Real Analysis 2

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Abstract

These are lecture notes for Real Analysis 2 (Math 328) at the University of Washington.

Contents

1 Course Structure

This class will cover, very roughly:

1. Continuity of functions
   - Uniform continuity
2. Integration
   - Improper integrals
   - Uniform convergence
3. Power series

2 Continuity

In this section, we examine what it means for a function to be continuous. This section will prioritize the understanding of the limit, and will also introduce some elementary topology.

2.1 The Limit

Definition. $\lim_{x \to x_0} f(x) = L$ means that, for every $\epsilon > 0$, there is a δ so that if $|x - x_0| < \delta$ one has that $|f(x) - L| < \epsilon$.

Proposition. $\lim_{x \to x_0} f(x) = \infty$ means that, for every $A > 0$, there is a δ so that if $|x - x_0| < \delta$ one has that $f(x) > A$.

Proposition. $\lim_{x \to \infty} f(x) = L$ means that, for every $\epsilon > 0$, there is an N so that if $x > N$ one has that $|f(x) - L| < \epsilon$.

Proposition. $\lim_{x \to \infty} f(x) = \infty$ means that, for every $A > 0$, there is an N so that if $x > N$ one has that $f(x) > A$.

Theorem. If f and g are functions defined in a neighborhood of x_0 and $\lim_{x \to x_0} f$ and $\lim_{x \to x_0} g$ exist and are finite, then

1. $\lim_{x \to x_0} (f + g) = \lim_{x \to x_0} f + \lim_{x \to x_0} g$
2. $\lim_{x \to x_0} (fg) = \left(\lim_{x \to x_0} f\right)\left(\lim_{x \to x_0} g\right)$
3. $\lim_{x \to x_0} c \cdot f = c \cdot \lim_{x \to x_0} f$ (given $c \in \mathbb{R}$)
4. $\lim_{x \to x_0} \left(\frac{f}{g}\right) = \frac{\lim_{x \to x_0} f}{\lim_{x \to x_0} g}$

Theorem. Let f, g, and h be functions defined in a neighborhood of x_0. For every x in this neighborhood, one has that $f(x) \le g(x) \le h(x)$. If $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} h(x) = L$, then $\lim_{x \to x_0} g(x) = L$.

Proof. We know:

- For all $\epsilon_f > 0$, there is a δ_f so that if $|x - x_0| < \delta_f$ then $|f(x) - L| < \epsilon_f$
- For all $\epsilon_h > 0$, there is a δ_h so that if $|x - x_0| < \delta_h$ then $|f(x) - L| < \epsilon_h$

Since we desire $|g(x) - L| < \epsilon$ and $f \le g \le h$, we see that $f(x) - L \le g(x) - L \le h(x) - L$. If we take $\epsilon_f = \epsilon_h = \epsilon$ and take $\delta = \min(\delta_f, \delta_h)$ so that, if $|x - x_0| < \delta$ then $|x - x_0| < \delta_f$ so $|f(x) - L| < \epsilon_f = \epsilon$ and $|x - x_0| < \delta_h$ so that $|h(x) - L| < \epsilon_h = \epsilon$. Thus, we find that $-\epsilon < f(x) - L < \epsilon$ and $-\epsilon < h(x) - L < \epsilon$, for which it follows that

$$\epsilon < f(x) - L \le g(x) - L \le h(x) - L < \epsilon$$

$$\implies -\epsilon < g(x) - L < \epsilon$$

$$\implies |g(x) - L| < \epsilon$$

as required. ■

2.2 Continuity

Definition. Let f be defined in a neighborhood of x_0. It is said that f is **continuous** about x_0 if $\lim_{x \to x_0} f(x) = f(x_0)$.

Remark. In the language of the limit we've presented, this states that given any $\epsilon > 0$, there is a δ so that if $|x - x_0| < \delta$ then $|f(x) - f(x_0)| < \epsilon$.

Proposition. A function f is continuous on an interval I if f is continuous about any $x_0 \in I$.

Theorem. Let f and g be functions that are assumed continuous about x_0. We have the following:

1. $h(x) = f(x) + g(x)$ is continuous at x_0.
2. $h(x) = f(x)g(x)$ is continuous at x_0.
3. $h(x) = cf(x)$ (for some $c \in \mathbb{R}$) is continuous at x_0.
4. $h(x) = \dfrac{f(x)}{g(x)}$ (for $g(x) \neq 0$) is continuous at x_0.
5. $h(x) = g \circ f(x)$ is continuous at x_0.

Proof. We will examine them case-by-case:

1. $\lim_{x \to x_0} h(x) = \lim_{x \to x_0} f(x) + g(x) = \lim_{x \to x_0} f(x) + \lim_{x \to x_0} g(x) = f(x_0) + g(x_0)$
2. $\lim_{x \to x_0} h(x) = \lim_{x \to x_0} f(x)g(x) = \left(\lim_{x \to x_0} f(x)\right)\left(\lim_{x \to x_0} g(x)\right) = f(x_0)g(x_0)$
3. $\lim_{x \to x_0} h(x) = \lim_{x \to x_0} cf(x) = c \lim_{x \to x_0} f(x) = cf(x_0)$
4. $\lim_{x \to x_0} h(x) = \lim_{x \to x_0} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to x_0} f(x)}{\lim_{x \to x_0} g(x)} = \dfrac{f(x_0)}{g(x_0)}$
5. (left as an exercise)

■

2.3 Differentiability

Definition. Given f defined in a neighborhood of x_0, f is said to be **differentiable** at x_0 if $\lim_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$ exists and is finite.

Proposition. If f is differentiable at x_0, then f is continuous at x_0.

Proof. Observe that $\lim_{x \to x_0} \left[\left(\dfrac{f(x) - f(x_0)}{x - x_0}\right)(x - x_0)\right] = \left[\lim_{x \to x_0} \left(\dfrac{f(x) - f(x_0)}{x - x_0}\right)\right]\left[\lim_{x \to x_0} (x - x_0)\right] = 0$, and so $\lim_{x \to x_0} (f(x) - f(x_0)) = 0$, as required. ■

2.4 Compact Sets

We ought to become familiar with the notion of the compact set, since it will set us up to study the Extreme Value Theorem and the Intermediate Value Theorem.

2.4.1 Coverings

Definition. Given a set $\mathcal{S}$, a **covering** of $\mathcal{S}$ is a collection of sets $\{\mathcal{U}_\alpha \mid \alpha \in I\}$ such that $\mathcal{S} \subset \bigcup_\alpha \mathcal{U}_\alpha$.

EXAMPLE 1. Let $\mathcal{S} = [0, 1]$ and observe that $\mathcal{U}_\alpha = (-\alpha, \alpha)$ for $\alpha \in \mathbb{R}^+$ is a covering of $\mathcal{S}$.

Proof. In particular, we may choose $\mathcal{U}_2 = (-2, 2)$ and we observe that $[0, 1] \subset (-2, 2)$. Since any $s \in \mathcal{S}$ is also in $\mathcal{U}_2$, it is certainly in the union over all $\alpha \in \mathbb{R}^+$, and thus $(-\alpha, \alpha)$ provides a covering for $\mathcal{S}$. ■

EXAMPLE 2. Let $\mathcal{S} = \mathbb{Z}$ and observe, once again, that $\mathcal{U}_\alpha = (-\alpha, \alpha)$ for $\alpha \in \mathbb{R}^+$ is a covering of $\mathcal{S}$.

Proof. For any $z \in \mathbb{Z}$, we observe that z is in at least one $\mathcal{U}_\alpha$, and is thus in the union over all $\alpha \in \mathbb{R}^+$. ■

Definition. Given a set $\mathcal{S}$, we say that $\{\mathcal{U}_\alpha \mid \alpha \in I\}$ is a covering of $\mathcal{S}$, and a **sub-covering** of $\mathcal{S}$ is a subset of $\{\mathcal{U}_\alpha \mid \alpha \in I\}$ that covers $\mathcal{S}$.

EXAMPLE. Let $\mathcal{S} = \mathbb{Z}$ and take $\mathcal{U}_\alpha = (\alpha - 1, \alpha + 1)$ for $\alpha \in \mathbb{R}$. Observe that $\mathcal{U}_\alpha$ provides a covering for $\mathcal{S}$. Now, note that defining $\mathcal{U}_\beta = (\beta - 1, \beta + 1)$ for $\beta \in \mathbb{Z}$ admits that $\mathcal{U}_\beta \subset \mathcal{U}_\alpha$, so $\mathcal{U}_\beta$ provides a sub-covering for $\mathcal{S}$.

