MATH 441: TOPOLOGY

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Contents

Disclaimer: This course used James Munkres' *Topology*, second edition.

1 Preliminaries

The study of topology cuts swiftly and deeply through several previously-touched areas of mathematics. To keep up, it is encouraged that one has at least seen, in some form, the following notions:

- Set theory (set notation, intersection, union, power set, Cartesian product, etc.)
- General proofwriting techniques (direct, contraposition, contradiction, etc.)
- Functions (injection, surjection, bijection, inverse, composition, etc.)
- Elementary abstract algebra (equivalence relations, partitioning of sets, groups, etc.)
- Elementary analysis (ϵ-δ arguments, continuity, convergence of sequences, sup and inf, etc.)

1.1 Finite, Countable, and Uncountable Sets

Definition. A *finite* set S is one that admits a bijection $f : S \to \{1, 2, \ldots, n\}$.

Definition. A *countable* set S is one that admits an injection $f : S \to \mathbf{Z}_+$.

Proposition. The set of integers $\mathbf{Z}$ is countable.

Proof. Let $f : \mathbf{Z} \to \mathbf{Z}_+$ be given by

$$f(n) = \begin{cases} 2n & \text{if } n \geq 0 \\ -2n+1 & \text{if } n \leq 0. \end{cases}$$

We now must show that f is an injection; that is, we must verify that, if $f(n) = f(m)$ for some integers n and m, then $n = m$. We thus develop four cases:

1. If $2n = 2m$, then certainly $n = m$.
2. If $-2n + 1 = -2m + 1$, then $n = m$.
3. If $-2n + 1 = 2m$, then $\frac{1}{2} = n + m$. Since n and m are integers, this is a contradiction.
4. If $2n = -2m + 1$, then $\frac{1}{2} = n + m$. Since n and m are integers, this is a contradiction.

Thus, in all valid cases, we have that $f(n) = f(m) \implies n = m$, as required. $\square$

Theorem. If S is infinite and countable, then there is a bijection $f : S \to \mathbf{N}$.

Theorem. The set of rationals $\mathbf{Q}$ is countable.

Proof. Note that there is a one-to-one correspondence between $\mathbf{Q}$ and the set $\mathbf{Z} \times \mathbf{Z}$ described by the map $\frac{a}{b} \mapsto (a, b)$. Thus, if we can prove that $\mathbf{Z} \times \mathbf{Z}$ is countable, then we've proven that $\mathbf{Q}$ is countable. By Cantor's diagonal argument, $\mathbf{Z} \times \mathbf{Z}$ is countable, and so we're done. $\square$

2 Introduction

Topology is the study of spaces and how they behave under deformations. It does this through a rigorous framework built upon notions that are very strongly set-theoretic. The word *topology* derives its name from the roots *topos*, meaning "place" or "region", and *logos*, meaning "the study of." It is important to note, however, that topology is *not* geometry! Generally speaking, topology *is* geometry, but only up to *continuous deformations*, meaning deformations that don't cut or puncture the surface or piece it back together – more succinctly, everything stays in one piece throughout while being deformed. This notion has earned topology the nickname "rubber sheet geometry."

2.1 Topologies and Topological Spaces

Definition. A *topology* on a set X is a collection of subsets denoted $\boxed{\tau}$, where $\boxed{\tau \subseteq \mathcal{P}(X)}$. We also have that τ satisfies the following properties:

(i) $\varnothing \in \tau$

(ii) $X \in \tau$

(iii) If $\{U_i\}_{i=1}^n$ is a finite collection of elements of τ, then so is $\displaystyle\bigcap_{i=1}^{n} U_i$.

(iv) If $\{U_i\}_{i\in I}$ is a finite collection of elements of τ, then so is $\displaystyle\bigcup_{i\in I} U_i$.

From this, we have that a topology is closed under finite intersection and under arbitrary union.

Definition. A pair $\boxed{(X, \tau)}$ is called a *topological space*.

Remark. The elements of τ are *open* sets.

2.2 Exploring Some Specific Topologies

Example. Let X be a set, and define $\tau_{\text{fin}} \subset \mathcal{P}(X)$ to be the collection of subsets $U \subseteq X$ such that $X \setminus U$ is either all of X or is finite. Show that τ_{fin} is a topology on X.

Proof. We first check the trivial cases: $\varnothing \in \tau_{\text{fin}}$ since $X \setminus \varnothing = X$ is, in fact, all of X. Alternatively, $X \in \tau_{\text{fin}}$ since $X \setminus X = \varnothing$ is finite[1]. We now check closure under union and intersection:

- Closure under arbitrary union: Let $\{U_i\}_{i\in I}$ be elements from τ_{fin}. We have that $X \setminus \displaystyle\bigcup_{i\in I} U_i = \bigcap_{i\in I}(X \setminus U_i)$, by DeMorgan's law. We now have to show that $\displaystyle\bigcap_{i\in I}(X \setminus U_i)$ is finite or all of X. It's trivial that it can be all of X, so we see that $\displaystyle\bigcap_{j\in I}(X \setminus U_j) \subset X \setminus U_j$ for all finite j. Since $X \setminus U_j$ is finite, it stands that $\displaystyle\bigcap_{j\in I}(X \setminus U_j)$ is finite.
- Closure under finite intersection: A similar result follows using DeMorgan's law as before.

From this, we see that τ_{fin} is a topology on X. $\square$

[1] The null set is finite since it isn't infinite.

Definition. Suppose that τ and τ' are topologies on X. If $\tau \subseteq \tau'$, we say that τ' is *finer* than τ. Alternatively, we say that τ is *coarser* than τ'.

Definition. A *basis* for a topology on X is a collection $\boxed{\mathcal{B} \subseteq \mathcal{P}(X)}$ such that:

(i) For every $x \in X$, there exists a $B \in \mathcal{B}$ such that $x \in B$.

(ii) If $x \in B_1 \cap B_2$, then there exists $B_3 \in \mathcal{B}$ such that $x \in B_3$ and $B_3 \subset (B_1 \cap B_2)$.

Definition. If $\mathcal{B} \subset \mathcal{P}(X)$ is a basis on X, then the topology generated by $\mathcal{B}$, denoted $\boxed{\tau_{\mathcal{B}}}$, is defined in the following way:

- The elements of $\tau_{\mathcal{B}}$ are precisely those $U \in \mathcal{P}(X)$ such that, for any $x \in U$, there exists a $B \in \mathcal{B}$ such that $x \in B$ and $B \subset U$.

Theorem. Let X be a set and $\mathcal{B}$ a basis on X. We have that $\tau_{\mathcal{B}}$ is precisely the collection of all unions of elements of $\mathcal{B}$.

Example. The collection of all open disks in the plane is a basis on $\mathbf{R}^2$.

Proposition. If $\mathcal{B}$ is a basis on X, then $\tau_{\mathcal{B}}$ is a topology on X.

Definition. A *sub-basis* is a collection of sets whose closure under finite intersection and arbitrary union is a topology.

Definition. The set of *all* open intervals $(a, b) \subset \mathbf{R}$ forms a basis for the *standard topology* on $\mathbf{R}$, denoted $\boxed{\mathbf{R}_{\text{std}}}$.

Definition. The set of *all* half-open intervals $[a, b) \subset \mathbf{R}$ forms a basis for the *lower limit topology*, denoted $\boxed{\mathbf{R}_{\ell}}$.

Definition. Let $K = \left\{ \frac{1}{n} \;\middle|\; n \in \mathbf{Z}_+ \right\}$, and let $\mathcal{B}$ be the collection of sets of the form $(a, b) \smallsetminus K$. We say that this forms a basis for the K - *topology* on $\mathbf{R}$, denoted $\boxed{\mathbf{R}_K}$.

Remark. Given two topologies τ_1 and τ_2 on X, there *always* exists a mutual "refinement", called τ_3, such that $\tau_3 = \tau_1 \cap \tau_2$. That is, given any two topologies on the same set, there's always a third topology that is *coarser* than both of the two topologies.

Remark. τ_3 (as defined above) is a topology on X.

Example. Let $M_{n \times n}(\mathbf{R})$ denote the space of $n \times n$ matrices with real coordinates. There exists a bijection φ between $M_{n \times n}(\mathbf{R})$ and the space $\mathbf{R}^{n^2}$. What is this bijection? Define φ via the following map:

$$\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \mapsto \begin{pmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{1n} \\ a_{21} \\ \vdots \\ a_{2n} \\ \vdots \\ a_{nn} \end{pmatrix}.$$

So, φ strips the coordinates from each row of the $n \times n$ matrix and stacks them vertically in the resulting vector, and repeats this until it exhausts all the coordinates of the matrix. The resulting vector consists of n vertical strips of n coordinates, so its dimension is n^2. Now, note that $\mathbf{R}^{n^2}$ is a topological space, in the standard sense. We define the topological structure of it in the following way: $U \subseteq M_{n\times n}(\mathbf{R})$ is open if $\varphi(U)$ is open. So, this topology is *induced* by the function φ.

3 Making New Topologies from Old Ones

3.1 The Product Topology

Definition. Let (X, τ_1) and (Y, τ_2) be topological spaces. The *product topology* on the set $X \times Y$ is the topology generated by the basis consisting of the sets $U \times V$ where $U \in \tau_1$ and $V \in \tau_2$.

Theorem. If $\mathcal{B}$ is a basis for τ_1 and $\mathcal{C}$ is a basis for τ_2, then $\mathcal{B} \times \mathcal{C}$ is a basis for the product topology on $X \times Y$.

Proof. exercise. $\square$

Example. We can recover the standard topology on $\mathbf{R}^2$ as the product topology on $\mathbf{R} \times \mathbf{R}$.

Remark. We can further define *infinite products* of topological spaces: let J be a set and let $\{X_\alpha\}_{\alpha\in J}$ be a set of topological spaces indexed by J. We have that

$$\prod_{\alpha\in J} X_\alpha = \left\{ \{x_a\}_{\alpha\in J} \mid x_\alpha \in X_\alpha \right\}.$$

The notation $\{x_\alpha\}_{\alpha\in J}$ is rather strange, especially since we're defining a "tuple" of sorts. We're trying to represent it in such a way that it denotes a collection of elements from each X_α, but they're not necessarily ordered, since J is an arbitrary set that may not even be countable. To avert the temptation to write something like $\{x_\alpha\}_{\alpha\in J} = (x_{\alpha_1}, x_{\alpha_2}, \ldots, x_{\alpha_j})$, we innocuously say $\{x_\alpha\}_{\alpha\in J} = \vec{x}$, instead.

Proposition. Let $\mathcal{B}$ and $\mathcal{C}$ be bases for the topologies $\tau_{\mathcal{B}}$ and $\tau_{\mathcal{C}}$, respectively, on a set X. The following are equivalent:

(i) $\tau_{\mathcal{C}}$ is finer than $\tau_{\mathcal{B}}$.

(ii) For each $x \in X$ and each $B \in \mathcal{B}$ such that $x \in B$, there exists a $C \in \mathcal{C}$ such that $x \in C \subset B$.

Proof. We'll prove both implications separately:

- (i) $\implies$ (ii): Suppose $U \in \tau_{\mathcal{B}}$, and let $x \in U$. Since $U \in \tau_{\mathcal{B}}$, there exists a $B \in \mathcal{B}$ such that $x \in B \subset U$. By (ii), there exists $C \in \mathcal{C}$ so that $x \in C \subset B \subset U$, and so $U \in \tau_{\mathcal{B}}$.
- (ii) $\implies$ (i): Let $x \in X$ and $B \in \mathcal{B}$ such that $x \in B$. Since τ_C is finer than $\tau_{\mathcal{B}}$, we have that $B \in \tau_{\mathcal{B}} \subseteq \tau_{\mathcal{C}}$, implying that $B \in \tau_{\mathcal{C}}$. So, by definition, there exists a $C \in \mathcal{C}$ such that $x \in C \subset B$.

