MATH 427

Problem Set #6

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Problems from text:

- §3.1: 15, 16
- §3.2: 1, 6, 8, 13, 15
- §3.3: 4, 5, 8, 16
- Additionally, prove Theorem 3.3.10

Solutions to §3.1 exercises:

15. *Solution.* Let $f_n(w) = \sum_{k=0}^{n} (-1)^k w^k$, and observe that

$$\begin{aligned}\int_0^z f_n(w)\ dw &= \int_0^z \left(\sum_{k=0}^{n} (-1)^k w^k\right)\ dw \\ &= \sum_{k=0}^{n} \left(\int_0^z (-1)^k w^k\ dw\right) \\ &= \sum_{k=0}^{n} \frac{(-1)^k w^{k+1}}{k+1},\end{aligned}$$

where the integral-summation swap performed between the first and second line is done since the sum is finite. Now, apply the limit to both sides to get:

$$\begin{aligned}\lim_{n\to\infty} \int_0^z f_n(w)\ dw &= \lim_{n\to+\infty} \sum_{k=0}^{n} \frac{(-1)^k w^{k+1}}{k+1} \\ &= \sum_{k=0}^{+\infty} \frac{(-1)^k w^{k+1}}{k+1}.\end{aligned}$$

However, $f_n(w)$ converges uniformly to $f(w) = 1/(1+w)$ on $D_1(0)$, so we can bring the limit inside the integral so that $\lim_{n\to+\infty} \int_0^z f_n(w)\ dw = \int_0^z \lim_{n\to+\infty} f_n(w)\ dw$, whence

$$\begin{aligned}\int_0^z \left(\sum_{k=0}^{+\infty} (-1)^k w^k\right)\ dw &= \int_0^z \frac{1}{1+w}\ dw \\ &= \boxed{\sum_{k=0}^{+\infty} \frac{(-1)^k w^{k+1}}{k+1}}.\end{aligned}$$

The above derivation justifies an integral-summation swap for the sum now being infinite. Via Theorem 3.1.11, the above equality holds for all $w \in D_R(0)$, where R is the radius of convergence of the original series for $f(w)$. Since the series for $f(w)$ converges for any $w \in D_1(0)$, we have that our new series also converges on $D_1(0)$, because of equality. □

16. *Solution.* Given $E(z) = \int_0^z e^{-w^2}\ dw$, we differentiate to obtain

$$E'(z) = e^{-z^2},$$

which follows by the Fundamental Theorem of Calculus. We've done this since e^{-w^2} can't be anti-differentiated in elementary terms, so we'll differentiate and work backward. Thus,

$$E'(z) = e^{-z^2} = \sum_{k=0}^{+\infty} \frac{(-1)^k z^{2k}}{k!}.$$

Anti-differentiating, we have

$$\begin{aligned} E(z) &= \int E'(z)\ dz \\ &= \int \left(\sum_{k=0}^{+\infty} \frac{(-1)^k z^{2k}}{k!} \right)\ dz \\ &= \sum_{k=0}^{+\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)k!} + C. \end{aligned}$$

Since $E(0) = 0$, we have that $C = 0$, and so the desired series is

$$\boxed{E(z) = \sum_{k=0}^{+\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)k!}},$$

which converges over all of $\mathbb{C}$. □

Solutions to §3.2 exercises:

1. *Solution.* We know that $\dfrac{1}{1-z} = \displaystyle\sum_{k=0}^{+\infty} z^k$ for $|z| < 1$. Now, note that $\dfrac{1}{(1-z)^2} = \left(\dfrac{1}{1-z}\right)\left(\dfrac{1}{1-z}\right)$. Since we know the power series expansion for each factor, we have that

$$\frac{1}{(1-z)^2} = \left(\sum_{k=0}^{+\infty} z^k \right) \left(\sum_{k=0}^{+\infty} z^k \right).$$