2.4.2 Compactness

Definition. A set $\mathcal{S}$ is said to be **compact** if, for any arbitrary covering of $\mathcal{S}$ by open sets[1], there exists a finite sub-covering of $\mathcal{S}$.

EXAMPLE. The set $\mathcal{S} = \{0, 1, 5\}$ is compact.

Proof. In particular, if we choose a covering $\mathcal{U}_\alpha$ of $\mathcal{S}$, then one particular $\mathcal{U}_\alpha$ in the covering contains 1 – likewise for 0 and 5. Thus, define the sets $\mathcal{U}_{\alpha_1}$ which contains 1, $\mathcal{U}_{\alpha_0}$ which contains 0, and $\mathcal{U}_{\alpha_5}$ which contains 5. Now construct the collection $\{\mathcal{U}_{\alpha_0}, \mathcal{U}_{\alpha_1}, \mathcal{U}_{\alpha_5}\}$ and observe that it is a subset of the covering we had declared. Also note that this collection covers $\mathcal{S}$, and is thus a sub-covering. Lastly, we see that this sub-covering is finite, and thus $\mathcal{S}$ is compact. ■

EXAMPLE 1. The interval $\mathcal{I} = (0, 1]$ is not compact.

Proof. Take the collection $\mathcal{C} = \left\{ \left(\frac{1}{n}, 2\right) \;\middle|\; n \in \mathbb{N} \right\}$ as a cover for $\mathcal{I}$. Note that, for any $x \in (0, 1]$, there is an n such that $nx > 1$ and so $2 > 1 \geq x > 1/n$. Thus, we've verified that there is an n so that $x \in \left(\frac{1}{n}, 2\right)$. This verifies that we have, in fact, identified a suitable cover. Now, we'll choose some subset of $\mathcal{C}$, namely $\mathcal{C}^* = \left\{ \left(\frac{1}{n_1}, 2\right), \left(\frac{1}{n_2}, 2\right), \ldots, \left(\frac{1}{n_p}, 2\right) \right\}$ for some finite p and assume that $n_1 > n_2 > \ldots > n_p$. We verify quite easily that

$$\bigcup_{i=1}^{p} \left(\frac{1}{n_i}, 2\right) = \left(\frac{1}{n_1}, 2\right)$$

For some $\frac{1}{2n_1} \in (0, 1]$ we see that, consequently, $\frac{1}{2n_1} \notin \bigcup_{i=1}^{p} \left(\frac{1}{n_i}, 2\right)$ – that is, we can see that $\mathcal{C}^*$ does not cover all of $\mathcal{I}$. Thus, any finite subset we choose is not a cover for $\mathcal{I}$. Thus, $\mathcal{I}$ does have a cover, but for such a cover, there is no finite sub-cover and so $\mathcal{I}$ is not compact. ■

EXAMPLE 2. The interval $\mathcal{I} = [0, 1]$ is compact.

Proof. Let $\{\mathcal{U}_\alpha\}$ be a cover of $[0, 1]$ by open sets. Let $\mathcal{I}_n$ be half of the interval $[0, 1]$ that doesn't have a finite sub-cover. Thus, $\mathcal{I}_{n+1}$ is half of $\mathcal{I}_n$, and it also doesn't have a finite sub-cover. Lastly, we note that $\mathcal{I}_{n+1} \subset \mathcal{I}_n$ and so, by the Nested Interval Theorem, we have that $\bigcap_{n \in \mathbb{N}} \mathcal{I}_n = \{x_0\}$. Thus $x_0 \in [0, 1]$ and x_0 is covered by $\{\mathcal{U}_\alpha\}$. So, there exists an open set $\mathcal{U}_{\alpha_0}$ so that $x_0 \in \mathcal{U}_{\alpha_0}$ (that is to say that there is at least one set that contains x_0). Thus, there exists a neighborhood of x_0 – call it $\mathbf{N}(x_0)$ – such that $\mathbf{N}(x_0) = (x_0 - h, x_0 + h)$ and $\mathbf{N}(x_0) \subset \mathcal{U}_{\alpha_0}$. Note that $\operatorname{diam}(\mathcal{I}_n) = 1/2^n$ and $x_0 \in \mathcal{I}_n$. So, if $1/2^n < h$, then $\mathcal{I}_n \subset (x_0 - h, x_0 + h)$, which yields that $\mathcal{I}_n \subset \mathcal{U}_{\alpha_0}$. Thus, $\mathcal{I}_n$ is covered by the sub cover $\mathcal{U}_{\alpha_0}$ – a contradiction. ■

Proposition. Any closed, real interval $[a, b] \subset \mathbb{R}$ is compact.

Theorem. (Heine-Borel) A set $\mathcal{C}$ is compact iff it is closed and bounded.

Proof. We prove two directions:

- $\boxed{\Rightarrow}$: Assume that $\mathcal{C}$ is compact – that is, for every cover of $\mathcal{C}$ by open sets, there exists a finite sub-cover. Note that, if we can cover all of $\mathbb{R}$, certainly we can cover all of $\mathcal{C}$. So choose $\{(\alpha - 1, \alpha + 1) \mid \alpha \in \mathbb{Z}\}$ as a cover for $\mathbb{R}$ – this

[1] See the course notes for Math 327 for a definition of the open set.

also acts as a cover for $\mathcal{C}$. Since $\mathcal{C}$ is compact, there exists a finite subset of our cover that also covers $\mathcal{C}$. Denote this subset by $\{(\alpha_1-1,\alpha_1+1),(\alpha_2-1,\alpha_2+1),\ldots,(\alpha_p-1,\alpha_p+1)\}$ for some finite p. Since there are finitely-many α_i, assume that $\alpha_1>\alpha_2>\ldots>\alpha_p$. Note that $\mathcal{C}\subset(\alpha_1-1,\alpha_1+1)\cup(\alpha_2-1,\alpha_2+1)\cup\ldots\cup(\alpha_p-1,\alpha_p+1)$, since this subset covers $\mathcal{C}$. We now note, in particular, that $\mathcal{C}$ is bounded below by α_p+1 and above by α_1-1. We must now show that $\mathcal{C}$ is closed – pick some $x\in\mathcal{C}^c$ and develop a cover of $\mathbb{R}$ that does not cover x. Choose this cover to be $\mathcal{U}_n=\left\{\left(-\infty,x-\frac{1}{n}\right)\cup\left(x+\frac{1}{n},\infty\right)\,\middle|\, n\in\mathbb{N}\right\}$. Note that this also covers $\mathcal{C}$. Since $\mathcal{C}$ is compact, there is a finite sub-cover of p – call it $\{\mathcal{U}_{n_1},\mathcal{U}_{n_2},\ldots,\mathcal{U}_{n_s}\}$ and assume that $n_1<n_2<\ldots<n_s$. Observe that $\mathcal{C}\subset\bigcup_{i=1}^{s}\mathcal{U}_{n_i}=\mathcal{U}_{n_s}$ and so $\left(x-\frac{1}{n_s},x+\frac{1}{n_s}\right)\subset\mathcal{C}^c$, and thus $\mathcal{C}$ is closed.

- $\boxed{\Leftarrow}$: Assume that $\mathcal{C}$ is closed and bounded. If $\mathcal{C}$ is bounded, then there exists an m and an M such that any $x\in\mathcal{C}$ obeys $m\le x\le M$. By the result that $[0,1]$ is compact, we see that $[m,M]$ is compact, as well. Since $\mathcal{C}$ is closed, we find that $\mathcal{C}^c$ is an open set, and so $\{\mathcal{C}^c,\{\mathcal{U}_\alpha\}\}$ is a cover of $[m,M]$, provided a cover of $\mathcal{C}$ by open sets $\{\mathcal{U}_\alpha\}$ (note that $\mathcal{C}\subset[m,M]$). Since $[m,M]$ is compact, there exists a finite sub-cover of $[m,M]$ – denote this sub-cover by $\{\mathcal{C}^c,\mathcal{U}_{\alpha_1},\mathcal{U}_{\alpha_2},\ldots,\mathcal{U}_{\alpha_p}\}$. Note that $\{\mathcal{U}_{\alpha_1},\mathcal{U}_{\alpha_2},\ldots,\mathcal{U}_{\alpha_p}\}$ is a cover of $\mathcal{C}$, so for every $x\in\mathcal{C}$ one has that $x\in[m,M]$ and so x is covered by $\{\mathcal{C}^c,\mathcal{U}_{\alpha_1},\mathcal{U}_{\alpha_2},\ldots,\mathcal{U}_{\alpha_p}\}$. Thus, $x\in\mathcal{C}\Rightarrow x\notin\mathcal{C}^c$ and so $x\in\{\mathcal{U}_{\alpha_1},\mathcal{U}_{\alpha_2},\ldots,\mathcal{U}_{\alpha_p}\}$, and thus there exists a finite sub-cover of $\mathcal{C}$, as required.

