Reliable Tensor Clusters

CC-BY 2021 James B. Wilson
Colorado State University

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Applications

Data
Results of measurement & computation, e.g.

Mercury is 2.5cm high in tube.

Information
data used to make a decision, e.g.

Thermometer reads $38^\circ F$*; I'll wear a coat.*

The Data Problem
Turn data into information.

Nutrition Matrix

Nutrients$\times$Produce data collected and used on diet problems.

	Apple	Beef	Egg	Filbert	$\cdots$	Strawberry
Carbs	13.8	1	2	4.8	$\cdots$	7.7
Protein	0.3	20.7	17.1	3.9		0.7
Fat	0.2	7.4	14.4	1.3		0.3
$\vdots$	$\vdots$					

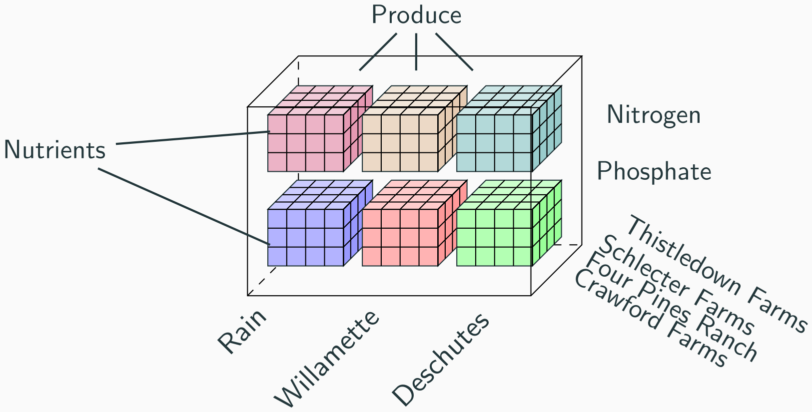
Substitute other examples

minerals$\times$ mines, pollutants$\times$water-source, keywords$\times$authors

Nutrition Tensor

Reality is more complex...

Nutrients $\times$ Produce $\times$ Farms $\times$ Water Source $\times$ Fertilizer



First Approximation of Tensor

Fox coefficients K and set $[n] = \{1, \ldots, n\}$.

A *hypermatrix* (some call it a "tensor") is a function
$\Gamma : [d_1] \times \cdots \times [d_\ell] \to K$.

$$K^{d_1 \times \cdots \times d_\ell} = \{\Gamma : [d_1] \times \cdots \times [d_\ell] \to K\}.$$

Technical matter: For general sets $X_1, \ldots, X_\ell$
$\Gamma : X_1 \times \cdots \times X_\ell \to K$ has *finite support* (only finitely many nonzero values).

Data collection chooses bases for convenience/practicality/instrumentation/safety/laws/...

- Measure each fruit, farm, fertilizer separately
- Choose nutrients a lab can measure (Carbs, Sugar, Vit. A, Vit. B...)

Data collection is basis dependent;
Likely some qualities of nutrition are basis independent.

The Tensor-Data Problem
Find basis invariant properties of tensor data.

(This is now a math problem!)

Second Approximation of Tensor

A function $\langle\Gamma| : K^{X_1} \times \cdots \times K^{X_\ell} \rightarrowtail K$ where

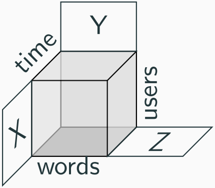
$$\langle\Gamma|u_1, \ldots, u_\ell\rangle = \sum_{x_1 \in X_1} \cdots \sum_{x_\ell \in X_\ell} \Gamma_{x_1 \cdots x_\ell} u_{1x_1} \cdots u_{\ell x_\ell}.$$

$\rightarrowtail$? Just reminds me to tell you this is same "multi-linear"

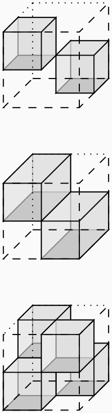
$$(\forall a) \qquad \langle t|u_a + \lambda\tilde{u}_a, u_{\bar{a}}\rangle = \langle t|u_a, u_{\bar{a}}\rangle + \lambda\langle t|\tilde{u}_a, u_{\bar{a}}\rangle$$

