Tensor Clusters:
The hard case

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Applications

Data
Results of measurement & computation, e.g.

Mercury is 2.5cm high in tube.

Information
data used to make a decision, e.g.

Thermometer reads 38°F; I'll wear a coat.

The Data Problem
Turn data into information.

Nutrition Matrix

Nutrients$\times$Produce data collected and used on diet problems.

	Apple	Beef	Egg	Filbert	$\cdots$	Strawberry
Carbs	13.8	1	2	4.8	$\cdots$	7.7
Protein	0.3	20.7	17.1	3.9		0.7
Fat	0.2	7.4	14.4	1.3		0.3
$\vdots$	$\vdots$					

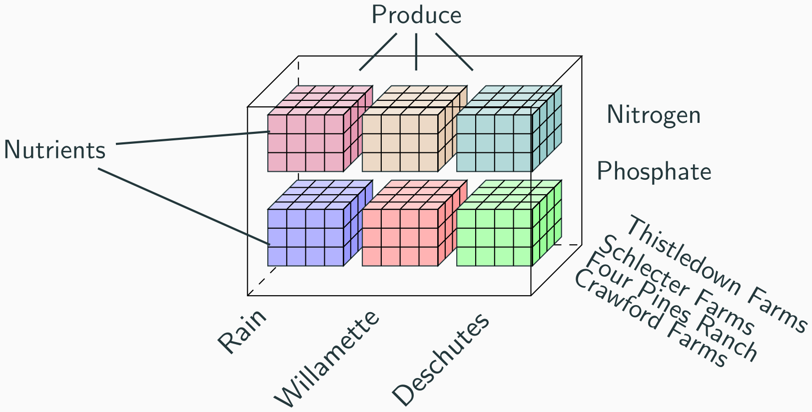
Substitute other examples

minerals$\times$ mines, pollutants$\times$water-source, keywords$\times$authors

Nutrition Tensor

Reality is more complex...

Nutrients $\times$ Produce $\times$ Farms $\times$ Water Source $\times$ Fertilizer



First Approximation of Tensor

Fox coefficients K and set $[n] = \{1, \ldots, n\}$.

A *hypermatrix* (some call it a "tensor") is a function $\Gamma : [d_1] \times \cdots \times [d_\ell] \to K$.

$$K^{d_1 \times \cdots \times d_\ell} = \{\Gamma : [d_1] \times \cdots \times [d_\ell] \to K\}.$$

Technical matter: For general sets $X_1, \ldots, X_\ell$
$\Gamma : X_1 \times \cdots \times X_\ell \to K$ has *finite support* (only finitely many nonzero values).

Data collection chooses bases for convenience/practicality/instrumentation/safety/laws/...

- Measure each fruit, farm, fertilizer separately
- Choose nutrients a lab can measure (Carbs, Sugar, Vit. A, Vit. B...)

Data collection is basis dependent;
Likely some qualities of nutrition are basis independent.

The Tensor-Data Problem
Find basis invariant properties of tensor data.

(This is now a math problem!)

Second Approximation of Tensor

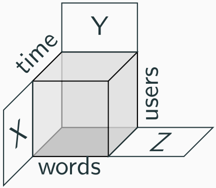
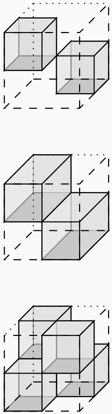
A function $\langle\Gamma| : K^{X_1} \times \cdots \times K^{X_\ell} \rightarrowtail K$ where

$$\langle\Gamma|u_1, \ldots, u_\ell\rangle = \sum_{x_1 \in X_1} \cdots \sum_{x_\ell \in X_\ell} \Gamma_{x_1 \cdots x_\ell} u_{1x_1} \cdots u_{\ell x_\ell}.$$

$\rightarrowtail$? Just reminds me to tell you this is same "multi-linear"

$$(\forall a) \qquad \langle t|u_a + \lambda\tilde{u}_a, u_{\bar{a}}\rangle = \langle t|u_a, u_{\bar{a}}\rangle + \lambda\langle t|\tilde{u}_a, u_{\bar{a}}\rangle$$

Example: Find change of basis matrices to get clusters

$(\exists X)(\exists Y)(\exists Z)$  $\stackrel{?}{=}$ 

Then ask: *What type? Are they unique? How to find them?*

The Tensor Product

Main Idea

$$\text{Data table} \equiv \text{Multiplication Table}$$

So study multiplication tables.

