MATH 340 - Spring 2017

Homework 6

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Problems Assigned:

- §5.1: 3d, 4c, 4d, 9, 17
- §5.2: 2b, 2d, 3a, 3b, 7, 11

§5.1

3.

(d) (i) We need to solve the equation $\det(A - \lambda I) = 0$ for λ, in which case we compute

$$\det\begin{pmatrix} 2-\lambda & 0 & -1 \\ 4 & 1-\lambda & -4 \\ 2 & 0 & -1-\lambda \end{pmatrix} = 0.$$

Computing the Laplace expansion along the second column (the one with the most zeroes), we obtain

$$-\lambda^3 + 2\lambda^2 - \lambda = 0$$
$$\implies -\lambda(\lambda-1)^2 = 0,$$

for which $\lambda_1 = 0$ and $\lambda_2 = \lambda_3 = 1$. (ii) We now need to calculate the eigenvectors of A associated to each λ_i. Doing so requires us to determine all vectors $\vec{x}$ such that $\vec{x} \in \mathrm{Null}(A - \lambda_i I)$.

- For $\lambda = 0$, we have that $A - \lambda I = A$, and so we have that $A\vec{x} = \vec{0} \implies \begin{pmatrix} 2 & 0 & -1 \\ 4 & 1 & -4 \\ 2 & 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \vec{0}$.
  This gives the following system of equations:
  $$\begin{aligned} 2x_1 - x_3 &= 0 \\ 4x_1 + x_2 - 4x_3 &= 0 \end{aligned}$$
  The first row gives $2x_1 = x_3$ for which we have $x_1 = 1$ and $x_3 = 2$. Substituting these into the second row gives $4 + x_2 - 8 = 0$, for which $x_2 = 4$. Thus, $\vec{x}_1 \in \mathrm{Span}\left\{\begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}\right\}$ is an eigenvector associated to $\lambda = 0$.
- For $\lambda = 1$, we have that $\vec{x}$ is an eigenvector if $\begin{pmatrix} 1 & 0 & -1 \\ 4 & 0 & -4 \\ 2 & 0 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \vec{0}$. Of course, the rows of $A - I$ are constant multiples of each other, so choosing $x_1 = x_3 = 1$ and $x_2 = 0$ is sufficient. Thus, $\vec{x}_2 \in \mathrm{Span}\left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right\}$ is an eigenvector associated to $\lambda = 1$.
- For $\lambda = 1$ (again), we need to find a *generalized* eigenvector, which requires us to solve $(A - I)\vec{x}_3 = \vec{x}_2$. Thus, $\begin{pmatrix} 1 & 0 & -1 \\ 4 & 0 & -4 \\ 2 & 0 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, for which we obtain $\vec{x}_3 \in \mathrm{Span}\left\{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right\}$ being the other eigenvector associated to $\lambda = 1$.

(iii) Since $\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$ is linearly independent and spans $\mathbf{R}^3$, we have that a basis for $\mathbf{R}^3$ is the set $\left\{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}\right\}$. (iv) If we choose $Q = (\vec{x}_1 \mid \vec{x}_2 \mid \vec{x}_3) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 4 & 1 \\ 1 & 2 & 0 \end{pmatrix}$, we have that

$Q^{-1}AQ =$
$\begin{pmatrix} 2 & 0 & -1 \\ -1 & 0 & 1 \\ 4 & 1 & -4 \end{pmatrix}\begin{pmatrix} 2 & 0 & -1 \\ 4 & 1 & -4 \\ 2 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 4 & 1 \\ 1 & 2 & 0 \end{pmatrix} =$
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = D$, as required.

4.

