MATH 340 - Spring 2017

Homework 4

Solutions written by Alex Menendez (1438704)

Problems Assigned:

- §2.2: 4, 16
- §2.3: 3, 13
- §2.4: 2a, 2c, 2e, 4
- §2.5: 3b, 3d, 10

§2.2

4. Let's first examine how T acts on the elements of β (i.e. how T maps each basis element of $\mathbf{M}_{2\times 2}(\mathbb{R})$). We'll denote the elements of β as $\beta = \{e_1, e_2, e_3, e_4\}$.

$$T\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} = 1 + 0x + 0x^2, \quad T\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} = 1 + 0x + 1x^2, \quad T\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} = 0 + 0x + 0x^2, \quad T\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} = 0 + 2x + 0x^2$$

Thus, when $T(e_j)$ is written as a linear combination of the elements of γ, the coefficients of the resultant polynomial give the entries in the j^{th} column of $[T]_\beta^\gamma$. Thus,

$$\boxed{[T]_\beta^\gamma = \begin{pmatrix}1 & 1 & 0 & 0\\ 0 & 0 & 0 & 2\\ 0 & 1 & 0 & 0\end{pmatrix}}$$

16.

Proof. Let $\beta = \{v_1, \ldots, v_n\}$ be a basis for V and let $\gamma = \{w_1, \ldots, w_n\}$ be a basis for W. Note that, since β and γ both contain n elements, $\dim V = \dim W$, as prescribed. Since $T : V \to W$, then – in particular – given some $v \in V$, and $w \in W$, $T(v) = w$. Since γ is a basis for W, then any $w \in W$ can be written uniquely as $w = \alpha_1 w_1 + \ldots + \alpha_n w_n$ and so $T(v) = \alpha_1 w_1 + \ldots + \alpha_n w_n$. Thus, for any $v_i \in V$, $T(v_i)$ has a unique representation as a linear combination of $w_1, \ldots, w_n$. So, we write $T(v_i) = \alpha_{1i} w_1 + \alpha_{2i} w_2 + \ldots + \alpha_{ni} w_i$. More explicitly,

$$T(v_1) = \alpha_{11} w_1 + \alpha_{21} w_2 + \ldots + \alpha_{n1} w_n$$

$$T(v_2) = \alpha_{12} w_1 + \alpha_{22} w_2 + \ldots + \alpha_{n2} w_n$$

$$\vdots$$

$$T(v_n) = \alpha_{1n} w_1 + \alpha_{2n} w_2 + \ldots + \alpha_{nn} w_n$$

such that the coefficients of $T(v_j)$ identify the j^{th} column of $[T]_\beta^\gamma$. Thus, examining the matrix $[T]_\beta^\gamma$, one has

$$[T]_\beta^\gamma = \begin{pmatrix}\alpha_{11} & \alpha_{12} & \ldots & \alpha_{1n}\\ \alpha_{21} & \alpha_{22} & \ldots & \alpha_{2n}\\ \vdots & & \ddots & \\ \alpha_{n1} & \alpha_{n2} & \ldots & \alpha_{nn}\end{pmatrix}$$

Thus, if $[T]_\beta^\gamma$ is to be diagonal, then we require that

$$\alpha_{ij} = \begin{cases} a \neq 0 & \text{if } i = j \\ 0 & \text{otherwise}\end{cases}$$

In which case $T(v_i) = \alpha_i w_i$ for some arbitrary scalar α_i. The problem is now reduced to finding out how to configure γ such that this is true. We'll proceed as follows: write $T(v_{k+1}) = w_{k+1}$. We see that $\{w_1, \ldots, w_k\}$ forms a basis for $\mathcal{N}(T)$ and $\{w_{k+1}, \ldots, w_n\}$ forms a basis for $\mathcal{R}(T)$ and so, by the rank-nullity theorem we can extend to a basis $\gamma = \{w_1, \ldots, w_n\}$ for W, which justifies what we had originally conjectured. $\square$

§2.3

3.

