MATH 441

Problem Set #6

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Problem 1. Prove the following theorem:

Theorem. A subspace $Y \subset X$ is connected (in the subspace topology) if it is *not* a separated subspace.

Proof. Suppose that there exists no separation of Y. Further suppose that Y is *not* connected. If Y is not connected, then there exist nonempty, disjoint, closed subsets U and V such that $Y = U \cup V$. Now, using basic set-theoretic operations,

$$\begin{aligned}\overline{U} \cap B &= \overline{U} \cap (Y \cap V) \\ &= (\overline{U} \cap Y) \cap V \\ &= U \cap V \\ &= \varnothing,\end{aligned}$$

where the second line to the third line follows from the fact that U is closed in Y. Similarly,

$$\begin{aligned}U \cap \overline{B} &= (U \cap Y) \cap \overline{V} \\ &= U \cap (Y \cap \overline{V}) \\ &= U \cap V \\ &= \varnothing.\end{aligned}$$

So, we've found nonempty subsets U and V for which $Y = U \cup V$, and $\overline{U} \cap V = U \cap \overline{V} = \varnothing$, which constitutes a separation of Y. This is a contradiction, since we had assumed that Y had no separation. $\square$

Problem 2. Let $\tilde{\mathbb{R}^n}$ be the collection of sequences $\{x_i\}_{i \in \mathbf{N}}$ such that $x_i = 0$ for $i > n$. Show that the map $f : \tilde{\mathbb{R}^n} \to \mathbb{R}^n$ given by $\{x_i\}_{i \in \mathbf{N}} \mapsto (x_1, \ldots, x_n)$ is a homeomorphism.

Proof. Before we really get into the proof, let's gain some intuition for what f does: if we set $n = 1$, then $\tilde{\mathbb{R}^n} = \tilde{\mathbb{R}}$, which is the set of sequences which, after the first term, are zero. What f will do, then, is take such a sequence and assign it to a point in $\mathbb{R}$. So, $f(\{x_1, 0, 0, \ldots\}) = x_1$. Likewise, for $n = 2$, we have that $f(\{x_1, x_2, 0, 0, \ldots\}) = (x_1, x_2)$. So, in general,

$$f(\{x_1, x_2, \ldots, x_n, 0, 0, \ldots\}) = (x_1, x_2, \ldots, x_n),$$

where f hacks off everything after the n^{th} term (all of these terms are, by assumption, zero) in the sequence and gives us a set of coordinates in $\mathbb{R}^n$ consisting of the first n terms, some of which could be zero. Okay, now we need to show that such an f is a homeomorphism. Let's begin with bijectivity: to show that f is injective, if we take two sequences $\{x_i\}$ and $\{y_i\}$ in $\tilde{\mathbb{R}^n}$ that are distinct, then we know that the two sequences differ by at least one term. Since we know they're both zero after the n^{th} term, they can't differ there, so they must differ somewhere in the first n terms. Now, $f(\{x_1, \ldots, x_n, 0, 0, \ldots\}) = (x_1, \ldots, x_n) = \vec{x}$ and $f(\{y_1, \ldots, y_n, 0, 0, \ldots\}) = (y_1, \ldots, y_n) = \vec{y}$. Since at least one of the first n terms for $\{x_i\}$ and $\{y_i\}$ is different, we have that $\vec{x} \neq \vec{y}$. Now, if we take any $\vec{t} = (t_1, \ldots, t_n) \in \mathbb{R}^n$, then certainly there exists a sequence $\{t_i\} = \{t_1, \ldots, t_n, 0, 0, \ldots\} \in \tilde{\mathbb{R}^n}$ so that, when we apply f to $\{t_i\}$, we get back $\vec{t}$. Thus, f is surjective and is hence bijective. From this, we now know, as well, that f^{-1} exists. What remains for us to do, now, is show that f and f^{-1} are continuous. Let's begin with the continuity of f: let $U = U_1 \times U_2 \times \ldots \times U_n \subset \mathbb{R}^n$ be a basis element, assuming the product topology on $\mathbb{R}^n$, and for any $\vec{x} = (x_1, \ldots, x_n)$ contained in U, we have that $x_i \in U_i$. Now, consider the preimage

