MATH 340 - Spring 2017

Homework 3

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Problems assigned:

- §1.3: 31
- §1.6: 35
- §2.1: 4, 6, 11, 17, 28, 40a, 40b

Solutions:

§1.3

31.

(a) *Proof.* We'll prove two directions:

- $\Rightarrow$: If $v + W$ is a subspace, then it contains zero – that is, $0 \in v + W$, or $0 = v + w$ for some $w \in W$. However, it's not hard to see that $w = -v$ in which case $-v \in W$. Since W is a subspace, it's closed under scaling, and so if $-v \in W$ then $v \in W$, as required.
- $\Leftarrow$: We'll assume that $v \in W$ and then show that $v + W = W$, in which case $v + W$ is a subspace of V since W is a subspace of V. Thus, we'll show a double subset inclusion to show equality: for some $x \in v + W$, $x = v + w$ for some fixed v and w. Since $v \in W$, however, and since W is a subspace and is closed under addition, $v + w \in W \Rightarrow x \in W$, and so $v + W \subset W$. For the other inclusion, take some $y \in W$ and write $y = w + (w - v)$ (i.e. $y = w$ so there's no ambiguity about it being an element of W). Since $v \in W$, as assumed, $w - v \in W$ (again, since W is a subspace) and so $y \in v + W \Rightarrow v + (w - v) \in v + W$. Thus, $W \subset v + W$ and so $v + W = W$, showing that it is a subspace of V, as required.

$\square$

(b) *Proof.* We need to show two implications:

- $\Rightarrow$: If $v_1 + W = v_2 + W$, then both $v_1 + W \subset v_2 + W$ and $v_1 + W \supset v_2 + W$. In either case, $v_1 + w_1 = v_2 + w_2$ for some $w_1, w_2 \in W$ (we can't assume we're adding the same element of W to v_1 and v_2). By algebra, $v_1 - v_2 = w_2 - w_1$. Since W is a subspace of V it's closed under scaling and addition, so $w_2 - w_1 \in W$ whence $v_1 - v_2 \in W$, as required.
- $\Leftarrow$: On a similar note, if $v_1 - v_2 \in W$, then $v_1 - v_2 = w - w'$ for some $w, w' \in W$. Thus, $v_1 + w' = v_2 + w$, whence both $v_1 + W \subset v_2 + W$ and $v_1 + W \supset v_2 + W$, implying that $v_1 + W = v_2 + W$, as required.

$\square$

(c) *Proof.* We know the following:

- $v_1 + W \subset v_1' + W$ and $v_1 + W \supset v_1' + W$
- $v_2 + W \subset v_2' + W$ and $v_2 + W \supset v_2' + W$

From this, we see that $(v_1 + W) + (v_2 + W) \subset (v_1' + W) + (v_2' + W)$ $(v_1 + W) + (v_2 + W) \supset (v_1' + W) + (v_2' + W)$, which implies that $(v_1 + W) + (v_2 + W) = (v_1' + W) + (v_2' + W)$, as required. Similarly, knowing that $x \in av_1 + W$, that implies that $x = av_1 + w_0$ for some $w_0 \in W$. But since $v_1 + W \subset v_1' + W$ we have that $x = av_1' + w_1$ for some $w_1 \in W$, and so $x \in av_1' + W$. Thus, $a(v_1 + W) \subset a(v_1' + W)$. By the same exact argument using the fact that $v_1 + W \supset v_1' + W$, we have that $a(v_1' + W) \subset a(v_1 + W)$. Therefore, $a(v_1 + W) = a(v_1' + W)$, as required. $\square$

(d) *Proof.* We'll go through and manually verify each axiom:

(a) Given $v_1 + W \in V/W$ and $v_2 + W \in V/W$, we have that $(v_1 + W) + (v_2 + W) = (v_1 + v_2) + W$. Alternatively, we can define $(v_2 + W) + (v_1 + W) = (v_2 + v_1) + W$. Now, since, for any $x \in (v_1 + v_2) + W$, $x = (v_1 + v_2) + w$ for some $w \in W$. But $v_1 + v_2 = v_2 + v_1$ so $x = (v_2 + v_1) + w$ for which $x \in (v_2 + v_1) + W$, and so $(v_1 + W) + (v_2 + W) \subset (v_2 + W) + (v_1 + W)$. By the exact same argument exploiting the commutativity of addition of v_1 and v_2, $(v_2 + W) + (v_1 + W) \subset (v_1 + W) + (v_2 + W)$ and so $(v_1 + W) + (v_2 + W) = (v_2 + W) + (v_1 + W)$.

