PROOF OF PARSEVAL'S IDENTITY

Parseval's Identity: Given $f(x) \in L^2([-L,L],\mathbb{R})$ which can be represented by its Fourier series $\mathscr{F}[f](x)$, we have the following equality:

$$\frac{1}{L}\int_{-L}^{L}(f(x))^2\ dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$$

Proof. We first recognize the integral $\int_{-L}^{L}(f(x))^2\ dx$ as an inner product on $L^2([-L,L],\mathbb{R})$. That is,

$$\int_{-L}^{L}(f(x))^2\ dx = \langle f, f\rangle .$$

Since f can be equivalently represented by its Fourier series, we can write the inner product as:

$$\left\langle \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right), f\right\rangle .$$

The inner product is linear in the first component (to be pedantic, it's linear in both components since we're working with real-valued functions), so it may be split up as:

$$\frac{a_0}{2}\langle 1, f\rangle + \sum_{n=1}^{\infty} a_n\left\langle \cos\left(\frac{n\pi x}{L}\right), f\right\rangle + \sum_{n=1}^{\infty} b_n\left\langle \sin\left(\frac{n\pi x}{L}\right), f\right\rangle .$$

Now, we'll evaluate the inner product termwise: we first recognize that the first term gives $\frac{a_0}{2}\langle 1, f\rangle = \int_{-L}^{L} f(x)\ dx = \frac{a_0^2 L}{2}$. The second term gives $\left\langle \cos\left(\frac{n\pi x}{L}\right), f\right\rangle = \int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\ dx = La_n$. Similarly, the third term gives $\left\langle \sin\left(\frac{n\pi x}{L}\right), f\right\rangle = \int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\ dx = Lb_n$. Putting it all back together, we now have:

$$\begin{aligned}\langle f, f\rangle &= \frac{a_0^2 L}{2} + \sum_{n=1}^{\infty} La_n(a_n) + \sum_{n=1}^{\infty} Lb_n(b_n)\\ &= \frac{a_0^2 L}{2} + \sum_{n=1}^{\infty} L\left(a_n^2 + b_n^2\right).\end{aligned}$$

Lastly, dividing both sides by L gives the result:

$$\boxed{\frac{1}{L}\int_{-L}^{L}(f(x))^2\ dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)}.$$

QED.