MATH 340 - Spring 2017

Homework 7

Solutions written by Alex Menendez (1438704)

Problems Assigned:

- §5.2: 20, 22
- §6.1: 2, 5, 15
- §6.2: 2d, 2h, 9, 11

§5.2

20.

Proof. We prove two directions:

- $\Longrightarrow$: Assume that V is a direct sum of $W_1, \ldots, W_k$. We thus have that $V = \sum_{i=1}^k W_i$. From this, it follows that $\dim(V) = \dim\left(\sum_{i=1}^k W_i\right) = \dim(W_1 + \ldots + W_k)$.

  Lemma 1. *For any two subspaces U and W of a vector space V, $\dim(U+W) = \dim(U) + \dim(W) + \dim(U \cap W)$.*

  Applying the results of the lemma, we have that $\dim(W_1 + \ldots + W_k) = \dim(W_1) + \ldots + \dim(W_k) + \dim\left(\bigcap_{i=1}^k W_i\right)$. since V is assumed to be a direct sum, however, $\bigcap_{i=1}^k W_i = \{0\}$ and so $\dim\left(\bigcap_{i=1}^k W_i\right) = 0$ and thus $\dim(W_1 + \ldots + W_k) = \dim(W_1) + \ldots + \dim(W_k) = \sum_{i=1}^k \dim(W_i) = \dim(V)$, as required.

- $\Longleftarrow$: Assuming $\dim(V) = \sum_{i=1}^k \dim(W_i)$, let β be an arbitrary basis for V and let γ_i be an arbitrary basis for W_i for each integer $i \in [1,k]$. Assume that $\dim(W_i) = m_i$, for which $\sum_{i=1}^k \dim(W_i) = \sum_{i=1}^k m_i$. Thus, $\dim(V) = m_1 + \ldots + m_k$. This claim can be fleshed out inductively: for $k=1$, we have that if $\dim(V) = \sum_{i=1}^1 m_i$, then $\dim(V) = m_1$ in which case $V = W_1$. Assuming that, for some $\ell \geq 1$, if $\dim(V) = \sum_{i=1}^{\ell} m_i$, then $V = \sum_{i=1}^{\ell} W_\ell$, then it follows that $\dim(V) = \sum_{i=1}^{\ell+1} m_i = \sum_{i=1}^{\ell} m_i + m_{\ell+1} = \dim(V) + m_{\ell+1}$, and so $V = W_1 + W_2 + \ldots + W_{\ell+1}$. Thus, we have that $V = \sum_{i=1}^k W_i$. Furthermore, for any $j \in [i,k]$, we see that $W_j \cap \sum_{i \neq j} W_i = \{0\}$. That is, for any $\vec{w} \in W_j$, we have that $\vec{w}$ can't be in the set $\left\{\sum_{i\neq j} w_i \;\middle|\; w_i \in W_i\right\}$, as well unless $\vec{w} = \vec{0}$. So we see that the conditions of the direct sum are satisfied, and so $V = \bigoplus_{i=1}^k W_i$, as required.

QED.

22. For some $\vec{u} \in \text{Span}(\vec{x})$, we have that $\vec{u} = \alpha\vec{x}$, for some scalar $\alpha \in \mathbf{F}$. Alternatively, if $\vec{u} \in \sum_{i=1}^k E_{\lambda_i}$, then $\vec{u} \in \{\vec{x}_1 + \ldots + \vec{x}_k \mid \vec{x}_i \in E_{\lambda_i}\}$. For each x_i, $([T] - \lambda_i I)x_i = \vec{0}$. Thus, for any one eigenspace, the only element common to that particular eigenspace and the sum of the remaining eigenspaces. Thus, $\vec{u} \in \sum_{i=1}^k E_{\lambda_i}$, as well, and this is tautologous with the fact that $\vec{u} = \alpha x$, and so $\text{Span}(\vec{x}) \subseteq \bigoplus_{i=1}^k E_{\lambda_i}$. Alternatively, if $\vec{u} \in \bigoplus_{i=i}^k E_{\lambda_i}$, then $\vec{u} \in \{\vec{x}_1 + \ldots + \vec{x}_k \mid \vec{x}_i \in E_{\lambda_i}\}$. Since $E_{\lambda_j} \cap \sum_{i\neq j} E_{\lambda_i} = \{\vec{0}\}$, we have that $\vec{u} \in \text{Span}(\vec{x})$, and so $\bigoplus_{i=i}^k E_{\lambda_i} \subseteq \text{Span}(\vec{x})$, as required. Putting this together yields that $\text{Span}(\vec{x}) = \bigoplus_{i=1}^k E_{\lambda_i}$.

