Math 328B: Winter 2017

Homework 1

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Exercise 1. Prove, using the definition of the limit, that $\lim_{x\to 4}\sqrt{x+5} = 3$.

Proof. Take $\delta = 3\epsilon$ and consider $|x-4| < \delta \Rightarrow |x-4| < 3\epsilon$. From this, we develop that $\frac{|x-4|}{3} < \epsilon \Rightarrow \left|\frac{x-4}{3}\right| < \epsilon$. Now observe that, since $\sqrt{x+5}+3 > 3$, one has that $\frac{x-4}{3} > \frac{x-4}{\sqrt{x+5}+3}$ and so $\left|\frac{x-4}{3+\sqrt{x+5}}\right| < \epsilon \Rightarrow \left|\frac{(x+5)-9}{3+\sqrt{x+5}}\right| < \epsilon$ $\Rightarrow \left|\frac{\left(\sqrt{x+5}-3\right)\left(\sqrt{x+5}+3\right)}{3+\sqrt{x+5}}\right| < \epsilon \Rightarrow \left|\sqrt{x+5}-3\right| < \epsilon$, as required. □

Exercise 2. Let f be defined as

$$f(x) = \begin{cases} 1/n & \text{if } x = 1/2^n \\ 0 & \text{otherwise} \end{cases}$$

If f continuous at 0?

Solution. f is continuous if $\lim_{x\to 0} f(x) = f(0)$. Since $0 \notin \left\{\frac{1}{2^n} \;\middle|\; n \in \mathbb{N}\right\}$, we see that $f(0) = 0$, as prescribed. Now, we observe that $\lim_{x\to 0} f(x) \equiv \lim_{n\to\infty} f(x)$ because $\lim_{n\to\infty} x = \lim_{n\to\infty}\frac{1}{2^n} = 0$, so as $n\to\infty$ one sees that $x \to 0$. Thus, we see that $\lim_{n\to\infty} f(x) = \lim_{n\to\infty}\frac{1}{n} = 0 = \lim_{x\to 0} f(x)$. Thus, $\lim_{x\to 0} f(x) = f(0) = 0$ and so f is continuous about 0.

Exercise 3. Let $f(x) = \frac{2x+3}{3x-1}$ for $x \in (-\infty, 1/3)$. For any $\epsilon > 0$, find a δ such that if $|x| < \delta$ and $x < 1/3$, then $|f(x)+3| < \epsilon$.

Solution. Let us first do some scratch work:

Observe

$$\left|\frac{2x+3}{3x-1}+3\right| < \epsilon \Rightarrow \left|\frac{11x}{3x-1}\right| < \epsilon \Rightarrow \frac{|11x|}{|3x-1|} < \epsilon$$

Now, since $x < 1/3$, we have two cases to consider – if $0 < x < 1/3$, then $-1 < 3x-1 < 0$, and if $-\infty < x < 0$, then $3x-1 < 0$. We needn't consider $3x-1 = 0$, because that implies that $x = 1/3$, whereas in the context of the problem, this is not allowed. In either case, though, we recognize that $3x-1 < 0$, and so $|3x-1| = -(3x-1) = 1-3x$. Thus, we examine

$$\frac{11|x|}{1-3x} < \epsilon$$

Since $11|x| < \frac{11|x|}{1-3x}$, we see that $11|x| < \epsilon$, and thus $|x| < \frac{\epsilon}{11}$.

Thus, we find $\delta = \epsilon/11$.

Exercise 4. Let f and g be two functions defined in a neighborhood of x_0 such that $\lim_{x\to x_0} f(x) = L_f$ and $\lim_{x\to x_0} g(x) = L_g$. Prove that $\lim_{x\to x_0} f(x) + g(x) = L_f + L_g$.

Proof. We know the following:

- For any $\epsilon_f > 0$, there is a δ_f so that if $|x - x_0| < \delta_f$ then $|f(x) - L_f| < \epsilon_f$.
- For any $\epsilon_g > 0$, there is a δ_g so that if $|x - x_0| < \delta_g$ then $|g(x) - L_g| < \epsilon_g$.

