# Nilpotence just beyond groups

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# Finite nilpotent groups

The most important theorem about finite nilpotent groups is of course:

**Theorem (Decomposition Theorem)**

*A finite group is nilpotent iff it is a direct product of $p$-groups.*

We therefore know everything about finite nilpotent groups, modulo our knowledge of $p$-groups.

We don't understand $p$-groups very much.

# Universal algebra

Universal algebraists came up with the “correct” notion of **nilpotence** for general algebraic structures (details later).

It turns out that (universal algebraic) nilpotence is too weak to furnish the analog of the Decomposition Theorem. There exist finite nilpotent algebraic structures very close to groups that are not direct products of algebras of prime power orders.

In 2010, Eichinger and Mudrinski identified a concept stronger than nilpotence, so-called **supernilpotence**, that precisely guarantees the Decomposition Theorem in reasonable varieties.

# Mal'cev term and permutable congruences

A **congruence** of an algebra $A$ is an equivalence relation on $A$ preserved by all operations of $A$.

Congruences can be **composed**: $x\alpha\beta y$ iff there is $z$ such that $x\alpha z\beta y$.

A variety $V$ is **congruence permutable** if $\alpha\beta = \beta\alpha$ for any $A \in V$ and any congruences $\alpha$, $\beta$ of $A$.

**Theorem (Mal'cev 1954)**

*A variety is congruence permutable iff it contains a* **Mal'cev term***, a term satisfying* $m(x, y, y) = x = m(y, y, x)$.

In groups, we can take $m(x, y, z) = xy^{-1}z$.

# Supernilpotence

A **polynomial** is a term with constants.

## Definition

Let $A$ be an algebra, $p(x_1, \ldots, x_n)$ a polynomial, $(e_1, \ldots, e_n) \in A^n$ and $e \in A$. Then $p$ is **absorbing at** $(e_1, \ldots, e_n)$ **into** $e$ if whenever $a_i = e_i$ for some $i$ then $p(a_1, \ldots, a_n) = e$.

## Definition

An algebra $A$ is $k$-**supernilpotent** if every polynomial of arity $n > k$ that is absorbing at some $(e_1, \ldots, e_n) \in A^n$ into some $e \in A$ is constant. An algebra is **supernilpotent** if it is $k$-supernilpotent for some $k$.

## Theorem (Eichinger+Mudrinski 2010)

*Let $V$ be a congruence permutable variety. Then a finite algebra in $V$ is supernilpotent iff it is a direct product of algebras of prime power order.*

# Absorption in groups

In groups:

- it suffices to consider absorption at $(1, \ldots, 1)$ into $1$:
  replace $p(x_1, \ldots, x_n)$ with $p(x_1^{-1}e_1, \ldots, x_n^{-1}e_n)e^{-1}$,
- the commutator $[x, y] = x^{-1}y^{-1}xy$ and all complex commutators are prototypical absorbing terms.

# Nilpotence vs. supernilpotence in groups

We still did not define nilpotence in general. Anyway:

**Theorem (Eichinger+Mudrinski, Moorhead)**

*In general, $k$-supernilpotence implies $k$-nilpotence. But nilpotence (even $2$-nilpotence) does not imply supernilpotence.*

**Theorem (folklore, E+M wrote it down, S+V gave a conceptually simple proof in 2023)**

*A group is $k$-nilpotent iff it is $k$-supernilpotent.*

# Main idea of the group proof

- If $G$ is $k$-supernilpotent then the absorbing commutator

$$[x_1, [x_2, [\ldots, [x_k, x_{k+1}] \ldots]]]$$

  is constant (thus trivial), and $G$ is $k$-nilpotent from upper central series considerations.

- The other direction is more complicated (about 3 pages now).

- Use a 1934 result of Phillip Hall: *In a $k$-nilpotent group all complex commutators of weight $k+1$ vanish*.

