Introduction to Real Analysis

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Abstract

These are lecture notes for Introductory Real Analysis (Math 327) at the University of Washington. Lectures were given by Fanny Dos Reis (fdr3@math.washington.edu).

Contents

1 Fields

1.1 The Commutative Field

Definition 1. $(\mathcal{F}, +, \cdot)$ *is a commutative field if it has the following properties:*

1. *Associativity of addition:* $\forall\ a, b, c \in \mathcal{F}$, *one has that* $(a + b) + c = a + (b + c)$.
2. *Associativity of multiplication:* $\forall\ a, b, c \in \mathcal{F}$, *one has that* $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
3. *Commutativity of addition:* $\forall\ a, b \in \mathcal{F}$, *one has that* $a + b = b + a$.
4. *Commutativity of multiplication:* $\forall\ a, b \in \mathcal{F}$, *one has that* $a \cdot b = b \cdot a$.
5. *Distributivity:* $\forall\ a, b, c \in \mathcal{F}$, *one has that* $a \cdot (b + c) = a \cdot b + a \cdot c$.
6. *There is an additive identity in* $\mathcal{F}$, *denoted* 0, *such that* $\forall a \in \mathcal{F}$, *one has that* $0 + a = a$.
7. *There is a multiplicative identity in* $\mathcal{F}$, *denoted* 1, *such that* $\forall a \in \mathcal{F}$, *one has that* $1 \cdot a = a$.
8. *There is an additive inverse in* $\mathcal{F}$, *denoted* $-a$, *such that* $\forall a \in \mathcal{F}$, *one has that* $(-a) + a = 0$.
9. *There is a multiplicative inverse in* $\mathcal{F}$, *denoted* a^{-1}, *such that* $\forall a \in \mathcal{F}$, *one has that* $a^{-1} \cdot a = 1$.

EXAMPLES:

1. $\mathbb{R}$ is a commutative field.
2. $\mathbb{Z}$ is not a commutative field; for any $x \in \mathbb{Z}$, one has that the multiplicative inverse $x^{-1} \notin \mathbb{Z}$.
3. $\mathbb{R}(x) = \left\{ \dfrac{a_0 + a_1 x + \ldots + a_n x^n}{b_0 + b_1 x + \ldots + b_n x^n} \;\middle|\; (a_0, a_1, \ldots, a_n, b_0, b_1, \ldots, b_n) \in \mathbb{R} \right\}$ is a commutative field.

   Proof. Let $p(x) = a_0 + a_1 x + \ldots + a_n x^n$ and $q(x) = b_0 + b_1 x + \ldots + b_n x^n$. Commutativity, associativity, and the distributive property are trivial. In particular,

   - There is an additive identity: $\dfrac{p(x)}{q(x)} + \dfrac{0}{q(x)} = \dfrac{p(x)}{q(x)}$
   - There is a multiplicative identity: $\left(\dfrac{p(x)}{q(x)}\right)\left(\dfrac{\bar{p}(x)}{\bar{q}(x)}\right) = \dfrac{p(x)}{q(x)}$ for $\bar{p}(x) = \bar{q}(x)$
   - There is an additive inverse: $\dfrac{p(x)}{q(x)} + \dfrac{-p(x)}{q(x)} = \dfrac{0}{q(x)} = 0$
   - There is a multiplicative inverse: $\left(\dfrac{p(x)}{q(x)}\right)\left(\dfrac{q(x)}{p(x)}\right) = 1$

   □

THM 1. *The additive identity is unique.*

Proof. Let 0_1 and 0_2 be additive identities. For any $a \in \mathcal{F}$, one has that $a + 0_1 = a$ and $a + 0_2 = a$. In particular, $0 = \mathcal{F}$, so $0 + 0_1 = 0$ and $0 + 0_2 = 0$ and so $0 + 0_1 = 0 + 0_2$. But, for any $a \in \mathcal{F}$, one finds that $0 + a = a$ and so $0 + 0_1 = 0 + 0_2 \Longrightarrow 0_1 = 0_2$, as required. □

THM 2. *The multiplicative dentity is unique.*

Proof. Let e and e' be multiplicative identities. By definition, one has that $e' = e \cdot e' = e' \cdot e = e$, as required. □

THM 3. *For any* $a \in \mathcal{F}$, *one has that* $a \cdot 0 = 0$.

Proof. $a \cdot 0 = a(0 + 0) = (a \cdot 0) + (a \cdot 0)$ and so $(-a \cdot 0) + (a \cdot 0) + (a \cdot 0) = (-a \cdot 0) + (a \cdot 0) \Longrightarrow a \cdot 0 = 0$. □

Remark 1. *This is a consequence of* $\mathcal{F}$ *being a commutative field; it is not a general property.*

THM 4. $(-1)(-1) = 1$

Proof. By the previous result, we know that $-1 \cdot 0 = 0$ and that $(-1) + 1 = 0$. Thus, $(-1) \cdot 0 = (-1)(-1+1) = (-1)(-1) + (-1)(1) = (-1)(-1) + (-1) = 0$. Adding 1 to both sides gives $(-1)(-1) = 1$, as required. □

Remark 2. *Once again, this is simply a consequence of $\mathcal{F}$ being commutative.*

