Math 328B: Winter 2017

Homework 2

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Exercise 1. Prove, using the definition of the compact set, that a closed subset of a compact set is compact.

Proof. Let S be a compact set, and let $X \subset S$ be closed. Since X is closed, it follows that $\{X^c, \{U_\alpha\}\}$ is an open cover of S (provided an open cover $\{U_\alpha\}_{\alpha \in I}$ which covers X). Since S is compact, there exists a finite sub-cover of S, denoted $\{X^c, U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_n}\}$ for some finite n. We will now show that $\{U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_n}\}$ is a finite cover for X – for every $x \in X$ one has that $x \in S$ and so x is covered by $\{X^c, U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_n}\}$. However, if $x \in X$, then $x \notin X^c$ and so $x \in \{U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_n}\}$, and thus there exists a finite sub-cover for X, as required. □

Exercise 2. The set of rationals $\mathbb{Q}$ is not compact because it is not closed or bounded. Find a cover of $\mathbb{Q}$ by open sets for which a finite sub-collection cannot cover $\mathbb{Q}$.

Solution. Let $\mathcal{O} = \{(\alpha - 1, \alpha + 1) \mid \alpha \in \mathbb{R}\}$ be an open cover of $\mathbb{Q}$. We can verify that $\mathcal{O}$ does, in fact, cover $\mathbb{Q}$, as, because between any two real numbers there exist infinitely many rational numbers, it holds that $\mathbb{Q} \subset \bigcup\limits_{\alpha \in \mathbb{R}} (\alpha - 1, \alpha + 1)$.
From $\mathcal{O}$ we may extract a finite sub-collection – $\bar{\mathcal{O}} = \{(\alpha_1 - 1, \alpha_1 + 1), (\alpha_2 - 1, \alpha_2 + 1), \ldots, (\alpha_n - 1, \alpha_n + 1)\}$ for some finite n. We will also assume that $\alpha_1 > \alpha_2 > \ldots > \alpha_n$. We also see that $\bigcup\limits_{i=1}^{n} (\alpha_i - 1, \alpha_i + 1) = (\alpha_1 - 1, \alpha_1 - 1)$. If $\bar{\mathcal{O}}$ is to cover $\mathbb{Q}$, then one would have that $\mathbb{Q} \subset \bigcup\limits_{i=1}^{n} (\alpha_i - 1, \alpha_i + 1)$, which is to say that $\mathbb{Q} \subset (\alpha_1 - 1, \alpha_1 - 1)$. However, between any two real numbers exist infinitely many rational numbers, so there exist, in particular, an $m_0 \in \mathbb{Z}$ and an $n_0 \in \mathbb{Z}$ such that $\dfrac{m_0}{n_0} < \alpha_1 - 1$, and so $\dfrac{m_0}{n_0} \notin (\alpha_1 - 1, \alpha_1 + 1)$ and thus $\mathbb{Q} \not\subset (\alpha_1 - 1, \alpha_1 + 1) \Rightarrow \mathbb{Q} \not\subset \bigcup\limits_{i=1}^{n} (\alpha_i - 1, \alpha_i + 1)$ and so $\bar{\mathcal{O}}$ does not cover $\mathbb{Q}$.

Exercise 3. The set $S = \bigcup\limits_{n=1}^{\infty} \left[\dfrac{1}{2n+1}, \dfrac{1}{2n}\right]$ is not compact because it is not closed. Find a cover of S for which a finite sub-collection cannot cover S.

