Math 441

Problem Set #5

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Problem 1. Prove that the collection

$$\mathcal{B} = \{B_d(x,\epsilon) \;:\; x \in X,\ \epsilon > 0\}$$

is a basis for the topology on any set X with metric d.

Proof. Let X be a topological space and let d be a metric on X. Let $B_d(x,\epsilon)$ be a ball centered at some $x \in X$ with $\epsilon > 0$ its radius. Given any $y \in X$, we can take any $\epsilon > d(x,y)$, say $\epsilon = d(x,y) + \frac{1}{2}$, so that $y \in B_d(x,\epsilon) = B_d\left(x, d(x,y) + \frac{1}{2}\right)$. Moreover, for x_1 and x_2 in X and ϵ_1 and ϵ_2 being strictly positive, real numbers, we consider the intersection of balls given in the following way:

$$B_d(x_1,\epsilon_1) \cap B_d(x_2,\epsilon_2).$$

There exists a $z \in B_d(x_1,\epsilon_1) \cap B_d(x_2,\epsilon_2)$ so that $d(z,x_1) < \epsilon_1$ and $d(z,x_2) < \epsilon_2$. Now, take some $\delta > 0$ so that $\delta < d(z,x_1) - \epsilon_1$ and $\delta < d(z,x_2) - \epsilon_2$. Now, for any $x \in B_d(z,\delta)$, we have that

$$\begin{aligned} d(z,x) &< \delta \\ &< d(z,x_1) - \epsilon_1 \\ &< d(z,x_1) \\ &< \epsilon_1 \end{aligned}$$

so $d(z,x) < \epsilon_1$, so $x \in B_d(x_1,\epsilon_1)$. Similarly,

$$\begin{aligned} d(z,x) &< \delta \\ &< d(z,x_2) - \epsilon_2 \\ &< d(z,x_2) \\ &< \epsilon_2 \end{aligned}$$

so $d(z,x) < \epsilon_2$, so $x \in B_d(x_2,\epsilon_2)$. Thus, $x \in B_d(x_1,\epsilon_1) \cap B_d(x_2,\epsilon_2)$, so $B_d(z,\delta) \subset B_d(x_1,\epsilon_1) \cap B_d(x_2,\epsilon_2)$, as required. Thus, $\mathcal{B}$ satisfies the conditions of being a basis. $\square$

Problem 2. Prove that the distance function on $\mathbf{R}^n$ generates the standard topology on $\mathbf{R}^n$.

Proof. First, let's show that $d(\vec{x},\vec{y})$ is a metric. Let $\vec{x} = (x_1,\ldots,x_n)$ and $\vec{y} = (y_1,\ldots,y_n)$ be points in $\mathbf{R}^n$. First of all, we have that the distance function is positive-definite: for $\vec{x} = \vec{y}$, we have that $x_i = y_i$ for all $i \in [1,n]$, so $d(\vec{x},\vec{y}) = \sqrt{\sum_{i=1}^{n}(y_i - x_i)^2} = 0$. Alternatively, if $\vec{x} \neq \vec{y}$, then there exists at least one $i \in [1,n]$ for which $x_i \neq y_i$. For this particular i, we see that $(y_i - x_i) \neq 0 \Longrightarrow (y_i - x_i)^2 > 0$, so $d(\vec{x},\vec{y}) = \sqrt{\sum_{i=1}^{n}(y_i - x_i)^2} > 0$, since the square-root function is strictly increasing. We can easily verify symmetry: $d(\vec{x},\vec{y}) = \sqrt{\sum_{i=1}^{n}(y_i - x_i)^2} = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2} = d(\vec{y},\vec{x})$. Now, it remains to show that the distance function satisfies the triangle inequality: let $\vec{x}$, $\vec{y}$, and $\vec{z}$ be points in $\mathbf{R}^n$ and observe that