$\square$

Proposition. As a consequence of the preceding proposition, the collection of open rectangles in the plane and the collection of open disks in the plane generate the same topology.

Pseudo-proof. Let τ_{rect} be the topology generated by the collection of open rectangles in the plane and let τ_{disk} be the topology generated by the collection of open disks in the plane. By the preceding argument, we have that $\tau_{\text{rect}} \subseteq \tau_{\text{disk}}$ and $\tau_{\text{disk}} \subseteq \tau_{\text{rect}}$. $\square$

Remark. The above problem can be tackled using classical techniques from analysis, but mathematicians often shy away from using pure analysis to justify properties unless it's absolutely necessary for them to do so. The problem becomes much more approachable when we phrase the problem as a topological matter.

3.2 Projection Maps

Definition. Let X and Y be sets equipped with arbitrary topologies. We define two functions, called *projection maps* or, simply, *projections*, defined by:

(i) $\pi_1 : X \times Y \to X$, where $(x, y) \mapsto x$,

(ii) $\pi_2 : X \times Y \to Y$, where $(x, y) \mapsto y$.

In other words, given a finite product of topological spaces $X_1 \times X_2 \times \ldots \times X_n$, we have that π_i assigns each tuple in the product space to its i^{th} coordinate.

Remark. More generally, we can define a projection that operates on the aforementioned infinite, indexed product space:

$$\pi_\beta : \prod_{\alpha \in J} X_\alpha \to X_\beta,$$

which assigns each element of the product space to its β^{th} coordinate.

Definition. Given $S \subset X$ and $T \subset Y$, we denote the *preimage of S under* π_1 as

$$\boxed{\pi_1^{-1}(S) = \{(x, y) \in X \times Y \mid \pi_1(x, y) \in S\}}.$$

Similarly, we denote the *preimage of T under* π_2 as

$$\boxed{\pi_2^{-1}(T) = \{(x, y) \in X \times Y \mid \pi_2(x, y) \in T\}}.$$

Remark. In other words, we're describing the sets of ordered pairs such that, when π_1 and π_2 are applied to them, they return single elements that belong to *subsets* of X or Y.

Remark. It's worth noting that π_1^{-1} and π_2^{-1} are **not** inverses, at least in the traditional sense of a function having an inverse. That is, we cannot say that $\pi_1^{-1}(\pi_1(x, y)) = (x, y)$ because it isn't true! We've already seen that π_1 and π_2 return single elements of either X or Y, so saying that $\pi_1^{-1}(\pi_1(x, y)) = (x, y)$ is equivalent to saying that $\pi_1^{-1}(x) = (x, y)$, but this is egregiously wrong because π_1^{-1} is defined on entire *subsets* of X, not on single elements of X. Likewise, π_1^{-1} doesn't return a single ordered pair in $X \times Y$, but returns an entire *set* of such ordered pairs.

Theorem. Let X and Y be topological spaces. The set

$$\boxed{\mathcal{S} = \left\{ \pi_1^{-1}(U) \;\middle|\; U \text{ open in } X \right\} \cup \left\{ \pi_2^{-1}(V) \;\middle|\; V \text{ open in } Y \right\}}$$

is a sub-basis for the product topology on $X \times Y$.

Proof. Let τ_{prod} be the product topology on $X \times Y$, and let $\tau_{\mathcal{S}}$ be the topology generated by $\mathcal{S}$. Notice that $\pi_1^{-1}(U)$ and $\pi_2^{-1}(V)$ are both elements of τ_{prod}, so they're both open sets. Thus, $\tau_{\mathcal{S}} \subset \tau_{\text{prod}}$. Alternatively, note that

$$U \times V = \pi_1^{-1}(U) \cap \pi_2^{-1}(V) = (X \cap U) \times (Y \cap V),$$

and so $\tau_{\mathcal{S}} \supset \tau_{\text{prod}}$. $\square$

Remark. Observe that $\mathcal{S}' = \{A \cap B \mid A, B \in \mathcal{S}\} \supset \{\text{basis for the product topology}\}$, and $\mathcal{S}'' = \left\{\bigcup_{\alpha \in I} U_\alpha \;\middle|\; U_\alpha \in \mathcal{S}'\right\} \supset \tau_{\text{prod}}$, and so $\tau_{\mathcal{S}} \supset \tau_{\text{prod}}$. The other inclusion works similarly.

Remark. The standard topology on $\mathbf{R}^n$ can be phrased as an iterated product topology: first consider the product topology on $\mathbf{R} \times \mathbf{R}$, and then the product topology on $(\mathbf{R} \times \mathbf{R}) \times \mathbf{R}$, and – ultimately – the product topology on $(\mathbf{R} \times \ldots \times \mathbf{R}) \times \mathbf{R}$.

3.3 The Subspace Topology

The general idea of this section is that, if (X, τ) is a topological space and $Y \subseteq X$, then Y *should* have some kind of topology on it similar to that defined on X. This is, perhaps, an intuitive notion in that, if X has some salient geometric properties, then those properties should, in some way, "inform" the geometry of Y. What we glean from this is that Y will "inherit" some sort of geometric structure from X in the form of a topology.

Definition. Let (X, τ) be a topological space. If $Y \subseteq X$, then the collection

$$\boxed{\tau_Y = \{Y \cap U \mid U \text{ open in } X\}}$$

is a topology on Y called the *subspace topology*, or *induced topology*.

Remark. The subspace topology on Y is a topology on Y.

Proof. exercise. □

Lemma. If $\mathcal{B}$ is a basis for the topology on X, then

$$\boxed{\mathcal{B}_Y = \{B \cap Y \mid B \in \mathcal{B}\}}$$

is a basis for τ_Y.

Lemma. If Y is an open set in X, then every open set in the subspace topology on Y is an open set in X.

Proof. If U is open in Y, then $U = Y \cap V$ for some $V \in X$, and so U is open in X. □

Theorem. If A is a subspace[2] of X and B is a subspace of Y, then the product topology on $A \times B$ is exactly the subspace topology on $(A \times B) \subset (X \times Y)$.

Remark. This is another case in which we take a certain topology which we've seen already and phrase it as a new topology.

4 Closed Sets

Definition. A set $A \subseteq X$, for some topological space X, is *closed* if $X \setminus A$ is an open set in X (i.e., it belongs to the topology on X).

Theorem. We have the following properties:

(i) X and $\varnothing$ are both closed in X.

(ii) A collection of closed sets is closed under arbitrary intersection and finite union.

Remark. We can define a topology on a set X using closed sets rather than open sets.

[2]By "subspace," we mean that, if A is a subspace of a topological space X, then $A \subseteq X$ and is, itself, a topological space equipped with the topology inherited from X.

Theorem. Let Y be a subspace of a topological space X. A subset $A \subset Y$ is *closed in* Y (in the subspace topology τ_Y) iff $A = Y \cap C$ for some closed $C \subset X$.

Proof. We prove two directions:

$\Longrightarrow$: Assume that $A = Y \cap C$ for C closed in X. It follows that $X \setminus C$ is open in X and $(X \setminus C) \cap Y$ is open in Y. Moreover, $(X \setminus C) \cap Y = Y \setminus A$ is open, so A is closed in Y.

$\Longleftarrow$: Conversely, assume that A is closed in Y. From this, it follows that $Y \setminus A = Y \cap U$ for some open $U \in X$. Thus, $X \setminus U$ is closed in X and $A = Y \cap (X \setminus U)$. Since $X \setminus U$ is closed in X, this completes the proof.

□

Theorem. If Y is closed in X and A is closed in Y, then A is closed in X.

Remark. This implies a "transitivity" about closed sets.

5 Closure and Interior

Definition. The *interior* of a subset $A \subset X$, where X is a given topological space, is defined as

$$\boxed{A^\circ := \bigcup_{\text{open } U \subset A} U}.$$

In essence, we are taking all of the open spaces inside of A and piecing them together to get a representation of the "total" open space inside of A.

Definition. The *closure* of a subset $A \subset X$, where X is a given topological space, is defined as

$$\boxed{\overline{A} := \bigcap_{\text{closed } C \supset A} C}.$$

Remark. If U is open, A is a subset of X, and C is closed, then the following are always true:

(i) $U^\circ = U$.

(ii) $\overline{C} = C$.

(iii) $A^\circ \subseteq A \subseteq \overline{A}$.

Remark. If Y is a subspace of a topological space X and $A \subset Y$, then the closure of A in Y and the closure of A in X can be different, by virtue of the fact that X and Y may very well be equipped with different topologies. This is, perhaps, best illustrated in the following example:

Example. Let $Y = (0, 1] \subset \mathbf{R}$ and let $\mathbf{R}$ be equipped with the standard topology. Furthermore, let $A = \left(0, \frac{1}{2}\right)$. From this, we see that the closure of A in Y is the set $\left(0, \frac{1}{2}\right]$ whilst the closure of A in $\mathbf{R}$ is given by the set $\left[0, \frac{1}{2}\right]$.

Remark. $\overline{A}$ is always closed due to the fact that closed sets are closed under arbitrary intersection. For the same reason, we also have that A° is always open.

Theorem. Let A be a subset of a topological space X.

(i) For any $x \in X$, we have that $x \in \overline{A}$ if and only if every open $U \in X$ (where $x \in U$) admits that $U \cap A$ is nonempty.

(ii) Suppose that $\mathcal{B}$ is a basis on X. We have that $x \in A$ if and only if every $B \in \mathcal{B}$ (where $x \in \mathcal{B}$) admits that $\mathcal{B} \cap A$ is nonempty.

Definition. If, $A \subset X$, define the *boundary* of A by

$$\boxed{\partial A = \overline{A} \cap \overline{(X \setminus A)}}.$$

Moreover, if A is open, then this reduces to $\boxed{\partial A = \overline{A} \setminus A}$.

Definition. Let W and Z be sets equipped with topologies, and define a function $f : W \to Z$. We say that f is an *open map* if, for some U open in W, $f(U)$ is open in Z.

Proposition. For any two topological spaces X and Y, the functions $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are open maps.

Theorem. Let A be a subset of a topological space X.

(i) For any $x \in X$, $x \in \overline{A}$ if and only if, for all open U containing x, $U \cap A = \varnothing$.

(ii) If $\mathcal{B}$ is a basis for X, then $x \in \overline{A}$ if and only if, for any $B \in \mathcal{B}$ containing x, $B \cap A = \varnothing$.

Proof. Proving each proposition separately:

(i) ($\Longrightarrow$) If $x \notin \overline{A}$, then $x \in U$ for $U = X \setminus A$, but $X \setminus \overline{A}$ is open. Thus, $U \cap A = \varnothing$, since $A \subset \overline{A}$. ($\Longleftarrow$) Suppose that there exists an open set U containing x such that $U \cap A = \varnothing$. It follows, then, that $X \setminus U$ is closed and also contains A. However, $x \in X \setminus U$ since $x \in U$. So $x \notin \overline{A}$.

