Math 328B: Winter 2017

Homework 4

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EXERCISE 1. Determine whether the following integrals are convergent or divergent:

1. $\displaystyle\int_5^\infty \frac{\arctan(x)}{x^2+3x+5}\,dx$
2. $\displaystyle\int_0^\infty \frac{\cos(t)}{\sqrt{e^t-1}}\,dx$
3. $\displaystyle\int_0^\infty \frac{\ln(x)}{x^2-1}\,dx$
4. $\displaystyle\int_0^\infty (x+2)-\sqrt{x^2+4x+1}\,dx$
5. $\displaystyle\int_0^1 \frac{1}{\sqrt[3]{x^2-x^3}}\,dx$

SOLUTION.

1. First, we'll begin by examining where any discontinuities may lie. First of all, the denominator is never zero on $[5,\infty)$. Secondly, $\arctan(x)$ is continuous everywhere, so we needn't split the integral into pieces, and can therefore dive into a direct comparison. Observe that $\displaystyle\lim_{x\to\infty}\arctan(x)=\frac{\pi}{2}$, so $\arctan(x)$ has a horizontal asymptote at $y=\pi/2$ and thus $\arctan(x)<\dfrac{\pi}{2}$ on $[5,\infty)$. Next, we observe that $x^2+3x+5\geq x^2$ and so
$$\frac{\arctan(x)}{x^2+3x+5}\leq\frac{\pi}{2}\left(\frac{1}{x^2}\right)$$
So $\displaystyle\frac{\pi}{2}\int_5^\infty\frac{1}{x^2}\,dx=\frac{\pi}{10}$, and thus $\displaystyle\int_5^\infty\frac{\arctan(x)}{x^2+3x+5}\,dx$ converges to a finite value by comparison.

2. For convenience, we'll split the integral at $t=\pi/2$, where it's guaranteed that $\cos(t)$ is positive in one of the sub-intervals. Thus,
$$\int_0^\infty\frac{\cos(t)}{\sqrt{e^t-1}}\,dt=\underbrace{\int_0^{\pi/2}\frac{\cos(t)}{\sqrt{e^t-1}}\,dt}_{I_1}+\underbrace{\int_{\pi/2}^\infty\frac{\cos(t)}{\sqrt{e^t-1}}\,dt}_{I_2}$$
Examining I_1, we see that $\cos(t)\leq 1$ for $t\in[0,\pi/2]$ and is non-negative, so in particular $\dfrac{\cos(t)}{\sqrt{e^t-1}}\leq\dfrac{1}{\sqrt{e^t-1}}$. Now, observe that $\displaystyle\int_0^{\pi/2}\frac{1}{\sqrt{e^t-1}}\,dt$ converges:
$$\int_0^{\pi/2}\frac{1}{\sqrt{e^t-1}}\,dt=\int_0^{\sqrt{e^{\pi/2}-1}}2e^{-\ln(u^2+1)}\,du=2\arctan\left(\sqrt{e^{\pi/2}-1}\right)$$
which results from letting $u=\sqrt{e^t-1}$ so that $du=e^t/2u$. Thus, since this integral converges to a finite value, so too does $\displaystyle\int_0^{\pi/2}\frac{\cos(t)}{\sqrt{e^t-1}}\,dt=I_1$. Now, it remains to examine whether I_2 converges or diverges. First, let's observe that
$$\left|\frac{\cos(t)}{\sqrt{e^t-1}}\right|\leq\left|\frac{t-\pi/2}{\sqrt{e^t-1}}\right|$$