THM 5. *The additive inverse of $(a + b)$ is $(-a) + (-b)$.*

Proof. Let I be the additive inverse of $(a + b)$. By definition, we know that $(a + b) + I = 0 \Longrightarrow a + b + I = 0$. Adding $(-a)$ to both sides yields $b + I = -a$, and adding (-b) to both sides yields $I = (-a) + (-b)$, as required. □

THM 6. *The multiplicative inverse of $a \cdot b$ is $\left(a^{-1}\right)\left(b^{-1}\right)$.*

Proof. The proof follows in exactly the same manner as in the proof of Theorem 5. □

1.2 The Totally-Ordered Field

Definition 2. *A commutative field $\mathcal{F}$ is totally-ordered if, for every $a \in \mathcal{F}$,*

1. *Exactly one of the following is true:*
   - $a = 0$
   - $a > 0$
   - $-a > 0$
2. *For every $a > 0$ and every $b > 0$, one has that $a + b > 0$.*
3. *For every $a > 0$ and every $b > 0$, one has that $a \cdot b > 0$.*

As a consequence, for every a and b within $\mathcal{F}$, one finds that

- $a > b$ if $a - b > 0$.
- $a < b$ if $b - a > 0$.
- $a \geq b$ if $a - b > 0$ or $a - b = 0$.
- $b \geq a$ if $b - a > 0$ or $b - a = 0$.
- If $a \neq 0$, $a^2 > 0$.

THM 7. *If $a > b$ and $c > 0$, then $ac > bc$.*

Proof. Since $a > b$, one has that $a-b > 0$. We also know that $c > 0$ in which case $c(a-b) > 0 \Longrightarrow ac-bc > 0 \Longrightarrow ac > bc$. □

THM 8. *If $a > b$, then for any $c \in \mathcal{F}$, one has that $a + c > b + c$*

Proof. Since $a > b$, one has that $a - b > 0$. It follows that $(a - b) + (c - c) > 0 \Longrightarrow (a + c) - (b + c) > 0 \Longrightarrow a + c > b + c$. □

THM 9. *If $a > b$ and $b > c$, then $a > c$.*

Proof. Since $a - b > 0$ and $b - c > 0$, we can take $(a - b) + (b - c) > 0 \Longrightarrow a + (b - b) - c > 0 \Longrightarrow a - c > 0 \Longrightarrow a > c$. □

Definition 3. *For any $a \in \mathcal{F}$, the absolute value of a, denoted $|a|$, is defined as:*

$$|a| = \begin{cases} a & \text{if } a > 0 \\ 0 & \text{if } a = 0 \\ -a & \text{if } a < 0 \end{cases}$$

Proposition 1. $|a| \cdot |b| = |ab|$

THM 10. *Triangle Inequality: given a and b in $\mathcal{F}$, we have $|a + b| \leq |a| + |b|$.*

Proof. We will break down the proof into cases:

- Take $a > 0$ and $b > 0$. Surely, then, $|a| = a$ and $|b| = b$ and so $|a| + |b| = a + b$. Similarly, since a and b are both positive, $(a + b) > 0$ and so $|a + b| = a + b$. Thus, we get $|a + b| = |a| + |b|$.

- Take $-a > 0$ and $-b > 0$. It follows that $|a| = -a$ and $|b| = -b$ and so $|a| + |b| = (-a) + (-b) = -(a + b)$. By the same token, since $-a > 0$ and $-b > 0$, it holds that $(-a) + (-b) > 0 \Longrightarrow -(a + b) > 0$ and so $|a + b| = -(a + b)$. Thus, $|a + b| = |a| + |b|$.

- Without loss of generality, take $a > 0$ and $-b > 0$. It follows that $|a| = a$ and $|b| = -b$ and so $|a| + |b| = a + (-b)$. Now, since $a > 0$ and $-b > 0$ we have to be careful about what assumptions we make about $|a + b|$ – in particular, we find

$$|a+b| = \begin{cases} a+b & \text{if } (a+b) > 0 \\ -(a+b) & \text{if } -(a+b) > 0 \end{cases}$$

  But, we do know that, since $a > 0$, one has $a > -a$ and similarly, since $-b > 0$, one has $-b > b$. Thus, $a + (-b) > a + b$ and $a + (-b) > (-a) + (-b) = -(a + b)$. In both cases, $|a| + |b| > |a + b|$ under these assumptions.

In summary, it is clear that $|a + b| \leq |a| + |b|$, as required. □

Corollary 1. $|a - b| \leq |a| + |b|$.

Corollary 2. $||a| + |b|| \leq |a - b|$.

THM 11. *Axiom of Continuity: Let $\mathcal{F}$ be a totally-ordered, commutative field. If there are sets L and R so that*

- *L and R are nonempty*
- *Any element of $\mathcal{F}$ is either in L or in R.*
- *For any $x \in L$ and $y \in R$, it holds that $x < y$*

Then there exists a cut number, c, so that every $x < c$ belongs to L and every $y > c$ belongs to R.