§6.1

2. Computing $\langle x, y\rangle$, we have that

$$\langle x, y\rangle = \sum_{i=1}^{3} x_i \overline{y_i}$$

$$= 2(2+i) + 2(1+i) + i(1-2i)$$

$$= \boxed{8+5i}.$$

Computing $\|x\|$, we have that

$$\|x\| = \sqrt{\sum_{i=1}^{3} x_i \overline{x_i}}$$

$$= \sqrt{4 + (1+i)(1-i) + i\bar{i}} = \boxed{\sqrt{7}}.$$

Computing $\|y\|$, we have that

$$\|y\| = \sqrt{\sum_{i=1}^{3} y_i \overline{y_i}}$$

$$= \sqrt{(2-i)(2+i) + 4 + (1+2i)(1-2i)} = \boxed{\sqrt{14}}.$$

Applying the Cauchy-Schwartz inequality, we see that $|\langle x, y\rangle| \leq \|x\|\|y\| \Longrightarrow |8+5i| \leq \sqrt{7}\sqrt{14} \Longrightarrow \sqrt{8^2+5^2} \leq \sqrt{98} \Longrightarrow \sqrt{89} \leq \sqrt{98}$, as expected. Applying the triangle inequality, we see that $\|x+y\| \leq \|x\| + \|y\| \Longrightarrow \sqrt{\sum_i^3 (x+y)_i \overline{(x+y)_i}} \leq \sqrt{7} + \sqrt{14} \Longrightarrow \sqrt{37} \leq \sqrt{7} + \sqrt{14} \Longrightarrow 6.082\ldots \leq 6.387\ldots$, as required.

5. We'll manually verify the conditions of the inner product:

- $\langle x+z, y\rangle = (x+z)Ay^* = xAy^* + zAy^* = \langle x, y\rangle + \langle z, y\rangle$.
- $\langle \alpha x, y\rangle = (\alpha x)Ay^* = \alpha(xAy^*) = \alpha\langle \alpha x, y\rangle$.
- $\overline{\langle x, y\rangle} = (xAy^*)^* = yA^*x^* = yAx^* = \langle y, x\rangle$. (Note that we've used the fact that $A^* = A$.)
- $\langle x, x\rangle = (x_1, x_2)A(x_1, x_2)^* = \|x_1\|^2 + 2\Re(ix_1\overline{x_2}) + 2\|x_2\|^2$, which is greater than zero if x_1 or x_2 isn't zero.

Given $x = (1-i, 2+3i)$ and $y = (2+i, 3-2i)$, we have that $\langle x, y\rangle = \begin{pmatrix} 1-i & 2+3i \end{pmatrix} \begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix} \begin{pmatrix} 2+i \\ 3-2i \end{pmatrix} = \boxed{6+21i}$.

15.

(a) If $|\langle x,y\rangle| = \|x\|\|y\|$, then $\|x\| = \frac{|\langle x,y\rangle|}{\|y\|}$. Dividing both sides by $\|y\|$, we have that $\frac{\|x\|}{\|y\|} = \frac{|\langle x,y\rangle|}{\|y\|^2}$. Now, let $a = \frac{|\langle x,y\rangle|}{\|y\|^2} = \frac{\|x\|}{\|y\|}$, for which we define $z = x - ay = x - \frac{|\langle x,y\rangle|}{\|y\|^2}y$. Now, we show that y and z are orthogonal:

$$\left\langle y, x - \frac{|\langle x,y\rangle|}{\|y\|^2}y \right\rangle = \overline{\left\langle x - \frac{|\langle x,y\rangle|}{\|y\|^2}y, y \right\rangle}$$

$$= \langle x,y\rangle - \frac{|\langle x,y\rangle|}{\|y\|^2}\langle y,y\rangle,$$

and since $\|y\|^2 = \langle y,y\rangle$, we get that this equals

$$\langle x,y\rangle - \langle x,y\rangle = 0,$$

which verifies that y and z are orthogonal.