We want to consolidate this sum so that it's written with only one $\sum$ symbol, so note that, for a product of power series, we have that

$$\left(\sum_{k=0}^{+\infty} \alpha_k z^k \right) \left(\sum_{k=0}^{+\infty} \beta_k z^k \right) = \sum_{k=0}^{+\infty} \gamma_k z^k,$$

where $\gamma_k = \displaystyle\sum_{j=0}^{k} \alpha_k \beta_{k-j}$. The coefficients for each factor are all 1, so

$$\gamma_k = \underbrace{1 + 1 + \ldots + 1}_{k+1 \text{ terms}} = k + 1,$$

so

$$\frac{1}{(1-z)^2} = \left(\sum_{k=0}^{+\infty} z^k \right) \left(\sum_{k=0}^{+\infty} z^k \right) = \boxed{\sum_{k=0}^{+\infty} (k+1) z^k}.$$

□

6. *Solution.* Let $f(z) = 1/\cos(z) = \sec(z)$. In a general sense, the power series for $\sec(z)$ assumes the form

$$\sec(z) = \sum_{k=0}^{+\infty} \frac{\sec^k(0) z^k}{k!},$$

where $\sec^k(0)$ denotes the k^{th} derivative of $\sec(z)$ at $z_0 = 0$. However, the poles of the function $\sec(z)$ and its subsequent derivatives occur precisely where the function $\cos(z)$ has its zeroes, which are at odd integer multiples of $\pi/2$. Let's formalize this argument. Suppose that the radius of convergence of the series is $\pi/2 + \epsilon$, for some $\epsilon > 0$. Certainly, then,

$$D_{\frac{\pi}{2}}(0) \subset D_{\frac{\pi}{2}+\epsilon}(0).$$

By convergence, for any $\epsilon' > 0$, there exists an M such that, if $n > M$, then

$$\left| \sum_{k=0}^{n} \frac{\sec^k(0) z^k}{k!} - \sec(z) \right| < \epsilon'.$$

By assumption, this is true for any $z \in D_{\frac{\pi}{2}+\epsilon}(0)$, but in particular, we can choose $z = \pi/2$, but that's precisely where $\sec(z)$ has a pole and diverges to $\pm\infty$. So the above bound doesn't hold when $|z| \geq \pi/2$, and so we require that $|z| < \pi/2$, so the radius of convergence is $\pi/2$, as required. □

8. *Solution.* Let

$$f(z) = \begin{cases} \sin(z)/z & \text{if } z \neq 0, \\ 1 & \text{if } z = 0. \end{cases}$$

Obviously, f is analytic at $z = 0$ since 1 is analytic. What concerns us is the analyticity of f on $\mathbb{C} \setminus \{0\}$, wherein we'll write $\sin(z)/z$ using power series methods. Let's begin by decomposing $\sin(z)/z$ in the following way:

$$\begin{aligned} \frac{\sin(z)}{z} &= \sin(z)\left(\frac{1}{z}\right) \\ &= \sin(z)\left(\frac{1}{1-(1-z)}\right). \end{aligned}$$

Writing each factor as its power series, we have that

$$\begin{aligned} \sin(z)\left(\frac{1}{1-(1-z)}\right) &= \left(\sum_{k=0}^{+\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!}\right)\left(\sum_{k=0}^{+\infty}(1-z)^k\right) \\ &= \sum_{k=0}^{+\infty} \frac{(-1)^k z^{2k}}{(2k+1)!}. \end{aligned}$$

Moreover,

$$\frac{d}{dz}\left(\sum_{k=0}^{+\infty} \frac{(-1)^k z^{2k}}{(2k+1)!}\right) = \sum_{k=0}^{+\infty} \frac{2k(-1)^k z^{2k-1}}{(2k+1)!},$$

so $\sin(z)/z$ is analytic, provided $z \in \mathbb{C} \setminus \{0\}$. □

13. *Proof.* Since f is analytic on $D_r(z_0)$, we have that the power series for f converges uniformly to f everywhere inside the disc. Now, the radius of convergence must be at least r. Supposing that the radius of convergence is $R = r - \epsilon$, then f is bounded, a contradiction. □

15. *Proof.* Suppose that $f_n \rightrightarrows f$ on every compact $K \subset U$. We'll use the derivative approximation formula given by:

$$f^k(z_0) = \frac{k!}{2\pi i}\int_{|w-z_0|=\epsilon} \frac{f(w)}{(w-z_0)^{k+1}}\, dw.$$

