Math 327A: Autumn 2016

Homework 1

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Exercise 1. Let $\mathcal{F}$ be the set of all the numbers of the form $p+\sqrt{3}q$, where p and q are rational numbers:

$$\mathcal{F}=\left\{p+\sqrt{3}q \;\middle|\; p,q\in\mathbb{Q}\right\}$$

1. Find the additive and multiplicative identities in the forms $p_0+\sqrt{3}q_0$ and $p_1+\sqrt{3}q_1$.
2. Given an element of $\mathcal{F}$, find its additive inverse in the form $p'+\sqrt{3}q'$, where p' and q' are rational numbers.
3. Given an element of $\mathcal{F}$, find its multiplicative inverse in the form $p''+\sqrt{3}q''$, where p'' and q'' are rational numbers.

Solution.

1. We will find both the additive and multiplicative identities summarily:
   - To find the additive identity, we desire $p_0+\sqrt{3}q_0$ so that $\left(p+\sqrt{3}q\right)+\left(p_0+\sqrt{3}q_0\right)=p+\sqrt{3}q$. Expanding the left hand side, one has
   $$(p+p_0)+\sqrt{3}\,(q+q_0)=p+\sqrt{3}q$$
   And so by direct comparison, $p_0=q_0=0$ so that the additive identity is $\boxed{0+0\cdot\sqrt{3}}$.
   - To find the multiplicative identity, we desire $p_1+\sqrt{3}q_1$ so that $\left(p+\sqrt{3}q\right)\left(p_1+\sqrt{3}q_1\right)=p+\sqrt{3}q$. As before, expanding the left hand side gives
   $$pp_1+\sqrt{3}q_1p+\sqrt{3}qp_1+3qq_1=pp_1+\sqrt{3}\,(q_1p+qp_1)+3qq_1=p+\sqrt{3}q$$
   Direct comparison reveals that $p_1=1$ and $q_1=0$ so that the multiplicative identity is $\boxed{1+0\cdot\sqrt{3}}$.
2. We desire $p'+\sqrt{3}q'$ so that $\left(p+\sqrt{3}q\right)+\left(p'+\sqrt{3}q'\right)=0$. Given $p+\sqrt{3}q$, we may define $\boxed{p'=-p}$ and $\boxed{q'=-q}$ so that $p'+\sqrt{3}q'=(-p)+\sqrt{3}(-q)$ and thus
   $$\left(p+\sqrt{3}q\right)+\left((-p)+\sqrt{3}(-q)\right)=\left((p+(-p))+\sqrt{3}(q+(-q))\right)=0+0\cdot\sqrt{3}=0$$
   This verifies that $(-p)+\sqrt{3}(-q)$ is the additive inverse for an element of $\mathcal{F}$, and since p and q are rational, $p'=-p$ and $q'=-q$ are rational.
3. We desire $p''+\sqrt{3}q''$ so that $\left(p+\sqrt{3}q\right)\left(p''+\sqrt{3}q''\right)=1$. Given $p+\sqrt{3}q$, we can first multiply by its conjugate to obtain $\left(p+\sqrt{3}q\right)\left(p-\sqrt{3}q\right)=p^2-3q^2$. We want the right hand side to be equal to 1, thus dividing by p^2-3q^2 gives
   $$\left(p+\sqrt{3}q\right)\left(\frac{p-\sqrt{3}q}{p^2-3q^2}\right)=\left(p+\sqrt{3}q\right)\underbrace{\left(\frac{p}{p^2-3q^2}+\sqrt{3}\frac{-q}{p^2-3q^2}\right)}_{p''+\sqrt{3}q''}=1$$
   whence $\boxed{p''=\dfrac{p}{p^2-3q^2}}$ and $\boxed{q''=\dfrac{-q}{p^2-3q^2}}$. Since p and q are rational, so too are p'' and q''.

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Exercise 2.

1. Prove, using the definition of the commutative field, that the multiplicative identity is unique.
2. Prove that $(a \cdot b)^{-1} = a^{-1} \cdot b^{-1}$ for any $a \neq 0$ and $b \neq 0$.
3. Prove that $\left(b^{-1}\right)^{-1} = b$ for any $b \neq 0$.

Solution.