■

Remark. Heine-Borel can be extended to subsets of $\mathbb{R}^n$.

Theorem. Let f be defined in a neighborhood of p. It is said that f is continuous at p iff, for any sequence $\{x_n\}$ that converges to p, $\{f(x_n)\}$ converges to $f(p)$.

Proof. We prove two implications:

- $\boxed{\Rightarrow}$: Assume that f is not continuous at p – that is, assume that there exists an $\epsilon>0$ such that, for every δ, there exists an x so that $|x-p|<\delta$ and $|f(x)-f(p)|\ge\epsilon$. We desire a sequence that converges to p, so choose $\delta=1/n$. Thus, for $\delta=1/n$ there exists an x_n such that $|x_n-p|<1/n$ and $|f(x_n)-f(p)|\ge\epsilon$. Thus, $\{x_n\}$ converges to p, but $\{f(x_n)\}$ does not converge to $f(p)$. This develops a contradiction, and we are done.

- $\boxed{\Leftarrow}$: [left as an exercise]

■

Theorem. Let S be a compact set, and let f be a continuous function on S (f is continuous at every $s\in S$). We have that $f(S)$ is compact.

Remark. What the above says, essentially, is that given a closed, bounded set, its image is also closed and bounded.

Proof. Let $\{\mathcal{U}_\alpha\}_{\alpha\in I}$ be an open cover of $f(S)$. We note that, for every $x\in S$, it follows that $y=f(x)\in f(S)$. Since $\{\mathcal{U}_\alpha\}_{\alpha\in I}$ covers $f(S)$, there exists a $\mathcal{U}_{\alpha_x}$ such that $f(x)\in\mathcal{U}_{\alpha_x}$. Since $\mathcal{U}_{\alpha_x}$ is open, there exists an ϵ_x so that $(f(x)-\epsilon_x,f(x)+\epsilon_x)\subset\mathcal{U}_{\alpha_x}$. We also know that f is continuous over S, so there exists a δ_x such that if $|t-x|<\delta_x$ then $|f(t)-f(x)|<\epsilon_x$. Since $|t-x|<\delta_x$, one has that $t\in(x-\delta_x,x+\delta_x)$. We can see rather easily that $\{(x-\delta_x,x+\delta_x)\mid x\in S\}$ is an open cover of S. Since S is known to be compact, there exists a finite sub-cover, which we write as $\{(x_1-\delta_{x_1},x_1+\delta_{x_1}),(x_2-\delta_{x_2},x_2+\delta_{x_2}),\ldots,(x_p-\delta_{x_p},x_p+\delta_{x_p})\}=\{\mathcal{U}_{\alpha_{x_1}},\mathcal{U}_{\alpha_{x_2}},\ldots,\mathcal{U}_{\alpha_{x_p}}\}$. Note that there exists a $x_0\in S$ so that $y=f(x_0)$ and there exists a q so that $x_0\in(x_q-\delta_{x_q},x_q+\delta_{x_q})$. What this implies is that $|x_0-x_q|<\delta_{x_q}$ which by the continuity of f admits that $|f(x_0)-f(x_q)|<\epsilon_{x_q}$. From this, it's trivial to notice that $f(x_0)\in(f(x_q)-\epsilon_{x_q},f(x_q)+\epsilon_{x_q})\subset\mathcal{U}_{\alpha_{x_q}}$, and so $\{\mathcal{U}_{\alpha_{x_1}},\mathcal{U}_{\alpha_{x_2}},\ldots,\mathcal{U}_{\alpha_{x_q}}\}$ is a cover for $f(S)$ and it is thus compact, as required. ■

2.5 Extreme and Intermediate Value Theorems

Theorem. (Extreme value) – Let S be a non-empty, closed, and bounded set (read: compact set) and let f be continuous on S. Since $f(S)$ is bounded[2], it has a greatest lower bound, m, and a least upper bound, M. There exists an $x_m\in S$ and an $x_M\in S$ such that $f(x_m)=m$ and $f(x_M)=M$.

[2] By the previous theorem*, given a compact set S and a function f continuous over S, it follows that $f(S)$ is compact $\Rightarrow f(S)$ is closed and bounded.

Proof. Since S is compact and continuous, $f(S)$ is closed and bounded. We also observe that, since S is non-empty, there exists an $x \in S$ such that $f(x) \in f(S)$, for which it follows that $f(S)$ is non-empty, so $\inf(f(S)) = m$ and $\sup(f(S)) = M$ exist. Note that, for any $n \in \mathbb{N}$, one has that $M - 1/n$ is not an upper bound of $f(S)$, so there exists a sequence $\{y_n\}$ such that $M - 1/n < y_n \leq M$. It is worth noting, too, that $\{y_n\}$ converges to M. Thus, since $y_n \in f(S)$, there exists a sequence $\{x_n\}$ such that $f(x_n) = y_n$. Since S is a bounded set, it has within it a convergent sub-sequence, $\{x_{n_k}\}$ – that is to say that $x_{n_k} \to x$ for sufficiently large n_k. We also note that S is closed and thus contains all of its accumulation points, so $x \in S$. Thus, since $\{x_{n_k}\}$ converges to $x \in S$ and f is continuous over S, it follows that $f(x_{n_k}) \to f(x)$, which is to say that $y_{n_k} \to M$, as from before. Putting this together yields that $f(x) = M$, as required. ■

Remark. We only carried out the proof for showing that an x_M exists for which $f(x_M) = M$, but the proof that an x_m exists so that $f(x_m) = m$ follows in the exact same fashion.

Theorem. (Intermediate value) – Let f be continuous over $[a, b]$. For any $f(a) \leq y \leq f(b)$, there exists a $c \in [a, b]$ such that $f(c) = y$.

Proof. Assume, without loss of generality, that $f(a) < f(b)$ and let $S = \{x \in [a, b] \mid f(x) < y \text{ on } [a, x]\}$.

- If $f(c) < y$, let $\epsilon = \dfrac{y - f(c)}{2}$. Since f is continuous at c, we have that there exists a δ so that, if $x \in (c - \delta, c + \delta)$, then $f(x) \in (f(c) - \epsilon, f(c) + \epsilon)$ for which we see that $f(c) + \epsilon = f(c) + \dfrac{y - f(c)}{2} = \dfrac{y + f(c)}{2} < y$. Now, observe that $c - \delta$ is not an upper bound of S so there exists a $p \in S$ so that $p > c - \delta$ – take $p = c + \dfrac{\delta}{2}$ and so $f(x) < y$ on $\left[a, c + \dfrac{\delta}{2}\right]$. What this implies is that $c + \dfrac{\delta}{2} \in S$ – that is, there exists an element of S greater than its upper bound. This develops a contradiction.

- If $f(c) > y$, we let $\epsilon = \dfrac{f(c) - y}{2}$ so that there is a δ such that if $|t - c| < \delta$ then $|f(t) - f(c)| < \epsilon$. This can be rephrased to say that, if $t \in (c - \delta, c + \delta)$, then, in particular, $f(t) - f(c) > -\epsilon$. Of course, this yields that

$$f(t) - f(c) > -\left(\frac{f(c) - y}{2}\right)$$

$$\Rightarrow f(t) > \frac{y + f(c)}{2} > y$$

$$\Rightarrow f(t) > y$$

  So, for every $x \in (c - \delta, c)$ there exists a $t = \dfrac{x + c - \delta}{2} > \dfrac{c - \delta}{2}$ so that $f(t) > y$. Thus, $x \in S$ – that is, there are no elements of S between $c - \delta$ and c. Now, we know that, for every $x \in S$, one has that $x \leq c$, and for every $x \in (c - \delta, c)$, one finds that $x \notin S$. So $x \leq c - \delta$. This, of course, implies that $c - \delta$ is an upper bound of S smaller than its least upper bound. Once again, this develops a contradiction.