Let $\epsilon > 0$ be given and take $\epsilon_f = \epsilon_g = \epsilon/2$. Also take $\delta = \min\{\delta_f, \delta_g\}$ so that, if $|x - x_0| < \delta$, then one has that $|x - x_0| < \delta_f$ for which it follows that $|f(x) - L_f| < \epsilon_f$. By the same token, one has that $|x - x_0| < \delta_g$ and thus it follows that $|g(x) - L_g| < \epsilon_g$. By the triangle inequality, one finds that

$$\underbrace{|f(x) - L_f|}_{<\epsilon/2} + \underbrace{|g(x) - L_g|}_{<\epsilon/2} \geq |(f(x) - L_f) + (g(x) - L_g)|$$

$$\implies |(f(x) - L_f) + (g(x) - L_g)| < \epsilon$$

$$\implies |(f(x) + g(x)) - (L_f + L_g)| < \epsilon$$

as required. □

Exercise 5. Prove that, if $\lim_{x \to x_0} f(x) = \infty$ and $\lim_{x \to x_0} g(x) = L$, then

1. $\lim_{x \to x_0} f(x) + g(x) = \infty$
2. $\lim_{x \to x_0} f(x)g(x) = \infty$
3. $\lim_{x \to x_0} \dfrac{g(x)}{f(x)} = 0$

1. *Proof.* We know:
   - For every $A_f > 0$, there is a δ_f so that if $|x - x_0| < \delta_f$ then $f(x) > A_f$
   - For every $\epsilon_g > 0$ there is a δ_g so that if $|x - x_0| < \delta_g$ then $|g(x) - L| < \epsilon_g$

   From this, we shall develop:

   - For every $A > 0$, there is a δ so that if $|x - x_0| < \delta$ then $f(x) + g(x) > A$

   Let $\delta = \min\{\delta_f, \delta_g\}$. Given that $|x - x_0| < \delta$, it follows that $|x - x_0| < \delta_f$ and $|x - x_0| < \delta_g$. From this, it follows that $f(x) > A_f$, given some $A_f > 0$, and $|g(x) - L| < \epsilon_g$, given some $\epsilon_g > 0$. Now, take $A = A_f + (L - \epsilon_g) > 0$. Since $|g(x) - L| < \epsilon_g$, we see that $L - \epsilon_g < g(x) < L + \epsilon_g$. In particular, we have that $g(x) > L + \epsilon_g$ and $f(x) > A_f$. From this, it follows that $f(x) + g(x) > A_f + (L - \epsilon_g)$, or $f(x) + g(x) > A$, as required. □

2. *Proof.* We know:
   - For every $A_f > 0$, there is a δ_f so that if $|x - x_0| < \delta_f$ then $f(x) > A_f$
   - For every $\epsilon_g > 0$ there is a δ_g so that if $|x - x_0| < \delta_g$ then $|g(x) - L| < \epsilon_g$

   From this, we shall develop:

   - For every $A > 0$, there is a δ so that if $|x - x_0| < \delta$ then $f(x)g(x) > A$

   Take $\delta = \min\{\delta_f, \delta_g\}$. Since $|x - x_0| < \delta$, it follows that $|x - x_0| < \delta_f \Rightarrow f(x) > A_f$ and $|x - x_0| < \delta_g \Rightarrow |g(x) - L| < \epsilon_g$. Now, let $A = A_f(L - \epsilon_g) > 0$. In particular, $g(x) > L - \epsilon_g$ and so $f(x)g(x) > A_f(L - \epsilon_g) \Rightarrow f(x)g(x) > A$, as required. □

3. *Proof.* We know:
   - For every $A_f > 0$, there is a δ_f so that if $|x - x_0| < \delta_f$ then $f(x) > A_f$
   - For every $\epsilon_g > 0$ there is a δ_g so that if $|x - x_0| < \delta_g$ then $|g(x) - L| < \epsilon_g$

   From this, we shall develop:

   - For every $\epsilon > 0$, there is a δ so that if $|x - x_0| < \delta$ then $\left|\dfrac{g(x)}{f(x)}\right| < \epsilon$.

Let $\epsilon = \dfrac{\epsilon_g - L}{A_f}$ and $\delta = \min\{\delta_f, \delta_g\}$. Also assume that $|f(x)| \leq M$ for all x. Thus, if $|x - x_0| < \delta$ then simultaneously $|x - x_0| < \delta_f$ and $|x - x_0| < \delta_g$ for which it follows that $f(x) > A_f$ and $|g(x) - L| < \epsilon_g$, which gives, in particular, that $g(x) > L - \epsilon_g$. Now, note that, given our choice of ϵ, we see that

$$\epsilon = \frac{\epsilon_g - L}{A_f}$$

$$\Longrightarrow L - \epsilon_g = -\epsilon A_f$$

Thus, $g(x) > L - \epsilon_g \Rightarrow g(x) > -\epsilon A_f \Rightarrow g(x) > -\epsilon M \Rightarrow g(x) > -\epsilon|f(x)|$. From this we glean that $|g(x)| < -\epsilon|f(x)|$, which gives $\left|\dfrac{g(x)}{f(x)}\right| < \epsilon$, as required. $\square$