Math 427

Problem Set #4

Problems from text:

- §2.3: 4, 5, 7, 12
- §2.4: 2, 8, 10, 11
- §2.5: 1, 4, 7, 9, 12
- §2.6: 2, 3, 5, 8, 13, 15

Solutions to §2.3:

4. Let $\gamma(t)$ be defined as follows:

$$\begin{aligned}\gamma(t) &= t(2-i) + (1-t)(-1+3i) \\ &= 2t - it - 1 + 3i + t - 3it \\ &= \boxed{(3t-1) + i(3-4t)}.\end{aligned}$$

We see that, for $t \in [0,1]$, γ describes a linear path from $2-i$ to $-1+3i$ since its real and imaginary components are linear in t.

5. (a) $\gamma(t) = z_0 + re^{it}$, where $t \in [0, 2\pi]$.
   (b) $\gamma(t) = z_0 + re^{-it}$, where $t \in [0, 2\pi]$.
   (c) $\gamma(t) = z_0 + re^{3it}$, where $t \in [0, 2\pi]$.

7. *Proof.* Let $f(t) = x(t) + iy(t)$ so that $f'(t) = x'(t) + iy'(t)$. Thus,

$$\begin{aligned}\int_a^b f'(t)\ dt &= \int_a^b x'(t)\ dt + i\int_a^b y'(t)\ dt \\ &= (x(b) - x(a)) + i(y(b) - y(a)) \\ &= (x(b) + iy(b)) - (x(a) + iy(a)) \\ &= f(b) - f(a).\end{aligned}$$

□

12. Since γ is defined piecewise, we'll split the integral into its constituent parts:

$$\int_\gamma \Im\left(z^2\right)\ dz = \int_{\gamma_1} \Im\left(z^2\right)\ dz + \int_{\gamma_2} \Im\left(z^2\right)\ dz + \int_{\gamma_3} \Im\left(z^2\right)\ dz.$$

Along the first contour, $\gamma_1(t) = t$, so $(\gamma_1)^2 = t^2$, and so $\Im\left(t^2\right) = 0$, so $\int_{\gamma_1} \Im\left(z^2\right)\ dz = 0$. Along the second contour, $\gamma_2(t) = (2-t) + i(t-1)$, so $(\gamma_2)^2 = \left((2-t)^2 - (t-1)^2\right) + i\left(2(2-t)(t-1)\right)$, thus $\Im\left(\left((2-t)^2 - (t-1)^2\right) + i\left(2(2-t)(t-1)\right)\right) = 2(2-t)(t-1)$. Moreover, $\gamma_2'(t) = i - 1$, so

$$\begin{aligned}\int_{\gamma_2} \Im\left(z^2\right)\ dz &= 2(i-1)\int_1^2 (2-t)(t-1)\ dt \\ &= \frac{i-1}{3}.\end{aligned}$$

Lastly, along the third contour, $\gamma_3(t) = i(3-t)$, so $(\gamma_3)^2 = -(3-t)^2$, whence $\Im\left(-(3-t)^2\right) = 0$, so $\int_{\gamma_3} \Im\left(z^2\right)\ dz = 0$. Thus, we arrive at a final answer of $\boxed{(i-1)/3}$.

Solutions to §2.4:

2. We'll first define our path: $\gamma(t) = e^{2it}$ for $t \in [0, 2\pi]$ is sufficient. Thus,
$$\begin{aligned}\int_\gamma \frac{1}{z}\ dz &= \int_0^{2\pi} e^{-2it}\left(2ie^{2it}\right)\ dt \\ &= 2i\int_0^{2\pi}\ dt \\ &= \boxed{4\pi i}.\end{aligned}$$

8. *Proof.* We'll begin by using the following result:

   Lemma. If $|z| = 1$, then $|\cos z| \leq e$.

