Group isomorphism is nearly-linear time for most orders

IEEE Foundations On Computer Science FOCS 2021

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Motivation

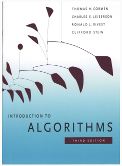
Outward Facing Motive: honest data types

Where in this...

.

Outward Facing Motive: honest data types

Where in this...



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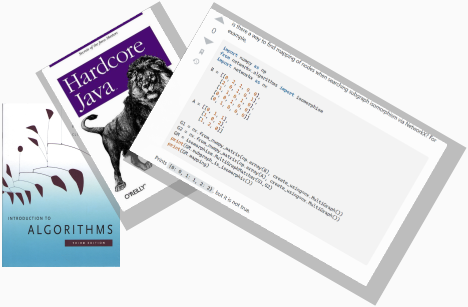

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Outward Facing Motive: honest data types

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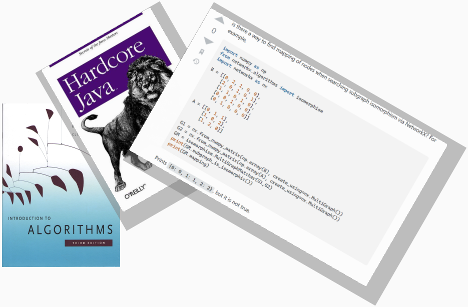

...do we send people to get help making this...

```
boolean equals(Object that) {
    // <this> can transform into <that>?
}
```

Why groups(oids)?

- **Transitive→ Partial Multiplication**

$$trans_{xyz} : (x \equiv y) \wedge (y \equiv z) \Rightarrow (x \equiv z)$$
$$* : Eq \times Eq \dashrightarrow Eq$$

- **Reflexive→ Identity**

$$\frac{\begin{aligned} refl_x &: x \Rightarrow (x \equiv x) \\ trans_{xxy} &: (x \equiv x) \wedge (x \equiv y) \Rightarrow (x \equiv y) \end{aligned}}{Identity : refl * evidence = evidence}$$

- **Symmetric→ Inverse**

$$\frac{\begin{aligned} sym_{xy} &: (x \equiv y) \Rightarrow (y \equiv x) \\ trans_{xyx} &: (x \equiv y) \wedge (x \equiv y) \Rightarrow (x \equiv x) \end{aligned}}{Inverses : evidence * (evidence)^{-1} = refl}$$

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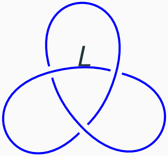
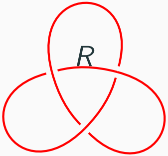
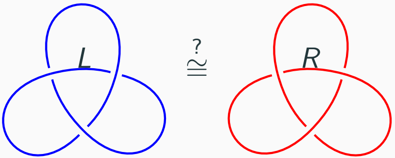
Anatomy of hard equality



$\stackrel{?}{\cong}$



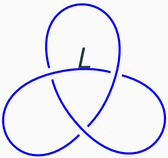
Anatomy of hard equality



Relax category until automorphisms computable.

$\pi_1(L) = \langle x, y \mid x^2 = y^3 \rangle$ $\qquad$ $\pi_1(R) = \langle x, y \mid x^2 = y^{-3} \rangle$

Anatomy of hard equality



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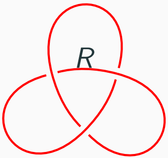

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$$\mathrm{Aut}_\chi \pi_1(L) \quad \not\cong_\chi \quad \mathrm{Aut}_\chi \pi_1(R)$$

Recursively refine comparing automorphisms with incrementally stricter properties. E.g. respect crossing number χ?

Inward facing Motive: equalivance surveys complexity

FPGroupIso Adjan, Rabin ‘50’s		Undecideable
PlaneGroupIso Dietrich et.al. STACS‘21		Σ_3^P
BlackBoxGroupIso Babai-Szemerédi FOCS‘84	**MatroidIso**	Σ_2^P
PermGroupIso Luks DIMACS		$\Sigma_1^P = NP$
CayleyGroupIso Tarjan	**GraphIso** Babai	$DTIME(2^{\log^c n})$
TableGroupIsoAbel Kivitha (nearly-linear in RAM model)		$DTIME(n^2 \log^c n)$
TableGroupIsoMostOrders, IsGroup This Talk		$DTIME(n \log^c n)$

Schreier-Sims

Problem: Transport

Given: A set Ω, allowed permutations X, $\omega, \omega' \in \Omega$

Return: decide if a string g over X maps ω to ω', written $\omega^g = \omega'$, and give all such g.[1]

[1] Give words W over X so that $\omega^h = \omega'$ implies $h = wg$ for a string w over W.

