Math 327A: Autumn 2016

Homework 4

Solutions written by Alex Menendez (1438704)

EXERCISE 1. Consider the sequence defined by

$$\begin{cases} x_1 = 4 \\ x_{n+1} = \dfrac{5}{6 - x_n} \end{cases}$$

1. Prove that x_n is decreasing and convergent.
2. Prove that if a sequence u_n is convergent to a limit, L, then the sequence $v_n = u_{n+1}$ is also convergent to L.
3. Using the result of the previous question, find the limit of x_n.

SOLUTION.

1. *Proof.* We will carry out the proof in two components:

   - **Decreasing:** To first show that x_n is decreasing, we must verify that, for every $n \in \mathbb{N}$, one has $x_{n+1} < x_n$. That is,

     $$\frac{5}{6 - x_n} < x_n$$

     Examining the case that $n = 1$, we verify

     $$\frac{5}{6-4} < 4 \Longrightarrow \frac{5}{2} < 4$$

     Now, assume that for some $k \geq 1$, it holds that $x_{k+1} < x_k$. From this, we should verify that $x_{(k+1)+1} < x_{k+1}$. by definition, we have $x_{(k+1)+1} = \dfrac{5}{6 - x_{k+1}}$. However, $x_{k+1} < x_k$ and so $6 - x_{k+1} > 6 - x_k$ and thus $\dfrac{5}{6 - x_{k+1}} < \dfrac{5}{6 - x_k}$. Since $x_{(k+1)+1} = \dfrac{5}{6 - x_{k+1}}$ and $x_{k+1} = \dfrac{5}{6 - x_k}$, this verifies that $x_{(k+1)+1} < x_{k+1}$, as required.

   - **Convergent:** Now, we must show that the sequence converges. Set $\lim\limits_{n\to\infty} x_{n+1} = \lim\limits_{n\to\infty} x_n = L$ so that

     $$L = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \left(\frac{5}{6 - x_n}\right) = \frac{\lim_{n\to\infty} 5}{\lim_{n\to\infty}(6 - x_n)} = \frac{5}{6 - L}$$

     For which solutions are $L = 1$ and $L = 5$. Since $x_1 = 4$ and decreases onward, we shall choose $L = 1$ as a lower bound on the sequence – that is to say, $\forall n \in \mathbb{N},\ x_n > 1$. In the case that $n = 1$, one has that $x_1 = 4$ and obviously $x_1 > 1$. Assume that, for some $k \geq 1$, one has that $x_k > 1$. From this, it follows that $6 - x_k < 5$ and thus

     $$\frac{5}{6 - x_k} > \frac{5}{5} = 1$$
     $$\Longrightarrow x_{k+1} > 1$$

     and thus x_n is bounded below. Since x_n is bounded below and – by the previous result – decreasing for all $n \in \mathbb{N}$, it follows that x_n converges.

   □

2. *Proof.* If u_n converges to L, then by definition, given some $\epsilon > 0$, there exists some $N \in \mathbb{N}$ such that, if $n > N$, one has that $|u_n - L| < \epsilon$, which is to say that $L - \epsilon < u_n < L + \epsilon$. Thus, $u_{N+1}, u_{N+2}, \ldots$ are bounded above and below by $L + \epsilon$ and $L - \epsilon$, respectively. Note that if we focus on $u_{N+2}, u_{N+3}, \ldots$, we see that, in particular, they are bounded in the exact same manner, which is to say that $u_{(N+1)+1}, u_{(N+1)+2}, \ldots$ are bounded. Since, by definition, $v_n = u_{n+1}$ what we have is that $v_{N+1}, v_{N+2}, v_{N+3}, \ldots$ are bounded above and below by $L - \epsilon$ and $L + \epsilon$, respectively. This implies that, for $n > N$, $L - \epsilon < v_n < L + \epsilon \Longrightarrow |v_n - L| < \epsilon$, given some ϵ. This verifies that v_n, as it is defined, converges to L. □

3. By part (1), we had discovered that x_{n+1} approaches 1 as $n \to \infty$. By (2), this implies that x_n also approaches 1. This agrees with our intuition, since each new value which is generated from the sequence, x_{n+1}, is plugged back into the sequence itself as

x_n for each iteration, and so as $x_{n+1} \to 1$, so too does $x_n \to 1$. Here is a simple Java program that verifies the sequence converges to 1:

```
public class Sequence{
    public static void main(String[] args){
        double x = 4.0; //initialize
        double threshold = Math.pow(10, -100); //set near-zero tolerance to guarantee
        //convergence
        boolean notConverging = true;
        while(notConverging){
            x = 5 / (6 - x);
            if(Math.abs(x - 1) < threshold){
                notConverging = false;
            }
        }
        System.out.print(x);
    }
}


output:  1.0
```

EXERCISE 2. Let S_n be a sequence of open sets.

