)+(c+d) = (a+c)+(b+d) \tag{Distributable}$$

Proposition 1.2. If + in both associative and commutative then it is distributable.

Conversely, if + is