Math 327A: Autumn 2016

Homework 2

Solutions written by Alex Menendez (1438704)

EXERCISE 1. Assume that the real numbers are a commutative field, totally ordered.

1. Given real numbers a, b, and c, if $a < b$ then $a + c < b + c$.
2. For any real numbers a and b such that $0 < a < b$, one has that $b^{-1} < a^{-1}$.
3. If $a < b$ and $c < 0$, then $ac > bc$.

SOLUTION.

1. *Proof.* We will go the route of proving the contrapositive. Assuming that $a + c$ is not less than $b + c$, there are two possibilities:

   - If $a + c = b + c$, then $(a + c) - (b + c) = 0$. By both the distributive and associative rules, the left hand side may be concatenated as $a + c - b - c = (a - b) + (c - c)$ so that $(a - b) + (c - c) = 0$. Since $c - c = 0$ for any $c \in \mathcal{F}$, one has that $a - b = 0$, or $a = b$.
   - Alternatively, if $a + c > b + c$, then by the same reasoning it follows that $(a + c) - (b + c) > 0$. By the distributive and associative rules, this is equivalent to $(a - b) + (c - c) > 0$. Since $c - c = 0$ for any $c \in \mathcal{F}$, it stands that $a - b > 0$ and so $a > b$.

   In either case, the assumption that $a + c$ is not less than $b + c$ has allowed us to prove that a is not less than b and thus the contrapositive is proven. $\square$

2. *Proof.* Beginning with $0 < a < b$, we may multiply the inequality through by b^{-1}, a legal move since $b > 0$. Doing so, we get $0 < a \cdot b^{-1} < b \cdot b^{-1}$, or

   $$0 < a \cdot b^{-1} < 1$$

   Multiplying through by a^{-1}, another legal move given $a > 0$, one has $0 < a^{-1} \cdot a \cdot b^{-1} < a^{-1} \cdot 1$, or

   $$0 < b^{-1} < a^{-1}$$

   Which is the desired result. $\square$

3. *Proof.* If $a < b$, then $a - b < 0$. Multiplying both sides of the inequality by c, one has that $c \cdot (a - b) < 0$. Now, we note that $c < 0$ as prescribed, and $a - b < 0$, as we deduced earlier. Since both factors are negative, $c \cdot (a - b)$ is greater than zero, or $c \cdot (a - b) > 0$. By the distributive and associative rules, the left hand side can be rephrased so that $ac - bc > 0$, and thus $ac > bc$. $\square$

EXERCISE 2. Assume that m is a positive, real number.

1. Let x and z be two real numbers. Prove that, if $x^2 < z^2$, then $x < z$.
2. Let $L = \{ x \mid x \leq 0, \text{ or } 0 < x \text{ and } x^2 < m \}$. Prove that $L \cap \{x > 0\}$ is not empty.
3. Let $R := L^c$. Prove that there exists a cut number c such that if $x < c$, then $x \in L$, and if $x > c$ then $x \in R$.
4. Prove that $c^2 = m$.

SOLUTION.

1. *Proof.* If $x^2 < z^2$, it follows that $x^2 - z^2 < 0$. Factoring the left hand side gives $(x + z)(x - z) < 0$ for which multiplying both sides by $(x + z)^{-1}$ gives[1] $(x + z)^{-1}(x + z)(x - z) < (x + z)^{-1}(0)$, which is $x - z < 0$ or $x < z$, as required. $\square$

[1] Note that $(x + z)^{-1}$ is positive because $x + z$ is positive – thus, the sign of the inequality is preserved.

2. *Proof.* Consider the following cases:

   - If $m > 1 > 0$, then choose $x = 1$ and certainly $x \in L$ but $x = 1$ is also in $\{x > 0\}$, so $L \cap \{x > 0\}$ is not empty.
   - If $0 < m < 1$, then choose $x = m$ so that $m^2 < m$. It follows that $x \in L$ and also $x \in \{x > 0\}$ so that $L \cap \{x > 0\}$ is not empty[2].
   - If $m = 1$, then choose any $0 < x < 1$ and $x^2 < m$ so that $x \in L$. Consequently, $x \in \{x > 0\}$ as well, so $L \cap \{x > 0\}$ is not empty.