From this, we conclude that $f(c) = y$. ■

2.6 Uniform Continuity

Definition. A function f is **uniformly continuous** on I if, for every $\epsilon > 0$, there exists a δ so that, for every $x \in I$ and every $t \in I$, if $|t - x| < \delta$ then $|f(t) - f(x)| < \epsilon$.

EXAMPLE 1. $f(x) = x^2$ is uniformly continuous on $[0, 5]$.

Proof. For every $\epsilon > 0$, there exists a $\delta = \epsilon/10$ so that, for every $x \in [0, 5]$ and every $t \in [0, 5]$, one has that if $|t - x| < \delta$ then $|t - x| < \dfrac{\epsilon}{10}$. But,

$$|t - x| \underbrace{|t + x|}_{\leq 10} < \epsilon$$

$$\Rightarrow |(t + x)(t - x)| < \epsilon$$

$$\Rightarrow |t^2 - x^2| < \epsilon$$

$$\Rightarrow |f(t) - f(x)| < \epsilon$$

as required. ■

EXAMPLE 2. $f(x) = \dfrac{1}{x}$ is uniformly continuous on $[1, \infty)$.

Proof. For every $\epsilon > 0$, let $\delta = \epsilon$ so that, for every $x \in [1, \infty)$ and every $t \in [1, \infty)$, if $|x - t| < \delta$, then $|x - t| < \epsilon$. However, we notice that since $t \geq 1$ and $x \geq 1$, surely $|xt| \geq 1$. So $\dfrac{|x-t|}{|xt|} \leq |x - t|$ and thus $\left|\dfrac{x-t}{xt}\right| < \epsilon \Rightarrow \left|\dfrac{1}{t} - \dfrac{1}{x}\right| < \epsilon$, as required. ■

Theorem. Let S be a compact set. If f is continuous over S, then f is uniformly continuous over S.

Proof. Since f is continuous for any $x \in S$, we have that for any $\epsilon > 0$, there is a δ_x so that if $|t - x| < \delta_x$ then $|f(t) - f(x)| < \epsilon$. The continuity of f also reveals to us that $\mathcal{O} = \left\{ \left(x - \dfrac{\delta_x}{2}, x + \dfrac{\delta_x}{2} \right) \middle| \ x \in S \right\}$ is an open cover of S. However, since S is compact, there is a finite sub-collection from $\mathcal{O}$ that covers S. Let the finite sub-collection be $\bar{\mathcal{O}} = \left\{ \left(x_1 - \dfrac{\delta_1}{2}, x_1 + \dfrac{\delta_1}{2} \right), \ldots, \left(x_p - \dfrac{\delta_p}{2}, x_p + \dfrac{\delta_p}{2} \right) \right\}$ for some finite p. Let $\delta = \min \left\{ \dfrac{\delta_i}{2} \right\}_{i \in [1,p]}$ (this will be useful later). Now, there exists an i so that $x \in \left(x_i - \dfrac{\delta_i}{2}, x_i + \dfrac{\delta_i}{2} \right)$ and thus, for this particular x, one finds that $|t - x| < \delta_i/2$, which ensures that $|t - x| < \delta$. By the triangle inequality, however, we see that $|x_i - t| < \delta_i$. Notice that $|f(x) - f(t)| = |f(x) - f(x_i) + f(x_i) - f(t)|$ so that invoking the triangle inequality gives

$$|f(x) - f(t)| \leq |f(t) - f(x_i)| + |f(x_i) - f(t)|$$

$$\Rightarrow |f(x) - f(t)| < \epsilon$$

as required. ■

3 Integrals

In this section, we study the properties of the integral and what exactly is meant for a function to be *integrable*.

3.1 Darboux Sums

To set the stage, let f be defined on an interval $[a, b]$ and let f be bounded over $[a, b]$ – that is, for every $x \in [a, b]$, one finds that $m \leq f \leq M$.

Definition. Take $\{x_0, x_1, \ldots, x_p\}$ to be a **partition** of the interval $[a, b]$ where $x_0 = a$, $x_p = b$, and $x_i < x_{i+1}$ everywhere else. It's also important to note that each sub-interval within $[a, b]$ does not necessarily have the same length.

Definition. Let R be the **upper Darboux sum** over $[a, b]$:

$$R = \sum_{i=0}^{p-1} (x_{i+1} - x_i) \sup_{x \in [x_i, x_{i+1}]} (f)$$

Definition. Let r be the **lower Darboux sum** over $[a, b]$:

$$r = \sum_{i=0}^{p-1} (x_{i+1} - x_i) \inf_{x \in [x_i, x_{i+1}]} (f)$$

Proposition. When adding points to the partition of $[a, b]$, the upper Darboux sum is non-increasing and the lower Darboux sum is non-decreasing.

Proof. Add a single point, p, to the standard partition of $[a, b]$ (described above) to obtain the partition

$$x_0 = a < x_1 < \ldots < x_j \leq p < x_{j+1} < \ldots < x_p = b$$

We can now compute an upper Darboux sum over this new partition, $\bar{R}$, where

$$\bar{R} = \sum_{i=0}^{j-1} (x_{i+1} - x_i) \sup_{x \in [x_i, x_{i+1}]} (f) + (p - x_j) \sup_{x \in [x_j, p]} (f) + (x_{j+1} - p) \sup_{x \in [x_p x_{j+1}]} (f) + \sum_{i=j+1}^{n-1} (x_{i+1} - x_i) \sup_{x \in [x_i, x_{i+1}]} (f)$$

Thus, we may compute the difference between the "modified" sum and the original sum:

$$\bar{R} - R = (p - x_j) \sup_{x \in [x_j, p]} (f) + (x_{j+1} - p) \sup_{x \in [x_p x_{j+1}]} (f) - (x_{j+1} - x_j) \sup_{x \in [x_j x_{j+1}]} (f)$$

And so

$$\bar{R} - R \leq ((p - x_j) + (x_{j+1} - p) - (x_{j+1} - x_j)) \sup_{x \in [x_p x_{j+1}]} (f)$$

$$\Rightarrow \bar{R} - R \leq 0$$

$$\Rightarrow \bar{R} \leq R$$

as required. ■

Remark. The exact same steps apply in proving that the lower Darboux sum is non-decreasing.

Proposition. Let $C_1 := x_0 < x_1 < \ldots < x_n$, where $x_0 = a$ and $x_n = b$ be a partition of $[a, b]$ and let R_1 and r_1 be the upper and lower Darboux sums over C_1, respectively. Furthermore, let $C_2 := y_0 < y_1 < \ldots < y_p$, where $y_0 = a$ and $y_p = b$ be a partition of $[a, b]$ and let R_2 and r_2 be the upper and lower Darboux sums over C_2, respectively. We have that $r_1 \leq R_2$.

Proof. Let $C_3 := z_0 < z_1 < \ldots < z_q$ be yet another partition of $[a, b]$ that contains the points of both C_1 and C_2. Note that since $\inf_{z \in [z_i, z_{i+1}]} (f) \leq \sup_{z \in [z_i, z_{i+1}]} (f)$, it's obvious that $r_3 \leq R_3$. However, C_3 is obtained by adding the points of C_2 to C_1. From this, it follows that $r_3 \geq r_1$. Similarly, C_3 is created by adding the points of C_1 to C_2 and so $R_3 \leq R_2$. Thus, $r_1 \leq r_3 \leq R_3 \leq R_2 \Rightarrow r_1 \leq R_2$, as required. ■

3.2 Darboux Integral

Let $I = \inf\{R \text{ for any possible partition of } [a, b]\}$ and $J = \sup\{r \text{ for any possible partition of } [a, b]\}$. We say that I is the upper Darboux integral of f over a partition P of $[a, b]$ whilst J is the lower Darboux integral of f over P. That is,

$$I = \overline{\int_a^b} f(t)\ dt$$

and

$$J = \underline{\int_a^b} f(t)\ dt$$

3.3 Reimann Sum

Definition. The Reimann sum of f over a partition P of $[a, b]$ is defined as

$$S(f, P) = \sum_{i=0}^{p-1} f(q_i)(x_{i+1} - x_i)$$

Where $q_i \in [x_i, x_{i+1}]$ is a chosen **quadrature point** in each sub-interval.

