MATH 340 / Spring 2017

Homework 1

Solutions written by Alex Menendez (1438704)

NOTE: All exercises are taken from Friedberg, Insel, and Spence's *Linear Algebra*.

PROBLEMS ASSIGNED:

- §1.2 – 1, 8, 13, 20
- §1.3 – 1, 6, 8, 21

SOLUTIONS:

§1.2

Exercise 1.

(a) TRUE; (b) FALSE; (c) FALSE; (d) FALSE; (e) TRUE; (f) FALSE; (g) FALSE; (h) FALSE; (i) TRUE; (j) TRUE; (k) TRUE

Exercise 8.

Proof. Since $a \in \mathbb{F}$ and $b \in \mathbb{F}$, certainly $a + b \in \mathbb{F}$, as well, so it may be treated as an entirely new scalar. Let $c = a + b$ where $c \in \mathbb{F}$. Thus, $(a + b)(\mathbf{x} + \mathbf{y}) = c(\mathbf{x} + \mathbf{y})$. By axiom 7, $c(\mathbf{x} + \mathbf{y}) = c\mathbf{x} + c\mathbf{y}$. Substituting $c = a + b$, one has $c\mathbf{x} + c\mathbf{y} = (a + b)\mathbf{x} + (a + b)\mathbf{y}$. Via axiom 8, $(a + b)\mathbf{x} + (a + b)\mathbf{y} = (a\mathbf{x} + b\mathbf{x}) + (a\mathbf{y} + b\mathbf{y})$. Finally, by axiom 2, we can concatenate terms to obtain $(a\mathbf{x} + b\mathbf{x}) + (a\mathbf{y} + b\mathbf{y}) = a\mathbf{x} + a\mathbf{y} + b\mathbf{x} + b\mathbf{y}$, as required. ■

Exercise 13.

We'll go through and manually check each axiom:

1. For any $(a_1, a_2) \in V$ and $(b_1, b_2) \in V$, we have that $(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_1 b_1)$ and $(b_1, b_2) + (a_1, a_2) = (b_1 + a_1, b_1 a_1)$. However, since addition and multiplication of real scalars commute, $a_1 + b_1 = b_1 + a_1$ and $a_1 b_1 = b_1 a_1$. Thus, $(a_1, a_2) + (b_1, b_2) = (b_1, b_2) + (a_1, a_2)$, as required.

2. To check associativity of addition, we'll declare $(a_1, a_2), (b_1, b_2)$, and (c_1, c_2) as elements of V and check the following:

$$(a_1, a_2) + ((b_1, b_2) + (c_1, c_2)) \stackrel{?}{=} ((a_1, a_2) + (b_1, b_2)) + (c_1, c_2)$$

$$\Rightarrow (a_1, a_2) + (b_1 + c_1, b_2 c_2) \stackrel{?}{=} (a_1 + b_1, a_2 b_2) + (c_1, c_2)$$

$$\Rightarrow (a_1 + (b_1 + c_1), a_2(b_2 c_2)) \stackrel{?}{=} ((a_1 + b_1) + c_1, (a_2 b_2) c_2)$$

by associativity, we're now examining

$$(a_1 + b_1 + c_1, a_2 b_2 c_2) \stackrel{?}{=} (a_1 + b_1 + c_1, a_2 b_2 c_2)$$

whence we see that the two are identical. Hence, associativity of addition holds.

3. For any $(a_1, a_2) \in V$, the element $(0, 1)$ satisfies the condition that $(a_1, a_2) + (0, 1) = (a_1 + 0, a_2) = (a_1, a_2)$. Thus, $(1, 0)$ acts as the zero element in V.

4. For any $(a_1, a_2) \in V$, the element $(1 - a_1, 0)$ satisfies the condition that $(a_1, a_2) + (1 - a_1, 0) = (a_1 + (1 - a_1), 0) = (1, 0)$. Thus, given any element of V, we can identify an element so that, when added to the original element, we obtain the zero element.

5. Given that $c(a_1, a_2) = (ca_1, a_2)$, if we take $c = 1$ then $1 \cdot (a_1, a_2) = (1 \cdot a_1, a_2) = (a_1, a_2)$.

6. Given some $c_1 \in \mathbb{R}$ and $c_2 \in \mathbb{R}$, we'll examine whether the following is true:

$$(c_1c_2)(a_1, a_2) \stackrel{?}{=} c_1 \left(c_2(a_1, a_2)\right)$$

The LHS gives $(c_1c_2a_1, a_2)$ whilst the RHS gives $c_1(c_2a_1, a_2) = (c_1c_2a_1, a_2)$. Thus, LHS = RHS and thus axiom 6 is satisfied.