(ii) ($\Longrightarrow$) If $x \in \overline{A}$, then every open set U continuing x admits that $U \cap A = \varnothing$. Since basis elements are always open by definition, we have that if $x \in B$, then $B \cap A = \varnothing$. ($\Longleftarrow$) Suppose that $x \in B$ so that $B \cap A = \varnothing$. However, since $x \in U$, there exists a $B \subset U$ with $x \in B$ and $B \cap A \neq \varnothing$, so $U \cap A \neq \varnothing$, and so $x \in \overline{A}$.

□

Definition. An open set U in a topological space X is called a *neighborhood* of $x \in X$ when $x \in U$.

Example. Consider the real line equipped with the standard topology. For some $x \in \mathbf{R}$ and an $\epsilon > 0$, the interval $(x - \epsilon, x + \epsilon)$ is a neighborhood of x.

Example. Consider $\mathbf{R}^3$ equipped with the standard topology (or, if one prefers, the product topology on $(\mathbf{R} \times \mathbf{R}) \times \mathbf{R}$). For some $\vec{x} \in \mathbf{R}^3$ and an $\epsilon > 0$, the open ball $B^3_\epsilon(\vec{x})$ is a neighborhood of $\vec{x}$.

Definition. Let X be a topological space and $A \subset X$ be some subset. If $\overline{A} = X$, then A is *dense* in X.

Example. The set of rationals $\mathbf{Q}$ is dense in $\mathbf{R}$. By extension, $\mathbf{Q}^n$ is dense in $\mathbf{R}^n$.

6 Limit Points

Definition. Let A be a subset of a topological space X. We say that $x \in X$ is a *limit point*, or *accumulation point*, of A if every neighborhood of x, denoted $U(x)$, has the property that $\boxed{(U(x) \setminus \{x\}) \cap A \neq \varnothing}$.

Remark. The above definition is equivalent to saying that $U(x)$ intersects A in a point that is not x itself. That is, $U(x) \cap A \neq x$.

Theorem. Let A' be the set of all limit points of A. We have that

$$\boxed{\overline{A} = A \cup A'}.$$

Proof. Proving both inclusions:

- $\supseteq$: Let $x \in \overline{A}$ be given. If $x \in A$, it's a trivial consequence that $x \in A \cup A'$. Conversely, suppose that $x \notin A$. Since $x \in \overline{A}$, it follows that every neighborhood of x intersects A; since $x \notin A$, any neighborhood of x must intersect A in a point different from x. Thus, $x \in A'$ and so $x \in A \cup A'$.
- $\subseteq$: If $x \in A'$, then every neighborhood of x intersects A in a point different from x. Thus, $x \in \overline{A}$, and so $A' \subseteq \overline{A}$. By definition, $A \subseteq \overline{A}$, and so it follows that $A \cup A' \subseteq \overline{A}$.

$\square$

Remark. This is equivalent to saying that there exist *no* limit points of A that don't belong to $\overline{A}$.

Definition. Let X be a topological space and let $(x_i)_{i=1}^{\infty}$ be a countable sequence in X (that is, for $x_i \in X$ and i a positive integer, the map $x_i \mapsto i$ is an injection) We say that $x_i \to y$ if, for all open sets U containing y, there exists an N such that $x_i \in U$ for all $i \geq N$.

7 Hausdorff Spaces

Definition. A topological space X is called a *Hausdorff space* (read: X is Hausdorff) if, for every $x_1, x_2 \in X$ such that $x_1 \neq x_2$, there exist neighborhoods U_1 and U_2 of x_1 and x_2, respectively, such that $U_1 \cap U_2 = \varnothing$.

Remark. Any Euclidean space is Hausdorff: for any $\vec{x}$ and $\vec{y}$ in $\mathbf{R}^n$, assume that $\text{dist}(\vec{x}, \vec{y}) = \epsilon$. If we let B_1 and B_2 be balls of radius ϵ/n centered at $\vec{x}$ and $\vec{y}$, respectively, where $n > 2$, then B_1 and B_2 don't intersect.

Remark. Any metric space is Hausdorff.

Theorem. In a Hausdorff space, a convergent sequence converges to *exactly* one point.

Proof. exercise[3]. $\square$

Example. Define the *spectrum of the integers*, $\boxed{\text{Spec}(\mathbf{Z})}$, as the set of all prime numbers, as well as 0. Define the topology on $\text{Spec}(\mathbf{Z})$ to be the closed sets indexed by the integers in the following way:

$$\boxed{C(n) = \{\text{all primes } \mathfrak{p} \text{ or zero} \mid \mathfrak{p} \text{ divides } n\}}.$$

It's immediately clear that $C(1) = \varnothing$ since no primes divide 1. Also observe that $C(0) = \text{Spec}(\mathbf{Z})$ and $C(\mathfrak{p}) = \{\mathfrak{p}\}$.

Claim. The closed sets of $\text{Spec}(\mathbf{Z})$ are closed under arbitrary intersection and finite union. Moreover, $\text{Spec}(\mathbf{Z})$ is *not* Hausdorff.

Proof. exercise. $\square$

[3] It's ultimately pointless to encourage this as an exercise since it will be proven shortly, anyways.

7.1 Spaces That Aren't Hausdorff

Here, we'll construct some spaces that are not Hausdorff spaces and show that unreasonable consequences can transpire in non-Hausdorff spaces. This isn't to say that being Hausdorff is a strict requirement for topological spaces – topologists work in non-Hausdorff spaces very frequently. This is simply to suggest that being Hausdorff is a "nice" restriction on topological spaces much like in abstract algebra how linearity is a sought-after property of maps.

Example. Consider $\mathbf{R}$ equipped with the finite complement topology. We claim that this space isn't Hausdorff. Suppose, however, that it is – our construction is as follows: consider the elements 0 and 1. Let U and V be open and declare that $0 \in U$ and $1 \in V$. Further suppose that $U \cap V = \varnothing$. From this, it arises that $U \subseteq V^c$, so U is a finite set. Since U is a finite set of reals, it follows that U^c is infinite, and so U isn't open, as originally claimed – this develops a contradiction.

Theorem. If X is Hausdorff, then any finite subset $\{x_1, \ldots, x_n\} \subset X$ is closed.

Proof. To reduce the complexity of the proof, we only need to verify that singleton subsets of X are closed since any subset of X can be expressed as a finite union of singleton sets. Consider $x \in X$, and observe that for any $y \in X$ where $y \neq x$ is not in the closure of the singleton set $\{x\}$. Thus, there exist disjoint neighborhoods $U \ni y$ and $V \ni x$, which shows that y has a neighborhood containing it that is disjoint from the set $\{x\}$. $\square$

Remark. There exist non-Hausdorff spaces in which finite sets are closed.

Example. Again, consider $\mathbf{R}$ with the finite complement topology. If $\{x_1, \ldots, x_n\} \subset \mathbf{R}$ is finite, then $\mathbf{R} \setminus \{x_1, \ldots, x_n\}$ is an open set, and so $\{x_1, \ldots, x_n\}$ is closed.

Remark. A topological space in which finite sets are closed is called a T1 space. As a sub-remark, it's worth noting that Hausdorff spaces are T2 and there further exist T3 and T4 spaces.

Theorem. Let X be T1 and take some subset $A \subset X$. We claim that x is a limit point of A if and only if every open neighborhood of x contains infinitely-many points of A.

Proof. ($\Longrightarrow$) If every neighborhood $U \ni x$ contains infinitely-many points of A, then $(U \setminus \{x\}) \cap A = \varnothing$. So, by definition, x is a limit point of A. ($\Longleftarrow$) Alternatively, suppose that x is a limit point of A and further suppose that $(U\ \{x\}) \cap A = (U \cap A) \setminus \{x\}$ is finite, so that $(U \cap A) \setminus \{x\} = \{x_1, \ldots, x_m\}$. From this, it follows that $U \setminus \{x_1, \ldots, x_m\} = U \cap (X \setminus \{x_1, \ldots, x_m\})$ is open, and $x \in U \setminus \{x_1, \ldots, x_m\}$. So now, observe that $((U \setminus \{x_1, \ldots, x_m\}) \setminus \{x\}) \cap A = ((U \setminus \{x\}) \cap A) \setminus \{x_1, \ldots, x_m\} = \varnothing$, which gives rise to a contradiction. $\square$

Theorem. If X is Hausdorff, then a sequence of points in X converges to *at most* one point.

Remark. This is a slightly different phrasing of a previous theorem that stated that sequences of points in some Hausdorff space X converge to *exactly one* point. It's simply stating that, if it converges, it converges to a single point. Alternatively, if it diverges, it converges to no points.

Proof. Suppose that $(x_n)_{n=1}^{\infty}$ is a sequence of points (hereinafter denoted (x_n)) in X that converges to a point z. For some $y \neq z$, there exist disjoint neighborhoods U and V of y and z, respectively. Thus, there exists an N such that, for any $n \geq N$, we have that $x_n \in V$. However, only a finite subset of (x_n) can be contained in U. Thus, (x_n) cannot converge to y. $\square$

Let's now examine one of the more notable curiosities that arise in non-Hausdorff spaces:

Example. Consider $\mathbf{R}$ equipped with the finite complement topology. Let $(x_n)_{n=1}^{\infty} = (n)_{n=1}^{\infty}$ be a sequence in $\mathbf{R}$. Take some $c \in \mathbf{R}$ for which U is a neighborhood containing it. We see that $\mathbf{R} \setminus U = \{u_1, \ldots, u_k\}$ is finite. Now, defining $M = \max(u_1, \ldots, u_k)$ allows us to take $N > M$ for which, if $n > N$, then $x_n \in U$. This is because, as n ranges over $[1, N]$, x_n exhausts all of $\mathbf{R} \setminus U$ and so must lie in U thereafter. Of course, the only restriction we placed on c was that it was a real number. Thus, $(x_n) = (n)$ converges to *every* real number, which is absolutely perverse.

Remark. While we've eeked out a technically correct result, it is nonetheless unreasonable and is an example of what happens when we don't require a space to be Hausdorff. Just before, we proved that any sequence in a Hausdorff space must converge to exactly one point – here, we've removed the requirement that our space is Hausdorff and no sooner had we done that did we discover a sequence that converges to infinitely-many points. This, of course, fundamentally violates the notion learned in analysis that the limit of a sequence is unique. That result, however, stems from the fact that real and complex analysis only work over $\mathbf{R}$ or $\mathbf{C}$ equipped with their standard topologies, making them Hausdorff, by definition.

8 Continuous Functions

In analysis, one devotes a lot of time to building the machinery needed to define continuity of functions. A recurring theme in topology, however, is that one can re-visit these notions formed in analysis and re-phrase them using the topological framework we've built so far. This doesn't always make matters any simpler or more complicated, but simply couches them in a much more set-theoretic manner that eschews using any calculus. The definition of continuity we'll be working with is as follows:

Definition. A function between topological spaces $f : X \to Y$ is *continuous* if, for all open sets $V \subset Y$, one has that $f^{-1}(V) = \{x \in X \mid f(x) \in V\}$ is open in X.

Remark. Note that the above definition doesn't rely on the notion of the limit or its ϵ-δ formulation – it simply states a relation between open sets in Y and open sets in X.

Remark. Recall the we previously defined an *open map* – that is, some function π defined so that, for some open subset of the domain, applying π to that open set produces an open subset of the codomain. That's not exactly what's happening here. In this definition, we're picking an arbitrary open subset of the codomain and finding that the preimage of that subset is open in the domain. So, a map being open is not tautologous with it being continuous.

Proposition. If $\mathcal{B}$ is a basis for the topology on Y and $f^{-1}(B)$ is open for every $B \in \mathcal{B}$, then f is continuous.