3.4 Integrability / The Definite Integral

Definition. A function f is **integrable** if $I = J$. This common value is $\displaystyle\int_a^b f(t)\ dt$ – the **definite integral** of f.

Remark. Note that $\displaystyle\int_a^b f(t)\ dt = \lim_{p \to \infty} \sum_{i=0}^{p-1} f(q_i)(x_{i+1} - x_i)$.

Theorem. f is integrable iff, for any $\epsilon > 0$, there exists a partition such that $0 \leq R - r < \epsilon$.

Proof. We prove two directions:

- $\boxed{\Leftarrow}$: Given any $\epsilon > 0$, there exists a partition such that $0 \leq R - r < \epsilon$. We know that I is a lower bound of all possible values of R – that is, for any R, $R \geq I$. Similarly, J is an upper bound of all values of r – that is, for any r, one has that $r \leq J$. Thus, $-r \geq -J \Rightarrow R - r \geq I - J$. So $I - J \leq R - r < \epsilon$. Thus, since $I - J \leq \epsilon$ for any $\epsilon > 0$, surely $I - J \leq 0 \Rightarrow I \leq J$. Also, given any partitions C_1 and C_2, one has that $r_1 \leq R_2$, so R_2 is an upper bound of all possible lower Darboux sums over C_1. Thus, $R_2 \geq J$. So J is an lower bound of all values of R_2. So, I is the **greatest** lower bound and so $I \geq J \Rightarrow I - J \geq 0$. And so, since $I - J \leq 0$ and $I - J \geq 0$, one has that $I - J = 0 \Rightarrow I = J$, as required.

- $\boxed{\Rightarrow}$: Since f is integrable, it follows that $I = J$. Recall that $I = \inf\{\text{all } R\}$ and $J = \sup\{\text{all } r\}$. Of course, for some $\epsilon > 0$, one has that $I + \epsilon/2$ is not a lower bound; that is, there exists a partition C_1 with a corresponding upper Darboux sum R_1 so that $R_1 < I + \epsilon/2$. Similarly, $J - \epsilon/2$ is not an upper bound and so there exists a partition C_2 with a corresponding lower Darboux sum r_2 so that $r_2 > J - \epsilon/2$. Now, consider a partition C_3 which includes the points of both C_1 and C_2. Since C_3 is obtained by adding the points of C_2 to C_1, it follows that $R_3 \leq R_1$. Since $R_1 < I + \epsilon/2$, however, it holds that $R_3 \leq R_1 < I + \epsilon/2$. Also, C_3 is obtained by adding the points from C_1 to C_2. Thus, $r_3 \geq r_2$. Since we know that $r_2 > J - \epsilon/2$ we have that $R_3 - r_3 < I + \epsilon/2 - J + \epsilon/2 \Rightarrow R_3 - r_3 < \epsilon$, since $I - J = 0$. This gives the required result.

■

EXAMPLE. Let f be defined as follows:

$$f(x) = \begin{cases} 0 \text{ if } x \neq 1/2 \\ 1 \text{ if } x = 1/2 \end{cases}$$

f is integrable on $[0, 1]$.

Proof. Partition $[0, 1]$ as $x_0 = 0$, $x_1 = \dfrac{1}{2} - \dfrac{\epsilon}{3}$, $x_2 = \dfrac{1}{2} + \dfrac{\epsilon}{3}$, $x_3 = 1$. That is, set the endpoints of $[0, 1]$ as the edges of the partition and place a point on either side of, and arbitrarily close to, $x = 1/2$. Thus, $R = 2\epsilon/3$ and $r = 0$, so that $R - r = 2\epsilon/3 < \epsilon$, as required. ■

Theorem. If f is continuous on $[a, b]$ then f is integrable on $[a, b]$.

Proof. Since $[a, b]$ is compact and f is continuous on $[a, b]$, one has that f is uniformly continuous over $[a, b]$. Thus, we have that for any $\epsilon_c > 0$ there is a δ_c so that, for any $x \in [a, b]$ and $y \in [a, b]$, if $|x - y| < \delta_c$ then $|f(x) - f(y)| < \epsilon_c$. Since we don't know what f is, we can't determine I or J. Therefore, our best approach is to identify a partition such that $R - r < \epsilon$. Thus, we have

$$R - r = \sum_{i=0}^{p-1} (x_{i+1} - x_i) \left(\sup_{x \in [x_i, x_{i+1}]} (f) - \inf_{x \in [x_i, x_{i+1}]} (f) \right)$$

Since f is continuous over $[a, b]$ it's certainly continuous over each sub-interval $[x_i, x_{i+1}]$. Thus, by the Extreme Value theorem, there exists an $x_{M_i} \in [x_i, x_{i+1}]$ so that $f(x_{M_i}) = \sup_{x \in [x_i, x_{i+1}]}(f)$ and similarly there exists an $x_{m_i} \in [x_i, x_{i+1}]$ so that $f(x_{m_i}) = \inf_{x \in [x_i, x_{i+1}]}(f)$. Thus,

$$R - r = \sum_{i=0}^{p-1} (x_{i+1} - x_i)(f(x_{M_i}) - f(x_{m_i}))$$

If $[x_i, x_{i+1}]$ has width less than δ_c than $|x_{M_i} - x_{m_i}| < \delta_c \Rightarrow |f(x_{M_i}) - f(x_{m_i})| < \epsilon_c$. So, let $\epsilon_c = \epsilon/(b - a)$. More formally, consider a partiton of $[a, b]$ into n sub-intervals of **equal** length so that $n > (b - a)/\delta_c$. Thus, $x_{i+1} - x_i = (b - a)/n$. Thus,

$$|x_{i+1} - x_i| < \delta_c \Rightarrow |x_{M_i} - x_{m_i}| < \delta_c$$

$$\Rightarrow R - r = \sum_{i=0}^{n-1} \left(\frac{b - a}{n} \right) (f(x_{M_i}) - f(x_{m_i})) \leq \sum_{i=0}^{n-1} \left(\frac{b - a}{n} \right) \left(\frac{\epsilon}{b - a} \right) = \sum_{i=0}^{n-1} \frac{\epsilon}{n} = \epsilon$$

$$\Rightarrow R - r < \epsilon$$

■

Theorem. Let f be a non-negative, continuous function on $[a, b]$. If $\int_a^b f(t)\ dt = 0$, then $f(t) = 0$ for every $t \in [a, b]$.

Theorem. If there exists a $c \in [a, b]$ so that $f(c) > 0$, then $\int_a^b f(t)\ dt > 0$.

Proof. If f is continuous at c then $\epsilon = f(c)/2$. There is an $h > 0$ so that if $t \in (c - h, c + h)$, then $|f(t) - f(c)| < f(c)/2 \Rightarrow f(c)/2 < f(t) < 3f(c)/2$. Now, partition $[a, b]$ as $P : x_0 = a, x_1 = c - h/2, x_2 = c + h/2$, and $x_3 = b$. Thus,

$$\int_a^b f(t)\ dt \geq r = (x_1 - x_0) \inf(f, [x_0, x_1]) + (x_2 - x_1) \inf(f, [x_1, x_2]) + (x_3 - x_2) \inf(f, [x_2, x_3])$$

However, since $f(t) > f(c)/2$, it stands that $f(c)/2$ is a lower bound of f, thus $\inf(f, [x_0, x_1]) \geq 0$, $\inf(f, [x_1, x_2]) \geq f(c)/2$, and $\inf(f, [x_2, x_3]) \geq 0$, so

$$\int_a^b f(t)\ dt \geq r > \frac{hf(c)}{2} > 0$$

as required. ■

Proposition. Let f be continuous on $[a,b]$. For any $c \in [a,b]$,

$$\int_a^b f(t)\ dt = \int_a^c f(t)\ dt + \int_c^b f(t)\ dt$$

whence it's clear that f should also be continuous on $[a,c]$ and $[c,b]$.

Theorem. (FTC) – Given a continuous function f on $[a,b]$,

$$F(x) = \int_a^x f(t)\ dt$$

is an antiderivative of f, and $F'(x) = f(x)$.