   Proof. For any $k \in [0, \infty)$, we have that
$$\frac{(-1)^k z^{2k}}{(2k)!} \leq \frac{z^k}{k!},$$
   and so
$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{(-1)^k z^{2k}}{(2k)!} \leq \sum_{k=0}^{\infty}\frac{z^k}{k!} \\ \implies &\left|\sum_{k=0}^{\infty}\frac{(-1)^k z^{2k}}{(2k)!}\right| \leq \left|\sum_{k=0}^{\infty}\frac{z^k}{k!}\right| \\ \implies &\sum_{k=0}^{\infty}\left|\frac{(-1)^k z^{2k}}{(2k)!}\right| \leq \sum_{k=0}^{\infty}\left|\frac{z^k}{k!}\right| = \sum_{k=0}^{\infty}\frac{|z|^k}{|k!|}.\end{aligned}$$
   When $|z| = 1$, the RHS is exactly the Taylor series representation of e, so $|\cos z| \leq e$, as required. □

   Proceeding with the main proof, we'll now define our path: let $\gamma(t) = e^{ikt}$ for $t \in \left[0, \frac{2\pi}{k}\right]$. Now, notice that
$$\left|\int_\gamma \frac{\cos z}{z}\ dz\right| \leq \int_\gamma \frac{|\cos z|}{|z|}\ dz.$$
   Since $|\cos z| \leq e$ and $|z| = 1$, we have that
$$\begin{aligned}\left|\int_\gamma \frac{\cos z}{z}\ dz\right| &\leq \int_\gamma e\ dz \\ &= \int_0^{2\pi/k} e\left(ike^{ikt}\right)\ dt \\ &= 2\pi e,\end{aligned}$$
   as required.

   □

10. *Proof.* Let $p(z) = a_0 + a_1 z + \ldots + a_n z^n = \sum_{k=0}^{n} a_k z^k$. Thus,
$$\begin{aligned}\int_\gamma p(z)\ dz &= \int_\gamma\left(\sum_{k=0}^{n} a_k z^k\right)\ dz \\ &= \sum_{k=0}^{n}\left(a_k\int_\gamma z^k\ dz\right) \\ &= \sum_{k=0}^{n}\left(a_k\int_0^{2\pi} e^{ikt}\left(ie^{it}\right)\ dt\right) \\ &= \sum_{k=0}^{n}\left(ia_k\int_0^{2\pi} e^{it(k+1)}\ dt\right) \\ &= \sum_{k=0}^{n}\left(a_k\left[\frac{1}{(k+1)}\left(e^{i(2\pi k+2\pi)} - 1\right)\right]\right),\end{aligned}$$

for which we see that $e^{i(2\pi k+2\pi)} = \cos(2\pi k + 2\pi) + i\sin(2\pi k + 2\pi) = 1$, so $\left[\frac{1}{(k+1)}\left(e^{i(2\pi k+2\pi)} - 1\right)\right] = 0$ for all $k \in \mathbf{Z}_{\geq 0}$, thus

$$\sum_{k=0}^{n}\left(a_k\left[\frac{1}{(k+1)}\left(e^{i(2\pi k+2\pi)} - 1\right)\right]\right) = 0,$$

which is the desired result. □

11. *Proof.* Given $R(z)$ as prescribed, we have that $|R(z)| = \left|\sum_{k=n+1}^{\infty}\frac{z^k}{k!}\right|$. Moreover,

$$\begin{aligned}|R(z)| &= \left|\sum_{k=n+1}^{\infty}\frac{z^k}{k!}\right| \\ &\leq \sum_{k=n+1}^{\infty}\left|\frac{z^k}{k!}\right| \\ &= \sum_{k=n+1}^{\infty}\frac{|z^k|}{|k!|} \\ &= \sum_{k=n+1}^{\infty}\frac{|z|^k}{|k!|} \\ &= \sum_{k=0}^{\infty}\frac{|z|^k}{|k!|} - \sum_{k=0}^{n}\frac{|z|^k}{|k!|}\end{aligned}$$