1. Prove that $\bigcup_{n=1}^{\infty} S_n$ is open.
2. Prove that the finite intersection $\bigcap_{n=1}^{N} S_n$ is open.
3. Is $\bigcap_{n=1}^{\infty} S_n$ open or closed? Prove the result, or provide a counter-example.

SOLUTION.

1. *Proof.* Let $\mathbf{S} = \bigcup_{n=1}^{\infty} S_n$. Take $x \in \mathbf{S}$, which of course implies that x belongs to a particular S_n. Since any given S_n is open, it follows that given some $x \in S_n$, there exists a $\delta > 0$ such that a neighborhood $(x - \delta, x + \delta)$ is entirely included in S_n. Since a particular S_n belongs to the union $\mathbf{S}$, however, it follows that $(x - \delta, x + \delta) \subset \mathbf{S}$, which implies that given an $x \in \mathbf{S}$, there exists a neighborhood of x entirely included in $\mathbf{S}$, and so $\mathbf{S}$ is open. □

2. *Proof.* Beginning with the case when $N = 1$, one has that $\bigcap_{n=1}^{1} S_n = S_1$, and so S_1 is open, by definition. We will now assume that, given some $k \geq 1$, it holds that $\bigcap_{n=1}^{k} S_n$ is open. Furthermore, we see that $\bigcap_{n=1}^{k+1} S_n = \left(\bigcap_{n=1}^{k} S_n \right) \cap S_{k+1}$.

   Lemma 1. *Let A and B be open sets. It follows that $A \cap B$ is open.*

   We see that $\bigcap_{n=1}^{k} S_n$ is open, as prescribed, as is S_{k+1}. By Lemma 1, it holds that their intersection, $\left(\bigcap_{n=1}^{k} S_n \right) \cap S_{k+1}$, is also open. This verifies that $\bigcap_{n=1}^{k+1} S_n$ is open, as desired. □

ALTERNATIVE PROOF:

Proof. Take $x \in \bigcap_{n=1}^{N} S_n$, which implies:

- $x \in S_1$, which is open, and so there exists a $\delta_1 > 0$ such that $(x - \delta_1, x + \delta_1) \subset S_1$.
- $x \in S_2$, which is open, and so there exists a $\delta_2 > 0$ such that $(x - \delta_2, x + \delta_2) \subset S_2$.

  $\vdots$
- $x \in S_N$, which is open, and so there exists a $\delta_N > 0$ such that $(x - \delta_N, x + \delta_N) \subset S_N$.

Now take $\delta = \min(\delta_1, \delta_2, \ldots, \delta_N)$ and so $(x - \delta, x + \delta)$ is entirely included in $S_1, S_2, \ldots, S_N$ and so is entirely included in $\bigcap_{n=1}^{N} S_n$, and thus the intersection is open. □

3. This intersection *can* be closed.

   Claim. Consider the sequence of *open* intervals $\left\{\left(-\frac{1}{n}, \frac{1}{n}\right)\right\}$. It holds that $\bigcap_{n=1}^{\infty}\left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$, which is closed.

   Proof. We must show a double subset inclusion:

   - $0 \in \left(-\frac{1}{n}, \frac{1}{n}\right)$ for every $n \in \mathbb{N}$ and so $0 \in \bigcap_{n=1}^{\infty}\left(-\frac{1}{n}, \frac{1}{n}\right)$. Since 0 is the only element of $\{0\}$, one finds that every element of $\{0\}$ is included in $\bigcap_{n=1}^{\infty}\left(-\frac{1}{n}, \frac{1}{n}\right)$ and so $\{0\} \subset \bigcap_{n=1}^{\infty}\left(-\frac{1}{n}, \frac{1}{n}\right)$.
   - Take some $x \in \bigcap_{n=1}^{\infty}\left(-\frac{1}{n}, \frac{1}{n}\right)$. It follows, then, that $x \in \left(-\frac{1}{n}, \frac{1}{n}\right)$ for every $n \in \mathbb{N}$. Ideally, what we want to show is that $x = 0$ so suppose, by contradiction, that $x \neq 0$. If this is the case, then $|x| > 0$ and by the Archimedean Law, there is a $n \in \mathbb{N}$ such that $n > |x|$, which implies that $\frac{1}{n} < \frac{1}{|x|}$ and so $x \notin \left(-\frac{1}{n}, \frac{1}{n}\right)$, a contradiction. Thus, $x = 0$ and so the only element of $\bigcap_{n=1}^{\infty}\left(-\frac{1}{n}, \frac{1}{n}\right)$ is 0, which shows that $\bigcap_{n=1}^{\infty}\left(-\frac{1}{n}, \frac{1}{n}\right) \subset \{0\}$.