Proof. We want to show that $\displaystyle\lim_{y\to x} \frac{F(y)-F(x)}{y-x} = f(x)$. That is,

$$\lim_{y\to x} \frac{1}{y-x}\left(\int_a^y f(t)\ dt - \int_a^x f(t)\ dt\right) = f(x)$$

$$\Rightarrow \lim_{y\to x} \frac{1}{y-x}\left(\int_x^y f(t)\ dt\right) = f(x) \quad \to \text{(because } \int_a^x f(t)\ dt + \int_x^y f(t)\ dt = \int_a^y f(t)\ dt)$$

Since f is continuous, for every $\epsilon > 0$, there is a δ so that if $|t-x| < \delta$, then $|f(t) - f(x)| < \epsilon \Rightarrow f(x) - \epsilon < f(t) < f(x) + \epsilon$, and so

$$|y-x|(f(x)-\epsilon) \le r = |y-x| \inf(f,[x,y]) \le \int_x^y f(t)\ dt \le R = |y-x|\sup(f,[x,y]) \le |y-x|(f(x)+\epsilon)$$

$$\Rightarrow f(x) - \epsilon \le \frac{1}{|y-x|}\int_x^y f(t)\ dt \le f(x) + \epsilon$$

$$\Rightarrow \left|\frac{1}{y-x}\int_x^y f(t)\ dt - f(x)\right| \le \epsilon$$

as required. ∎

3.5 Improper Integrals

Definition. Let f be a continuous function on $[a,\infty)$ for some $a \in \mathbb{R}$. For every $s \in [a,b]$, f is continuous on $[a,s]$. Thus, define

$$\int_a^\infty f(t)\ dt = \lim_{s\to\infty} \int_a^s f(t)\ dt$$

which is only true if the limit exists. This is said to be an **improper integral of the first kind**.

Remark. If the limit above exists and is finite, then we say that the improper integral **converges**. Otherwise, it **diverges**.

Definition. Let f be a continuous function on $[a,b)$ (note: non-inclusive). For every $s \in [a,b)$, f is continuous on $[a,s]$. Thus, define

$$\int_a^b f(t)\ dt = \lim_{s\to b} \int_a^s f(t)\ dt$$

which is only true if the limit exists. This is said to be an **improper integral of the second kind**.

Example 1.

$$\int_1^\infty t^p\ dt = \lim_{s\to\infty}\int_1^s t^p\ dt = \lim_{s\to\infty} \frac{s^{p+1}-1}{p+1} = \begin{cases} -1/(p+1) & \text{if } p+1 < 0 \\ \infty & \text{if } p+1 > 0 \\ \infty & \text{if } p+1 = 0 \end{cases}$$

So, the integral converges for $p < -1$ and diverges otherwise.

Example 2. $\int_0^1 t^p\ dt$ is improper for $p < 0$, so

$$\lim_{s\to 0}\int_s^1 t^p\ dt = \lim_{s\to 0}\frac{1-s^{p+1}}{p+1} = \begin{cases} 1/(p+1) & \text{if } p+1 > 0 \\ \infty & \text{if } p+1 < 0 \\ \infty & \text{if } p+1 = 0 \end{cases}$$

So, the integral converges for $p > -1$ and diverges otherwise.

Theorem. Let f be a non-negative function on $[a, b)$ where b is either a finite value or infinite. We have that $\int_a^b f(t)\ dt$ is convergent iff $\int_a^x f(t)\ dt$ is bounded for all $x \in [a, b)$.

Proof. (exercise) ■

Theorem. Let $f(t)$ and $g(t)$ be two non-negative functions on $[a, b)$ and assume that they are both integrable on some interval $[a, c] \subset [a, b)$ given some $c \in [a, b)$. Lastly, assume that $f \leq g$ for every $t \in [a, b)$. We have the following properties:

- $\int_a^b g(t)\ dt$ converges $\Rightarrow \int_a^b f(t)\ dt$ converges
- $\int_a^b f(t)\ dt$ diverges $\Rightarrow \int_a^b g(t)\ dt$ diverges

Remark. The above integrals are improper and are of the **second kind**.

Proof. To show that $\int_a^b f(t)\ dt$ converges, we must show that $\int_a^x f(t)\ dt$ is bounded (cf. previous theorem). Proceeding, we see that

$$\int_a^x f(t)\ dt \leq \int_a^x g(t)\ dt$$

However, since $\int_a^b g(t)\ dt$ converges, it holds that $\int_a^x g(t)\ dt$ is bounded above[3] by M. That is, $\int_a^x g(t)\ dt \leq M$. Thus,

$$\int_a^x f(t)\ dt \leq \int_a^x g(t)\ dt \leq M$$

$$\Rightarrow 0 \leq \int_a^x f(t)\ dt \leq M$$

and so $\int_a^x f(t)\ dt$ is bounded and thus $\int_a^b f(t)\ dt$ converges, as required. Note that we needn't prove the second property since it's the contrapositive of the one we just proved. ■

Corollary. If $\lim\limits_{t \to b} \dfrac{f(t)}{g(t)} = L \neq 0$, then either...

- ... $\int_a^b f(t)\ dt$ and $\int_a^b g(t)\ dt$ are both convergent, or...
- ... $\int_a^b f(t)\ dt$ and $\int_a^b g(t)\ dt$ are both divergent

Proof. (exercise) ■

Definition. Let f be defined on $[a, b)$, and assume that f is integrable on every $[a, c] \subset [a, b)$. It is said that $\displaystyle\int_a^b f(t)\ dt$ is **absolutely convergent** if the integral $\displaystyle\int_a^b |f(t)|\ dt$ is convergent.

Proof. Break f into sub-functions so as to focus only on where f is positive and where it's negative:

$$f^+(t) = \begin{cases} f(t) & \text{if } f(t) \geq 0 \\ 0 & \text{if } f(t) < 0 \end{cases}$$

$$f^-(t) = \begin{cases} -f(t) & \text{if } f(t) < 0 \\ 0 & \text{if } f(t) \geq 0 \end{cases}$$

It follows that $f(t) = f^+(t) - f^-(t) \Rightarrow |f(t)| = f^+(t) + f^-(t)$. Now, we know that $0 \leq f^+(t) \leq |f(t)|$, and $\displaystyle\int_a^b |f(t)|\ dt$ converges, so by comparison $\displaystyle\int_a^b f^+(t)\ dt$ converges. By the exact same argument, $\displaystyle\int_a^b f^-(t)\ dt$ converges, as well. Thus, $\displaystyle\int_a^b f^+(t)\ dt - \int_a^b f^-(t)\ dt$

$= \displaystyle\int_a^b f^+(t)\ dt - f^-(t)\ dt = \int_a^b f(t)\ dt$ converges, and we are done. ■

Proposition. Let $\Phi(t)$ be differentiable on $[a, b)$ and assume that $\Phi'(t)$ is non-negative and continuous on $[a, \infty)$. Also, assume that $\lim\limits_{t \to \infty} \Phi(t) = 0$. Let f be an integrable function on any $[a, c] \subset [a, \infty)$ so that $\int_a^x f(t)\ dt$ is bounded on $[a, \infty) \Rightarrow \int_a^\infty f(t)\Phi(t)\ dt$ is convergent.

[3]Since $g(t) \geq 0$ for all $t \in [a, b)$, it holds that $\int_a^x g(t)\ dt \geq \int_a^x 0\ dt = 0$, so the lower bound is trivial.

3.6 Functions Defined by Integrals

Consider an interval $[a, b)$ where b is either finite or infinite. Define a new function as follows:

$$F(x) = \int_a^b f(t, x)\ dt$$

It is said that $F(x)$ is convergent on I if, for every $x \in I$, it holds that $\lim_{c \to b} \int_a^c f(t, x)\ dt = \int_a^b f(t, x)\ dt$. More formally, F converges if, for every $x \in I$ and for every $\epsilon > 0$, there exists a neighborhood of b such that if c is within this neighborhood, then $\left|\int_b^c f(t, x)\ dt\right| < \epsilon$.

Remark. This notion of functions being defined by an integral is not very foreign – a common example seen in differential equations is the Laplace transform:

$$\mathcal{L}\left\{f(t)\right\}(s) = \int_0^\infty e^{-st} f(t)\ dt$$

Theorem. If $|f(t, x)| \leq g(t)$ for every $x \in I$ and $\int_a^b g(t)\ dt$ is convergent, then $\int_a^b f(t, x)\ dt$ is **uniformly convergent** on I.