For finite sets

$$\boxtimes : R^m \times R^n \rightarrowtail R^{m\times n} \quad u \boxtimes v = uv^\dagger = \begin{bmatrix} u_1 v_1 & \cdots & u_1 v_n \\ \vdots & & \vdots \\ u_m v_1 & \cdots & u_m v_n \end{bmatrix}.$$

In general possibly infinite matrices, but finite support

$$\boxtimes : R^X \times R^Y \rightarrowtail R^{X\times Y} \qquad\qquad (u \boxtimes v)_{xy} = u_x v_y.$$

The Tensor Product

A right R-modules has a presentation

$$U_R = \operatorname{Pres}_R\langle X \mid S\rangle = R^X / \operatorname{Span} S$$

A left R-module has a presentation

$${}_R V = \operatorname{Pres}_R\langle Y \mid T\rangle = R^Y / \operatorname{Span} T$$

Their **Tensor Product** is

$$U \otimes_R V = R^{X\times Y}/(R^X \boxtimes T + S \boxtimes R^Y)$$

along with the induced function

$$\begin{gathered} \otimes : U \times V \rightarrowtail U \otimes_R V \\ x \otimes y = x \boxtimes y + (R^X \boxtimes T + S \boxtimes R^Y). \end{gathered}$$

$$U_R = \mathrm{Pres}_R\langle X \mid S\rangle = R^X / \operatorname{Span} S;$$
$$_R V = \mathrm{Pres}_R\langle Y \mid T\rangle = R^Y / \operatorname{Span} T$$

$$\begin{tikzcd}
\operatorname{Span} S \arrow[r, hook] & R^X \arrow[r, two heads] & U_R = \operatorname{Span} X \\[-1em]
\times & \times & \times \\[-1em]
\operatorname{Span} T \arrow[r, hook] \arrow[d, tail, "\boxtimes"] & R^Y \arrow[r, two heads] \arrow[d, tail, "\boxtimes"] & {}_R V = \operatorname{Span} Y \arrow[d, tail, "\otimes"] \\
R^X \boxtimes T + S \boxtimes R^Y \arrow[r, hook] & R^{X\times Y} \arrow[r, two heads] & U \otimes_R V
\end{tikzcd}$$

Example

$$(\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}) \otimes (\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z})$$

Solution.

$$\begin{aligned}
&\begin{bmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{bmatrix} \Bigg/ \left(\begin{bmatrix} \mathbb{Z} \\ \mathbb{Z} \\ \mathbb{Z} \end{bmatrix} \begin{bmatrix} 4\mathbb{Z} & 8\mathbb{Z} \end{bmatrix} + \begin{bmatrix} 2\mathbb{Z} \\ 6\mathbb{Z} \\ 0\mathbb{Z} \end{bmatrix} \begin{bmatrix} \mathbb{Z} & \mathbb{Z} \end{bmatrix} \right) \\
&= \begin{bmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{bmatrix} \Bigg/ \begin{bmatrix} 4\mathbb{Z} + 2\mathbb{Z} & 8\mathbb{Z} + 2\mathbb{Z} \\ 4\mathbb{Z} + 6\mathbb{Z} & 8\mathbb{Z} + 6\mathbb{Z} \\ 4\mathbb{Z} + 0\mathbb{Z} & 8\mathbb{Z} + 0\mathbb{Z} \end{bmatrix} \\
&= \begin{bmatrix} \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} \\ \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} \\ \mathbb{Z}/4\mathbb{Z} & \mathbb{Z}/8\mathbb{Z} \end{bmatrix}
\end{aligned}$$

$\mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Q}$

Solution

$\mathbb{Q} = \left\langle \frac{1}{1!}, \frac{1}{2!}, \frac{1}{3!}, \ldots \right\rangle = \langle e_1, e_2, \ldots \mid e_n = (n+1)e_{n+1} \rangle$

$$\mathbb{Z}^{[1]\times\mathbb{N}} \bigg/ \left(2\mathbb{Z} \boxtimes \mathbb{Z}^{\mathbb{N}} + \mathbb{Z} \boxtimes \begin{bmatrix} 1 & -2 & & \\ & 1 & -3 & \\ & & \ddots & \ddots \end{bmatrix} \right)$$

$$\begin{bmatrix} 1 & -2 & & \\ & 1 & -3 & \\ & & \ddots & \ddots \\ 2 & 0 & & \\ & 2 & 0 & \\ & & \ddots & \ddots \end{bmatrix}$$

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$\mathbb{Q} = \left\langle \frac{1}{1!}, \frac{1}{2!}, \frac{1}{3!}, \dots \right\rangle = \langle e_1, e_2, \dots \mid e_n = (n+1)e_{n+1} \rangle$

