Math 327A: Autumn 2016

Homework 3

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EXERCISE 1. Given two bounded, non-empty sets of real numbers A and B, the greatest lower bound of a set C is denoted $\inf C$ and the least upper bound is denoted $\sup C$.

1. If $A + B = \{a + b \mid a \in A \text{ and } b \in B\}$, prove that $\inf(A + B) = \inf A + \inf B$.
2. Let $-A$ be the set of additive inverses of the elements of A. Prove that $\inf(-A) = -\sup A$.
3. If A is a non-empty subset of B, prove that $\inf B \leq \inf A \leq \sup A \leq \sup B$.

SOLUTION.

1. *Proof.* The most streamlined way of performing this proof is to use inequalities[1] to show that $\inf(A + B) \leq \inf A + \inf B$ and $\inf(A + B) \geq \inf A + \inf B$, in which case $\inf(A + B) = \inf A + \inf B$.

   - For any $a \in A$ and any $b \in B$, it holds that $a \geq \inf A$ and $b \geq \inf B$. It follows, then, that $a + b \geq \inf A + \inf B$. However, $a + b \geq \inf(A + B)$ for all $a + b \in A + B$, so $\inf(A + B) \geq \inf A + \inf B$.
   - Since, for any $a + b \in A + B$, one has that $a + b \geq \inf(A + B)$, it follows that $\inf(A + B) - a \leq b$. Since $b \geq \inf B$, certainly $\inf(A + B) - a \leq \inf B$. Alternatively, we now have that $\inf(A + B) - \inf B \leq a$, and by the same token, $a \geq \inf A$, so it follows that $\inf(A + B) - \inf B \leq \inf A$. From this it follows that $\inf(A + B) \leq \inf A + \inf B$.

   Since it can be shown that both $\inf(A + B) \geq \inf A + \inf B$ and $\inf(A + B) \leq \inf A + \inf B$, we have that $\inf(A + B) = \inf A + \inf B$, as required. □

2. *Proof.* Since $a \leq \sup A$ for any $a \in A$, one has that $-a \geq -\sup A$ for any $-a \in -A$. This, of course, implies that $-\sup A$ is a lower bound on $-A$. What we now need to show is that $-\sup A$ is the *greatest* lower bound on $-A$, in which case we are done. This can be shown by proving that $-\sup A$ is greater than or equal to any general lower bound of $-A$. Let β be a lower bound on $-A$. This implies that $\beta \leq -a$ for any $-a \in -A$. Of course, this yields that $-\beta \geq a$ for any $a \in A$, but since A is a bounded set[2], $\sup A \in A$, and so it follows that $-\beta \geq \sup A$ and thus $\beta \leq -\sup A$. This verifies that $-\sup A$ is, in fact, the *greatest* lower bound on $-A$, which is exactly $\inf(-A)$. This gives $-\sup A = \inf(-A)$, as required. □

3. *Proof.* We can examine this long chain of inequalities by first examining three smaller, intermediate inequalities:

   - $\inf B \leq \inf A \quad (1)$
   - $\inf A \leq \sup A \quad (2)$
   - $\sup A \leq \sup B \quad (3)$

   We will go for the lowest-hanging fruit first; we can show (2) quite easily – for any $a \in A$, it follows that $a \geq \inf A$ and $a \leq \sup A$, in which case $\inf A \leq a \leq \sup A$, and so $\inf A \leq \sup A$, as required. We can now turn our attention to (3) – first note that since $A \subset B$, it follows that for any $a \in A$, it is also true that $a \in B$. Note that, since A is non-empty and bounded (see the footnote in the previous problem), it follows that $\sup A \in A$ and thus $\sup A \in B$. Since any $b \in B$ obeys the property that $b \leq \sup B$ and since $\sup A \in B$, one has that $\sup A \leq \sup B$, as required. Lastly, we can prove (1) in the same way as (3) – observe that any $b \in B$ boys the property that $b \geq \inf B$, but since $\inf A \in A$ and $A \subset B$, one has that $\inf A \in B$, so $\inf A \geq \inf B$, as required. Naturally, putting this all together gives $\inf B \leq \inf A \leq \sup A \leq \sup B$. □

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[1] i.e., we exploit the fact that, if $x \leq a$ and $x \geq a$ simultaneously, then the only conclusion is that $x = a$.

[2] Technically, $\sup A \in A$ isn't true for **any** bounded set A, but we are working with the definition of a bounded set presented by Dr. Dos Reis in class: $A = \{x \in \mathbb{R} \mid M \leq x \leq N\}$, in which case $\sup A = N$ and $N \in A$, so $\sup A \in A$. For similar reasons, we can say that $\inf A \in A$.

EXERCISE 2. Let $\{a_n\}$ be a convergent sequence.

1. Prove that the limit of $\{a_n\}$ is unique.
2. Prove that the set $S = \{a_n \mid n \in \mathbb{N}\}$ is bounded.

SOLUTION.

