Math 328B: Winter 2017

Homework 3

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EXERCISE 1. Let f be an increasing function on $[a, b]$ (for any $x \in [a, b]$ and $y \in [a, b]$, if $x < y$ then $f(x) \leq f(y)$). Prove that f is integrable on $[a, b]$.

Proof. Since f is continuous on $[a, b]$, it holds that for any $\epsilon' > 0$, there is a δ' so that, if $|x_{i+1} - x_i| < \delta'$ then $|f(x_{i+1}) - f(x_i)| < \epsilon'$. Thus, let $\epsilon > 0$ be given and let $\epsilon' = \epsilon/(b-a)$. Now, partition $[a, b]$ into p sub-intervals, each of length $(b-a)/p$, where $p > (b-a)/\delta'$ for some δ' given by the continuity of f. Since f is increasing on $[a, b]$ we may write the upper and lower sums, respectively, as

$$R = (x_1 - x_0)f(x_1) + (x_2 - x_1)f(x_2) + \ldots + (x_p - x_{p-1})f(x_p)$$

and

$$r = (x_1 - x_0)f(x_0) + (x_2 - x_1)f(x_1) + \ldots + (x_p - x_{p-1})f(x_{p-1})$$

for which we see that

$$R - r = \sum_{i=0}^{p-1} (x_{i+1} - x_i)\left(f(x_{i+1}) - f(x_i)\right)$$

If $[x_i, x_{i+1}]$ has length less than δ', then $|x_{i+1} - x_i| < \delta'$, but since $\delta' > (b-a)/p$ it stands that $|x_{i+1} - x_i| \leq (b-a)/p \Rightarrow x_{i+1} - x_i \leq (b-a)/p$. Furthermore, since f is continuous we find that $f(x_{i+1}) - f(x_i) < \epsilon/(b-a)$. Thus,

$$R - r \leq \sum_{i=0}^{p-1} \left(\frac{b-a}{p}\right)\left(\frac{\epsilon}{b-a}\right) = \sum_{i=0}^{p-1} \frac{\epsilon}{p} = \epsilon$$

$$\Rightarrow R - r < \epsilon$$

as required. Thus, f is integrable on $[a, b]$. ■

EXERCISE 2. Let f and g be two integrable functions on $[a, b]$. Prove that $f + g$ is integrable and that $\int_a^b f(t) + g(t)\, dt = \int_a^b f(t)\, dt + \int_a^b g(t)\, dt$.

Proof. We will first show that $f + g$ is integrable; this can be achieved by showing that the difference of the upper and lower sums for $f + g$ over a given partition are less than a given $\epsilon > 0$. That is, we must show that

$$R_{f+g} - r_{f+g} < \epsilon$$

Since f is integrable, it holds that, for any given $\epsilon_f > 0$, there exists a partition so that $R_f - r_f < \epsilon_f$ (where R_f and r_f are the upper and lower sums of f over the given partition). Similarly, since g is integrable, it holds that, for any given $\epsilon_g > 0$, there exists a partition so that $R_g - r_g < \epsilon_g$. We can partition $[a, b]$ so that $x_0 = a < x_1 < \ldots < x_p = b$ and summarize this in the following way:

$$R_f - r_f = \sum_{i=0}^{p-1} (x_{i+1} - x_i)(\sup(f, [x_i, x_{i+1}]) - \inf(f, [x_i, x_{i+1}]))$$

$$R_g - r_g = \sum_{i=0}^{p-1} (x_{i+1} - x_i)(\sup(g, [x_i, x_{i+1}]) - \inf(g, [x_i, x_{i+1}]))$$

Now, observe that

$$(R_f - r_f) + (R_g - r_g) = \sum_{i=0}^{p-1} (x_{i+1} - x_i)((\sup(f, [x_i, x_{i+1}]) + \sup(g, [x_i, x_{i+1}])) - (\inf(f, [x_i, x_{i+1}]) + \inf(g, [x_i, x_{i+1}])))$$

If f and g are bounded, then $\sup(f) + \sup(g) = \sup(f + g)$ and $\inf(f) + \inf(g) = \inf(f + g)$ so that

$$(R_f - r_f) + (R_g - r_g) = \sum_{i=0}^{p-1} (x_{i+1} - x_i)(\sup(f + g, [x_i, x_{i+1}]) - \inf(f + g, [x_i, x_{i+1}]))$$

Given some $\epsilon > 0$, taking $\epsilon_f = \epsilon_g = \epsilon/2$ gives

$$\sum_{i=0}^{p-1}(x_{i+1}-x_i)(\sup(f+g,[x_i,x_{i+1}]) - \inf(f+g,[x_i,x_{i+1}])) < \epsilon/2 + \epsilon/2 = \epsilon$$

$$\Rightarrow R_{f+g} - r_{f+g} < \epsilon$$

as required. Furthermore, let $I_f = \inf\{R_f \text{ for any partition of } [a,b]\}$, $I_g = \inf\{R_g \text{ for any partition of } [a,b]\}$, and $I_{f+g} = \inf\{R_{f+g}$ for any par
Similarly, let $J_f = \sup\{r_f \text{ for any partition of } [a,b]\}$, $J_g = \sup\{r_g \text{ for any partition of } [a,b]\}$, and $J_{f+g} = \sup\{r_{f+g} \text{ for any partition of } [a,b]\}$
Since $I_{f+g} = J_{f+g} = I_f = J_f = I_g = J_g \Rightarrow \int_a^b f+g\ dt = \int_a^b f\ dt + \int_a^b f\ dt$. ■

EXERCISE 3. Prove that

$$f(x) = \begin{cases} x \text{ if } x \in \mathbb{Q} \\ 0 \text{ if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

is not integrable on $[0,1]$.