$$\begin{aligned}
d(\vec{x},\vec{y}) + d(\vec{y},\vec{z}) &= \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2} + \sqrt{\sum_{i=1}^{n}(y_i - z_i)^2} \\
&\geq \sqrt{\sum_{i=1}^{n}\left[(x_i - y_i)^2 + (y_i - z_i)^2\right]} \\
&\geq \sqrt{\sum_{i=1}^{n}(x_i - z_i)^2} \\
&= d(\vec{x},\vec{z}),
\end{aligned}$$

where the second line follows from Minkowski's inequality and the third line follows from the fact that $(x_i - y_i)^2 + (y_i - z_i)^2 \geq (x_i - z_i)^2$, since the metric $\tilde{d}(x,y) = (x-y)$ satisfies the triangle inequality trivially. Alright, now we know the distance function is a metric. Let's now show that the metric topology it generates can be identified with the standard topology on $\mathbf{R}^n$. Letting τ_d be the metric topology generated by the Euclidean metric and letting τ be the standard topology on $\mathbf{R}^n$, we must show that $\tau = \tau_d$, or that a set being open in one topology implies that it's open in the other. Let's begin by showing that an open set in the metric topology is open in the standard topology. Given some open $U \in \tau_d$, we have that U can be written as an arbitrary union of basis elements, so $U = \bigcup B_d(\vec{x},\epsilon)$, where $\vec{x} \in \mathbf{R}^n$ and $\epsilon > 0$. So, in particular, we see that, given some $\vec{y} \in U$, we have that $\vec{y} \in \bigcup B_d(\vec{x},\epsilon)$, so $\vec{y}$ is contained in at least one open ϵ-ball centered at some point in $\mathbf{R}^n$. Thus, $d(\vec{x},\vec{y}) < \epsilon \implies \sqrt{(x_1-y_1)^2 + \ldots + (x_n - y_n)^2} < \epsilon$. In the case that $n = 1$, we have that $|y - x| < \epsilon$, so $y \in (x - \epsilon, x + \epsilon)$. For arbitrary $n \geq 1$ we have that $\sqrt{\sum_{i=1}^{n}(x_i - y_i)^2} < \epsilon$, for which $\sum_{i=1}^{n}(x_i - y_i)^2 < \epsilon^2$. Now, for some $k \in [1,n]$ we have that $(x_k - y_k)^2 < \epsilon^2 - \sum_{i \neq k}(x_i - y_i)^2$, and so

$$|y_k - x_k| < \sqrt{\epsilon^2 - \sum_{i\neq k}(x_i - y_i)^2}$$

$$\implies y_k \in \left(x_k - \sqrt{\epsilon^2 - \sum_{i\neq k}(x_i - y_i)^2}, x_k + \sqrt{\epsilon^2 - \sum_{i\neq k}(x_i - y_i)^2} \right).$$

So, for each $k \in [1,n]$ we have that y_k is inside some neighborhood of x_k. Let the neighborhood of x_k be called (a_k, b_k) and observe that, since $y_k \in (a_k, b_k)$, we have that $\vec{y} \in (a_1,b_1) \times (a_2,b_2) \times \ldots \times (a_n,b_n)$. So $\vec{y}$ belongs to some basis element of the product topology on $(\mathbf{R} \times \mathbf{R} \times \ldots \times \mathbf{R}) \times \mathbf{R} \equiv \mathbf{R}^n$, so it's in a basis element of the standard topology on $\mathbf{R}^n$, and so U is open in the standard topology. □

Problem 3. Describe an ellipse in the square and taxicab metrics.

Solution. Given two fixed foci f_1 and f_2 and some point p, we have that p is on the ellipse if $d(f_1,p) + d(f_2,p) = k$, for some constant k. Thus, an ellipse is given by:

$$E = \{\vec{p} \in \mathbf{R}^2 \;:\; d(f_1,\vec{p}) + d(f_2,\vec{p}) = k\}.$$

Letting $\vec{p} = (x,y)$, $f_1 = (a_1,b_1)$ and $f_2 = (a_2,b_2)$, we have that, in the square metric,

$$E = \{(x,y) \;:\; \max\{|x - a_1|, |y - b_1|\} + \max\{|x-a_2|, |y-b_2|\} = k\}.$$

Similarly, in the taxicab metric, we have

$$E = \{(x,y) \;:\; |x-a_1| + |y-b_1| + |x-a_2| + |y-b_2| = k\}.$$

□

Problem 4. Show that, if $d : X \times X \to \mathbf{R}$ is a metric, then d, as a function of topological spaces, (where $X \times X$ has the product topology induced by the metric topology) is continuous.