(a) We'll go through and compute each matrix:

- To compute $[U]_\beta^\gamma$, we'll first observe how U acts on the basis elements of $P_2(\mathbb{R})$. Thus,

$$U(1) = (1,0,1) = e_1 + e_3$$

$$U(x) = (1,0,-1) = e_1 - e_3$$

$$U\left(x^2\right) = (0,1,0) = e_2$$

Reading off the coefficients of each linear combination as the columns of $[U]_\beta^\gamma$, we have

$$\boxed{[U]_\beta^\gamma = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}}$$

- To compute $[T]_\beta$, we'll observe how T acts on the basis elements of $P_2(\mathbb{R})$:

$$T(1) = (1)'(3+x) + 2 = 2$$

$$T(x) = (x)'(3+x) + 2x = 3 + x + 2x = 3 + 3x$$

$$T\left(x^2\right) = \left(x^2\right)'(3+x) + 2x^2 = 2x(3+x) + 2x^2 = 6x + 2x^2 + 2x^2 = 4x^2 + 6$$

Reading off the coefficients of each linear combination as the columns of $[T]_\beta$, we have

$$\boxed{[T]_\beta = \begin{pmatrix} 2 & 3 & 6 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}}$$

- To compute $[UT]_\beta^\gamma$, we'll observe how T acts on each basis element of $P_2(\mathbb{R})$ and then how U acts on each resulting element. We already know $T(1), T(x)$, and $T\left(x^2\right)$ from before, so applying U to each result gives:

$$U(T(1)) = U(2) = 2e_1 + 2e_3$$

$$U(T(x)) = U(3+3x) = 6e_1$$

$$U\left(T\left(x^2\right)\right) = U\left(6 + 4x^2\right) = 6e_1 + 4e_2 + 6e_3$$

Reading off the coefficients of each linear combination identifies each column of $[UT]_\beta^\gamma$, so

$$\boxed{[UT]_\beta^\gamma = \begin{pmatrix} 2 & 6 & 6 \\ 0 & 0 & 4 \\ 2 & 0 & 6 \end{pmatrix}}$$

By Theorem 2.11, the matrix $[UT]_\beta^\gamma$ is simply the result of multiplying the matrix representations of U and T together, and – as expected – one has

$$\underbrace{\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}}_{[U]_\beta^\gamma} \underbrace{\begin{pmatrix} 2 & 3 & 6 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}}_{[T]_\beta} = \underbrace{\begin{pmatrix} 2 & 6 & 6 \\ 0 & 0 & 4 \\ 2 & 0 & 6 \end{pmatrix}}_{[UT]_\beta^\gamma}$$

13. We'll first prove the proposition that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$.

Proof. Denote $A = a_{ij}$ and $B = b_{ij}$. From the definition of matrix multiplication, one has that $(AB)_{ii} = \sum_{k=1}^{n} a_{ik}b_{ki}$, which allows us to identify the diagonal entries of AB. Since the trace is simply defined as the sum of AB's diagonal entries, we have that $\operatorname{Tr}(AB) = \sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk}b_{kj}$. In a similar vein, $(BA)_{ii} = \sum_{k=1}^{n} b_{ik}a_{ki}$, which identifies the diagonal entries of BA. Thus, as before, $\operatorname{Tr}(BA) = \sum_{j=1}^{n}\sum_{k=1}^{n} b_{jk}a_{kj}$. We can see, rather heuristically, that these two sums are identical – but for clarity's sake, we have:

$$\operatorname{Tr}(AB) = a_{11}b_{11} + a_{22}b_{22} + \ldots + a_{nn}b_{nn} = \sum_{i=1}^{n} a_{ii}b_{ii} = \sum_{i=1}^{n} b_{ii}a_{ii} = b_{11}a_{11} + b_{22}a_{22} + \ldots + b_{nn}a_{nn} = \operatorname{Tr}(BA)$$

as expected. □

Now, we'll show the second proposition – that $\operatorname{Tr}(A) = \operatorname{Tr}\left(A^\top\right)$.

Proof. As before, we recall the definition of trace: given $A = a_{ij}$, define $\operatorname{Tr}(A) = \sum_{i=1}^{n} a_{ii}$. However, under transposition, the diagonal elements of a matrix remain unaffected. That is, under transposition, $a_{ij} \mapsto a_{ji}$. When we're on the diagonal, however, this reads as $a_{ii} \mapsto a_{ii}$. Thus, $\operatorname{Tr}(A) = \sum_{i=1}^{n} a_{ii} = \operatorname{Tr}\left(A^\top\right)$, as required. □

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§2.4

2.