(b) Given $v_1 + W$, $v_2 + W$, and $v_3 + W$ in V/W, we have $v_1 + W + ((v_2 + W) + (v_3 + W)) = v_1 + (v_2 + v_3) + W$. Alternatively, $((v_1 + W) + (v_2 + W)) + (v_3 + W) = (v_1 + v_2) + v_3 + W$. If we take any $x \in v_1 + (v_2 + v_3) + W$, we have that $x = v_1 + (v_2 + v_3) + w$ for some $w \in W$. By associativity of addition, $x = (v_1 + v_2) + v_3 + w$ and so $x \in (v_1 + v_2) + v_3 + W$. Thus, $v_1 + (v_2 + v_3) + W \subset (v_1 + v_2) + v_3 + W$. By the exact same argument flipped the other way around, $(v_1 + v_2) + v_3 + W \subset v_1 + (v_2 + v_3) + W$, and so $v_1 + W + ((v_2 + W) + (v_3 + W)) = ((v_1 + W) + (v_2 + W)) + v_3 + W$, as required.

(c) We desire an element of V/W such that, given some arbitrary $v + W \in V/W$, $(v + W) + (\text{that element}) = v + W$. That is, we want an element that will give us a double subset inclusion. Note that, if we take W, certainly it's in V/W and $(v + W) + W = v + W$. That is, $(v + W) + W \subset v + W$ and $(v + W) + W \supset v + W$. (Also, with how we've defined addition in V/W, the equality is obvious.)

(d) For any $v + W \in V/W$, we can take $-v + W$, in which case $(v + W) + (-v + W) = (v - v) + W = W$.

(e) Given $v + W \in V/W$, we have that $1(v + W) = 1 \cdot v + W = v + W$.

(f) Given some $a, b \in \mathbb{F}$, we have that $(ab)(v + W) = (ab)v + W$ and $a(b(v + W)) = a(bv + W)$. Given some $x \in (ab)v + W$, we have that $x = (ab)v + w$ for some $w \in W$. Of course, by associativity of scalar multiplication, $x = a(bv) + w$ for which $x \in a(b(v + W))$ and so $(ab)(v + W) \subset a(b(v + W))$. By the exact same argument, $(ab)(v + W) \supset a(b(v + W))$, and so $(ab)(v + W) = a(b(v + W))$.

(g) Given some $a \in \mathbb{F}$ and $v_1 + W$ and $v_2 + W$ in V/W, we have that any $x \in a\left((v_1 + W) + (v_2 + W)\right)$, $x = a((v_1 + v_2) + W) = (av_1 + av_2) + W$ and so $x \in (av_1 + W) + (av_2 + W)$. Thus, $a\left((v_1 + W) + (v_2 + W)\right) \subset (av_1 + W) + (av_2 + W)$. By the same token, $a\left((v_1 + W) + (v_2 + W)\right) \supset (av_1 + W) + (av_2 + W)$ and so $a\left((v_1 + W) + (v_2 + W)\right) = (av_1 + W) + (av_2 + W)$, as required.

(h) Given $a, b \in \mathbb{F}$, we have that $(a + b)(v + W) = (a + b)v + W = (av + bv) + W$, and so, by a double subset inclusion (as before), $(a + b)(v + W) = (av + W) + (bv + W)$.

□

§1.6

35.

(a) *Proof.* We first need to show that $\{u_{k+1} + W, \ldots, +u_n + W\}$ is linearly independent: let $\alpha_{k+1}, \ldots, \alpha_n$ be scalars such that $\sum_{i=k+1}^{n} \alpha_i(u_i + W) = 0 \Rightarrow \alpha_{k+1} = \ldots = \alpha_n = 0$. Secondly, $\text{Span}\{u_{k+1} + W, \ldots, +u_n + W\} = \alpha_{k+1}u_{k+1} + \ldots + \alpha_n u_n$ whence $\text{Span}\{u_{k+1} + W, \ldots, +u_n + W\} \subset V/W$. Moreover, given any $v + W \in V/W$, we have that $v + W \in \text{Span}\{u_{k+1} + W, \ldots, +u_n + W\}$ and so $V/W \subset \text{Span}\{u_{k+1} + W, \ldots, +u_n + W\}$ for which $\text{Span}\{u_{k+1} + W, \ldots, +u_n + W\} = V/W$, as required. □

(b) *Claim.* $\dim V = \dim W + \dim V/W$.

Proof. Given a linear transformation $T : V \to V/W$, we see that $\mathcal{N}(T) = W$, $\mathcal{R}(T) = W$, and so by the rank-nullity theorem, $\dim \mathcal{R}(T) + \dim \mathcal{N}(T) = \dim V \Rightarrow \dim V/W + \dim W = \dim V$. □

§2.1

4.