(c) If we examine how T acts on the basis elements of $\mathbf{R}^3$, we see that

$$T(\vec{e}_1) = -4\vec{e}_1 + 6\vec{e}_2 + 6\vec{e}_3$$
$$T(\vec{e}_2) = 3\vec{e}_1 - 7\vec{e}_2 - 6\vec{e}_3$$
$$T(\vec{e}_3) = -6\vec{e}_1 + 12\vec{e}_2 + 11\vec{e}_3,$$

whence we obtain the matrix

$$[T]_\beta = \begin{pmatrix} -4 & 3 & -6 \\ 6 & -7 & 12 \\ 6 & -6 & 11 \end{pmatrix}.$$

Computing the eigenvalues, we have that $\lambda_1 = 2$ and $\lambda_2 = \lambda_3 = -1$. If $[T]_\beta$ is to be diagonal, then we require that every element off the diagonal is equal to zero. Thus, in particular we have that, for the first column we must solve the system

$$\begin{aligned} -4a + 3b - 6c &= -4 \\ 6a - 7b + 12c &= 0 \\ 6a - 6b + 11c &= 0 \end{aligned}$$

which, when solved via direct elimination, gives

$$\begin{aligned} a &= 10 \\ b &= -12 \\ c &= -12 \end{aligned}$$

Performing the same calculation for rows 2 and 3 of $[T]_\beta$, we generate

$$\begin{aligned} a &= -21/2 \\ b &= 28 \\ c &= -21 \end{aligned}$$

and

$$\begin{aligned} a &= -33 \\ b &= 66 \\ c &= 55 \end{aligned}$$

Thus, the ordered basis for $\mathbf{R}^3$ we're after is the set $\beta = \left\{ \begin{pmatrix} 10 \\ -12 \\ -12 \end{pmatrix}, \begin{pmatrix} -\frac{21}{2} \\ 28 \\ 21 \end{pmatrix}, \begin{pmatrix} -33 \\ 66 \\ 55 \end{pmatrix} \right\}$. Indeed, β forms a basis due to the fact that its elements are linearly independent and span $\mathbf{R}^3$. We'll show this directly:

- Linear Independence: The system

$$A \begin{pmatrix} 10 \\ -12 \\ -12 \end{pmatrix} + B \begin{pmatrix} -\frac{21}{2} \\ 28 \\ 21 \end{pmatrix} + C \begin{pmatrix} -33 \\ 66 \\ 55 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

corresponds to the matrix

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

for which $A = B = C = 0$, and thus the linear combination above can only be satisfied trivially.

- Spans: We compute $\text{Span}(\beta) = \left\{ \begin{pmatrix} 10A - \frac{21}{2}B - 33C \\ -12A + 28B + 66C \\ -12A + 21B + 55C \end{pmatrix} \middle| \; A, B, C \in \mathbf{R} \right\}$, for which we see that, given any $\vec{u} \in \mathbf{R}^3$, it holds that $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ where $u_1, u_2, u_3 \in \mathbf{R}$. Since $A, B, C \in \mathbf{R}$, it follows that $10A - \frac{21}{2}B - 33C \in \mathbf{R}$, $-12A + 28B + 66C \in \mathbf{R}$, and $-12A + 21B + 55C \in \mathbf{R}$. It's a natural consequence, then, that for any $\vec{u} \in \mathbf{R}^3$, $\vec{u} \in \text{Span}(\beta)$. So $\mathbf{R}^3 \subseteq \text{Span}(\beta)$. Alternatively, any $\vec{u} \in \text{Span}(\beta)$ is written as a triplet of real numbers, for which we have that $\vec{u} \in \mathbf{R}^3$, and so $\text{Span}(\beta) \subseteq \mathbf{R}^3$. Thus, $\text{Span}(\beta) = \mathbf{R}^3$, as required.

(d) Applying T to the standard basis elements of $P_1(\mathbf{R})$, (that is, $\beta = \{1, x\}$)we have that

$$T(1) = 2x + 2$$

$$T(x) = -6x - 6$$

for which we have that

$$[T]_\beta = \begin{pmatrix} 2 & -6 \\ 1 & -6 \end{pmatrix}.$$

Computing the eigenvalues directly, we have that $\det \begin{pmatrix} 2 - \lambda & -6 \\ 1 & -6 - \lambda \end{pmatrix} = 0 \implies \lambda^2 + 4\lambda - 6 = 0$, for which $\lambda_1 = -2 + \sqrt{10}$ and $\lambda_2 = -2 - \sqrt{10}$. If T is to be diagonal, then we require a basis of $P_1(\mathbf{R})$ so that $[T]_{11} = \lambda_1$, $[T]_{22} = \lambda_2$, and $[T]_{ij} = 0$ everywhere else. Applying the exact same methodology as in part (c), we must solve the following sets of simultaneous equations,