Before we proceed with the proof, let's build up some machinery we'll need to do so:

Claim. For any collection of subsets $A_i \subset Y$ for $i \in I$, we have that

$$f^{-1}\left(\bigcup_{i \in I} A_i\right) = \bigcup_{i \in I} \left(f^{-1}(A_i)\right).$$

Proof of claim. Take $x \in f^{-1}\left(\bigcup_{i \in I} A_i\right)$ for which $f(x) \in \bigcup_{i \in I} A_i$. Thus, $f(x) \in A_j$ for some $j \in I$. Of course, this means that $x \in f^{-1}(A_j)$, so $x \in \bigcup_{i \in I} \left(f^{-1}(A_i)\right)$. Conversely, suppose that $x \in \bigcup_{i \in I} f^{-1}(A_i)$. This implies that $x \in f^{-1}(A_j)$ for some $j \in I$, but this further implies that $f(x) \in A_j$, and so

$f(x) \in \bigcup_{i\in I} A_i$. It follows from this that $x \in f^{-1}\left(\bigcup_{i\in I} A_i\right)$. This proves the claim so that we can now proceed with the main proof.

$\rightsquigarrow$ Suppose that f is continuous and take $B \in \mathcal{B}$. By definition, B is open; couple this with the fact that f is continuous and we have that $f^{-1}(B)$ is open. Conversely, suppose that $f^{-1}(B)$ is open for all $B \in \mathcal{B}$. Let $U \subset Y$ be an open set. Observe that

$$U = \bigcup_{\alpha} B_\alpha,$$

for all $B_\alpha \in \mathcal{B}$. This implies that

$$f^{-1}(U) = f^{-1}\left(\bigcup_{\alpha} B_\alpha\right) = \bigcup_{\alpha} f^{-1}(B_\alpha),$$

which follows via the previous claim. Of course, the rightmost expression is open, so f is continuous. This completes the proof. □

Now, we'll ask ourselves, *how well does this definition recover the one learned in analysis?* Here, we'll consider continuity in the context of classical analysis and use the techniques we've just developed to connect it to the topological definition:

Proposition. If $X = Y = \mathbf{R}$, then $f : X \to Y$ is continuous if and only if f is continuous by the ϵ-δ argument.

Proof. Suppose that f is continuous by the ϵ-δ definition. That is, suppose that for all $x_0 \in \mathbf{R}$ and $\epsilon > 0$, there exists a $\delta > 0$ so that if $|x - x_0| < \delta$ then $|f(x) - f(x_0)| < \epsilon$. Now, let $V = (a, b)$ be an open interval in $\mathbf{R}$ and take $x \in f^{-1}(V) = f^{-1}((a, b))$. Clearly, we have that $f(x) \in (a, b)$. Let $\epsilon = \min\left(|f(x) - a|, |f(x) - b|\right)$ so that there exists a δ such that if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$. From this, it follows that $f\left((x - \delta, x + \delta)\right) \subset (a, b)$, and so $(x - \delta, x + \delta) \subset f^{-1}((a, b)) = U$. The converse is left as an exercise. □

Does a function's continuity depend on the topologies its domain and codomain are equipped with? Yes – consider the following example:

Example. Consider the *identity function* $\mathrm{id} : \mathbf{R} \to \mathbf{R}$ given by the map $x \mapsto x$. Let the domain be equipped with the standard topology and let the codomain be equipped with the lower limit topology. Note that $\mathrm{id}^{-1}\left([a, b)\right)$ isn't open in $\mathbf{R}$ with respect to the standard topology – this means that $\mathrm{id}(x)$ isn't continuous when we map from the reals equipped with the standard topology into the reals equipped with the lower limit topology. Here's where it gets even more interesting: now, equip the domain with the lower limit topology and equip the codomain with the standard topology. Under these stipulations, we now see that $\mathrm{id}^{-1}\left((a, b)\right)$ is open in $\mathbf{R}$ with respect to the lower limit topology. Thus, $\mathrm{id}(x)$ is now continuous.

Theorem. Let $f : X \to Y$ be a function between topological spaces. TFAE:

(i) f is continuous.

(ii) For every subset $A \subset X$, we have $f\left(\overline{A}\right) = \overline{f(A)}$.

(iii) For any closed set $C \subset Y$, $f^{-1}(C)$ is closed.

(iv) For each $x \in X$ and every neighborhood $V \ni f(x)$, there exists a neighborhood $U \ni x$ so that $f(U)$ is contained in V.

Proof. We'll prove (i) $\implies$ (ii) $\implies$ (iii) $\implies$ (i), and then separately prove (i) $\iff$ (iv).

(i) $\implies$ (ii): Let $x \in \overline{A}$ be given and let V be a neighborhood of $f(x)$. Since f is continuous, $f^{-1}(V)$ is open and contains x. Moreover, since $x \in \overline{A}$, $f^{-1}(V) \cap A \neq \varnothing$. Let $y \in f^{-1}(V) \cap A$ and observe that $V \cap f(A) \neq \varnothing$.

(ii) $\implies$ (iii): Let $C \subset Y$ be closed. Let $A = f^{-1}(C)$. We now need to show that $A = \overline{A}$ so A is closed. Note that $f(A) = f\left(f^{-1}(C)\right) \subseteq C$. So if $x \in \overline{A}$ then $f(x) \in f\left(\overline{A}\right) \subseteq \overline{f(A)}$, by (ii) and $\overline{f(A)} \subset \overline{C} = C$. So $x \in f^{-1}(C) = A$.

(iii) $\implies$ (i): Let V be open in Y. Let $C = Y \setminus V$. We have that $f^{-1}(C) = f^{-1}(Y) \setminus f^{-1}(V) = X \setminus f^{-1}(V)$, which is closed, and so $f^{-1}(V)$ is open.

(i) $\iff$ (iv): Suppose that $x \in X$ and $V \ni f(x)$ is a neighborhood. Take $U = f^{-1}(V)$ since $f\left(f^{-1}(V)\right) \subseteq V$. Conversely, let V be an open set in Y and let $x \in f^{-1}(V)$ so that $f(x) \in V$. By assumption, there exists some $U_x \ni x$ so that $f\left(U_x\right) \subset V$.

$\square$

Theorem. Let X, Y, and Z be topological spaces.

(i) If $f : X \to Y$ is given by $\boxed{x \mapsto y_0}$ for some fixed $y_0 \in Y$, then f is continuous, in which case f is a *constant function.*

(ii) If $A \subset X$ is a subspace of X, then the inclusion map $\boxed{\iota : A \hookrightarrow X}$ is continuous.

(iii) If $f : X \to Y$ and $g : Y \to Z$ are continuous functions, then $\boxed{(g \circ f) : X \to Z}$ is continuous.

(iv) If $f : X \to Y$ is continuous and $A \subset X$ is a subspace, then the *restriction* $\boxed{f|_A : A \to Y}$ is continuous.

(v) Let $f : X \to Y$ be continuous. If $Z \subset Y$ and $\operatorname{im}(f) \subseteq Z$, then $\boxed{g : X \to Z}$ is continuous. Moreover, if $Z \supset Y$, then the map $\boxed{X \to Y \to Z}$ is continuous.

8.1 Homeomorphisms

Definition. A map between topological spaces $f : X \to Y$ is called a *homeomorphism* if:

(i) f is bijective (so indeed f^{-1} exists),

(ii) f is continuous,

(iii) f^{-1} is continuous.

Remark. *Homeomorphism* sounds eerily like *homomorphism*, but the two shan't be confused. A *homomorphism* is a structure-preserving map used almost exclusively in algebra and combinatorics. Typically, we speak of homomorphisms between groups, rings, lattices, and graphs. Alternatively, a homomorphism between topological spaces is exactly the aforementioned *open map* which is not terribly useful in the grand scheme of mathematics. Rather, a homeomorphism is a *continuous map*, which isn't always structure-preserving, as we'll see in some upcoming examples.

Remark. We'll occasionally write $X \cong Y$ to say that "X is homeomorphic to Y."

Remark. If X and Y are homeomorphic, then they have the same number of "holes." We'll formalize what this means later. Conversely, if X and Y aren't homeomorphic, then they have a different number of holes. This intuition will break down when we dive into the quotient topology, but that's a separate story.

Example. Let $f : \mathbf{R} \to \mathbf{R}$ be given by $x \mapsto 3x + 1$. Assume that the domain and codomain of f are equipped with their standard topologies. We can verify that f is a homeomorphism because, first of all, it's bijective due to it having a two-sided inverse: let $g(y) = (y-1)/3$ and observe that $f(g(y)) = y$ and $g(f(x)) = x$. Moreover, f is continuous and so is its inverse – this follows from basic calculus. Thus, f is a homeomorphism.

Remark. A quick musing about the preceding example: note that a homeomorphism only preserves the topological structure between X and Y but doesn't preserve their structure as *vector spaces*, as one would see in linear algebra. Note that, in particular, $f(0) = 1$, and so zero doesn't get sent to zero. Also, *distances* in $\mathbf{R}$ aren't preserved under f – observe that $f(0) = 1$ and $f(1) = 4$. We see that $|1 - 0| = 1$ and $|4 - 1| = 3$.

Example. Let $\gamma : (-1, 1) \to \mathbf{R}$ be given by $x \mapsto \dfrac{x}{x^2 + 1}$. It's left to the reader to verify that γ is a homeomorphism.

Example. Let $\rho : [0, 1) \to \mathbf{R}^2$ be given by $x \mapsto (\cos(2\pi x), \sin(2\pi x))$. Note that the image of ρ is the circle of radius 1, also known as S^1, the 1-torus. We can verify that $\rho : [0, 1) \to S^1$ is a bijection, but doesn't constitute a homeomorphism. Why?

8.2 Embeddings

Definition. Let $f : X \to Y$ be injective, and suppose that $Z = \operatorname{im}(X) \subset Y$. If the map

$$\boxed{f' : X \to Z}$$

is a homeomorphism and Z has the subspace topology, then f is called an *embedding* of X in Y.

Remark. What we observe above is a bijection between the topological data of X and the topological data in a subset of Y. Essentially, we're able to represent a full copy of the topological data in X in some part of Y, so the data gets *embedded* into Y.

9 Metric Spaces

9.1 Metrics

Topology is largely focused on studying the geometric properties of spaces, and one of the ways we can "enrich" the geometry of a space is by endowing it with some measure of "distance" within the space. Enter the metric:

Definition. Let X be a set. A *metric* on X is a map $\boxed{d : X \times X \to \mathbf{R}}$ given by $\boxed{(x, y) \mapsto d(x, y)}$ that satisfies the following properties:

(i) Positive-definite: $d(x, y) \geq 0$ for all x and y in X. More specifically, $d(x, y) > 0$ for $x \neq y$ and $d(x, y) = 0$ for $x = y$.

(ii) Symmetric: $d(x, y) = d(y, x)$.

(iii) Triangle inequality: $d(x, y) + d(y, z) \geq d(x, z)$.

Remark. Generally speaking, a metric describes the "distance" between points in X, and the preceding properties reflect exactly that.

9.2 The Metric Topology

Definition. Given a set X, a metric d on X, and some $\epsilon > 0$, we define the *open ball of radius* ϵ *centered at* $x \in X$ to be:

$$\boxed{B_d(x, \epsilon) = \{y \in X \mid d(x, y) < \epsilon\}}.$$

Definition. If X is a set with metric d, then the *metric topology* on X is the topology whose basis is given by

$$\boxed{\mathcal{B} = \{B_d(x, \epsilon) \mid x \in X,\ \epsilon > 0\}}.$$

Corollary. A set $U \subset X$ is said to be *open with respect to the metric topology* if, for any $x \in U$, there exists a $\delta > 0$ so that $B_d(x, \delta) \subset U$.