Example: Find change of basis matrices to get clusters

$(\exists X)(\exists Y)(\exists Z)$

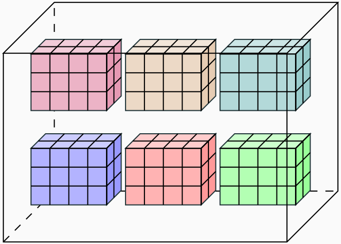


$\stackrel{?}{=}$



Then ask: *What type? Are they unique? How to find them?*

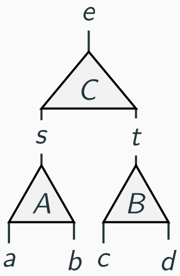
Tensors can store structured probabilities even non-stochastic such a quantum states.



$$\longleftrightarrow \quad \Gamma_{abcde} = \sum_{s} \sum_{t} A_{abs} B_{cdt} C_{ste}$$

Tensors can store structured probabilities even non-stochastic such a quantum states.

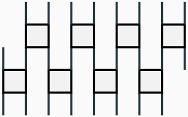
$$\Gamma_{abcde} = \sum_s \sum_t A_{abs} B_{cdt} C_{ste} \longleftrightarrow$$



Now we've actually encoded particles whose states are entangled.

Boundry Physics of Quantum Matter

What is the boundary physics of a *Quantum Material*: a network of entangled particles?



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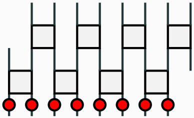

Graphic depicts a quantum *conductor* because quantum operator in red moves through material.

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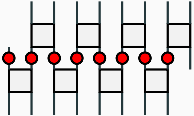

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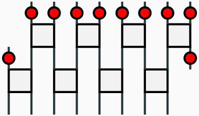

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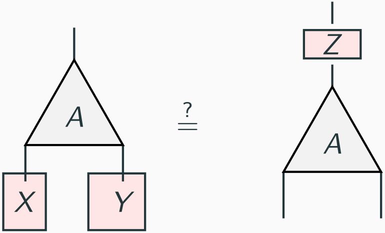
Boundary physics

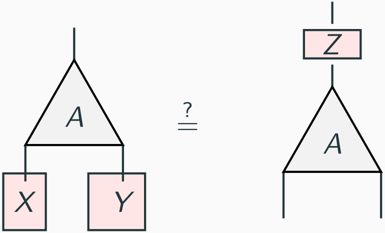


Is just another way of asking if tensors are isomorphic:

In product notation

$$(Xu) * (Yv) = Z(u * v)$$

Boundary physics



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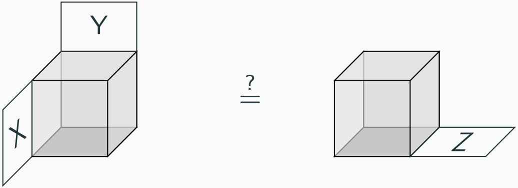

In product notation

$$(Xu) * (Yv) = Z(u * v)$$

The Tensor Product

Main Idea

$$\text{Data table} \equiv \text{Multiplication Table}$$

So study multiplication tables.

For finite sets

$$\boxtimes : R^m \times R^n \rightarrowtail R^{m\times n} \quad u \boxtimes v = uv^\dagger = \begin{bmatrix} u_1 v_1 & \cdots & u_1 v_n \\ \vdots & & \vdots \\ u_m v_1 & \cdots & u_m v_n \end{bmatrix}.$$

In general possibly infinite matrices, but finite support

$$\boxtimes : R^X \times R^Y \rightarrowtail R^{X\times Y} \qquad (u \boxtimes v)_{xy} = u_x v_y.$$

The Tensor Product

A right R-modules has a presentation

$$U_R = \operatorname{Pres}_R\langle X \mid S\rangle = R^X / \operatorname{Span} S$$

A left R-module has a presentation

$${}_R V = \operatorname{Pres}_R\langle Y \mid T\rangle = R^Y / \operatorname{Span} T$$