when $|z| \leq 1$, we have

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{|z|^k}{|k!|} - \sum_{k=0}^{n}\frac{|z|^k}{|k!|} &\leq \sum_{k=0}^{\infty}\frac{1}{|k!|} - \sum_{k=0}^{n}\frac{1}{|k!|} \\ &= e - \sum_{k=0}^{n}\frac{1}{|k!|} \\ &\leq e - \sum_{k=1}^{n+1}\frac{1}{|k!|} \\ &= e - \left(1 + \frac{1}{2} + \ldots + \frac{1}{(n+1)!}\right) \\ &\leq e - \frac{1}{(n+1)!} \\ &\leq \frac{e-1}{(n+1)!}\end{aligned}$$

as required. □

Solutions to §2.5:

1. *Proof.* If $K \subset \mathbf{R}^n$ is compact, then it's closed and bounded. Thus, if $(\vec{x}_n) \subset K$, then $(\vec{x}_n)$ is bounded, as well. Thus, it suffices to show that a bounded sequence in $\mathbf{R}^n$ has a convergent subsequence. We'll begin in the base case when $n = 1$. Suppose that (x_n) is a bounded sequence of real numbers. Thus, the terms of (x_n) must lie in some interval $[m, M]$, where m and M are lower and upper bounds, respectively. Divide the interval $[m, M]$ into two halves, each of which have length $(M-m)/2$. One of these halves must contain infinitely-many terms of (x_n) – call this half I_1. Now, let $(x_{n'})$ be the subsequence of (x_n) consisting of all the terms inside I_1. Now, divide I_1 once again and note that the resultant halves have length $(M-m)/4$. By construction, one of the halves must contain infinitely-many terms of $(x_{n'})$ – call this half I_2. Now, let $(x_{n''})$ be the subsequence of terms in I_2. If we perform this division iteratively, we obtain a sequence of

nested intervals $I_1 \supset I_2 \supset \ldots I_n \supset \ldots$, where the length of I_n is $(M-m)/2^n$. Now, let (y_n) be the subsequence of (x_n) where the $y_n \in I_n$. Note that, for any natural numbers p and q, we have that $|y_p - y_q| < 1/2^N$ for some natural number N. Thus, (y_n) is Cauchy and thus converges.
Now, we'll extend this argument to subsets of $\mathbf{R}^n$. Let $(\vec{x}^n)$ be a bounded sequence of n-tuples. Let $(\vec{x}_i^n)$ be the sequence of i^{th} coordinates of each n-tuple. Each of these sequences is a bounded sequence of real numbers, so by the preceding argument, for $(\vec{x}_i^n)$ there exists a convergent subsequence $(\vec{x}_i^{n_k})$. This, however, is true for each coordinate, so certainly there exists a subsequence $(\vec{x}^{n_k})$ that converges in $\mathbf{R}^n$, as desired. $\square$

4. *Proof.* Let U be an open subset of $\mathbf{C}$, and suppose that γ is differentiable at every $t_0 \in [a,b]$ and that $g(z)$ is differentiable at every $z_0 \in U$, where $\gamma(t_0) = z_0$. Define

$$h(z) = \begin{cases} (g(z) - g(z_0))/(z - z_0) & \text{if } z \neq z_0, \\ g'(z_0) & \text{if } z = z_0. \end{cases}$$

We have that h is continuous at z_0, since $g'(z_0) = \lim_{z \to z_0} \dfrac{g(z) - g(z_0)}{z - z_0}$. Moreover,

$$\frac{g \circ \gamma(t) - g \circ \gamma(t_0)}{t - t_0} = h(\gamma(t)) \frac{\gamma(t) - \gamma(t_0)}{t - t_0}$$

for all $t \in \gamma^{-1}([a,b]) \setminus \{t_0\}$. Taking the limit on both sides and using the fact that g and h are continuous at z_0 and that γ is continuous at t_0, we have that $(g \circ \gamma)'(t_0) = g'(\gamma(t_0))\gamma'(t_0)$. $\square$