String Isomorphism

"Eighth" $==$ "HeigHt"

String Isomorphism

- **Given** strings $s, t : I \to \Sigma$ allowed permutations $G = \langle g_k \rangle \leq \mathrm{Sym}_I$, $H = \langle h_k \rangle \leq \mathrm{Sym}_\Sigma$
- **Return** strings $g = g_{a_1} \cdots g_{a_u}$ and $h = h_{b_1} \cdots h_{b_v}$ where $h(s_i) = t_{g(i)}$; or prove impossible.

$$\begin{array}{ccc} I & \xrightarrow{s} & \Sigma \\ \downarrow g & & \downarrow h \\ I & \xrightarrow{t} & \Sigma \end{array}$$

Theorem. (Babai 2016+) If Σ fixed, $\textsc{StringIso}$ is in Quasipolynomial $n^{O((\log n)^c)}$-time.

($\textsc{GraphIso} \leq_P \textsc{StringIso}$)

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$$\begin{tikzcd} I \arrow[r, "s"] \arrow[d, "g"] & \Sigma \arrow[d, "h"] \\ I \arrow[r, "t"] & \Sigma \end{tikzcd}$$

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Snapshot of solving a hard isomorphism problem

Math Facts

Schreier-Sims

‘65

Snapshot of solving a hard isomorphism problem

Math Facts

Exponential
Babai-Luks
STOC83

Bounded Valence
Luks ‘82

Color Refinement
Weisfeiler-Leman,
Babai-Erdős-
Selkow,... 70-80’s

Schreier-Sims
‘65

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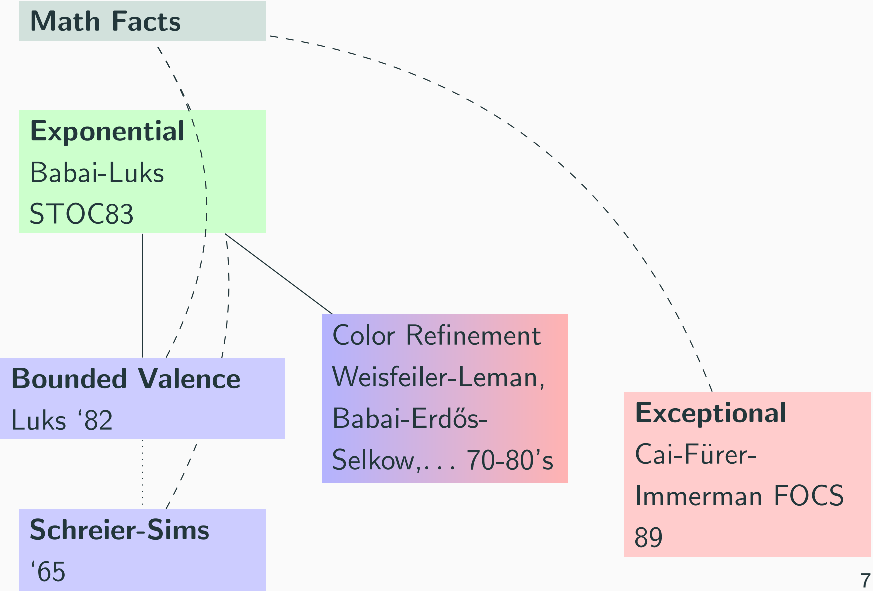

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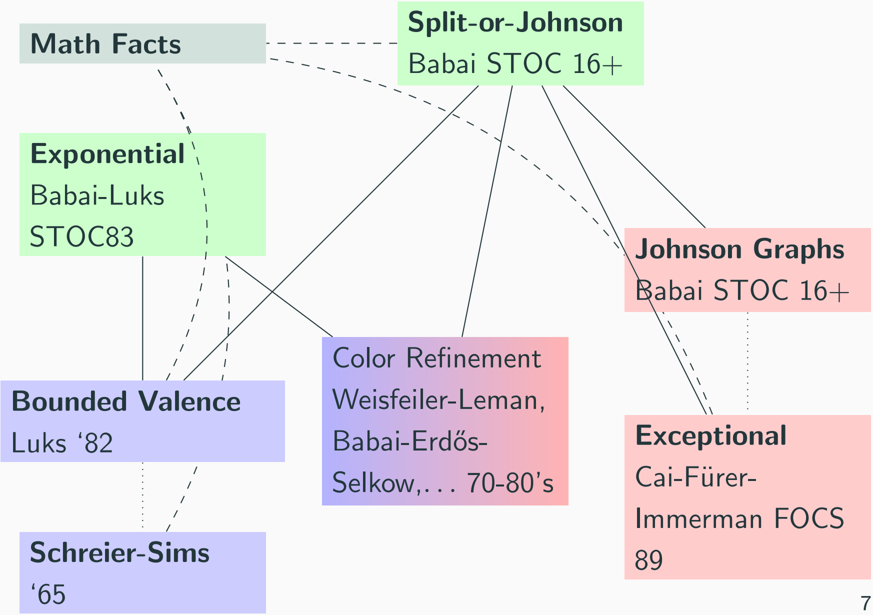