Proof. If we consider $T\begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{pmatrix} + T\begin{pmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \end{pmatrix}$ we obtain $T\begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{pmatrix} + T\begin{pmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \end{pmatrix} =$ $\begin{pmatrix} 2a_1 - a_2 & a_3 + 2a_2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 2b_1 - b_2 & b_3 + 2b_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2(a_1 + b_1) - (a_2 + b_2) & (a_3 + b_3) + 2(a_2 + b_2) \\ 0 & 0 \end{pmatrix}$. Alternatively, $T\left(\begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{pmatrix} + \begin{pmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \end{pmatrix}\right) = T\begin{pmatrix} a_1 + b_1 & a_2 + b_2 & a_3 + b_3 \\ a_4 + b_4 & a_5 + b_5 & a_6 + b_6 \end{pmatrix} = \begin{pmatrix} 2(a_1 + b_1) - (a_2 + b_2) & (a_3 + b_3) + 2(a_2 + b_2) \\ 0 & 0 \end{pmatrix}$. Indeed, they're identical, so T is additive. Furthermore, given some $\alpha \in \mathbb{F}$, $T\left(\alpha\begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{pmatrix}\right) = T\begin{pmatrix} \alpha a_1 & \alpha a_2 & \alpha a_3 \\ \alpha a_4 & \alpha a_5 & \alpha a_6 \end{pmatrix} = \begin{pmatrix} 2\alpha a_1 - \alpha a_2 & \alpha a_3 + 2\alpha a_2 \\ 0 & 0 \end{pmatrix}$, and $\alpha T\begin{pmatrix} \alpha a_1 & \alpha a_2 & \alpha a_3 \\ \alpha a_4 & \alpha a_5 & \alpha a_6 \end{pmatrix} = \alpha\begin{pmatrix} 2a_1 - a_2 & a_3 + 2a_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2\alpha a_1 - \alpha a_2 & \alpha a_3 + 2\alpha a_2 \\ 0 & 0 \end{pmatrix}$, whence T obeys scalar multiplication, as well, and is thus linear. □

- To compute a basis for $\mathcal{N}(T)$, we'll examine $T\begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, which follows from the definition of the null space. From the definition of T itself, we obtain $2a_1 - a_2 = 0 \Rightarrow 2a_1 = a_2$ and $a_3 + 2a_2 \Rightarrow a_3 = -2a_2$. Thus, we can feed any matrix of the form $\begin{pmatrix} a & 2a & -4a \\ b & c & d \end{pmatrix}$ for $a, b, c, d \in \mathbb{F}$ into T and obtain the zero matrix in $\mathbb{F}^{2\times 2}$. What we glean from this is that

$$\begin{pmatrix} a & 2a & -4a \\ b & c & d \end{pmatrix} = a\begin{pmatrix} 1 & 2 & -4 \\ 0 & 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and so $\mathcal{N}(T) = \text{Span}\left\{\begin{pmatrix} 1 & 2 & -4 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\}$ and thus we have found a basis for $\mathcal{N}(T)$.

- Similarly, $\mathcal{R}(T) = \text{Span}\left\{T\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, T\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, T\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, T\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, T\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, T\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\} =$ $\text{Span}\left\{\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right\} = \text{Span}\left\{\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right\}$.

From this, we see that $\dim(\mathcal{N}(T)) = 4$ and $\dim(\mathcal{R}(T)) = 2$, whence $\dim(\mathbb{F}^{2\times 3}) = 6$, thus verifying the dimension theorem. We also see that T is neither one-to-one or onto, since $\mathcal{N}(T) \neq \{0\}$ and $\mathcal{R}(T) \neq \mathbb{F}^{2\times 2}$.

6.

Proof. First, we see that T satisfies additivity: let A and B be matrices in $\mathbb{F}^{n\times n}$. It follows that $T(A+B) = \text{Tr}(A+B)$; however, the trace is linear, and so $\text{Tr}(A+B) = \text{Tr}(A) + \text{Tr}(B) = T(A) + T(B)$, as required. Furthermore, given some $\alpha \in \mathbb{F}$, we have that $T(\alpha A) = \text{Tr}(\alpha A) = \alpha \text{Tr}(A) = \alpha T(A)$, as expected. $\square$

- To find a basis for $\mathcal{N}(T)$, we desire a basis for the kernel of the map $A \mapsto \text{Tr}(A)$ – that is, a basis for the set of all matrices A such that $\text{Tr}(A) = 0$. In general, any matrix A in $\mathbb{F}^{n\times n}$ can be written as $A = D + S$, where D is the matrix consisting of A's diagonal elements and zero everywhere else while S consists of A's off-diagonal elements and zero down the diagonal. Thus, $\text{Tr}(A) = \text{Tr}(D+S) = \text{Tr}(D) + \text{Tr}(S)$. But, since S has zeroes down its diagonal, $\text{Tr}(S) = 0$, so $\text{Tr}(A) = 0 \Rightarrow \text{Tr}(D) + 0 = 0$. Thus, the problem of finding a basis for matrices A such that $\text{Tr}(A) = 0$ amounts to finding a basis for the set of diagonal matrices D such that $\text{Tr}(D) = 0$, which is markedly simpler.

