MATH 340 / Spring 2017

Homework 2

Solutions written by Alex Menendez (`1438704`)

Note: All exercises are taken from Friedberg, Insel, and Spence's *Linear Algebra*.

Problems Assigned:

- §1.4 – 2(b, d), 7
- §1.5 – 4, 13(a, b)
- §1.6 – 3(a, b, c, d, e), 8, 16, 28

Solutions:

§1.4

Exercise 2.

(b) This system is equivalent to the matrix equation $\begin{pmatrix} 3 & -7 & 4 \\ 1 & -2 & 1 \\ 2 & -1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 10 \\ 3 \\ 6 \end{pmatrix}$. Writing out the augmented matrix, one has

$$\left(\begin{array}{ccc|c} 3 & -7 & 4 & 10 \\ 1 & -2 & 1 & 3 \\ 2 & -1 & -2 & 6 \end{array}\right) \overset{-3R_2+R_1\to R_2}{\rightsquigarrow} \left(\begin{array}{ccc|c} 3 & -7 & 4 & 10 \\ 0 & -1 & 1 & 1 \\ 2 & -1 & -2 & 6 \end{array}\right) \overset{-\frac{3}{2}R_3+R_1\to R_3}{\rightsquigarrow} \left(\begin{array}{ccc|c} 3 & -7 & 4 & 10 \\ 0 & -1 & 1 & 1 \\ 0 & -\frac{11}{2} & 7 & 1 \end{array}\right) \overset{2R_3\to R_3}{\rightsquigarrow}$$

$$\left(\begin{array}{ccc|c} 3 & -7 & 4 & 10 \\ 0 & -1 & 1 & 1 \\ 0 & -11 & 14 & 2 \end{array}\right) \overset{-11R_2\to R_2}{\rightsquigarrow} \left(\begin{array}{ccc|c} 3 & -7 & 4 & 10 \\ 0 & 11 & -11 & -11 \\ 0 & -11 & 14 & 2 \end{array}\right) \overset{R_2+R_3\to R_3}{\rightsquigarrow} \left(\begin{array}{ccc|c} 3 & -7 & 4 & 10 \\ 0 & 11 & -11 & -11 \\ 0 & 0 & 3 & -9 \end{array}\right).$$

The bottom row reveals that $3x_3 = -9 \Rightarrow x_3 = -3$ wherein moving to the next row yields $11x_2 - 11x_3 = -11 \Rightarrow 11x_2 = -44 \Rightarrow x_2 = -4$, and finally $3x_1 - 7x_2 + 4x_3 = 10 \Rightarrow 3x_1 = -6 \Rightarrow x_1 = -2$. Thus, the solution set is $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \\ -3 \end{pmatrix}$.

(d) This system gives the augmented matrix $\left(\begin{array}{cccc|c} 1 & 2 & 2 & 0 & 2 \\ 1 & 0 & 8 & 5 & -6 \\ 1 & 1 & 5 & 5 & 3 \end{array}\right)$. Row reducing, we obtain the matrix $\left(\begin{array}{cccc|c} 1 & 2 & 2 & 0 & 2 \\ 0 & 2 & -6 & -5 & 8 \\ 0 & 0 & 0 & 5 & 10 \end{array}\right)$. The bottom row reveals that $x_4 = 2$, and now if we are to set $x_2 = t$ for some $t \in \mathbb{R}$, we obtain $x_3 = \frac{t}{3} - 3$ and $x_1 = 8\left(1 - \frac{t}{3}\right)$. Thus, the solution set is $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 8 \\ 0 \\ -3 \\ 2 \end{pmatrix} + t\begin{pmatrix} -8/3 \\ 1 \\ 1/3 \\ 0 \end{pmatrix}$ for some real t.

Exercise 7.

Proof. First, we compute $\mathrm{Span}(e_1, \ldots, e_n) = \{\alpha_1 e_1 + \ldots + \alpha_n e_n \mid \alpha_i \in \mathbb{F}\} = \{(\alpha_1, \alpha_2, \ldots, \alpha_n) \mid \alpha_i \in \mathbb{F}\}$. If we take any $v \in \mathbb{F}^n$ it stands that $v \in \mathrm{Span}(e_1, \ldots, e_n)$ and so $\mathbb{F}^n \subset \mathrm{Span}(e_1, \ldots, e_n)$. Likewise, for any $v \in \mathrm{Span}(e_1, \ldots, e_n)$, it stands that v is an arbitrary n-tuple with coordinates that are in $\mathbb{F}^n$ and so $v \in \mathbb{F}^n$ which gives $\mathrm{Span}(e_1, \ldots, e_n) \subset \mathbb{F}^n$. Putting this together yields $\mathrm{Span}(e_1, \ldots, e_n) = \mathbb{F}^n$, and so $\{e_1, \ldots, e_n\}$ generates $\mathbb{F}^n$, as required. ∎

§1.5

Exercise 4.