Example. Let X be a set and define $d(x, y)$ as

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

We can verify that $d(x, y)$ is a metric:

Solution. First of all, it's positive-definite – for $x = y$, we have $d = 0$ and for $x \neq y$, we have $d = 1 > 0$. Secondly, it's symmetric because $d(x, y) = d(y, x)$. Lastly, it satisfies the triangle inequality – we have that $d(x, y) + d(y, z) = 0$ if and only if $x = y = z$, and $d(x, z) = 0$, as well, so $d(x, y) + d(y, z) \geq d(x, z) \to 0 + 0 \geq 0$. Alternatively, if $d(x, y) + d(y, z) \geq 1$ then we have two possibilities: if $d(x, y) + d(y, z) = 1$, then one of the summands must be 1 while the other is 0. Without loss of generality, suppose that $d(x, y) = 1$ and $d(y, z) = 0$ in which case $x \neq y$ and $y = z$. By transitivity, we must then have that $x \neq z$, so $d(x, z) = 1$, and so we have $d(x, z) \leq d(x, y) + d(y, z)$. A similar case is made for when $d(x, y) + d(y, z) > 1$. □

Remark. The preceding metric is often referred to as the *discrete metric*, and the metric topology generated by it is the *discrete topology* where $\tau = \mathcal{P}(X)$.

Example. Let $X = \mathbf{R}$ and define $d(x, y) = |x - y|$. This is often referred to as the *Euclidean metric* on $\mathbf{R}$ and the metric topology it generates is the standard topology on $\mathbf{R}$.

Remark. Given $\vec{x} = (x_1, x_2, \ldots, x_n)$ and $\vec{y} = (y_1, y_2, \ldots, y_n)$, the Euclidean metric in $\mathbf{R}^n$ is given by:

$$\begin{aligned} d(\vec{x}, \vec{y}) &= \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2 + \ldots + (y_n - x_n)^2} \\ &= \boxed{\sqrt{\sum_{i=1}^{n} (y_i - x_i)^2}} \end{aligned}$$

Definition. A topological space (X, τ) is called a *metric space* if it has a metric d defined on it and the metric topology that d generates is the same as the topology on X.

Remark. A space X can have a metric and a topology defined on it that don't agree. That's fine, it just means that X is not a metric space.

Definition. Let X be a metric space and take $A \subset X$. We say that A is *bounded* if there exists a real number M such that $\boxed{d(a_1, a_2) \leq M}$ for any pair of points a_1 and a_2 in A.
Moreover, if A is bounded, then the *diameter* of A is given by
$\boxed{\operatorname{diam}(A) = \sup\{d(a_1, a_2) \mid a_1, a_2 \in A\}}$.

Remark. We can have two metric spaces (X, d, τ_d) and $(X, d', \tau_{d'})$ where $d \neq d'$ but $\tau_d = \tau_{d'}$. That is, different metrics on the same space X can induce equal, or homeomorphic, topologies.

Theorem. Let X be a metric space with metric d and define

$$\boxed{\overline{d}(x, y) = \min\{d(x, y), 1\}}.$$

We claim that $\overline{d}$ is a metric and that it generates the same topology on X as d does.

Proof. We'll first show that $\overline{d}$ is a metric. The fact that $\overline{d}$ is positive-definite and symmetric is rather trivial, so we'll focus mostly on the triangle inequality: first suppose that $d(x, y) \geq 1$ or $d(y, z) \geq 1$. We thus have that, in the inequality

$$\overline{d}(x, z) \leq \overline{d}(x, y) + \overline{d}(y, z),$$

RHS ≥ 1, so $\overline{d}(x, z) \leq 1$, as required. Conversely, suppose that $d(x, y) < 1$ and $d(y, z) < 1$. It follows that

$$d(x, z) \geq \underbrace{d(x, y)}_{<1} + \underbrace{d(y, z)}_{<1},$$

so

$$d(x, z) \leq \overline{d}(x, y) + \overline{d}(y, z)$$

and, by definition, $\overline{d}(x, z) \leq d(x, z)$, so $\overline{d}(x, z) \leq \overline{d}(x, y) + \overline{d}(y, z)$, as required.
Now, we'll investigate the topology that $\overline{d}$ generates. First note that

$$\{B_d(x, \epsilon) \mid x \in X,\ \epsilon \leq 1\} = \{B_{\overline{d}}(x, \epsilon) \mid x \in X,\ \epsilon \leq 1\}.$$

Let τ_d be the metric topology generated by d, and let $U \in \tau_d$ be open. For any $x \in U$, there exists an $\epsilon > 0$ such that $B_d(x, \epsilon) \subseteq U$, by definition of an open set. We can further locate a $\delta < 1$ with $\delta < \epsilon$ so that $B_d(x, \delta) \subset B_d(x, \epsilon)$, but $B_{\overline{d}}(x, \delta) = B_d(x, \delta)$, and so we're done. □

Definition. Given two points $\vec{x} = (x_1, y_1)$ and $\vec{y} = (x_2, y_2)$, we define the *square metric* on $\mathbf{R}^2$ as:

$$\boxed{\rho(\vec{x}, \vec{y}) = \max\{|x_1 - y_1|, |x_2 - y_2|\}}.$$

Of course, this can be generalized to points in $\mathbf{R}^n$ in the following way:

$$\boxed{\rho(\vec{x}, \vec{y}) = \max\{|x_1 - y_1|, |x_2 - y_2|, \ldots, |x_n - y_n|\}},$$

where $\vec{x}$ and $\vec{y}$ are n-tuples.

Remark. The visualization of this metric is easiest in $\mathbf{R}^2$: given $\vec{x}$ and $\vec{y}$ lying in the plane, we can draw a rectangle with $\vec{x}$ and $\vec{y}$ lying on its opposite corners. What ρ gives us, then, is the length of the longer side of the rectangle.

Remark. Moreover, ρ generates the same metric topology on $\mathbf{R}^n$ as the standard Euclidean metric.

Example. So, using the square metric, what does a circle of radius 1 centered at the origin in $\mathbf{R}^2$ look like? It's no surprise that it looks like a square. By definition,

$$S^1(0, 1) = \{\vec{y} \in \mathbf{R}^2 \mid \operatorname{dist}(0, \vec{y}) = 1\}.$$

However, since we're measuring distance via the square metric, this means that

$$S^1(0, 1) = \{\vec{y} \in \mathbf{R}^2 \mid \rho(0, \vec{y}) = 1\} = S^1(0, 1) = \{(x, y) \in \mathbf{R}^2 \mid \max\{|x|, |y|\} = 1\},$$

which is the set of points lying along the square of side length 1.

Lemma. Let d and d' be metrics on X and τ and τ' are the respective metric topologies. We have that $\tau \subseteq \tau'$ if and only if, for every $x \in X$ and every $\epsilon > 0$, there exists a $\delta > 0$ such that $B_{d'}(x, \delta) \subset B_d(x, \epsilon)$.

Proof. Suppose $\tau \subseteq \tau'$. Let $B_d(x, \epsilon)$ be a basis element of τ. Since $B_d(x, \epsilon)$ is open in τ, it's also open in τ', by assumption. Now, since $B_d(x, \epsilon) \in \tau'$, there exists $B' \subseteq B_d(x, \epsilon)$, where B' is a basis element of τ', so that $x \in B'$. Assume that ϵ' is the radius of B' and let $r = d'(x, y)$ and $\delta = (\epsilon' - r)/2$. Taking some $z \in B_{d'}(x, (\epsilon' - r)/2)$, we have that

$$\begin{aligned} d(z, y) &\leq d(z, x) + d(x, y) \\ &< \left(\frac{\epsilon' - r}{2}\right) + r = \frac{\epsilon' + r}{2} \\ &< \frac{2\epsilon'}{2} = \epsilon', \end{aligned}$$

as required. So $B_{d'}(x, \delta) \subset B_d(x, \epsilon)$. The other direction is left as an exercise. □

Lemma. If (X, d) is a metric space and U is open in the metric topology, then, for any $x \in U$, there exists an $\epsilon > 0$ such that $B_d(x, \epsilon) \subset U$.

Definition. Given $\vec{x} = (x_1, \ldots, x_n)$ and $\vec{y} = (y_1, \ldots, y_n)$ in $\mathbf{R}^n$, we define the *taxicab metric* as

$$\boxed{d_t(\vec{x}, \vec{y}) = \sum_{i=1}^{n} |x_i - y_i|}.$$

Remark. Some intuition behind this metric: note that it sums up the distance between i^{th} coordinates, so what it's doing is it's first traversing the distance between x_1 and y_1, then between x_2 and y_2, and so on. So, it's only measuring the distance between $\vec{x}$ and $\vec{y}$ as the sum of the individual distances between each of their respective coordinates. Also note that, for $n = 1$, we recover just the Euclidean metric on $\mathbf{R}$.

Example. As always, this is most easily visualized in $\mathbf{R}^2$: given the point $\vec{x} = (0, 0)$ and $\vec{y} = (1, 2)$, applying the taxicab metric gives $d_t(\vec{x}, \vec{y}) = |1 - 0| + |2 - 0| = 3$. This doesn't agree with the classic "draw a right triangle" approach because that would give us $\sqrt{5}$. No, this is different because what's happening is that the taxicab metric is first telling us that, beginning at the origin, we need to move one unit along the x-axis, stop, turn 90° counterclockwise, and then move two units vertically to the point $(1, 2)$, giving a total of three units travelled.

Remark. This is why this metric is given such a name: because urban taxis usually have to navigate along a grid-like pattern of streets where they can only turn left or right, instead of just taking a straight line to where they need to go – that's dangerous and illegal.

9.3 Isometries

Definition. A continuous function of metric spaces $f : (X, d) \to (Y, d')$ is called an *isometry* if $d(x, y) = d'(f(x), f(y))$. That is, the mapping between metric spaces preserves the distances in each of the spaces.

9.4 Summary of Metrics

Let's recap the metrics we've seen on $\mathbf{R}^n$:

(a) EUCLIDEAN METRIC: $d(\vec{x}, \vec{y}) = \sqrt{\sum_{i=1}^{n} (y_i - x_i)^2}$ and $\overline{d} = \min\{d(\vec{x}, \vec{y}), 1\}$.

(b) SQUARE METRIC: $\rho(\vec{x}, \vec{y}) = \max\{|x_1 - y_1|, |x_2 - y_2|, \ldots, |x_n - y_n|\}$ and $\overline{\rho} = \min\{\rho(\vec{x}, \vec{y}), 1\}$.

(c) TAXICAB METRIC: $d_t(\vec{x}, \vec{y}) = \sum_{i=1}^{n} |x_i - y_i|$ and $\overline{d_t} = \min\{d_t(\vec{x}, \vec{y}), 1\}$.

Obviously, there are far more than these alone.

Remark. Note that all six of these metric generate the same metric topologies, but all of them induce different geometries. Of course, this sounds strange, but remember in the very beginning where we said that topology is *not* geometry? That's what's going on here. By declaring different means of measuring distances, we reveal quite a bit about the "geometric" structure of the metric space, but we actually reveal very little about the topological structure, since all of these metrics generate the same topology. This really pounds into us the notion that topology is *kind of* like geometry, but only slightly since it observes the "geometry" of a space in a theoretical, detached fashion. For example, we could ask: *what do the conic sections look like under each of these metrics?* For one, we know that a circle is abstractly defined as *all points equidistant from a center*. Well, in each of these metrics, that could mean many different things and present a very different geometric picture – a circle would look like a circle, a square, or a diamond in the Euclidean, square, or taxicab metrics, respectively.