11. We'll begin by trying to boil everything down so that it's in terms of the standard basis vectors e_1, e_2 in $\mathbb{R}^2$. Since $T(1,1) = T(e_1 + e_2) = T(e_1) + T(e_2)$ and $T(2,3) = T(2e_1 + 3e_2) = 2T(e_1) + 3T(e_2)$, we obtain the following system of equations:

$$\begin{cases} T(e_1) + T(e_2) = (1,0,2) \\ 2T(e_1) + 3T(e_2) = (1,-1,4) \end{cases}$$

which, by elimination, yields

$$\begin{cases} T(e_1) = (2,1,2) \\ T(e_2) = (-1,-1,0) \end{cases}$$

Thus, for any $(x,y) \in \mathbb{R}^2$, $T(x,y) = xT(e_1) + yT(e_2) = x(2,1,2) + y(-1,-1,0) = (2x-y, x-y, 2x)$. This gives the desired transformation. We now know that $T(8,11) = (5,-3,16)$.

17.

(a) *Proof.* We'll go the route of contraposition – if T is assumed to be onto then $\mathcal{R}(T) = W$ in which case, by the rank-nullity theorem, we have that $\dim \mathcal{R}(T) + \dim \mathcal{N}(T) = \dim V \Rightarrow \dim V = \dim \mathcal{N}(T) + \dim W$. Thus, if $\mathcal{N}(T) = \{0\}$, we have that $\dim V = \dim W$ and $\dim V > \dim W$ otherwise. This, of course, gives $\dim V \geq \dim W$, the desired result. $\square$

(b) *Proof.* Assume that $\dim V > \dim W$. Because $\dim \mathcal{R}(T)$ is a subspace of W, it follows that $\dim \mathcal{R}(T) \leq \dim W < \dim V$, so $\dim \mathcal{N}(T) = \dim V - \dim \mathcal{R}(T)$ (by the rank-nullity theorem), and thus $\mathcal{N}(T) \neq \text{Span}\{0\}$, so T isn't one-to-one. $\square$

28.

- $\{0\}$ is T-invariant: given any $x \in \{0\}$, it holds that $x = 0$ and so, because T is linear, $0 \overset{T}{\mapsto} 0$ and so $T(x) \in \{0\}$.

- V is T-invariant: given any $x \in V$, it holds that, since T is defined so that $V \xrightarrow{T} V$, $T(x) \in V$.

- $\mathcal{R}(T)$ is T-invariant: given any $x \in \mathcal{R}(T)$, there exists a $v \in V$ so that $T(v) = x$. It follows that, since $V \xrightarrow{T} V$, $T(v) \in V$ and so $T(x) = T(T(v))$ is well-defined. It follows from the definition of T that $T(T(v)) \in \mathcal{R}(T)$ because $T(v) \in \mathcal{R}(T)$.

- $\mathcal{N}(T)$ is T-invariant: given any $x \in \mathcal{N}(T)$, it follows that $T(x) = 0$. However, since T is linear, $T(0) = 0$ and so $0 \in \mathcal{N}(T) \Rightarrow T(x) \in \mathcal{N}(T)$.

40.

(a) *Proof.* We can prove the linearity of η in one fell swoop by first computing $\eta(\alpha v_1 + \beta v_2) = (\alpha v_1 + \beta v_2) + W = \alpha v_1 + W + \beta v_2 + W$. Secondly, $\alpha\eta(v_1) + \beta\eta(v_2) = \alpha(v_1 + W) + \beta(v_2 + W) = \alpha v_1 + W + \beta v_2 + W$. Thus, $\eta(\alpha v_1 + \beta v_2) = \alpha\eta(v_1) + \beta\eta(v_2)$ and so η is linear, as required. We can now show, rather trivially, that η is onto – for any $v + w \in v + W$, there exists a $v \in V$ so that $\eta(v) = v + W$. Furthermore, $\mathcal{N}(\eta) = \{v \in V \mid v + W = W\} = W$. $\square$

(b) Since η is onto, it holds that $\mathcal{R}(\eta) = V/W$. Furthermore, $\mathcal{N}(T) = W$. By the rank-nullity theorem, $\dim \mathcal{N}(T) + \dim \mathcal{R}(T) = \dim V$. Of course, this implies that $\dim W + \dim V/W = \dim V$, as required.