1. *Proof.* Let e and e' be multiplicative identities. It follows, by definition, that $e = e' \cdot e = e \cdot e' = e'$. Thus we have exploited the fact that multiplication is commutative to prove necessarily that $e = e'$ and so the multiplicative identity is unique, as required. $\square$

2. *Proof.* We know that
$$(a \cdot b) \cdot (a \cdot b)^{-1} = 1 \tag{1}$$
which follows from the definition of the multiplicative inverse. More generally, for a field $\mathcal{F}$, an element $x \in \mathcal{F}$ is a multiplicative inverse if $(a \cdot b) \cdot x = 1$. Since we know from (1) that $x = (a \cdot b)^{-1}$, we should deduce algebraically that $x = a^{-1} \cdot b^{-1}$. Knowing $(a \cdot b) \cdot x = 1$, we can isolate x as follows:
$$a^{-1}((a \cdot b) \cdot x) = a^{-1} \cdot 1 = a^{-1}$$
$$\implies \underbrace{a^{-1} \cdot (a \cdot b) \cdot x = \left(a^{-1} \cdot a\right) \cdot (b \cdot x)}_{\text{Associative rule}} = a^{-1}$$
$$\implies 1 \cdot (b \cdot x) = b \cdot x = a^{-1}$$
$$\implies b^{-1}(b \cdot x) = b^{-1} \cdot a^{-1}$$
$$\implies \left(b^{-1} \cdot b\right) \cdot x = b^{-1} \cdot a^{-1}$$
$$\implies 1 \cdot x = x = b^{-1} \cdot a^{-1}$$
Lastly, applying the associative rule on the right hand side gives $x = a^{-1} \cdot b^{-1}$, as required. $\square$

3. *Proof.* In a manner similar to that used to prove problem (2), we will begin by making the assumption that $\left(b^{-1}\right)^{-1}$ denotes the inverse of b^{-1} so that
$$\left(b^{-1}\right) \cdot \left(b^{-1}\right)^{-1} = 1 \tag{2}$$
More generally, (2) can be rephrased as $b^{-1} \cdot x = 1$, where $x = \left(b^{-1}\right)^{-1}$, by definition. Our aim is to deduce algebraically that $x = b$. It follows that
$$b \cdot \left(b^{-1} \cdot x\right) = b \cdot 1 = b$$
By the associative rule, re-grouping the parentheses on the left hand side gives
$$\left(b \cdot b^{-1}\right) \cdot x = b$$
$$\implies 1 \cdot x = x = b$$
And so $x = \left(b^{-1}\right)^{-1} = b$, as required. $\square$

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Exercise 3. Prove, using axiom 2, that

1. Given real numbers a, b, and c, if $a < b$ then $a + c < b + c$.
2. For any real numbers a and b such that $0 < a < b$, one has that $b^{-1} < a^{-1}$.
3. If $a < b$ and $c < 0$, then $ac > bc$.

Solution.

1. *Proof.* We will go the route of proving the contrapositive. Assuming that $a + c$ is not less than $b + c$, there are two possibilities:

   - If $a + c = b + c$, then $(a + c) - (b + c) = 0$. By both the distributive and associative rules, the left hand side may be concatenated as $a + c - b - c = (a - b) + (c - c)$ so that $(a - b) + (c - c) = 0$. Since $c - c = 0$ for any $c \in \mathcal{F}$, one has that $a - b = 0$, or $a = b$.
   - Alternatively, if $a + c > b + c$, then by the same reasoning it follows that $(a + c) - (b + c) > 0$. By the distributive and associative rules, this is equivalent to $(a - b) + (c - c) > 0$. Since $c - c = 0$ for any $c \in \mathcal{F}$, it stands that $a - b > 0$ and so $a > b$.

   In either case, the assumption that $a + c$ is not less than $b + c$ has allowed us to prove that a is not less than b and thus the contrapositive is proven. □

2. *Proof.* Beginning with $0 < a < b$, we may multiply the inequality through by b^{-1}, a legal move since $b > 0$. Doing so, we get $0 < a \cdot b^{-1} < b \cdot b^{-1}$, or

   $$0 < a \cdot b^{-1} < 1$$

   Multiplying through by a^{-1}, another legal move given $a > 0$, one has $0 < a^{-1} \cdot a \cdot b^{-1} < a^{-1} \cdot 1$, or

   $$0 < b^{-1} < a^{-1}$$

   Which is the desired result. □

3. *Proof.* If $a < b$, then $a - b < 0$. Multiplying both sides of the inequality by c, one has that $c \cdot (a - b) < 0$. Now, we note that $c < 0$ as prescribed, and $a - b < 0$, as we deduced earlier. Since both factors are negative, $c \cdot (a - b)$ is greater than zero, or $c \cdot (a - b) > 0$. By the distributive and associative rules, the left hand side can be rephrased so that $ac - bc > 0$, and thus $ac > bc$. □