Proposition 2. *Archimedean Law: For any $a \in \mathbb{R}$ and $b \in \mathbb{R}$ where $a > 0$ and $b > 0$, there exists an $n \in \mathbb{N}$ so that $na > b$.*

Proposition 3. *The cut number c is unique.*

Proof. Assume there are two cut numbers c_1 and c_2 and, without loss of generality, that $c_1 < c_2$. It follows that $c_1 < \frac{c_1 + c_2}{2} < c_2$. Now, since $c_1 < \frac{c_1 + c_2}{2}$, one finds that $\frac{c_1 + c_2}{2} \in R$. Alternatively, since $\frac{c_1 + c_2}{2} < c_2$, we see that $\frac{c_1 + c_2}{2} \in L$. This develops a contradiction, since we have identified an element of $\mathcal{F}$ that is simultaneously in L and in R, which is impossible. □

Proposition 4. *If $\mathcal{F}$ is a commutative, totally-ordered field that obeys the axiom of continuity, then $\mathcal{F}$ is isomorphic to $\mathbb{R}$.*

2 Defining $\mathbb{N}$

The natural numbers, denoted $\mathbb{N}$, are often thought of as the "positive integers" or "counting numbers" – that is, $1, 2, 3, \ldots$

Definition 4. *$\mathbb{N}$ is the smallest set of numbers such that:*

- $1 \in \mathbb{N}$
- if $n \in \mathbb{N}$, then $n + 1 \in \mathbb{N}$.

Proposition 5. *Any natural number is greater than or equal to 1.*

Proof. By definition, $1 \geq 1$. Now assume that $n \geq 1$. Since $1 \geq 0$, we find that $n + 1 \geq 1 + 0$, which is to say that $n + 1 \geq 1$, as required. □

Proposition 6. *For any $p \in \mathbb{N}$ and $m \in \mathbb{N}$ where $m < p$ there exists a $k \in \mathbb{N}$ so that $p = m + k$.*

Proposition 7. *Let S be a nonempty set of natural numbers. S has a smallest element.*

Proof. Let $T = \{n \mid \forall s \in S,\ n < s\}$. Since S is nonempty, there is a $s_0 \in S$ but in particular $s_0 \notin T$. So $T \neq \mathbb{N}$ – that is to say that T doesn't satisfy the definition of the natural numbers. So either $1 \notin T$ or there is a $p \in \mathbb{N}$ so that $p + 1 \notin T$. But $1 \in T$ so it must be that there is a $p \in T$ so that $p + 1 \notin T$. Possibly, however, $p + 1 \in S$. We do know that for any $s \in S$, $p < s$. So there is a $s_1 \in S$ so that $p + 1 \geq s_1$. So for any $s \in S$ $p < s$ and thus there is no natural number between p and $p + 1$. Also, for any $s \in S$, $s \geq p + 1 \geq s_1$ and so s_1 is the smallest element. □

3 Defining $\mathbb{Q}$

Definition 5. *The rational numbers are the set of elements of the form $\pm p/q$ where $p \in \mathbb{N}$ and $q \in \mathbb{N}$. As a convention, 0 is a rational number.*

THM 12. *For any $a \in \mathbb{R}$ and for any $b \in \mathbb{R}$ so that $a < b$ there exists a rational number between a and b.*

4 The Notion of $\sup$ and $\inf$

Definition 6. *A bounded set is a set S such that for every $s \in S$, one has that $M \leq s \leq N$, or $M < s < N$.*

Definition 7. *Let S be a nonempty set.*

- *The least upper bound (LUB) of S, also called $\sup S$, is*
  - *an upper bound β. That is, for any $s \in S$, $s \leq \beta$.*
  - *β is the smallest upper bound. That is, any element smaller than β is not an upper bound.*
- *The greatest lower bound (GLB) of S, also called $\inf S$, is*
  - *a lower bound α. That is, for any $s \in S$, $s \geq \alpha$.*
  - *α is the largest lower bound. That is, any element larger than α is not a lower bound.*

THM 13. *If S is a nonempty, bounded set, then $\inf S$ and $\sup S$ are unique.*

Proof. Sketch:

- To show that $\sup S$ is unique, use axiom of continuity with $L = \{x \mid x \text{ is not an upper bound of } S\}$ and $R = \{y \mid y \text{ is an upper bound of } S\}$.
- To show that $\inf S$ is unique, use axiom of continuity with $L = \{x \mid x \text{ is a lower bound of } S\}$ and $R = \{y \mid y \text{ is not a lower bound of } S\}$.

Show that, in each case, the cut number c is $\sup S$ or $\inf S$, and then show uniqueness. □

5 Sequences

As a sequence is a function, it is necessary to first define a function:

Definition 8. *A function $f : A \to B$ is a collection of (x, y) pairs such that $x \in A$ and $y \in B$.*

- *Any x belongs to at most one ordered pair.*
- *The domain of f is the set of all x that belong to an ordered pair.*
- *The range of f is the set of all y that belong to an ordered pair. That is, $\text{Range}(f) = \{y \mid y = f(x)\}$.*

Definition 9. *A sequence is a function $f : \mathbb{N} \to \mathbb{R}$ that consists of ordered pairs where the first coordinate is the term, whilst the second coordinate is the value of the term itself. That is, $(1, u_1), (2, u_2), \ldots, (n, u_n)$.*

Proposition 8. *A $u_n \in \mathbb{R}$ must be signed to every $n \in \mathbb{N}$, so a sequence $\{u_n\}$ is infinite.*

5.1 Convergence

What does it mean when a sequence converges to a finite limit? To answer this question, we'll examine the meaning of

$$\lim_{n\to\infty} u_n = L$$

Definition 10. *Convergence: For every $\epsilon > 0$, there exists an $M \in \mathbb{N}$ so that if $n > M$, one finds that $|u_n - L| < \epsilon$. That is, for large values of n beyond a specific threshold that depends on ϵ, u_n can be bounded to within ϵ units of L.*

EXAMPLE. To show that $\lim_{n\to\infty} \frac{1}{n} = 0$, we must find an M so that for any ϵ, if $n > M$ then $\left|\frac{1}{n}\right| < \epsilon$.