- Rewrite any absorbing polynomial, attempting to order commutators first by their support and then by complexity. This will succeed, using the fact that $(\mathbb{N}^m, \leq_{\mathrm{lex}})$ has no infinite descending chains and commutators of weight $> k$ vanish.

- Apply a technical result (next page).

# A technical result

## Lemma

*Let* $(A, , \cdot, 1, \dots)$ *be an algebra such that* $1$ *is the identity element with respect to the binary operation* $\cdot$*. Let* $p$ *be an n-ary polynomial on A with support* $\{x_1, \dots, x_n\}$ *that is absorbing at* $(e_1, \dots, e_n)$ *into 1. Assume that* $p$ *is equivalent to*

$$\prod_{S \in \mathcal{S}} p_S,$$

*where the product (in some order and in some parenthesizing) ranges over a subset* $\mathcal{S}$ *of proper subsets of* $\{1, \dots, n\}$*, and where every* $p_S$ *is a polynomial with support* $\{x_i : i \in S\}$ *that is absorbing at* $(e_i : i \in S)$ *into* $1$*. Then* $p$ *is constant.*

# Loops

## Definition

A **loop** is a groupoid $(Q, \cdot)$ with identity element $1$ such that all translations $L_x : y \mapsto xy$, $R_x : y \mapsto yx$ are permutations of $Q$.

Groups are associative loops (two-sided inverses follow).

The space between general loops and groups is vast.

A **Moufang loop** is a loop satisfying $x(y(xz)) = ((xy)x)z$. Think octonions.

# Nilpotence in loops: fortunately like in groups!

**Definition**

The **center** $Z(Q)$ of a loop consists of all elements of $Q$ that commute and *associate* with all other elements.

Normal subloop $=$ kernel of homomorphisms. Factor loop is defined as usual.

Upper central series and $k$-nilpotence are then defined as in groups.

# Commutators and associators

These are unique elements $[x, y]$, $[x, y, z]$ such that

$$yx = (xy)[x, y], \quad x(yz) = ((xy)z)[x, y, z].$$

Note that they are absorbing.

# $k$-supernilpotent loops for $k = 1, 2$

**Theorem**

*For $k \in \{1, 2\}$, a loop is $k$-supernilpotent iff it is a $k$-nilpotent group.*

There exists a $2$-nilpotent loop (of order $8$) that is not supernilpotent.

So, in loops, $k$-nilpotence and $k$-supernilpotence diverge already at $k = 2$.

# 3-supernilpotent loops

## Theorem (S+V 2023)

*A loop is 3-supernilpotent iff the following identities hold:*

$$\begin{aligned}
1 &= [x,[y,u,v]],\\
1 &= [x,y,[u,v,w]] = [x,[u,v,w],y] = [[u,v,w],x,y],\\
1 &= [x,y,[u,v]] = [x,[u,v],y] = [[u,v],x,y],\\
1 &= [x,[y,[u,v]]] = [x,[[u,v],y]],\\
1 &= [[y,[u,v]],x] = [[[u,v],y],x],\\
1 &= [[x,y],[u,v]],\\
[xy,u,v] &= [x,u,v]\,[y,u,v],\\
[u,xy,v] &= [u,x,v]\,[u,y,v],\\
[u,v,xy] &= [u,v,x]\,[u,v,y].
\end{aligned}$$

# $k$-supernilpotent loops with $k > 3$?

The general theory guarantees that there is an equational basis for $k$-supernilpotent algebras $V_k$ in a given variety $V$, constructed from an equational basis for $V$.

If $V$ is finitely based, it is not clear if $V_k$ is finitely based.

**Conjecture**

*$k$-supernilpotent loops are finitely based.*

That should not be too hard to prove. Finding a small basis will be hard.

# Centralizing a congruence

In 1978, Freese and McKenzie developed commutator theory for congruence modular varieties.