Remark. This is analogous to uniform convergence of series: if $|u_n(x)| \leq M_n$ for every $x \in I$ and $\sum M_n$ is convergent, then $\sum u_n(x)$ is uniformly convergent on I.

Example. $\int_0^\infty e^{-t(1+x)}\ dt$ is uniformly convergent on $[0, \infty)$.

Proof. Observe that $e^{-t(1+x)} \leq e^{-t}$ for all $x \in [0, \infty)$, and $\int_0^\infty e^{-t}\ dt$ is convergent, and thus $\int_0^\infty e^{-t(1+x)}\ dt$ is uniformly convergent. ■

Proposition. Let $f(t, x)$ is continuous on $[a, b) \times I$. If $F(x) = \int_a^b f(t, x)\ dt$ is uniformly convergent on I, then F is continuous on I.

Proof. First, note that $\int_a^b f(t, y)\ dt$ is within $\epsilon/3$ of $\int_a^c f(t, y)\ dt$ and $\int_a^b f(t, x)\ dt$ is within $\epsilon/3$ of $\int_a^c f(t, x)\ dt$. So, ideally what we want is to show that

$$\left|\int_a^c f(t, y)\ dt - \int_a^c f(t, x)\ dt\right| < \frac{\epsilon}{3}$$

Since $\int_a^b f(t, x)\ dt$ is uniformly convergent, there exists a neighborhood of b such that, for any c within this neighborhood and for any $x \in I$,

$$\left|\int_a^b f(t, x)\ dt - \int_a^c f(t, x)\ dt\right| < \frac{\epsilon}{3}$$

Now, pick some c within this neighborhood. For any $y \in I$,

$$\left|\int_a^b f(t, y)\ dt - \int_a^c f(t, x)\ dt\right| < \frac{\epsilon}{3}$$

Since f is continuous on $[a, b) \times I$, there is a δ so that we obtain $|f(t, x) - f(t, y)| < \epsilon/3(c-a) \Rightarrow \left|\int_a^c f(t, x) - f(t, y)\ dt\right| < \epsilon/3$. So, putting it together,

$$\left|\int_a^b f(t, x)\ dt - \int_a^b f(t, y)\ dt\right| \leq \underbrace{\left|\int_a^b f(t, x)\ dt - \int_a^c f(t, x)\ dt\right|}_{<\epsilon/3} + \underbrace{\left|\int_a^c f(t, x)\ dt - \int_a^c f(t, y)\ dt\right|}_{<\epsilon/3} + \underbrace{\left|\int_a^b f(t, y)\ dt - \int_a^b f(t, y)\ dt\right|}_{<\epsilon/3}$$

$$\Rightarrow \left|\int_a^b f(t, x)\ dt - \int_a^b f(t, y)\ dt\right| < \epsilon$$

as required. ■

Example. For which value of x is $F(x) = \int_0^\infty \frac{\sin(xt)}{1+t^2}\ dt$ convergent?

Solution. We observe that $\left|\frac{\sin(xt)}{1+t^2}\right| \leq \frac{1}{1+t^2}$ for every $x \in \mathbb{R}$, and since $\int_0^\infty \frac{1}{1+t^2}\ dt$ is convergent, we see that $F(x)$ is convergent for all $x \in \mathbb{R}$.

Theorem. Let $f(t,x)$ be continuous on $[a,b)\times I$ and let $\int_a^b f(t)\ dt$ be uniformly convergent on I; also let $F(x)=\int_a^b f(t,x)\ dt$. For any $c\in I$ and $d\in I$, we have that

$$\int_a^b\int_c^d f(t,x)\ dx\ dt=\int_c^d\int_a^b f(t,x)\ dt\ dx$$

which is formally known as **Fubini's theorem**.

Theorem. If the following three conditions are satisfied:

1. $\displaystyle\int_a^b f(t,x)\ dt$ is convergent on I
2. $\displaystyle\int_a^b \frac{\partial f}{\partial x}(t,x)\ dt$ is uniformly convergent on I
3. $\displaystyle\frac{\partial f}{\partial x}$ is continuous on $[a,b)\times I$

then $F(x)=\displaystyle\int_a^b f(t,x)\ dt$ is differentiable and $\displaystyle\frac{d}{dx}\left(F(x)\right)=\int_a^b\frac{\partial f}{\partial x}(t,x)\ dt$.

Remark. Extending the theorem above, one has that $\displaystyle\int_{x_0}^x\int_a^b\frac{\partial f}{\partial x}(t,x)\ dt=\int_a^b\int_{x_0}^x\frac{\partial f}{\partial x}(t,x)\ dt=\int_a^b f(t,x)-f(t,x_0)\ dt=$ $F(x)-F(x_0)$.

Definition. Let the **gamma function** be denoted by

$$\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}\ dt$$

Remark. The Γ function has the property that $\Gamma(x+1)=x\Gamma(x)$.

Proof. $\displaystyle\Gamma(x+1)=\int_0^\infty t^xe^{-t}\ dt=\underbrace{\int_\alpha^\beta t^xe^{-t}\ dt=-t^xe^{-t}\Big|_\alpha^\beta+\int_\alpha^\beta xt^{x-1}e^{-t}\ dt}_{\alpha\to0\ ,\ \beta\to\infty}=x\int_0^\beta t^{x-1}e^{-t}\ dt=x\Gamma(x).$ ∎

Remark. As a result of the aforementioned property, $\Gamma(2)=1\cdot\Gamma(1)=1$, $\Gamma(3)=2\cdot\Gamma(2)=2\cdot1=2$, $\Gamma(4)=3\Gamma(3)=3\cdot2\cdot1=6$. By extension, $\Gamma(n)=(n-1)!$ for all $n\in\mathbb{N}$.

Proof. We proceed by induction on n. Beginning at $n=1$, one has that $\Gamma(1)=0!=\displaystyle\int_0^\infty e^{-t}=\ dt=1$. Now assume that, for some $k\geq1$, $\Gamma(k)=(k-1)!$. We now see that $\Gamma(k+1)=k\Gamma(k)$, which, by our induction hypothesis, is $k(k-1)!$. Of course, $k(k-1)!=k!$ and so $\Gamma(k+1)=k!$, as required. This ends the proof. ∎

Remark. $\Gamma(x)$ is continuous and differentiable on $(0,\infty)$.

4 Power Series

Definition. A **power series** is a function of the form $f(x)=\displaystyle\sum_{n=0}^\infty a_nx^n=a_0+a_1x+a_2x^2+\ldots+a_kx^k+\ldots$

Theorem. Given a power series $f(x)=\displaystyle\sum_{n=0}^\infty a_nx^n$, if there exists an a such that $f(a)$ is convergent, then for any $x\in(-a,a)$ the series $f(x)$ is absolutely convergent.

4.1 Radius of Convergence

Theorem. Given a power series $f(x)=\displaystyle\sum_{n=0}^\infty a_nx^n$, there exists a positive, real R called the **radius of convergence** such that, for any $x\in(-R,R)$, $f(x)$ converges.

Proposition. We have the following properties:

- If $f(x)$ converges for all $x \in \mathbb{R}$, then the radius of convergence is infinite.
- If $f(x)$ converges only if $x = 0$, then the radius of convergence is zero.
- If $x \in (-R, R)$, then $f(x)$ is absolutely convergent.
- If $x \in (-\infty, -R) \cup (R, \infty)$, then $f(x)$ is divergent.

4.2 Ratio Test

Proposition. Given a power series of the form $\sum_{n=0}^{\infty} a_n x^n$, if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$ (for some finite L), then the radius of convergence is given by $R = 1/L$.

Remark. If $L = 0$, then $R \to \infty$ and so the radius of convergence is infinite; likewise, if $L \to \infty$ then $R = 0$.

EXAMPLE 1. Given $\sum_{n=0}^{\infty} x^n$, we see that $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$ and so $R = 1$. Thus, this series converges for $|x| < 1$.

EXAMPLE 2. Given $\sum_{n=0}^{\infty} \frac{x^n}{n!}$, we see that $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0$ and so $R \to \infty$. Thus, this series converges for all $x \in \mathbb{R}$.

Remark. This is the Taylor series for $f(x) = e^x$.