10 The Quotient Topology

10.1 Quotient Maps

Definition. A surjective, continuous map of topological spaces is called a *quotient map* if we have the following property:

$$\boxed{U \subset Y \text{ open} \iff f^{-1}(U) \subset X \text{ open}}.$$

Remark. Note that the ($\Longrightarrow$) direction is simply the definition of continuity itself.

Remark. A surjective, continuous map that is *open* or *closed* is a quotient map.

10.2 Defining the Quotient Topology

Definition. Suppose X is a topological space, A is a set, and $f : X \to A$ is a function. Define the *quotient topology with respect to* f on A by saying that $B \subseteq A$ is open if $f^{-1}(B)$ is open.

Remark. We can verify that this is, in fact, a topology: $f^{-1}(\varnothing) = \varnothing$, and $\varnothing$ is open in A. Secondly, $f^{-1}(A) = X$ and A is open in A. Moreover, we have that

$$f^{-1}\left(\bigcup_{\alpha \in J} B_\alpha\right) = \bigcup_{\alpha \in J} f^{-1}(B_\alpha),$$

and

$$f^{-1}\left(\bigcap_{\alpha \in J} B_\alpha\right) = \bigcap_{\alpha \in J} f^{-1}(B_\alpha).$$

Note that, in the case of intersection, we have open-ness only if J is finite.

Proposition. If $f : X \to A$ is a surjective map of sets, then f is a quotient map when A is equipped with the quotient topology with respect to f. Moreover, the quotient topology is the *unique* topology on A that makes f a quotient map.

Proof. Take $U \subset A$ and assume that $f^{-1}(U) \subset X$ is open. By construction, $U \subset A$ open, so $f^{-1}(U) \subset X$ is open, so f is a quotient map. Now, we show uniqueness: let τ be a topology on A for which f is a quotient map. We have that $U \subset A$ is in τ if and only if $f^{-1}(U)$ is open in X if and only if U is open in the quotient topology. So τ is the quotient topology. $\square$

In retrospect, the preceding proof may not have even been worth taking the time to write out, since all it came down to was working by definition and exploiting the logical connectives between them.

Example. Let X be a set and $\sim$ an equivalence relation on X. For $x \in X$, denote the *equivalence class* of x as

$$\boxed{[x] = \{y \in X \mid x \sim y\}}.$$

There is a surjective function $p : X \to X/_\sim$, where $X/_\sim\ = \{[x] \mid x \in X\}$. So, if X is a space, then we can equip $X/_\sim$ with the quotient topology with respect to p. We'll call it the *quotient space* of X associated to the relation $\sim$.

Example. Let $X = [0, 1]$ be equipped with the subspace topology and define

$$[x] = \begin{cases} \{x\} & \text{if } x \neq 0,\ x \neq 1, \\ \{0, 1\} & \text{if } x = 0,\ x = 1. \end{cases}$$

We have that $X/_\sim$ is homeomorphic to the set $\{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 = 1\}$.

Remark. Any equivalence relation on X gives a *partition* of X, and vice-versa.

Example. Let $D = \{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 \leq 1\}$ and define

$$D^* = \{\{(x, y)\} \mid x^2 + y^2 < 1\} \cup \{\{(x, y) \mid x^2 + y^2 = 1\}\}.$$

There exists a surjective function $p : D \to D^*$ given by

$$(x, y) \mapsto \begin{cases} \{(x, y)\} & \text{if } x^2 + y^2 < 1, \\ \{(x, y) \mid x^2 + y^2 = 1\} & \text{if } x^2 + y^2 = 1. \end{cases}$$

Moreover, D^* with the quotient topology with respect to p is homeomorphic to the sphere of radius 1 in $\mathbf{R}^3$.

Theorem. Let $p : X \to Y$ be a quotient map of topological spaces and let Z be a space. Let $g : X \to Z$ be a function such that, if $p(x_1) = p(x_2)$ for $x_1, x_2 \in X$, then $g(x_1) = g(x_2)$. In other words, if p takes two points to the same place in Y, then g takes those two points to the same place in Z. More formally, we say that g is constant on sets of the form $p^{-1}(y)$ for $y \in Y$.
If the above is given, then there exists a function $f : Y \to Z$ such that $f \circ p = g$. Moreover, f is continuous if and only if g is continuous and f is a quotient map if and only if g is a quotient map.

Proof. For each $y \in Y$, $g(p^{-1}(y))$ is a single point in Z. Now, define $f(y) = g(p^{-1}(y))$. Clearly, f is a function and $f(p(x)) = g(p^{-1}(p(x))) = g(x)$. So the composition is satisfied. It's worth noting that $p^{-1}(p(x))$ is not necessarily a single point, but applying g, by assumption, ensures we get a single point. So, we've found f – we now need to verify that f satisfies the following properties:

- Continuity: Suppose that f is continuous. Since $g = f \circ g$ and p is continuous, so is g. Conversely, suppose that g is continuous. Let $V \subset Z$ be open, so $g^{-1}(V) \subset X$ is open, by assumption. But, $g^{-1}(V) = p^{-1}(f^{-1}(V))$. Because Y has the quotient topology (p is a quotient map), $p^{-1}(f^{-1}(V))$ is open if and only if $f^{-1}(V)$ is open, by definition. But $g^{-1}(V)$ is open and $g^{-1} = p^{-1}(f^{-1}(V))$ so $f^{-1}(V)$ is open, so f is continuous.
- Quotient map: Suppose that f is a quotient map. We have that $g = f \circ p$, and p is a quotient map by assumption, and so is f, thus g is a quotient map. Conversely, suppose that g is a quotient map. Since g is surjective and $g = f \circ p$, we have that f must also be surjective. Take $V \subset Z$ and suppose that $f^{-1}(V)$ is open. Since $f^{-1}(V)$ is open, $p^{-1}(f^{-1}(V)) \subset X$ is open. Moreover, $p^{-1}(f^{-1}(V)) = g^{-1}(V)$, so $g^{-1}(V)$ is open and thus V is open since g is a quotient map.

□

Corollary. Let $g : X \to Z$ be a surjective, continuous map which is not necessarily a quotient map. Define

$$X^* := \{g^{-1}(z) \mid z \in Z\},$$

which is a partition of X. Give X^* the quotient topology, and we have that:

(a) There exists a bijective, continuous map $f : X^* \to Z$ which is a homeomorphism if and only if g is a quotient map.

(b) If Z is Hausdorff, then so is X^*.

Proof. (a) By the preceding theorem, since $p : X \to X^*$ is a quotient map, by construction, we get a map $f : X^* \to Z$ which is continuous. Moreover, f is a bijection. Once again, by the theorem, if g is a quotient map, then so is f. So $A \subset Z$ is open if and only if $f^{-1}(A)$ is open, so f is a homeomorphism.

(b) Suppose Z is Hausdorff. Let $y \neq y'$ be points in X^*. Since f is a bijection, $f(y)$ and $f(y')$ are distinct in Z. So since Z is Hausdorff, there exist open $U \ni f(y)$ and $V \ni f(y')$ so that $U \cap V = \varnothing$. Since f is continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are open and since f is a bijection, $f^{-1}(U) \cap f^{-1}(V) = \varnothing$.

□

Example. Recall the map $g : [0, 1] \to S^1$, where $t \mapsto (\cos(2\pi t), \sin(2\pi t))$. From this we automatically get a map $f : [0, 1]/_{0\sim 1} \to S^1$. One can show that g is a quotient map, and it turns out that f is a homeomorphism.

Example. Take $\mathbf{R}^2/_\sim$ to be defined by $(x, y) \sim (x + n, y + m)$ for $n, m \in \mathbf{Z}$. What this does is slices the plane into an integer lattice and then mirrors points from each section. As a result, $\mathbf{R}^2$ can be identified with the torus $\mathbf{T}$ via this lattice.

11 Connectedness

11.1 Connected Spaces

Definition. A space X is *separated* if $\boxed{X = U \cup V}$ for open sets U and V where $\boxed{U \neq \varnothing}$, $\boxed{V \neq \varnothing}$, and $\boxed{U \cap V = \varnothing}$. In this case, U and V are a *separation* of X, or X can be cut into two nonempty sets which have empty intersection.

Definition. A space X is *connected* if it has no separation.

Theorem. If $f : X \to Y$ is a homeomprhism, then X is connected if and only if Y is connected.

Proof. Let $X = U \cup V$ be a separation of X. This holds if and only if U and V are disjoint, nonempty, open sets in X. Since f is a homeomorphism, $f(U)$ and $f(V)$ are a separation of Y. So X has a separation if and only if Y does. $\square$

Remark. If $Y \subset X$ is a subspace, then showing that Y is connected is "easier" than showing that X is connected.

Definition. If $Y \subset X$ is a subspace, then we say that Y is a *separated subspace* of X if there are disjoint, nonempty subsets of Y (call them A and B) such that $Y = A \cup B$, and $A \cup \overline{B} = \overline{A} \cup B = \varnothing$.

Remark. In this way, we aren't explicitly requiring that the subsets be open, as before.

Theorem. A subspace $Y \subset X$ is connected (in the subspace topology) if it is not a separated subspace.

Proof. exercise. $\square$

Lemma. Let C and D form a separation of X. If Y is a connected subspace of X, then either $Y \subset C$ or $Y \subset D$.

Proof. Consider the inclusion map $\iota : Y \hookrightarrow X$. Recall that this map is continuous. Moreover, note that $Y = \iota^{-1}(C) \cup \iota^{-1}(D)$. Suppose that $y_0 \neq y_1$ (in Y) where $y_0 \in C$ and $y_1 \in D$. We have that $\iota^{-1}(C) \cup \iota^{-1}(D)$ is a separation of Y. However, Y is connected by assumption, so either $Y \subset C$ or $Y \subset D$, as required. $\square$

Lemma. Let $\{Y_\alpha\}_{\alpha \in J}$ be a collection of connected subsets of X. Further, suppose that

$$\bigcap_{\alpha \in J} Y_\alpha \neq \varnothing.$$

We have that $\displaystyle\bigcup_{\alpha \in J} Y_\alpha$ is connected.

Proof. Let $p \in \displaystyle\bigcap_{\alpha \in J} Y_\alpha$ and suppose that $C \cup D \in \displaystyle\bigcap_{\alpha \in J} Y_\alpha$ is a separation. Well, we know that $p \in Y_\alpha$ for some $\alpha \in J$ and that Y_α is connected. By the preceding lemma, we have, then, that $pinC$ or $p \in D$. Without loss of generality, assume that $p \in C$. However, we also have that $p \in Y_\alpha$ for all α. So, $Y_\alpha \cap C \neq \varnothing$ for all α. Moreover, $Y_\alpha \subset C$ for all α. That means, however, that $D \cap \left(\displaystyle\bigcup_{\alpha \in J} Y_\alpha \right) = \varnothing$ since C and D are disjoint. Thus, $C \cup D$ is not a separation, as originally assumed. Thus, $\displaystyle\bigcup_{\alpha \in J} Y_\alpha$ is connected. $\square$

Example. $\mathbf{R}^n$ is connected. Let $\mathbf{R}^n = U \cup V$ for some nonempty, open, disjoint U and V. Let $\vec{u} \in U$ and $\vec{v} \in V$. We then have that the straight-line segment $\ell : [0,1] \to \mathbf{R}^n$ given by $t \mapsto t\vec{u} + (1-t)\vec{v}$ is continuous. So, $\ell^{-1}(U) \cup \ell^{-1}(V) = [0,1]$. However, $\mathbf{R}$ itself, as well as any intervals in $\mathbf{R}$, are connected. So the fact that we've found a separation of $[0,1]$ is a contradiction – thus there exists no separation of $\mathbf{R}^n$.