Proof. Suppose that d is a metric on X. Let $\epsilon > 0$ and take some $(x, y) \in X \times X$. Moreover, let $(a, b) \subset \mathbf{R}$ be open and suppose $d(x, y) \in (a, b)$. If we let

$$U = B_d\left(x, \frac{\epsilon}{2}\right) \times B_d\left(y, \frac{\epsilon}{2}\right),$$

we have that U is a neighborhood of (x, y). Moreover,

$$d(U) \subseteq (d(x, y) - \epsilon, d(x, y) + \epsilon) \subset (a, b),$$

which follows by the triangle inequality. Thus, $U \subseteq d^{-1}(a, b) \implies B_d\left(x, \frac{\epsilon}{2}\right) \times B_d\left(y, \frac{\epsilon}{2}\right) \subseteq d^{-1}(a, b)$. So $d^{-1}(a, b)$ is open, as required. □

Problem 5. Let (X, τ) be a topological space and let X be equipped with a metric d. Show that, if $d : X \times X \to \mathbf{R}$ is continuous, then τ is finer than the metric topology.

Proof. To show that the topology on X is finer than the metric topology on X, we need to show that an open set in the metric topology is open in the topology on X. The continuity of d implies that the set

$$U = \{(u, v) \in X \times X \ : \ d(u, v) < \epsilon\}$$

or simply the preimage of the interval $(-\infty, \epsilon)$ under d, is open in $X \times X$. Now, fix some $x_0 \in X$ and take $\epsilon > 0$. We want to show that, for any $y \in B_d(x_0, \epsilon)$ (i.e. an open set in the metric topology), there exists an open V so $y \in V \subseteq B_d(x_0, \epsilon)$. Since $y \in B_d(x_0, \epsilon)$, we have that $(x_0, y) \in U$. Since U is open in $X \times X$ by assumption, there are open sets S and T so that $(x_0, y) \in S \times T \subset U$. Note that, in particular, T is an open set continuing y, which is what we wanted. Thus, we are done. □

Problem 6. Prove the following:

Proposition. If $A \subset X$ and X is a metric space, then the subspace topology on A is equal to the metric topology on A generated by $d|_A$.

Proof. Let X be a metric space with metric $d : X \times X \to \mathbf{R}$. To show that these topologies are equal, it's sufficient to show that they have the same basis. Let $\mathcal{B}$ be a basis for the subspace topology on A and let $\mathcal{C}$ be a basis for the metric topology on A generated by $d|_A$. In particular, we want to show that these two bases are equal, so we must show that $\mathcal{B} \subseteq \mathcal{C}$ and $\mathcal{C} \subseteq \mathcal{B}$. We'll show the following first:

Lemma. For some $x \in X$ and $\epsilon > 0$, $B_d(x, \epsilon) \cap A = B_{d|_A}(x, \epsilon)$.

Proof (of lemma). Suppose that $t \in B_{d|_A}(x, \epsilon)$. By definition, this means that $d|_A(t, x) < \epsilon$. However, $d|_A$ is defined so that $d|_A : A \times A \to \mathbf{R}$. So by definition of $d|_A$, t must live in A. However, $A \subset X$, so for any $u, v \in X$, we have that $\{d|_A(u, v)\} \subset \{d(u, v)\}$. So $t \in B_d(x, \epsilon)$, as well, and so $t \in B_d(x, \epsilon) \cap A$. Conversely, Suppose $t \in B_d(x, \epsilon) \cap A$. From this, we have that $d(t, x) < \epsilon$ and $t \in A$. Since d takes ordered pairs from $X \times X$, but $t \in A$ and $A \subset X$, this is equivalent to saying that $d|_A(t, x) < \epsilon$, so $t \in B_{d|_A}(x, \epsilon)$, as required. □