   □

EXERCISE 3. For each of the following sets: $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$,

1. Is the set open?
2. Is the set closed?
3. Find the accumulation point of the set.

SOLUTION.

- $\mathbb{Z}$:
  1. $\mathbb{Z}$ is not open. Given any $x \in \mathbb{Z}$, we can locate non-integers in any neighborhood of x.
  2. $\mathbb{Z}$ is closed – that is to say $\mathbb{R} \setminus \mathbb{Z}$ is open. This is because, for any $x \in \mathbb{R} \setminus \mathbb{Z}$, there is a $\delta > 0$ such that $(x - \delta, x + \delta) \subset \mathbb{R} \setminus \mathbb{Z}$. That is, given any real number that is not an integer, we can construct a neighborhood around that number whose elements are all real numbers, but none of which are integers[1].
  3. $\mathbb{Z}$ has no accumulation points. Take $x \in \mathbb{R}$. We know that x is an accumulation point of $\mathbb{Z}$ if, for any $\delta > 0$, the neighborhood $(x - \delta, x + \delta)$ contains a point in $\mathbb{Z}$ other than x itself. Since $x \in \mathbb{R}$, given some $n \in \mathbb{Z}$, it holds that $n < x < n + 1$ (between any two integers, there is a real number). Now take $\delta \leq \min(|x - n|, |x - (n + 1)|)$ and so $(x - \delta, x + \delta)$ is guaranteed to contain no integers.

[1] e.g., take $x = 2.3$ and $\delta = 0.1$. Observe the interval $(2.2, 2.4)$ contains no integers.

- $\mathbb{Q}$:

  1. $\mathbb{Q}$ is not open. If $\mathbb{Q}$ is open, then for any $x \in \mathbb{Q}$, there exists a $\delta > 0$ so that the neighborhood $(x-\delta, x+\delta)$ is fully contained in $\mathbb{Q}$. Since $x \in \mathbb{Q}$, one has that $x = p/q$, for $q \neq 0$. Thus, what we have is that $\left(\frac{p}{q} - \delta, \frac{p}{q} + \delta\right) \subset \mathbb{Q}$. That is, every rational number belongs to an open interval containing *only* rational numbers. This is not the case, so $\mathbb{Q}$ is not open.
  2. $\mathbb{Q}$ is not closed, either. If $\mathbb{Q}$ were closed, then each irrational number would be contained inside of an open interval that *only* contains irrational numbers. That is, for every $x \in \mathbb{R} \setminus \mathbb{Q}$, there is a $\delta > 0$ so that $(x-\delta, x+\delta) \subset \mathbb{R} \setminus \mathbb{Q}$. This is not the case, either, and thus $\mathbb{Q}$ is closed.
  3. The set of accumulation points of $\mathbb{Q}$ is $\mathbb{R}$. Take $x \in \mathbb{R}$. For x to be an accumulation point of $\mathbb{Q}$, we require that for any $\delta > 0$, the interval $(x-\delta, x+\delta)$ contains a point in $\mathbb{Q}$ that is not x itself. We now recall that, between any two real numbers, there is a rational number. That is, between $x-\delta$ and $x+\delta$, there is a rational number. But, of course, this can be extended to show that $(x-\delta, x+\delta)$ contains infinitely many rational numbers. Because of this, there certainly exists a rational number in $(x-\delta, x+\delta)$ that is different than x. However, this applies for every $x \in \mathbb{R}$, and so every $x \in \mathbb{R}$ is an accumulation point.

- $\mathbb{R}$:

  1. $\mathbb{R}$ is open. Take some $x \in \mathbb{R}$. Then there is a $\delta > 0$ such that $(x-\delta, x+\delta) \subset \mathbb{R}$. That is, given some real number, we can find an interval that contains that particular real number, and all of the elements of that interval are real numbers.
  2. $\mathbb{R}$ is closed. If $\mathbb{R}$ is closed, then its complement is open. The complement of $\mathbb{R}$ is $\mathbb{R} \setminus \mathbb{R} = \varnothing$. Take some $x \in \varnothing$. Then there is a $\delta > 0$ so that $(x-\delta, x+\delta) \subset \varnothing$. Since $\varnothing$ contains nothing, then since $x \in \varnothing$, one finds that $(x-\delta, x+\delta)$ also contains nothing and thus $(x-\delta, x+\delta)$ is entirely included in $\varnothing$, so $\varnothing$ is open and thus $\mathbb{R}$ is closed.
  3. The set of accumulation points of $\mathbb{R}$ is $\mathbb{R}$. Take some $x \in \mathbb{R}$, and x is an accumulation point of $\mathbb{R}$ if for every $\delta > 0$, one finds that $(x-\delta, x+\delta)$ contains a point in $\mathbb{R}$ that is not x itself. This is the case for every $x \in \mathbb{R}$ and so the set of accumulation points is $\mathbb{R}$.