Since $t \geq \pi/2$, it stands that $\left|\dfrac{t-\pi/2}{\sqrt{e^t-1}}\right| = \dfrac{t-\pi/2}{\sqrt{e^t-1}}$ if $t > \pi/2$ and 0 if $t = \pi/2$. Obviously, $\lim\limits_{t\to\infty} 0 = 0$ and $\lim\limits_{t\to\infty} \dfrac{t-\pi/2}{\sqrt{e^t-1}} = 0$ since the exponential term in the denominator dominates as $t \to \infty$. Also, since $f(t) = \dfrac{t-\pi/2}{\sqrt{e^t-1}} \geq 0$ for all $t \geq \pi/2$ and since $f(t)$ has a limit of zero as t tends to infinity, it stands that $\displaystyle\int_{\pi/2}^{\infty} \left|\frac{t-\pi/2}{\sqrt{e^t-1}}\right| \, dt$ converges, in which case $\displaystyle\int_{\pi/2}^{\infty} \left|\frac{\cos(t)}{\sqrt{e^t-1}}\right| \, dt$ converges, which implies that $\displaystyle\int_{\pi/2}^{\infty} \frac{\cos(t)}{\sqrt{e^t-1}} \, dt$ converges, as well. Thus, since I_1 and I_2 converge, so too does $I_1 + I_2$.

3. There are discontinuities at $x = 0$, $x = 1$, and $x = \infty$. Thus, split the integral as follows:

$$\int_0^{\infty} \frac{\ln(x)}{x^2-1} \, dx = \underbrace{\int_0^{1/2} \frac{\ln(x)}{x^2-1} \, dx}_{I_1} + \underbrace{\int_{1/2}^{1} \frac{\ln(x)}{x^2-1} \, dx}_{I_2} + \underbrace{\int_1^{2} \frac{\ln(x)}{x^2-1} \, dx}_{I_3} + \underbrace{\int_2^{\infty} \frac{\ln(x)}{x^2-1} \, dx}_{I_4}$$

It's worth noting that $\ln(x)/(x^2+1)$ is always positive on $(0,\infty)$ and on each subsequent sub-interval. On each sub-interval, the integral $\int_a^x \frac{\ln(x)}{x^2-1} \, dx$ is bounded, and thus $I_1 + I_2 + I_3 + I_4$ converges.

4. First, let's do some re-writing:

$$\int_0^{\infty} (x+2) - \sqrt{x^2+4x+1} \, dx = \underbrace{\int_0^{\infty} x+2 \, dx}_{I_1} + \underbrace{\int_0^{\infty} \sqrt{x^2+4x+1} \, dx}_{I_2}$$

Focusing on the first summand, we examine

$$\lim_{N\to\infty} \int_0^N x+2 \, dx = \lim_{N\to\infty} \left(\frac{x^2}{2} + 2x \bigg|_0^N \right) = \lim_{N\to\infty} \frac{N^2}{2} + 2N = \infty$$

whence we immediately see that I_1 diverges and thus the entire integral diverges.

5. Since there are discontinuities at $x = 0$ and $x = 1$, we'll split the integral as follows:

$$\int_0^1 \frac{1}{\sqrt[3]{x^2-x^3}} \, dx = \underbrace{\int_0^{1/2} \frac{1}{\sqrt[3]{x^2-x^3}} \, dx}_{I_1} + \underbrace{\int_{1/2}^{1} \frac{1}{\sqrt[3]{x^2-x^3}} \, dx}_{I_2}$$

Examining I_1 first, we see that $\sqrt[3]{x-1} \leq \sqrt[3]{x^2-x^3}$ for $x \in \left(0, \frac{1}{2}\right]$ and so

$$-\frac{1}{\sqrt[3]{x^2-x^3}} \leq \frac{1}{\sqrt[3]{x-1}}$$

We are considering $-f(x) = -\frac{1}{\sqrt[3]{x^2-x^3}}$ because $f(x)$ is negative for $x > 0$. Observe that the integral $-\int_0^{1/2} \frac{1}{\sqrt[3]{x-1}} \, dx$ converges – perform the substitution $t = x-1$ so as to obtain $\int_{-1}^{-1/2} \frac{1}{\sqrt[3]{t}} \, dt = \frac{3}{2}\left(\left(-\frac{1}{2}\right)^{2/3} - (-1)^{2/3}\right)$. Thus, by comparison, $-\int_0^{1/2} \frac{1}{\sqrt[3]{x^2-x^3}} \, dx$ converges. Now, examining I_2 we can perform the *exact* same calculation: observe that

$$\frac{1}{x} \leq -\frac{1}{\sqrt[3]{x^2-x^3}}$$

on $x \in \left[\frac{1}{2}, \infty\right)$, and since $\int_{1/2}^{\infty} \frac{1}{x} \, dx$ diverges, so too does I_2 by comparison. Thus, since I_1 is convergent and I_2 is divergent, it stands that $I_1 + I_2$ is divergent.