Isomorphism of Tables

Code Equivalence

$$\boxed{\begin{array}{cc} L & i \\ v & e \end{array}} == \boxed{\begin{array}{cc} E & v \\ i & l \end{array}}$$

Code Equivalence[2]

- **Given** $s, t : I \times J \to \Sigma$, (generators for) permutations $R \leq \mathrm{Sym}_I$, $C \leq \mathrm{Sym}_J$, & $V \leq \mathrm{Sym}_\Sigma$
- **Return** $\sigma \in R$, $\tau \in C$, $\mu \in V$,

$$\begin{array}{ccccc} I & \times & J & \xrightarrow{s} & \Sigma \\ \downarrow \sigma & & \downarrow \tau & & \downarrow \mu \\ I & \times & J & \xrightarrow{t} & \Sigma \end{array}$$

$$\mu(s_{ij}) = t_{\sigma(i)\tau(j)}$$

Babai-Codenotti-Grochow-Qiao $2^{O(n)}$-time bound for constant alphabet Σ (SODA '11)

Builds on Luks $2^{O(n)}$-hypergraph isomorphism, FOCS '99.

[2] Non-linear twisted, with variable alphabet.

Algebra Isomorphism

$$\begin{array}{|ccccc|}\hline 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 & 3 \\ 5 & 1 & 2 & 3 & 4 \\ \hline\end{array} == \begin{array}{|ccccc|}\hline 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 5 & 3 \\ 3 & 1 & 5 & 2 & 4 \\ 4 & 5 & 2 & 3 & 1 \\ 5 & 3 & 4 & 1 & 2 \\ \hline\end{array}$$

Algebra Isomorphism

- **Given** $s, t : I \times I \to I$, (generators for) permutations $G \leq \mathrm{Sym}_I$,
- **Return** $\sigma \in G$,

$$\begin{array}{ccccc} I & \times & I & \xrightarrow{s} & I \\ \downarrow\sigma & & \downarrow\sigma & & \downarrow\sigma \\ I & \times & I & \xrightarrow{t} & I \end{array}$$

$$\sigma(s_{ij}) = t_{\sigma(i)\sigma(j)}$$

Group Isomorphism Strategy

Math Facts

Generator Test
Tarjan, Miller
STOC78

Group Isomorphism Strategy

Data Types

Math Facts

(Nearly) Abelian
Kivitha '07, LeGall

Adjoint Tensor
Lewis-W. '12

Semisimplicty
Babai-Codenotti-Qiao ICALP '11

Generator Test
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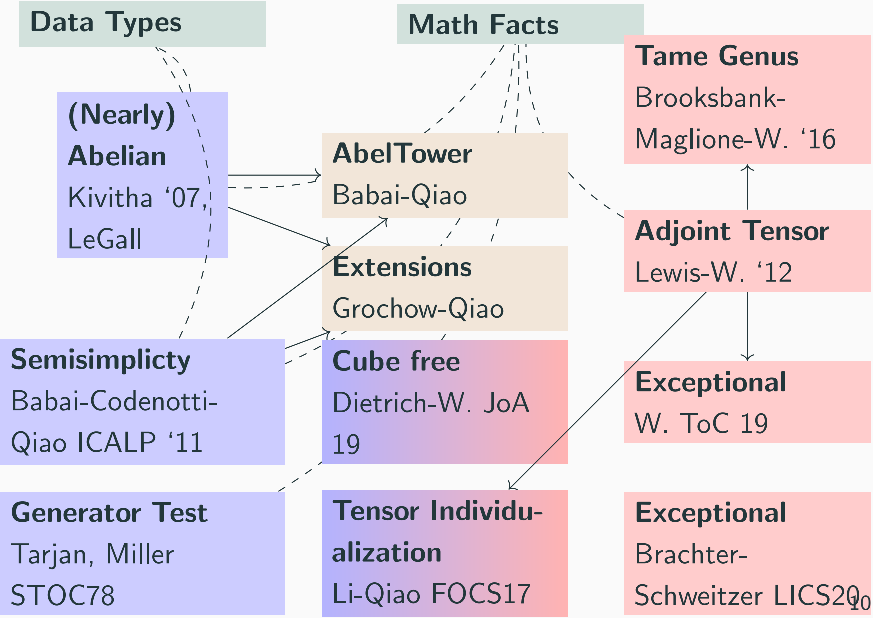
Group Isomorphism Strategy



