Math 327A: Autumn 2016

Homework 5

Solutions written by Alex Menendez (1438704)

EXERCISE 1. Let S be a set and let A be the set of all of the accumulation points of S. Prove that A is closed.

Proof. To show that A is closed, we will go the route of showing that A^c is open. For any $x \in A^c$, one finds that x is not an accumulation point of S. That is, there is a $\delta > 0$ so that $(x-\delta, x+\delta)$ does not contain an element of S that is not x itself. We now need to show that, for every $p \in (x-\delta, x+\delta)$, one has that $p \in A^c$.

- If $p = x$, then obviously p is not an accumulation point, and so $p \in A^c$.
- If $p \neq x$, then pick $\alpha = \min(|x-p|, |p-(x-\delta)|, |p-(x+\delta)|)$ and construct the neighborhood $(p-\alpha, p+\alpha)$ which certainly contains no elements of S and certainly not x. Thus, $(p-\alpha, p+\alpha) \subset (x-\delta, x+\delta) \setminus \{x\}$, and so $p \in A^c$.

Since every $p \in (x-\delta, x+\delta)$ is also in A^c, one finds that for any $x \in A^c$, there is a $\delta > 0$ so that $(x-\delta, x+\delta) \subset A^c$, and thus A^c is open and A is closed, as required. □

EXERCISE 2. Given a sequence a_n with distinct terms, prove that a_n converges to ℓ if and only if ℓ is the only accumulation point of the set $S = \{a_n | n \in \mathbb{N}\}$.

Proof. We will prove two directions:

- $\boxed{\Rightarrow}$: Since a_n converges to ℓ, one has that $\forall \delta > 0$, $\exists N$ so that if $n > N$, it holds that $|a_n - \ell| < \delta$. Of course, this implies that $\ell - \delta < a_n < \ell + \delta$ and so $a_n \in (\ell-\delta, \ell+\delta)$. We now must find an element inside this open interval that is not ℓ itself. In particular, we can see that $a_{N+1} \neq \ell$, and that, since the terms of a_n are distinct, $a_{N+2} \neq a_{N+1} \neq \ell$. So indeed we can locate these points quite easily. Now, take some $p \neq \ell$ – we must show that p is *not* an accumulation point. That is, $\exists \delta > 0$ so that $(p-\delta, p+\delta)$ does not contain an element of S that is not p itself. Since $p \neq \ell$, we have that $a_n \notin (p-\delta, p+\delta)$. Now, take $d_k = |p - a_k|$ (the distance between any value of the sequence and p) and choose $\delta = \min(d_1, d_2, \ldots, d_k)$ and so we can guarantee that $(p-\delta, p+\delta)$ does not contain an element of S that is not p itself. This verifies that ℓ is, in fact, the only accumulation point of S.
- $\boxed{\Leftarrow}$: Assume that S has two accumulation points, ℓ_1 and ℓ_2.
  - If ℓ_1 is an accumulation point of S, then for any $\epsilon_1 > 0$, one has that the neighborhood $(\ell_1 - \epsilon_1, \ell_1 + \epsilon_1)$ contains a point in S that is not ℓ_1 itself. That is, for some $N_1 \in \mathbb{N}$, if $n > N_1$, then $a_n \in (\ell_1 - \epsilon_1, \ell_1 + \epsilon_1)$ and $a_n \neq \ell_1$. Thus, $\ell_1 - \epsilon_1 < a_n < \ell_1 + \epsilon_1 \Longrightarrow |a_n - \ell_1| < \epsilon_1$, for which we can see that a_n converges to ℓ_1.
  - If ℓ_2 is an accumulation point of S, then for any $\epsilon_2 > 0$, one has that the neighborhood $(\ell_2 - \epsilon_2, \ell_2 + \epsilon_2)$ contains a point in S that is not ℓ_2 itself. That is, for some $N_2 \in \mathbb{N}$, if $n > N_2$, then $a_n \in (\ell_2 - \epsilon_2, \ell_2 + \epsilon_2)$ and $a_n \neq \ell_2$. Thus, $\ell_2 - \epsilon_2 < a_n < \ell_2 + \epsilon_2 \Longrightarrow |a_n - \ell_2| < \epsilon_2$, for which we can see that a_n converges to ℓ_2.

  This develops a contradiction, since the limit of a_n is unique. Hence, we must have that there is only one accumulation point which a_n converges to.

□

EXERCISE 3. The goal of this exercise is to prove that $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^2}$ is convergent.

