Math 427

Problem Set #3

Problems from text:

- §2.2: 8, 10, 11, 12.

Solutions to §2.2:

8. On the right half-plane (hereinafter RHP), $\log(z)/z$ is given by

$$\frac{\log(x+iy)}{x+iy} = \frac{\ln\left(x^2+y^2\right)}{2(x+iy)} + i\frac{\tan^{-1}(y/x)}{x+iy},$$

where $u = (1/2)\ln\left(x^2+y^2\right)\ (x+iy)$ and $\tan^{-1}(y/x)/(x+iy)$, so $u_x = v_y$ and $u_y = -v_x$. Thus, $\log(z)/z$ satisfies the Cauchy-Riemann (hereinafter CR) equations on the RHP. Furthermore, for some z not on the negative real axis and an angle $\varphi = \pm\pi/2$, we have that $e^{i\varphi}z$ gets rotated into the RHP. So

$$\frac{\log(z)}{z} \equiv \frac{\log\left(e^{i\varphi}z\right) - i\varphi}{e^{i\varphi}z}.$$

$\log(z)/z$ has derivative $(1-\log(w))/w^2$ so, by the chain rule, we have that its derivative is $(1-\log(z))/z^2$.

10. *Proof.* Suppose that $f(z) = z^2$ and that $z = x+iy$ is a point in $\mathbb{C}$. Expanding f, we have that

$$z^2 = (x+iy)^2 = x^2 + 2ixy - y^2 = \left(x^2-y^2\right) + i(2xy),$$

so

$$\begin{cases} u(x,y) = x^2 - y^2 \\ v(x,y) = 2xy. \end{cases}$$

Invoking the CR equations, we have that

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y},$$

and

$$\frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}.$$

So f satisfies the CR equations at $z = x+iy$ and so f is analytic at z. Since z is arbitrary, we're done. □

Moreover, $f(z) = f(x,y) = x^2 + 2ixy - y^2$, and since f satisfies the CR equations at z, we have that $f' = f_x = -if_y$, where

$$f_x = -if_y = 2x + 2iy$$

11. Since f can be decomposed as a function on some subset of $\mathbb{R}^2$, we have that, for $z = x+iy$, $f(z) \equiv f(x,y) = u(x,y) + iv(x,y)$. So, for f to be real-valued, we require $\Im(f(x,y)) = v(x,y) = 0$. Since f is required to be analytic, it must satisfy the CR equations at

z. Thus, $u_x = v_y$ and $u_y = -v_x$. However, since $v \equiv 0$, we have that $u_x = v_y = 0$ and $u_y = -v_x = 0$, so that $u_x = u_y = v_y = v_x = 0$. Thus,

$$\int \left(\frac{\partial u}{\partial x}\right) dx = \int 0 \, dx$$
$$\implies u(x,y) = g(y).$$

Alternatively,

$$\int \left(\frac{\partial u}{\partial y}\right) dy = \int 0 \, dy$$
$$\implies u(x,y) = h(x).$$

So, $u(x,y) = g(y) = h(x)$. Moreover, $g(y) = h(x) = K$, for some real constant K. That is, $g(y)$ and $h(x)$ can only agree if they're both constant. So $u(x,y)$ is a constant function. Thus, the function $f : U \longrightarrow \mathbb{R}$ (U being a subset of $\mathbb{C}$) that we're after is

$$\boxed{f(z) = K},$$

where $K \in \mathbb{R}$. Once again, we also know that f is analytic on $\mathbb{C}$ because it satisfies CR.

12. Since $x = r\cos(\theta)$ and $y = r\sin(\theta)$, we have that $x = x(r,\theta)$ and $y = y(r,\theta)$ and so

$$u(x,y) = u(x(r,\theta), y(r,\theta)).$$

Differentiating, we have

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta},$$
$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r},$$
$$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \theta},$$
$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial r}.$$

This gives

$$\frac{\partial u}{\partial \theta} = u_x(-r\sin(\theta)) + u_y(r\cos(\theta)) = -yu_x + xu_y,$$
$$\frac{\partial u}{\partial r} = u_x(\cos(\theta)) + u_y(\sin(\theta)) = \left(\frac{x}{r}\right)u_x + \left(\frac{y}{r}\right)u_y,$$
$$\frac{\partial v}{\partial \theta} = v_x(-r\sin(\theta)) + v_y(r\cos(\theta)) = -yv_x + xv_y,$$
$$\frac{\partial v}{\partial r} = v_x(\cos(\theta)) + v_y(\sin(\theta)) = \left(\frac{x}{r}\right)v_x + \left(\frac{y}{r}\right)v_y.$$