7. Given $(a_1, a_2) \in V$ and $(b_1, b_2) \in V$ as well as some $c \in \mathbb{R}$, we'll examine whether the following is true:

$$c\left((a_1, a_2) + (b_1, b_2)\right) \stackrel{?}{=} c(a_1, a_2) + c(b_1, b_2)$$

The LHS gives $c\left((a_1, a_2) + (b_1, b_2)\right) = c(a_1 + b_1, a_2b_2) = (ca_1 + cb_1, a_2b_2)$ whilst the RHS gives $c(a_1, a_2) + c(b_1, b_2) = (ca_1, a_2) + (cb_1, b_2) = (ca_1 + cb_1, a_2b_2)$. Thus, it's clear that the LHS and RHS are identical so axiom 7 is satisfied.

8. Given any $c_1 \in \mathbb{R}$ and $c_2 \in \mathbb{R}$ and $(a_1, a_2) \in V$, we'll examine:

$$(c_1 + c_2)(a_1, a_2) \stackrel{?}{=} c_1(a_1, a_2) + c_2(a_1, a_2)$$

The LHS gives $(c_1 + c_2)(a_1, a_2) = ((c_1 + c_2)a_1, a_2) = (c_1a_1 + c_2a_1, a_2)$ whilst the RHS gives $c_1(a_1, a_2) + c_2(a_1, a_2) = (c_1a_1, a_2) + (c_2a_1, a_2) = (c_1a_1 + c_2a_1, a_2)$. Since the LHS and RHS are identical, we see that axiom 8 is satisfied.

Since V satisfies each of the eight axioms of a vector space, V is thus a vector space.

Exercise 20.

Proof. Once again, we'll manually verify each axiom:

1. $\{a_n\} + \{b_n\} = \{a_n + b_n\} = \{b_n + a_n\} = \{b_n\} + \{a_n\}$.
2. $\{a_n\} + (\{b_n\} + \{c_n\}) = \{a_n\} + \{b_n + c_n\} = \{a_n + (b_n + c_n)\} = \{(a_n + b_n) + c_n\} = \{a_n + b_n\} + \{c_n\} = (\{a_n\} + \{b_n\}) + \{c_n\}$.
3. $\{a_n\} + \{0\} = \{a_n + 0\} = \{a_n\}$, where $\{0\}_{n \geq 1}$ is the zero sequence ($a_n = 0$ for all $n \geq 1$).
4. $\{a_n\} + \{-a_n\} = \{a_n - a_n\} = \{0\}$.
5. $1 \cdot \{a_n\} = \{1 \cdot a_n\} = \{a_n\}$.
6. Given $t \in \mathbb{R}$ and $s \in \mathbb{R}$, $(ts)\{a_n\} = \{(ts)a_n\} = \{t(sa_n)\} = t\left(s\{a_n\}\right)$.
7. Given $t \in \mathbb{R}$,
$t\left(\{a_n\} + \{b_n\}\right) = t\{a_n + b_n\} = \{t(a_n + b_n)\} = \{ta_n + tb_n\} = \{ta_n\} + \{tb_n\} = t\{a_n\} + t\{b_n\}$.
8. Given $t \in \mathbb{R}$ and $s \in \mathbb{R}$, $(t + s)\{a_n\} = \{(t + s)a_n\} = \{ta_n + sa_n\} = \{ta_n\} + \{sa_n\} = t\{a_n\} + s\{a_n\}$.

whence V is a vector space. ∎

§1.3

Exercise 1.

(a) FALSE; (b) FALSE; (c) TRUE; (d) FALSE; (e) TRUE; (f) FALSE; (g) FALSE

Exercise 6.

Proof. The trace of a matrix is just the sum of its diagonal elements, so

$$\mathrm{Tr}(a\mathbf{A} + b\mathbf{B}) = \sum_{i=1}^{n}(a\mathbf{A} + b\mathbf{B})_{ii} = \sum_{i=1}^{n} a\mathbf{A}_{ii} + b\mathbf{B}_{ii} = a\sum_{i=1}^{n}\mathbf{A}_{ii} + b\sum_{i=1}^{n}\mathbf{B}_{ii} = a\mathrm{Tr}(\mathbf{A}) + b\mathrm{Tr}(\mathbf{B}).$$ ■

Exercise 8.