Their **Tensor Product** is

$$U \otimes_R V = R^{X\times Y}/(R^X \boxtimes T + S \boxtimes R^Y)$$

along with the induced function

$$\begin{aligned} \otimes : U \times V &\rightarrowtail U \otimes_R V \\ x \otimes y &= x \boxtimes y + (R^X \boxtimes T + S \boxtimes R^Y). \end{aligned}$$

$$U_R = \mathrm{Pres}_R\langle X \mid S\rangle = R^X / \operatorname{Span} S;$$
$${}_R V = \mathrm{Pres}_R\langle Y \mid T\rangle = R^Y / \operatorname{Span} T$$

$$\begin{tikzcd}
\operatorname{Span} S \arrow[r, hook] & R^X \arrow[r, two heads] & U_R = \operatorname{Span} X \\
\times & \times & \times \\
\operatorname{Span} T \arrow[r, hook] \arrow[d, tail, "\boxtimes"] & R^Y \arrow[r, two heads] \arrow[d, tail, "\boxtimes"] & {}_R V = \operatorname{Span} Y \arrow[d, tail, "\otimes"] \\
R^X \boxtimes T + S \boxtimes R^Y \arrow[r, hook] & R^{X \times Y} \arrow[r, two heads] & U \otimes_R V
\end{tikzcd}$$

Example

$$(\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}) \otimes (\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z})$$

Solution.

$$\begin{bmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{bmatrix} \Bigg/ \left(\begin{bmatrix} \mathbb{Z} \\ \mathbb{Z} \\ \mathbb{Z} \end{bmatrix} \begin{bmatrix} 4\mathbb{Z} & 8\mathbb{Z} \end{bmatrix} + \begin{bmatrix} 2\mathbb{Z} \\ 6\mathbb{Z} \\ 0\mathbb{Z} \end{bmatrix} \begin{bmatrix} \mathbb{Z} & \mathbb{Z} \end{bmatrix} \right)$$

$$= \begin{bmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{bmatrix} \Bigg/ \begin{bmatrix} 4\mathbb{Z} + 2\mathbb{Z} & 8\mathbb{Z} + 2\mathbb{Z} \\ 4\mathbb{Z} + 6\mathbb{Z} & 8\mathbb{Z} + 6\mathbb{Z} \\ 4\mathbb{Z} + 0\mathbb{Z} & 8\mathbb{Z} + 0\mathbb{Z} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} \\ \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} \\ \mathbb{Z}/4\mathbb{Z} & \mathbb{Z}/8\mathbb{Z} \end{bmatrix}$$

$\mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Q}$

Solution
$\mathbb{Q} = \left\langle \frac{1}{1!}, \frac{1}{2!}, \frac{1}{3!}, \ldots \right\rangle = \langle e_1, e_2, \ldots \mid e_n = (n+1)e_{n+1} \rangle$

$$\mathbb{Z}^{[1]\times\mathbb{N}} \Big/ \left(2\mathbb{Z} \boxtimes \mathbb{Z}^{\mathbb{N}} + \mathbb{Z} \boxtimes \begin{bmatrix} 1 & -2 & & \\ & 1 & -3 & \\ & & \ddots & \ddots \end{bmatrix} \right)$$

$$\begin{bmatrix} 1 & -2 & & \\ & 1 & -3 & \\ & & \ddots & \ddots \\ 2 & 0 & & \\ & 2 & 0 & \\ & & \ddots & \ddots \end{bmatrix}$$

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$$\mathbb{Z}^{1\times\mathbb{N}} \Big/ \mathbb{Z} \boxtimes \begin{bmatrix} 1 & -2 & & \\ & 1 & -3 & \\ & & \ddots & \ddots \end{bmatrix} \cong \mathbb{Q}$$

Theory

Tensor products are distributive

$$(u + \tilde{u}) \otimes v = u \otimes v + \tilde{u} \otimes v$$
$$u \otimes (v + \tilde{v}) = u \otimes v + u \otimes \tilde{v}$$
$$(U \oplus \tilde{U}) \otimes_R V = U \otimes_R V \oplus \tilde{U} \otimes_R V$$
$$U \otimes_R (V \oplus \tilde{V}) = U \otimes_R V \oplus U \otimes \tilde{V}$$

Theory

Tensor products have a 1.

$$U \otimes_R R \cong U \qquad\qquad R \otimes_R V \cong V$$

Theory

Sometimes it is said tensor products are associative and commutative, *that is nonsense*

$$\mathbb{R}^{\otimes}$$

But in special cases, e.g. if R is *commutative*

$$U \otimes_R (V \otimes_R W) \cong (U \otimes_R V) \otimes_R W$$
$$U \otimes_R V \cong V \otimes_R U.$$

But be warned $U_1 \otimes \cdots \otimes U_n$ is generally not defined!

Universal Mapping Property

Further fact $ur \boxtimes v = (ur)v^\dagger = u(rv)^\dagger = u \boxtimes rv$;
so $ur \otimes v = u \otimes rv$.