We'll begin by showing that $\mathcal{B} \subseteq \mathcal{C}$. For some $U \in \mathcal{B}$, we have that $U = B_d(x, \epsilon) \cap A$ for some $x \in X$ and $\epsilon > 0$. So, in particular, $U \subseteq B_d(x, \epsilon) \cap A$ and $U \supseteq B_d(x, \epsilon) \cap A$. For any $s \in U$, we have that $x \in B_d(x, \epsilon) \cap A$, so $s \in B_d(x, \epsilon)$ for $x \in X$, $\epsilon > 0$, and $s \in A$. So s is within the ϵ-ball around x, but s is also in A, so $d|_A(s, x) < \epsilon$ and so $s \in B_{d|_A}(x, \epsilon)$. Thus, $U \subseteq B_{d|_A}(x, \epsilon)$. Conversely, for some $s \in B_{d|_A}(x, \epsilon)$, we have that $d|_A(s, x) < \epsilon$, so $s \in A$ and $s \in B_d(x, \epsilon)$. Thus, $B_{d|_A}(x, \epsilon) \subseteq U$. So $U = B_{d|_A}(x, \epsilon)$ and so $\mathcal{B} \subseteq \mathcal{C}$. On the other hand, given some $U \in \mathcal{C}$, we have that $U = B_{d|_A}(x, \epsilon)$ for some $x \in X$ and $\epsilon > 0$ and so $U \subseteq B_{d|_A}(x, \epsilon)$ and $U \supseteq B_{d|_A}(x, \epsilon)$. For some $s \in U$, it follows that $s \in B_{d|_A}(x, \epsilon)$, so once again $s \in B_d(x, \epsilon) \cap A$. So $U \subseteq B_d(x, \epsilon) \cap A$. Alternatively, for $s \in B_d(x, \epsilon) \cap A$, we have that $s \in B_{d|_A}(x, \epsilon)$, so $s \in U$. Thus, $B_d(x, \epsilon) \cap A \subseteq U$ and so $U = B_d(x, \epsilon) \cap A$. So $\mathcal{C} \subseteq \mathcal{B}$, giving us that $\mathcal{B} = \mathcal{C}$, as required. □

Problem 7. Show that every metric space is Hausdorff.

Proof. Let X be a metric space endowed with the metric d. Suppose that x and y are distinct points in X. By definition of the metric, we have that $d(x, y) > 0$. Now, choose $\epsilon = d(x, y)/4$ and create the balls $B_d(x, \epsilon)$ and $B_d(y, \epsilon)$. We have that

$$\begin{aligned} B_d(x,\epsilon) \cap B_d(y,\epsilon) &= \{u \in X \ : \ d(u,x) < \epsilon\} \cap \{v \in X \ : \ d(v,y) < \epsilon\} \\ &= \left\{u \in X \ : \ d(u,x) < \frac{d(x,y)}{4}\right\} \cap \left\{v \in X \ : \ d(v,y) < \frac{d(x,y)}{4}\right\}. \end{aligned}$$

Suppose that this intersection is nonempty. For any t in this intersection, we have that both $d(t, x) < d(x, y)/4$ and $d(t, y) < d(x, y)/4$. By symmetry, we may write $d(t, x) \equiv d(x, t)$. Since d is a metric, it must obey the triangle inequality. Thus,

$$\begin{aligned} d(x,y) &\leq d(x,t) + d(t,y) \\ &< \frac{d(x,y)}{4} + \frac{d(x,y)}{4} \\ &= \frac{d(x,y)}{2}, \end{aligned}$$

but since $d(x, y) > 0$ this is a contradiction. Thus, the intersection must be empty and so X is Hausdorff, as required. □