Exercise 2. Prove that the integral $\displaystyle\int_0^{\infty} \frac{t\ln(t)}{(t^2+1)^2} \, dt$ is convergent and evaluate the integral.

Proof. Begin by splitting the integral at $t = 1$ as follows:

$$I = \int_0^{\infty} \frac{t\ln(t)}{(t^2+1)^2} \, dt = \underbrace{\int_0^{1} \frac{t\ln(t)}{(t^2+1)^2} \, dt}_{I_1} + \underbrace{\int_1^{\infty} \frac{t\ln(t)}{(t^2+1)^2} \, dt}_{I_2}$$

Examining the second integral, Let $\Phi(t) = \dfrac{t}{(t^2+1)^2}$ and let $f(t) = \ln(t)$. We can see that $f(t) = \ln(t)$ is continuous over $[1, \infty)$ and that $F(x) = \displaystyle\int_1^x \ln(t)\ dt$ is bounded. Now, note that $\Phi'(t) = \dfrac{(t^2+1)^2 - 4t^2(t^2+1)}{(t^2+1)^4}$ is continuous and non-negative over $[1, \infty)$. Lastly,

$$\lim_{t\to\infty} \Phi'(t) = \lim_{t\to\infty} \frac{(t^2+1)^2 - 4t^2(t^2+1)}{(t^2+1)^4} = 0$$

Thus, I_2 converges. Now, it remains to show that I_1 converges. Since the upper limit of integration is finite, we can't use the same method in determining the convergence of I_1 as with I_2. For I_2, we first observe that $\ln(t) \le 0$ for $(0, 1]$ and so $\dfrac{-t\ln(t)}{(t^2+1)^2}$ is non-negative on $(0, 1]$. Furthermore, we see that $\displaystyle\int_0^x \frac{-t\ln(t)}{(t^2+1)^2}\ dt$ is bounded for any $x \in [0, 1]$ and so $\displaystyle\int_0^1 \frac{-t\ln(t)}{(t^2+1)^2}\ dt$ converges $\Rightarrow \displaystyle\int_0^1 \frac{t\ln(t)}{(t^2+1)^2}\ dt$ converges. Since I_1 and I_2 both converge, so too does $I = I_1 + I_2$. ■

Now, it remains to evaluate the integral itself. Just as we verified convergence for each piece of the integral, we'll evaluate the integral by evaluating its two components. We begin by evaluating the indefinite integral so as to focus on the limits of integration later – integrating by parts with $u = \ln(t) \Rightarrow 1/t\ dt$ and $dv = t/(t^2+1)^2\ dt \Rightarrow v = -1/2(t^2+1)$. Thus,

$$\int \frac{t\ln(t)}{(t^2+1)^2}\ dt = -\frac{\ln(t)}{2(t^2+1)} + \int \frac{1}{2t(t^2+1)}\ dt$$

whence performing partial fraction decomposition on the rightmost integral gives

$$\begin{aligned}
\int \frac{t\ln(t)}{(t^2+1)^2}\ dt &= -\frac{\ln(t)}{2(t^2+1)} + \frac{1}{2}\int \frac{1}{t} - \frac{t}{t^2+1}\ dt \\
&= -\frac{\ln(t)}{2(t^2+1)} + \frac{1}{2}\left(\int \frac{1}{t}\ dt - \int \frac{t}{t^2+1}\ dt\right) \\
&= \frac{1}{2}\left(\ln|t| - \frac{1}{2}\ln|t^2+1|\right) - \frac{\ln(t)}{2(t^2+1)} + C
\end{aligned}$$