Theorem
If $* : U \times V \rightarrowtail W$ is distributive and $\forall r \in R$, $ur * v = u * rv$ then $\exists! \widehat{*} : U \otimes_R V \to W$ where

$$u * v = \widehat{*}(u \otimes v).$$

In particular $U \otimes_R V$ does not depend on choice of presentations.

Whitney Tensor Product

Every module has its “regular” presentation

$$U_R = \mathrm{Pres}_R\langle e_u, u \in U \mid e_{u+\tilde{u}} = e_u + e_{\tilde{u}}, e_{ur} = e_u r\rangle$$

$$_R V = \mathrm{Pres}_R\langle e_v, v \in V \mid e_{v+\tilde{v}} = e_v + e_{\tilde{v}}, e_{rv} = re_v\rangle$$

With these presentations we recover Whitney's definition (i.e. your textbook definition)

$$U \otimes_R V = R^{U\times V} \Bigg/ \left\langle \begin{array}{c} e_{u+\tilde{u}} \otimes e_v = e_u \otimes e_v + e_{\tilde{u}} \otimes e_v, \\ e_u \otimes e_{v+\tilde{v}} = e_u \otimes e_v + e_u \otimes e_{\tilde{v}}, \\ e_{ur} \otimes e_v = e_u \otimes e_{rv} \end{array} \right\rangle$$

So established methods are a special case.

Similar ideas picking up on matrix-vector product $R^{m \times n} \times R^n \rightarrowtail R^m$ give rise to

$$U \oslash V \times V \rightarrowtail U$$

Which behave like fractions, e.g.

$$A \oslash K \cong A \qquad A \to K \oslash (K \oslash A)$$

$$A \oslash (B \otimes C) = A \oslash B \oslash C$$

I.e.

$$\hom(C \otimes B, A) \cong \hom(C, \hom(B, A))$$

Solving for Coordinates

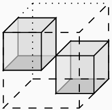
Applications
We *have* a tensor. Why would we want a tensor product?

What if $R = R_1 \oplus R_2$?

$$\begin{tikzcd}
(R_1 \oplus R_2)^X \arrow[r, two heads] & U_R = \operatorname{Span} X \\
\times & \times \\
(R_1 \oplus R_2)^Y \arrow[r, two heads] \arrow[d, tail, "\boxtimes"] & {}_R V = \operatorname{Span} Y \arrow[d, tail, "\otimes"] \\
(R_1 \oplus R_2)^{X \times Y} \arrow[r, two heads] & U \otimes_{R_1 \oplus R_2} V
\end{tikzcd}$$

What if $R = R_1 \oplus R_2$?

$$\begin{tikzcd} R_1^X \oplus R_2^X \arrow[r, two heads] & U_1 \oplus U_2 \\ \times & \times \\ R_1^Y \oplus R_2^Y \arrow[r, two heads] \arrow[d, tail, "\boxtimes"] & V_1 \oplus V_2 \arrow[d, tail, "\otimes"] \\ R_1^{X\times Y} \oplus R_2^{X\times Y} \arrow[r, two heads] & (U_1 \otimes_{R_1} V_1) \oplus (U_2 \otimes_{R_2} V_2) \end{tikzcd}$$

That's one of these:  ...for which people are hunting!

Solve for **universal** R!

$$\begin{array}{ccccc}
(?)^X & \twoheadrightarrow & U_? = \operatorname{Span} X & = & U \\
\times & & \times & & \times \\
(?)^Y & \twoheadrightarrow & {}_?V = \operatorname{Span} Y & = & V \\
\downarrow \boxtimes & & \downarrow \otimes & & \downarrow * \\
(?)^{X\times Y} & \twoheadrightarrow & U \otimes_? V & \longrightarrow & W
\end{array}$$

Solution is the **Centroid**

$$C(*) = \{(X, Y, Z) \mid Xu * v = Z(u * v) = u * (Yv)\}$$

Cluster Algorithm (W. 2008)

$$C(*) = \{(X, Y, Z) \mid Xu * v = Z(u * v) = u * (Yv)\}$$

Random $\alpha \in C(*)$,
if $\min_\alpha(x) = a(x)b(x)$ with $\deg a(x), \deg b(x) > 0$ and

$$1 = \mathrm{GCD}(a(x), b(x)) = s(x)a(x) + t(x)b(x)$$

Then $e = s(\alpha)a(\alpha)$; $f = t(\alpha)b(\alpha)$
Return $U = eU \oplus fU$, $V = eV \oplus fV$, $W = eW \oplus fW$.
Else try another α.