Proof. We want $\dfrac{1}{n} < \epsilon$, which is to have $n > \dfrac{1}{\epsilon}$. So, given some $\epsilon > 0$ let $M = \frac{1}{\epsilon}$. Thus, if $n > M$, then $n > \dfrac{1}{\epsilon}$ which is to say that $\dfrac{1}{n} < \epsilon \implies \left|\dfrac{1}{n} - 0\right| < \epsilon$, as required. □

EXAMPLE. $\lim_{n\to\infty} \dfrac{2n+1}{3n+1} = \dfrac{2}{3}$.

Proof. Let $\epsilon > 0$ be given and take $M = \dfrac{1}{3}\left(\dfrac{1}{3\epsilon} - 1\right)$. Thus, if $n > M$, then

$$n > \frac{1}{3}\left(\frac{1}{3\epsilon} - 1\right) \implies 3n+1 > \frac{1}{3\epsilon} \implies \frac{1}{3(3n+1)} < \epsilon \implies \left|\frac{1}{3(3n+1)}\right| < \epsilon \implies \left|\frac{(6n+3)-(6n+2)}{3(3n+1)}\right| < \epsilon$$

$$\implies \left|\frac{2n+1}{3n+1} - \frac{2}{3}\right| < \epsilon$$

as required. □

5.2 Divergence

Definition 11. *Divergence: For every $A > 0$, there exists an $M \in \mathbb{N}$ so that if $n > M$, one finds that $u_n > A$.*

EXAMPLE. $\lim_{n\to\infty} n^2 + 5n = \infty$.

Proof. Let $A > 0$ be given and take $M = A$ for which it follows that if $n > M$ then $n > A \implies 5n > A \implies n^2 + 5n > A$, as required. □

5.2.1 Special Case

Let $a \in \mathbb{R}$ be given so that

$$\lim_{n\to\infty} a^n = \begin{cases} \infty & \text{if } a > 1 \\ 0 & \text{if } a < 1 \\ 1 & \text{if } a = 1 \end{cases}$$

Proof. For the case when $a = 1$, take $M = 0$ and apply limit definition. For case where $a > 1$, let $a = 1 + \delta$ and use Bernoulli principle that $a^n \geq 1 + n\delta$. For case where $a < 1$, apply limit definition. □

5.3 Properties of Limits

Let $\{a_n\}$ and $\{b_n\}$ be two sequences such that $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist and are finite. We then have the following properties:

1. $\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$
2. $\lim_{n\to\infty} \lambda a_n = \lambda \lim_{n\to\infty} a_n$ for some $\lambda \in \mathbb{R}$
3. $\lim_{n\to\infty} a_n b_n = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right)$
4. $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ granted $\lim_{n\to\infty} b_n \neq 0$

Proof. We will prove some:

1. Take $\epsilon_b = \epsilon_a = \frac{\epsilon}{2}$ and $M = \max(M_a, M_b)$ and analyze $|(a_n - \alpha) + (b_n - \beta)| \geq |a_n - \alpha| + |b_n - \beta|$.

2. Know that $\lim_{n\to\infty} a_n = \alpha$, so for every $\epsilon_a > 0$ there is an M_a so that if $n > M_a \implies |a_n - \alpha| < \epsilon_a$. We want to prove that for every $\epsilon > 0$, there is an M so that if $n > M \implies |ca_n - c\alpha| < \epsilon$. This gives $|a_n - \alpha| < \frac{\epsilon}{|c|}$ for which we can take $\epsilon_a = \frac{\epsilon}{|c|}$ and $M = M_a$ and the result follows.

□

THM 14. *Given three sequences $\{u_n\}$, $\{v_n\}$, and $\{w_n\}$ where $u_n \leq v_n \leq w_n$ and $\lim_{n\to\infty} u_n = \lim_{n\to\infty} w_n = L$, one has that $\lim_{n\to\infty} v_n = L$.*

Definition 12. *A sequence is*

- *increasing if for every $n \in \mathbb{N}$, $u_{n+1} > u_n$*
- *decreasing if for every $n \in \mathbb{N}$, $u_{n+1} < u_n$*
- *non-increasing if for every $n \in \mathbb{N}$, $u_{n+1} \leq u_n$*
- *non-decreasing if for every $n \in \mathbb{N}$, $u_{n+1} \geq u_n$*

Definition 13. *If a sequence is increasing or decreasing, it is called monotonic.*

Proposition 9. *If a sequence is monotonic and bounded, it is convergent.*

6 Open, Closed Sets

Definition 14. *A set U is open if, for every $x \in U$, there is a $\delta > 0$ so that $(x - \delta, x + \delta) \subset U$.*

Definition 15. *A set V is closed if V^c is open.*

THM 15. *If U and V are open sets, then $U \cup V$ is open.*

Proof. We consider two possibilities:

- If $x \in U$, then there exists a $\delta_u > 0$ so that $(x - \delta_u, x + \delta_u) \subset U$ since U is open. But since $U \subset U \cup V$, one finds that $(x - \delta_u, x + \delta_u) \subset U \cup V$.
- If $x \in V$, then there exists a $\delta_v > 0$ so that $(x - \delta_v, x + \delta_v) \subset V$ since V is open. But since $V \subset U \cup V$, one finds that $(x - \delta_v, x + \delta_v) \subset U \cup V$.