Evaluating the antiderivative at ∞ and 1 gives $\ln(2)/4$ whilst evaluating it at 0 and 1 gives $-\ln(2)/4$, so

$$\int_0^\infty \frac{t\ln(t)}{(t^2+1)^2}\ dt = \frac{\ln(2)}{4} - \frac{\ln(2)}{4} = 0$$

EXERCISE 3. For which value of x is the integral $\displaystyle\int_0^\infty \frac{t^x + t^{2-x}}{t^3 + \sqrt{t}}\ dt$ convergent?

SOLUTION. Let $\Phi(t) = t^x + t^{2-x}$ and $f(t) = \dfrac{1}{t^3 + \sqrt{t}}$. First, observe that $f(t)$ is integrable on any closed subset of $(0, \infty)$. For any finite a and b within $(0, \infty)$, we know that $\int_a^b f(t)\ dt$ exists and is finite because, by comparison, $\int_a^b \frac{1}{\sqrt{t}}\ dt = 2(\sqrt{b} - \sqrt{a})$, thus verifying it is convergent and finite – not only is it convergent and finite, but $\int_a^b f(t)\ dt$ is also bounded. We now examine $\Phi'(t) = xt^{x-1} + (2-x)t^{1-x}$. Ideally, what we want is

$$\lim_{t\to\infty} \Phi'(t) = \lim_{t\to\infty} xt^{x-1} + (2-x)t^{1-x} = 0$$

$$\Rightarrow \lim_{t\to\infty}\left(\frac{x}{t^{1-x}} + \frac{(2-x)}{t^{x-1}}\right) = 0$$

which holds for any $\boxed{x \in (0,1)}$.

EXERCISE 4. Let $f(t)$ and $g(t)$ be two positive, continuous functions on $[a, \infty)$ and assume that $\int_a^\infty f(t)\ dt$ and $\int_a^\infty g(t)\ dt$ are both convergent. Prove that, if $\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$, then $\displaystyle\lim_{x\to\infty} \frac{\int_x^\infty f(t)\ dt}{\int_x^\infty g(t)\ dt} = 1$.

Proof. Since $\lim_{x\to\infty} \dfrac{f(x)}{g(x)} = 1$, it holds that, for any $\epsilon > 0$ there is an N so that if $x > N$ then $(1-\epsilon)g(x) < f(x) < (1+\epsilon)g(x)$. We can now re-write this inequality as

$$(1-\epsilon)\frac{d}{dx}\left(\int_a^\infty g(t)\ dt - \int_a^x g(t)\ dt\right) < \frac{d}{dx}\left(\int_a^\infty f(t)\ dt - \int_a^x f(t)\ dt\right) < (1+\epsilon)\frac{d}{dx}\left(\int_a^\infty g(t)\ dt - \int_a^x g(t)\ dt\right)$$

This done specifically because $\int_a^\infty f(t)\ dt$ and $\int_a^\infty g(t)\ dt$ converge and thus produce constant values, so they vanish upon differentiation. Likewise, by the FTC, $\frac{d}{dx}\int_a^x f(t)\ dt = f(x)$ and $\frac{d}{dx}\int_a^x g(t)\ dt = g(x)$. From this, we have that

$$(1-\epsilon)\left(\int_a^\infty g(t)\ dt - \int_a^x g(t)\ dt\right) < \left(\int_a^\infty f(t)\ dt - \int_a^x f(t)\ dt\right) < (1+\epsilon)\left(\int_a^\infty g(t)\ dt - \int_a^x g(t)\ dt\right)$$

$$\Rightarrow (1-\epsilon)\int_x^\infty g(t)\ dt < \int_x^\infty f(t)\ dt < (1+\epsilon)\int_x^\infty g(t)\ dt$$

$$\Rightarrow \left|\frac{\int_x^\infty f(t)\ dt}{\int_x^\infty g(t)\ dt} - 1\right| < \epsilon$$

as required. ■