   □

3. *Proof.* To show the existence of a cut number, we will cite the Axiom of Continuity:

   - Certainly $x = -1$ is an element of L and, likewise, $y = m$ is an element of R since $m > 0$ and $m^2 \geq m$. Thus, both L and R are not empty.
   - For any x, if one considers $x \leq 0$ then $x \in L$. Likewise, if $x > 0$, then we consider two possibilities:
     - $x^2 \geq m$, in which case $x \in R$.
     - $x^2 < m$, in which case $x \in L$.

     And thus any x is contained in either L or R.
   - Consider $x \in L$ and $y \in R$. We must deduce that $x < y$.
     - If $x \leq 0$ and $y > 0$ then certainly $x < y$.
     - If $x > 0$ and $y > 0$, then $x^2 < m \leq y^2$ so that $x^2 < y^2$. However, it follows from part (1) that $x^2 < y^2 \Rightarrow x < y$, as desired.

   As a consequence of these conditions being satisfied, there exists a cut number c such that any $x < c$ is contained in L and any $x > c$ is contained in R. □

4. *Proof.* Assume the contrary; that is, consider the following two cases:

   - If $c^2 < m$, then there exists some $x > 0$ such that $c^2 < x^2 < m$. Since $c^2 < x^2$, one has that $c < x$. However, since $c < x$ implies $x \in R$ and $x^2 < m$ implies $x \in L$, we have arrived at a contradiction.
   - If $c^2 > m$, then in much the same way, there exists some $y > 0$ such that $c^2 > y^2 > m$. Since $c^2 > y^2$, it follows that $c > y$. Observe once again that, since $c > y$, it must be that $y \in L$. Alternatively, since $y^2 > m$, it stands that $y \in R$. Once again, a contradiction has developed.

   As a consequence, it follows that $c^2 = m$, as required. □

EXERCISE 3. Prove the Archimedean Law is equivalent to:

1. For any $z \in \mathbb{R}^+$, there exists an $n \in \mathbb{N}$ such that $n > z$.
2. For any $z \in \mathbb{R}^+$, there exists a rational number of the form $\frac{1}{n}$ such that $0 < \frac{1}{n} < z$.

SOLUTION.

1. *Proof.* Take $a, b \in \mathbb{R}$ where $a > 0$ and $b > 0$. Also take $n \in \mathbb{N}$ and $z \in \mathbb{R}^+$. The Archimedean Law has it that, for any $n \in \mathbb{N}$, one has that $na > b$. Since this applies for any $a \in \mathbb{R}$, we may let $a = 1$ so that $n > b$. Lastly, since b is defined to be positive, it stands that $b \in \mathbb{R}^+$. Letting $b = z$, we have that $n > z$ for some $n \in \mathbb{N}$ and $z \in \mathbb{R}^+$, the desired result. □

2. *Proof.* Recall the Archimedean property $na > b$ for $a > 0$, $b > 0$, and $n \in \mathbb{N}$. Letting $a = z$ and $b = 1$ gives $nz > 1$. Multiplying both sides of the inequality by the multiplicative inverse of n, one has $n^{-1}nz > n^{-1} \cdot 1$, or $z > \frac{1}{n}$. Now, we must show that $n^{-1} > 0$. This follows from previous results. The result of part (1) has that $n > z$ for some $n \in \mathbb{N}$ and $z \in \mathbb{R}^+$. From part (2) of exercise (1) has that, since $n > z$, it follows that $z^{-1} > n^{-1} > 0$. Since $n^{-1} = \frac{1}{n} > 0$, we now have that $0 < \frac{1}{n} < z$, as required. □

[2] e.g., take $m = 3/4$ and certainly $9/16 < 3/4$. It follows that $x = 3/4$ is in L and also $\{x > 0\}$.