1. Prove that $S_n = \displaystyle\sum_{k=1}^{n}\frac{1}{k(k+1)}$ is a Cauchy sequence.
2. Prove that the series $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^2}$ is convergent.
3. Prove that the series $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^p}$ is convergent for any real $p \geq 2$.

SOLUTION.

1. *Proof.* The most streamlined way of showing S_n is Cauchy is to show that S_n converges to a finite limit. First decompose $\dfrac{1}{k(k+1)}$ as $\dfrac{1}{k} - \dfrac{1}{k+1}$ so that

$$S_n = \sum_{k=1}^{n}\left(\frac{1}{k} - \frac{1}{k+1}\right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right) = \frac{n}{n+1}$$

Now, note that $\displaystyle\lim_{n\to\infty} S_n = \lim_{n\to\infty}\frac{n}{n+1} = 1$. And so S_n converges and is thus Cauchy. □

2. *Proof.* First note that $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^2} = 1 + \sum_{k=2}^{\infty}\frac{1}{k^2}$. This way, we can examine the series $\displaystyle\sum_{k=2}^{\infty}\frac{1}{k^2}$ for convergence; this will indicate if the series $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^2}$ converges, since adding 1 to a convergent series will not affect its convergence. We will show that $\displaystyle\sum_{k=2}^{\infty}\frac{1}{k^2}$ converges by comparison to $\displaystyle\sum_{k=2}^{\infty}\frac{1}{k^2-k}$. In particular, note that, for $k \geq 2$,

$$0 \leq \frac{1}{k^2} \leq \frac{1}{k^2-k}$$

Decompose $\dfrac{1}{k^2-k}$ as $\dfrac{1}{k-1} - \dfrac{1}{k}$ so that

$$S_n = \sum_{k=2}^{n}\left(\frac{1}{k-1} - \frac{1}{k}\right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots + \left(\frac{1}{n-1} - \frac{1}{n}\right) = \frac{n-1}{n}$$

Furthermore, we find that $\displaystyle\lim_{n\to\infty} S_n = \lim_{n\to\infty}\frac{n-1}{n} = 1$. Thus, we see that $\displaystyle\sum_{k=2}^{\infty}\frac{1}{k^2-k}$ converges and so, by comparison, $\displaystyle\sum_{k=2}^{\infty}\frac{1}{k^2}$ converges as well. Of course, it follows that $\displaystyle\sum_{k=2}^{\infty}\frac{1}{k^2} + 1$ converges, which is to say that $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^2}$ converges, as required. □

3. *Proof.* For $p = 2$, one has that $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^p} = \sum_{k=1}^{\infty}\frac{1}{k^2}$. By the previous result, it is known that $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^2}$ converges. For some $p > 2$, we find that

$$\sum_{k=1}^{\infty}\frac{1}{k^p} < \sum_{k=1}^{\infty}\frac{1}{k^2} < 1 + \sum_{k=2}^{\infty}\frac{1}{k^2-k} = 2$$

which follows by the previous result, as well. This indicates that the sum $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^p}$ is bounded above by 2 for any $p > 2$. We can also show that, for $p > 2$, the sequence $S_n = \displaystyle\sum_{k=1}^{n}\frac{1}{k^p}$ is monotonically increasing. That is, we would like to show that, for any

$n \geq 1$ and $p > 2$, it holds that $\sum_{k=1}^{n+1} \frac{1}{k^p} > \sum_{k=1}^{n} \frac{1}{k^p}$. We first observe that for $n = 1$, one has that $\sum_{k=1}^{1} \frac{1}{k^p} + \frac{1}{(n+1)^p} > \sum_{k=1}^{1} \frac{1}{k^p}$ because $1 + \frac{1}{2^p} > 1$ given $p > 2$. Assuming that, for some $n \geq 1$, $\sum_{k=1}^{n+1} \frac{1}{k^p} > \sum_{k=1}^{n} \frac{1}{k^p}$, we find that

$$\sum_{k=1}^{n+1} \frac{1}{k^p} + \frac{1}{(n+2)^p} > \sum_{k=1}^{n} \frac{1}{k^p} + \frac{1}{(n+2)^p}$$

Which is amply sufficient to have $\sum_{k=1}^{n+1} \frac{1}{k^p} + \frac{1}{(n+2)^p} > \sum_{k=1}^{n} \frac{1}{k^p} + \frac{1}{(n+1)^p}$, which is to say that $\sum_{k=1}^{n+2} \frac{1}{k^p} > \sum_{k=1}^{n+1} \frac{1}{k^p}$, as required. Since for $p \geq 2$, S_n increases and is bounded above, it is convergent. $\square$