Remark. Since the definition of connectedness itself is a negative definition (i.e. connected $\iff \neg$(separated)), all of these proofs of connectedness proceed in the same fashion: we assume that there is, in fact, a separation, and then derive a contradiction.

Example. $\mathbf{Q}$ is not connected. Take some set $A \subset \mathbf{Q}$ and choose come $p, q \in A$ so that $p < q$. Then there exists an $r \in \mathbf{R} \setminus \mathbf{Q}$ such that $p < r < q$ and so $(-\infty, r) \cup (r, \infty)$ forms a separation of $\mathbf{Q}$.

Definition. A space X in which the only connected subspaces of X are singleton sets is called *totally disconnected*.

Remark. Some examples of totally disconnected spaces are $\mathbf{R} \setminus \mathbf{Q}$ as well as any set equipped with the discrete topology.

Example. Let $A \subset \mathbf{R}$ be given by $A = [-1, 0) \cup (0, 1]$. Note that $\overline{[-1, 0)} \cap \overline{(0, 1]} = \{0\}$, but this isn't really enough for us to conclude connectedness with respect to subspaces. This is because we still have that $\overline{[-1, 0)} \cap (0, 1] = [-1, 0) \cap \overline{(0, 1]} = \varnothing$, so A is still separated.

Theorem. Let A be a connected subspace of X. If $A \subset B \subset \overline{A}$, then B is also connected.

Proof. Suppose that $B = C \cup D$ is a separation of B. From before, either $A \subset C$ or $A \subset D$. Without loss of generality, suppose that $A \subset C$. From this, it follows that $\overline{A} \subset \overline{C}$ (this was never actually formally stated or proven, but it's true). Since $C \cup D$ is a separation of B, it must follow from before that $\overline{C} \cap D = \varnothing$. So, since $B \subset \overline{A} \subset \overline{C}$ and $\overline{C} \cap D = \varnothing$, we have that $B \cap D = \varnothing$. As was expected, this is a contradiction, so we're done. □

Theorem. The image of a connected space (under a continuous map) is connected.

Proof. Let $f : X \to Y$ be continuous and assume that X is connected. Let $Z = \operatorname{im}(X)$. By restriction of range, we have that the map $g : X \to Z$, where $x \mapsto f(x)$, is continuous and (obviously) surjective. Now, suppose that Z can be separated as $Z = C \cup D$. We can verify that $g^{-1}(C)$ and $g^{-1}(D)$...

(i) ...are nonempty, on account of g being surjective,

(ii) ...are open, as g is continuous,

(iii) ...are disjoint.

So $X = g^{-1}(C) \cup g^{-1}(D)$. Once again, this gives a contradiction, since X was assumed connected, so Z can't be separated. □

So, from this immediately follows:

Corollary. If $f : X \to Y$ is a surjective, continuous map, then Y is connected if X is connected.

Theorem. A finite product of connected spaces is connected.

Proof. As a reduction, we'll just prove this for products of two spaces that take the form $X \times Y$. From this, it will follow inductively that higher-order products are also connected. So, pick some point $(a, b) \in X \times Y$. Now, take the sets $\{x\} \times Y$ and $X \times \{y\}$. Note that $\{x\} \times Y \cong Y$ for all $x \in X$ and $X \times \{y\} \cong X$ for all $y \in Y$. Further define

$$\mathcal{T}_x = (X \times \{b\}) \cup (\{x\} \times Y) \subseteq X \times Y.$$

We see that $\mathcal{T}_x$ is connected since it's a union of connected sets (which are, in their own right, connected because they're homeomorphic to connected spaces). We then have that

$$X \times Y = \bigcup_{x \in X} \mathcal{T}_x,$$

where $(a, b) \in \bigcap_{x \in X} \mathcal{T}_x$. So, from before we have that $X \times Y$ is connected, and we are done. □

Theorem. Arbitrary products of connected spaces are connected: if $\{X_\alpha\}_{\alpha \in J}$ is a collection of connected spaces, then $\prod_{\alpha \in J} X_\alpha$ is connected (in the product topology).

Proposition. $\mathbf{R}^\omega = \prod_{n \in \mathbf{N}} \mathbf{R}$ is connected (we'll take as a fact that $\mathbf{R}$ is connected).

Proof. Here's the idea: we'll show that the set

$$\mathbf{R}^\infty = \{\{x_n\}_{n \in \mathbf{N}} \mid \text{there exists an } N \text{ such that } x_n = 0 \text{ for all } n > N\}$$

(i.e., the set of sequences in $\mathbf{R}$ that eventually exhaust themselves and become zero) is connected. From there, we'll use the fact that $\overline{\mathbf{R}^\infty} = \mathbf{R}^\omega$, and we'll be done.
$\rightsquigarrow$ So, to show that $\mathbf{R}^\infty$ is connected, we'll let $\tilde{\mathbf{R}^n} \subset \mathbf{R}^\infty$ be the collection of sequences $\{x_i\}_{i \in \mathbf{N}}$ such that $x_i = 0$ if $i > n$. Note that

$$\mathbf{R}^\infty = \bigcup_{n \in \mathbf{N}} \tilde{\mathbf{R}^n},$$

and that

$$\{0, 0, \ldots, 0\} \in \bigcap_{n \in \mathbf{N}} \tilde{\mathbf{R}^n}.$$

So, by a previous result, $\mathbf{R}^\infty$ is connected since it's the union of connected subspaces that share a point. So, $\overline{\mathbf{R}^\infty} = \mathbf{R}^\omega$ and the result follows. $\square$

Theorem. Convex subsets of $\mathbf{R}$ are connected.

Proof. First, let's lay down some groundwork from analysis that will come in handy:

(i) Given x and y in $\mathbf{R}$ such that $x < y$, there exists a $z \in \mathbf{R}$ so that $z \in (x, y)$.

(ii) $\mathbf{R}$ has the least upper bound property (for any bounded $A \subset \mathbf{R}$, $\sup(A)$ exists and is finite).

Let $Y \subseteq \mathbf{R}$ be convex and suppose that $Y = A \cup B$ is a separation, defined in the usual way. Since A and B are nonempty by assumption, take some $a \in A$ and $b \in B$ so that $a \neq b$ and, without loss of generality, suppose $a < b$. From this, it follows that $[a, b] \subseteq Y$, by the fact that Y is convex. Moreover, we'll suppose that $[a, b]$ can be separated in the following way:

$$[a, b] = A_0 \cup B_0 = (A \cap [a, b]) \cup (B \cap [a, b]).$$

We have that A_0 is bounded above by b, so A_0 has a least upper bound. We'll formally define it as $c = \sup(A_0)$. Our goal is to show that $c \notin A_0$ and $c \notin B_0$, in which case $A_0 \cup B_0 \neq [a, b]$ and we'll have reached a contradiction. Let's proceed case-by-case:

- Case 1: Suppose that $c \in B_0$. Since B_0 is open in $[a, b]$, there exists an interval $(d, c] \subset B_0$. Either $c = b$, in which case $(d, c]$ is open in B_0 or $c \neq b$, in which case there exists an interval $(d, d') \subset B_0$ containing c so then $(d, c] \subset (d, d') \subset B_0$. Let's go into more depth: if $c = b$, then d is a *smaller* upper bound for A_0. This isn't immediately clear, so suppose the contrary – there exists some $a_0 > d$, implying that $a_0 \in (d, c] \subset B_0$, a contradiction. Alternatively, if $c \neq b$, then d is, once again, a smaller upper bound on A_0. This is because if there exists an $a_0 \in A_0$ so $a_0 > d$, then one of two things is possible: either $a_0 \in (d, c]$ or $a_0 \in (c, b)$. These are both contradictions; the former because $(d, c] \subset B_0$, and the latter because $a_0 \leq c = \sup(A_0)$. So, $c \notin B_0$ since, if it were, it wouldn't be an upper bound.

- Case 2: Suppose that $c \in A_0$. From this, $c \neq b$, so either $c = a$ or $c \in (a, b)$. Since A_0 is open, c is contained in some interval $[c, e) \subset A_0$. Further, there exists a $z \in \mathbf{R}$ so that $z \in (c, e)$. This, however, tells us that $z \in A_0$ and $z > c$, so c isn't an upper bound.

So, $c \notin A_0$ and $c \notin B_0$, giving the contradiction we sought and ending the proof. $\square$

11.2 Path-Connected Spaces

Definition. Given a space X and points $x, y \in X$, we define a *path* from x to y in X as a continuous function $\boxed{\varphi : [a,b] \to X}$ such that $\boxed{\varphi(a) = x}$ and $\boxed{\varphi(b) = y}$.

Remark. The use of prepositions brings up some subtleties one should beware of: note that our path is defined *from x to y*. That is, it must start at x and end at y. If we're simply defining a path *between* x and y, then it doesn't matter if the path goes from x to y, or vice-versa.

Definition. A space X is *path-connected* if, for all x and y in X, there exists a path between x and y.

Proposition. Every *path-connected* space is *connected*, but the converse isn't true. That is, we can find connected spaces that aren't path-connected.

Example. As an illustration of the above proposition (and one of the more tired examples in topology), take the subset of the plane defined as

$$S = \left\{ \left(x, \sin\left(\frac{1}{x}\right)\right) \in \mathbf{R}^2 \;\middle|\; x \in (0,1] \right\},$$

often called the *topologist's sine curve*. We know, from before, that $(0,1]$ is connected and that its image under a continuous map is connected, so S is connected. From this, $\overline{S}$ is connected. However, $\overline{S}$ is not path-connected. Why? Consider the following argument:

Proof. The idea is that we'll attempt to construct a path from the origin to a point in S. Suppose that such a path exists; then there is a continuous map $\tilde{f} : [a,b] \to \overline{S}$ such that $\tilde{f}(a) = (0,0)$ and $\tilde{f}(b) \in S$. Let I be the segment of the y-axis given by $-1 \le y \le 1$. Note that $\tilde{f}^{-1}(I)$ is closed in $[a,b]$, so there is a maximum,[4] called c. We can further construct a path $f : [c,b] \to \overline{S}$ such that $f(c) \in I$ and $f(t) \in S$ for all $t > c$. Since $[0,1] \cong [a,b]$, we can assume that such an f can be defined as $f : [0,1] \to \overline{S}$, where $f(0) \in I$ and $f(t) \in S$ for all $t > 0$. Let $f(t) = (x(t), y(t))$, where

$$x(t) = \begin{cases} 0 & \text{if } t = 0 \\ x(t) > 0 & \text{if } t > 0 \end{cases}$$

and

$$y(t) = \sin\left(\frac{1}{x(t)}\right), \qquad \text{for } t > 0.$$

Now, our goal is to construct a sequence that converges in $[0,1]$ but doesn't converge in $\overline{S}$ – this will negate the continuity of f and give us a contradiction. So, let $(t_n)_{n=1}^{\infty}$ in $[0,1]$ be defined in such a way that $\lim_{n\to\infty} t_n = 0$, but $(f(t_n))$ doesn't converge. Let $n \in \mathbf{Z}_+$ be given. Since $1/n > 0$, we have that $x(1/n) > 0$. So we can find a $u_n \in (0, x(1/n))$ such that

$$\sin\left(\frac{1}{u_n}\right) = (-1)^n.$$

Since $x(0) = 0$ and $x(1/n) > u_n$, by the Intermediate Value Theorem, there exists a $t_n \in (0, 1/n)$ such that $x(t_n) = u_n$, and so $f(t_n) = (u_n, (-1)^n)$. Finally, we see that $t_n \to 0$ since $t_n < 1/n$ for all n, and $(f(t_n))$ doesn't converge, by merit of how it's defined. This contradicts the assumed continuity of f, and so we are done. □

[4] This is kind of a leap; saying there exists a maximum evokes some notions of compactness. However, this hasn't been covered, so it still follows, at least logically, that the preimage of I has a maximum in $[a,b]$.