(a) Under the map $(a, b) \mapsto (a_1 - a_2, a_2, 3a_1 + 4a_2)$, we'll apply T to each basis element of $\mathbb{R}^2$ and produce linear combinations of the basis elements of $\mathbb{R}^3$. Doing so, one has that

$$T(e_1) = e_1 + 3e_3$$

$$T(e_2) = -e_1 + e_2 + 4e_3$$

Reading off the coefficients of each linear combinations as the columns of $[T]_{\{e_1,e_2\}}^{\{e_1,e_2,e_3\}}$, we have that

$$[T]_{\{e_1,e_2\}}^{\{e_1,e_2,e_3\}} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 3 & 4 \end{pmatrix}$$

Of course, since $[T]_{\{e_1,e_2\}}^{\{e_1,e_2,e_3\}} \in \mathbb{R}^{3\times 2}$, it's not square, and so isn't invertible. Since, if T were invertible, there would exist a map $T^{-1} : \mathbb{R}^3 \to \mathbb{R}^2$ represented by the matrix $\left[T^{-1}\right]_{\{e_1,e_2,e_3\}}^{\{e_1,e_2\}} = \left([T]_{\{e_1,e_2\}}^{\{e_1,e_2,e_3\}}\right)^{-1}$, but since the desired inverse doesn't exist, T is not invertible.

(c) As before, we apply T to each basis element of $\mathbb{R}^3$:

$$T(e_1) = 3e_1 + 3e_3$$

$$T(e_2) = e_2 + 4e_3$$

$$T(e_3) = -2e_1$$

And so

$$[T] = \begin{pmatrix} 3 & 0 & -2 \\ 0 & 1 & 0 \\ 3 & 4 & 0 \end{pmatrix}$$

Since the columns of $[T]$ are linearly independent, $[T]$ is invertible, and so $([T])^{-1} = \left[T^{-1}\right]$ exists, so the map $T^{-1} : \mathbb{R}^3 \to \mathbb{R}^3$ is well-defined. Thus, T is invertible.

(e) Applying T to the basis elements of $\mathbb{R}^{2\times 2}$, we have:

$$T\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} = 1, \qquad T\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} = 2x, \qquad T\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} = x^2, \qquad T\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} = x^2$$

And so

$$[T]_{\{e_1,e_2,e_3,e_4\}}^{\{1,x,x^2\}} = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 2 & 0 & 0\\ 0 & 0 & 1 & 1\end{pmatrix}$$

But, since $[T]_{\{e_1,e_2,e_3,e_4\}}^{\{1,x,x^2\}}$ is non-square, it isn't invertible, and so T isn't invertible.

4.

Proof. If AB is invertible, then there exists a matrix $(AB)^{-1}$ such that the following is true:

$$(AB)^{-1}AB = I_n \tag{1}$$

Where I_n denotes the $n \times n$ identity matrix. Since B is invertible, there exists a matrix B^{-1} such that $BB^{-1} = I_n$, so multiplying both sides of (1) by B^{-1} gives:

$$(AB)^{-1}ABB^{-1} = I_nB^{-1} \implies (AB)^{-1}A = B^{-1} \tag{2}$$

Now, since A is invertible, there exists a matrix A^{-1} such that $AA^{-1} = I_n$. Thus, multiplying both sides of (2) by A^{-1} gives:

$$(AB)^{-1}AA^{-1} = B^{-1}A^{-1} \implies \boxed{(AB)^{-1} = B^{-1}A^{-1}}$$

as required. □

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§2.5

3.

(a) Writing the elements of β' in terms of the elements of β, one has that

$$a_2x^2 + a_1x + a_0 = (a_2)x^2 + (a_1)x + (a_0)1$$

$$b_2x^2 + b_1x + b_0 = (b_2)x^2 + (b_1)x + (b_0)1$$
$$c_2x^2 + c_1x + c_0 = (c_2)x^2 + (c_1)x + (c_0)1$$

Reading off the coefficients of each linear combination as the columns of the change-of-coordinate matrix, one has that

$$\boxed{Q = \begin{pmatrix}a_2 & b_2 & c_2\\ a_1 & b_1 & c_1\\ a_0 & b_0 & c_0\end{pmatrix}}$$

(b) Since this problem is identical to problem (a) but the elements of β are listed in the opposite order, the columns of Q will be identical, but in the opposite order. Thus,

$$\boxed{Q = \begin{pmatrix}a_0 & b_0 & c_0\\ a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\end{pmatrix}}$$

10.

Proof. If A and B are similar, then there exists a matrix C such that $B = C^{-1}AC$. Applying the trace, one has that $\mathrm{Tr}(B) = \mathrm{Tr}\left(C^{-1}AC\right)$. Letting $D = AC$ yields $\mathrm{Tr}(B) = \mathrm{Tr}\left(C^{-1}D\right)$. As shown in exercise 13 of §2.3, $\mathrm{Tr}\left(C^{-1}D\right) = \mathrm{Tr}\left(DC^{-1}\right)$. However, $\mathrm{Tr}\left(DC^{-1}\right) = \mathrm{Tr}\left(ACC^{-1}\right) = \mathrm{Tr}(A)$, as required. □