Group Isomorphism of most orders

Divide and conquer

Isomorphism of $G = A \times B$, i.e.

$$(a, b)(\tilde{a}, \tilde{b}) = (a\tilde{a}, b\tilde{b})$$

reduces to isomorphism of A and B in parallel.[3]

Isomorphism of $G = A \ltimes_\theta B$, i.e.

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reduces to isomorphism of A and B sequentially, plus adjusting θ.[4]

[3] Technical detail: decompose maximally and adjust non-unique decompositions by Krull-Schmidt; see W. 2008.

[4] Lemma 2.1

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Division Graph: Erdős-Pálfy

Factor n into a *graph* $\Gamma(n)$.

Edge $(p_i^{e_i}, p_j^{e_j})$ where $p_i | p_j^k - 1$ for some $k \leq e_j$, & symmetrically.

Erdős-Pálfy, 1999
A group G of order n factors as

$$N_1 \times \cdots \times N_\ell,$$

$|N_i| = n_i$, order of connected components of $\Gamma(n)$.

Example $n = 1785$

$|G| = n \Rightarrow N_{3\cdot 7} \times N_5 \times N_{17}$

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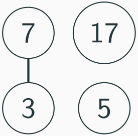

$$|G| = n \Rightarrow N_{3\cdot 7} \times N_5 \times N_{17}$$

Extending implications

Factor n into a *direct hypergraph* $\mathcal{H}(n)$. (i) *Oriented* Erdős-Pálfy *hyper*-edges, (ii) exceptions for finite nonabelian simple groups.

Proposition
A group G of order n factors as

$$N_0 \ltimes (N_1 \ltimes \cdots \ltimes N_\ell),$$

$|N_i| = n_i$, where n_i is order of interconnected components of $\mathcal{H}(n)$.

Example $n = 5,810,340$

$|G| = n \Rightarrow N_{60} \ltimes (N_{179} \times N_{547})$

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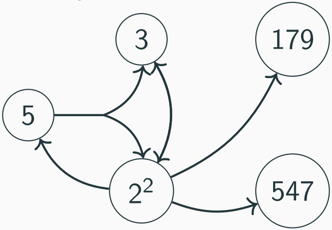

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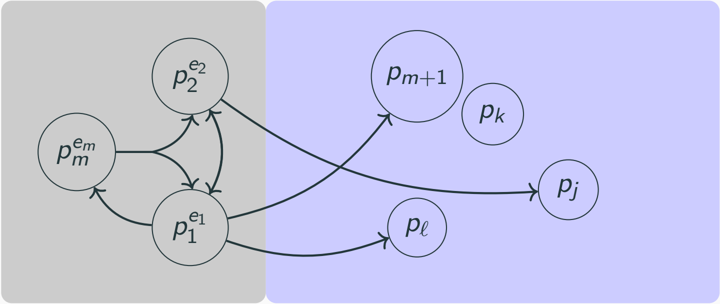
Most Orders



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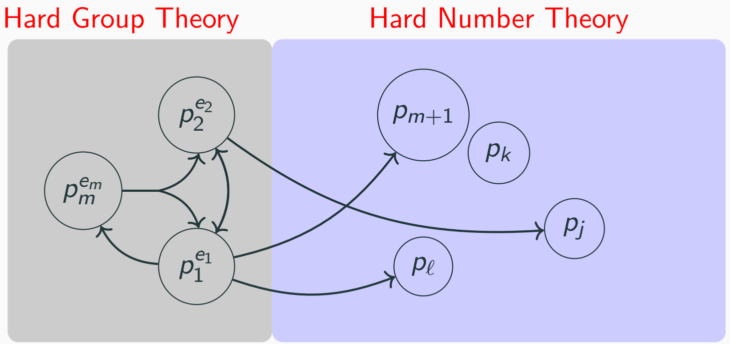
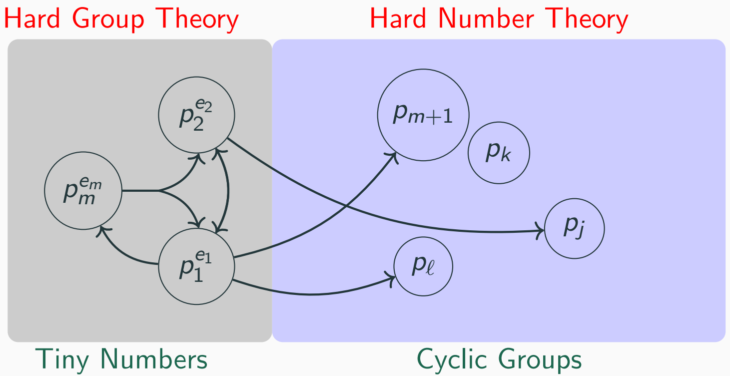