Now, observe that

$$u_r = \left(\frac{x}{r}\right)u_x + \left(\frac{y}{r}\right)u_y = \frac{1}{r}\left(xu_x + yu_y\right),$$

and by the CR equations, $u_x = v_y$ and $u_y = -v_x$, so

$$u_r = \left(\frac{x}{r}\right)u_x + \left(\frac{y}{r}\right)u_y = \frac{1}{r}\left(xv_y - yv_x\right) = \frac{v_\theta}{r},$$

as required. Similarly,

$$u_\theta = -yu_x + xu_y = r\left(-\frac{y}{r}\right)u_x + r\left(\frac{x}{r}\right)u_y,$$

and once again by the CR equations,

$$u_\theta = r\left(-\frac{y}{r}\right)v_y - r\left(\frac{x}{r}\right)v_x = -r\left(\left(\frac{x}{r}\right)v_x + \left(\frac{y}{r}\right)v_y\right) = -rv_r,$$

and we have the desired result.

Solutions to additional problems:

1. (a) For $f(z) = \cos(z)$, we have

$$\begin{aligned}\cos(z) &= \frac{1}{2}\left(e^{iz} + e^{-iz}\right)\\ &= \frac{1}{2}\left(e^{i(x+iy)} + e^{-i(x+iy)}\right)\\ &= \frac{1}{2}\left(e^{-y+ix} + e^{y-ix}\right)\\ &= \frac{1}{2}\left(e^{-y}\left[\cos(x) + i\sin(x)\right] + e^{y}\left[\cos(x) - i\sin(x)\right]\right)\\ &= \frac{1}{2}\left(\cos(x)\left[e^{y} + e^{-y}\right] - i\sin(x)\left[e^{y} - e^{-y}\right]\right)\\ &= \cos(x)\cosh(y) - i\sin(x)\sinh(y),\end{aligned}$$

whence we see that $\boxed{u(x,y) = \cos(x)\cosh(y)}$ and $\boxed{v(x,y) = -\sin(x)\sinh(y)}$.

(b) For $g(z) = ze^{\overline{z}}$, we have that

$$\begin{aligned}ze^{\overline{z}} &= (x+iy)e^{x-iy}\\ &= (x+iy)e^{x}(\cos(y) - i\sin(y))\\ &= (xe^{x} + iye^{x})(\cos(y) - i\sin(y))\\ &= xe^{x}\cos(y) - ixe^{x}\sin(y) + iye^{x}\cos(y) + ye^{x}\sin(y)\\ &= xe^{x}\cos(y) + ye^{x}\sin(y) + i\left(ye^{x}\cos(y) - xe^{x}\sin(y)\right)\\ &= e^{x}\left(x\cos(y) + y\sin(y)\right) + i\left(e^{x}\left[y\cos(y) + x\sin(y)\right]\right),\end{aligned}$$

for which $\boxed{u(x,y) = e^{x}\left(x\cos(y) + y\sin(y)\right)}$ and $\boxed{v(x,y) = e^{x}\left(y\cos(y) + x\sin(y)\right)}$.

2. (a) For $f(z) = \cos(z)$ as decomposed above, we have that

$$u_x = -\sin(x)\sinh(y),$$

$$u_y = \cos(x)\cosh(y),$$

$$-v_x = \cos(x)\sinh(y),$$

$$v_y = -\sin(x)\cosh(y),$$

for which we see that the CR equations aren't satisfied since $-\sin(x)\sinh(y) \neq -\sin(x)\cosh(y)$ and $\cos(x)\cosh(y) \neq \cos(x)\sinh(y)$.

(b) For $g(z) = ze^{\overline{z}}$, we have that

$$u_x = e^x(x\cos(y) + y\sin(y)) + e^x cos(y),$$

$$v_y = e^x(\cos(y) - y\sin(y) - x\cos(y)),$$
$$u_y = e^x(-x\sin(y) - \sin(y) - y\cos(y)),$$
$$v_x = e^x(\cos(y) - x\sin(y)) - e^x\sin(y),$$

for which, by the CR equations,
$e^x(x\cos(y) + y\sin(y)) + e^x\cos(y) \neq e^x(\cos(y) - y\sin(y) - x\cos(y))$ and
$e^x(-x\sin(y) - \sin(y) - y\cos(y)) \neq -e^x(\cos(y) - x\sin(y)) + e^x\sin(y)$, so the CR equations aren't satisfied.