$$S_1 : \begin{cases} -6a + 2b &= 0 \\ -6a + b &= -2 + \sqrt{10} \end{cases}$$

$$S_2 : \begin{cases} -6a + 2b &= -2 - \sqrt{10} \\ -6a + b &= 0 \end{cases}$$

which can easily solved via elimination. Doing so, we have that S_1 is satisfied by $\left(\frac{2\sqrt{10}-4}{-6}, 2 - \sqrt{10}\right)$ and S_2 is satisfied by $\left(\frac{2+\sqrt{10}}{-6}, -2 - \sqrt{10}\right)$. Thus, we have that a suitable basis is

$$\beta = \left\{ \left(\frac{2\sqrt{10} - 4}{-6}\right) x + \left(2 - \sqrt{10}\right), \left(\frac{2 + \sqrt{10}}{-6}\right) x + \left(-2 - \sqrt{10}\right) \right\}.$$

9.

Proof. Let M be an $n \times n$ upper-triangular matrix, and, without loss of generality, let the diagonal entries of M be denoted m_{ii}. By definition, computing the eigenvalues of M requires us to solve the equation $\det(M - \lambda I) = 0$, where I is the identity matrix. The diagonal entries of $M - \lambda I$ are then given by $m_{ii} - \lambda$. Since the determinant of an upper-triangular matrix is simply the product of the elements down its diagonal, it follows that

$$\det(M - \lambda I) = \prod_{i=1}^{n} (m_{ii} - \lambda).$$

Thus, if $\det(M - \lambda I) = 0$, then

$$\prod_{i=1}^{n} (m_{ii} - \lambda) = 0 \implies (m_{11} - \lambda)(m_{22} - \lambda) \ldots (m_{nn} - \lambda) = 0,$$

which is satisfied if λ is equal to any one of $m_{11}, m_{22}, \ldots, m_{nn}$ since, if any one of the factors is zero, the entire product is zero. Thus, we must have that $\lambda_i = m_{ii}$, as required. **QED.**

17.

(a) *Proof.* Applying T to the standard basis elements of $\mathbf{R}^{2\times 2}$, we have that

$$T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$T \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$T \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

which, when expressed as linear combinations of the standard basis elements of $\mathbf{R}^{2\times 2}$, gives the matrix

$$[T] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We can swap rows 2 and 3 to obtain the diagonal matrix

$$[\tilde{T}] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

for which computing the eigenvalues of $[\tilde{T}]$ requires us to solve

$$\det \begin{pmatrix} 1-\lambda & 0 & 0 & 0 \\ 0 & 1-\lambda & 0 & 0 \\ 0 & 0 & 1-\lambda & 0 \\ 0 & 0 & 0 & 1-\lambda \end{pmatrix} = 0.$$

Of course, $[\tilde{T}] - \lambda I$ is diagonal, so its determinant is simply the product of its diagonal elements, so we obtain $-(\lambda - 1)^4 = 0$, where the sign change was induced through swapping rows. Of course, this equation is only satisfied for $\lambda = \pm 1$. **QED.**

(b) A is an eigenvector of T if $T(A) = \lambda A$, where λ is an eigenvalue. Since we already know that $\lambda = \pm 1$, we have that A is an eigenvector of T if $T(A) = \pm A$. Of course, knowing the definition of T yields $A^\top = \pm A$. If $A^\top = A$, then A is symmetric. Alternatively, if $A^\top = -A$, then A is skew-symmetric. Thus, the eigenvectors of T are any $n \times n$ symmetric or skew-symmetric matrices.

(c) If we take the set $\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$, we have the desired result.

(d) Let E^{ij} be the matrix with the entry at position $E_{ij} = 1$, and 0 otherwise. Taking

$$\beta = \{E_{ii}\} \cup \{E_{ij} + E_{ji}\} \cup \{E_{ij} - E_{ji}\}$$

as a basis for $T(A) = A^\top$ is sufficient.

§5.2

2.