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$$\begin{bmatrix} 1 & -2 & & \\ & 1 & -3 & \\ & & \ddots & \ddots \\ 2 & 0 & & \\ & 2 & 0 & \\ & & \ddots & \ddots \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & & \\ & 1 & -3 & \\ & & \ddots & \ddots \\ 0 & 4 & & \\ & 2 & 0 & \\ & & \ddots & \ddots \end{bmatrix}$$

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Solution

$\mathbb{Q} = \left\langle \frac{1}{1!}, \frac{1}{2!}, \frac{1}{3!}, \ldots \right\rangle = \langle e_1, e_2, \ldots \mid e_n = (n+1)e_{n+1} \rangle$

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$$\begin{bmatrix} 1 & -2 & & \\ & 1 & -3 & \\ & & \ddots & \ddots \\ 2 & 0 & & \\ & 2 & 0 & \\ & & \ddots & \ddots \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & & \\ & 1 & -3 & \\ & & \ddots & \ddots \\ 0 & 0 & & \\ & 0 & 6 & \\ & & \ddots & \ddots \end{bmatrix}$$

$\mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Q}$

Solution

$\mathbb{Q} = \left\langle \frac{1}{1!}, \frac{1}{2!}, \frac{1}{3!}, \ldots \right\rangle = \langle e_1, e_2, \ldots \mid e_n = (n+1)e_{n+1} \rangle$

$$\mathbb{Z}^{[1]\times\mathbb{N}} \Big/ \left(2\mathbb{Z} \boxtimes \mathbb{Z}^{\mathbb{N}} + \mathbb{Z} \boxtimes \begin{bmatrix} 1 & -2 & & \\ & 1 & -3 & \\ & & \ddots & \ddots \end{bmatrix} \right)$$

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$\mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Q}$

Solution
$\mathbb{Q} = \left\langle \frac{1}{1!}, \frac{1}{2!}, \frac{1}{3!}, \dots \right\rangle = \langle e_1, e_2, \dots \mid e_n = (n+1)e_{n+1} \rangle$

$$\mathbb{Z}^{[1]\times\mathbb{N}} \Bigg/ \left(2\mathbb{Z} \boxtimes \mathbb{Z}^{\mathbb{N}} + \mathbb{Z} \boxtimes \begin{bmatrix} 1 & -2 & & \\ & 1 & -3 & \\ & & \ddots & \ddots \end{bmatrix} \right)$$

$$\mathbb{Z}^{1\times\mathbb{N}} \Big/ \mathbb{Z} \boxtimes \begin{bmatrix} 1 & -2 & & \\ & 1 & -3 & \\ & & \ddots & \ddots \end{bmatrix} \cong \mathbb{Q}$$

Theory

Tensor products are distributive

$$(u + \tilde{u}) \otimes v = u \otimes v + \tilde{u} \otimes v$$

$$u \otimes (v + \tilde{v}) = u \otimes v + u \otimes \tilde{v}$$

$$(U \oplus \tilde{U}) \otimes_R V = U \otimes_R V \oplus \tilde{U} \otimes_R V$$

$$U \otimes_R (V \oplus \tilde{V}) = U \otimes_R V \oplus U \otimes \tilde{V}$$

Theory

Tensor products have a 1.

$$U \otimes_R R \cong U \qquad\qquad R \otimes_R V \cong V$$

Theory

Sometimes it is said tensor products are associative and commutative, *that is nonsense*

$$\mathbb{R}^{\otimes}$$

But in special cases, e.g. if R is *commutative*

$$U \otimes_R (V \otimes_R W) \cong (U \otimes_R V) \otimes_R W$$
$$U \otimes_R V \cong V \otimes_R U.$$

But be warned $U_1 \otimes \cdots \otimes U_n$ is generally not defined!

Universal Mapping Property

Further fact $ur \boxtimes v = (ur)v^\dagger = u(rv)^\dagger = u \boxtimes rv$;
so $ur \otimes v = u \otimes rv$.

Theorem
If $* : U \times V \rightarrowtail W$ is distributive and $\forall r \in R$, $ur * v = u * rv$ then $\exists! \widehat{*} : U \otimes_R V \to W$ where

$$u * v = \widehat{*}(u \otimes v).$$

In particular $U \otimes_R V$ does not depend on choice of presentations.