Thus, for any $x \in U \cup V$, there is a $\delta > 0$ so that $(x - \delta, x + \delta) \subset U \cup V$. □

The following results follow in exactly the same manner.

Proposition 10. *If U and V are open sets, then $U \cap V$ is open.*

Proposition 11. *If U and V are closed sets, then $U \cup V$ is closed.*

Proposition 12. *If U and V are closed sets, then $U \cap V$ is closed.*

6.1 Accumulation Points

Definition 16. *Let S be a set. A point p is an accumulation point of S if, for every $\delta > 0$, the neighborhood $(p - \delta, p + \delta)$ contains an element of S that is not p itself.*

Remark 3. *p is not necessarily in S.*

EXAMPLE. The accumulation points of $S = \left\{ (-1)^n \left(1 + \frac{1}{n}\right) \,\middle|\, n \in \mathbb{N} \right\}$ are ± 1.

THM 16. *Bolzano-Weierstrass: If S is an infinite, bounded set, then S has at least one accumulation point.*

Proof. Let m be a lower bound on S and M be an upper bound. Create a sequence of nested intervals $I_n = [a_n, b_n]$ where $I_0 = [a_0, b_0] = [m, M]$. Take $c_n = \dfrac{a_n + b_n}{s}$. Noting that $I_n \cap S$ has infinitely many elements, one finds that $([a_n, c_n] \cup [c_n, b_n]) \cap S$ has infinitely many elements. That is to say that at least one of $[a_n, c_n] \cap S$ or $[c_n, b_n] \cap S$ has infinitely many elements. If $[a_n, c_n] \cap S$ has infinitely many elements, than $I_{n+1} = [a_n, c_n]$. Otherwise, $I_{n+1} = [c_n, b_n]$. Thus, $b_n - a_n = \dfrac{M-m}{2^n}$ and so $\lim\limits_{n\to\infty} b_n - a_n = 0$ and so $\bigcap\limits_{n\in\mathbb{N}} I_n = \{p\}$. We must now show that p is an accumulation point. For any neighborhood $(p-\delta, p+\delta)$, insert a small interval $I_n \subset (p-\delta, p+\delta)$ so that I_n has infinitely many elements. By the definition of the limit, we know that $\dfrac{M-m}{2^n} \to 0$ implies that we can take $\epsilon = \delta$ for which there exists an M so that if $n > M$, one has that $\left|\dfrac{M-m}{2^n} - 0\right| < \delta$. For $n_0 > M$, I_{n_0} contains infinitely many elements of S, and since $\bigcap\limits_{n\in\mathbb{N}} I_n = \{p\}$, $p \in I_n$ for all $n \in \mathbb{N}$. Surely, then, $p \in I_{n_0}$. Also, the width of I_{n_0} is strictly less than δ, and so $I_{n_0} \subset (p-\delta, p+\delta)$. And so $(p-\delta, p+\delta)$ has infinitely many elements of S contained within it. Pick some arbitrary $s_1 \in S \cap (p-\delta, p+\delta)$:

- If $s_1 \neq p$, then $s_1 \in S$ and $s_1 \in (p-\delta, p+\delta)$, so p is an accumulation point.
- If $s_1 = p$, then we may pick some $s_2 \neq s_1$ where $s_2 \in S$ and $s_2 \in (p-\delta, p+\delta)$. Certainly $s_2 \neq p$ so p is once again an accumulation point.

□

THM 17. *p is an accumulation point of S iff there exists a sequence of distinct terms that converge to p.*

EXAMPLE. We will re-visit $S = \left\{ (-1)^n \left(1 + \dfrac{1}{n}\right) \;\middle|\; n \in \mathbb{N} \right\}$. Take n to be even, in which case

$$\lim_{n\to\infty} (-1)^n \left(1 + \frac{1}{n}\right) = 1$$

Alternatively, taking n odd gives

$$\lim_{n\to\infty} (-1)^n \left(1 + \frac{1}{n}\right) = -1$$

And so ± 1 are the accumulation points of S.

Proof. We prove two directions:

- $\boxed{\Rightarrow}$: If p is an accumulation point of S then for every $\delta > 0$ there exists an element within $(p-\delta, p+\delta)$ that is not p. Call this element s_1. Now examine the neighborhood $\left(p - \dfrac{|p-s_1|}{2}, p + \dfrac{|p-s_1|}{2}\right)$. There exists an s_2 within this neighborhood so that $s_2 \in S$ and $s_2 \neq s_1$. Thus, $s_1 \notin \left(p - \dfrac{|p-s_1|}{2}, p + \dfrac{|p-s_1|}{2}\right)$ and $\dfrac{|p-s_1|}{2} < \dfrac{1}{2}$. Now use induction to create a sequence s_n so that $\delta_n = \dfrac{|p-s_n|}{2}$ which implies that $\delta_n < \dfrac{1}{2^{n-1}}$. So there exists an $s_n \in (p-\delta_n, p+\delta_n)$ where $s_n \in S$ and $s_n \neq p$. So in particular $s_n \neq s_k$ for $k < n$. Thus, since $|p - s_n| < \dfrac{1}{2^{n-1}}$ and so by squeezing, we have that $p - x_n$ converges to 0 and so x_n converges to p.
- $\boxed{\Leftarrow}$: For every $\delta > 0$, there exists an M so that $n > M$, $|s_n - p| < \delta$, which implies that $p - \delta < s_n < p + \delta$. In particular, $s_{N+1} \in S \cap (p-\delta, p+\delta)$.
  - If $s_{N+1} \neq p$ then $s_{N+1} \in S \cap (p-\delta, p+\delta) \setminus \{p\}$.
  - If $s_{N+1} = p$ then $s_{N+2} \neq s_{N+1}$ and so $s_{N+2} \neq p$. Thus, $s_{N+2} \in S \cap (p-\delta, p+\delta) \setminus \{p\}$.

  In either case, we have found an element in the desired interval that reveals that p is an accumulation point.

□

THM 18. *Any bounded sequence has a convergent sub-sequence.*

7 Cauchy Sequences

Definition 17. *A sequence $\{u_n\}$ is Cauchy if for every $\epsilon > 0$, there is an $M \in \mathbb{N}$ so that if $n > M$ and $p > M$, it holds that $|u_n - u_p| < \epsilon$.*

Proposition 13. *A sequence is convergent iff it is Cauchy.*

Proof. We prove two directions:

- $\boxed{\Rightarrow}$: Assume that $\{u_n\}$ is convergent – that is, $\lim_{n\to\infty} u_n = L$. More formally, this indicates that for every $\epsilon_L > 0$ there is an M_L so that if $n > M_L$ one has that $|u_n - L| < \epsilon_L$. In particular, re-name n as p in the premise so that if $p > M_L$ we have that $|u_p - L| < \epsilon_L$. Note that $u_n - u_p = (u_n - L) - (u_p - L)$ and by the triangle inequality, one has that $|u_n - u_p| \leq |u_n - L| + |u_p - L|$ which implies that $|u_n - u_p| < 2\epsilon_L$. Take $\epsilon_L = \frac{\epsilon_c}{2}$ so that $|u_n - u_p| < \epsilon_c$, as required.

- $\boxed{\Leftarrow}$: Take $\{u_n\}$ to be Cauchy so that for every $\epsilon_c > 0$, there is an $M_c > 0$ so that if $n > M_c$ and $p > M_c$ then $|u_n - u_p| < \epsilon_c$. We first verify that $\{u_n\}$ is bounded. Take $\epsilon_c = L$, in which case there is an M_1 so that if $n > M_1$ and $p = M_1 + 1$ we have that $|u_n - u_{M_1+1}| < 1$ so $u_{M_1+1} - 1 < u_n < u_{M_1+1} + 1$ and so $\{u_n\}$ is bounded. We now verify that there is a convergent subsequence of $\{u_n\}$, which we shall call $\{(u_n)_k\}$. For every $\epsilon > 0$, there is an M_k so that if $n > M_k$ one has that $|(u_n)_k - L| < \epsilon$. By the triangle inequality, we see that $|u_n - L| = |\,(u_n - (u_n)_k) + ((u_n)_k - L)\,| \leq |u_n - L| + |(u_n)_k - L| < \epsilon$, as required.

$\square$

This result is very powerful, as we can now show that a sequence $\{u_n\}$ is Cauchy simply by showing convergence, rather than working off of the definition.

EXAMPLE. $\left\{\frac{1}{n}\right\}_{n\in\mathbb{N}}$ is a Cauchy sequence.

Proof. We know that $\lim_{n\to\infty} \frac{1}{n} = 0$, and so since the sequence converges, it is Cauchy. $\square$

8 Series

8.1 Partial Sums

Let $\{u_n\}$ be a sequence and define $\{S_n\}$ as a sequence of partial sums where $S_n = \displaystyle\sum_{k=1}^{n} u_k = u_1 + u_2 + \ldots + u_n$. Note that this sum has finitely-many terms.

8.2 Defining a Series

To define a series, consider the partial sum S_n and let n tend to infinity: $\displaystyle\lim_{n\to\infty}\sum_{k=1}^{n} u_k = \sum_{n=1}^{\infty} u_n$. If it so happens that if $\displaystyle\lim_{n\to\infty} S_n$ is finite, then $\displaystyle\lim_{n\to\infty} S_n = \sum_{n=1}^{\infty} u_n$.

THM 19. *Given a sequence $\{u_n\}$, if $\displaystyle\sum_{n=1}^{\infty} u_n$ converges, then $\displaystyle\lim_{n\to\infty} u_n = 0$.*

Remark 4. *We also find that, if $\displaystyle\lim_{n\to\infty} u_n \neq 0$, then $\displaystyle\sum_{n=0}^{\infty} u_n$ is divergent.*

Proof. If $\displaystyle\sum_{n=1}^{\infty} u_n$ converges, then the sequence $S_n = \displaystyle\sum_{k=1}^{n} u_k$ converges to a finite limit. Thus, $\displaystyle\lim_{n\to\infty} S_n = L$ and so $\displaystyle\lim_{n\to\infty} S_{n-1} = L$. So,

$$\lim_{n\to\infty} (S_n - S_{n-1}) = L - L = 0$$

But also

$$S_n - S_{n-1} = (u_1 + u_2 + \ldots + u_{n-1} + u_n) - (u_1 + u_2 + \ldots + u_{n-1}) = u_n$$