(b) We compute the eigenvalues of A so as to obtain $\det \begin{pmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{pmatrix} = 0 \Longrightarrow \lambda^2 - 2\lambda - 8 = 0$, for which $\lambda_1 = 4$ and $\lambda_2 = -2$. Thus, A has 2 distinct eigenvalues, and thus it is diagonalizable. Now, our task is dredging up a matrix S such that $S^{-1}AS = D$. We'll first compute the eigenvectors associated to each eigenvalue:

- For $\lambda = 4$, we solve $\begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. The rows of the coefficient matrix are scalar multiples of one another (they differ by a sign change), so we can choose $x_1 = x_2$ for which we have that $\vec{x}_1 \in \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ is an eigenvector.
- For $\lambda = -2$, we solve $\begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. The rows of the coefficient matrix are identical, so we can choose $x_1 = -x_2$ for which we have that $\vec{x}_2 \in \operatorname{Span}\left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$ is an eigenvector.

Thus, we've found that $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ are eigenvectors of A, so we'll let $S = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$. We see that S is invertible since $\det(S) = -2 \neq 0$, and $S^{-1} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$. It remains for us to check:

$$\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} -2 & 0 \\ 0 & 4 \end{pmatrix}}_{D},$$

as required.

(d) As is typical, we'll begin by computing the eigenvalues of A, thus solving

$$\det \begin{pmatrix} 7-\lambda & -4 & 0 \\ 8 & -5-\lambda & 0 \\ 6 & -6 & 3-\lambda \end{pmatrix} = 0.$$

Expanding along the 3$^{\text{rd}}$ column, we have

$$\det(A - \lambda I) = (3-\lambda) \det \begin{pmatrix} 7-\lambda & -4 \\ 8 & -5-\lambda \end{pmatrix}$$

$$= (3-\lambda)(\lambda+1)(\lambda-3) = 0.$$

This is only satisfied if $\lambda = 3$ or $\lambda = -1$. Since $A \in \mathbf{R}^{3\times 3}$ and has three *non-distinct* eigenvalues (i.e. $\lambda = 3$ has multiplicity 2), it's not diagonalizable.

3.

(a) Applying T to the elements of the standard basis $\beta = \{1, x, x^2, x^3\}$, we have

$$T(1) = 0$$

$$T(x) = 1$$

$$T\left(x^2\right) = 2x + 2$$

$$T\left(x^3\right) = 3x^2 + 6x,$$

for which the matrix $[T]$ is given by

$$[T] = \begin{pmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Computing the eigenvalues yields the characteristic polynomial $\lambda^4 = 0$, for which the only root is $\lambda = 0$ (with multiplicity 4). Now, we have to solve $T(f(x)) = 0$, for which it follows, by the definition of T, that $f'(x) + f''(x) = 0$. This can easily be done using basic techniques to solve second-order, linear, homogeneous ODEs; doing so, we obtain

$f(x) = A + Be^{-x}$, for some real constants A and B. So $\text{Null}(T - \lambda I) = \text{Null}(T)$ consists of only exponentials of the form given in the previous step, and so $\{1, e^{-x}\}$ is a basis for E_λ, so $\dim(E_\lambda) = 2$. As a consequence, we can't write a general element of $P_3(\mathbf{R})$ as a linear combination of the basis elements 1 and e^{-x}, so T isn't diagonalizable.

(b) We have that

$$T(1) = x^2$$
$$T(x) = x$$
$$T\left(x^2\right) = 1,$$

so as to obtain the matrix

$$[T] = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Computing the eigenvalues yields the characteristic polynomial $(1 - \lambda)\left(\lambda^2 - 1\right) = 0$, for which $\lambda = \pm 1$ is a root. Now, it remains to examine $T(f(x)) = \pm f(x) \Longrightarrow cx^2 + bx + a = \pm(ax^2 + bx + c)$. This gives rise to two cases: if $cx^2 + bx + a = ax^2 + bx + c$, then by direct comparison, we see that $c = a$ is necessary. Alternatively, if $cx^2 + bx + a = -ax^2 - bx - c$, then $c = -a$ and $b = -b$ are necessary. Of course, the only real value of b that satisfies $b = -b$ is $b = 0$, so we gather that

- The eigenvectors associated to the eigenvalue 1 are the quadratic polynomials $f(x) = ax^2 + bx + a = a(x^2 + 1) + bx$.
- The eigenvectors associated to the eigenvalue -1 are the quadratic polynomials $f(x) = ax^2 - a = a(x^2 - 1)$.