However, this integral is taken over a disc, so we need to square away some logistical issues first – we can't take any z_0 on the "corners" of K, otherwise the disc containing z_0 will have to include points outside of K. So, let $\tilde{K}$ be the "thickened" version of K. We can enlarge K to account for this because $\tilde{K}$ will still have the same cover as K and thus still be compact. So, using the approximation formula to write the derivatives $f_n^k(z_0)$ and $f^k(z_0)$, we have that:

$$\begin{aligned} |f_n^k(z_0) - f^k(z_0)| &= \left| \frac{k!}{2\pi i}\int_{|w-z_0|=\epsilon} \frac{f_n(w)}{(w-z_0)^{k+1}}\, dw - \frac{k!}{2\pi i}\int_{|w-z_0|=\epsilon} \frac{f(w)}{(w-z_0)^{k+1}}\, dw \right| \\ &\leq \frac{k!}{2\pi}\int_{|w-z_0|=\epsilon} \left| \frac{f_n(w) - f(w)}{(w-z_0)^{k+1}} \right| dw \\ &= \frac{k!}{2\pi}\int_{|w-z_0|=\epsilon} \frac{|f_n(w) - f(w)|}{|w-z_0|^{k+1}}\, dw. \end{aligned}$$

□

We know that $f_n \rightrightarrows f$ on all $\tilde{K}$, so certainly f_n converges uniformly to f on any disc inside $\tilde{K}$. So, for any $\epsilon' > 0$, there exists an $N > 0$ such that, if $n > N$, then $|f_n - f| < \epsilon'$. So in particular, we have an upper bound for the integrand along the path that the integral is taken over:

$$\frac{|f_n(w) - f(w)|}{(w-z_0)^{k+1}} < \frac{\epsilon'}{|w-z_0|^{k+1}}.$$

However, $|w - z_0| = \epsilon$, so by Cauchy's estimate we have that

$$\frac{|f_n(w) - f(w)|}{(w-z_0)^{k+1}} < \frac{\epsilon'}{\epsilon^{k+1}}.$$

Lastly, we know that any path integral is bounded above by the supremum of the integrand multiplied by the arc length of the path, so in particular,

$$\begin{aligned}\frac{k!}{2\pi}\int_{|w-z_0|=\epsilon}\frac{|f_n(w)-f(w)|}{|w-z_0|^{k+1}}\,dw &\leq (2\pi\epsilon)\frac{k!}{2\pi}\frac{\epsilon'}{\epsilon^{k+1}}\\ &= \frac{k!\epsilon'}{\epsilon^k},\end{aligned}$$

and so we have that he difference $|f_n^k - f^k|$ is bounded uniformly on each compact $\tilde{K} \subset U$ and hence each compact $K \subset U$, so $f_n^k \rightrightarrows f^k$, as required.

Solutions to §3.3 exercises:

Theorem (Theorem 3.3.10 from Taylor)**.** An entire function f is a polynomial of degree at most n if and only if there are positive constants A and B such that

$$|f(z)| \leq A + B|z|^n$$

for all $z \in \mathbb{C}$.

Proof. The stipulation that f being a polynomial implies the given bound on f has been proven in the text, so we'll prove the converse. Suppose that f is an entire function and that there exist real, positive constants A and B so that $|f(z)| \leq A + B|z|^n$ for all z. We need to show that this inequality can only be satisfied if f is a polynomial of degree $\leq n$. We know that f is entire, so it's analytic on all of $\mathbb{C}$. Certainly, this means that f is analytic on any open $U \subset \mathbb{C}$ containing a disc $\overline{D_R(z_0)}$. For any z on the boundary of this disc, called z_b, we have that

$$|f(z_b)| \leq A + B|z_b|^n,$$

which gives us a bound on f along the boundary. Then, using Cauchy's estimate, we then have that the n^{th} derivative of f evaluated at z_0 is bounded in the following way:

$$|f^n(z_0)| \leq \frac{n!M}{R^n},$$

where M is the maximum of f on the boundary. However, we know what this maximum is, so more explicitly we have that

$$|f^n(z_0)| \leq \frac{n!\,(A + B|z_b|^n)}{R^n}.$$

Of course, $R = |z_0 - z_b|$, so by substitution, we obtain

$$|f^n(z_0)| \leq \frac{n!\,(A + B|z_b|^n)}{|z_0 - z_b|^n}.$$

Estimating the $(n+1)^{\text{th}}$ derivative, we have that

$$|f^{n+1}(z_0)| \leq \frac{(n+1)!\,\left(A + B|z_b|^{n+1}\right)}{|z_0 - z_b|^{n+1}}.$$

However, this is true for the entire plane, so letting $|z_0 - z_b|$ tend to ∞ gives that $|f^{n+1}(z_0)|$ is bounded above by zero, so $f^{n+1}(z_0) = 0$. This, of course, implies that f must be a polynomial of degree at most n, which is what we wanted to show. □

4. *Proof.* Suppose that f is entire and that $|f(z)| \geq 1$ for all $z \in \mathbb{C}$. Now, consider the function $1/|f(z)|$. Since f is entire, it's analytic over all of $\mathbb{C}$ and $|f| \geq 1$ everywhere in the plane (so in particular $|f|$ is not zero), so $1/|f(z)|$ is entire, as well. Since $|f(z)| \geq 1$, it follows by inequality that

$$\frac{1}{|f(z)|} \leq 1$$

for all $z \in \mathbb{C}$. So, we have an entire, bounded function, so invoking Liouville's Theorem, we have that $1/f(z)$ must be constant everywhere. This implies that $f(z)$ must be constant, and so we are done. □

5. *Proof.* f is entire, by assumption, so if we can show that f is bounded, then by Liouville's Theorem, f has to be constant, and we'll be done. So, we can express f in the following way:

$$f(z) = \Re(f(z)) + i\Im(f(z)).$$

However, $\Re(f(z)) \leq a$ for some finite a, so

$$f(z) = \Re(f(z)) + i\Im(f(z)) \leq a + i\Im(f(z)).$$

Taking the modulus on both sides, we get:

$$\begin{aligned} |f(z)| &\leq |a + i\Im(f(z))| \\ &= \sqrt{a^2 + \Im(f(z))^2}, \end{aligned}$$

which is a real number. So, we've found a real bound for $|f(z)|$ and so f is bounded. Together with the fact that f is entire, Liouville's Theorem tells us that f is constant, as required. □

8. *Proof.* Suppose that $|f(z)| \leq K|e^z|$ for every $z \in \mathbb{C}$ and that $K > 0$ is a real number. Dividing both sides, we have that

$$\begin{aligned} \frac{|f(z)|}{|e^z|} &= \left|\frac{f(z)}{e^z}\right| \\ &\leq K. \end{aligned}$$

Since f is entire and e^z is entire, letting $g(z) = f(z)/e^z$, we see that g is entire, as well. In the inequality above, however, we've found that $|g(z)| \leq K$, so by Liouville's Theorem, $g(z)$ must be constant, so

$$\frac{f(z)}{e^z} = C,$$

for some real constant C, whence re-arranging gives $f(z) = Ce^z$, as required. □

16. *Solution.* Suppose that such functions f do exist. For such f, we have – by assumption – that $|f| \leq A + B\log|z|$. However, for $|z| \gg 1$, the $\log|z|$ term grows much slower than the polynomial term $|z|$, so certainly $|f| \leq A + B|z|$ for $|z| \gg 1$. To be more explicit, this holds for $|z| \geq R$, for some sufficiently large R. For $|z| < R$, we have that $|f| \leq C + D|z|$, for all $z \in \mathbb{C}$ (provided different constants C and D). So, for all z, we have that f behaves polynomially, so $f(z) = a_1 z + a_0$. However, if f is a polynomial, then $|f| \leq A + B\log|z|$ for $|z| \geq 1$ if and only if $a_1 = 0$, in which case f has to be constant. This gives a contradiction, since f must be non-constant. So, the answer is *no* and we are done. □