And so $\displaystyle\lim_{n\to\infty} (S_n - S_{n-1}) = 0 \Longrightarrow \lim_{n\to\infty} u_n = 0$. $\square$

THM 20. *Given a sequence $\{u_n\}$ with non-negative terms, the series $\sum_{n=1}^{\infty} u_n$ converges iff $S_n = \sum_{k=1}^{n} u_k$ is bounded.*

Proof. We prove two directions:

- $\boxed{\Rightarrow}$: $\sum_{n=1}^{\infty} u_n$ converges, and so S_n converges. Thus, for every $\epsilon > 0$, there is an M so that if $n > M$ one has that $|S_n - L| < \epsilon$. This implies that any S_n with $n > M$ is bounded between $L - \epsilon$ and $L + \epsilon$, and any S_n with $n \leq M$ is a member of a finite set, so obviously there are finitely many of those. Thus, S_n is bounded.
- $\boxed{\Leftarrow}$: S_n is bounded and so $u_n \geq 0$. Note that $S_n = u_1 + u_2 + \ldots + u_n$ and since S_n is bounded $m \leq S_n \leq M$. Observe that
$$S_{n+1} - S_n = (u_1 + u_2 + \ldots + u_n + u_{n+1}) - (u_1 + u_2 + \ldots + u_n) = u_{n+1}$$
But since $\{u_n\}$ only has non-negative terms, we see that $u_{n+1} \geq 0$ and so $S_{n+1} - S_n \geq 0 \Longrightarrow S_{n+1} \geq S_n$. Since S_n is bounded and monotone, we see that S_n converges to a finite limit.

$\square$

THM 21. *Let $\{u_n\}$ and $\{v_n\}$ be sequences with non-negative terms where, for every n, one has that $u_n \leq v_n$. We have the following:*

- *If $\sum v_n$ converges, then $\sum u_n$ converges.*
- *If $\sum u_n$ diverges, then $\sum v_n$ diverges.*

THM 22. *Let $\{u_n\}$ and $\{v_n\}$ be two non-negative sequences. If $\lim_{n\to\infty} \left(\frac{u_n}{v_n}\right) = L \neq 0$, then $\sum u_n$ and $\sum v_n$ are either both convergent or both divergent.*

THM 23. *If $\{u_n\}$ and $\{v_n\}$ are sequences with positive terms, then given that $\frac{u_{n+1}}{u_n} < \frac{v_{n+1}}{v_n}$ we then have*

- *$\sum v_n$ converges $\Longrightarrow \sum u_n$ converges.*
- *$\sum u_n$ diverges $\Longrightarrow \sum v_n$ diverges.*

Definition 18. *A geometric series takes the form $\sum_{n=0}^{\infty} a^n$.*

- *This series converges for $|a| < 1$.*
- *This series converges for $|a| \geq 1$.*

Note that $S_n = \sum_{k=0}^{\infty} a^k = \frac{1 - a^{n+1}}{1 - a}$. Further observe that $\lim_{n\to\infty} S_n = \frac{1}{1-a}$. So, as a general rule,

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$$

Corollary 3. *Let $\{u_n\}$ be a sequence with positive terms. If there is an $r < 1$ so that if $0 < \frac{u_{n+1}}{u_n} < r < 1$, then $\sum u_n$ converges. Alternatively, if there is an $R > 1$ so that if $\frac{u_{n+1}}{u_n} > R > 1$, then $\sum u_n$ diverges.*

Corollary 4. *Let $\{u_n\}$ be a sequence with positive terms, and assume that $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = L$.*

- *If $L < 1$, then $\sum u_n$ converges.*
- *If $L > 1$, then $\sum u_n$ diverges.*
- *If $L = 1$, then nothing useful can be concluded.*

Proposition 14. *If f is integrable on any closed interval $[0, c]$ and f is non-increasing (that is, $\forall\, x \le y$ one has $f(x) \ge f(y)$) then $\sum_{n=1}^{\infty} f(n)$ and $\int_0^{\infty} f(t)\, dt$ are either both convergent or both divergent.*

Proposition 15. *The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for any $p \ge 2$.*

Definition 19. *A series $\sum u_n$ is absolutely convergent if $\sum |u_n|$ is convergent.*

THM 24. *A series that is absolutely convergent is convergent.*

Definition 20. *If $\sum u_n$ is convergent but not absolutely convergent, then it is conditionally convergent.*

9 Pointwise and Uniform Convergence

Definition 21. *Let $u_n(x)$ be a sequence of functions. We say $u_n(x)$ is pointwise convergent to $L(x)$ on an interval I if, for every $x \in I$ and for every $\epsilon > 0$ there is an $M(x, \epsilon)$ so that if $n > M$ then $|u_n(x) - L(x)| < \epsilon$.*

Definition 22. *Let $u_n(x)$ be a sequence of functions. We say $u_n(x)$ is uniformly convergent to $L(x)$ on an interval I if, for every $\epsilon > 0$, there is an $M(\epsilon)$ so that if $n > M$ then for every $x \in I$ one has $|u_n(x) - L(x)| < \epsilon$.*

Example.