Hard Group Theory
Hard Number Theory
$p_2^{e_2}$
$p_m^{e_m}$
$p_1^{e_1}$
p_{m+1}
p_k
p_j
p_ℓ

Most Orders



$$G = H \ltimes (\mathbb{Z}_{p_{m+1}} \times \cdots \times \mathbb{Z}_{p_\ell})$$

Is it a Group Table?

Promise-to-decision

A great many computational algebra are analyzed as **promise problems** not **decision problems**.

Identity testing is needed to remove the promise; unsolvable in general (word problem), but on tables at least brute-force.

Theorem Rajagopalan-Schulman, 2000
Given $* : [n] \times [n] \to [n]$, test associativity (and other identities) in nearly-linear time $\tilde{O}(n^2)$ in RAM model (constant time ops and memory access). Also can test if a group.

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RAM-to-TM

At larger scales Turing Machine (TM) model better match to computations that are communication bounded (typical in practice).

RAM -¿ TM at most a quadratic blow-up.

Corollary
Nearly Quadratic-time $\tilde{O}(n^4)$ on multi-tape Turing Machine (TM).

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Given $* : [n] \times [n] \to [n]$, test if a group in time nearly-linear time $\tilde{O}(n^2)$ on deterministic multi-tape TM.

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IsGroup

•	1	2	3	4	5
1	1	2	3	4	5
2	2	1	4	5	3
3	3	5	1	2	4
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5	5	4	2	3	1

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$$\rho(2) = \begin{array}{|ccccc|}\hline 1 & 2 & 3 & 4 & 5 \\ \hline 2 & 1 & 4 & 5 & 3 \\ \hline\end{array} = (1,2)(3,5,4)$$

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$$G_1 = \langle (354) \rangle$$

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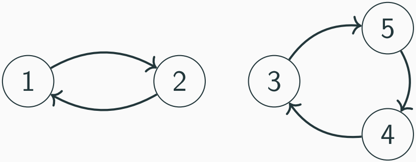

$G_1 = \langle (354) \rangle$ $\quad$ $G_{13} = \{1\}$

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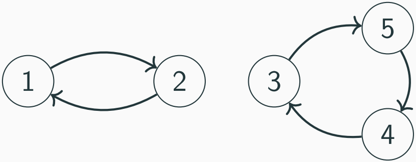

$G_1 = \langle (354) \rangle$ $\quad$ $G_{13} = \{1\}$

$|G| = [G : G_1][G_1 : G_{13}]$
$= 2 \cdot 3 = 6.$

Should be 5,
not a group.

IsGroup

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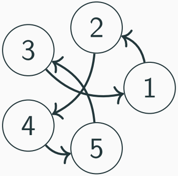

$$|G| = [G : G_1| = 5$$

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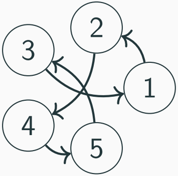

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$*$	1	2	3	4	5
$\rho(2)^0$	1	2	3	4	5
$\rho(2)^1$	2	4	1	5	3
$\rho(2)^2$	4	5	2	3	1

$\rho(2)^2 = 45231 \neq T_4 = 41532$

Not a group.

Summary

Summary

IsGroup nearly linear time
From $\tilde{O}(n^4)$ to $\tilde{O}(n^2)$: Promise-to-decision by transferring group to permutation model.

GroupIso most orders nearly linear time
From $n^{O(\log n)}$ to $\tilde{O}(n^2)$: Split group into

$$\text{hard group} \ltimes \text{hard numbers} = \text{tiny numbers} \ltimes \text{cyclic groups}.$$

Then standard divide-and-conquer.

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