MATH 340 - Spring 2017

Homework 5

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Problems Assigned:

- §4.1: 5, 9
- §4.2: 2, 4
- §4.3: 13, 15
- §4.4: 4f, 4g, 6

§4.1

5.

Proof. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, for which we define $B = \begin{pmatrix} c & d \\ a & b \end{pmatrix}$. From this, we see $\det(B) = bc - ad = -(ad - bc) = -\det(A)$, as required. **QED.**

9.

Proof. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$. From these, we can define $AB = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$. It follows that $\det(AB) = (ae+bg)(cf+dh) - (af+bh)(ce+dg) = adeh - adfg - bceh + bcfg = (ad-bc)(eh-fg) = \det(A)\det(B)$, as required. **QED.**

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§4.2

2. Let $M = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$. We can see that $\begin{pmatrix} 3a_1 & 3a_2 & 3a_3 \\ 3b_1 & 3b_2 & 3b_3 \\ 3c_1 & 3c_2 & 3c_3 \end{pmatrix} = 3\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = 3M$, whence $\det(3M) = 3^3\det(M)$. Thus, $\boxed{k = 27}$.

4. We'll compute k by exploiting the linearity of the determinant on rows. Our steps in decomposing are as follows:

$$① : \det\begin{pmatrix} b_1+c_1 & b_2+c_2 & b_3+c_3 \\ a_1+c_1 & a_2+c_2 & a_3+c_3 \\ a_1+b_1 & a_2+b_2 & a_3+b_3 \end{pmatrix} = \det\begin{pmatrix} b_1 & b_2 & b_3 \\ a_1+c_1 & a_2+c_2 & a_3+c_3 \\ a_1+b_1 & a_2+b_2 & a_3+b_3 \end{pmatrix} + \det\begin{pmatrix} c_1 & c_2 & c_3 \\ a_1+c_1 & a_2+c_2 & a_3+c_3 \\ a_1+b_1 & a_2+b_2 & a_3+b_3 \end{pmatrix}$$

$$② : \det\begin{pmatrix} b_1 & b_2 & b_3 \\ a_1+c_1 & a_2+c_2 & a_3+c_3 \\ a_1+b_1 & a_2+b_2 & a_3+b_3 \end{pmatrix} = \det\begin{pmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ a_1+b_1 & a_2+b_2 & a_3+b_3 \end{pmatrix} + \det\begin{pmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1+b_1 & a_2+b_2 & a_3+b_3 \end{pmatrix}$$

$$③ : \det\begin{pmatrix} c_1 & c_2 & c_3 \\ a_1+c_1 & a_2+c_2 & a_3+c_3 \\ a_1+b_1 & a_2+b_2 & a_3+b_3 \end{pmatrix} = \det\begin{pmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ a_1+b_1 & a_2+b_2 & a_3+b_3 \end{pmatrix} + \det\begin{pmatrix} c_1 & c_2 & c_3 \\ c_1 & c_2 & c_3 \\ a_1+b_1 & a_2+b_2 & a_3+b_3 \end{pmatrix}$$

$$④ : \det\begin{pmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ a_1+b_1 & a_2+b_2 & a_3+b_3 \end{pmatrix} = \det\begin{pmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \end{pmatrix} + \det\begin{pmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$

$$⑤ : \det\begin{pmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1+b_1 & a_2+b_2 & a_3+b_3 \end{pmatrix} = \det\begin{pmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{pmatrix} + \det\begin{pmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$

$$⑥ : \det\begin{pmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ a_1+b_1 & a_2+b_2 & a_3+b_3 \end{pmatrix} = \det\begin{pmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \end{pmatrix} + \det\begin{pmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$

$$(7) : \det \begin{pmatrix} c_1 & c_2 & c_3 \\ c_1 & c_2 & c_3 \\ a_1 + b_1 & a_2 + b_2 & a_3 + b_3 \end{pmatrix} = \det \begin{pmatrix} c_1 & c_2 & c_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{pmatrix} + \det \begin{pmatrix} c_1 & c_2 & c_3 \\ c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$

Thus, we've decomposed the original determinant into eight simpler determinants. That is,

$$\det \begin{pmatrix} b_1 + c_1 & b_2 + c_2 & b_3 + c_3 \\ a_1 + c_1 & a_2 + c_2 & a_3 + c_3 \\ a_1 + b_1 & a_2 + b_2 & a_3 + b_3 \end{pmatrix} = \det \cancel{\begin{pmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \end{pmatrix}} + \det \cancel{\begin{pmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}} + \det \begin{pmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{pmatrix} + \det \cancel{\begin{pmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \end{pmatrix}} + \ldots$$

$$\ldots + \det \cancel{\begin{pmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \end{pmatrix}} + \det \begin{pmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} + \det \cancel{\begin{pmatrix} c_1 & c_2 & c_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{pmatrix}} + \det \cancel{\begin{pmatrix} c_1 & c_2 & c_3 \\ c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \end{pmatrix}}$$

Examining the individual terms that didn't vanish, we have:

$$\det \begin{pmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{pmatrix} = \det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

because of two row swaps being made; similarly,

$$\det \begin{pmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} = \det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

again, because of two row swaps. Thus,

$$\det \begin{pmatrix} b_1 + c_1 & b_2 + c_2 & b_3 + c_3 \\ a_1 + c_1 & a_2 + c_2 & a_3 + c_3 \\ a_1 + b_1 & a_2 + b_2 & a_3 + b_3 \end{pmatrix} = \det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} + \det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = 2 \det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

So $\boxed{k = 2}$.