- $f_n(x) = \sin\left(\frac{x}{n}\right)$ converges uniformly to $f(x) = 0$ on $[0, 1]$.
- $g_n(x) = \left(x + \frac{1}{n}\right)^2$ converges pointwise to $g(x) = x^2$ on $[0, 2]$.

THM 25. *Let $u_n(x)$ be a sequence of continuous functions on an interval I. Also assume that $u_n(x)$ is also uniformly convergent to $L(x)$ on I. If these conditions are satisfied, then $L(x)$ is continuous.*

Definition 23. *$u_n(x)$ is continous at $x = x_0$ if $\lim_{x \to x_0} u_n(x) = u_n(x_0)$. That is to say that $\forall\, \epsilon > 0$ there exists an $\alpha > 0$ so that, if $|x - x_0| < \alpha$, then $|u_n(x) - u_n(x_0)| < \epsilon$.*

Proof. Since $u_n(x)$ is uniformly convergent on I, we have that for every $\epsilon_1 > 0$, there is an M_1 so that if $n > M_1$ then for every $x \in I$ one has $|u_n(x) - L(x)| < \epsilon_1$. Now, take $\epsilon_1 = \epsilon/3$ and pick $n = M + 1$. Thus, for every $x \in I$, we see that $|u_n(x) - L(x)| < \epsilon/3$. In particular, this holds for $x = x_0$ so that $|u_n(x_0) - L(x_0)| < \epsilon/3$. But, we know that $u_n(x)$ is continuous at $x = x_0$ so for$\epsilon_2 = \epsilon/3$ there is an $\alpha > 0$ so that if $|x - x_0| < \alpha$ then $|u_n(x) - L(x)| < \epsilon/3$. Also observe that

$$|L(x) - L(x_0)| = |L(x) - u_n(x) + u_n(x) - u_n(x_0) + u_n(x_0) - L(x_0)|$$

$$\implies |L(x) - L(x_0)| \le \underbrace{|L(x) - u_n(x)|}_{< \epsilon/3} + \underbrace{|u_n(x) - u_n(x_0)|}_{< \epsilon/3} + \underbrace{|u_n(x_0) - L(x_0)|}_{< \epsilon/3}$$

$$\implies |L(x) - L(x_0)| < \epsilon$$

□

THM 26. *Let $u_n(x)$ be a sequence of functions on an interval I. If there is an M_n so that $|u_n(x)| \le M_n$ for every $x \in I$ and the series $\sum M_n$ converges, then the series $\sum u_n(x)$ is uniformly convergent on I.*

Example. Consider the series $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2}$ for $x \in \mathbb{R}$. Note that since $|\sin(nx)| \le 1$, one has that $\frac{|\sin(nx)|}{|n^2|} \le \frac{1}{|n^2|} \implies \left|\frac{\sin(nx)}{n^2}\right| \le \frac{1}{n^2}$. Thus, it is clear that $M_n = 1/n^2$ and observe that $\sum M_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, and thus $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2}$ converges uniformly on $\mathbb{R}$.

Proposition 16. *Let $S_n(x) = \sum_{k=1}^{n} u_k(x)$. One has that $S_n(x)$ is a uniform Cauchy sequence if and only if $S_n(x)$ converges uniformly.*

THM 27. *If $f_n(x)$ converges to $f(x)$ uniformly on I, and $f_n(x)$ is continuous on I for every n, then for every $a \in I$ and every $b \in I$, one has*

$$\int_a^b f(x)\ dx = \lim_{n\to\infty} \int_a^b f_n(x)\ dx$$

COUNTEREXAMPLE. Let $f_n(x) = (n+1)x^n$ for $x \in [0,1]$. Observe that $\int_0^1 (n+1)x^n\ dx = 1$ and that

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 0 \\ \infty \end{cases}$$

So that $\int_0^1 \lim_{n\to\infty} f_n(x)\ dx = 0$. Since $0 \neq 1$, we have a counter-example and thus we can verify that this property doesn't hold for functions $f_n(x)$ that do not converge uniformly.

THM 28. *If:*

- *$f_n(x)$ is continuous in I*
- *$f_n(x)$ converges uniformly to $L(x)$*
- *$f_n'(x)$ is continuous on I*
- *$f_n'(x)$ converges uniformly to $G(x)$*

then $L(x)$ is differentiable on I and $L'(x) = G(x)$.

This results from the fact that $\frac{d}{dx}\left(\lim_{n\to\infty} f_n(x)\right) = \lim_{n\to\infty} \frac{d}{dx}\left(f_n(x)\right) \implies \frac{d}{dx}(L(x)) = \lim_{n\to\infty} f_n'(x) \implies L'(x) = G(x)$. Alternatively, take some $a \in I$ and some $t \in I$ and

$$\lim_{n\to\infty} \int_a^t f_n'(x)\ dx = \int_a^t \lim_{n\to\infty} f_n'(x)\ dx$$

$$\implies \lim_{n\to\infty} \left(f_n(t) - f_n(a)\right) = \int_a^t G(x)\ dx$$

$$\implies \lim_{n\to\infty} f_n(t) - \lim_{n\to\infty} f_n(a) = \int_a^t G(x)\ dx$$

$$\implies L(t) - L(a) = \int_a^t G(x)\ dx$$

which verifies that $G(x)$ is the derivative of $L(x)$.