Theorem W. The resulting clusters are unique.

Other observation: Writing a product over larger coefficients makes for smaller dimensions.

Isomorphism testing speeds up by orders of magnitude.

Wakeup Pure Algebra; you're directly useful!

For data science tensors use is still spartan.

Different decompositions call for solving for different coordinate types, adjoints, nuclei, derivations, ...

Such ideas essentially the same arose in

- Kaplansky and Jacobson non-associative algebras.
- Mal'cev and Miyasnikov on Groups of finite Morley Rank.
- Finite Simple groups, e.g. Parker-Norton MeatAxe and Schneider's work
- W. Direct and central product decomps. of p-groups.

Look back...did R have to be a ring?

$$\begin{tikzcd}
\operatorname{Span} S \arrow[r, hook] & R^X \arrow[r, two heads] & U_R = \operatorname{Span} X \\
\times & \times & \times \\
\operatorname{Span} T \arrow[r, hook] \arrow[d, tail, "\boxtimes"] & R^Y \arrow[r, two heads] \arrow[d, tail, "\boxtimes"] & {}_R V = \operatorname{Span} Y \arrow[d, tail, "\otimes"] \\
R^X \boxtimes T + S \boxtimes R^Y \arrow[r, hook] & R^{X \times Y} \arrow[r, two heads] & U \otimes_R V
\end{tikzcd}$$

This quotient exists because we create a two-sided ideal. No use of associative ring R.

You can use **any non-associative algebra** A.

Assume Lie, e.g. $[a, b] = ab - ba$ in a ring R.

$$\begin{tikzcd}
\operatorname{Span} S \arrow[r, hook] & A^X \arrow[r, two heads] & U_A = \operatorname{Span} X \\
\times & \times & \times \\
\operatorname{Span} T \arrow[r, hook] \arrow[d, tail, "\boxtimes"] & A^Y \arrow[r, two heads] \arrow[d, tail, "\boxtimes"] & {}_A V = \operatorname{Span} Y \arrow[d, tail, "\otimes"] \\
R^X \boxtimes T + S \boxtimes R^Y \arrow[r, hook] & A^{X \times Y} \arrow[r, two heads] & U \otimes_a V
\end{tikzcd}$$

With a lie algebra $ua \otimes v \neq u \otimes av$ but instead

$$[a, u \otimes v] = [a, u] \otimes v + u \otimes [a, v]$$

Lie tensor products

Definability
Whitney tensor product $U_0 \otimes_{R_1} \cdots \otimes_{R_n} U_n$ not generally well-defined (need bimodules or commutativity, i.e. constraints).

Lie tensor product $\llparenthesis U_1, \ldots, U_n \rrparenthesis_L$ exists for arbitrary numbers of modules.

Lie tensor products

Global reach
Whitney tensor product $U_0 \otimes_{R_1} \cdots \otimes_{R_n} U_n$ (when defined) compares nearest neighbor relationships.

Lie tensor product $\llparenthesis U_1, \ldots, U_n \rrparenthesis_L$ compares all modules to all others.

Lie tensor products

Size
Whitney Tensor Product

$$\dim U_0 \otimes_{R_1} \cdots \otimes_{R_n} U_n \approx \frac{\dim U_1 \cdots \dim U_n}{\dim R_1 \cdots \dim R_n}$$

Lie tensor product

$$\dim \llparenthesis U_1, \ldots, U_n \rrparenthesis_L$$

given by subtle Clebsch-Gordan formulas, can actually be bounded by a constant while $\dim U_a$ go to infinity.

Universality of Derivation Tensors "Densors"

Theorem First-Maglione-W.
If a tensor T factors through a tensor product over any nonassociative algebra A then A then the tensor product $\otimes_A : U \times V \rightarrowtail U \otimes_A V$ factors through the Lie tensor product over the derivations of T.

In formally the Lie tensor product over derivations is universal.

Limits on Associativity

Theorem FMW
Over fields with some nondegeneracy conditions. If $\Omega \subset \mathrm{End}(U_1) \times \cdots \times \mathrm{End}(U_n)$ is closed to composition then the operators in Ω are limited to nearest neighbor interactions governed by binomials

$$X^{e_1} - X^{f_1}, \dots, X^{e_n} - X^{f_n}$$

where the neighborhood is given by the support of (e_i, f_i)