Proof. If we consider the linear combination $\alpha_1 e_1 + \ldots + \alpha_n e_n = 0$, we have that $\alpha_1(1, 0, \ldots, 0) + \alpha_2(0, 1, \ldots, 0) + \ldots + \alpha_n(0, 0, \ldots, 1) = (\alpha_1, \alpha_2, \ldots, \alpha_n) = (0, 0, \ldots, 0)$ whence $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$, the trivial solution. Thus, 0 can only be trivially represented by $e_1, \ldots, e_n$ and so $\{e_1, \ldots, e_n\}$ is linearly independent. ∎

Exercise 13.

(a)

Proof. We prove two cases:

- $\Rightarrow$: Suppose that $\{u, v\}$ is linearly independent. If we consider $\alpha_1(u + v) + \alpha_2(u - v) = 0$ this implies that $(\alpha_1 + \alpha_2)u + (\alpha_1 - \alpha_2)v = 0$. Since u and v are linearly independent, $\alpha_1 + \alpha_2 = 0$ and $\alpha_1 - \alpha_2 = 0$ and so $\alpha_1 = \alpha_2 = 0$, and so $\{u + v, u - v\}$ is linearly independent, as required.
- $\Leftarrow$: Suppose $\{u + v, u - v\}$ is linearly independent. This implies that the linear combination $\alpha_1(u + v) + \alpha_2(u - v) = 0$ only has the trivial solution $\alpha_1 = \alpha_2 = 0$. Of course, $\alpha_1(u + v) + \alpha_2(u - v) = 0 \Rightarrow (\alpha_1 + \alpha_2)u + (\alpha_1 - \alpha_2)v = 0$ and since $\alpha_1 = \alpha_2 = 0$, then $\alpha_1 + \alpha_2 = 0$ and $\alpha_1 - \alpha_2 = 0$, and so u and v are linearly independent, as required.

∎

(b)

Proof. We prove two cases:

- $\Rightarrow$: Suppose that $\{u, v, w\}$ is linearly independent. Thus, given some $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{F}$, the linear combination $(\alpha_1 + \alpha_2)u + (\alpha_1 + \alpha_3)v + (\alpha_2 + \alpha_3)w = 0$ is only satisfied when $\alpha_1 + \alpha_2 = \alpha_1 + \alpha_3 = \alpha_2 + \alpha_3 = 0$. Naturally, by elimination this implies that $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Now observe that $(\alpha_1 + \alpha_2)u + (\alpha_1 + \alpha_3)v + (\alpha_2 + \alpha_3)w = 0 \Rightarrow \alpha_1(u + v) + \alpha_2(u + w) + \alpha_3(v + w) = 0$, and since $\alpha_1 = \alpha_2 = \alpha_3 = 0$, we have that $\{u + v, u + w, v + w\}$ is linearly independent, as required.
- $\Leftarrow$: Suppose that $\{u + w, u + v, v + w\}$ is linearly independent. That is, the linear combination $\alpha_1(u + w) + \alpha_2(u + v) + \alpha_3(v + w) = 0$ is only satisfied when $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Of course, $\alpha_1(u + w) + \alpha_2(u + v) + \alpha_3(v + w) = (\alpha_1 + \alpha_2)u + (\alpha_1 + \alpha_3)v + (\alpha_2 + \alpha_3)w = 0$. Since $\alpha_1 = \alpha_2 = \alpha_3 = 0$, it follows that $\alpha_1 + \alpha_2 = \alpha_1 + \alpha_3 = \alpha_2 + \alpha_3 = 0$, and so $\{u, v, w\}$ is linearly independent, as required.

∎

§1.6

Exercise 3.

(a) The set $\beta = \{-1 - x + 2x^2, 2 + x - 2x^2, 1 - 2x + 4x^2\}$ is linearly dependent, hence it is not a basis for $P_2(\mathbb{R})$.

(b) The set $\beta = \{1 + 2x + x^2, 3 + x^2, x + x^2\}$ linearly independent and $\text{Span}(\beta) = P_2(\mathbb{R})$, so it forms a basis for $P_2(\mathbb{R})$.

(c) The set $\beta = \{1 - 2x - 2x^2, -2 + 3x - x^2, 1 - x + 6x^2\}$ is linearly independent and $\text{Span}(\beta) = P_2(\mathbb{R})$, so it forms a basis for $P_2(\mathbb{R})$.

(d) The set $\beta = \{-1 + 2x + 4x^2, 3 - 4x - 10x^2, -2 - 5x - 6x^2\}$ is linearly independent and $\text{Span}(\beta) = P_2(\mathbb{R})$, so it forms a basis for $P_2(\mathbb{R})$.