$$f^{-1}(U) = \{\{x_i\} \in \tilde{\mathbb{R}^n} \ : \ f(\{x_i\}) \in U\}.$$

We need to show that this is open in $\tilde{\mathbb{R}^n}$, which amounts to showing that, for any sequence in $f^{-1}(U)$, there's a basis element of $\tilde{\mathbb{R}^n}$ containing that sequence. What do the open sets in $\tilde{\mathbb{R}^n}$ look like? Well, $\tilde{\mathbb{R}^n}$ has basis elements that look like $\tilde{\mathbb{R}^n} \cap V$, where V is open in $\mathbb{R}^\infty$. So, $\tilde{\mathbb{R}^n}$ is a subspace of $\mathbb{R}^\infty$ and so has the subspace topology inherited from $\mathbb{R}^\infty$. So, for U as prescribed, we have that the preimage of U under f is the collection of sequences that get sent into U. Let $\{x_i\} \in \tilde{\mathbb{R}^n}$ be arbitrary, and since f sends $\{x_i\}$ into U, we know that, for $i \in [1, n]$, $x_i \in U_i$, an open set. We see that $\{x_i\} \in \mathbb{R}^\infty$, since $x_i = 0$ for $i > n$, so certainly there exists an N for which $x_i > N$ is zero afterward − it's just $N = n$.

So $\{x_i\}$ is contained in a basis element in $\mathbb{R}^\infty$, so $\{x_i\} \in \tilde{\mathbb{R}^n} \cap V$. Now, we need to verify the continuity of f^{-1}. Suppose that $\mathbb{R}^n \cap V$ (where V is open in $\mathbb{R}^\infty$) is open in $tilde\mathbb{R}^n$, and take some $\vec{x} \in f(\tilde{\mathbb{R}^n} \cap V)$. For each coordinate of $\vec{x}$, we have that $x_i \in U_i$, for some open U_i, so $\vec{x} \in U_1 \times \ldots \times U_n$, which is a basis element of $\mathbb{R}^n$, so $f(\tilde{\mathbb{R}^n} \cap V)$ is open in $\mathbb{R}^n$, and so f^{-1} is continuous. From this, f is a homeomorphism. □

Problem 3. Let $X = \prod_{\alpha \in J} X_\alpha$, where each X_α is connected, and fix a point $a = \{a_\alpha\}_{\alpha \in J}$.

(i) Given a finite set $K \subset J$, let X_K denote all $\vec{x} = \{x_\alpha\}_{\alpha \in J}$ such that $x_\alpha = a_\alpha$ for all $\alpha \notin K$. Show that X_K is connected.

(ii) Show that $Y = \bigcup_{K \subset J} X_K$ is connected (for finite K).

(iii) Show that $X = \overline{Y}$ and deduce that X is connected.

(i) *Proof.* We know that finite products of connected spaces are connected, and since K is finite, certainly $\prod_{\alpha \in K} X_\alpha$ is connected. Thus, if we can show that X_K is homeomorphic to $\prod_{\alpha \in K} X_\alpha$, then we'll be done. Let's try to identify this homeomorphism. The most natural map we can choose is

$$\varphi : X_K \to \prod_{\alpha \in K} X_\alpha.$$

To streamline things, say that $K = \{1, 2, \ldots, n\}$. We're allowed to say that K has countable indices since it's finite – outside of K is where that becomes a problem, but that doesn't matter. So, our map is

$$\varphi : X_K \to \prod_{i=1}^{n} X_i,$$

where, for $\vec{x} \in X_K$, φ removes the coordinates of $\vec{x}$ that don't agree with those of a. So, if $a = (a_1, a_2, \ldots)$ and $\vec{x} = (x_1, \ldots, x_n, a_{n+1}, \ldots)$, then $\varphi(\vec{x}) = (x_1, \ldots, x_n)$. Likewise, φ^{-1} fills the missing coordinates back in.

Let's verify the continuity of φ: suppose that U is open in $\prod_{i=1}^{n} X_i$. Then $U = U_1 \times \ldots \times U_n$, where U_i is open in X_i. Because, as said before, φ^{-1} replaces the disagreeing coordinates, we have that

$$\varphi^{-1}(U) = U_1 \times \ldots \times U_n \times \{a_{n+1}\} \times \ldots$$

so that, for any $\vec{x} \in \varphi^{-1}(U)$, we have $\vec{x} \in X_K$ trivially, but also $\vec{x} \in V = \prod_{\alpha \in J} V_\alpha$, where V is open in X. So, $\vec{x}$ is contained in a basis element for the subspace topology on X_K (X_K is a subspace of X), so $\varphi^{-1}(U)$ is open, and thus φ is continuous. Now, define

$$\varphi^{-1} : \prod_{i=1}^{n} X_i \to X_K$$

by the map $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, a_{n+1}, \ldots)$. We have

$$\varphi^{-1}(\varphi((x_1, \ldots, x_n, a_{n+1}, \ldots))) = \varphi^{-1}((x_1, \ldots, x_n)) = (x_1, \ldots, x_n, a_{n+1}, \ldots),$$

and

$$\varphi(\varphi^{-1}((x_1, \ldots, x_n))) = \varphi((x_1, \ldots, x_n, a_{n+1}, \ldots)) = (x_1, \ldots, x_n).$$