Whitney Tensor Product

Every module has its "regular" presentation

$$U_R = \mathrm{Pres}_R\langle e_u, u \in U \mid e_{u+\tilde{u}} = e_u + e_{\tilde{u}}, e_{ur} = e_u r\rangle$$

$$_R V = \mathrm{Pres}_R\langle e_v, v \in V \mid e_{v+\tilde{v}} = e_v + e_{\tilde{v}}, e_{rv} = re_v\rangle$$

With these presentations we recover Whitney's definition (i.e. your textbook definition)

$$U \otimes_R V = R^{U \times V} \Big/ \left\langle \begin{array}{c} e_{u+\tilde{u}} \otimes e_v = e_u \otimes e_v + e_{\tilde{u}} \otimes e_v, \\ e_u \otimes e_{v+\tilde{v}} = e_u \otimes e_v + e_u \otimes e_{\tilde{v}}, \\ e_{ur} \otimes e_v = e_u \otimes e_{rv} \end{array} \right\rangle$$

So established methods are a special case.

Similar ideas picking up on matrix-vector product $R^{m \times n} \times R^n \rightarrowtail R^m$ give rise to

$$U \oslash V \times V \rightarrowtail U$$

Which behave like fractions, e.g.

$$A \oslash K \cong A \qquad A \to K \oslash (K \oslash A)$$

$$A \oslash (B \otimes C) = A \oslash B \oslash C$$

I.e.

$$\hom(C \otimes B, A) \cong \hom(C, \hom(B, A))$$

Solving for Coordinates

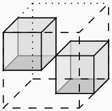
Applications
We *have* a tensor. Why would we want a tensor product?

What if $R = R_1 \oplus R_2$?

$$\begin{tikzcd}
(R_1 \oplus R_2)^X \arrow[r, two heads] & U_R = \operatorname{Span} X \\
\times & \times \\
(R_1 \oplus R_2)^Y \arrow[r, two heads] \arrow[d, tail, "\boxtimes"] & {}_R V = \operatorname{Span} Y \arrow[d, tail, "\otimes"] \\
(R_1 \oplus R_2)^{X \times Y} \arrow[r, two heads] & U \otimes_{R_1 \oplus R_2} V
\end{tikzcd}$$

What if $R = R_1 \oplus R_2$?

$$
\begin{tikzcd}
R_1^X \oplus R_2^X \arrow[r, two heads] & U_1 \oplus U_2 \\
\times & \times \\
R_1^Y \oplus R_2^Y \arrow[r, two heads] \arrow[d, tail, "\boxtimes"] & V_1 \oplus V_2 \arrow[d, tail, "\otimes"] \\
R_1^{X \times Y} \oplus R_2^{X \times Y} \arrow[r, two heads] & (U_1 \otimes_{R_1} V_1) \oplus (U_2 \otimes_{R_2} V_2)
\end{tikzcd}
$$

That's one of these:  ...for which people are hunting!

Solve for **universal** R!

$$
\begin{tikzcd}
(?)^X \arrow[r, two heads] & U_? = \operatorname{Span} X \arrow[r, equal] & U \\
\times & \times & \times \\
(?)^Y \arrow[r, two heads] \arrow[d, tail, "\boxtimes"] & {}_?V = \operatorname{Span} Y \arrow[r, equal] \arrow[d, tail, "\otimes"] & V \arrow[d, tail, "*"] \\
(?)^{X\times Y} \arrow[r, two heads] & U \otimes_? V \arrow[r] & W
\end{tikzcd}
$$

Solution is the **Centroid**

$$C(*) = \{(X, Y, Z) \mid Xu * v = Z(u * v) = u * (Yv)\}$$

Cluster Algorithm (W. 2008)

$C(*) = \{(X, Y, Z) \mid Xu * v = Z(u * v) = u * (Yv)\}$

Random $\alpha \in C(*)$,
if $\min_\alpha(x) = a(x)b(x)$ with $\deg a(x), \deg b(x) > 0$ and

$$1 = \mathrm{GCD}(a(x), b(x)) = s(x)a(x) + t(x)b(x)$$

Then $e = s(\alpha)a(\alpha)$; $f = t(\alpha)b(\alpha)$
Return $U = eU \oplus fU$, $V = eV \oplus fV$, $W = eW \oplus fW$.
Else try another α.

Theorem W. The resulting clusters are unique.

Adjoints

Universal Property

Theorem Brooksbank-Wilson.
The **adjoint algebra** is the largest set of operators over which we can for tensor product.

Proof of Universality of Adjoints

Proof.
Suppose $\Omega \to \mathrm{End}(U) \times \mathrm{End}(V)^{op}$ $\square$