EXERCISE 4. Find the accumulation points of each set:

1. $\left\{ (-1)^n \left(\frac{2n+5}{n+2}\right) \,\middle|\, n \in \mathbb{N} \right\}$
2. $\left\{ \frac{n}{n+m} \,\middle|\, n, m \in \mathbb{N} \right\}$

SOLUTION.

1. Let's figure out what the sequence $(-1)^n \left(\frac{2n+5}{n+2}\right)$ converges to.

   - If n is taken to be even, then in particular the constant $(-1)^n$ will always be 1, so we now examine

   $$\lim_{n\to\infty} \frac{2n+5}{n+2} = \lim_{n\to\infty} \frac{2+\frac{5}{n}}{1+\frac{2}{n}} = \frac{\lim_{n\to\infty} 2+\frac{5}{n}}{\lim_{n\to\infty} 1+\frac{2}{n}} = 2$$

   - If n is taken to be odd, then in particular the constant $(-1)^n$ will always be -1, so we now examine

   $$\lim_{n\to\infty} -\frac{2n+5}{n+2} = \lim_{n\to\infty} -\frac{2+\frac{5}{n}}{1+\frac{2}{n}} = -\frac{\lim_{n\to\infty} 2+\frac{5}{n}}{\lim_{n\to\infty} 1+\frac{2}{n}} = -2$$

   So ± 2 are candidates for the accumulation points.

   - To show that 2 is an accumulation point, for every $\delta > 0$, there is an even n so that $n > \frac{3}{\delta} - 2 \Rightarrow \delta > \frac{3}{n+2} \Rightarrow 2 + \delta > \frac{2n+5}{n+2}$ but the sequence $(-1)^n \left(\frac{2n+5}{n+2}\right)$ decreases toward 2 and since $2+\delta > 2$, one has that

   $$2 + \delta > \frac{2n+5}{n+2} > 2$$

   And so there are infinitely many (even) values of n so that $(-1)^n \left(\frac{2n+5}{n+2}\right) \in (2-\delta, 2+\delta)$ and in particular $(-1)^n \left(\frac{2n+5}{n+2}\right) \neq 2$ and thus 2 is an accumulation point.

- To show that -2 is an accumulation point, for every $\delta > 0$, there is an odd n so that $\delta > \frac{4n+9}{n+2} \Rightarrow -2+\delta > \frac{2n+5}{n+2}$ but the sequence $(-1)^n\left(\frac{2n+5}{n+2}\right)$ decreases toward -2 and since $-2+\delta > -2$, one has that

$$-2+\delta > \frac{2n+5}{n+2} > -2$$

  And so there are infinitely many (odd) values of n so that $(-1)^n\left(\frac{2n+5}{n+2}\right) \in (-2-\delta, -2+\delta)$ and in particular $(-1)^n\left(\frac{2n+5}{n+2}\right) \neq -2$ and thus -2 is an accumulation point.

2. Let's find out what the sequence $\frac{n}{n+m}$ converges to. One one hand,

$$\lim_{n\to\infty} \frac{n}{n+m} = \lim_{n\to\infty} \frac{1}{1+\frac{m}{n}} = \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty} 1+\frac{m}{n}} = 1$$

   And on the other hand,

$$\lim_{m\to\infty} \frac{n}{n+m} = \lim_{m\to\infty} \frac{\frac{n}{m}}{1+\frac{n}{m}} = \frac{\lim_{m\to\infty} \frac{n}{m}}{\lim_{m\to\infty} 1+\frac{n}{m}} = 0$$

   So 1 and 0 are candidates for the accumulation points.

   - To show that 0 is an accumulation point, for every $\delta > 0$, there is an m so that $m > \frac{n}{\delta} \Rightarrow \delta > \frac{n}{m}$. But since $\frac{n}{m} > \frac{n}{n+m}$ one finds that $\delta > \frac{n}{n+m}$ and since $\frac{n}{n+m} > 0$ for any $n, m \in \mathbb{N}$.

$$0 < \frac{n}{n+m} < \delta$$

     And so $\frac{n}{n+m} \in (-\delta, \delta)$ and since $\frac{n}{n+m} \neq 0$ for any $n, m \in \mathbb{N}$, one finds that 0 is indeed an accumulation point.
   - To show that 1 is an accumulation point, choose $\delta > 0$ so that $1+\delta > 1$. But certainly $\frac{n}{n+m} > \frac{n}{n} = 1$ for any $n, m \in \mathbb{N}$ and so $1+\delta > \frac{n}{n+m} > 1$ and thus $\frac{n}{n+m} \in (1-\delta, 1+\delta)$ and since $\frac{n}{n+m} \neq 1$ for any $n, m \in \mathbb{N}$, we find that 1 is an accumulation point, as required.