Math 427

Problem Set #5

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Problems from text:

- §2.7: 2, 9
- §3.1: 2, 5, 7, 9, 11, 12

Solutions to §2.7 exercises:

2. *Proof.* Let $U \subset \mathbf{C}$ be open, and let C be a component of U, defined as

$$C = \bigcup_{F \subset U} F,$$

where each F is connected. For any $x \in \bigcup_{F \subset U} F$, we have that x resides in at least one connected set F, call it $\overline{F}$. However, by assumption, $\overline{F} \subset U$ and $\overline{F} \subset \bigcup_{F \subset U} F$, so for any $x \in C$, there exists a connected neighborhood of x entirely contained in C, so C is open. □

9. See the back for an attached picture.

Solutions to §3.1 exercises:

2. *Proof.* Let $\{f_n\} = \left\{\sin\left(\frac{x}{n}\right)\right\}$ and $f = 0$. For any $0 \leq x \leq k$, we have that $\sin\left(\frac{x}{n}\right) \leq \frac{x}{n}$. Thus,

$$\begin{aligned} |f_n - f| &= \left|\sin\left(\frac{x}{n}\right)\right| \\ &\leq \left|\frac{x}{n}\right| \\ &\leq \left|\frac{k}{n}\right|. \end{aligned}$$

Let $\{g_n\} = \left\{\frac{k}{n}\right\}$ and observe that $\lim_{n\to\infty} g_n = \lim_{n\to\infty} \frac{k}{n} = 0$. Thus, we've found a g_n for which $|f_n - f| \leq |g_n|$ and $g_n \to 0$, so $\sin\left(\frac{x}{n}\right) \rightrightarrows 0$ on $[0,k]$. Alternatively, now consider the interval $[0,\infty)$. As before, we'd like to bound the distance $|f_n(x) - f(x)|$, which equates to bounding $|\sin(x/n)|$ for all $0 \leq x < \infty$. Observe that

$$\sup_{x\in[0,\infty)} \left|\sin\left(\frac{x}{n}\right)\right| = 1,$$

where we attain the value 1 for any x of the form $x = n\pi/2$, for $n \in \mathbf{Z}_{\geq 0}$. So, $\|\sin(x/n)\|_\infty = 1$, which doesn't converge to 0 as $n \to \infty$, so the sequence diverges on $[0,\infty)$. □

5. *Proof.* Let $f_k(z) = \frac{k+z}{k^3+1}$. Our first task is to bound $f_k(z)$. Doing so, we find that

$$\begin{aligned} |f_k(z)| &= \left|\frac{k+z}{k^3+1}\right| \\ &\leq \left|\frac{k+z}{k^3}\right| \\ &\leq \left|\frac{z}{k^3}\right| \\ &= \frac{|z|}{k^3}. \end{aligned}$$

Now, since $z \in \overline{D_1(0)}$, we have that $|z| \leq 1$, so $\dfrac{|z|}{k^3} \leq \dfrac{1}{k^3}$. We'll choose $M_k = 1/k^3$ so that

$$\sum_{k=1}^{\infty} M_k = \sum_{k=1}^{\infty} \frac{1}{k^3}$$

converges via the p-test. Thus, $|f_k(z)| \leq 1/k^3$ for all $z \in \overline{D_1(0)}$ and $\sum \dfrac{1}{k^3}$ converges, so $\sum f_k(z) = \sum \dfrac{k+z}{k^3+1}$ converges uniformly. □

7. *Proof.* Let $f_k(z) = k^{-z}$ and let's examine $\sum_{k=1}^{\infty} f_k(z)$. Generally, we know that $z = \Re(z) + i\Im(z)$, so $k^{-z} = k^{-\Re(z)}k^{-i\Im(z)}$. However, $\Re(z) > s$ and $s > 1$, so $\Re(z) > 1$. Thus, $k^{-z} = k^{-\Re(z)}k^{-i\Im(z)} < k^{-1}k^{-i\Im(z)}$. So,

$$\begin{aligned} \left|\frac{1}{k^z}\right| &< \left|\frac{1}{k^{1+i\Im(z)}}\right| \\ &< \left|\frac{1}{k^{i\Im(z)}}\right| \\ &= \frac{1}{\left|k^{i\Im(z)}\right|} \end{aligned}$$

Now, note that $k = e^{\log(k)}$, so $k^{i\Im(z)} = e^{i\log(k)\Im(z)}$. In this way, $\left|k^{i\Im(z)}\right| = \left|e^{i\log(k)\Im(z)}\right| = 1$. So,

$$\frac{1}{\left|k^{i\Im(z)}\right|} < 1.$$

What this now tells us is that $|k^{-z}| < 1$, so $\sum_{k=1}^{\infty} |f_k(z)|$ converges, and so $\sum_{k=1}^{\infty} f_k(z)$ converges uniformly. □