## Definition

Let $A$ be an algebra and $\alpha$, $\beta$, $\delta$ congruences of $A$. Then $\alpha$ **centralizes** $\beta$ **over** $\delta$, written $C(\alpha,\beta;\delta)$, if

$$t(a,u_1,\dots,u_n)\delta t(a,v_1,\dots,v_n) \;\Rightarrow\; t(b,u_1,\dots,u_n)\delta t(b,v_1,\dots,v_n)$$

whenever $t$ is a term, $a\alpha b$, and $u_i\beta v_i$.

## Definition

The **commutator** $[\alpha,\beta]$ **of congruences** is the smallest congruence $\delta$ such that $C(\alpha,\beta;\delta)$.

# Series

Let $0_A = \{(a,a) : a \in A\}$ be the smallest and $1_A = A \times A$ the largest congruence of $A$.

An algebra $A$ is **nilpotent** if the "lower central series"

$$\gamma_0 = 1_A, \quad \gamma_{i+1} = [\gamma_i, 1_A]$$

reaches $0_A$ in finitely many steps.

An algebra $A$ is **solvable** if the "derived series"

$$\gamma^0 = 1_A, \quad \gamma^{i+1} = [\gamma^i, \gamma^i]$$

reaches $0_A$ in finitely many steps.

# Congruence commutators in groups and loops

Normal subloops $=$ congruence classes containing $1$.

Deviations from commutativity and associativity:

$$T_a(x) = a\backslash(xa), \quad L_{a,b}(x) = (ab)\backslash(a(bx)), \quad R_{a,b}(x) = ((xa)b)/(ab).$$

**Theorem (S+V 2014, improved by Barnes 2021)**

*Let $\alpha$, $\beta$ be congruences of a loop $Q$. Then $[\alpha,\beta]$ is the congruence generated by*

$$(T_{u_1}(a), T_{v_1}(a)),\ (L_{u_1,u_2}(a), L_{v_1,v_2}(a)),\ (R_{u_1,u_2}(a), R_{v_1,v_2}(a)),$$

*where $1\alpha a$ and $u_i\beta v_i$.*

# Fun with conjugation in groups

$$
\begin{aligned}
T_u(a)^{-1}T_v(a) &= T_u(a)^{-1}T_{bu}(a) = (u^{-1}au)^{-1}(bu)^{-1}abu \\
&= u^{-1}a^{-1}uu^{-1}b^{-1}abu = u^{-1}a^{-1}b^{-1}abu = [a,b]^u = [a^u, b^u].
\end{aligned}
$$

This pretty much shows that we recover $[A, B] = \langle [a, b] : a \in A, b \in B \rangle$ in groups.

# Congruence vs. classical nilpotence/solvability

In groups, congruence nilpotence $=$ classical nilpotence, and congruence solvability $=$ classical solvability.

In loops, congruence nilpotence $=$ classical nilpotence, but congruence solvability is strictly stronger than classical solvability.

*Classical solvability*: $1 = A_0 \leq A_1 \leq \cdots \leq A_n = A$, where each factor $A_{i+1}/A_i$ is an abelian group.

*Congruence solvability*: $1 = A_0 \leq A_1 \leq \cdots \leq A_n = A$, where each factor $A_{i+1}/A_i$ induces an abelian congruence of $A/A_i$. Here, a congruence $\alpha$ is **abelian** if $[\alpha, \alpha] = 0_A$.

# Open problems

- For which varieties of loops the two solvability theories coincide?
- In which varieties of loops does $A \trianglelefteq Q$ imply that $A$ induces an abelian congruence of $Q$?
- Moufang loops do not satisfy the second, but *might* satisfy the first.

# Some "odd" results

**Theorem (Glaubermann 1968)**

*Moufang loops of odd order are classically solvable.*

**Theorem (Drápal + V 2023)**

*Moufang loops of odd order are congruence solvable.*

**Theorem (Drápal + V, submitted)**

*The two theories of solvability coincide in 6-divisible Moufang loops.*

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