7. By a previous homework result, we found that $(\log(z))' = 1/z$ anywhere in $\mathbf{C}$ that isn't on the negative real axis. Thus, for any $z \in \mathbf{C} \setminus (-\infty, 0]$, we have that the antiderivative of $1/z$ is $\log(z)$. Moreover, given some path $\gamma : [a,b] \to \mathbf{C} \setminus (-\infty, 0]$,

$$\int_\gamma \frac{1}{z}\, dz = \log(\gamma(b)) - \log(\gamma(a)).$$

All we know of this path is that it starts at $-i$ and ends at i, so $\gamma(a) = -i$ and $\gamma(b) = i$, thus

$$\int_\gamma \frac{1}{z}\, dz = \log(-i) - \log(i).$$

We have that $\log(i) = \pi/2$ and $\log(-i) = -\pi/2$, so

$$\int_\gamma \frac{1}{z}\, dz = \boxed{-\pi}.$$

9. We'll first verify that $\frac{2}{3}e^{3\log(z)/2}$ is an antiderivative for $e^{\log(z)/2}$:

$$\begin{aligned} \frac{d}{dz}\left(\frac{2}{3}e^{3\log(z)/2}\right) &= \left(\frac{2}{3}e^{3\log(z)/2}\right)\left(\frac{3}{2z}\right) \\ &= e^{-\log z} e^{3\log(z)/2} \\ &= e^{\log(z)/2}. \end{aligned}$$

Thus, $\int_\gamma \sqrt{z}\, dz = (2/3)e^{3\log(z)/2}$. Moreover, $\gamma(a) = -1$ and $\gamma(b) = 1$, so we have that

$$\int_\gamma \sqrt{z}\, dz = \frac{2}{3}e^{3\log(1)/2} - \frac{2}{3}e^{3\log(-1)/2} = \boxed{0}.$$

12. Let Δ be the triangle and let S be the square. Note that Δ and S both share an edge (as depicted below). Thus, the difference $\int_{\partial\Delta} f\ dz - \int_{\partial S} f\ dz$ gives zero. The reasoning is as follows: consider the regions I and II. Along ∂(I), we have that the integral gives zero, and likewise along ∂(II).

Solutions to §2.6:

2. We'll begin by splitting the integrand:

$$\int_\gamma \frac{1}{z^2-4}\ dz = -\frac{1}{4}\int_\gamma \frac{1}{z+2}\ dz + \frac{1}{4}\int_\gamma \frac{1}{z-2}\ dz.$$

We can find antiderivatives for each term; this evaluates to

$$-\frac{1}{4}\int_\gamma \frac{1}{z+2}\ dz + \frac{1}{4}\int_\gamma \frac{1}{z-2}\ dz = -\frac{1}{4}\log|z+2| + \frac{1}{4}\log|z-2|.$$

Now, since γ traverses the unit circle once, it starts at $\Re(z) = 1$ and ends in the same place. So, the integral gives

$$\left(-\frac{1}{4}\log|1+2| + \frac{1}{4}\log|1+2|\right) + \left(\frac{1}{4}\log|1-2| - \frac{1}{4}\log|1-2|\right) = \boxed{0}.$$

Of course, we could have straight away said the answer was zero since γ was a closed contour and $f(z) = \left(z^2-4\right)^{-1}$ is analytic everywhere in the interior of the unit circle.