(e) The set $\beta = \{1 + 2x - x^2, 4 - 2s + x^2, -1 + 18x - 9x^2\}$ is linearly dependent, hence it is not a basis for $P_2(\mathbb{R})$.

Exercise 8. Stack the vectors as columns to form the matrix $M = [u_1 \mid u_2 \mid u_3 \mid u_4 \mid u_5 \mid u_6 \mid u_7 \mid u_8]$. If we are to find a basis for W, then it suffices to find a basis for the column space of M. Thus, row reduction yields:

$$M = \begin{pmatrix} 2 & -6 & 3 & 2 & -1 & 0 & 1 & 2 \\ -3 & 9 & -2 & -8 & 1 & -3 & 0 & -1 \\ 4 & -12 & 7 & 2 & 2 & -18 & -2 & 1 \\ -5 & 15 & -9 & -2 & 1 & 9 & 3 & -9 \\ 2 & -6 & 1 & 6 & -3 & 12 & -2 & 7 \end{pmatrix} \overset{\text{rref}}{\rightsquigarrow} \tilde{M} = \begin{pmatrix} 1 & -3 & 0 & 4 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -4 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Since there's a leading one in the 1st, 3rd, 5th, and 7th columns of $\tilde{M}$, the corresponding columns of M form a basis for $\text{col}(M)$. Thus, the subset we're after is $\{u_1, u_3, u_5, u_7\}$.

Exercise 16. First, let's examine a small, familiar case. Consider the set of 2×2 upper-triangular matrices of the form $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$. A basis for such a set would be $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ since $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. By extension, the set $\{1^{ij} \mid 1 \leq i \leq j \leq n\}$ where 1^{ij} is the matrix whose only non-zero coordinate is 1 at position a_{ij} forms a basis for the set of upper-triangular $n \times n$ matrices. We can also calculate the dimension – in our case study of the set of 2×2 upper-triangular matrices, $\dim(W) = 3$. In a similar vein, if we restrict ourselves to 3×3 upper-triangular matrices, $\dim(W) = 6$ (which can be manually verified in the exact same fashion). Thus, we can inductively say that for the set of $n \times n$ upper-triangular matrices, $\dim(W) = \frac{n(n+1)}{2}$.

Exercise 28.

Proof. We need to show that any arbitrary basis of V, if taken over the real numbers, contains $2n$ elements. Let $\beta = \{v_1, \ldots, v_n\}$ be a basis for V as a vector space over $\mathbb{C}$ and take $\gamma = \{v_1, iv_1, \ldots, v_n, iv_n\}$ to be a basis for V as a vector space over $\mathbb{R}$ where $i = \sqrt{-1}$. It's quite easy to see that γ has n elements copied twice, so it contains $2n$ elements. We now need to show that γ is, indeed, a basis for V over $\mathbb{R}$. First, we'll show linear independence: let x_j and y_j be real constants such that $x_1v_1 + y_1(iv_1) + \ldots + x_nv_n + y_n(iv_n) = 0$. If we simplify our notation further and define $c_j = x_j + iy_j$, then this becomes $c_1v_1 + \ldots + c_nv_n = 0$. Since $\{v_1, \ldots, v_n\}$ is already known to be a basis for V over $\mathbb{C}$, $v_1, \ldots, v_n$ are all linearly independent, which implies that $c_1 = c_2 = \ldots = c_n = 0 \Rightarrow x_j = y_j = 0$ for every $j \in \{1, \ldots, n\}$. Thus, the only representation of 0 in $\mathbb{R}$ is the trivial one, so γ is linearly independent in $\mathbb{R}$. Now, it remains to show that γ spans V in $\mathbb{R}$. Because β is a basis for V over $\mathbb{C}$, there exist scalars $c_1, \ldots, c_n$ so that, for some $w \in V$, $w = c_1v_1 + \ldots + c_nv_n$. Once again, if we define $c_j = x_j + iy_j$ for some real x_j and y_j, $i \in \{1, \ldots, n\}$, one has that, for some $w = c_1 v_1 + \ldots + c_n v_n = x_1 v_1 + y_1(iv_1) + \ldots + x_n v_n + y_n(iv_n)$ whence $w \in \mathrm{Span}(\gamma)$. Thus, $V \subset \mathrm{Span}(\gamma)$. However, by the same token, given any $w \in \mathrm{Span}(\gamma)$ it follows that $w \in V$, and so $\mathrm{Span}(\gamma) \subset V$. Thus, γ spans V in $\mathbb{R}$, as required. So, γ is a basis for V in $\mathbb{R}$ and it contains $2n$ elements, so $\dim(V) = 2n$, as required. ■