Thus, φ^{-1} exists, and so φ is bijective. We now need to verify that φ^{-1} is continuous: suppose that $X_K \cap V$ is open in X_K (where V is open in X). For any $(x_1, \ldots, x_n) \in \varphi(X_K \cap V)$, we have that $x_i \in U_i$, where U_i is open in X_i. Thus, $(x_1, \ldots, x_n) \in U_1 \times \ldots \times U_n$, which is a basis element for the product topology on $\prod_{i=1}^{n} X_i$. Thus, φ^{-1} is continuous, and so φ is a homeomorphism. Since $X_K \cong \prod_{\alpha \in K} X_\alpha$, we have that X_K is connected. □

(ii) *Proof.* We must prove that, for all finite $K \subset J$, the union $\bigcup X_K$ is connected. First of all, we know that, for all K, each X_K has a in it, so

$$a \in \bigcap X_K,$$

and so the intersection of all X_K is nonempty. Suppose that $\bigcup X_K$ is not connected; that is, let

$$\bigcup X_K = A \cup B,$$

for A and B being disjoint, nonempty, open subsets. Now, fix some $\vec{x} \in \bigcap X_K$ and, without loss of generality, suppose that $\vec{x} \in A$. By assumption, B is nonempty, so pick some $\vec{u} \in B$. We have that there is some X_K so that $\vec{u} \in X_K$, and as assumed $\vec{x} \in X_K$, as well. So, $\vec{x} \in A \cap X_K$ and $\vec{u} \in B \cap X_K$, so that $A \cap X_K$ and $B \cap X_K$ are both nonempty, a contradiction. Thus, $\bigcup X_K$ must be connected. □

(iii) *Proof.* Take some point $\vec{x} = \{x_\alpha\} \in X$. Then there exists an open neighborhood containing $\vec{x}$, called $V = \prod_{\alpha \in J} V_\alpha$, where each V_α is open in X_α. We see that V further contains another neighborhood $U = \prod_\alpha U_\alpha$, where $U_\alpha = X_\alpha$ for $\alpha \notin K$. Letting

$$c_\alpha = \begin{cases} x_\alpha & \text{if } \alpha \in K \\ a_\alpha & \text{if } \alpha \notin K \end{cases}$$

we have that $\{c_\alpha\} \in (X_K \cap U)$, so $Y \cap U \neq \varnothing$, so $\{x_\alpha\} \in \overline{Y}$, as required. □

Problem 4. Prove that a path-connected space is connected.

Proof. Let X be a space, and suppose that X is path-connected. Let x_1 and x_2 be distinct points in X and, since X is path-connected, let $\varphi : [a, b] \to X$ be the continuous path joining x_1 and x_2. Without loss of generality, assume that $\varphi(a) = x_1$ and $\varphi(b) = x_2$. Now, suppose that $X = U \cup V$ is a separation and that $x_1 \in U$ and $x_2 \in V$. Since U and V are open in X, by assumption, and φ is continuous, $\varphi^{-1}(U)$ and $\varphi^{-1}(V)$ are open in $[a, b]$. Moreover, they're also disjoint since U and V are disjoint (preimages preserve "disjointness"). Lastly, U and V are obviously nonempty. Since $\varphi(a) = x_1$ and $x_1 \in U$, we have that $\varphi(a) \in U$, so $a \in \varphi^{-1}(U)$. Likewise, $\varphi(b) = x_2$ and $x_2 \in V$, we have that $\varphi(b) \in V$, so $b \in \varphi^{-1}(V)$. Thus, $\varphi^{-1}(U)$ and $\varphi^{-1}(V)$ are nonempty. Now, we note that $[a, b] = \varphi^{-1}(U) \cup \varphi^{-1}(V)$, which gives a separation of $[a, b]$. This, however, is a contradiction since $[a, b]$ is a subset of $\mathbb{R}$, and is connected. Thus, X has to be connected, and we are done. □