9. *Solution.* Before we proceed, let's establish that

$$(-1)^k = \begin{cases} -1 & \text{if } k \text{ is odd,} \\ 1 & \text{if } k \text{ is even.} \end{cases}$$

Thus, we instantly get an upper bound in that, for any k, it holds that $(-1)^k \leq 1$. It follows, then, that $1 + (-1)^k + \frac{1}{k} \leq 2 + \frac{1}{k}$. Now, let

$$S_n = \left\{1 + (-1)^k + \frac{1}{k} \ : \ k \geq n\right\}.$$

Let's go through some cases to get a feel for what's happening:

- For S_1: for $k = 1$, we have that $1 + (-1)^k + \frac{1}{k} = 1$. For $k \geq 2$ we have that, from before, $1 + (-1)^k + \frac{1}{k} \leq 2 + \frac{1}{k} \leq \frac{5}{2}$. So $\sup(S_1) = \frac{5}{2}$.
- For S_2: for $k = 2$, we have that $1 + (-1)^k + \frac{1}{k} = \frac{5}{2}$. For $k \geq 3$ we have that, from before, $1 + (-1)^k + \frac{1}{k} \leq 2 + \frac{1}{k} \leq \frac{7}{3}$. So $\sup(S_2) = \frac{5}{2}$.
- For S_3: for $k = 3$, we have that $1 + (-1)^k + \frac{1}{k} = \frac{1}{3}$. For $k \geq 4$ we have that, from before, $1 + (-1)^k + \frac{1}{k} \leq 2 + \frac{1}{k} \leq \frac{9}{4}$. So $\sup(S_3) = \frac{9}{4}$.
- For S_4: for $k = 4$, we have that $1 + (-1)^k + \frac{1}{k} = \frac{9}{4}$. For $k \geq 5$ we have that, from before, $1 + (-1)^k + \frac{1}{k} \leq 2 + \frac{1}{k} \leq \frac{11}{5}$. So $\sup(S_3) = \frac{9}{4}$.

So, if we let $u_n = \sup(S_n)$, we have that

$$\boxed{u_n = \begin{cases} (2n+3)/(n+1) & \text{if } n \text{ is odd,} \\ (2n+1)/n & \text{if } n \text{ is even.} \end{cases}}$$

□

11. *Proof.* Take $\epsilon > 0$ and choose M so that $1/M < \epsilon$. For $k \geq M$, we have that $(k+1)/k < 1+\epsilon$, so

$$M + 1 \leq M(1+\epsilon),$$

which, for any $\ell \geq 1$, extends to

$$M + \ell \leq (M + \ell + 1)(1+\epsilon) \leq M(1+\epsilon)^{\ell}.$$

For $k \geq M$, $k = M + (k-M) \leq M(1+\epsilon)^{k-M}$, or

$$k \leq \left(\frac{M}{(1+\epsilon)^M}\right)(1+\epsilon)^k.$$

Thus, for $k \geq M$,

$$k^{1/k} \leq \left(\frac{M}{(1+\epsilon)^M}\right)^{1/k}(1+\epsilon)$$

and $\lim_{k\to\infty}\left(\frac{M}{(1+\epsilon)^M}\right)^{1/k} = 1$, so

$$\limsup_{k\to\infty} k^{1/k} \leq 1+\epsilon.$$

As ϵ is arbitrary, we have, in particular, that $\limsup_{k\to\infty} k^{1/k} \leq 1$. In a similar way, $\liminf_{k\to\infty} k^{1/k} \geq 1$, so $\lim_{k\to\infty} k^{1/k} = 1$. □

12. *Proof.* We ideally want to bound the difference $\sup\{a_k b_k\} - ab$, so let's begin by rewriting it in a more tangible way:

$$\sup\{a_k b_k\} - ab = \sup\{a_k b_k\} - a_k b + a_k b - ab.$$

Now, taking the norm, we have

$$\begin{aligned} |\sup\{a_k b_k\} - ab| &= |\sup\{a_k b_k\} - a_k b + a_k b - ab| \\ &\leq |\sup\{a_k b_k\} - a_k b| + |a_k b - ab| \\ &= |\sup\{a_k b_k\} - a_k b| + |b||a_k - a| \end{aligned}$$

We know, by assumption, that $|a_k - a| \to 0$ for $k \to \infty$. Our focus is really on bounding the first summand. Note that, for large enough k, we have that $\sup\{b_k\} \sim b_k$, so $a_k \sup\{b_k\} \sim a_k b_k$ and thus $\sup\{a_k b_k\} \sim a_k b_k$. Moreover, $b_k \leq \sup\{b_k\}$, which behaves on the order of a constant for large enough k. So, $\sup\{a_k b_k\} \sim a_k b_k \leq a_k M$, for some finite M. So $|\sup\{a_k b_k\} - a_k b| \leq |M(a_k - a)|$, which tends to zero for $k \to \infty$, so we're done. □