EXAMPLE 3. Given $\sum_{n=0}^{\infty} \frac{x^{3n+1}}{n}$, we see that $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = |x^3|$ and so if $|x^3| < 1$ the series is convergent and if $|x^3| > 1$ the series diverges.

Remark. This is the "ratio test" from Math 327 – it doesn't reveal any information about the radius of convergence but it does tell whether the series converges or not.

4.3 The Root Test

Definition. $\overline{\lim_{n\to\infty}} u_n = \lim_{n\to\infty} \sup\{u_k \mid k \geq n\}$

Remark. $\{L_n\} = \sup\{u_k \mid k \geq n\}$ is a decreasing sequence. Either its limit is finite or infinite.

Remark. The inferior limit is defined similarly: $\underline{\lim_{n\to\infty}} u_n = \lim_{n\to\infty} \inf\{u_k \mid k \geq n\}$

Theorem. Given a power series $\sum_{n=0}^{\infty} a_n x^n$, $\overline{\lim_{n\to\infty}} |a_n|^{1/n} = 1/R$.

4.4 Continuity, Differentiability, and Integrability of Power Series

Theorem. Given a power series $\sum_{n=0}^{\infty} a_n x^n$ with a radius of convergence $R > 0$, there exists an r so that $0 < r < R$ and $\sum_{n=0}^{\infty} a_n x^n$ is uniformly convergent on $[-r, r]$.

Theorem. Given $\sum_{n=0}^{\infty} a_n x^n$ with a radius of convergence $R > 0$, the series is continuous on $(-r, r)$.

Theorem. Given $\sum_{n=0}^{\infty} a_n x^n$ with $R > 0$, for every a and b in $(-R, R)$, one has that $\int_a^b \sum_{n=0}^{\infty} a_n x^n \, dx = \sum_{n=0}^{\infty} a_n \left(\frac{b^{n+1}}{n+1} - \frac{a^{n+1}}{n+1}\right)$.

Theorem. Given $\sum_{n=0}^{\infty} a_n x^n$ with $R > 0$, the series is differentiable at every $x \in (-R, R)$ and its derivative is $\frac{d}{dx}\left(\sum_{n=0}^{\infty} a_n x^n\right) = \sum_{n=0}^{\infty} \frac{d}{dx} a_n x^n = \sum_{n=0}^{\infty} n a_n x^{n-1}$.

Proposition. Differentiating a power series does not affect its radius of convergence.

Proposition. Given $f(x) = \sum_{n=0}^{\infty} a_n x^n$, we see that $f(0) = a_0$. Similarly, $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ and $f'(0) = a_1$. By the same construction, $f''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$ and $f''(0) = 2a_2$. Thus, $f^n(0) = a_n n! \Rightarrow a_n = \frac{f^n(0)}{n!}$. So, $f(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \left(\frac{f^n(0)}{n!}\right) x^n$.

Proposition. Given $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ where, in a neighborhood of 0, $f(x) = g(x)$, one has that for every n, $a_n = b_n$.

4.5 Analytic Functions

Definition. Given a function f that has derivatives of every order on an interval I, we say that f is **analytic** on I if, for every $x_0 \in I$, $f(x) = \sum_{n=0}^{\infty} \frac{f^n(x_0)}{n!}(x - x_0)^n$ on some neighborhood of x_0. That is, a function is analytic on an interval if it's exactly equal to its Taylor series at every interior point of that interval.

EXAMPLE 1. Let f be defined as

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ e^{-1/x^2} & \text{if } x \neq 0 \end{cases}$$

Note that $f^n(x)$ is always zero at zero, which means that the Taylor series of f is the zero function $0(x) = 0$. However, f is zero *only* at $x = 0$. Thus, f isn't analytic at $x = 0$. For it to be analytic at $x = 0$, we require $f = 0(x)$ on a neighborhood of zero, but $f = 0(x)$ if and only if $x = 0$.

EXAMPLE 2. $f(x) = e^x$ is analytic on $\mathbb{R}$.

Proof. Let $S(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ and observe that $S'(x) = \sum_{n=0}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = S(x)$. From this it's obvious that $f(x) \equiv S(x)$ has derivatives of every order that exist. Furthermore, we know that $S(0) = f(0) = 1$, and so it's obvious that f is analytic at $x = 0$. It remains to show that f is analytic for every $x_0 \in \mathbb{R}$. For every $x_0 \in \mathbb{R}$, we can write $e^x = e^{(x-x_0)+x_0} = e^{(x-x_0)}e^{x_0}$. We write in this (crude, albeit obvious) manner so as to force a factor of $(x - x_0)$ to appear so it fits the Taylor series definition. Thus,

$$e^{(x-x_0)}e^{x_0} = e^{x_0}\left(\sum_{n=0}^{\infty} \frac{(x-x_0)^n}{n!}\right) = \sum_{n=0}^{\infty} \frac{e^{x_0}(x-x_0)^n}{n!}$$

which is just the Taylor series of e^x center at x_0. ■

Remark. Note that the functions $\sin(x), \cos(x), \sinh(x), \cosh(x), \tanh(x)$, etc. are "built" off of the exponential function is some way, so by the analyticity of e^x, the analyticity of these functions follows.

EXAMPLE 3. Consider the geometric series $S(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $|x| < 1$. Note that $\frac{1}{1-x}$ is analytic at $x = 0$ since it's exactly equal to its Taylor series $S(x)$ on $(-1, 1)$ as prescribed. We can also ask, for novelty's sake, *what is the value of* $f^{10,000}(0)$? Note that the coefficient in front of x^n is always 1, so indeed $\frac{f^{10,000}(0)}{(10,000)!} = 1$ and thus $f^{10,000}(0) = 10,000!$

Example 4. Is $f(x) = \dfrac{1}{1-x}$ analytic at any $x_0 \neq 1$? Well, first let's pull the same trick of writing $x = (x - x_0) + x_0$ so we get

$$\frac{1}{1-x} = \frac{1}{1-((x-x_0)+x_0)} = \frac{1}{(1-x_0)-(x-x_0)} = \frac{1}{1-x_0}\left(\frac{1}{1-\left(\frac{x-x_0}{1-x_0}\right)}\right)$$

$$= \frac{1}{1-x_0}\left(\sum_{n=0}^{\infty}\left(\frac{x-x_0}{1-x_0}\right)^n\right) = \sum_{n=0}^{\infty}\left(\frac{1}{1-x_0}\right)^{n+1}(x-x_0)^n$$

which is a cleaner and more obvious way of writing the sum. So indeed, it is equal to its Taylor series for any $x_0 \neq 1$ so it's analytic on $\mathbb{R} \setminus \{1\}$.

As a quick note on analyticity, we see that differentiating and integrating an analytic function doesn't change *if* it's analytic and *where* it's analytic. As an example, consider $f(x) = \dfrac{1}{1+x^2}$. It's analytic on $\mathbb{R}$, and note that $\displaystyle\int \frac{1}{1+x^2} = \arctan(x) = \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$. Thus, $g(x) = \arctan(x)$ is analytic on $\mathbb{R}$.

4.6 Products and Quotients of Power Series

Let $\displaystyle\sum_{n=0}^{\infty} a_n x^n$ and $\displaystyle\sum_{n=0}^{\infty} b_n x^n$ be two power series with radii of convergence $R > 0$. We have the following properties:

1. $\displaystyle\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n$, where $\displaystyle c_n = \sum_{i=0}^{n} a_i b_{n-i}$.

2. $\dfrac{\sum_{n=0}^{\infty} a_n x^n}{\sum_{n=0}^{\infty} b_n x^n} = \displaystyle\sum_{n=0}^{\infty} c_n x^n$, from which we see that $\displaystyle\sum_{n=0}^{\infty} a_n x^n = \left(\sum_{n=0}^{\infty} b_n x^n\right)\left(\sum_{n=0}^{\infty} c_n x^n\right)$. This gives us a product, allowing us to work with the formula from before; doing so develops the following system of equations:

$$\begin{cases} a_0 = c_0 b_0 \\ a_1 = c_1 b_0 + c_0 b_1 \\ a_2 = c_2 b_0 + c_1 b_1 + c_0 b_2 \\ \qquad\vdots \end{cases}$$

which is a triangular system of equations. Thus, solving such a system will give us the appropriate values of c_i.

END OF COURSE MATERIAL