(a) YES

- CONTAINS ZERO: Let $(a_1, a_2, a_3) = (0, 0, 0)$. Certainly $0 = 3(0)$ and $0 = -0$.
- CLOSED UNDER ADDITION: Take (a_1, a_2, a_3) where $a_1 = 3a_2$ and $a_3 = -a_2$ and (b_1, b_2, b_3) where $b_1 = 3b_2$ and $b_3 = -b_2$. It holds that $(a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$, whence $a_1 + b_1 = 3a_2 + 3b_2 = 3(a_2 + b_2)$ and $a_3 + b_3 = -a_2 - b_2 = -(a_2 + b_2)$.
- CLOSED UNDER SCALAR MULTIPLICATION: Given $\lambda \in \mathbb{R}$, $\lambda(a_1, a_2, a_3) = (\lambda a_1, \lambda a_2, \lambda a_3)$, whence $\lambda a_1 = \lambda(3a_2) = 3(\lambda a_2)$ and $\lambda a_3 = \lambda(-a_2) = -(\lambda a_2)$.

(b) NO; W_2 doesn't contain zero because $a_1 = a_3 + 2 \Rightarrow 0 = 0 + 2$, which is false.

(c) YES

- CONTAINS ZERO: Given $(a_1, a_2, a_3) = (0, 0, 0)$, it's obvious that $2(0) - 7(0) + 0 = 0$.
- CLOSED UNDER ADDITION: Given (a_1, a_2, a_3) where $2a_1 - 7a_2 + a_3 = 0$ and (b_1, b_2, b_3) where $2b_1 - 7b_2 + b_3 = 0$, it holds that $(a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$. Furthermore, $2(a_1 + b_1) - 7(a_2 + b_2) + (a_3 + b_3) = 2a_1 + 2b_1 - 7a_2 - 7b_2 + a_3 + b_3 =$ $(2a_1 - 7a_2 + a_3) + (2b_1 - 7b_2 + b_3) = 0 + 0 = 0$.
- CLOSED UNDER SCALAR MULTIPLICATION: Given some $\lambda \in \mathbb{R}$ and (a_1, a_2, a_3), it holds that $\lambda(a_1, a_2, a_3) = (\lambda a_1, \lambda a_2, \lambda a_3)$. Moreover, $2(\lambda a_1) - 7(\lambda a_2) + \lambda a_3 = \lambda(2a_1 - 7a_2 + a_3) = \lambda(0) = 0$.

(d) YES

- CONTAINS ZERO: $(a_1, a_2, a_3) = (0, 0, 0) \Rightarrow 0 - 4(0) - 0 = 0$.
- CLOSED UNDER ADDITION: $(a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$. Moreover, $(a_1+b_1)-4(a_2+b_2)-(a_3+b_3) = a_1+b_1-4a_2-4b_2-a_3-b_3 = (a_1-4a_2-a_3)+(b_1-4b_2-b_3) = 0+0 = 0$.
- CLOSED UNDER SCALAR MULTIPLICATION: Given $\lambda \in \mathbb{R}$, one has that $\lambda(a_1, a_2, a_3) = (\lambda a_1, \lambda a_2, \lambda a_3)$. Moreover, $\lambda a_1 - 4(\lambda a_2) - \lambda a_3 = \lambda(a_1 - 4a_2 - a_3) = 0$.

(e) NO; W_5 doesn't contain zero because if $(a_1, a_2, a_3) = (0, 0, 0)$ then $a_1 + 2a_2 - 3a_3 = 1 \Rightarrow 0 = 1$, which is false.

(f) NO; W_6 isn't closed under addition. For example, $\left(0, \sqrt{2}, 1\right) \in W_6$ and $\left(\sqrt{2}, \sqrt{\frac{10}{3}}, 0\right) \in W_6$, but $\left(\sqrt{2}, \sqrt{2} + \sqrt{\frac{10}{3}}, 1\right) \notin W_6$ because $5\left(\sqrt{2}\right)^2 - 3\left(\sqrt{2} + \sqrt{\frac{10}{3}}\right)^2 + 6(1)^2 \neq 0$.

Exercise 21.

Proof. We verify three conditions:

1. CONTAINS ZERO: Let $\{a_n\} = \{0\}_{n\geq 1}$. Obviously, $\lim_{n\to\infty} 0 = 0$, and so the zero sequence has a finite limit.

2. CLOSED UNDER ADDITION: Take $\{a_n\}$ and $\{b_n\}$ so that $\lim_{n\to\infty} a_n = L_1$ and $\lim_{n\to\infty} b_n = L_2$. By addition, we have that $\{a_n\} + \{b_n\} = \{a_n + b_n\}$ for which it follows that $\lim_{n\to\infty}(a_n + b_n) = (\lim_{n\to\infty} a_n) + (\lim_{n\to\infty} b_n) = L_1 + L_2$, which is a finite limit.

3. CLOSED UNDER SCALAR MULTIPLICATION: Given $t \in \mathbb{R}$, one has that $t\{a_n\} = \{ta_n\}$ and that $\lim_{n\to\infty} ta_n = t\lim_{n\to\infty} a_n = tL_1$, a finite limit.

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