Thus, a candidate for our new basis would be $\bar{\beta} = \left\{a(x^2 + 1) + bx, a(x^2 - 1)\right\}$. We see that, since the sum

$$A_1\left(a(x^2 + 1) + bx\right) + A_2\left(a(x^2 - 1)\right) = 0$$

is satisfied if and only if $A_1 = A_2 = 0$, we have linear independence. Secondly, $\text{Span}(\bar{\beta}) \subseteq P_2(\mathbf{R})$ and $P_2(\mathbf{R}) \subseteq \text{Span}(\bar{\beta})$, so $\bar{\beta}$ spans, and thus $\bar{\beta}$ forms a basis for $P_2(\mathbf{R})$ such that $[T]$ is diagonal.

7. If A is diagonalizable, then there exists S such that $S^{-1}AS = D$, where the eigenvectors associated to the eigenvalues of A are the columns of S and D is the matrix with A's eigenvalues down the diagonal and 0 everywhere off the diagonal. Of course, if $S^{-1}AS = D$, then solving for A, we have that $A = SDS^{-1}$. Thus,

$$A^n = \left(SDS^{-1}\right)^n = \underbrace{\left(SDS^{-1}\right)\left(SDS^{-1}\right)\ldots\left(SDS^{-1}\right)}_{n \text{ times}}.$$

But $S^{-1}S = I$, so this yields

$$A^n = SD^nS^{-1}.$$

Thus, it suffices to determine the eigenvalues and eigenvectors of A in order to compute A^n. Computing the eigenvalues of A, we have that $\lambda_1 = 5$ and $\lambda_2 = -1$ for which the eigenvectors are $\vec{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\vec{x}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$, respectively. Thus, we have that $D = \begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix}$ and $S = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} \Longrightarrow S^{-1} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix}$. Thus,

$$\boxed{A^n = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 5^n & 0 \\ 0 & (-1)^n \end{pmatrix}\begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix}}$$

$$= \boxed{\begin{pmatrix} \frac{5^n + 2(-1)^n}{3} & \frac{2(5^n)}{3} \\ \frac{5^n - (-1)^n}{3} & \frac{2(5^n) + (-1)^n}{3} \end{pmatrix}},$$

where $n \in \mathbf{Z}_{>0}$.

11.

(a) *Proof.* If A and T are similar, then we have that, via a previous exercise, that $\text{Tr}(A) = \text{Tr}(T)$. Since A and T are related by the equation $A = S^{-1}TS$, for some change of coordinate matrix S, and since A has distinct eigenvalues, T is a diagonal matrix and has the eigenvalues of A down its diagonal. Thus,

$$\text{Tr}(T) = \underbrace{(\lambda_1 + \ldots + \lambda_1)}_{m_1 \text{ terms}} + \underbrace{(\lambda_2 + \ldots + \lambda_2)}_{m_2 \text{ terms}}$$
$$+ \ldots + \underbrace{(\lambda_k + \ldots + \lambda_k)}_{m_k \text{ terms}}$$
$$= \sum_{i=1}^{k} m_i\lambda_i.$$

Since $\text{Tr}(T) = \text{Tr}(A)$, the result follows. **QED.**

(b) *Proof.* By a previous exercise, since A and T are similar, $\det(A) = \det(T)$. Since, as described in part (a), T has the eigenvalues of A down its diagonal, and T is upper-triangular, we have that $\det(T) = \prod_{i=1}^{n} T_{ii}$. So $\det(T) = \underbrace{(\lambda_1\lambda_1\ldots\lambda_1)}_{m_1 \text{ factors}}\underbrace{(\lambda_2\lambda_2\ldots\lambda_2)}_{m_2 \text{ factors}}\ldots\underbrace{(\lambda_k\lambda_k\ldots\lambda_k)}_{m_k \text{ factors}} = \lambda_1^{m_1}\lambda_2^{m_2}\ldots\lambda_k^{m_k}$. And, since $\det(T) = \det(A)$, the result follows. **QED.**