The Relative Eigenvector and Pure Tensor Problems

Alex Ryba,
Queens College, CUNY

Groups, Nilpotence and Tensors 2023
Colorado State University, 04/29/2023

Relative Eigenvectors

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Definition

If matrices A, B over k have same size, v is a relative eigenvector if vA and vB lie in a 1-dimensional space.

The difference is that we add the nullspace of A to the set of relative eigenvectors.

Simultaneous Eigenvectors

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A simultaneous eigenvector v for $A_1, A_2, \ldots, A_n$ satisfies:

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$$xa_1 = \lambda_1 y, \qquad xa_2 = \lambda_2 y, \qquad \ldots, \qquad xa_n = \lambda_n y.$$

If a_1 has a right inverse r, this reduces to a simultaneous eigenvector problem. Here, $yr = \frac{xa_1 r}{\lambda_1} = \frac{x}{\lambda_1}$. Hence, x is a simultaneous eigenvector for the maps $a_2 r$, $a_3 r$, $\ldots a_n r$.

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Given k-vector spaces Y and Z and $X \leq Y \otimes Z$. Find all pure tensors in $X \otimes \overline{k}$.

Let $b_1, \ldots, b_n$ be a basis of the dual space of Z. We have maps:

$$a_i = 1 \otimes b_i : Y \otimes Z \to Y \otimes k \cong Y.$$

A pure tensor $x = y \otimes z$ is a relative eigenvector for the maps $a_1, \ldots a_n$. Because images $xa_i = (a_i(z))y$ are proportional. Conversely, if an element of $Y \otimes Z$ has proportional images under these maps then it is a pure tensor.

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A similar approach detects embeddings of finite groups into exceptional groups of Lie type.

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$mn - l$ (quadratic) polynomial conditions on $n + m$ unknowns.

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A (linear) recursion: In the base case, map a has a right inverse r. We have the eigenvector problem for the map br.

Otherwise, write N for the non-zero nullspace of a and M for its image Nb. Both a and b map N to M, so there are induced maps $[a], [b] : X/N \to Y/M$. A relative eigenvector x for a and b maps to a relative eigenvector $x + N$ for $[a]$ and $[b]$. The following lemma shows that relative eigenvectors lift back from $[a]$ and $[b]$ to a and b.

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Lemma

Suppose that $x + N$ is a relative eigenvector for $[a]$ and $[b]$ with $(x + N)[b] = \lambda(x + N)[a]$. Then the set of lifting vectors $x^\dagger \in x + N \subset X$ for which $x^\dagger b = \lambda x^\dagger a$ is a (non-empty) coset of $N(a) \cap N(b)$.

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Proof. Let x^* be any lift of $x + N$ to X. Let $y = x^* a$. We have

$$(x + N)[b] = \lambda(x + N)[a] = \lambda(y + M).$$

So, $x^* b = \lambda y + m$, with $m \in M = Nb$. Pick $n \in N$ with $nb = m$. Let $x^\dagger = x^* - n$. Then $x^\dagger a = y$ and $x^\dagger b = x^* b - nb = \lambda y + m - m = \lambda x^\dagger a$. $\square$

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Given a set of matrices with entries in the field k, does there exists a relative eigenvector whose first coordinate is 1 and whose other coordinates belong to k?

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This clearly belongs to NP. To prove that it's in NPC we can construct a reduction from:

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A formula in CNF is a conjunction of clauses, each of which is a disjunction of terms. The terms are selected from n variables and their negations. The following is an example with 2 clauses, 4 variables and 5 terms:

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From a formula that uses n variables, m clauses and ℓ terms we construct an instance of the Relative Eigenvector Decision Problem that uses $1 + n + \ell + m$ matrices with size $(\ell + 1)(2n + 1) \times (2n + 2)$ and entries in the field F_2.

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Is the following NPC? Does it lead to special relative eigenvector problems?

Problem (Tensor Decomposition)

Given an absolutely irreducible matrix group, does there exist a tensor decomposition?