(b)

§6.2

2.

(d) Applying Gram-Schmidt, we have that $\langle(1,i,0),(1-i,2,4i)\rangle \neq 0$, and so β isn't orthogonal, however, we compute

$$\left\{x, y - \frac{\langle y,x\rangle}{\|x\|^2}x\right\} = \{(1,i,0),(1+i,1-i,4i)\}.$$

After normalizing, we have that

$$\beta = \left\{\frac{(1,i0)}{\sqrt{2}}, \frac{(1+i,1-i,4i)}{2\sqrt{17}}\right\}.$$

Furthermore, the desired Fourier coefficients are:

$$\left\langle (3+i,4i,-4), \frac{(1,i0)}{\sqrt{2}} \right\rangle = (7+i)/\sqrt{2},$$

and

$$\left\langle (3+i,4i,-4), \frac{(1+i,1-i,4i)}{2\sqrt{17}} \right\rangle = \sqrt{17}i.$$

(h) Applying Gram-Schmidt, we obtain the orthonormal basis

$$\beta = \left\{\begin{pmatrix} 2/\sqrt{13} & 2/\sqrt{13} \\ 2/\sqrt{13} & 1/\sqrt{13} \end{pmatrix}, \begin{pmatrix} 5/7 & -2/7 \\ -4/7 & 2/7 \end{pmatrix}, \right.$$

$$\left.\begin{pmatrix} 8/\sqrt{373} & -8/\sqrt{373} \\ 7/\sqrt{373} & -14/\sqrt{373} \end{pmatrix}\right\},$$

and obtain the Fourier coefficients

$$\left\langle \begin{pmatrix} 8 & 6 \\ 25 & -13 \end{pmatrix}, \begin{pmatrix} 2/\sqrt{13} & 2/\sqrt{13} \\ 2/\sqrt{13} & 1/\sqrt{13} \end{pmatrix} \right\rangle = 5\sqrt{13},$$

$$\left\langle \begin{pmatrix} 8 & 6 \\ 25 & -13 \end{pmatrix}, \begin{pmatrix} 5/7 & -2/7 \\ -4/7 & 2/7 \end{pmatrix} \right\rangle = -14,$$

and

$$\left\langle \begin{pmatrix} 8 & 6 \\ 25 & -13 \end{pmatrix}, \begin{pmatrix} 8/\sqrt{373} & -8/\sqrt{373} \\ 7/\sqrt{373} & -14/\sqrt{373} \end{pmatrix} \right\rangle = \sqrt{373}.$$

9. We can see that $\beta = \left\{(i,0,1), \frac{(i,0,1)}{\sqrt{2}}\right\}$ is an orthogonal basis for W. Now, if we're to find a basis for $W^\perp$, we have to solve the equation $(a,b,c)\cdot(i,0,1) = 0$ (that is, finding a vector that is orthogonal to $(i,0,1)$ and any multiples of it). This gives $ai + c = 0$, for which we obtain

$$\beta = \{(i,0,1),(0,1,0)\}$$

as a basis for $W^\perp$. Now, the last thing we have to do is normalize, for which we have

$$\beta = \left\{\frac{(i,0,1)}{\sqrt{2}}, \frac{(0,1,0)}{\sqrt{2}}\right\}$$

as an orthonormal basis for $W^\perp$.

11. First, observe that $(AA^*)_{ij} = \langle v_i, v_j\rangle$, where v_k is the k^{th} row vector of A. Of course, if $AA^* = I$, then we have that

$$AA^* = I = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} = \delta_{ij}.$$

So $AA^* = I$ if and only if

$$\langle v_i, v_j\rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$

Observe that, for all i, $\|v_i\| = 1$, and $v_i \perp v_j$ for all $i \neq j$. Thus the rows of A are of unit length and are orthogonal, and since they are linearly independent and span $\mathbf{C}^n$, they form a basis.