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§4.3

13.

(a) *Proof.* Writing out the cofactor expansion of M along a chosen row i, we have that

$$\det(M) = \sum_{j=1}^{n} (-1)^{i+j} m_{ij} \det(M_{ij}),$$

where M_{ij} denotes the matrix M with the i^{th} row and j^{th} column removed and $m_{ij} \in \mathbb{C}$. Taking the complex conjugate of $\det(M)$, we have that

$$\overline{\det(M)} = \overline{\sum_{j=1}^{n} (-1)^{i+j} m_{ij} \det(M_{ij})},$$

$$= \sum_{j=1}^{n} \overline{(-1)^{i+j} m_{ij} \det(M_{ij})}.$$

For any $\alpha \in \mathbb{R}$ and $x \in \mathbb{C}$, we have that $\overline{\alpha x} = \alpha \overline{x}$ and – for any $x, y \in \mathbb{C}$ – one has that $\overline{xy} = \overline{x}\,\overline{y}$. This allows us to write the above sum in the form:

$$\overline{\det(M)} = \sum_{j=1}^{n} (-1)^{i+j} \overline{m_{ij}}\ \overline{\det(M_{ij})}$$

$$= \sum_{j=1}^{n} (-1)^{i+j} \overline{m}_{ij}\ \det(\overline{M}_{ij}) = \det(\overline{M}),$$

as required. **QED.**

Proof. If Q is unitary, then there exists Q^* such that $QQ^* = I$. It thus follows that $\det(QQ^*) = \det(I) = 1$. Moreover, we have that $\det(QQ^*) = \det(Q)\det(Q^*)$. Since $Q^* = \overline{Q^\top}$, we have that $\det(Q) \det\left(\overline{Q^\top}\right) = 1$. By part (a), this implies that $\det(Q)\overline{\det\left(Q^\top\right)} = 1$. The determinant of a matrix transpose is equal to the transpose of the matrix itself, thus, $\det(Q)\overline{\det(Q)} = 1 \implies (\det(Q))^2 = 1$, from which it follows that $|\det(Q)| = 1$, as required. **QED.**

15.

Proof. If A and B are similar, then there exists Λ such that $B = \Lambda^{-1}A\Lambda$. From this, we have that $\det(B) = \det\left(\Lambda^{-1}A\Lambda\right)$. Of course, we have that $\det\left(\Lambda^{-1}\right) = 1/\det(\Lambda)$, so

$$\det(B) = \frac{1}{\det\Lambda}\det(A)\det(\Lambda),$$

via the fact that the determinant is multiplicative; this gives

$$\det(B) = \det(A),$$

as required. **QED.**

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§4.4

4.

(f) We'll expand along the third column (owing to the fact that it has the simplest entries), which gives

$$\det\begin{pmatrix} -1 & 2+i & 3 \\ 1-i & i & 1 \\ 3i & 2 & -1+i \end{pmatrix} = 3\det\begin{pmatrix} 1-i & i \\ 3i & 2 \end{pmatrix} - \det\begin{pmatrix} -1 & 2+i \\ 3i & 2 \end{pmatrix} + \det\begin{pmatrix} -1 & 2+i \\ 1-i & i \end{pmatrix}$$

$$\ldots = 3(2-2i+3) - (-2-6i+3) + (-1+i)(-i-(2-2i+i+1)) = \boxed{-11+5i}$$

(g) Expanding along the second column, we have

$$\det\begin{pmatrix} 1 & 0 & -2 & 3 \\ -3 & 1 & 1 & 2 \\ 0 & 4 & -1 & 1 \\ 2 & 3 & 0 & 1 \end{pmatrix} = \det\begin{pmatrix} -3 & 1 & 2 \\ 0 & -1 & 1 \\ 2 & 0 & 1 \end{pmatrix} - \det\begin{pmatrix} 1 & -2 & 3 \\ 0 & -1 & 1 \\ 2 & 0 & 1 \end{pmatrix} - 4\det\begin{pmatrix} 1 & -2 & 3 \\ -3 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix} - 3\det\begin{pmatrix} 1 & -2 & 3 \\ -3 & 1 & 2 \\ 0 & -1 & 1 \end{pmatrix} = \boxed{95}$$

6.

Proof. Suppose that $A \in \mathbb{F}^{d\times d}$. By using row reduction on rows 1 through d of M, we can write A in its corresponding upper-triangular form, $\tilde{A}$. Thus, M becomes $\tilde{M} = \begin{pmatrix} \tilde{A} & B \\ 0 & C \end{pmatrix}$. Now, using row reduction on rows $d+1$ through n of $\tilde{M}$, C can also be written in upper-triangular form so as to obtain the matrix $\hat{M} = \begin{pmatrix} \tilde{A} & B \\ 0 & \tilde{C} \end{pmatrix}$. Since $\hat{M}$ is upper triangular, its determinant is simply the product of the diagonal entries, thus

$$\det(\hat{M}) = \det(\tilde{A})\det(\tilde{C})$$

Since row operations don't affect the determinant, however, we have that $\det(\hat{M}) = \det(M)$, $\det(\tilde{A}) = \det(A)$, and $\det(\tilde{C}) = \det(C)$, for which we have that

$$\det(M) = \det(A)\det(C),$$

as required. **QED.**