Solution. We first note that, in order to cover S, it is sufficient to cover the interval $(0, 1/2]$. Let $\mathcal{O} = \left\{ \left(\dfrac{1}{\alpha}, 1\right) \;\middle|\; \alpha > 1 \right\}$ be an open cover of $(0, 1/2]$. Note that we may extract from $\mathcal{O}$ a finite sub-collection –
$\bar{\mathcal{O}} = \left\{ \left(\dfrac{1}{\alpha_0}, 1\right), \left(\dfrac{1}{\alpha_1}, 1\right), \ldots, \left(\dfrac{1}{\alpha_n}, 1\right) \right\}$ for some finite n. Also assume that $\alpha_0 < \alpha_1 < \ldots < \alpha_n$. It follows, naturally[1], that $\bigcup\limits_i \left(\dfrac{1}{\alpha_i}, 1\right) = \left(\dfrac{1}{\alpha_0}, 1\right)$. If $\bar{\mathcal{O}}$ is to be a cover of $(0, 1/2]$ then $(0, 1/2] \subset \bigcup\limits_i \left(\dfrac{1}{\alpha_i}, 1\right)$ for which it is implied that $(0, 1/2] \subset \left(\dfrac{1}{\alpha_0}, 1\right)$. Note, however, that for some $1/2n \in (0, 1/2]$, there exists an $N \in \mathbb{N}$ so that $1/2N < 1/\alpha_0$ – this is to say that $(0, 1/2] \not\subset \left(\dfrac{1}{\alpha_0}, 1\right) \Rightarrow (0, 1/2] \not\subset \bigcup\limits_i \left(\dfrac{1}{\alpha_i}, 1\right)$ and thus $\bar{\mathcal{O}}$ does not cover $(0, 1/2]$ and so does not cover S, either.

Exercise 4. Prove that the union of two compact sets is compact.

Proof. Let A and B be compact sets, and let $C = A \cup B$. Let $\{U_C\}_{C \in I}$ be an open cover of C. Since $C = A \cup B$, we see that $\{U_C\}_{C \in I}$ also covers A and B. Given that A is compact, for any open cover of A, there exists a finite sub-cover – let

[1] That is, we can verify that $\bigcup\limits_i \left(\dfrac{1}{\alpha_i}, 1\right) \subset \left(\dfrac{1}{\alpha_0}, 1\right)$ and $\left(\dfrac{1}{\alpha_0}, 1\right) \subset \bigcup\limits_i \left(\dfrac{1}{\alpha_i}, 1\right)$.

$\{U_{C_1}, U_{C_2}, \ldots, U_{C_n}\}$ (for a finite n) be a sub-cover of A. Similarly, since B is compact, for any open cover of B, there exists a finite sub-cover – let $\{U_{C_{n+1}}, U_{C_{n+2}}, \ldots, U_{C_d}\}$ (for a finite $d > n+1$) be a sub-cover of B. Now, we see that $A \subset \bigcup_{i=1}^{n} U_{C_i}$ and $B \subset \bigcup_{j=n+1}^{d} U_{C_j}$ for which it follows that $A \cup B \subset \left(\bigcup_{i=1}^{n} U_{C_i}\right) \cup \left(\bigcup_{j=n+1}^{d} U_{C_j}\right)$, or $C \subset \bigcup_{k=1}^{d} U_{C_k}$, verifying that $\{U_{C_1}, U_{C_2}, \ldots, U_{C_d}\}$ is a finite sub-cover of C, as required. □

Exercise 5. Prove that, if A is closed and B is compact, then $B \cap A$ is compact.

Proof. Let $\mathcal{O} = \{U_i\}$ be an open cover of $B \cap A$. We first note that, for any $x \in B \cap A$, it holds that $x \in B$, and so $B \cap A \subset B$. So $\mathcal{O}$ may or may not be a cover of B, but for assurance, we take $\mathcal{O} \cup A^c$ as an open cover of B. Since B is compact, there exists a finite sub cover – let $\bar{\mathcal{O}} = \{A^c, U_1, \ldots, U_n\}$ for some finite n. We now see that, since $B \cap A \subset B$, it follows that $\{A^c, U_1, \ldots, U_n\}$ covers $B \cap A$. Finally, note that for any $x \in B \cap A$, $x \in B$ and $x \in A$, so in particular $x \notin A^c$ and thus $x \in \{U_1, \ldots, U_n\}$, which reveals that $\{U_1, \ldots, U_n\}$ is a finite cover of $B \cap A$, as required. □