1. *Proof.* Assume that $\{a_n\}$ converges to L_1 and L_2, where $L_1 \neq L_2$ and take $\epsilon = |L_1 - L_2|/4$. We may write $L_1 - L_2 = L_1 - a_n + a_n - L_2$ for algebraic reasons – doing so allows us to apply the triangle inequality:

$$|L_1 - L_2| = |L_1 - a_n + a_n - L_2| \leq |L_1 - a_n| + |L_2 - a_n|$$

However, since $\{a_n\}$ converges to L_1 and L_2, it follows that $|a_n - L_1| < \epsilon$ and $|a_n - L_2| < \epsilon$, provided n is sufficiently large. Thus,

$$|L_1 - L_2| = |L_1 - a_n + a_n - L_2| \leq \underbrace{|L_1 - a_n|}_{<\epsilon} + \underbrace{|L_2 - a_n|}_{<\epsilon}$$

$$\implies |L_1 - L_2| < 2\epsilon = 2\left(\frac{|L_1 - L_2|}{4}\right)$$

$$\implies |L_1 - L_2| < \frac{|L_1 - L_2|}{2}$$

However, this is impossible since $|L_1 - L_2| > 0$, and thus develops a contradiction. From this it follows that $L_1 = L_2$, as required. □

2. *Proof.* Since $\{a_n\}$ is convergent, then given some $\epsilon > 0$, we can find some $N \in \mathbb{N}$ such that, if $n > N$, one finds that $|a_n - L| < \epsilon$. By the definition of the absolute value, this implies

$$-\epsilon < a_n - L < \epsilon$$

$$\implies L - \epsilon < a_n < L + \epsilon$$

So provided $n > N$, we can find an explicit upper and lower bound on a_n. However, just because we can say $a_{N+1}, a_{N+2}, \ldots$ are bounded doesn't say anything meaningful about $a_1, a_2, \ldots, a_N$. We can easily resolve this by knowing that $\{a_1, a_2, \ldots, a_N\}$ form a *finite* set, so obviously there is a smallest and largest element. Thus, $\{a_1, a_2, \ldots, a_N\}$ is bounded and so is $\{a_{N+1}, a_{N+2}, \ldots\}$ and thus S is bounded. □

EXERCISE 3. If a sequence $\{a_n\}$ converges to $A > 0$, prove that there exists N such that $a_n > 0$ for any $n > N$.

Proof. If $\{a_n\}$ converges to A, then for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that, for any $n > N \implies |a_n - A| < \epsilon$. By definition, this gives

$$-\epsilon < a_n - A < \epsilon$$

$$\implies A - \epsilon < a_n < A + \epsilon$$

Since we may choose any $\epsilon > 0$, take $\epsilon = A$ because $A > 0$ and so

$$\implies 0 < a_n < 2A$$

In particular, we have $a_n > 0$, as required. □

EXERCISE 4. Given two convergent sequences $\{a_n\}$ and $\{b_n\}$ with $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$, show that the product $\{a_n b_n\}$ is convergent and that $\lim_{n\to\infty}(a_n b_n) = AB$.

Proof. Since $\{a_n\}$ and $\{b_n\}$ both converge, we know:

- For every $\epsilon_a > 0$, there exists an $N_a \in \mathbb{N}$ such that if $n > N_a$, one has that $|a_n - A| < \epsilon_a$.
- For every $\epsilon_b > 0$, there exists an $N_b \in \mathbb{N}$ such that if $n > N_b$, one has that $|b_n - B| < \epsilon_b$.

We must show that, for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that if $n > N$, one has that $|a_nb_n - AB| < \epsilon$. Using the fact that $a_nb_n - AB = a_nb_n - a_nB + a_nB - AB$, we may say

$$|a_nb_n - AB| = |a_nb_n - a_nB + a_nB - AB| = |a_n(b_n - B) + B(a_n - A)| < \epsilon$$

scratch work...

From this, we may say that $a_n(b_n - B) < \epsilon/2$ and $B(a_n - A) < \epsilon/2$ so that $(b_n - B) < \epsilon/2M$ and $(a_n - A) < \epsilon/2B$, where M denotes the largest value of a_n. So we find that $\epsilon_a = \epsilon/2B$ and $\epsilon_b = \epsilon/2M$. Lastly, we can take N to be $N = \max\{N_a, N_b\}$.

Now, take $\epsilon_a = \epsilon/2B$ and $\epsilon_b = \epsilon/2M$, where $M = \max_{n\in\mathbb{N}}\{a_n\}$, and take $N = \max\{N_a, N_b\}$. Now, if $n > N$ then certainly $n > N_a$ which implies that for every $\epsilon_a > 0$, we have $|a_n - A| < \epsilon_a$, which gives $a_n - A < \epsilon/2B$. Similarly, one has that $n > N_b$ and so for every $\epsilon_b > 0$, we have $|b_n - B| < \epsilon_b$, which gives $b_n - B < \epsilon/2M$. It follows that $B(a_n - A) < \epsilon/2$ and $M(b_n - B) < \epsilon/2$. Certainly, then,

$$|M(b_n - B) + B(a_n - A)| < \epsilon/2 + \epsilon/2 = \epsilon$$

Since $M(b_n - B) < \epsilon/2$ and M is the maximum value of a_n, then certainly $a_n(b_n - B) < \epsilon/2$ so that

$$|a_n(b_n - B) + B(a_n - A)| < \epsilon$$

$$\implies |a_nb_n - AB| < \epsilon$$

which verifies that, given our choice of N, ϵ_a, and ϵ_b, we have shown explicitly that $\lim\limits_{n\to\infty} a_nb_n = AB$, as required. $\square$