3. We begin by rewriting:

$$\begin{aligned}\int_\gamma \frac{1}{1-e^z}\ dz &= \int_\gamma \frac{e^z}{1-e^z}\ dz + \int_\gamma\ dz \\ &= \log|1-e^z| + z\end{aligned}$$

Since $\gamma(0) = \gamma(2\pi)$, we have that

$$\int_\gamma \frac{1}{1-e^z}\ dz = \boxed{0}.$$

5. We have that

$$\int_\gamma \frac{1}{z^2}\ dz = -\frac{1}{z},$$

and, once again, since $\gamma(a) = \gamma(b)$, along with the fact that $f(z) = 1/z$ is analytic on $\mathbf{C} \setminus \{0\}$,

$$\int_\gamma \frac{1}{z^2}\ dz = 0.$$

8. We know already that $\int_\gamma 1/z\ dz$ around a circle centered at the origin is equal to $2\pi i$, so choose such a circle, called C, with small enough radius so that it lies completely within Δ. Next, join C to Δ by drawing straight line segments from C to each of the vertices of Δ. This creates 3 closed paths, each of which consists of a third of C, a line segments, a side of Δ, and another line segment, which puts us back at the starting point. Moreover, each of these three paths are contained in convex, open sets in which $f(z) = 1/z$ is analytic, so on each path, $\int_\gamma 1/z\ dz = 0$. Moreover, the sum of the three integrals is the difference between the integral along the circle and the integral along the circle, since the line segments cancel each other. Thus, we have that $\int_{\partial\Delta} 1/z\ dz = 2\pi i$, as required.

13. *Proof.* $|z-1| = 1$ describes the circle of radius 1 centered at $(1, 0)$. Thus, our path is a closed, circular contour. Now, we'll decompose the integral itself:

$$\int_{|z-1|=1} \frac{1}{z^2-1}\ dz = \int_{|z-1|=1} \frac{1}{2}\left(\frac{1}{z-1} - \frac{1}{z+1}\right)\ dz.$$

The integrand has two singularities: one at $z = 1$ and one at $z = -1$. Since our path is a circle centered about $z = 1$, we have that one singularity lies inside the circle whilst the other doesn't. Thus, for $z = 1$, $\text{Ind}(z) = 0$ and for $z = -1$, $\text{Ind}(z) = 1$. Thus, evaluating the integral term-by-term, we have that

$$\int_{|z-1|=1} \frac{1}{z-1}\ dz = 2\pi i \cdot \text{Ind}(z) = 0,$$

and

$$\int_{|z-1|=1} \frac{1}{z-1}\ dz = 2\pi i \cdot \text{Ind}(z) = 2\pi i.$$

Of course, taking the sum of the two and multiplying by $1/2$ gives πi, as required. Alternatively, along the path $|z+1| = 1$, we have that the values of the index numbers is switched (that is, $\text{Ind}(-1) = 0$ and $\text{Ind}(1) = 1$), inducing a sign change and giving $-\pi i$. ☐

15. *Proof.* Let's begin by figuring out what $f(0)$ is. By Cauchy's formula,

$$\text{Ind}_T(z) f(z) = \frac{1}{2\pi i}\int_T \frac{f(w)}{w-z}\ dw.$$

Letting $z = 0$, we have that $\text{Ind}_T(0) = 1$, since our curve is a closed circle. Moreover, we have that

$$f(0) = \frac{1}{2\pi i}\int_T \frac{f(w)}{w}\ dw.$$

We can bound the integral in the following way:

$$|f(0)| = \left|\frac{1}{2\pi i}\int_T \frac{f(w)}{w}\ dw\right| \le \frac{1}{2\pi}\int_T \left|\frac{f(w)}{w}\right|\ dw.$$

Evaluating the integral directly, we have that

$$\begin{aligned}\frac{1}{2\pi}\int_T \left|\frac{f(w)}{w}\right|\ dw &= \frac{1}{2\pi}\int_0^{2\pi} \left|\frac{f\left(e^{it}\right)}{e^{it}}\left(ie^{it}\right)\right|\ dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} \left|f\left(e^{it}\right)\right|\ dt\end{aligned}$$

However, since M is the maximum of f on T, we have that $|f| \le M$ on T, so

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} \left|f\left(e^{it}\right)\right|\ dt &\le \frac{1}{2\pi}\int_0^{2\pi} M\ dt \\ &= M,\end{aligned}$$

so $|f(0)| \le M$, as required. ☐