11.3 Connected Components

When a space X isn't connected, it can be separated. Naturally, such a separation "cuts" X into constituent parts. If we zoom into any one of these "pieces" of X, are they, at least, connected? If they aren't, can we continue zooming in enough so that we'll find a piece that is connected?

Definition. Define the following equivalence relation on X:

$$\boxed{x \sim y \text{ if there exists a connected subspace of } X \text{ containing } x \text{ and } y.}$$

The equivalence classes of this relation are the *connected components* or, simply, *components* of X.

Remark. $x \sim y$ is an equivalence relation: reflexivity and symmetry are trivial; for transitivity, suppose $x \sim y$ and $y \sim z$. There exists a connected subspace A so that $x, y \in A$ and B so that $y, z \in B$. Thus, $y \in A \cap B$, so $A \cup B$ is connected and finally $x, z \in A \cup B$.

Theorem. The components of X are connected, disjoint subspaces of X whose union is X. Moreover, if $Y \subset X$ is a connected subspace, then Y is contained in *exactly one* component of X.

Proof. exercise. □

Example. The set $\mathrm{O}(n) \subset M_{n\times n}(\mathbf{R})$ is known as the set of $n \times n$ *orthogonal matrices*, which are exactly the matrices M with $\det(M) = \pm 1$. Given $\mathrm{O}(n)$, we get a linear transformation $L_M : \mathbf{R}^n \to \mathbf{R}^n$ that preserves *volumes*[5]. Also, note that $\mathrm{O}(n)$ preserves the unit sphere.

Proposition. The set $\mathrm{O}(n)$ has two connected components.

Remark. For the algebraically-inclined, it's worth noting that $\mathrm{O}(n)$ satisfies the axioms of a group. In fact, it's a Lie group.

Theorem. The connected components of a space X are *always* closed in X.

Proof. Let $Y \subseteq X$ be a connected component. From this, we have that $\overline{Y}$ is connected, and $\overline{Y} \cap Y \neq \varnothing$, so $\overline{Y} \subseteq Y$. Hence, $Y = \overline{Y}$, so Y is closed. □

Remark. In the case that X has a *finite* number of components, each component is also open.

Proof. We typically don't follow remarks with proofs, but this one deserves an argument: given that Y is closed, as before, we have that $X \setminus Y$ is open. However, $X \setminus Y$ constitutes "the rest" of the connected components in X, so it's a finite union of closed sets, so $X \setminus Y$ is also closed. Of course, this tells us that Y is also open[6]. □

Remark. The preceding remark isn't necessarily true for path-connected components.

Example. $\mathbf{Q}$ has countably-many connected components, all of which are closed and none of which are open.

Definition. For a topological space X, let $\pi_0(X)$ denote the cardinality of the set of connected components of X. That is, $\pi_0(X) = |X/_\sim|$.

Example. $\pi_0(\mathbf{Q}) = |\mathbf{Q}|$.

[5] The most tangible example of this is in multivariable calculus when one performs a change of variables in double- or triple-integrals. This change of variables is often a linear transformation. When re-writing the volume (or area) element dV (or dA), we multiply by the absolute value of the *Jacobian* of the transformation, which is a 3×3 (or 2×2) determinant that serves as a "correction factor." In the case that such a determinant is ± 1, then its absolute value is simply 1, so we end up multiplying by 1, thus preserving the volume or area.

[6] Topologists often run into sets that are simultaneously open and closed, and have given them the informal name *clopen sets*.

11.4 Local Connectedness

Definition. A space X is called *locally connected at* $x_0 \in X$ if, for every neighborhood $U \ni x_0$, there exists a neighborhood $V \ni x_0$ such that $V \subset U$ and V is connected. Furthermore, we say that X is *locally connected* if it's locally connected at *every* $x_0 \in X$[7].

Definition. If we replace the word "connected" in the above definition with "path-connected," then X is called *locally path-connected.*

Example. $\mathbf{R}^n$, as well as any subset of $\mathbf{R}^n$, is locally path-connected.

Example. $\mathbf{Q}$ is not locally connected, nor is any set equipped with the discrete topology.

Theorem. A space X is (path) connected if and only if, for every open $U \subset X$, each (path) connected component of U is open in X.

Proof. We'll prove the theorem in the *connected* case only. Suppose that X is locally connected. Let $U \subseteq X$ be open, and let C be a component of U (with respect to the subspace topology). If $x \in C$, then there exists a neighborhood $V \ni x$ such that $V \subseteq U$, since X is locally connected. Since V is connected, we also have that $V \subseteq C$. So, what we've done is found that, for any $x \in C$, there exists an open set containing x which is, itself, further contained in C, so C is open in X. Conversely, let $U \subseteq X$ be open, and assume that every connected component of U is open in X. Take $x \in X$ so that $x \in U$. From this, we have that $x \in C \subseteq U$, for some component of U. By assumption, C is open. So, X is locally connected, and we are done. $\square$

Theorem. If X is a space, then each path-connected component of X is contained in a connected component of X.

Proof. exercise. $\square$

Remark. Spaces that are not locally connected (resp. not locally path-connected) are known as *exotic.*

12 Compactness

If nothing else, topology is largely concerned with classifying topological spaces and understanding restrictions on them that make them "well-behaved." One of the more notable examples we've touched on is Hausdorff spaces, and how a space being "non-Hausdorff"[8] often causes it to become ill-behaved. Another measure of how nicely-behaved spaces are is *compactness.* First, let's square away some preliminary definitions:

Definition. Let X be a space. A collection $\mathcal{U} = \{U_\alpha\}_{\alpha \in J}$, where J is an arbitrary indexing set, is called a *cover* of X if

$$\boxed{X \subseteq \bigcup_{\alpha \in J} U_\alpha}.$$

Moreover, $\mathcal{U}$ is an *open cover* of X if U_α is open for all $\alpha \in J$.

Remark. By default, we'll be working with open covers unless otherwise noted.

[7] We're defining local connectedness in more-or-less a pointwise fashion. This shan't be foreign, though, for in calculus we define continuity the same way: $f(x)$ is *continuous* if it's continuous at every point in its domain.

[8] This isn't to be confused with *anti-Hausdorff*. Just because a space isn't Hausdorff doesn't mean it qualifies as anti-Hausdorff.

Example. Consider the real line and let $\mathcal{U} = \{(-\alpha, \alpha) \mid \alpha \in \mathbf{Z}_+\}$. We see that $\mathcal{U}$ is an open cover of $\mathbf{R}$ since

$$\mathbf{R} \subseteq \bigcup_{\alpha \in \mathbf{Z}_+} (-\alpha, \alpha).$$

Example. Consider the interval $(0, 1)$ and let $\mathcal{U} = \{(1/\alpha, 1) \mid \alpha \in \mathbf{Z}_+\}$. We see that $\mathcal{U}$ is an open cover of $(0, 1)$ since

$$(0, 1) \subseteq \bigcup_{\alpha \in \mathbf{Z}_+} \left(\frac{1}{\alpha}, 1\right).$$

Definition. If $I \subseteq J$ and $\mathcal{V} = \{U_\alpha\}_{\alpha \in I}$ provides an open cover of X, then $\mathcal{V}$ is called a *subcover* of X.

Now, we can properly define compactness:

Definition. A space X is called *compact* if, for every open cover of X, there exists a *finite* subcover.

Example. The set $S = \{1, 2, 3\}$ is compact. Let $\mathcal{U}$ be an open cover of S. Since S has an open cover, we know, in particular, that there are open sets in $\mathcal{U}$ that contain 1, 2, and 3, respectively. Let's label these sets U_{α_1}, U_{α_2}, and U_{α_3} so that $1 \in U_{\alpha_1}$, $2 \in U_{\alpha_2}$, $3 \in U_{\alpha_3}$. Now, note that the collection $\mathcal{V} = \{U_{\alpha_1}, U_{\alpha_2}, U_{\alpha_3}\}$ is not only finite, but is a subset of $\mathcal{U}$. Now, it remains for us to verify that

$$\{1, 2, 3\} \subseteq (U_{\alpha_1} \cup U_{\alpha_2} \cup U_{\alpha_3}).$$

Thus, $\mathcal{V}$ is a finite subcover of S and as $\mathcal{U}$ was arbitrary, we see that S is compact.

Definition. A space X is called a *topological n-manifold* if X admits an open subcover $\{U_\alpha\}_{\alpha \in I}$ such that $U_\alpha \cong \mathbf{R}^n$ for a fixed n. We're assuming that each U_α has the subspace topology.

Theorem. A subspace of $\mathbf{R}^n$ is compact if and only if it is closed and bounded (with respect to the Euclidean metric).

Proposition. If X is compact and $f : X \to Y$ is a homeomorphism, then the space Y is compact.

Remark. From this, we glean that compactness is *homeomorphism invariant*. This is the third property we've seen that is homeomorphism invariant; the other two have been Hausdorff-ness and connectedness.

Proposition. If X is any space such that $|X| < \infty$, then X is compact.

Theorem. Let Y be a subspace of X. We have that Y is compact if and only if every collection of open sets in X whose union contains Y has a finite sub-collection whose union also contains Y.

Proof. exercise. □

Theorem. Every closed subspace of a compact space is compact.

Proof. Let X be compact and let Y be a closed subspace of X. Let $\mathcal{U}$ be an open cover of Y via open sets of X. We have that $\mathcal{U} \cup \{X \smallsetminus Y\}$ is an open cover of X. Since X is compact, there exists a finite subcover $\tilde{\mathcal{V}} = \mathcal{V} \cup \{X \smallsetminus Y\}$, where $\mathcal{V}$ is a finite subset of $\mathcal{U}$. But then this means that $\mathcal{V}$ is a finite collection of open sets in X whose union contains Y. So, Y is compact, as required. □

13 Homotopy Equivalence

Definition. Let X and Y be topological spaces. If $f : X \to Y$ and $g : X \to Y$ are both continuous maps, then it is said that f is *homotopic* to g if there exists a continuous map $h : X \times I \to Y$ so that $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$ for each $x \in X$. By convention, $I = [0, 1]$. Moreover, the map h is called a *homotopy* between f and g.

Remark. If g is a constant map, then f is *nulhomotopic*.

Definition. Let X and Y be topological spaces, and let $f : X \to Y$ and $g : Y \to X$ be continuous maps. If $\boxed{g \circ f : X \to X}$ is homotopic to the identity map of X ($x \mapsto x$ for all $x \in X$), and the map $\boxed{f \circ g : Y \to Y}$ is homotopic to the identity map of Y ($y \mapsto y$ for all $y \in Y$), then f and g are called *homotopy equivalences*, whence each is called a *homotopy inverse* of the other.