Proof. For each sub-interval $[x_i, x_{i+1}]$ corresponding to a partition P of $[0,1]$, it's obvious that $\inf(f,[x_i,x_{i+1}]) = 0$, so $r(f,P) = 0$ and thus $J = \sup_P\{r(f,P)\} = 0$. Also, for every sub-interval $[x_i,x_{i+1}]$ corresponding to a partition P of $[0,1]$, $\sup(f,[x_i,x_{i+1}]) = \sup(g,[x_i,x_{i+1}])$ and so $R(f,P) = R(g,P)$. Thus, $I = \inf_P\{R(f,P)\} = \inf_P\{R(g,P)\} = 1/2$. Thus, $I \neq J$ and so f is not integrable. ■

EXERCISE 4. Let f be a non-negative, continuous function on $[a,b]$. Prove that if there exists a $c \in [a,b]$ so that $f(c) > 0$, then $\int_a^b f(t)\ dt > 0$.

Proof. Let $R(f,P)$ and $r(f,P)$ denote the upper and lower sums of f over the partition P. Clearly, we have that

$$m(b-a) \leq r(f,P) \leq R(f,P) \leq M(b-a)$$

given $m \leq f(x) \leq M$ for all $x \in [a,b]$. Since $f(x) \leq 0$ for every $x \in [a,b]$, then clearly $0 \leq r(f,P) \leq R(f,P) \leq M(b-a)$. Also, for every partition of $[a,b]$, there exists an A so that $r(f,P) \leq A \leq R(f,P)$ and $A = \int_a^b f(x)\ dx$, so $0 \leq r(f,P) \leq A \leq R(f,P) \leq M(b-a) \Rightarrow \int_a^b f(x)\ dx > 0$, as required. ■

EXERCISE 5. Let f be an integrable, bounded function on $[a,b]$. Let c be between a and b and prove that $\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx$.

Proof. Partition $[a,b]$ as $P : x_0 = a < x_1 < \ldots < x_{j-1} < x_j = c < x_{j+1} < \ldots < x_n = b$. Observe that

$$R_{[a,b]} = \sum_{i=0}^{p-1}(x_{i+1}-x_i)\sup(f,[x_i,x_{i+1}]) = R_{[a,c]} = \sum_{i=0}^{j-1}(x_{i+1}-x_i)\sup(f,[x_i,x_{i+1}]) + R_{[c,b]} = \sum_{i=j}^{p-1}(x_{i+1}-x_i)\sup(f,[x_i,x_{i+1}])$$

$$r_{[a,b]} = \sum_{i=0}^{p-1}(x_{i+1}-x_i)\inf(f,[x_i,x_{i+1}]) = r_{[a,c]} = \sum_{i=0}^{j-1}(x_{i+1}-x_i)\inf(f,[x_i,x_{i+1}]) + r_{[c,b]} = \sum_{i=j}^{p-1}(x_{i+1}-x_i)\inf(f,[x_i,x_{i+1}])$$

Let $I_1 = \inf_P\{R_{[a,c]}\}$, $I_2 = \inf_P\{R_{[c,b]}\}$, and $I_3 = \inf_P\{R_{[a,b]}\}$. Similarly, let $J_1 = \sup_P\{r_{[a,c]}\}$, $J_2 = \sup_P\{r_{[c,b]}\}$, and $J_3 = \sup_P\{r_{[a,b]}\}$. Observe that $I_1 + I_2 = I_3$ and $J_1 + J_2 = J_3$, which verifies that $\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx$. ■

EXERCISE 6. Prove that

$$f(x) = \begin{cases} 1 \text{ if } x = 1/n,\ n \in \mathbb{N} \\ 0 \text{ if } x \neq 1/n \end{cases}$$

Proof. For every finite partition P, at least one sub-interval $[x_i,x_{i+1}]$ contains an infinite number of points where $f(x) = 0$ and an invite number of points where $f(x) = 1$. Thus, for various selections of points within $[0,1]$, the upper and lower sums will be different, and thus f will not be integrable. More generally, for any $\epsilon > 0$, let n be such that $1/n < \epsilon \leq 1/(n-1)$. Choose the partition $P : x_0 < 1/n < 1/(n-1) < \ldots < x_p$. The maximum value of f but the first interval will be 0, and the maximum value in the first interval is 1. So $R(f,P) = 1/n < \epsilon$ and $r(f,P) = 0$ for every partition P. Thus, $R(f,P) - r(f,P) < \epsilon$, as required. ■